General Relativity

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1 Aside: A Comment on $R^{\rho}{}_{\sigma\nu\mu}$

Clearly if we are in flat space, we can choose coordinates such that:

$$g_{\mu\nu} = \eta_{\mu\nu} \text{ and } \partial_\rho g_{\mu\nu} = 0 \text{ everywhere in } \mathcal{M}$$

In other words, $g_{\mu\nu}$ are all constant everywhere in $\mathcal{M}$

* so $\Gamma^{\rho}_{\mu\nu} = 0 \Rightarrow R^{\mu}{}_{\nu\rho\sigma} = 0$ in these coordinates.

But $R^{\mu}{}_{\nu\rho\sigma} = 0$ is a tensor equation, so it must be true in <u>all coordinates.</u>

* Therefore, if $\mathcal{M}$ is flat space $\Rightarrow R^{\mu}{}_{\nu\rho\sigma} = 0$

Now, what about the converse? If we find $R^{\mu}{}_{\nu\rho\sigma} = 0$, are we in flat space (can we find coordinates where $g_{\mu\nu}$ is constant)?

* Yes, $\rightarrow R^{\mu}{}_{\nu\rho\sigma} = 0 \Rightarrow g_{\hat{\mu}\hat{\nu}} =$ const for some $x^{\hat{\mu}}$, everywhere in $\mathcal{M}$ (not just at a point).

* See Carroll Section 3.6 for a quick proof.

So $R^{\mu}{}_{\nu\rho\sigma}$ can distinguish flat from curved spacetimes, and actually it is the defining feature of spacetime manifolds.

2 Integration on Manifolds

You may have noticed that the volume element in curvilinear coordinates looks different from the one in Cartesian coordinates:

$$dV = dx\,dy\,dz = r^2 sin\theta\,dr\,d\theta\,d\phi \text{ for example:}$$

* notice, dV = $\sqrt{|det(g_{ij})|}dr\,d\theta\,d\phi$ since $g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{bmatrix}$

* Indeed, this is the correct rule for curved spacetimes:

The volume element, dV, is equal to $\sqrt{|g|}\,d^n x$ in any coordinates. There is a reason for this. Formally, on curved manifolds, integration is a map from objects into real numbers. In this case, it is n-forms (for dim = n) into reals:

$$\int_U : \Lambda^n \rightarrow \mathbb{R},$$

$$\int_U \underset{\approx}{\omega} = a,$$

where U is a region of $\mathcal{M}$

* recall, n-forms are just of the form

$$\underset{\approx}{\omega} = f(x)\,\underset{\approx}{\epsilon} \qquad \text{since}$$

$$\underset{\sim}{\epsilon} = \frac{\sqrt{|g|}}{n!}\tilde{\epsilon}_{\mu_1\ldots\mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu^n} \qquad \leftarrow \text{as a form} \tag{2.1}$$

$$= \sqrt{|g|}\, dx^0 \wedge dx^1 \wedge \ldots \wedge dx^{n-1} \qquad \leftarrow \text{exploit antisymmetry} \tag{2.2}$$

$$= \sqrt{|g|}\,\tilde{\epsilon}_{\mu_1\ldots\mu_n} dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_n} \qquad \leftarrow \text{as a tensor} \tag{2.3}$$

and there is only 1 basis element of n-forms

* recall, the only basis element for n-forms is

$$\underset{\sim}{dx}^0 \wedge \underset{\sim}{dx}^1 \wedge \ldots \wedge \underset{\sim}{dx}^{n-1}$$

so:

$$\underset{\sim}{\omega} \propto \underset{\sim}{\epsilon} = \frac{1}{n!}\sqrt{|g|}\,\tilde{\epsilon}_{\mu_1\ldots\mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n} \tag{2.4}$$

$$= \sqrt{|g|}\, dx^0 \wedge dx^1 \wedge \ldots \wedge dx^n \qquad \leftarrow \text{antisymmetry of } \hat{\epsilon}_{\mu_1} \text{ and } \wedge \text{ product} \tag{2.5}$$

$$:= \sqrt{|g|}\, d^n x \tag{2.6}$$

* With this association between $d^n x$ and the ordered wedge product we have a fully covariant expression for integration on $\mathcal{M}$:

$$\boxed{\int_U \underset{\sim}{\omega} = \int_U f(x)\,\underset{\sim}{\epsilon} = \int_{\emptyset(U)} f(x)\,\sqrt{|g|}\, d^n x}$$

Now the integrand is manifestly covariant. Note that with a chart $\emptyset : \mathcal{M} \to \mathbb{R}$ we can evaluate integral using our usual methods.

At some point we will want to discuss hypersurfaces in $\mathcal{M}$. For now let's just mention that we have Stokes Theorem on Manifolds:

$$\boxed{\int_U \nabla_\mu V^\mu \sqrt{|g|}\, d^n x = \int_{\partial U} n_\mu V^\mu \sqrt{|\gamma|}\, d^{n-1} y}$$

with γ_{ij} the "induced metric" on the lower dimensional space ∂U,
y^k the coordinates we use on this boundary, and
n^μ the spacetime vectors normal to the surface.

