THE HEBREW UNIVERSITY OF JERUSALEM

Electricity and Magnetism

Numerical Analysis of Wien Filter Velocity Selector

Amit Roth
Amit Harel

1. Introduction

We will solve at first the following problem.

Suppose we have a charged particle with a charge q, moving under the influence of a constant electric field and a constant magnetic field.

$$\begin{cases} \overrightarrow{E}(\vec{r}) = E\hat{y} \\ \overrightarrow{B}(\vec{r}) = -B\hat{x} \end{cases}$$

In $t = 0$, the particle has the following velocity, $v(t = 0) = u\hat{z}$ and $u = \dfrac{3E}{B}$.

Solution

Let's start calculating the force on such a particle. The magnetic field points in $\hat{x}$ and the velocity in $\hat{z}$. We can reduce our problem in to 2-D problem on the zy plane. We will get:

$$\begin{cases} \overrightarrow{v} = v_y\hat{y} + v_z\hat{z} \\ \overrightarrow{B} = -B\hat{x} \end{cases} \quad \Rightarrow \quad q\overrightarrow{v} \times \overrightarrow{B} = qv_yB\hat{z} - qv_zB\hat{y}$$

So the net force will be

$$\overrightarrow{F} = (qE - qv_zB)\hat{y} + qv_yB\hat{z}$$

And from newton's second law

$$\begin{cases} m\dot{v}_y = qE - qv_zB \\ m\dot{v}_z = qBv_y \end{cases}$$

We can derive from the second equation

$$m\ddot{v}_z = qB\dot{v}_y \Rightarrow \dot{v}_y = \frac{m}{qB}\ddot{v}_z$$

We will substitute our result for $\dot{v}_z$ in the first equation and we will get

$$\frac{m^2}{q^2B^2}\ddot{v}_z = \frac{E}{B} - v_z$$

We got an harmonic oscillator for v_z, the solution is

$$v_z = Acos(\omega t + \phi) + \frac{E}{B}$$

We know that $v(t = 0) = u\hat{z}$, and we know that $\dot{v}_z(t = 0) = 0$. So the full solution for v_z is

$$v_z = \frac{2E}{B}cos(\frac{qB}{m}t) + \frac{E}{B}$$

From here we can derive v_y instantly using:

$$v_y = \frac{m}{qB}\dot{v}_z = \frac{m}{qB}\frac{-2E}{B}\frac{qB}{m}sin(\frac{qB}{m}t)$$

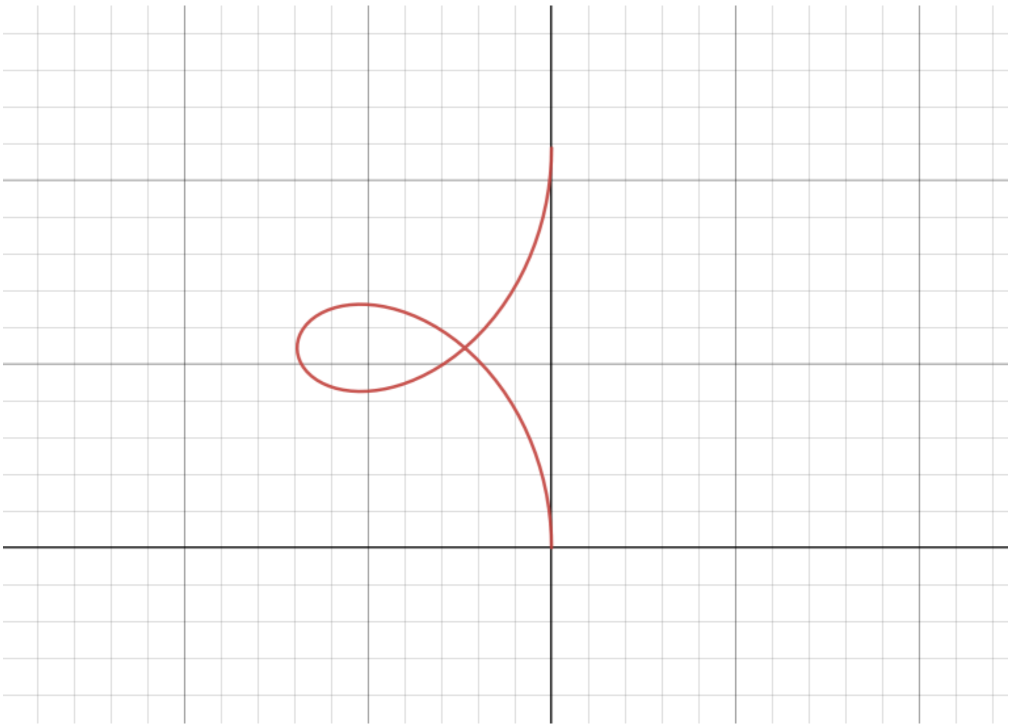
$$\Rightarrow v_y = \frac{-2E}{B}sin(\frac{qB}{m}t)$$

