CS499 Homework 9 (First Draft)

Intersteller

Exercise 9.1

We define $f_1 : \mathbb{N} \to \mathbb{N}$

$f_1(0) = 0, f_1(1) = 1, \cdots, f_1(n) = n.$

We define $f_2 : \mathbb{N} \to \mathbb{N}^2$ based on this graph:

	0	1	2	3	4	5	···
0	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	···
1	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	···
2	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	···
3	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	···
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Figure 1:

$f_2(0) = (0,0), f_2(1) = (0,1), f_2(2) = (1,0) \cdots$

We define $f_3 : \mathbb{N} \to \mathbb{N}^3$ based on this graph:

	0	1	2	3	4	5	···
$f_2(0)=(0,0)$	(0,0,0)	(0,0,1)	(0,0,2)	(0,0,3)	(0,0,4)	(0,0,5)	···
$f_2(1)=(0,1)$	(0,1,0)	(0,1,1)	(0,1,2)	(0,1,3)	(0,1,4)	(0,1,5)	···
$f_2(2)=(1,0)$	(1,0,0)	(1,0,1)	(1,0,2)	(1,0,3)	(1,0,4)	(1,0,5)	···
$f_2(3)=(2,0)$	(2,0,1)	(2,0,1)	(2,0,2)	(2,0,3)	(2,0,4)	(2,0,5)	···
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Figure 2:

$f_2(0) = (0,0,0)\,, f_2(1) = (0,0,1)\,, f_2(2) = (0,1,0)\cdots$

And so on, we can define f_k, $k \in \mathbb{N}$. Now we can define a bijection $\mathbb{N} \to \mathbb{N}^*$ base on this graph:

	0	1	2	3	4	5	$\cdots$
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	$f_1(4)$	$f_1(5)$	$\cdots$
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	$f_2(4)$	$f_2(5)$	$\cdots$
f_3	$f_3(0)$	$f_3(1)$	$f_3(2)$	$f_3(3)$	$f_3(4)$	$f_3(5)$	$\cdots$
f_4	$f_4(0)$	$f_4(1)$	$f_4(2)$	$f_4(3)$	$f_4(4)$	$f_4(5)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 3:

We have $0 \to f_1(0)\,, 1 \to f_1(1)\,, 2 \to f_2(0) \cdots$. This is a bijection $\mathbb{N} \to \mathbb{N}^*$.

Exercise 9.2

We can define a bijection from $\{0,1\}^{\mathbb{N}}$ to $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ as follows.
Given $A = (a_1 a_2 a_3 a_4 \cdots, b_1 b_2 b_3 b_4 \cdots)$, we define $f(A) = a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 \cdots$. To be more precisely,

$$f(A)[i] = \begin{cases} A[1][\dfrac{i+1}{2}], \ i \ is \ odd \ number \\[2mm] A[2][\dfrac{i}{2}], \ i \ is \ even \ number \end{cases}$$

Obviously, for each $A \in \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$, there is only one $f(A) \in \{0,1\}^{\mathbb{N}}$. For each $B \in \{0,1\}^{\mathbb{N}}$, there is only one $B = f^{-1}(B) \in \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. Therefore, f is a bijection and $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.
Using the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$, we can get $\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$.

Exercies 9.3

We use the Cantor's method to proove that. For any $A \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$, we define that $f(A)$ is the $\{0,1\}^{\mathbb{N}}$ sequence we get by following the blue line as follows.
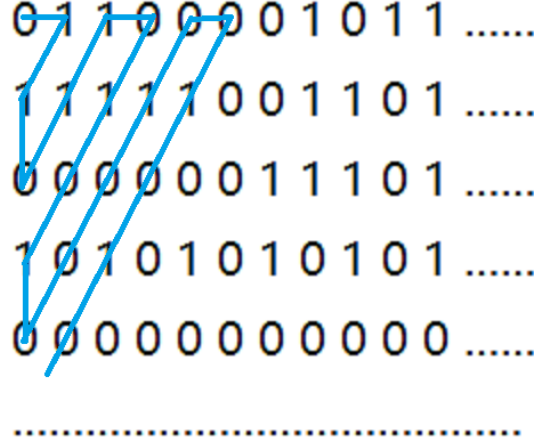
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Figure 4:

It is obvious that for any $B \in \{0,1\}^{\mathbb{N}}$, we can get to $f^{-1} \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ by writing it down following the blue line. Therefore, f is a bijection and $\{0,1\} \cong (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$. Using the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$, we can get $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$.

