Linear Algebra 1: TASK #11

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Problem 1

section 1:
$2z^2 - 2iz - 1 = 0$
$z_{1,2} = \frac{2i \pm \sqrt{-4+8}}{4} = \frac{i \pm 1}{2}$
$z_1 = \frac{i}{2} + \frac{1}{2}, \quad z_2 = \frac{i}{2} - \frac{1}{2}$

$w = \frac{i}{2} - \frac{1}{2}$
We will represent w using trigonometric representation.
$w = \frac{1}{\sqrt{2}}\left(cos(\frac{3}{4}\pi) + i \cdot sin(\frac{3}{4}\pi)\right)$
$w^{23} = \left(\frac{1}{\sqrt{2}}\right)^{23}\left(cos(\frac{3}{4}\pi) + i \cdot sin(\frac{3}{4}\pi)\right)^{23}$
De Moivre's formula $\rightarrow$
$w^{23} = \left(\frac{1}{2^{11}\sqrt{2}}\right)\left(cos(\frac{3}{4}23\pi) + i \cdot sin(\frac{3}{4}23\pi)\right), \qquad cos(x) = cos(x + 2\pi k), \quad sin(x) = sin(x + 2\pi k)$
$w^{23} = 2^{-11.5}\left(cos(1.25\pi) + i \cdot sin(1.25\pi)\right)$
$w^{23} = -2^{-11.5} \cdot \frac{\sqrt{2}}{2} - i \cdot 2^{-11.5} \cdot \frac{-\sqrt{2}}{2}$
$w^{23} = -\frac{1}{4096} - \frac{i}{4096}$

section 2:
$A = \{1 - \sqrt{3}, 2\sqrt{3} + 5\}$
From the conclusion in chapter 2 page 184. A group with 2 vectors are linear dependant if and only if one vector is a linear combination of the other.
Well see whether $a \in \mathbb{R}$ or / and $a \in \mathbb{Q}$.

$1 - \sqrt{3} = a(2\sqrt{3} + 5)$

$\frac{1-\sqrt{3}}{2\sqrt{3}+5} = a$

$\frac{1-\sqrt{3}}{2\sqrt{3}+5} \cdot \frac{1+\sqrt{3}}{1+\sqrt{3}} = a$

$\frac{1-3}{2\sqrt{3}+5+12+5\sqrt{3}} = a$

$\frac{-2}{7\sqrt{3}+17} = a$

We can see that $a \notin \mathbb{Q}$, but $a \in \mathbb{R}$ therefore A is liner dependent above $\mathbb{R}$ but not above $\mathbb{Q}$

Problem 2

$v_1...v_k, u, w \in V$
The euqation $x_1v_1 + ... + x_kv_k = u$ has 1 solution.
There is no solution to the equation $x_1v_1 + ... + v_kv_k = w$.
Find $dim(Sp\{v_1...v_k, w\})$.

We will show that $\{v_1...v_k\}$ is linear independent.
We will assume that $\{v_1...v_k\}$ is linear dependent, therefore exists vector v_m so that $1 \leq m \leq k$ and $y_1v_1 + ... + y_{m-1}v_{m-1} + y_{m+1}v_{m+1}... + y_kv_k = v_m$.
we can plot v_m in the first equation:
$x_1v_1 + ... + x_{m-1}v_{m-1} + x_{m+1}v_{m+1}... + x_kv_k + y_1v_1 + ... + y_{m-1}v_{m-1} + y_{m+1}v_{m+1}... + y_kv_k = u$
$(x_1 + y_1)v_1 + ... + (x_{m-1} + y_{m-1})v_{m-1} + (x_{m+1} + y_{m-1})v_{m+1}... + (x_k + y_k)v_k = u$

we got another solution to the equation $x_1v_1 + ... + x_kv_k = u$, in contradiction to the given.
Therefore $\{v_1...v_k\}$ is linear independent.
There is no solution to the equation $x_1v_1 + ... + v_kv_k = w$. therefore w is not a linear combination of $\{v_1...v_k\}$ so $\{v_1...v_k, w\}$ is linear independent.
$U = Sp\{v_1...v_k, w\}$. $\{v_1...v_k, w\}$ spans U and linear independent therefore $\{v_1...v_k, w\}$ is a base for U.