Now, in order to get r_y and r_z we will just integrate the velocities, using the given $\vec{r}(t = 0) = 0$.

$$r_y = \int v_y dt = \frac{2mE}{qB^2} cos(\frac{qB}{m}t) - \frac{2mE}{qB^2}$$

$$r_z = \int v_z dt = \frac{2mE}{qB^2} sin(\frac{qB}{m}t) + \frac{E}{B}t$$

The graph for $\vec{r}(t)$ for $0 \leq t \leq \dfrac{2\pi}{\omega}$ is



2. Numerical Integration

Taylor First Order

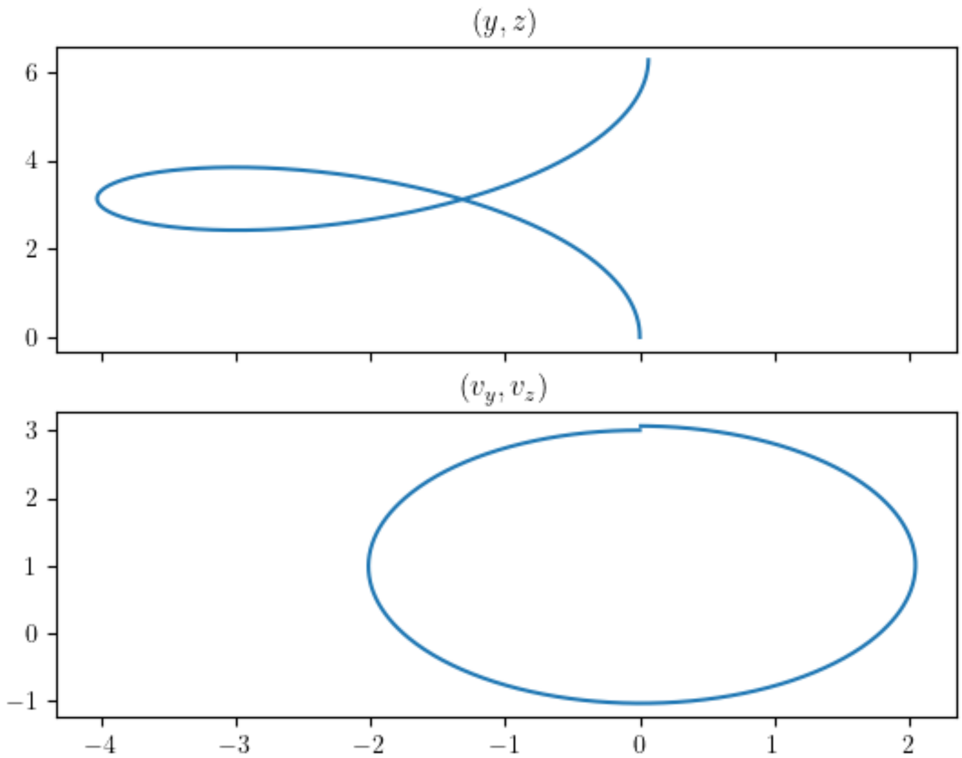
We will now solve our problem using numerical methods. Using first order Taylor approximation we will get the following relations:

