Mathematical Foundations of Computer Science

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Spring 2019

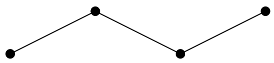
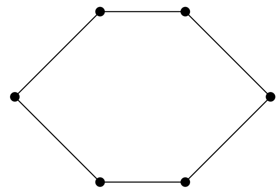
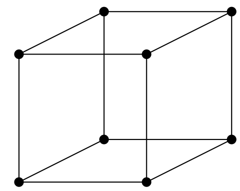
6 Graph Automorphisms

- Homework assignment published on Monday, 2019-04-08
- Submit first solutions and questions by Sunday, 2019-04-14, 12:00, by email to dominik.scheder@gmail.com and to the TAs.
- You will receive feedback by Wednesday, 2019-04-17.
- Submit final solution by Sunday, 2019-04-21 to me and the TAs.

Let $G = (V, E)$ and $H = (V', E')$ be two graphs. A *graph isomorphism* from G to H is a bijective function $f : V \to V'$ such that for all $u, v \in V$ it holds that $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E'$. If such a function exists, we write $G \cong H$ and say that G and H are *isomorphic*. In other words, G and H being isomorphic means that they are identical up to the names of its vertices.

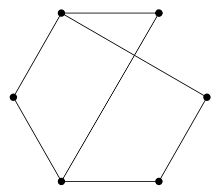
Obviously, every graph G is isomorphic to itself, because the identity function $f(u) = u$ is an isomorphism. However, there might be several isomorphisms f from G to G itself. We call such an isomorphism from G to itself an *automorphism* of G.

Exercise 6.1. For each of the graphs below, compute the number of automorphisms it has.

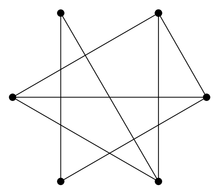
  

Justify your answer!

For a graph $G = (V, E)$, let $\bar{G} := \left(V, \binom{V}{2} \setminus E\right)$ denote its *complement graph*.

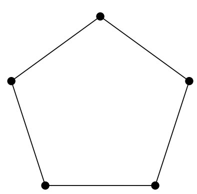


A graph H on six vertices

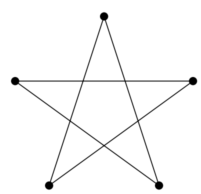


Its complement $\bar{H}$.

We call a graph *self-complementary* if $G \cong \bar{G}$. The above graph is not self-complementary. Here is an example of a self-complementary graph:



The pentagon G.



$\bar{G}$, the pentagram.

Exercise 6.2. Show that there is no self-complementary graph on 999 vertices.

Exercise 6.3. Characterize the natural numbers n for which there is a self-complementary graph G on n vertices. That is, state and prove a theorem of the form "There is a self-complementary graph on n vertices if and only if n <put some simple criterion here>."

Exercise 6.4. Show that for every $k \in \mathbb{N}$, there is some graph $G = (V, E)$ with exactly k automorphisms. **Hint.** Start with $k = 3$. Once you get this, the rest is somewhat easy.

Exercise 6.5. Show that for every $n \geq 6$, there is an asymmetric graph on n vertices.

Exercise 6.6. Show that for every "sufficiently large n" (I guess $n \geq 6$ works), there is a graph on $3n$ vertices with *exactly* 2^n automorphisms.

Exercise 6.7. Let $p \geq 7$ be a prime. From Exercise **??**, we know that there is a graph with exactly p automorphisms. However, this graph (at least in my construction) has roughly $6p$ vertices. Can you do better? Can you find such a graph on $O(\sqrt{p})$ vertices? Or even $O(\log(p))$ vertices? **Warning:** I don't know the answer; this might be very hard.