THE HEBREW UNIVERSITY OF JERUSALEM

Electricity and Magnetism

Numerical Analysis of Wien Filter Velocity Selector

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1. Introduction

We will solve at first the following problem.

Suppose we have a charged particle with a charge q, moving under the influence of a constant electric field and a constant magnetic field.

$$\begin{cases} \overrightarrow{E}(\vec{r}) = E\hat{y} \\ \overrightarrow{B}(\vec{r}) = -B\hat{x} \end{cases}$$

In $t = 0$, the particle has the following velocity, $v(t = 0) = u\hat{z}$ and $u = \dfrac{3E}{B}$.

Solution

Let's start calculating the force on such a particle. The magnetic field points in $\hat{x}$ and the velocity in $\hat{z}$. We can reduce our problem in to 2-D problem on the zy plane. We will get:

$$\begin{cases} \overrightarrow{v} = v_y\hat{y} + v_z\hat{z} \\ \overrightarrow{B} = -B\hat{x} \end{cases} \quad \Rightarrow \quad q\overrightarrow{v} \times \overrightarrow{B} = qv_yB\hat{z} - qv_zB\hat{y}$$

So the net force will be

$$\overrightarrow{F} = (qE - qv_zB)\hat{y} + qv_yB\hat{z}$$

And from newton's second law

$$\begin{cases} m\dot{v}_y = qE - qv_zB \\ m\dot{v}_z = qBv_y \end{cases}$$

We can derive from the second equation

$$m\ddot{v}_z = qB\dot{v}_y \Rightarrow \dot{v}_y = \frac{m}{qB}\ddot{v}_z$$

We will substitute our result for $\dot{v}_z$ in the first equation and we will get

$$\frac{m^2}{q^2B^2}\ddot{v}_z = \frac{E}{B} - v_z$$

We got an harmonic oscillator for v_z, the solution is

$$v_z = Acos(\omega t + \phi) + \frac{E}{B}$$

We know that $v(t = 0) = u\hat{z}$, and we know that $\dot{v}_z(t = 0) = 0$. So the full solution for v_z is

$$v_z = \frac{2E}{B}cos(\frac{qB}{m}t) + \frac{E}{B}$$

From here we can derive v_y instantly using:

$$v_y = \frac{m}{qB}\dot{v}_z = \frac{m}{qB}\frac{-2E}{B}\frac{qB}{m}sin(\frac{qB}{m}t)$$

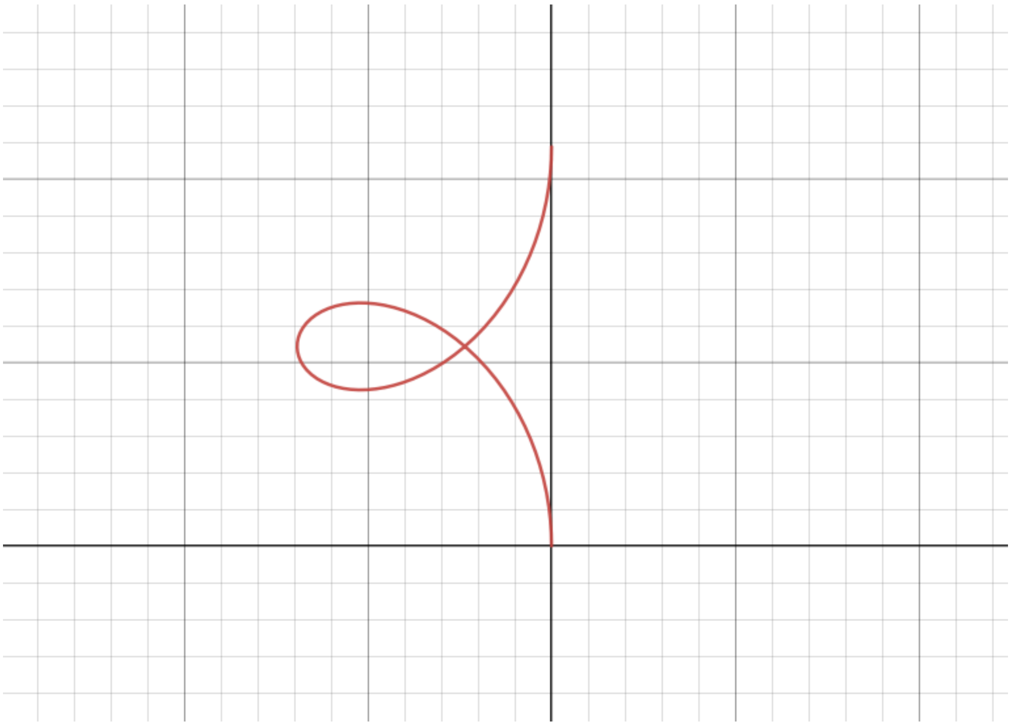
$$\Rightarrow v_y = \frac{-2E}{B}sin(\frac{qB}{m}t)$$

