General Relativity

Class 31

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Contents

PART

I

Black Holes: Conformal Diagrams and Causality

A black hole is set of events (an event defined as a point in the spacetime manifold) that can never communicate with asymptotic infinity. This means that regions in the manifold cannot communicate with regions that are arbitrarily far away. Communication is done by sending a causal curve, timelike or null, out to infinity. The boundary of a black hole is called the event horizon.

1 Conformal Transformations

A metric is conformally related to another metric if

$$\widetilde{g_{uv}} = \omega(x^u)^2 g_(uv) \tag{1.1}$$

Where ω is a function of spacectime. ω is essentially re-scaling the proper times and proper distances in the spacetime in a position dependent manner. The resulting $\widetilde{g_{uv}}$ is a conformal transform of g_{uv}. $\widetilde{g_{uv}}$ is oft4en viewed as unphysical, since g_{uv} solves Einstein's field equations for some sources, but $\widetilde{g_{uv}}$ will not solve Einstein's field equations with those sources. $\widetilde{g_{uv}}$ is not a physical metric, simply a tool that can be used to understand spacetime. For example, the Weyl tensor can be calculated with g_{uv} or $\widetilde{g_{uv}}$, while both options are equal, they are noted by

$$\widetilde{C^u_{v\rho\sigma}} = C^u_{v\rho\sigma} \tag{1.2}$$

Where the following is not equal to each other

$$\widetilde{C_{uv\rho\sigma}} \neq C_{uv\rho\sigma} \tag{1.3}$$

This is because the index is lowered using $\widetilde{g_{uv}}$ and g_{uv} respectively. In other words,

$$\widetilde{g_{u\alpha}}\widetilde{C^\alpha_{v\rho\sigma}} \neq g_{u\alpha}C^\alpha_{v\rho\sigma} \tag{1.4}$$

Another fact that makes conformal transformations useful in understanding a metric that is physical is, if given a null vector k^u (where $k^u k^v \widetilde{g_{uv}} = 0$) then k^u is null with respect to $\widetilde{g_{uv}}$:

$$k^u k^v \widetilde{g_{uv}} = \omega(k^u k^v g_{ev})^2 = 0 \tag{1.5}$$

Conformal transformations conveniently preserve the light cone structure. If two events are related by a null trajectory in g_{uv}, then they are also connected by a null trajectory in $\widetilde{g_{uv}}$.

2 Conformal Diagrams

A goal of conformal diagrams is to take a spacetime and move to another set of coordinates chosen specifically so that light cones in the $\widetilde{T}$ and $\widetilde{R}$ plane retain the same shape as in flat space, in 45°, and such that $\widetilde{T}$ and $\widetilde{R}$ range over finite values an $t \to \pm\infty$ and $r \to \pm\infty$. In the left hand side of the figure below, every point corresponds to a closed surface such as a sphere, or a point.

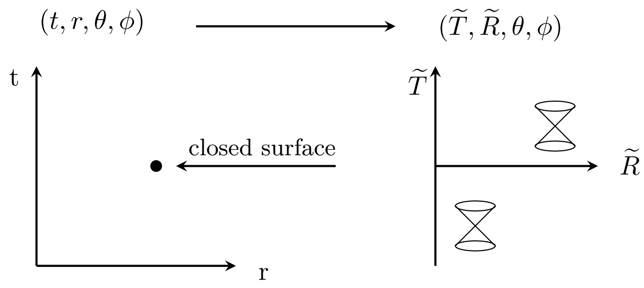

Figure 1. Goal of Conformal Diagram

The goal is to draw a spacetime diagram of all the spacetime, plus infinity. For example, lets look at Minkowski Spacetime. First, the line element is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{2.1}$$

Now the coordinates can be transformed to $\widetilde{T}$ and $\widetilde{R}$, with $0 \leq \widetilde{R} < \pi$ and $-\pi < \widetilde{T} < \pi$, and where $|\widetilde{T}| + \widetilde{R} < \pi$. With these coordinates, the Minkowski line element becomes

$$ds^2 = \frac{1}{(cos{-}\tilde{T} + cos\tilde{R})^2}(d\tilde{T} + d\tilde{R} + sin(\tilde{R})^2 d\omega^2) \tag{2.2}$$

