Mathematical Foundations of Computer Science

CS 499, Shanghai Jiaotong University, Dominik Scheder

Spring 2019

5 The Graph Score Theorem

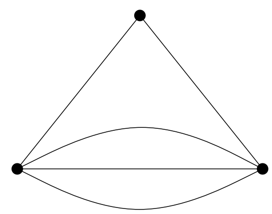
- Homework assignment published on Thursday, 2019-03-27.
- Submit questions and first solution by Wednesday, 2019-04-03, by email to dominik.scheder@gmail.com and the TAs.
- You will receive feedback by Monday, 2019-04-08.
- Submit your final solution by Sunday, 2019-04-14 to me and the two TAs.

Exercise 5.1. Describe, in simple sentences with a minimum of mathematical formalism, (1) the score of a graph, (2) what the graph score theorem is, (3) the idea of the graph score algorithm, (4) where the difficult part of its proof is. Imagine you have a friend who does not take this class, and think about how to answer the above questions to them.

5.1 Alternative Graphs

Now we will look at different notions of graphs. As defined in class and in the video lectures, a graph is a pair $G = (V, E)$ where V is a (usually finite) set, called the *vertices*, and $E \subseteq \binom{V}{2}$, called the set of *edges*.

Multigraphs. A *multigraph* is like a graph, but you can have several parallel edges between two vertices. You cannot, however, have self-loops. That is, there cannot be an edge from u to u itself. This is an example of a multigraph:



We can define degree and score for multigraphs, too. For example, this multigraph has score $(4, 4, 2)$. Obviously no graph can have this score.

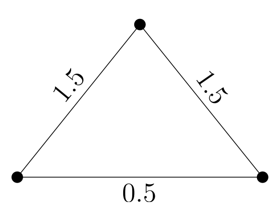
Exercise 5.2. State a score theorem for multigraphs. That is, something like

Theorem 5.3 (Multigraph Score Theorem)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{N}_0^n$. *There is a multigraph with this score if and only if* `<fill in some simple criterion here>`.

Remark. This is actually simpler than for graphs.

Exercise 5.4. Prove your theorem.

Weighted graphs. A weighted graph is a graph in which every edge e has a non-negative weight w_e. In such a graph the *weighted degree* of a vertex u is $\mathrm{wdeg}(u) = \sum_{\{u,v\} \in E} w_{\{u,v\}}$.



This is an example of a weighted graph, which has score $(3, 2, 2)$. Obviously no graph and no multigraph can have this score.

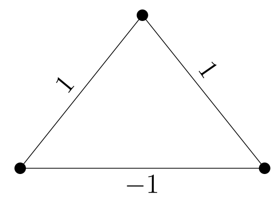
Exercise 5.5. State a score theorem for weighted graphs. That is, state something like

Theorem 5.6 (Weighted Graph Score Theorem)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{R}_0^n$*. There is a weighted graph with this score if and only if* `<fill in some simple criterion here>`.

Remark. This is actually even simpler.

Exercise 5.7. Prove your theorem.

Allowing negative edge weights. Suppose now we allow negative edge weights, like here:



This "graph with real edge weights" has score $(2, 0, 0)$. This score is impossible for graphs, multigraphs, and weighted graphs with non-negative edge weights.

Exercise 5.8. State a score theorem for weighted graphs when we allow negative edge weights. That is, state a theorem like

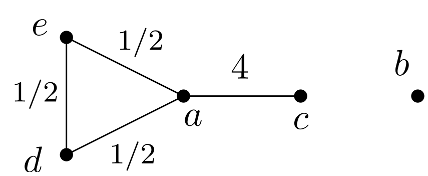
Theorem 5.9 (Score Theorem for Graphs with Real Edge Weights)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{R}^n$*. There is a graph with real edge weights with this score if and only if* `<fill in some simple criterion here>`.

Exercise 5.10. Prove your theorem.

Exercise 5.11. For each student ID $(a_1, \ldots, a_n)$ in your group, check whether this is (1) a graph score, (2) a multigraph score, (3) a weighted graph score, or (4) the score of a graph with real edge weights.

Whenever the answer is *yes*, show the graph, when it is *no*, give a short argument why.

Example Solution. My work ID is 50411. This is a weighted graph score, as shown by this picture:



This settles (3). It is not a multigraph score, because BLABLABLA. I won't give more details, as it might give too many hints about Exercise 7.2. Alright, this settles (2). Note that I do *not* need to answer (4), as this is already answered by (3). Neither do I need to answer (1), as a "no" for (2) implies a "no" for (1).