By def 8.3.3 $dim(Sp\{v_1...v_k, w\}) = k + 1$.

Problem 3

Section 1:

H **:**

$H = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R) | a \in \mathbb{Q}\}$
H is not a linear space because: for $k \in \mathbb{R} \setminus \mathbb{Q}$ and $v \in H$
$v \cdot k \notin H$ because $a \cdot k \notin \mathbb{Q}$

A **:**
$A \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} A, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix} = \begin{pmatrix} a & b \\ 2c & 2d \end{pmatrix}$

$$\begin{cases} a+3b = a \\ 2a+4b = b \\ c+3d = 2c \\ 2c+4d = 2d \end{cases}$$

from the first equation we get that $b = 0$. from plottin $b = 0$ in the second equation we get that also $a = 0$. and from the third and fourth equation we get that also $c = 0$ and $d = 0$.

Therefore $K = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} \rightarrow$ We will prove that K is a linear space.
From theorem (7.3.2'): $K \neq \phi$ and for every $k_2, k_1 \in K$ and for every $\alpha_1, \alpha_2 \in \mathbb{R}$

$\alpha_1 k_1 + \alpha_2 k_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in K$

Therefore K is a linear space.

Problem 3 continued on next page...

M **:**
$M = \{p(x) \in \mathbb{R}_4[x] | \forall x \big(p(x) = p(x+2)\big)\}$

$p(x) = p(x+2)$ if and only if p is a periodic function with period of 2.
The only finite polynoms which are periodic are constant functions from the form $p(x) = \alpha$ and α is a scalar.
We will prove that claim for $\mathbb{R}_4[x]$ algebraic:
$p(x) = p(x+2)$ The solution should be the vector $(\alpha, \beta, \gamma, \delta)$

$\alpha + \beta x + \gamma x^2 + \delta x^3 = \alpha + \beta(x+2) + \gamma(x+2)^2 + \delta(x+2)^3$
$\beta x + \gamma x^2 + \delta x^3 = \beta x + 2\beta + \gamma(x^2+4x+4)\delta(x^3+6x^2+12x+8)$
$\gamma x^2 + \delta x^3 = 2\beta + \gamma x^2 + \gamma 4x + \gamma 4 + \delta x^3 + \delta 6x^2 + \delta 12x + \delta 8$
$0 = 2\beta + \gamma 4x + \gamma 4 + \delta 6x^2 + \delta 12x + \delta 8$
$2\beta + 4\gamma + 8\delta + x(4\gamma + 12\delta) + \delta 6x^2$

$$\begin{cases} 2\beta + 4\gamma + 8\delta = 0 \\ 4\gamma + 12\delta = 0 \\ 6\delta = 0 \end{cases} \quad \to \delta = 0, \gamma = 0, \beta = 0$$

The solutions is $(\alpha, 0, 0, 0)$ for $\alpha \in \mathbb{R}$
$M = \{p(x) \in \mathbb{R}_1[x]\}$ therefore M is a linear space.

Section 2:

$K = Sp\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$

$M = Sp\{1\}$

Problem 4

Direction 1:

V is a n-dimension space
$U = Sp\{u_1...u_k\}$
$W = Sp\{v_1...v_{n-k}\}$
We will assume that $V = U \oplus W$ and we will see wether $\{u_1...u_k, v_1...v_{n-k}\}$ is linear independent
from conclusion (8.3.7) $\rightarrow dimV = dimU + dimW = n$

$dimU \leq k$ because the base of U is definitely smaller or equal from k (the size of the span).
$dimW \leq n-k$ because the base of W is definitely smaller or equal from n-k (the size of the span).
From the equation we can tell that $\rightarrow dimV = n \quad dimU = k \quad dimW = n-k$

$\{u_1...u_k\}, \quad \{v_1...v_{n-k}\}$ are linear independent because they span U and V and the size of the span equal to the dimension of the spaces.

