Linear Algebra 1: TASK #11

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Problem 1

Part One: Calculate a $= (3(-5)^{-1} - 2^{-1}6^2)^{-1}$ In $\mathbb{Z}_7$.

Solution:
$(3 \cdot 2^{-1} - 2^{-1} \cdot 6^2)^{-1} =$
$(3 \cdot 4 - 4 \cdot 1)^{-1} =$
$(5 - 4)^{-1} =$
$(5 + 3)^{-1} =$
$1^{-1} = 1$

Part Two: Solve in $\mathbb{Z}_5$: $2x^3 - x + 3 = \frac{1}{4}$

Solution:
$2x^3 - x + 3 = 4^{-1}$
$2x^3 - x + 3 = 4$
$2x^3 - x + 4 = 0$

We Will plot every object in $\mathbb{Z}_5$
x=0: $2 \cdot 0^3 - 0 + 4 = 0 + 0 + 4 = 4$
x=1: $2 \cdot 1^3 - 1 + 4 = 2 - 1 + 4 = 0$
x=2: $2 \cdot 2^3 - 2 + 4 = 1 + 3 + 4 = 3$
x=3: $2 \cdot 3^3 - 3 + 4 = 4 + 2 + 4 = 0$
x=4: $2 \cdot 4^3 - 4 + 4 = 3 + 1 + 4 = 3$

We can see that in $\mathbb{Z}_5$ the equation has 2 solutions: $x = 1, 3$.

Problem 2

Part One: Solve the following System of equations above $\mathbb{R}$.

$$\begin{cases} 3x - y + z + 7w = 13 \\ -2x + y - z - 3w = -9 \\ -2x + y - 7w = -8 \end{cases}$$

Solution:
We will swipe between the x's and the y's in the equations:

$$\begin{cases} -y + 3x + z + 7w = 13 \\ y - 2x - z - 3w = -9 \\ y - 2x - 7w = -8 \end{cases} \xrightarrow{R_2 \leftrightarrow R_1} \begin{cases} y - 2x - z - 3w = -9 \\ -y + 3x + z + 7w = 13 \\ y - 2x - 7w = -8 \end{cases} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \begin{cases} y - 2x - z - 3w = -9 \\ x + 4w = 4 \\ z - 4w = 1 \end{cases}$$

We got a system of equations which the first variable in each equation is not found in the equations bellow. We will express each variable with w.

$z = 1 + 4w$
$x = 4 - 4w$
$y = -9 + 2x + z + 3w = -9 + 8 - 8w + 1 + 4w + 3w = -w$

Every quartet $(4 - 4w, -w, 1 + 4w, w)$ solves the system of equations.

Part Two: Find the value of a so the system of equations has solutions, above $\mathbb{Z}_7$.

$$\begin{cases} 2x - y + z + w = 1 \\ x + 2y - z + 4w = 3 \\ x - 4z + 4w + a \end{cases}$$

Calculate the general solution, how many solutions exists?

Solution:
We will write the coefficient matrix of the system.

$$\begin{pmatrix} 2 & -1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 3 \\ 1 & 0 & -4 & 4 & a \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & -1 & 4 & 3 \\ 2 & -1 & 1 & 1 & 1 \\ 1 & 0 & -4 & 4 & a \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1}$$

$$\begin{pmatrix} 1 & 2 & 6 & 4 & 3 \\ 0 & 2 & 3 & 0 & 2 \\ 0 & 5 & 4 & 0 & a+4 \end{pmatrix} \xrightarrow{R_3 \to 6 \cdot R_3} \begin{pmatrix} 1 & 2 & 6 & 4 & 3 \\ 0 & 2 & 3 & 0 & 2 \\ 0 & 2 & 3 & 0 & 6a+3 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 2 & 6 & 4 & 3 \\ 0 & 2 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 6a+1 \end{pmatrix}$$

Problem 2 continued on next page. . .

To get a solution to our system, we need to get true statement in R_3.
$6a + 1 = 0$
$6a = 6$
$a = 1$

for $a = 1$ there are solutions to the system.

$$\begin{pmatrix} 1 & 2 & 6 & 4 & 3 \\ 0 & 2 & 3 & 0 & 2 \end{pmatrix} \xrightarrow{R_2 \to 4 \cdot R_2} \begin{pmatrix} 1 & 2 & 6 & 4 & 3 \\ 0 & 1 & 5 & 0 & 1 \end{pmatrix}$$

$y = 1 + 2z$
$x = 3 + 5y + z + 3w = 1 + 4z + w$

Every quatret $(1 + 4z + w, 1 + 2z, z, w)$ solves the system.
In the solution exists 2 free variables therefore there are $7^2 = 49$ solutions.

Problem 3

{u,v,w} is linear independent in F^n.
Determine whether the following sets are linear independent.
$A = \{$u-4v+4w, u+2v+3w, 4u+3v+2w$\}$ $\quad B = \{$u+3v-w, u+v-3w, v+w$\}$

Part One: $F = \mathbb{R}$.

