Applied Quantitative Logistics

Integer Programming and Formulation

Majid Sohrabi

National Research University Higher School of Economics

January 14, 2026

Operations Research: The Science of Better

Operations research encompasses a wide range of advanced analytical methods applied to help solve challenging decision-making problems

One of its important areas of applications deals with the management and efficient use of scarce resources to increase productivity

Most optimization problem arising in such applications are naturally stated as the minimization (or maximization) of a function of many variables over a discrete structure

Integer programming has emerged as one of the most important tools to tackle such discrete optimization problems

Applications in Discrete Optimization

Several areas within **Industrial Engineering**:

- Design and analysis of manufacturing systems
- Production planning and inventory control
- Machine and personnel scheduling
- Transportation and logistics
- Quality control
- Health care

Applications in Discrete Optimization

Several areas within **Industrial Engineering**:

- Design and analysis of manufacturing systems
- Production planning and inventory control
- Machine and personnel scheduling
- Transportation and logistics
- Quality control
- Health care

There are other fields of knowledge in which it can be employed:

- **Computer Science**
- Economics
- Statistics
- Mathematics
- Computational Biology
- Sports
- …

Scope of Discrete Optimization

Some examples in which Discrete Optimization is used are:

- Locating a warehouse or distributions center to deliver materials to reduce transportation costs
- Scheduling: personnel, machines, project tasks
- Managing the flow of raw material and products in a supply chain
- Designing the layout of a factory for efficient flow of materials
- Constructing a telecommunications network
- Determining the routes of snow plowing vehicles to minimize the time required to clear snow and ice from the streets
- Designing the layout of a computer chip to reduce manufacturing time
- Optimizing a portfolio of investments

Mathematical Modeling in Optimization

The modeling and analysis of models evolves through several stages:

1. **Problem definition:** study of the system; data collection; identification of specific problem that needs to be analyzed.
2. **Devising a mathematical formulation:** construction of a model that satisfactorily represents the system while keeping the model tractable.
3. **Solving the formulation:** use/develop a proper technique that exploits any special structure of the model.
4. **Testing, analysis, and restructuring the model:** examination of model solution and its sensitivity to system parameters; study of various what-if types of scenarios; enrich the model further or simplify the model.
5. **Implementation:** development of a decision support system to aid in the decision-making process.

The interested reader is referred to: Brown and Rosenthal, *Optimization Tradecraft: Hard-Won Insights from Real-Worls Decision Support*, Interfaces, 38(5), 2008.

What is an Integer Program?

Consider the following **linear program**:

$$
\text{(LP)} \quad \begin{aligned} \text{minimize} \quad & cx \\ \text{subject to} \quad & \boldsymbol{A}x \le b \\ & x \ge 0 \end{aligned}
$$

where $\boldsymbol{A}$ is an $m \times m$ matrix, $\boldsymbol{c}$ is a n-vector, $\boldsymbol{b}$ is a m-vector, and $\boldsymbol{x}$ is a n-vector of decision variables.

What is an Integer Program?

If we add the restriction that variables must take integer values we obtain the (linear) **integer program**:

(LP) minimize cx

subject to $\boldsymbol{A}x \leq b$

$x \geq 0$ and integer

What is an Integer Program?

If we have a mixture of continuous and integer variables we obtain (linear) **mixed integer program**:

$$
\begin{array}{lll}
\text{(MIP)} & \text{minimize} & cx + \mathrm{hy} \\
 & \text{subject to} & \boldsymbol{A}x + \boldsymbol{G}y \leq b \\
 & & y \geq 0 \\
 & & x \geq 0 \text{ and integer}
\end{array}
$$

What is an Integer Program?

If all the variables are restricted to 0-1 values we obtain the (linear) **binary integer program**:

$$
\begin{array}{lll}
\text{(BIP)} & \text{minimize} & cx \\
 & \text{subject to} & \boldsymbol{A}x \leq b \\
 & & x \in \{0, 1\}^n
\end{array}
$$

What is a Combinatorial Optimization Problem?

Given:

- A finite set $N = \{1, \ldots, N\}$
- Weights c_j for each $j \in N$
- A set F of feasible subsets of N

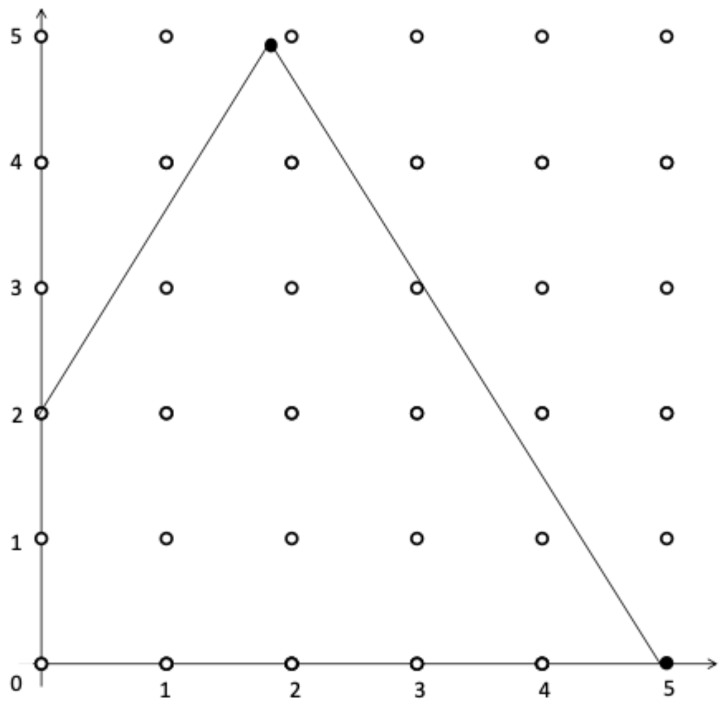
The problem of finding a minimum weight feasible subset is a Combinatorial Optimization Problem:

$$\text{(COP)} \qquad \min_{S \subseteq N} \left\{ \sum_{j \in S} c_j : S \in \mathcal{F} \right\}$$

Solving Integer Programs

Consider the following integer program:

$$
\begin{array}{ll}
\text{Maximize} & 1.00x_1 + 0.6x_2 \\
\text{Subject to} & 50x_1 + 31x_2 \le 250 \\
 & 3x_1 - 2x_2 \le -4 \\
 & x_1, x_2 \ge 0 \text{ and integer}
\end{array}
$$



Optimal solution of linear relaxation:

(376/193, 950/193) = (1.948, 4.92)

Optimal solution of integer program: (5, 0)

Mathematical Modeling in Optimization

Translating a problem description into a formulation should be done systematically.

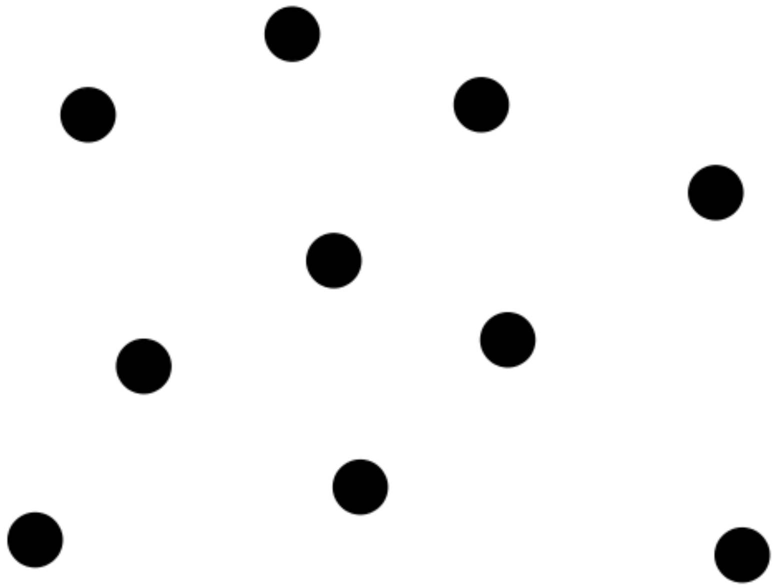
1. Define what appears to be the necessary decision variables
2. Use these variables to define a set of constraints so that the feasible points correspond to the feasible solutions of the problem
3. Use these variables to define the objective function

If difficulties arise, define an additional or alternative set of variables and iterate.

Important: A clear distinction should be made between the data of the problem and the decision variables used in the model.

Note: This modeling recipe has been provided by Lawrence Wolsey in Integer Programming, Wiley, 1998.

Example: The Traveling Salesman Problem



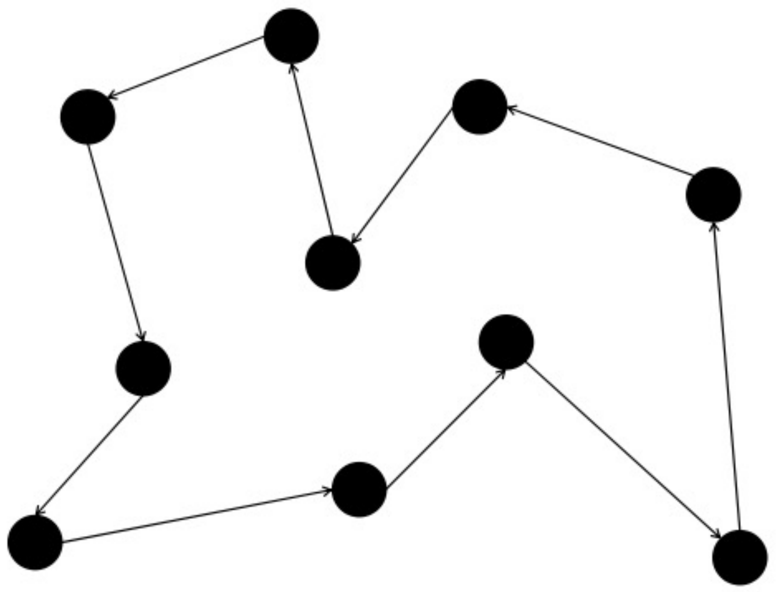
Input:

- N: set of cities ($|N| = n$)
- c_{ij}: travel time between i and j

The traveling salesman problem (TSP):

- A salesman must visit each city exactly once and then return to his starting point
- Objective: Minimize the total travel time of the tour

Example: The Traveling Salesman Problem



Input:

- N: set of cities ($|N| = n$)
- c_{ij}: travel time between i and j

The traveling salesman problem (TSP):

- A salesman must visit each city exactly once and then return to his starting point
- Objective: Minimize the total travel time of the tour

Example: The Traveling Salesman Problem

Routing variables:

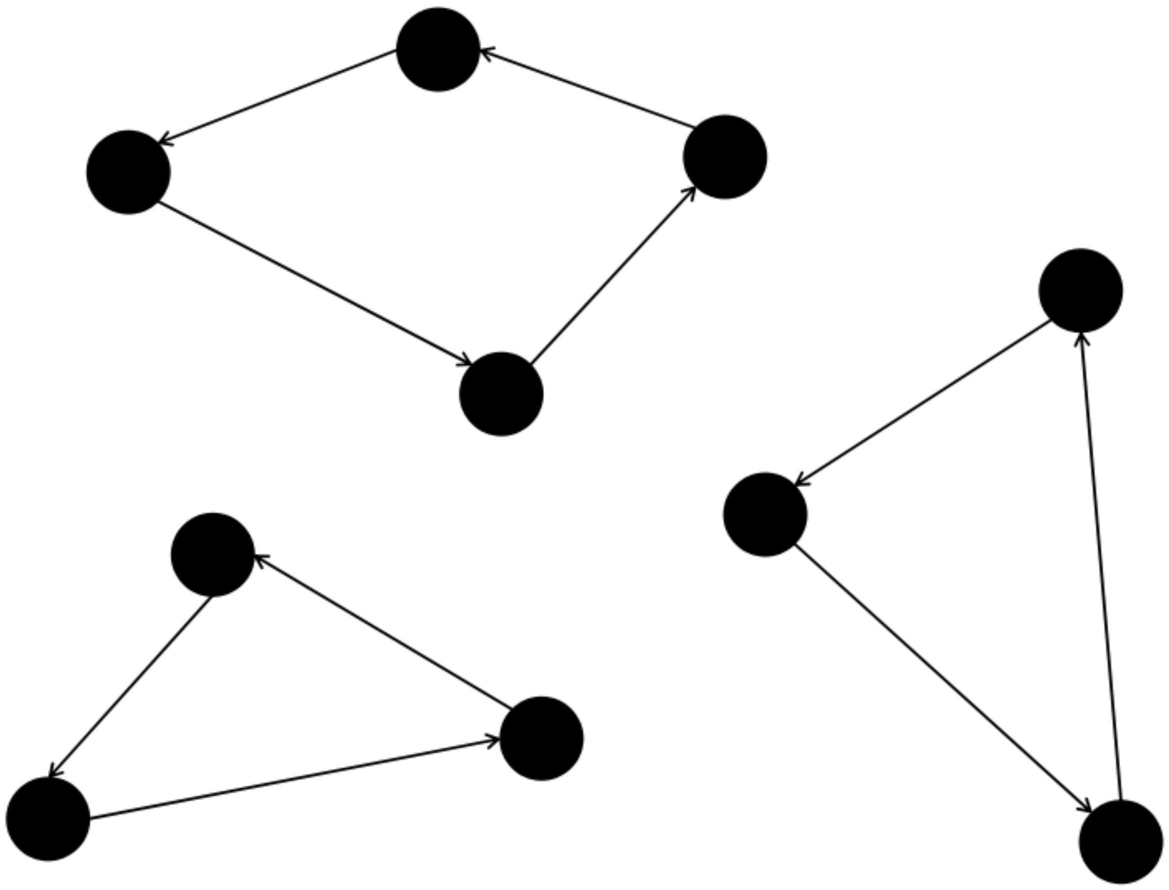
$$x_{ij} = \begin{cases} 1, & if\ the\ salesman\ goes\ directly\ from\ city\ i\ to\ city\ j; \\ 0, & otherwise \end{cases}$$