3 Einstein's equation from an Action Principle

Notice $[S] = [energy][time] = (length)^2$
$[\sqrt{|g|}\, d^4 x] = L^4$ (think about flat space, Cartesian coordinates)
So, for

$$S = \int \sqrt{-g}\, d^4 x\ \mathcal{L}$$

we want $[\mathcal{L}] = \frac{1}{L^2}$

* We need a scalar quantity, preferably with no more than second derivatives in $g_{\mu\nu}$, with the right units so we don't have to introduce new constants.

* Ricci scalar R fits the bill.

$$\boxed{S = \int R\,\sqrt{-g}\,d^4x}$$

is the Einstein-Hilbert action

* To get field equations for $g_{\mu\nu}$, we can vary S with respect to $g_{\mu\nu}$ and minimize this action.

* Actually it is equivalent, but easier, to vary $g^{\mu\nu}$

$$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$$

$S \to S + \delta S \quad \longleftarrow$ find δ S via expanding $S[g^{\mu\nu} + \delta g^{\mu\nu}]$ and set $\delta S = 0$ for arbitrary variations of $g^{\mu\nu}$, vanishing on the boundary of $\mathcal{M}$

* Note $\delta(g^{\mu\alpha}g_{\alpha\nu}) = (\delta g^{\mu\alpha})g_{\alpha\nu} + g^{\mu\alpha}\delta g_{\alpha\nu} = \delta(\delta^{\mu}_{\nu}) = 0$

$\Rightarrow g^{\mu\alpha}\delta g_{\alpha\nu} = -g_{\alpha\nu}\delta g^{\mu\alpha} \Rightarrow g_{\beta\mu}g^{\mu\alpha}\delta g_{\alpha\nu} = \delta^{\alpha}_{\beta}\delta g_{\alpha\nu} = \delta g_{\beta\nu} = -g_{\beta\mu}g_{\alpha\nu}\delta g^{\mu\alpha}$

$$\Rightarrow \boxed{\delta g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\partial g^{\alpha\beta}}$$

$$\delta S = \int d^n x \delta(\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu})$$

$$= \int d^n x\sqrt{-g}\, R_{\mu\nu}\,\delta g^{\mu\nu} + \int d^n x\sqrt{-g}\, g^{\mu\nu}(\delta R_{\mu\nu}) + \int d^n x R\,\delta(\sqrt{-g})$$

The first term has the desired form. The second term becomes a boundary term via Stokes' Theorem:

$$\int d^n x\sqrt{-g}\, g^{\mu\nu}(\delta R_{\mu\nu}) = d^n x\sqrt{-g}\,\nabla_\alpha[g_{\mu\nu}\nabla^\alpha\,\delta g^{\mu\nu} - \nabla_\beta\,\delta g^{\alpha\beta}]$$

With $V^\alpha = g_{\mu\nu}\nabla^\alpha\,\delta g^{\mu\nu} - \nabla_\beta\,\delta g^{\alpha\beta}$

$$\int_{\mathcal{M}} d^n x\sqrt{-g}\,\nabla_\alpha V^\alpha = \int_{\partial\mathcal{M}} d^{n-1}\sqrt{\gamma}\,n_\mu V^\mu$$

and we set $\delta g^{\mu\nu} = 0$ on $\partial\mathcal{M}$ as usual for variational problems.

For the last term, note $\ln(\det \underline{\mathrm{M}}) = \mathrm{Tr}(\ln \mathrm{M})$, where the first "ln" is a regular log, and the second one is a matrix log defined via infinite series.

* $\delta ln(det\, M) = \frac{1}{det\, M} = Tr(\underline{M}^{-1}\delta\underline{M}) = Tr(\delta\underline{M}\,\underline{M}^{-1})$

which applied to g, we get

$\delta g = g(g^{\alpha\beta}\delta g_{\beta\alpha}) = g(-g_{\alpha\beta}\delta g^{\alpha\beta})$

* $\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}(-\delta g) = -\frac{(-g)}{2\sqrt{-g}} g_{\mu\nu}\delta g^{\mu\nu} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}$

$$\Rightarrow \delta S = \int d^n x \sqrt{-g}[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R]\delta g^{\mu\nu} = 0 \qquad \forall \delta g^{\mu\nu}$$

$$\Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0} \qquad \text{(no matter)}$$

With matter, we add $S_{matter} = S_M$, and vary with respect to $\delta g^{\mu\nu}$:

$\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}} = G_{\mu\nu}$ and if $S = \frac{1}{16\pi G} S_H + S_M$,

$\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}} = 0$

where $\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}}$ defines $T_{\mu\nu}$, and so:

$$\boxed{T_{\mu\nu} = -\frac{1}{2\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}}}$$

This gives us a way to construct $T_{\mu\nu}$ given a field Lagrangian.