$$\boldsymbol{r}(t+\Delta t) = \mathbf{r}(t) + \frac{d\boldsymbol{r}}{dt}\Delta t + O(\Delta t^2) = \mathbf{r}(t) + \boldsymbol{v}(t)\Delta t + O(\Delta t^2)$$

$$\boldsymbol{v}(t+\Delta t) = \boldsymbol{v}(t) + \frac{d\boldsymbol{v}}{dt}\Delta t + O(\Delta t^2) = \boldsymbol{v}(t) + \boldsymbol{a}(t)\Delta t + O(\Delta t^2)$$

The code is implemented in the file numerical_integration.py, and also appended below in the code appendix.

The output graphs are:



We can see that the graph (y, x) has the same form as the graph that we got from the analytical solution.

Midpoint

We will now use a more precise approximation. Using the midpoint technique, with:

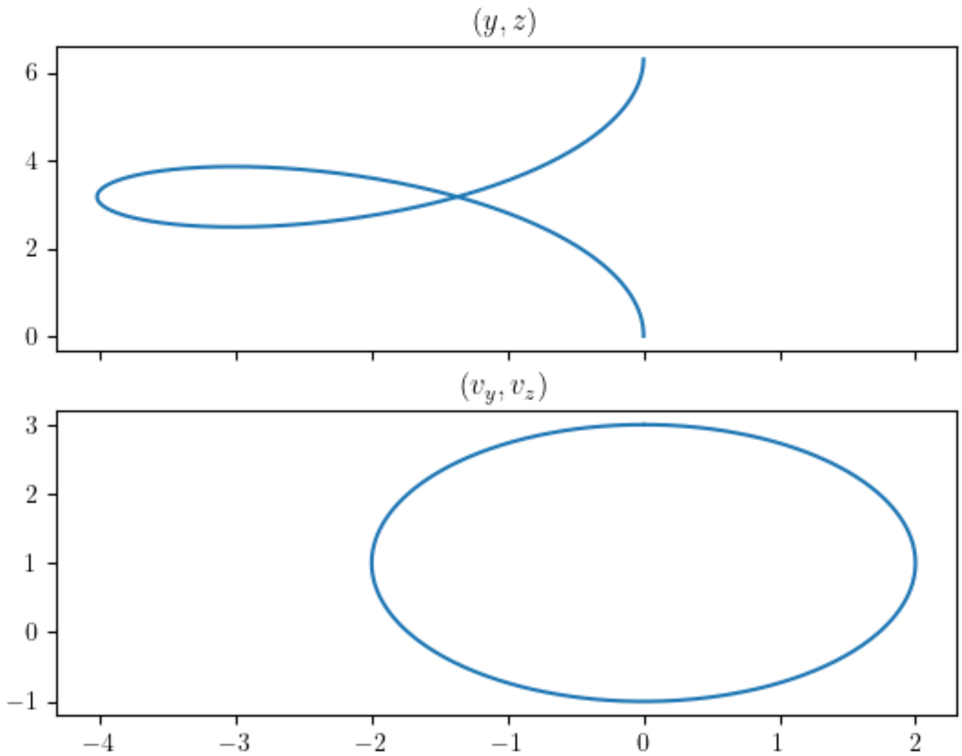
$$\begin{cases} f(t, y) = \frac{\partial f}{\partial t} \\ k_1 = \Delta t \cdot f(t, y(t)) \\ k_2 = \Delta t \cdot f(t + \frac{\Delta t}{2}, y(t + \frac{k_1}{2})) \end{cases}$$

And we will calculate the result using

$$y_{n+1} = y_n + k_2 + O(\Delta t^3)$$

The code is implemented in the file numerical_integration.py, and also appended below in the code appendix.

