

Linear Algebra 1: TASK #11

Due on 16.07.2020

Amit Roth

Problem 1

Part One: Calculate $a = (3(-5)^{-1} - 2^{-1}6^2)^{-1}$ In \mathbb{Z}_7 .

Solution:

$$(3 \cdot 2^{-1} - 2^{-1} \cdot 6^2)^{-1} =$$

$$(3 \cdot 4 - 4 \cdot 1)^{-1} =$$

$$(5 - 4)^{-1} =$$

$$(5 + 3)^{-1} =$$

$$1^{-1} = 1$$

Part Two: Solve in \mathbb{Z}_5 : $2x^3 - x + 3 = \frac{1}{4}$

Solution:

$$2x^3 - x + 3 = 4^{-1}$$

$$2x^3 - x + 3 = 4$$

$$2x^3 - x + 4 = 0$$

We Will plot every object in \mathbb{Z}_5

$$x=0: \quad 2 \cdot 0^3 - 0 + 4 = 0 + 0 + 4 = 4$$

$$x=1: \quad 2 \cdot 1^3 - 1 + 4 = 2 - 1 + 4 = 0$$

$$x=2: \quad 2 \cdot 2^3 - 2 + 4 = 1 + 3 + 4 = 3$$

$$x=3: \quad 2 \cdot 3^3 - 3 + 4 = 4 + 2 + 4 = 0$$

$$x=4: \quad 2 \cdot 4^3 - 4 + 4 = 3 + 1 + 4 = 3$$

We can see that in \mathbb{Z}_5 the equation has 2 solutions: $x = 1, 3$.

Problem 2

Part One: Solve the following System of equations above \mathbb{R} .

$$\begin{cases} 3x - y + z + 7w = 13 \\ -2x + y - z - 3w = -9 \\ -2x + y - 7w = -8 \end{cases}$$

Solution:

We will swipe between the x's and the y's in the equations:

$$\begin{cases} -y + 3x + z + 7w = 13 \\ y - 2x - z - 3w = -9 \\ y - 2x - 7w = -8 \end{cases} \xrightarrow{R_2 \leftrightarrow R_1} \begin{cases} y - 2x - z - 3w = -9 \\ -y + 3x + z + 7w = 13 \\ y - 2x - 7w = -8 \end{cases} \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 + R_1} \begin{cases} y - 2x - z - 3w = -9 \\ x + 4w = 4 \\ z - 4w = 1 \end{cases}$$

We got a system of equations which the first variable in each equation is not found in the equations bellow. We will express each variable with w.

$$z = 1 + 4w$$

$$x = 4 - 4w$$

$$y = -9 + 2x + z + 3w = -9 + 8 - 8w + 1 + 4w + 3w = -w$$

Every quartet $(4 - 4w, -w, 1 + 4w, w)$ solves the system of equations.

Part Two: Find the value of a so the system of equations has solutions, above \mathbb{Z}_7 .

$$\begin{cases} 2x - y + z + w = 1 \\ x + 2y - z + 4w = 3 \\ x - 4z + 4w + a \end{cases}$$

Calculate the general solution, how many solutions exists?

Solution:

We will write the coefficient matrix of the system.

$$\begin{pmatrix} 2 & -1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 3 \\ 1 & 0 & -4 & 4 & a \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & -1 & 4 & 3 \\ 2 & -1 & 1 & 1 & 1 \\ 1 & 0 & -4 & 4 & a \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - R_1]{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 2 & -1 & 4 & 3 \\ 0 & -5 & 3 & -7 & -5 \\ 0 & -2 & -3 & 0 & a-3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 6 & 4 & 3 \\ 0 & 2 & 3 & 0 & 2 \\ 0 & 5 & 4 & 0 & a+4 \end{pmatrix} \xrightarrow{R_3 \rightarrow 6 \cdot R_3} \begin{pmatrix} 1 & 2 & 6 & 4 & 3 \\ 0 & 2 & 3 & 0 & 2 \\ 0 & 2 & 3 & 0 & 6a+3 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{pmatrix} 1 & 2 & 6 & 4 & 3 \\ 0 & 2 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 6a+1 \end{pmatrix}$$