We will assume that $\{u_1...u_k, v_1...v_{n-k}\}$is linear dependant
WLOG $u_m = \alpha_1 v_1...\alpha_{n-k} v_{n-k}$
But than
$U \cap W = \{0\}$
In contrast to our assumption $\{u_1...u_k, v_1...v_{n-k}\}$ is linear independent

Direction 2:

We will assume that $\{u_1...u_k, v_1...v_{n-k}\}$ is linear independent .
There fore $\{u_1...u_k, v_1...v_{n-k}\}$ is a base for $Sp\{u_1...u_k, v_1...v_{n-k}\}$
$\{u_1...u_k, v_1...v_{n-k}\}$ contains n vectors and linear independent, $U + W \subseteq V$
given $dinV = n$
From 8.3.2c $W + U = \{u_1...u_k, v_1...v_{n-k}\} = V$

if $W \cap U \neq \{0\}$ would exists a vector $j \in W$ and $j \in U$ bu then $\{u_1...u_k, v_1...v_{n-k}\}$ is linear dependent in contradiction to our fundings. therefore $W \cap U = \{0\}$
and $V = U \oplus W$

Problem 5

Section 1:

We will find a base for U
$U = Sp\{\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\-1 & 0 & 0\end{pmatrix}\}$

We will find a base for W. We will check wether his span is linear independent

$W = Sp\{\begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 0 & 1\\2 & 2 & 3\end{pmatrix}, \begin{pmatrix}-1 & 1 & -1\\-3 & -2 & -3\end{pmatrix}\}$
We can see that: $\begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\end{pmatrix} - 2\begin{pmatrix}1 & 0 & 1\\2 & 2 & 3\end{pmatrix} = \begin{pmatrix}-1 & 1 & -1\\-3 & -2 & -3\end{pmatrix}$

the third vector is a linear combination of the others.

And also $\{\begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 0 & 1\\2 & 2 & 3\end{pmatrix}\}$ is linear independent.
$W = Sp\{\begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 0 & 1\\2 & 2 & 3\end{pmatrix}\}$
This span is linear independent therefore it is also the base of W

We will find a base for $W + U$

$W + U = Sp\{\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\-1 & 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 0 & 1\\2 & 2 & 3\end{pmatrix}\}$
We will check wether the span is linear independent.

$a\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1 & 0\\-1 & 0 & 0\end{pmatrix} + c\begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\end{pmatrix} + d\begin{pmatrix}1 & 0 & 1\\2 & 2 & 3\end{pmatrix} = 0$

We can see that $\begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\end{pmatrix} - \begin{pmatrix}0 & 1 & 0\\-1 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 1\\2 & 2 & 3\end{pmatrix}$
Therefore linear combination.

We will check wether:

$\{\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\-1 & 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1 & 1\\1 & 2 & 3\end{pmatrix}\}$
is linear independent

Problem 5 continued on next page. . .

We will write the coordinate vectors by the base :
$\{\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\end{pmatrix},\}$

$$\begin{pmatrix}1&0&0&0&1&0\\0&1&0&-1&0&0\\1&1&1&1&2&3\end{pmatrix}\xrightarrow{R_3-R_1}\begin{pmatrix}1&0&0&0&1&0\\0&1&0&-1&0&0\\0&1&1&1&1&3\end{pmatrix}\xrightarrow{R_3-R_2}\begin{pmatrix}1&0&0&0&1&0\\0&1&0&-1&0&0\\0&0&1&2&1&3\end{pmatrix}$$

By theorem 8.4.4 if the coordiantes are linear independent then also the vectors set.

We got that $\{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&1&0\\-1&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\2&1&3\end{pmatrix}\}$ linear independent and spans $U+W$
$\to\{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&1&0\\-1&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\2&1&3\end{pmatrix}\}$ Base for $U+W$

Section 2:

