Mathematical Foundations of Computer Science

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- Homework assignment published on Thursday, 2019-03-07
- Submit questions or first draft solutions by Sunday, 2019-06-10, by email to the TA and to me (dominik.scheder@gmail.com)
- We will discuss some problems on Monday, 2019-03-11.
- You will receive feedback by Wednesday, 2019-03-13.
- Revise your solution and hand in your final submission by Sunday, 2019-03-17.

2 Fibonacci Numbers and Other Recurrences

2.1 Identities among the Fibonacci Numbers

Exercise 2.1. Prove the following identity:

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 \ .$$

Actually, prove it twice:

1. Give an inductive proof.
2. Give a *combinatorial* argument. Remember that $F_{n+2} = |A_n|$, where $A_n := \{x \in \{0,1\}^n \mid x \text{ does not contain } 11\}$. Find a way to partition A_n into sets such that the identity above becomes obvious.

Exercise 2.2. Prove the following identity:

$$F_1 + F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2} \ .$$

Again, give two proofs, one using induction on n and one using a combinatorial argument involving the sets A_i.

2.2 General Linear Recurrences

For $a_1, \ldots, a_k \geq 0$ we can consider the recursively defined numbers:

$$\begin{array}{rll} F_n = & f_n & \text{if } n < k \\ F_n = & a_1 F_{n-1} + a_2 F_{n-2} + \cdots + a_k F_{n-k} & \text{if } n \geq k \ . \end{array}$$

The values $f_0, \ldots, f_{k-1}$ are the "start values" of the recurrence. For example, if we set $k = 1$, $f_0 = 1$, and $a_1 = 2$ then $F_n = 2^n$; setting $k = 2$, $f_0 = 0$, $f_1 = 1$, and $a_2 = a_1 = 1$ yields the Fibonacci numbers. As with the Fibonacci numbers, we can write the recursion in matrix-vector form:

$$\begin{pmatrix} F_n \\ F_{n-1} \\ \vdots \\ F_{n-k+1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{k-2} & a_{k-1} & a_k \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} F_{n-1} \\ F_{n-2} \\ \vdots \\ F_{n-k} \end{pmatrix}$$

Let us denote the matrix by A.

Exercise 2.3. Show that λ is an eigenvalue of A if and only if

$$\lambda^k = a_1 \lambda^{k-1} + a_2 \lambda^{k-2} + \cdots + a_{k-2}\lambda^2 + a_{k-1}\lambda + a_k \ . \qquad (1)$$

For an eigenvalue λ, show what the corresponding eigenvector is. **Hint.** You can do this by computing $\det(A - \lambda I)$. But there is a simpler way by thinking directly in terms of what eigenvectors are.

Exercise 2.4. Recall that $a_1, \ldots, a_k \geq 0$. Assume further that $a_1 + \cdots + a_k > 1$. Show that among the solutions to (1), there is exactly one solution λ_1 with $\lambda_1 > 0$, and this λ_1 is actually greater than 1.