General Relativity

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Contents

PART

I

Singularity Theorems

1 Singularity Theorem

So far, we have covered the main tools we will use to prove a singularity theorem. We covered how Raychaudhuri's equation combined with some energy conditions on spacetime which tell how neighboring geodesics can collide with each other. We can determine that a congruence of such geodesics begins to converge. This can occur if spacetime has an attractive curvature, the geodesics will collide within a finite amount of time.

From global definitions of causality, we looked at the boundary of the future of the set. The boundary of the future of the set must be made of null geodesics that don't collide with other geodesics. We can use this to provide singularity theorems.

Schwarzschild's solution as well as the Friedmann-Robertson-Walker metric, along with other solutions which have been discovered indicate that singularities exist. Oppenheimer-Snyder collapse (Figure 1) can be visualized at occuring at some initial time on a hypersurface Σ with a spherically symmetric star which collapses. The resulting spacetime shows a finite event horizon emerging, which the star will fall behind and form a singularity, which is a point of infinite curvature.

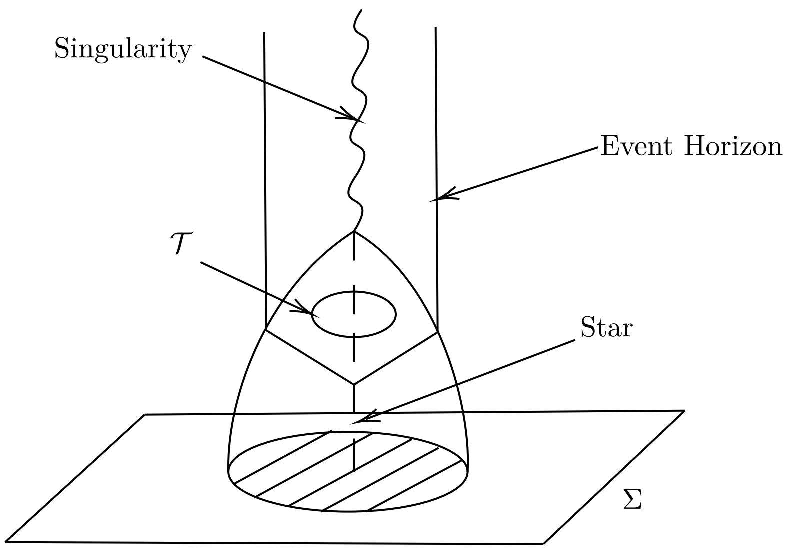

Figure 1. Oppenheimer-Snyder collapse of a star. Where $\mathcal{T}$ is a trapped surface.

We have shown with the conformal diagram (Figure 2) that the singularity sits in the future of all observers.

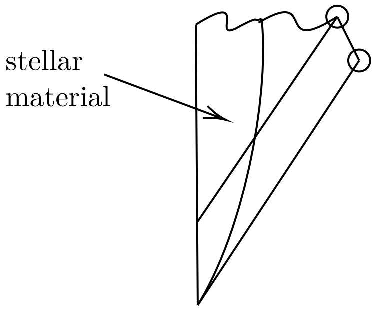

Figure 2. Conformal Diagram of a Singularity

Singularities do not occur only in special fine-tuned situations, but are generic. A scenario to prove this would be a trapped surface ($\mathcal{T}$), which is a 2-D surface which is closed and finite in size (compact).

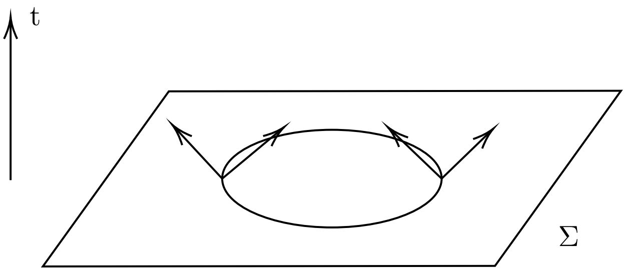

Figure 3. Hypersurface (Σ) with two families of normal null vectors.