We will take a vector $v\in U\cap W$

$v\in Sp\{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&1&0\\-1&0&0\end{pmatrix}\}$ and $v\in Sp\{\begin{pmatrix}1&1&1\\1&2&3\end{pmatrix},\begin{pmatrix}1&0&1\\2&2&3\end{pmatrix}\}$
$v=a\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}+b\begin{pmatrix}0&1&0\\-1&0&0\end{pmatrix}=c\begin{pmatrix}1&1&1\\1&2&3\end{pmatrix}+d\begin{pmatrix}1&0&1\\2&2&3\end{pmatrix}$

$$a\begin{pmatrix}-1&0&0\\0&-1&0\end{pmatrix}+b\begin{pmatrix}0&-1&0\\1&0&0\end{pmatrix}+c\begin{pmatrix}1&1&1\\1&2&3\end{pmatrix}+d\begin{pmatrix}1&0&1\\2&2&3\end{pmatrix}=0$$

From theorem 8.4.4 and from 8.1.1:

$$\begin{pmatrix}1&1&-1&0&0\\0&1&0&-1&0\\1&1&0&0&0\\2&1&0&1&0\\2&2&-1&0&0\\3&3&0&0&0\end{pmatrix}\xrightarrow{R_3\leftrightarrow R_1}\begin{pmatrix}1&1&0&0&0\\0&1&0&-1&0\\1&1&-1&0&0\\2&1&0&1&0\\2&2&-1&0&0\\3&3&0&0&0\end{pmatrix}\xrightarrow{R_3\to R_3-R_1,R_4\to R_4-2R_1,R_5\to R_5-2R_1,R_6\to R_6-3R_1}$$

$$\begin{pmatrix}1&1&0&0&0\\0&1&0&-1&0\\0&0&-1&0&0\\0&-1&0&1&0\\0&0&-1&0&0\\0&0&0&0&0\end{pmatrix}\xrightarrow{R_4\to R_4+R+2,R_5\to R_5-R_3}\begin{pmatrix}1&1&0&0&0\\0&1&0&-1&0\\0&0&-1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

$a = 0 \to d = b \to c = -d = -b$ We got $a = 0$ therefore the vectors that in $U \cap W$ are from the form $b\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$U \cap W = Sp\{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\}$

Section 3:

We will find a linear space T so that $M_{2x3}(R) = W \oplus T$.

$W = Sp\{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \end{pmatrix}\}$

We will choose $T = Sp\{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$ And we will check if $W \oplus T = M_{2x3}(R)$

$T + W = Sp\{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \end{pmatrix}\} = M_{2x3}(R)$

From 8.3.7 $dimT + dimW = dim(w + t)$ Therefore $T \oplus W = M_{2x3}(R)$

Problem 6

Section 1:

$AB = I_m$ From questions 8.5.7, 8.5.6

$$\begin{cases} p(I_m) \leq min\{p(A), p(B)\} \\ p(A) \leq min\{m, n\} \\ p(A) \leq min\{m, n\} \end{cases}$$

$p(I_m) = m$ because I_m has m non zeros lines in the canonical representation

$$\begin{cases} m \leq p(A) \leq min\{m, n\} \rightarrow p(A) = m \\ m \leq p(B) \leq min\{m, n\} \rightarrow p(B) = m \end{cases}$$

Section 2:
We will prove that the kernel $\vec{x_1}$ of BA,
equals to the kernel $\vec{x_2}$ of A

$$\begin{cases} BA\vec{x_1} = \vec{0} \\ A\vec{x_2} = \vec{0} \end{cases}$$

$$\begin{cases} ABA\vec{x_1} = A\vec{0} \\ A\vec{x_2} = \vec{0} \end{cases} \quad \rightarrow AB = I_m, A\vec{0} = \vec{0}$$

$$\begin{cases} A\vec{x_1} = \vec{0} \\ A\vec{x_2} = \vec{0} \end{cases}$$

$A\vec{x_1} = A\vec{x_2}$
$\vec{x_1} = \vec{x_2}$

From theorem 8.6.1

$$\begin{cases} dim(\vec{x_1}) = n - m \\ dim(\vec{x_2}) = n - p(BA) \end{cases} \rightarrow \begin{cases} dim(\vec{x_1}) = n - m \\ p(BA) = n - dim(\vec{x_2}) \end{cases} \quad \rightarrow from(8.3.4)dim(\vec{x_1}) = dim(\vec{x_2})$$

$p(BA) = n - (n - m) = m$