To get a solution to our system, we need to get true statement in R_3 .

$$6a + 1 = 0$$

$$6a = 6$$

$$a = 1$$

for $a = 1$ there are solutions to the system.

$$\begin{pmatrix} 1 & 2 & 6 & 4 & 3 \\ 0 & 2 & 3 & 0 & 2 \end{pmatrix} \xrightarrow{R_2 \rightarrow 4 \cdot R_2} \begin{pmatrix} 1 & 2 & 6 & 4 & 3 \\ 0 & 1 & 5 & 0 & 1 \end{pmatrix}$$

$$y = 1 + 2z$$

$$x = 3 + 5y + z + 3w = 1 + 4z + w$$

Every quatret $(1 + 4z + w, 1 + 2z, z, w)$ solves the system.

In the solution exists 2 free variables therefore there are $7^2 = 49$ solutions.

Problem 3

$\{u, v, w\}$ is linear independent in F^n .

Determine whether the following sets are linear independent.

$$A = \{u-4v+4w, u+2v+3w, 4u+3v+2w\} \quad B = \{u+3v-w, u+v-3w, v+w\}$$

Part One: $F = \mathbb{R}$.

Solution:

From definition (2.6.1) \rightarrow A is linear independent if there is only 1 solution to the equation:

$$a_1v_1 + a_2v_2 + a_3v_3 = 0$$

which is $v_1 = v_2 = v_3 = 0$

We will write this vectoric equation with a matrix:

$$\begin{pmatrix} 1 & 1 & 4 & 0 \\ -4 & 2 & 3 & 0 \\ 4 & 3 & 2 & 0 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - 4 \cdot R_1]{R_2 \rightarrow R_2 + 4 \cdot R_1} \begin{pmatrix} 1 & 1 & 4 & 0 \\ 0 & 6 & 19 & 0 \\ 0 & -1 & -14 & 0 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + \frac{1}{6}R_2} \begin{pmatrix} 1 & 1 & 4 & 0 \\ 0 & 6 & 19 & 0 \\ 0 & 0 & -10\frac{2}{3} & 0 \end{pmatrix}$$

From the coefficient matrix we can see that $v_3 = 0 \rightarrow v_2 = 0 \rightarrow v_1 = 0$.

The only solution is trivial therefore A is linear independent.

As we did with A, we will solve the vectoric equation with a matrix.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ -1 & -3 & 1 & 0 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 + R_1]{R_2 \rightarrow R_2 - 3 \cdot R_1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{pmatrix}$$

We can see that $R_2 = R_3$ therefore we can ignore from 1 equation.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \end{pmatrix}$$

We have 3 variables, 2 equations and an homogeneous system then from theorem (1.13.1) \rightarrow we have a non trivial solution. From definition (2.6.2) \rightarrow B is linear dependent.

Part Two: $F = \mathbb{Z}_5$.

Solution:

$$\begin{pmatrix} 1 & 1 & 4 & 0 \\ 1 & 2 & 3 & 0 \\ 4 & 3 & 2 & 0 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 + R_1]{R_2 \rightarrow R_2 + 4 \cdot R_1} \begin{pmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 4 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Every triple (a_1, a_2, a_3) solves $R_3 \rightarrow$ then we can ignore from R_3 .

We have an homogeneous system of 2 equations with 3 variables.

From theorem (1.13.1) \rightarrow we have a non trivial solution.

From definition (2.6.2) A is linear dependent.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 2 & 1 & 0 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 + R_1]{R_2 \rightarrow R_2 + 2 \cdot R_1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 4 & 3 & 1 & 0 \end{pmatrix}$$

We can see that $R_2 = R_3$ therefore we can ignore from 1 equation.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{pmatrix}$$

We have an homogeneous system of 2 equations with 3 variables.

From theorem (1.13.1) \rightarrow we have a non trivial solution.