For each surface (Figures 3 and 4), there are two null normal vectors to the surface, which can be ingoing our outgoing.

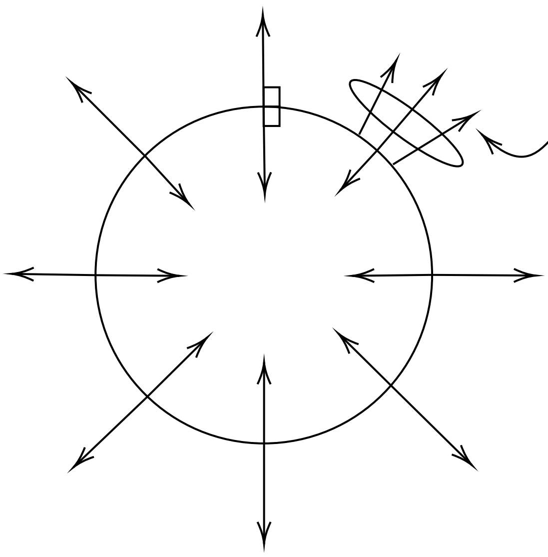

Figure 4. A hyper surface with a congruence of outgoing null normal geodesics of positive expansion.

The outgoing null geodesics will form a congruence which form a positive expansion ($\theta > 0$) on a typical surface. On a trapped surface $\mathcal{T}$ (Figure 5), both the inward and outward null normal geodesics have negative expansion ($\theta < 0$).

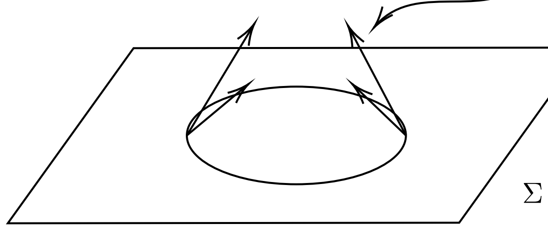

Figure 5. Outward null normal geodesics have $\theta < 0$ on a trapped surface $\mathcal{T}$

This can occur in any situation where gravity is strong enough at any instance of time on the surface. Penrose proved that if there is a trapped surface, there will inevitably be a singularity in the future of that trapped surface.

Theorem (Penrose 1965):

1. Manifold $\mathfrak{M}$ is globally hyperbolic with non-compact Cauchy surface.
2. Matter obeys the Null Energy Condition (NEC) (1.1).

$$\begin{aligned} T^{\mu\nu}k^{\mu}k^{\nu} &\geq 0 \\ R^{\mu\nu}k^{\mu}k^{\nu} &\geq 0 \end{aligned} \tag{1.1}$$

1

[1] *(1.1)Positive definite* $\forall k^{\mu}$

3. Spacetime contains a trapped surface $\mathcal{T}$.

If these three conditions are satistfied, then at least one null geodesic emanating from $\mathcal{T}$ has finite length.

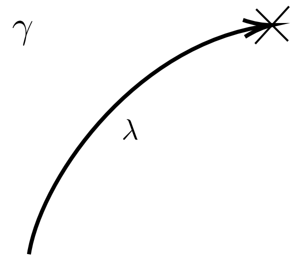

Figure 6. Geodesic γ must end in a proper affine distance (λ)

The geodesic must end at some affine distance, although we don't necessarily know what happens to it, but that it must end. This is a proof by contradiction. This theorem applies to any spacetime, which will inevitably terminate one of the geodesics. However, we still would not know what kind of singularity or the properties of the singularity the geodesic terminates in or the full number of terminating geodesics.