Using these variables, we could model the TSP as:

$$\begin{array}{lll} \text{minimize} & \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij} & \\ \text{subject to} & \sum_{j \in N: j \neq i} x_{ij} = 1 & j \in N \\ & \sum_{i \in N: i \neq j} x_{ij} = 1 & i \in N \\ & x_{ij} \in \{0, 1\} & i \in N, j \in N \end{array}$$

Is this enough to model the problem?

Example: The Traveling Salesman Problem



We have to eliminate subtours = obtain one single connected component!

Example: The Traveling Salesman Problem

To achieve it we need additional constraints that guarantee connectivity.

For that, we have at least two options:

Cut-set constraints:

$$\sum_{i\in S}\sum_{j\notin S} x_{ij} \geq 1 \qquad \forall S \subset N, S \neq \emptyset$$

Subtour elimination constraints:

$$\sum_{i\in S}\sum_{j\in S} x_{ij} \leq |S| - 1 \qquad \forall S \subset N, 2 \leq |S| \leq n - 1$$

Which formulation is better? The one that uses cut-set inequalities or the one that uses subtour elimination constraints?

Example: The Traveling Salesman Problem

To achieve it we need additional constraints that guarantee connectivity.

For that, we have at least two options:

Cut-set constraints:

$$\sum_{i \in S} \sum_{j \notin S} x_{ij} \geq 1 \qquad \forall S \subset N, S \neq \emptyset$$

Subtour elimination constraints:

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1 \qquad \forall S \subset N, 2 \leq |S| \leq n - 1$$

Which formulation is better? The one that uses cut-set inequalities or the one that uses subtour elimination constraints?

Answer: They are equivalent!

The Combinatorial Explosion

Discrete optimization problems are combinatorial in the sense that the optimal solution is some subset of a finite set

In principle, these problems can be solved by enumeration

However, the number of feasible solutions to a particular problem may be exponential

Knapsack and Location problems: in both cases, the number of subsets is 2^n

Traveling Salesman Problem: there are $(n-1)!$ Feasible tours

n	$\log n$	$\sqrt{n}$	n^2	2^n	$n!$
10	3.32	3.16	10^2	1.02×10^3	3.6×10^6
100	6.64	10.00	10^4	1.27×10^{30}	9.33×10^{157}
1000	9.97	31.62	10^6	1.07×10^{301}	4.02×10^{2567}

Table 1.1 Some typical functions

Thank you!