Exercise 9.4

We know that one continuous function can be expressed by an infinite sequence of real numbers. Thus we have $\mathcal{F} \cong \mathbb{R}^{\mathbb{N}}$. According to **Exercise 9.3**, $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. So we have $\mathcal{F} \cong \mathbb{R}^{\mathbb{N}} \cong \mathbb{R}$.

Exercise 9.5

$000\cdots, 100\cdots, 1100\cdots, 11100\cdots$ According to this rule, the first n bits of the n_{th} sequence are 1, and the remaining bits are 0. Obviously, these sequences constitute a countably infinite chain.

Exercise 9.6

$100\cdots, 0100\cdots, 00100\cdots, 000100\cdots$ According to this rule, the n_{th} bit of the n_{th} sequence is 1, and the remaining bits are 0. Obviously, these sequences constitute a countably infinite antichain.

Exercise 9.7

We can define A using a bijection f from $\{0,1\}^{\mathbb{N}}$ to A, which is a subset of $\{0,1\}^{\mathbb{N}}$ as follows.

Assuming a string s is an element of $\{0,1\}^{\mathbb{N}}$ and $f(s) = t$, let s_k determines t_{2k-1} and t_{2k} by the following rule.

If $s_k = 0$, then $t_{2k-1} := 0$, $t_{2k} := 1$. If $s_k = 1$, then $t_{2k-1} := 1$, $t_{2k} := 0$.

For example:

$$s = 1011010........$$
$$f(s) = 10, 01, 10, 10, 01, 10, 01, ...$$

Obviously, f is a bijection and A is uncountable. Also, any two elements t_a, t_b of A is not comparable since "01" and "10" is not comparable. Thus, A is an required antichain.

Exercise 9.8

We can define A using a bijection f from $\{0,1\}^{\mathbb{N}}$ to A, which is a subset of $\{0,1\}^{\mathbb{N}}$ as follows.

Assuming $s \in \{0,1\}^{\mathbb{N}}$ and $f(s) = t$, the first k digit of s (we call it $s_{1:k}$) determines $t_{(2^k+1):(2^{k+1})}$ by the following rule.

Consider the $s_{1:k}$ as a binary number a_k, then $t_{(2^k+1):(2^k+a_k)} := 1$ and the $t_{(2^k+a_k+1):(2^{k+1})} := 0$. Specially, we define that the first 2 digits of t are always 0.

For example:

$$s = 1011......$$
$$f(s) = 00, 10, 1100, 11111000, 1111111111100000, ...$$

Obviously, f is a bijection and A is uncountable. Also, any two elements $t_a, t_b \in A$ is comparable. Assuming that two elements s_a, s_b is different, their first different digit is the k_{th} digit and the k_{th} digit of s_a is 1, then for any m such that $m \geq k$, the binary number of the first m digit of s_a is greater than that of s_b, which leads to the conclusion that string t_a is "greater" than t_b. Thus, A is the required chain.

Exercise 9.9

For every sequence of $\{0,1\}^{\mathbb{N}}$, form a set $x_i = s_1, s_2, s_3, ...$ with all its prefixion as follow: assume that its first n digit is $a_1, a_2, ..., a_n$, $s_n = \sum_{i=1}^{N}(a_i + 1) \times 3^{i-1}$. Apparently every set x_i is infinite and $x_i \in 2^N$. Call such bijection from a sequence of $\{0,1\}^N$ to a set f. Then we get a set $X = \{f(x) | x \in \{0,1\}^N\}$ which satisfy the demand. As $\{0,1\}^N$ is uncountable, X is uncountable. And $f(x)$ is infinite. Whenever distinct $x, y \in X (x = \{s_{x1}, s_{x2}, ...\})$, suppose $m = f^{-1}(x), n = f^{-1}(y)$ and assume the first different digit between m and n is the k-th digit. Then $x \cap y = s_{x1}, s_{x2}, ..., s_{x(k-1)}$, which is finite.

question

How to prove that R is smaller than 2^R.

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