From definition (2.6.2) B is linear dependent.

Problem 4

Part One: Solve the following System of equations above \mathbb{R} .

$$\begin{cases} 2x + 3y + mz = k \\ -x - 5y + 3z = -2m \\ x - 9y + 5z = 0 \end{cases}$$

For which (k, m) there is a single solution, infinite solutions or no solution?
Calculate the general solution in case of infinite solutions.

Solution:

We will write the coefficients matrix.

$$\begin{pmatrix} 2 & 3 & m & k \\ -1 & -5 & 3 & -2m \\ 1 & -9 & 5 & 0 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & -9 & 5 & 0 \\ -1 & -5 & 3 & -2m \\ 2 & 3 & m & k \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - 2 \cdot R_1]{R_2 \rightarrow R_2 + R_1}$$

$$\begin{pmatrix} 1 & -9 & 5 & 0 \\ 0 & -14 & 8 & -2m \\ 0 & 21 & m - 10 & k \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_3 + 1.5 \cdot R_1} \begin{pmatrix} 1 & -9 & 5 & 0 \\ 0 & -14 & 8 & -2m \\ 0 & 0 & m + 2 & k - 3m \end{pmatrix}$$

We will look on R_3 :

for $m = -2$ and $k \neq -6 \rightarrow$ there is no solution.

for $m = -2$ and $k = -6 \rightarrow$ there are infinite solution.

for $m \neq -2 \rightarrow$ there is one solution.

We will plot $m = -2$ and $k = -6$

$$\begin{pmatrix} 1 & -9 & 5 & 0 \\ 0 & -14 & 8 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

z is a free variable, the general solution is:

$$y = \frac{4}{7}z - \frac{2}{7}$$

$$x = \frac{1}{7}z - \frac{18}{7}$$

Problem 5

let $v_1 = (1, 3, 4)$ $v_2 = (2, -1, -1)$ $v_3 = (-3, 5, a^2 - 2)$ $v_4 = (4, 2, a + 4)$ in \mathbb{R}^3 and $a \in \mathbb{R}$.

Part one

for which a values the set $\{v_1, v_2, v_3, v_4\}$ spans \mathbb{R}^3 .

Solution:

Let $v \in \mathbb{R}^3$ $v = (\alpha, \beta, \gamma)$.

We will check for which a values we can express v as a linear combination of $\{v_1, v_2, v_3, v_4\}$

$$x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = v$$

We will write the coefficients matrix:

$$\begin{pmatrix} 1 & 2 & -3 & 4 & \alpha \\ 3 & -1 & 5 & 2 & \beta \\ 4 & 1 & a^2 - 2 & a + 4 & \gamma \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - 4 \cdot R_1]{R_2 \rightarrow R_2 - 3 \cdot R_1} \begin{pmatrix} 1 & 2 & -3 & 4 & \alpha \\ 0 & -7 & 14 & -10 & \beta - 3\alpha \\ 0 & -7 & a^2 + 10 & a - 12 & \gamma - 4\alpha \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2}$$

$$\begin{pmatrix} 1 & 2 & -3 & 4 & \alpha \\ 0 & -7 & 14 & -10 & \beta - 3\alpha \\ 0 & 0 & a^2 - 4 & a - 2 & \gamma - \alpha - \beta \end{pmatrix}$$

We will look on R_3

$$(a^2 - 4)x_3 + (a - 2)x_4 = \gamma - \alpha - \beta$$

for $a = 2$ we can't express any vector v as a linear combination because we get:

$$0 = \gamma - \alpha - \beta$$

From theorem (2.53) \rightarrow there is a solution to the equation if and only if v is a linear span of $\{v_1, v_2, v_3, v_4\}$.

For any other a value we can solve R_3 and then to express the rest of the variables.

$a \neq 2$ if and only if $\{v_1, v_2, v_3, v_4\}$ spans \mathbb{R}^3 .

Part Two

For which a values $\{v_1, v_2, v_3\}$ is a basis of \mathbb{R}^3 ?