2 Sketch of Singularity Theorem Proof

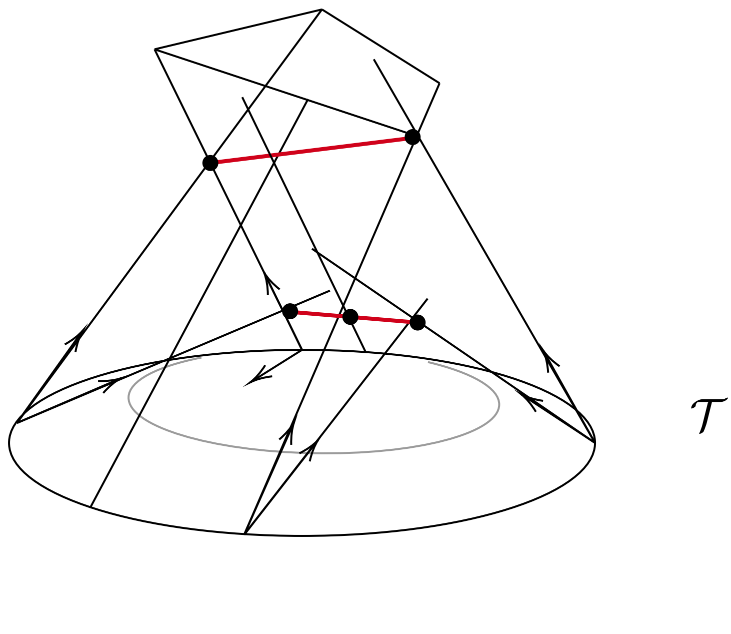

Figure 7. Negative expansion of null geodesics with length λ forming B

Assume we can continue all in-going and out-going normal geodesics a certain length.

Let θ_0 be $\max(\theta)$ of the expansion of all the geodesic congruences.

$\theta_0 <$ but close to 0, because all geodesics have negative expansion.

Extend all geodesics with affine paramter λ where $\lambda \in [0, \frac{2}{|\theta_0|}]$

Then every geodesic hits a conjugate point and leaves the boundary of the future of the trapped set (B) (2.1).

$$\dot{I}^+(\mathcal{T}) = B \tag{2.1}$$

The set A can be defined as:

$$A : \mathcal{T}_{-x}[0, \frac{2}{|\theta_0|}] \cup \mathcal{T}_{+x}[0, \frac{2}{|\theta_0|}] \tag{2.2}$$

Where $\mathcal{T}_{-x}[0, \frac{2}{|\theta_0|}]$ represents the lower sheet created by the in-going geodesics (gray area in Figure 7) and $\mathcal{T}_{+x}[0, \frac{2}{|\theta_0|}]$ is the upper sheet created by out-going geodesics.

A is compact, or a finite set of points that are closed on the manifold.

B is contained in A

B starts at the trapped surface $\mathcal{T}$ and stops where the geodesics collide (red lines in Figure 7).

$\Rightarrow B$ is compact.

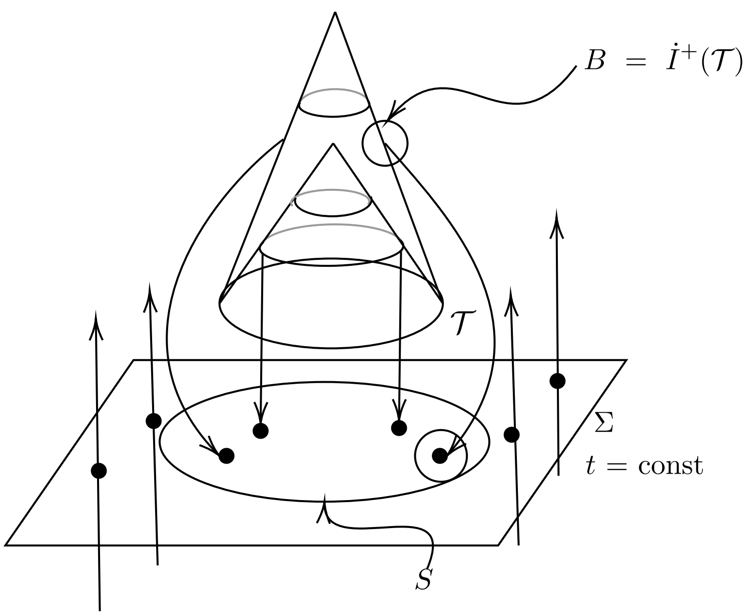

Figure 8. Representation of B, where S is a set of points in the Cauchy surface Σ. Integral curves give a map Ψ from B to $S \subset \Sigma$.