The output graphs are here:



We can see that the graphs are smoother.

Runge-Kutta

A more accurate technique can be implemented using:

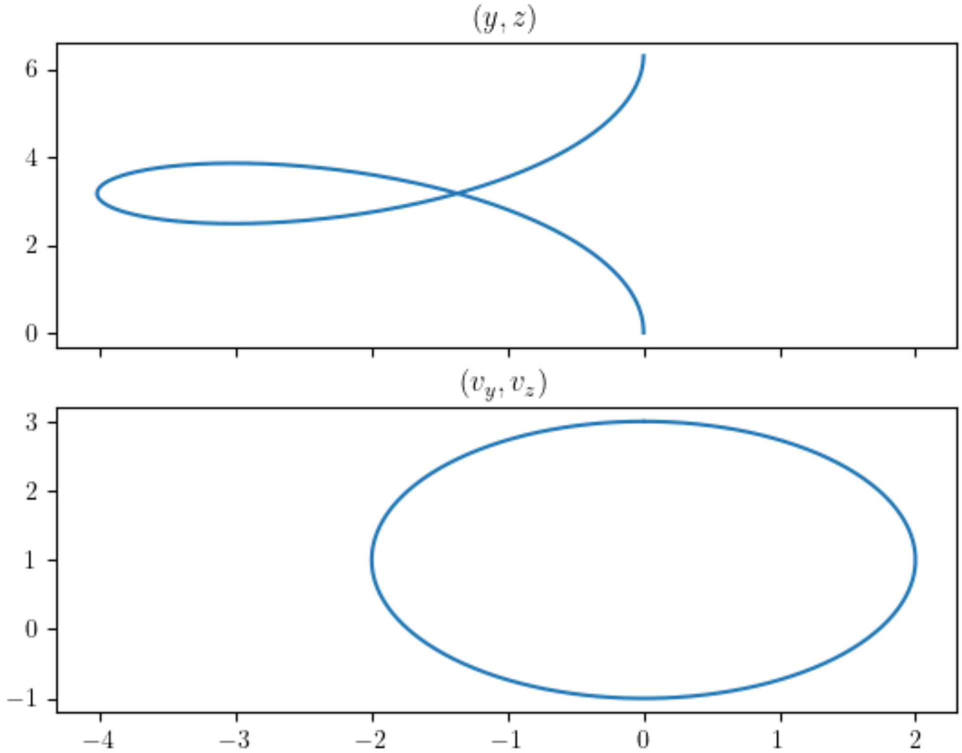
$$\begin{cases} k_1 = \Delta t \cdot f(t, y(t)) \\ k_2 = \Delta t \cdot f(t + \frac{\Delta t}{2}, y(t + \frac{k_1}{2})) \\ k_3 = \Delta t \cdot f(t + \frac{\Delta t}{2}, y(t + \frac{k_2}{2})) \\ k_3 = \Delta t \cdot f(t + \Delta t, y(t + k_3)) \end{cases}$$

And we will get

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + O(\Delta t^5)$$

The code is implemented as well in the file numerical_integration.py, and also appended below in the code appendix.

The output graphs are similar to the midpoint graphs:



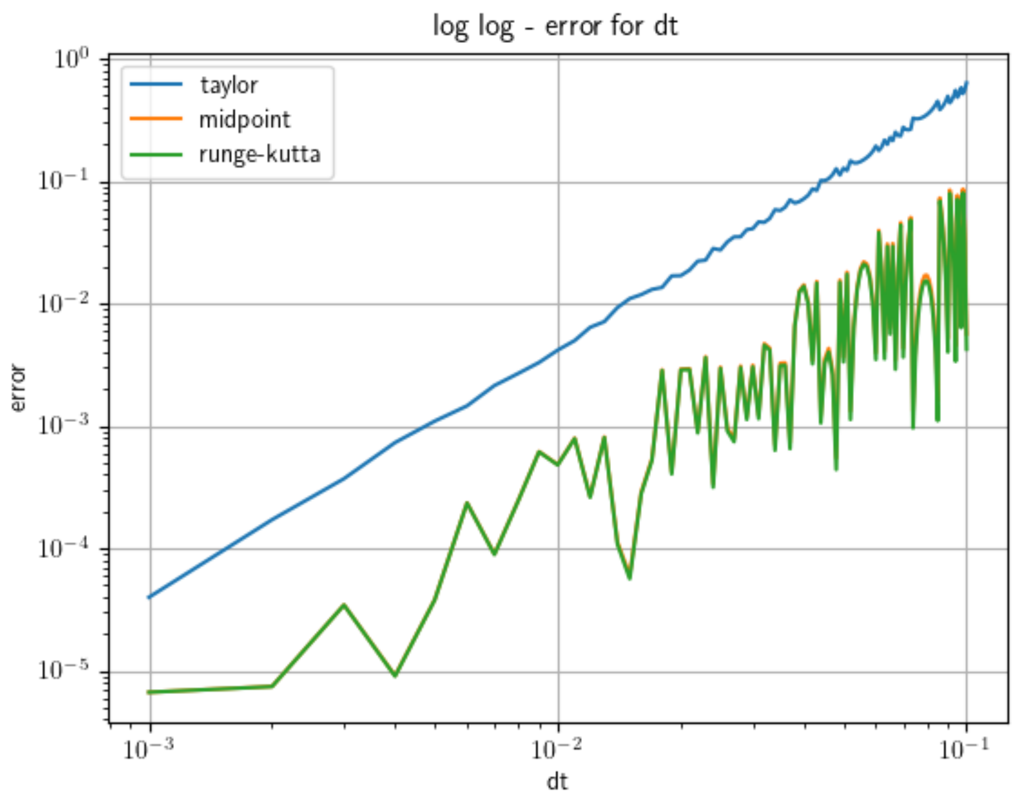
Benchmakrs

We will calculate the point that the particle will be in $T = \frac{2\pi}{\omega}$.

In the analytical solution we get:

$$\begin{cases} r_z(T) = \frac{2mE\pi}{qB^2} \\ r_y(T) = 0 \end{cases}$$

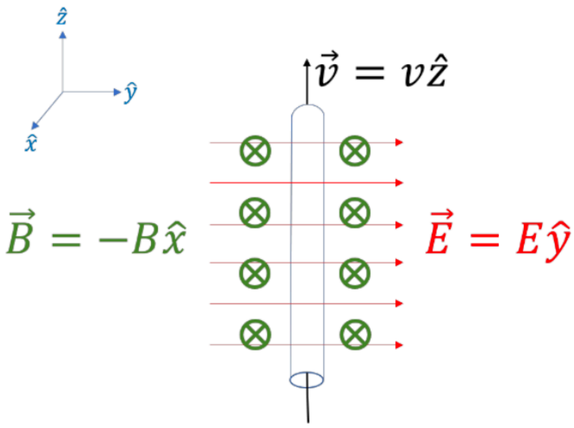
We will calculate the error (euclidian distance from the analytical to the numeric) and plot it against dt. The graph in log log scale is:



We can see that midpoint and range have similar results, but Taylor is much worse.

3. Wien Filter Velocity Selector

We will now deal with the following problem of Wien filter velocity selector.



We already solved the problem analytically in the first section.

Assume we have beam of protons traveling with average kinetic energy $E_0 = 5Mev = 8.0109 \cdot 10^{-13}J$, and pipe of length $l = 1m$ and radius $r = 3mm$.

The ratio $\frac{E}{B}$

The initial velocity we need to set in order to let the protons' beam to pass can be derived instantly from our solution and is $v_0 = \frac{E}{B}\hat{z}$

Code Appendix