Now, in order to get r_y and r_z we will just integrate the velocities, using the given $\vec{r}(t=0)=0$.

$$r_y = \int v_y dt = \frac{2mE}{qB^2} cos(\frac{qB}{m}t) - \frac{2mE}{qB^2}$$

$$r_z = \int v_z dt = \frac{2mE}{qB^2} sin(\frac{qB}{m}t) + \frac{E}{B}t$$

The graph for $\vec{r}(t)$ for $0 \leq t \leq \frac{2\pi}{\omega}$ is



2. Numerical Integration

Taylor First Order

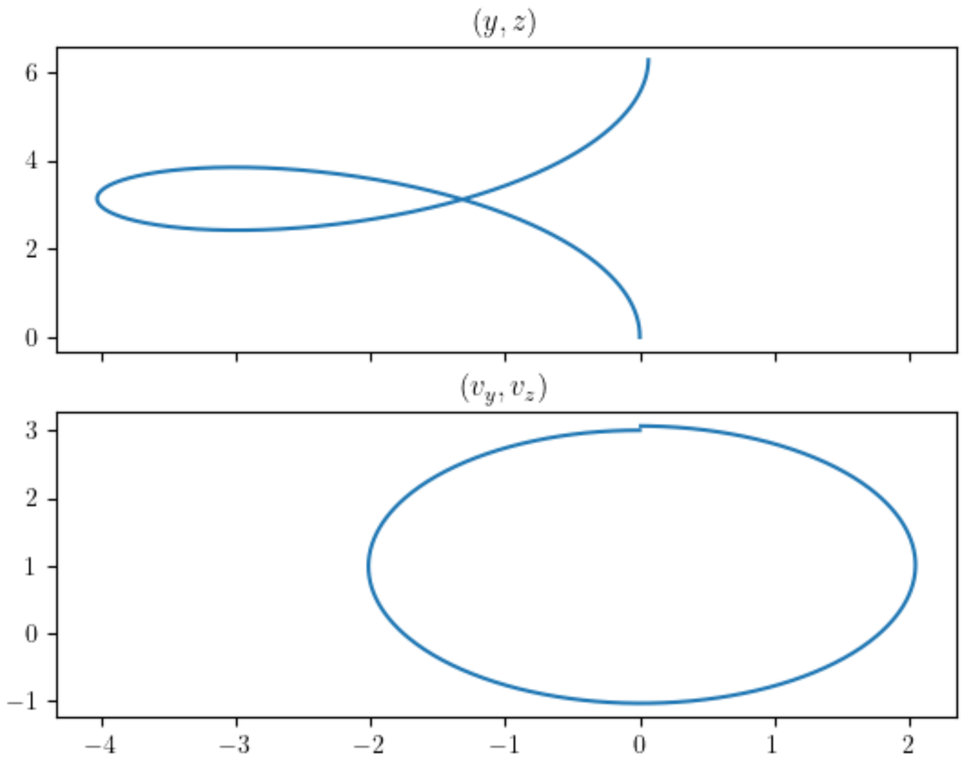
We will now solve our problem using numerical methods. Using first order Taylor approximation we will get the following relations:

$$\boldsymbol{r}(t+\Delta t) = \mathbf{r}(t) + \frac{d\boldsymbol{r}}{dt}\Delta t + O(\Delta t^2) = \mathbf{r}(t) + \boldsymbol{v}(t)\Delta t + O(\Delta t^2)$$

$$\boldsymbol{v}(t+\Delta t) = \boldsymbol{v}(t) + \frac{d\boldsymbol{v}}{dt}\Delta t + O(\Delta t^2) = \boldsymbol{v}(t) + \boldsymbol{a}(t)\Delta t + O(\Delta t^2)$$

The code is implemented in the file numerical_integration.py, and also appended below in the code appendix.

The output graphs are:



We can see that the graph (y, x) has the same form as the graph that we got from the analytical solution.