Solution:

The number of the vectors in $\{v_1, v_2, v_3\}$ equals to the power of the Field (\mathbb{R}^3).

From theorem (2.7.8) \rightarrow if $\{v_1, v_2, v_3\}$ linear independent then it spans \mathbb{R}^3 .

And from definition (2.7.6) We will get that $\{v_1, v_2, v_3\}$ is a basis of \mathbb{R}^3

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 3 & -1 & 5 & 0 \\ 4 & 1 & a^2 - 2 & 0 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - 4 \cdot R_1]{R_2 \rightarrow R_2 - 3 \cdot R_1} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -7 & 14 & 0 \\ 0 & -7 & a^2 + 10 & 0 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -7 & 14 & 0 \\ 0 & 0 & a^2 - 4 & 0 \end{pmatrix}$$

We can see that for $a = \pm 2$ we get infinite solutions.

Therefore for every $a \neq \pm 2$ $\{v_1, v_2, v_3\}$ is liner independent.

$a \neq \pm 2$ if and only if $\{v_1, v_2, v_3\}$ is a basis of \mathbb{R}^3 .

Part three

For which a values v_4 is a linear combination of $\{v_1, v_2, v_3\}$?
 set $a = -1$. express v_4 as a linear combination of $\{v_1, v_2, v_3\}$.

Solution:

We will check for which a values v_4 as a linear combination of $\{v_1, v_2, v_3\}$.

$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2 - 2 & a + 4 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 - 4 \cdot R_1]{R_2 \rightarrow R_2 - 3 \cdot R_1} \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2 + 10 & a - 12 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2 - 4 & a - 2 \end{pmatrix}$$

We will look on R_3

$$X_3(a - 2)(a + 2) = (a - 2)$$

When $a = -2$ there is no solution to the equation.

v_4 is a linear combination of $\{v_1, v_2, v_3\}$ if and only if $a \neq -2$.

We will plot $a = -1$

$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & -3 & -3 \end{pmatrix}$$

We will express v_4 as:

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = v_4$$

from the matrix we cant express x_1, x_2, x_3

$$x_3 = 1$$

$$7x_2 = 10 + 14x_3 \quad x_2 = 4$$

$$x_1 = 4 - 2x_2 + 3x_3 = -1$$

$$v_4 = -v_1 + 4v_2 + v_3$$

Problem 6

let a_1, a_2, \dots, a_m, b vectors in \mathbb{R}^n

Part One

Prove that if $m \leq n$ and for all $c \in \mathbb{R}^n$ there is a solution to the equation:

$x_1 a_1 + \dots + x_m a_m = c$ then $\{a_1, \dots, a_m\}$ is a basis for \mathbb{R}^n

Solution:

By definition (2.5.3) if for every c exists solution then for every vector in $\mathbb{R}^n \rightarrow$ it can be expressed as a linear combination of $\{a_1, \dots, a_m\}$.

Then from definition (2.7.1) the vector series in length m spans \mathbb{R}^n .

from conclusion (2.7.4) $\rightarrow m \geq n$. Given $n \geq m$ therefore $m = n$

From theorem (2.7.8) $\{a_1, \dots, a_m\}$ is linear independent.

By definition (2.7.8) $\{a_1, \dots, a_m\}$ is a basis for \mathbb{R}^n .

Part Two

Prove that: if $x_1 a_1 + \dots + x_m a_m = b$ has a solution and $\{a_1, \dots, a_m\}$ is linear independent then the solution is singular.

Solution:

From theorem (2.7.8) $\{a_1, \dots, a_m\}$ spans \mathbb{R}^n .

From definition (2.5.3) b is a linear combination of $\{a_1, \dots, a_m\}$.

From definition (2.7.6) $\{a_1, \dots, a_m\}$ basis of \mathbb{R}^n . From theorem (2.7.9) this linear combination is singular.

Therefore the solution to the equation $x_1 a_1 + \dots + x_m a_m = b$ is singular.