This is a huge problem for B to be compact. Because the spacetime is globally hyperbolic, the Cauchy surface is like a moment in constant time. And in globally hyperbolic spacetime, there is a set of integral curves of timelike vector fields which pierce the Cauchy surface.

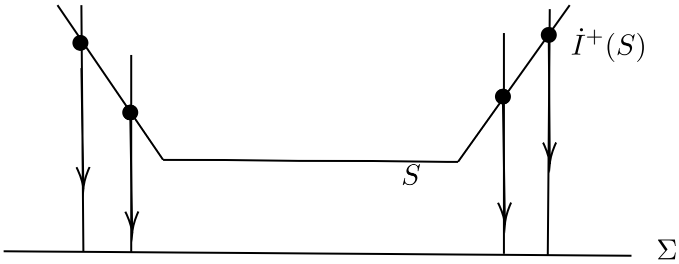

Figure 9. Set of integral curves piercing S from Cauchy surface Σ.

It must be true that the set of integral curves flowing from Σ pierces each point of S once and only once. Integral curves of t^μ associated with t on $\mathfrak{M}$ give a map:

$$\Psi : B \Rightarrow S \subset \Sigma \tag{2.3}$$

However, it is impossible to take a compact surface formed by B and flatten it in a consistent way onto a $S \subset \Sigma$. B is compact and Ψ is a homeomorphism (topology-conserving), S must be compact or closed. Every point in B has an open neighborhood where S must be open.

Therefore, S must be closed and open only if $S = \Sigma$

However S is compact (finite) and Σ is non-compact (infinite), which means that S cannot be closed and open, which is a contradiction!

Another picture to guide your intuition can be described by stereographic projection of a sphere.

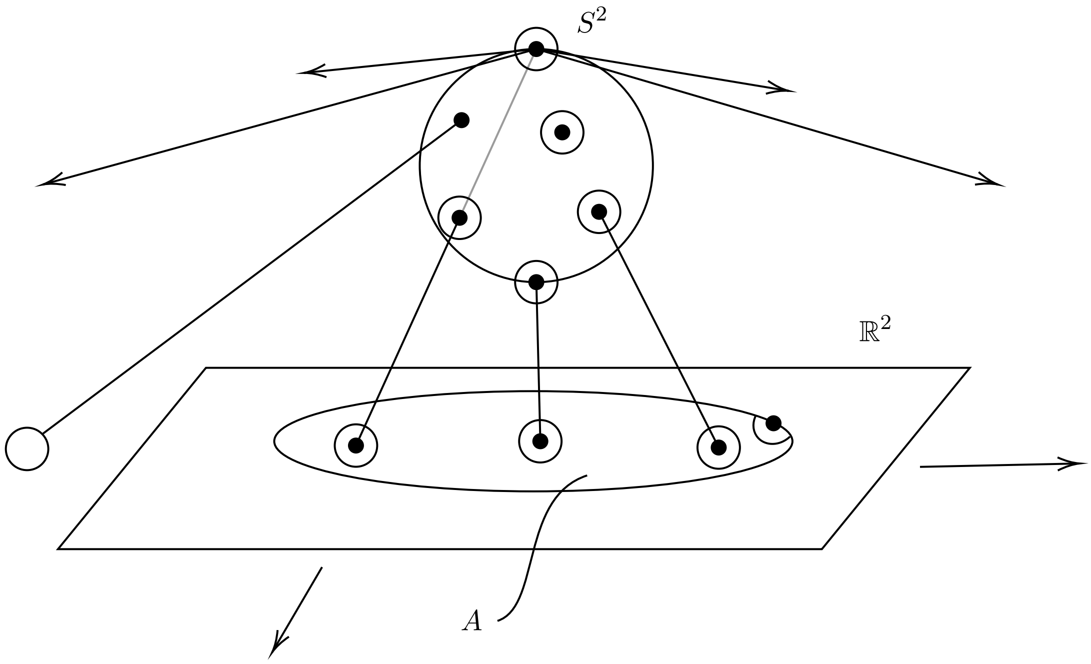

Figure 10. Stereographic projection of a sphere, where A is a compact subet of $\mathbb{R}^2$

Recall this example (Figure 10), which is to map every point on a sphere onto a point on a plane. We cannot cover the sphere with a single chart of a flat space. The mappings we must generate need to take an entire neighborhood and map it to a neighborhood of $\mathbb{R}^2$. This is possible, except at the north pole, which is mapped to infinity. S is a compact surface with every point having an open neighborhood. A is a compact subset of $\mathbb{R}^2$ that cannot have an open neighborhood for every point in A.

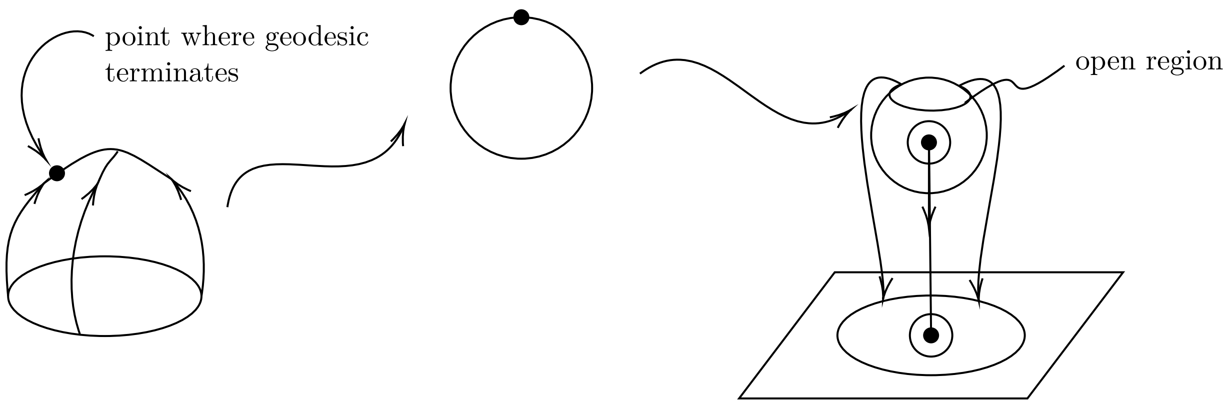

Figure 11. Mapping of

A single geodesic that hits a singularity can save us from the contradiction. In Figure 11, it is shown that one of the geodesics hits a singularity and terminates. This is the same as a sphere with a single point removed. This point can be widened out so that the sphere has an open region and can be flattened and mapped onto the plane. The points on the caps can be mapped onto the boundary of the disk, while the usual points can still be mapped to the an open neighborhood on the Cauchy surface plane.

What about trapped surfaces? Can we form them? Imagine a trapped surface such as a distribution of matter (Figure 12).

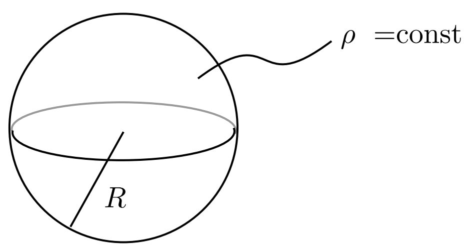

Figure 12. Sphere of constant density ρ

[2] How large do we need to make a ball of water before it becomes a trapped surface?

[2] *(2.4)* $\rho = 1g/cm^3$*, the density of water*

Example.

$$\begin{aligned} R &< \frac{2GM}{c^2} = \frac{2G}{c^2}\frac{4\pi}{3}\rho R^3 \\ \rho &> \frac{c^2}{G}\frac{3}{8\pi}R^{-2} \\ R &> \sqrt{\frac{c^2}{2G}\frac{3}{8\pi}\frac{1}{\rho}} \\ &= 3\times 10^{10} cm/s \sqrt{\frac{1}{1g/cm^3}\frac{g/cm^3 s}{6.67\times 10^{-8}}} \\ &\approx 10^{14} cm \end{aligned} \tag{2.4}$$

$10^{14} cm \Rightarrow$ Orbit of Neptune.