This is a conformal factor multiplied by a simple metric, so if the first term were to instead be written as ω, a conformal factor, then

$$d\tilde{s}^2 = \omega^2 ds^2 = (-d\tilde{T}^2 + d\tilde{R}^2 + sin(\tilde{R})^2 d\omega^2) \tag{2.3}$$

The metric $d\tilde{s}$ is a simpler metric that can now be drawn with a conformal diagram, with the restricted ranges of $|\widetilde{T}| + \widetilde{R} < \pi$. The $\widetilde{T} = \mathit{constant}$ surfaces cna never be inside the light cone, while $\widetilde{R} = \mathit{constant}$ surfaces are required to be inside a light cone.

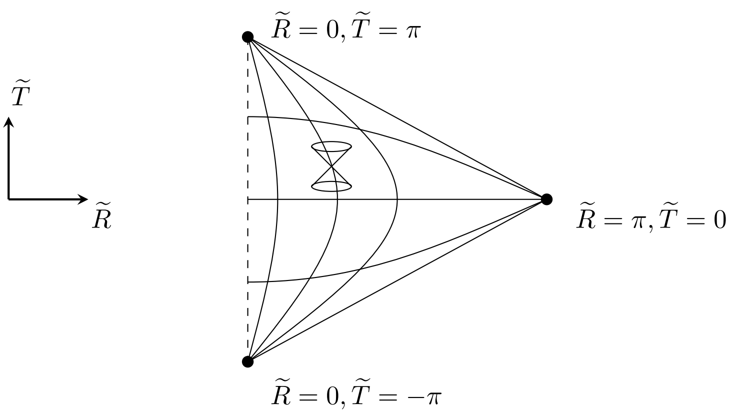

Figure 2. Conformal Diagram of Minkowski Space: Restricted by $|\widetilde{T}| + \widetilde{R} < \pi$

Steps to derive conformal diagram for Minkowski (Can be found in Appedix H of Carroll)

1. use null coordinates

$$\begin{aligned} u &= t - r & v &= t + r \\ t &= \frac{v+u}{2} & r &= \frac{v-u}{2} \\ -\infty &> u > \infty & -\infty &> v > \infty \\ & & v &\geq u \end{aligned} \tag{2.4}$$

Where the $v \geq u$ condition comes from the fact that r must be positive. In these coordinates, the metric can be written as

$$ds^2 = \frac{-1}{2}(dudv + dvdu)^2 + r^2\Omega^2 \tag{2.5}$$

Remembering that $dudv + dvdu = 2dudv$

2. Compactify coordinates

$$\begin{aligned} u &= tan(\widetilde{U}) & v &= tan(\widetilde{V}) \\ -\frac{\pi}{2} &< tan(\widetilde{U}) < \frac{\pi}{2} & -\frac{\pi}{2} &< tan(\widetilde{V}) < \frac{\pi}{2} \end{aligned} \tag{2.6}$$

Now the coordinate system is such that whole metric can be written in a finite range of coordinates.

$$du = sec^2(\widetilde{U}d\widetilde{U}) \quad dv = sec^2(\widetilde{V}d\widetilde{V}) \tag{2.7}$$

$$r^2 = (\frac{sin(\widetilde{V} - \widetilde{U}}{2cos(\widetilde{V}cos(\widetilde{U}))})^2 \tag{2.8}$$

Now

$$ds^2 = \frac{1}{4cos^2\widetilde{U}cos^2\widetilde{V}}(-4d\widetilde{U}d\widetilde{V} + sin^2(\widetilde{V} - \widetilde{U})d\Omega^2) \tag{2.9}$$

3. define $(\widetilde{T}, \widetilde{R})$, $\widetilde{R} = \widetilde{v} - \widetilde{R}$, and $\widetilde{R} = \widetilde{V} + \widetilde{U}$ Resulting in

$$\begin{aligned} ds^2 &= \frac{1}{(4cos\widetilde{U}cos\widetilde{V})^2} \quad (-d\widetilde{T}^2 + d\widetilde{R}^2 + sin^2\widetilde{R}d\Omega^2) \\ 0 &\leq \widetilde{R} < \pi \qquad\qquad |\widetilde{T}| + \widetilde{R} < \pi \end{aligned} \tag{2.10}$$