Midpoint

We will now use a more precise approximation. Using the midpoint technique, with:

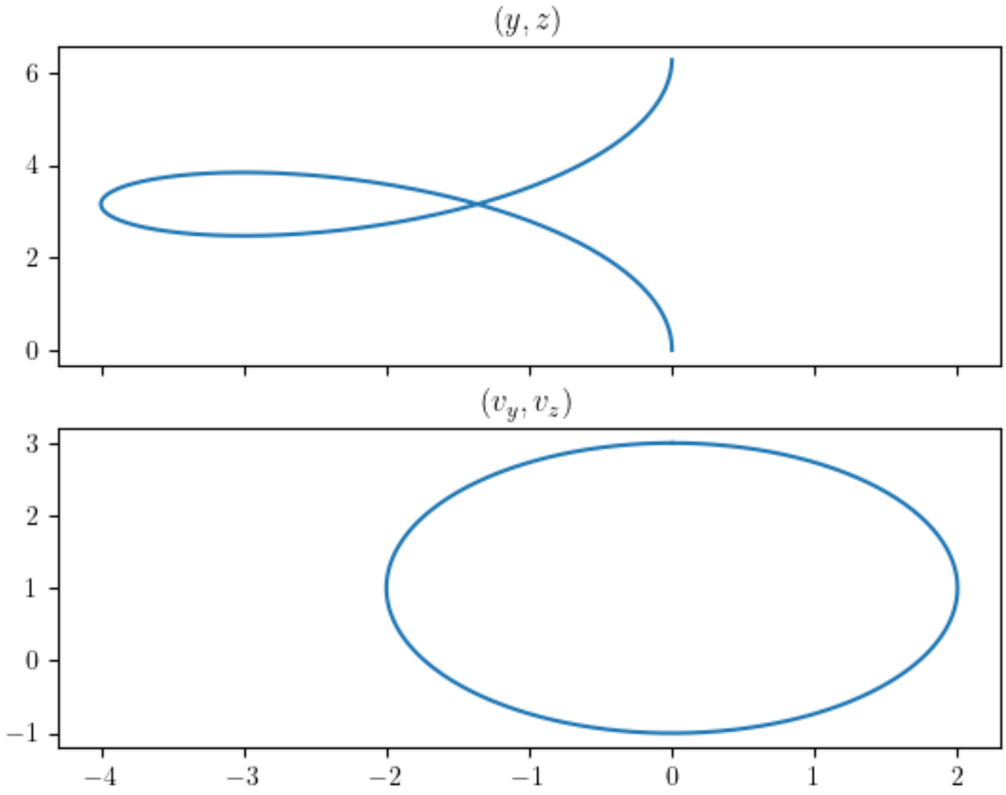
$$\begin{cases} f(t,y) = \frac{\partial f}{\partial t} \\ k_1 = \Delta t \cdot f(t, y(t)) \\ k_2 = \Delta t \cdot f(t + \frac{\Delta t}{2}, y(t + \frac{k_1}{2})) \end{cases}$$

And we will calculate the result using

$$y_{n+1} = y_n + k_2 + O(\Delta t^3)$$

The code is implemented in the file numerical_integration.py, and also appended below in the code appendix.

The output graphs are here:



We can see that the graphs are smoother.

Runge-Kutta

A more accurate technique can be implemented using:

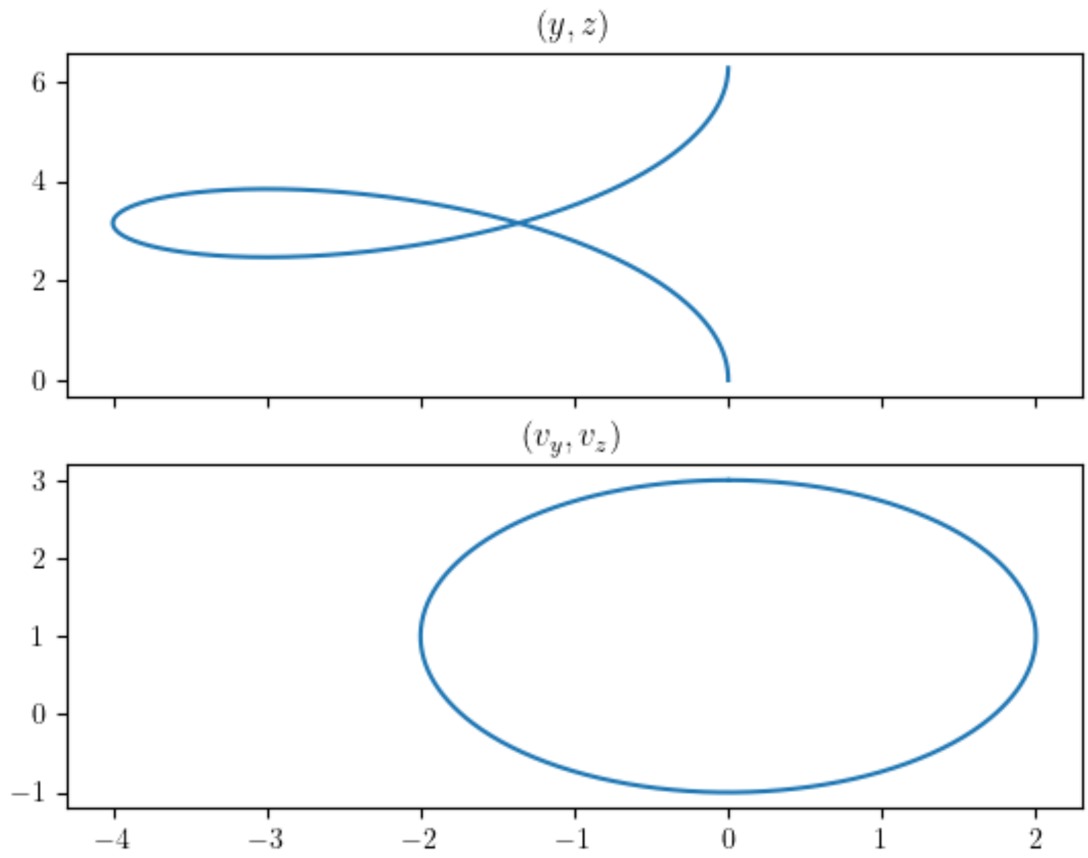
$$\begin{cases} k_1 = \Delta t \cdot f(t, y(t)) \\ k_2 = \Delta t \cdot f(t + \frac{\Delta t}{2}, y(t + \frac{k_1}{2})) \\ k_3 = \Delta t \cdot f(t + \frac{\Delta t}{2}, y(t + \frac{k_2}{2})) \\ k_3 = \Delta t \cdot f(t + \Delta t, y(t + k_3)) \end{cases}$$

