CS499 Homework 9 (First Draft)

Intersteller

Exercise 9.1

We define $f_1 : N \to N$

$f_1(0) = 0, f_1(1) = 1, \cdots, f_1(n) = n.$

We define $f_2 : N \to N^2$ based on this graph:

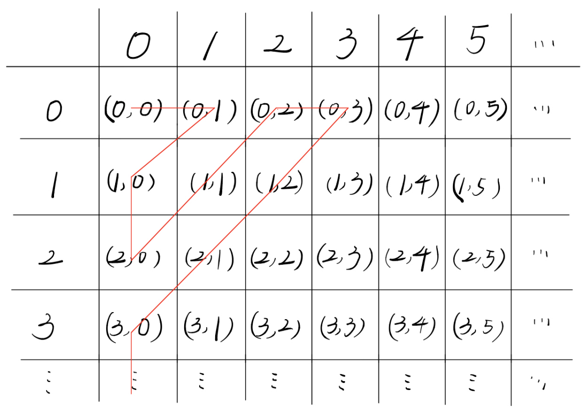

Figure 1:

$f_2(0) = (0,0), f_2(1) = (0,1), f_2(2) = (1,0) \cdots$

We define $f_3 : N \to N^3$ based on this graph:

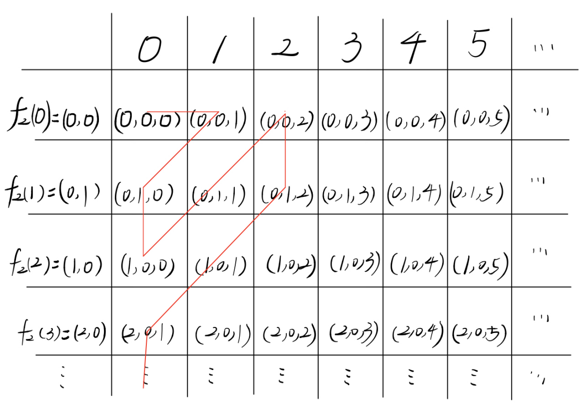

Figure 2:

$f_2(0) = (0,0,0)\,, f_2(1) = (0,0,1)\,, f_2(2) = (0,1,0) \cdots$

And so on, we can define $f_k, k \in N$. Now we can define a bijection $N \to N^*$ base on this graph:

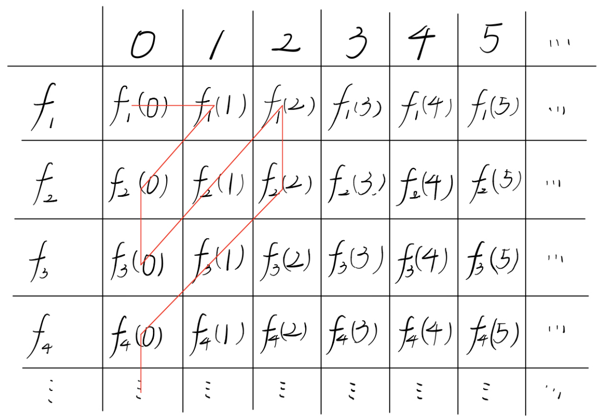

Figure 3:

We have $0 \to f_1(0)\,, 1 \to f_1(1)\,, 2 \to f_2(0) \cdots$. This is a bijection $N \to N^*$.

Exercise 9.2

We can define a bijection from $\{0,1\}^N$ to $\{0,1\}^N \times \{0,1\}^N$ as follows.
Given $A = (a_1a_2a_3a_4\cdots, b_1b_2b_3b_4\cdots)$, we define $f(A) = a_1b_1a_2b_2a_3b_3a_4b_4\cdots$. To be more precisely,

$$f(A)[i] = \begin{cases} A[1][\dfrac{i+1}{2}],\ i\ is\ odd\ number \\ A[2][\dfrac{i}{2}],\ i\ is\ even\ number \end{cases}$$

Obviously, for each $A \in \{0,1\}^N \times \{0,1\}^N$, there is only one $f(A) \in \{0,1\}^N$. For each $B \in \{0,1\}^N$, there is only one $B = f^{-1}(B) \in \{0,1\}^N \times \{0,1\}^N$. Therefore, f is a bijection and $\{0,1\}^N \cong \{0,1\}^N \times \{0,1\}^N$.
Using the fact that $R \cong \{0,1\}^N$, we can get $R \cong R \times R$.

Exercies 9.3

We use the Cantor's method to proove that. For any $A \in (\{0,1\}^N)^N$, we define that $f(A)$ is the $\{0,1\}^N$ sequence we get by following the blue line as follows.

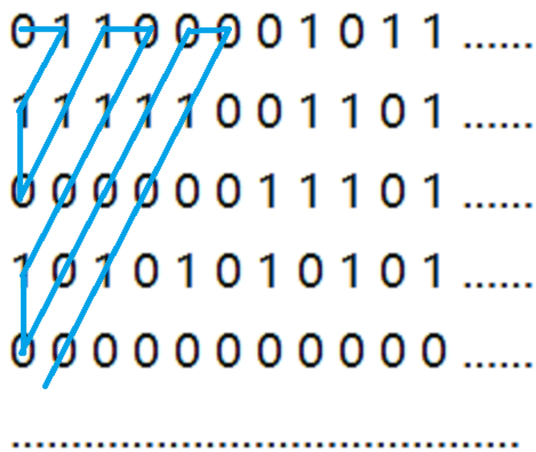

Figure 4:

It is obvious that for any $B \in \{0,1\}^N$, we can get to $f^{-1} \in (\{0,1\}^N)^N$ by writing it down following the blue line. Therefore, f is a bijection and $\{0,1\} \cong (\{0,1\}^N)^N$. Using the fact that $R \cong \{0,1\}^N$, we can get $R \cong R^N$. **Exercise 9.4**

We know that one continuous function can be expressed by an infinite sequence of real numbers. Thus we have $\mathcal{F} \cong \mathbb{R}^{\mathbb{N}}$. According to **Exercise 9.3**, $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. So we have $\mathcal{F} \cong \mathbb{R}^{\mathbb{N}} \cong \mathbb{R}$.

Exercise 9.5

$000\cdots, 100\cdots, 1100\cdots, 11100\cdots$ According to this rule, the first n bits of the n_{th} sequence are 1, and the remaining bits are 0. Obviously, these sequences constitute a countably and infinite chain.

Exercise 9.6

$100\cdots, 0100\cdots, 00100\cdots, 000100\cdots$ According to this rule, the n_{th} bit of the n_{th} sequence is 1, and the remaining bits are 0. Obviously, these sequences constitute a countably and infinite antichain. **Exercise 9.7**

We can form a bijection f from $\{0,1\}^{\mathbb{N}}$ to set A, which is a subset of $\{0,1\}^{\mathbb{N}}$ as follow: Assume a string s is an element of $\{0,1\}^{\mathbb{N}}$ and $t = f(s)$, the k digit of s determines the $(2k-1)$-th to $2k$-th digits of t as the following rule:

If k digit of s is 0, then $(2k-1)$-th to $2k$-th digits of t are 01. If k digit of s is 1, then $(2k-1)$-th to $2k$-th digits of t are 10.

Here is an example :

$$s = 1011010........$$

$$f(s) = 10, 01, 10, 10, 01, 10, 01, ...$$

Obviously, f is a bijection and A is uncountable. Also, any two elements t_1, t_2 of A is not comparable since 01 and 10 can not be comparable. Thus A is antichain required.

Exercise 9.8

We can form a bijection f from $0,1^N$ to set A, which is a subset of $0,1^N$ as follow: Assume a string s is an element of $0,1^N$ and $t = f(s)$, the first k digit of s_1(we call it s_k)determines the $(2^k+1)-th$ to $2^{k+1}-th$ digits of t(we call them $t_{2^k+1} to t_{2^{k+1}}$ as the following rule:

Consider the first k digits as a binary number a_k, then t_{2^k+1} to $t_{2^k+a_k}$ are 1 and the $t_{2^k+a_k+1}$ to $t_{2^{k+1}}$ are 0. Specially, we define that the first 2 digits of t are always $'00'$.

Here is an example:

$$s = 1011......$$

$$f(s) = 00, 10, 1100, 11111000, 1111111111100000, ...$$

Obviously, f is a bijection and A is uncountable. Also, any two elements t_1, t_2 of A is comparable. Assume two elements s_1, s_2 is different and their first different digit is the k-th digit. The k-th element of s_1 is 1. Then for any m such that $m \geq k$,the binary number of the first m digit of s_1 is greater than that of s_2, which leads to the conclusion that the according string of t_1 is "greater" than t_2. Thus A is the chain required.