Linear Algebra 1: TASK #12

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Problem 1

Given two system of equations (O), (M).
$(1, 0, 1)$ and $(-1, 1, 1)$ are solutions of (O) and $(2, -3, 1)$ is a solution of (M)
Find the general solution of (O) and (M)

$$(O)\begin{cases} ax+by+cz=0 \\ fx+gy+hz=0 \end{cases} \qquad (M)\begin{cases} ax+by+cz=d \\ fx+gy+hz=k \end{cases}$$

Solution:

From plotting the solutions in equation (O)

$$\begin{cases} a+c=0 \\ -a+b+c=0 \end{cases} \implies \begin{cases} c=-a \\ b=2a \end{cases}$$

$$\begin{cases} f+h=0 \\ -f+g+h=0 \end{cases} \implies \begin{cases} h=-f \\ g=2f \end{cases}$$

We can write a coefficients matrix:

$$A = \begin{pmatrix} a & 2a & -a \\ f & 2f & -f \end{pmatrix}$$

(A) is not a zero matrix because its given that (M) has a solution, therefore $a \neq 0$ or $f \neq 0$.
w.l.o.g $a \neq 0$

$$\begin{pmatrix} a & 2a & -a \\ f & 2f & -f \end{pmatrix} \xrightarrow{R_1 \to \frac{1}{a} R_1} \begin{pmatrix} 1 & 2 & -1 \\ f & 2f & -f \end{pmatrix} \xrightarrow{R_2 \to R_2 - f \cdot R_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We can see that there is 2 free variables z and y and the general solution for (O) is
$s, t \in \mathbb{R}$ and $z = s$, $y = t \to (s - 2t, t, s)$

From Question (3.7.1c) (M) has a general solution $(s - 2t + 2, t - 3, s + 1)$.

Problem 2

A, B are squared Matrices in same size.

Part One
prove that if A, B, A+B are invertible then $A^{-1}+B^{-1}$ is also invertible and $(A^{-1}+B^{-1})^{-1} = A(A+B)^{-1}B$.

Solution:

We will show that $A^{-1} + B^{-1}$ is invertible by multiple it by the inverse matrix.
If the product will be I $\rightarrow$ then $A^{-1} + B^{-1}$ is invertible.

$(A^{-1} + B^{-1})A(A + B)^{-1}B =$
$(A^{-1}A + B^{-1}A)(A + B)^{-1}B =$
$(I + B^{-1}A)(A + B)^{-1}B =$
$B^{-1}(B + A)(A + B)^{-1}B =$ Addition is commutative $\rightarrow$
$B^{-1}IB =$
$B^{-1}B = I$
By definition (3.8.2) $A^{-1} + B^{-1}$ is invertible.

Part Two
Prove that if $A, I + AB$ invertible so also $I + BA$ invertible and $(I + AB)^{-1}A = A(I + BA)^{-1}$

Solution:
We will show that $I + BA$ is invertible by multiple it by the inverse matrix.
$(I + BA)A^{-1}(I + AB)^{-1}A =$
$(IA^{-1} + BAA^{-1})(I + AB)^{-1}A =$
$(A^{-1} + B)(I + AB)^{-1}A =$
$A^{-1}A(A^{-1} + B)(I + AB)^{-1}A =$
$A^{-1}(I + AB)(I + AB)^{-1}A =$
$A^{-1}IA = I$

By definition (3.8.2) $I + BA$ is invertible.

Problem 3

Part 1:
Prove that if A, B are n-dimensional squared matrices and $A^2 + AB + I = 0$ then $AB = BA$

Solution:

$A^2 + AB + I = 0$
$I = -A^2 - AB$
$A(-A - B) = I \quad \rightarrow \quad A^{-1} = -A - B$ From (4.5.2)

$A^2 + AB + I = 0$
$A^2 + AB + (-A - B)A = 0$
$A^2 + AB - A^2 - BA = 0$
$AB = BA$

Part 2:
$A \in M_n$ and for every $B \in M_n$, $AB \neq 0$. Prove that A is invertable.

Solution:

$Ax = 0$ is an homogenous system, therefore exists at least 1 solution.
Given that $AB \neq 0$ we can tell that the only solution for the equation is $x = 0$.
From (3.10.2) $\rightarrow$ A is invertable.

Problem 4

Calculate the following determinant of the following matrix.

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & ... & 1 \\ 1 & 1 & 1 & 1 & 1 & ... & 1 \\ 1 & 1 & 2 & 1 & 1 & ... & 1 \\ 1 & 1 & 1 & 3 & 1 & ... & 1 \\ ... & & & & & & ... \\ 1 & 1 & 1 & 1 & 1 & 1 & n-1 \end{pmatrix}$$

Solution:

For $n > 1$, for all $1 \geq i \geq n$, $i \neq 2$ we will do the following elementary operation $R_i \to R_i - R_2$.

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & ... & 1 \\ 1 & 1 & 1 & 1 & 1 & ... & 1 \\ 0 & 0 & 1 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & 2 & 0 & ... & 1 \\ ... & & & & & & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & n-2 \end{pmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & ... & 1 \\ 0 & 1 & 1 & 1 & 1 & ... & 1 \\ 0 & 0 & 1 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & 2 & 0 & ... & 1 \\ ... & & & & & & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & n-2 \end{pmatrix}$$

After these elementary operations which the determinant did not change we got a triangular matrix.
We can calculate the determinant by multiplying all the objects in the main diagonal.
$det(A) = -1 * 1 * 1 * 2 * 3....(n-2) = -(n-2)!$

Problem 5

Prove the following equation:

$$\begin{pmatrix} a_1 + b_1x & a_1x + b_1 & c_1 \\ a_2 + b_2x & a_2x + b_2 & c_2 \\ a_3 + b_3x & a_3x + b_3 & c_3 \end{pmatrix} = (1 - x^2) \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} a_1 + b_1x & a_1x + b_1 & c_1 \\ a_2 + b_2x & a_2x + b_2 & c_2 \\ a_3 + b_3x & a_3x + b_3 & c_3 \end{pmatrix} \xrightarrow{4.3.1} \begin{pmatrix} a_1 + b_1x & a_2 + b_2x & a_3 + b_3x \\ a_1x + b_1 & a_2x + b_2 & a_3x + b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \xrightarrow{R_1 \to R_1 + R_2}$$

$$\begin{pmatrix} (a_1 + b_1)(x+1) & (a_2 + b_2)(x+1) & (a_2 + b_2)(x+1) \\ a_1x + b_1 & a_2x + b_2 & a_3x + b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \xrightarrow{4.3.3} (x+1) \begin{pmatrix} a_1 + b_1 & a_2 + b_2 & a_2 + b_2 \\ a_1x + b_1 & a_2x + b_2 & a_3x + b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1}$$

$$(x+1) \begin{pmatrix} a_1 + b_1 & a_2 + b_2 & a_2 + b_2 \\ a_1(x-1) & a_2(x-1) & a_3(x-1) \\ c_1 & c_2 & c_3 \end{pmatrix} \xrightarrow{4.3.3} (x+1)(x-1) \begin{pmatrix} a_1 + b_1 & a_2 + b_2 & a_2 + b_2 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_2}$$

$$(x+1)(x-1) \begin{pmatrix} b_1 & b_2 & b_2 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1, 4.3.2} (x^2 - 1) \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_2 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Problem 6

Let A be invertible matrix 6X6 that $A^4 + 2A = 0$, calculate $|\ A\ |$.

Solution:

$A^4 = -2A$
$A^3 = -2I$
$|A^3| = |-2I|$, $|-2I| = (-2)^n = 2^6 = 64$, Because I is a triangular matrix
$|A^3| = 64$
$|A| = 4$ From (4.5.3)