Solution:
From definition (2.6.1) $\rightarrow$ A is linear independent if there is only 1 solution to the equation:
$a_1v_1 + a_2v_2 + a_3v_3 = 0$
which is $v_1 = v_2 = v_3 = 0$

We will write this vectoric equation with a matrix:

$$\begin{pmatrix} 1 & 1 & 4 & 0 \\ -4 & 2 & 3 & 0 \\ 4 & 3 & 2 & 0 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 4\cdot R_1]{R_2 \to R_2 + 4\cdot R_1} \begin{pmatrix} 1 & 1 & 4 & 0 \\ 0 & 6 & 19 & 0 \\ 0 & -1 & -14 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 + \frac{1}{6}R_2} \begin{pmatrix} 1 & 1 & 4 & 0 \\ 0 & 6 & 19 & 0 \\ 0 & 0 & -10\frac{2}{3} & 0 \end{pmatrix}$$

From the coefficient matrix we can see that $v_3 = 0 \rightarrow v_2 = 0 \rightarrow v_1 = 0$.
The only solution is trivial therefore A is linear independent.

As we did with A, we will solve the vectoric equation with a matrix.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ -1 & -3 & 1 & 0 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 3\cdot R_1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{pmatrix}$$

We can see that $R_2 = R_3$ therefore we can ignore from 1 equation.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \end{pmatrix}$$

We have 3 variables, 2 equations and an homogeneous system then from theorem (1.13.1) $\rightarrow$ we have a non trivial solution. From definition (2.6.2) $\rightarrow$ B is linear dependent.

Problem 3 continued on next page. . .

Part Two: $F = \mathbb{Z}_5$.

Solution:

$$\begin{pmatrix} 1 & 1 & 4 & 0 \\ 1 & 2 & 3 & 0 \\ 4 & 3 & 2 & 0 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + 4 \cdot R_1} \begin{pmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 4 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 1 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Every triple (a_1, a_2, a_3) solves $R_3 \to$ then we can ignore from R_3.
We have an homogeneous system of 2 equations with 3 variabels.
From theorem (1.13.1) $\to$ we have a non trivial solution.
From defenition (2.6.2) A is linear dependent.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 2 & 1 & 0 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + 2 \cdot R_1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 4 & 3 & 1 & 0 \end{pmatrix}$$

We can see that $R_2 = R_3$ therefore we can ignore from 1 equation.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{pmatrix}$$

We have an homogeneous system of 2 equations with 3 variabels.
From theorem (1.13.1) $\to$ we have a non trivial solution.
From defenition (2.6.2) B is linear dependent.

Problem 4

Part One: Solve the following System of equations above $\mathbb{R}$.

$$\begin{cases} 2x+3y+mz=k \\ -x-5y+3z=-2m \\ x-9y+5z=0 \end{cases}$$

For which (k,m) there is a single solution, infinite solutions or no solution?
Calculate the general solution in case of infinite solutions.

Solution:

We will write the coefficients matrix.

$$\begin{pmatrix} 2 & 3 & m & k \\ -1 & -5 & 3 & -2m \\ 1 & -9 & 5 & 0 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & -9 & 5 & 0 \\ -1 & -5 & 3 & -2m \\ 2 & 3 & m & k \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2\cdot R_1]{R_2 \to R_2 + R_1}$$

$$\begin{pmatrix} 1 & -9 & 5 & 0 \\ 0 & -14 & 8 & -2m \\ 0 & 21 & m-10 & k \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_3 + 1.5\cdot R_1} \begin{pmatrix} 1 & -9 & 5 & 0 \\ 0 & -14 & 8 & -2m \\ 0 & 0 & m+2 & k-3m \end{pmatrix}$$

We will look on R_3:
for $m=-2$ and $k \neq -6 \to$ there is no solution.
for $m=-2$ and $k=-6 \to$ there are infinite solution.
for $m \neq -2 \to$ there is one solution.

We will plot $m=-2$ and $k=-6$

$$\begin{pmatrix} 1 & -9 & 5 & 0 \\ 0 & -14 & 8 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

z is a free variable, the general solution is:

$y = \frac{4}{7}z - \frac{2}{7}$

$x = \frac{1}{7}z - \frac{18}{7}$

Problem 5

let $v_1 = (1,3,4) \quad v_2 = (2,-1,-1) \quad v_3 = (-3,5,a^2-2) \quad v_4 = (4,2,a+4) \quad$ in $\mathbb{R}^3$ and $a \in \mathbb{R}$.

Part one
for which a values the set $\{v_1, v_2, v_3, v_4\}$ spans $\mathbb{R}^3$.

Solution:

Let $v \in \mathbb{R}^3 \quad v = (\alpha, \beta, \gamma)$.
We will check for which a values we can express v as a linear combination of $\{v_1, v_2, v_3, v_4\}$
$x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = v$
We will write the coefficients matrix:

$$\begin{pmatrix} 1 & 2 & -3 & 4 & \alpha \\ 3 & -1 & 5 & 2 & \beta \\ 4 & 1 & a^2-2 & a+4 & \gamma \end{pmatrix} \xrightarrow[R_3 \to R_3 - 4\cdot R_1]{R_2 \to R_2 - 3\cdot R_1} \begin{pmatrix} 1 & 2 & -3 & 4 & \alpha \\ 0 & -7 & 14 & -10 & \beta - 3\alpha \\ 0 & -7 & a^2+10 & a-12 & \gamma - 4\alpha \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2}$$

$$\begin{pmatrix} 1 & 2 & -3 & 4 & \alpha \\ 0 & -7 & 14 & -10 & \beta - 3\alpha \\ 0 & 0 & a^2-4 & a-2 & \gamma - \alpha - \beta \end{pmatrix}$$

We will look on R_3
$(a^2-4)x_3 + (a-2)x_4 = \gamma - \alpha - \beta$
for $a = 2$ we can't express any vector v as a linear combination because we get:
$0 = \gamma - \alpha - \beta$

From theorem (2.53) $\rightarrow$ there is a solution to the equation if and only if v is a linear span of $\{v_1, v_2, v_3, v_4\}$.
For any other a value we can solve R_3 and then to express the rest of the variables.
$a \neq 2$ if and only if $\{v_1, v_2, v_3, v_4\}$ spans $\mathbb{R}^3$.

Part Two
For which a values $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$?

Solution:

The number of the vectors in $\{v_1, v_2, v_3\}$ equals to the power of the Field ($\mathbb{R}^3$).
From theorem (2.7.8) $\rightarrow$ if $\{v_1, v_2, v_3\}$ linear independent then it spans $\mathbb{R}^3$.
And from definition (2.7.6) We will get that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 3 & -1 & 5 & 0 \\ 4 & 1 & a^2-2 & 0 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 4\cdot R_1]{R_2 \to R_2 - 3\cdot R_1} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -7 & 14 & 0 \\ 0 & -7 & a^2+10 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -7 & 14 & 0 \\ 0 & 0 & a^2-4 & 0 \end{pmatrix}$$

We can see that for $a = \pm 2$ we get infinite solutions.
Therefore for every $a \neq \pm 2$ $\{v_1, v_2, v_3\}$ is liner independent.
$a \neq \pm 2$ if and only if $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

Problem 5 continued on next page...

Part three

For which a values v_4 is a linear combination of $\{v_1, v_2, v_3\}$?
set $a = -1$. express v_4 as a linear combination of $\{v_1, v_2, v_3\}$.

Solution:

We will check for which a values v_4 as a linear combination of $\{v_1, v_2, v_3\}$.

$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-2 & a+4 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 4\cdot R_1]{R_2 \to R_2 - 3\cdot R_1} \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2+10 & a-12 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-4 & a-2 \end{pmatrix}$$

We will look on R_3
$X_3(a-2)(a+2) = (a-2)$
When $a = -2$ there is no solution to the equation.
v_4 is a linear combination of $\{v_1, v_2, v_3\}$ if and only if $a \neq -2$.

We will plot $a = -1$

$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & -3 & -3 \end{pmatrix}$$

We will express v_4 as:
$x_1v_1 + x_2v_2 + x_3v_3 = v_4$
from the matrix we can express x_1, x_2, x_3
$x_3 = 1$
$7x_2 = 10 + 14x_3 \quad x_2 = \frac{24}{7}$
$x_1 = 4 - 2x_2 + 3x_3 = -1$

$v_4 = -v_1 + \frac{24}{7}v_2 + v_3$

Problem 6

let $a_1, a_2, ..., a_m, b$ vectros in $\mathbb{R}^n$

Part One
Prove that if $m \leq n$ and for all $c \in \mathbb{R}^n$ there is a solution to the equation:
$x_1 a_1 ...... x_m a_m = c$ then $\{a_1 ..... a_m\}$ is a basis for $\mathbb{R}^n$

Solution:

By definition (2.5.3) if for every c exists solution then for every vector in $\mathbb{R}^n \rightarrow$ it can be expressed as a linear combination of $\{a_1 ..... a_m\}$.
Then from definition (2.7.1) the vector series in length m spans $\mathbb{R}^n$.
from conclusion (2.7.4) $\rightarrow m \geq n$. Given $n \geq m$ therefore $m = n$
From theorem (2.7.8) $\{a_1 ..... a_m\}$ is linear independent.
By definition (2.7.8) $\{a_1 ..... a_m\}$ is a basis for $\mathbb{R}^n$.

Part Two
Prove that: if $x_1 a_1 ...... x_m a_m = b$ has a solution and $\{a_1 ..... a_m\}$ is linear independent then the solution is singular.

Solution:
Given that $\{a_1 ..... a_m\}$ is linear independent, from proposition (2.6.5) the homogeneous coefficients matrix has 1 trivial solution.
From theorem (2.7.10) any matrix from the form $x_1 a_1 ...... x_m a_m = b$ has 1 solution.
Therefore the solution is singular.