And we will get

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + O(\Delta t^5)$$

The code is implemented as well in the file numerical_integration.py, and also appended below in the code appendix.

The output graphs are similar to the midpoint graphs:



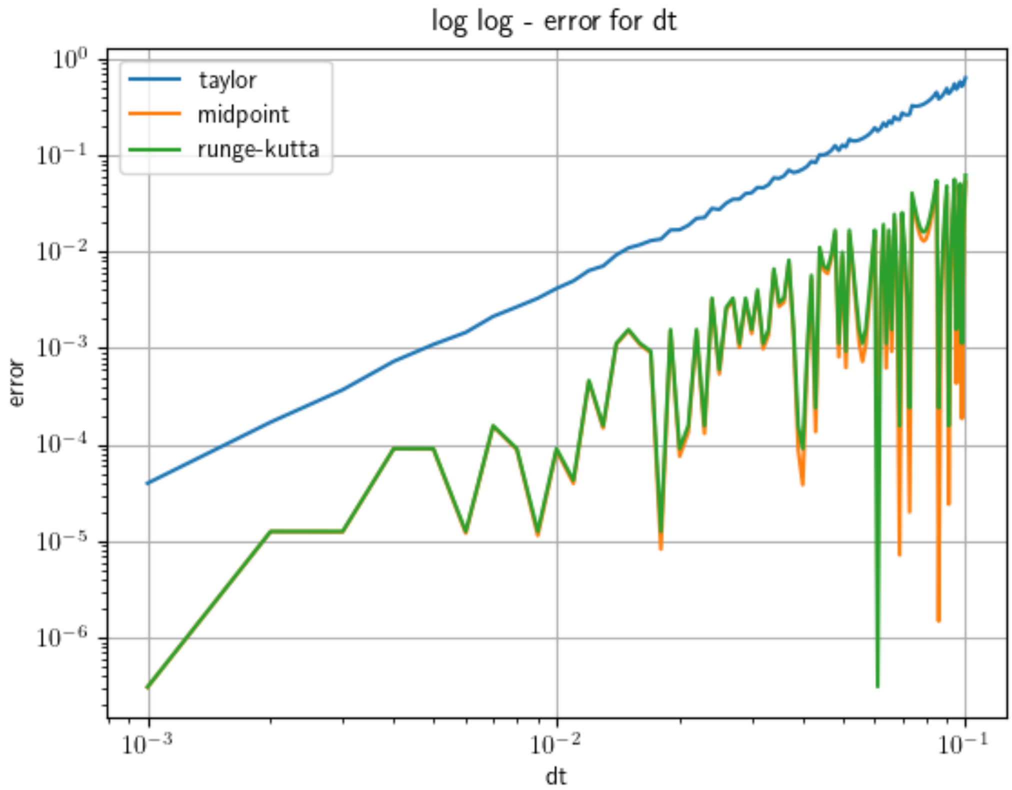
Benchmakrs

We will calculate the point that the particle will be in $T = \dfrac{2\pi}{\omega}$.

In the analytical solution we get:

$$\begin{cases} r_z(T) = \frac{2mE\pi}{qB^2} \\ r_y(T) = 0 \end{cases}$$

We will calculate the error (euclidian distance from the analytical to the numeric) and plot it against dt. The graph in log log scale is:



We can see that midpoint and range have similar results, but Taylor is much worse.

Code Appendix