Making R large enough will naturally generate a trapped surface, even with a small density. Inevitably, this will collapse and form a singularity. It is known that stellar collapse will form a trapped surface, so the trapped surface hypothesis is not problematic.

Are the assumptions that global hyperbolicity and the Null Energy Condition realistic? The NEC is believed to be realistic, and has been extended into the Average Null Energy Condition (ANEC).

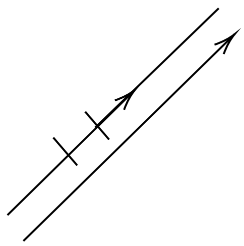

Figure 13. $\langle T_{\mu\nu}k^\mu k^\nu\rangle \geq 0$

Focus on the higher ray and average the energy content all along the ray, and you want the average number to be greater than 0. In Wald section 9.5, Hawking and others attempt to relax the globally hyperbolic requirement.

3 Cosmological Singularity Theorems

Consider a point P and a point Q where Q exists in the causal future of P.

$$Q \in J^+(P) \tag{3.1}$$

Where J^+ is like I, but can have either null or timelike curves. We can define a new kind of space that includes all the possible causal curves that join P and Q. The space of all curves is called $C(P,Q)$ (Figure 14).

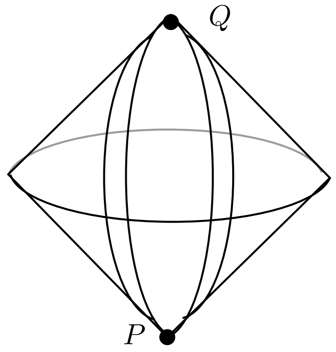

Figure 14. Diagram representing $C(P, Q)$ or topology of $P + Q$

In Wald, theorem 8.3.9: If $\mathfrak{M}$ is globally hyperbolic, $\subset (P, Q)$ is compact for all P, Q and the set should be closed and fill a finite volume of spacetime.

Idea: A continuous function on a compact space attain a maximum and a minimum. Imagine a function over x and a compact interval over x (Figure 15). If function is defined over the interval, it may have a max and a minimum (blue curve).

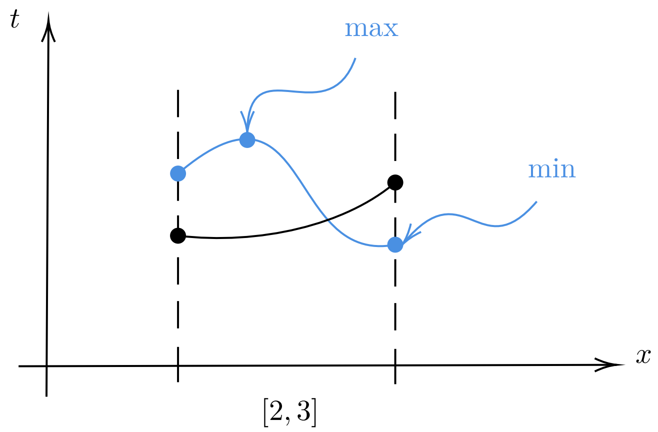

Figure 15. Continuous function on a compact space

τ is a function over a curve or set of curves (Figure 16).

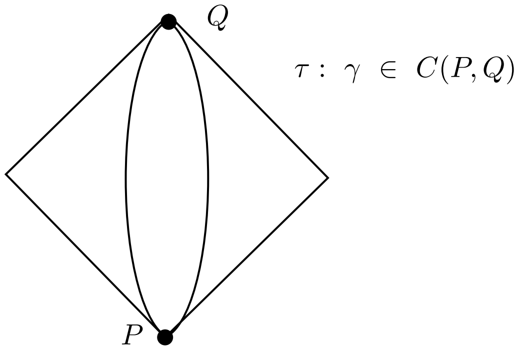

Figure 16. τ is not continuous on $C(P,Q)$

τ is a function over all the curves of the set on $C(P,Q)$. τ is not continuous on $C(P,Q)$.

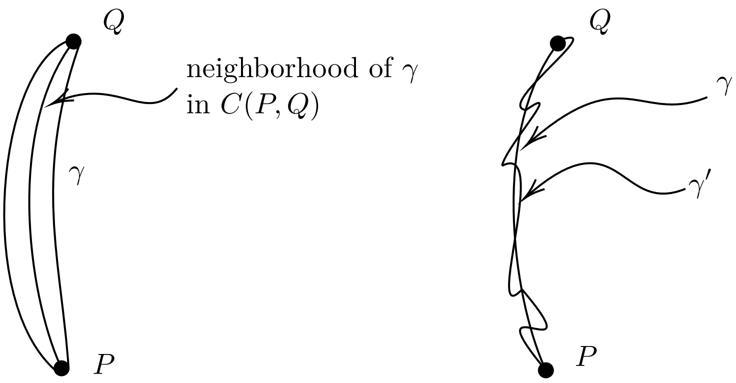

Figure 17. Neighborhood of γ where $\tau(\gamma)'$ can be arbitrarily small

Close to γ in the topology $C(P,Q)$ are a subset of curves contained around in the spacetime region around γ, or a neighborhood of γ. For that γ, there is a γ' with a length that is arbitrarily small. The function on these curves is not a continuous function. One must show τ is upper semi-continuous on $C(P,Q)$. We make make the neighborhood small enough so that there are no curves in the neighborhood longer than γ.

This proof is in Wald 9.4.1.

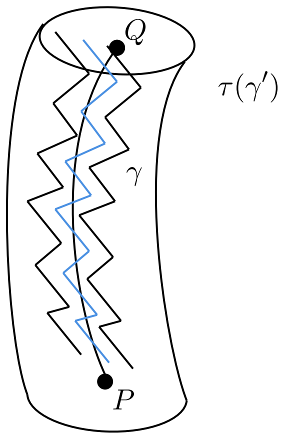

Figure 18. Neighborhood of null segmented curves around γ

In Figure 18, γ is made of null segments. We can create a special neighborhood (outlined in Wald) which excludes arbitrarily long curves. It must be true that if there exists $\tau(\gamma)$, then there must also exist:

$$\tau(\gamma') \leq \tau(\gamma') + \epsilon \tag{3.2}$$

for any ϵ.

Then τ attains a max in (P, Q).

This leads to Wald theorem 9.4.4: Let $\mathfrak{M}$ be globally hyperbolic.

$$Q \in J^+(P) \tag{3.3}$$

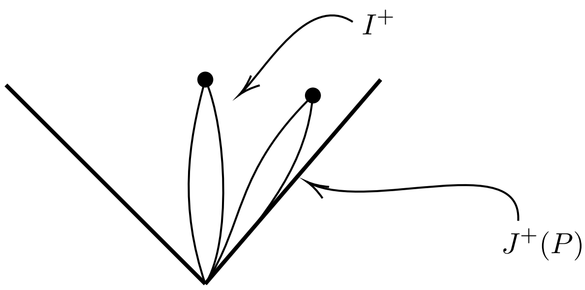

Figure 19. Where I^+ curve and J^+ is any causal or null curve

If Q is in the causal future of P, then there exists a curve between $P + Q$ which has a max value of τ.

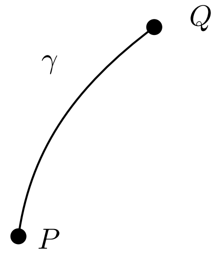

Figure 20. Curve from P to Q

$\tau(\gamma)$ is a max of τ on all curves $C(P,Q)$. Before, we saw that a curve with max length τ is a geodesic with no conjugate points between P and Q. In a globally hyperbolic space time, maximal length curves must exist, which is in conflict with previous ideas that conjugate points appear easily and geodesics cannot exist if there is a conjugate point along a geodesic (Figure 21).

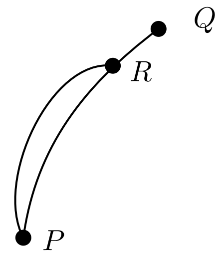

Figure 21. Geodesic with neighboring geodesic colliding at a conjugate point