4. Consider the conformally rescaled metric, then draw the spacetime diagrams

$$d\widetilde{S}^2 = \omega^2 ds^2 \quad \omega = cos\widetilde{T} + cos\widetilde{R} \tag{2.11}$$

$$d\widetilde{S} = -d\widetilde{T}^2 + d\widetilde{R}^2 + sin^2(\widetilde{R})d\Omega^2 \tag{2.12}$$

We can now draw the allowed portions such a spacetime, remembering the constraint $|\widetilde{T}| + \widetilde{R} < \pi$, which confines the diagram to the non shaded region in the diagram. This cylinder can be 'unrolled' , with only the allowed portions for Minkowski Space, resulting in the conformal diagram from Figure 3. In these conformal diagrams, it is important to remember that the boundaries of the diagram and the points are not included in the space.

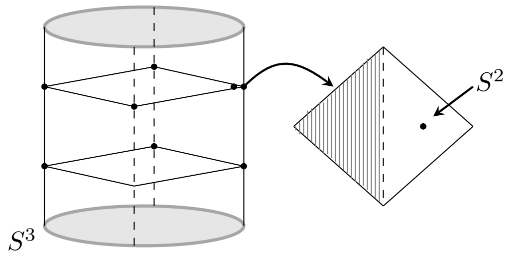

Figure 3. Conformal Diagram Creation: from S^3 to S^2 space

3 Conformal Diagrams and Infinities

Light cones are at 45° on this diagram, at the expense of other coordinates being bent in an odd way. This spacetime diagram is drawn on the conformally transformed metric, and the three points on the diagram are not points of the Minkowski Metric, rather they are different aspects of infinity. These points are placed on the diagram so that it becomes a manifold with a boundary.
This boundary contains five separate components, as labelled in figure 4. These are the infinities you would reach if you stayed at a fixed $\widetilde{R}$ in Minkowski Space. However, a curve at constant time would eventually reach spacelike infinity i^0. The last two components are $\mathscr{I}^+$ and $\mathscr{I}^-$. These are the solid boundaries that complete the triangle of the conformal diagram, and represent future and past null infinity respectively. All future directed light rays arrive at $\mathscr{I}^+$, and all past directed light rays at $\mathscr{I}^-$. In other words, $\mathscr{I}^-$is where all null rays come from as they enter into the spacetime, and $\mathscr{I}^+$ is where they arrive in the future.
Illustrated in blue is a representation of the trajectory of an observer on a surface of very large $\widetilde{R}. This trajectory only deviates from$ I$^+$ and $\mathscr{I}^-$ in the middle of the diagram. This makes the boundaries of $\mathscr{I}^+$ and $\mathscr{I}^-$ and idealization of a real observer's trajectory that is at a very far distance from the origin.

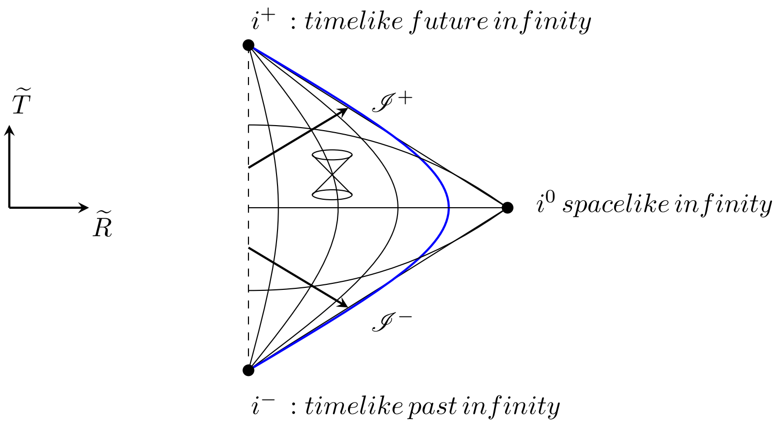

Figure 4. Conformal Diagram of Minkowski Space: Restricted by $|\widetilde{T}| + \widetilde{R} < \pi$

Lets look at another conformal diagram, the one for the Schwarzschild Black Hole. In a previous class, the cruscal the conveniently had light rays at 45°

However, these cruzcal coordinates still need some changes to create the conformal diagram. These coordinates need to be compactified. U and V are rescalings of u and v

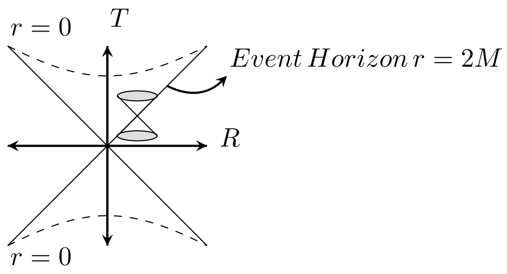

Figure 5. SBH digram in cruzcal coords

coordinates

$$\begin{aligned} u &= t - r_* \qquad v = t + r_* \\ T &= \frac{V+U}{2} \qquad R = \frac{V-U}{2} \\ U &= -e^{\frac{-u}{4m}} \qquad V = e^{\frac{v}{4m}} \end{aligned} \tag{3.1}$$

Define compactified coordinates

$$V = tan\widetilde{V} \quad U = tan\widetilde{U} \tag{3.2}$$

The same steps can be taken to transform the metric, identify a conformal factor, multiplying $d\widetilde{V}d\widetilde{U}$, then finally define new coordinates,

$$\widetilde{T} = \widetilde{V} + \widetilde{U} \qquad \widetilde{R} = \widetilde{V} - \widetilde{U} \tag{3.3}$$

then get rid of conformal factor, to look at the conformally related metric.

$$d\widetilde{s}^2 = -d\widetilde{T}^2 + d\widetilde{R}^2 + (function\, of\, \widetilde{T}, \widetilde{R})d\Omega^2 \tag{3.4}$$

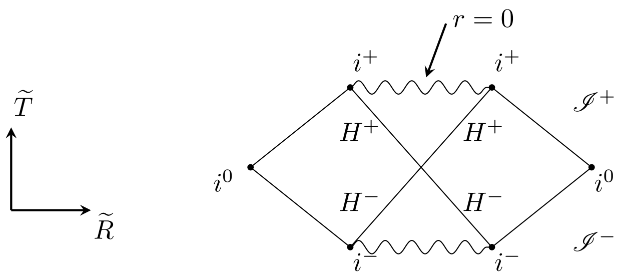

Figure 6. Conformal Diagram for Schwarzschild

The conformal diagram for a spherically symmetric star is
Looking at X of the diagram, it looks like region I of the Schwarzschild Black hole.
If a star were to collapse to form a black hole,the following conformal diagram could describe the space, with the shaded region being the stellar interior.

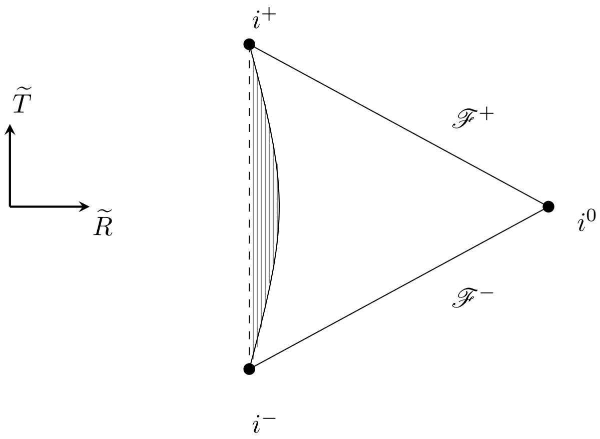

Figure 7. What a conformal diagram will look like for our coordinates

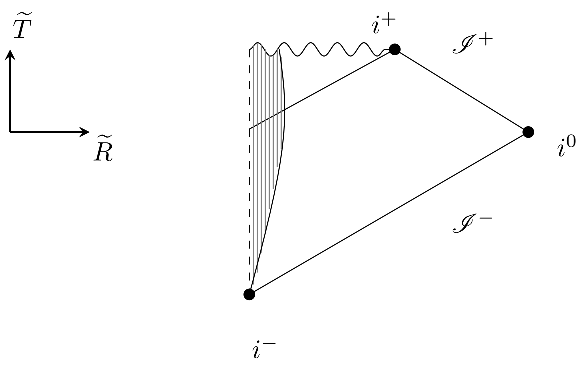

Figure 8. Conformal Diagram of Star Collapsing to Form Black Hole