Mathematical Foundations of Computer Science

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- Homework assignment published on Thursday, 2019-04-25.
- Work on it and submit a first solution or questions by Wednesday, 2019-05-01, 12:00 by email to me and the TAs.
- You will receive feedback by Sunday, 2019-05-05
- Submit your final solution by Wednesday, 2019-05-08 to me and the TAs.

8 Partial Orderings

Recall the definition of a *partial ordering* and an *equivalence relation*. A relation $\preceq$ on a set X is a partial ordering if (1) if it is *reflexive*, i.e., $x \preceq x$ for all $x \in X$; (2) *transitive*, i.e., $x \preceq y$ and $y \preceq z$ together imply $x \preceq z$; and (3) *anti-symmetric*, i.e., $x \preceq y$ and $y \preceq x$ only hold if $x = y$. A relation $\sim$ is an equivalence relation if it is (1) reflexive, (2) transitive, and (3) *symmetric*, i.e., $x \sim y$ if and only if $y \sim x$.

8.1 Equivalence Relations as a Partial Ordering

A *partition* $\mathcal{P}$ of V is a set $\{V_1, \dots, V_k\}$ where (1) $V_1 \cup \cdots \cup V_k = V$ and (2) the V_i are pairwise disjoint, i.e., $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq k$. For example, $\{\{1\}, \{2,3\}, \{4\}\}$ is a partition of $\{1,2,3,4\}$ but $\{\{1\}, \{2,3\}, \{1,4\}\}$ is not.

A partition $\mathcal{P}$ of V defines an equivalence relation $\sim$ on V: set $x \sim y$ if and only if x, y are in the same part of $\mathcal{P}$; conversely, an equivalence relation

$\sim$ defines a partition: throw equivalent elements into the same set; formally,

$$\begin{aligned} E(x) &:= \{y \in V \mid x \sim y\} && \text{(the set of elements equivalent to } x\text{)} \\ \mathcal{P} &:= \{E(x) \mid x \in V\} \ . \end{aligned}$$

Thus, equivalence relations and partitions are basically the same thing, just represented in a different way. For example, the partition $\{\{1\}, \{2, 3\}, \{4\}\}$ induces an equivalence relation R on $\{1, 2, 3, 4\}$ in which $2 \sim 3$ are equivalent but all other elements are not. Formally, written in set notation, we get

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 2)\} \ .$$

Exercise 8.1. Let E_4 be the set of all equivalence relations on $\{1, 2, 3, 4\}$. Note that E_4 is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 \mid R_1 \subseteq R_2\})$$

is a partial ordering.

1. Draw the Hasse diagram of this partial ordering in a nice way.
2. What is the size of the largest chain?
3. What is the size of the largest antichain?

8.2 Chains and Antichains

Define the partially ordered set $(\mathbb{N}_0^n, \leq)$ as follows: $x \leq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. For example, $(2, 5, 4) \leq (2, 6, 6)$ but $(2, 5, 4) \not\leq (3, 1, 1)$.

Exercise 8.2. Consider the infinite partially ordered set $(\mathbb{N}_0^n, \leq)$.

1. Which elements are minimal? