CS499 Homework 2 (First Draft)

Intersteller

1 Fibonacci Numbers and Other Recurrences

1.1 Identities among the Fibonacci Numbers

Exercise 2.1

Exercise 2.2

1.2 General Linear Recurrences

Exercise 2.3

solution 1:

For each eigenvalue of A, we have $|\lambda I - A| = 0$, which is

$$\begin{vmatrix} \lambda - a_1 & -a_2 & -a_3 & \cdots & -a_{k-1} & -a_k \\ -1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & -1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & \cdots & -1 & \lambda \end{vmatrix}$$

Let

$$row_{k-1} + \frac{1}{\lambda} row_k \to row_{k-1}$$

$$row_{k-2} + \frac{1}{\lambda} row_{k-1} \to row_{k-2}$$

$$\vdots$$

$$row_1 + \frac{1}{\lambda} row_2 \to row_1$$

We get

$$\begin{vmatrix} \lambda - a_1 - \frac{a_2}{\lambda} - \cdots - \frac{a_k}{\lambda^{k-1}} & \cdots & \cdots & \cdots & \cdots \\ 0 & \lambda & 0 & \cdots & \cdots \\ 0 & 0 & \lambda & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - a_1 - \frac{a_2}{\lambda} - \cdots - \frac{a_k}{\lambda^{k-1}})\dot{\lambda}^{k-1} = 0$$

$$\Rightarrow \lambda^k = a_1 \lambda^{k-1} + a_2 \lambda^{k-2} + \cdots + a_{k-1}\lambda + a_k$$

solution 2:

We suppose that for each enigenvalue λ, the enigenvector is

$$\begin{pmatrix} \lambda^{k-1} \\ \lambda^{k-2} \\ \vdots \\ 1 \end{pmatrix}$$

Thus,

$$A\alpha = \begin{pmatrix} a_1\lambda^{k-1} + a_2\lambda^{k-2} + \cdots + a_k \\ \lambda^{k-1} \\ \vdots \\ \lambda \end{pmatrix}$$

$$\lambda\alpha = \begin{pmatrix} \lambda^k \\ \lambda^{k-1} \\ \vdots \\ \lambda \end{pmatrix}$$

According to the definition of enigenvalue, $A\alpha = \lambda\alpha$. So, if $\lambda^k = a_1\lambda^{k-1} + a_2\lambda^{k-2} + \cdots + a_{k-1}\lambda + a_k$ **(1)**, λ is an eigenvalue of A, and $\alpha = \begin{pmatrix} \lambda^{k-1} \\ \lambda^{k-2} \\ \vdots \\ 1 \end{pmatrix}$ is its corresponding eigenvector.

Also, n-dimension matrix has n eigenvalues, and equation **(1)** has n roots. So, λ is an eigenvalue of A if and only if equation **(1)** is satisfied.

Exercise 2.4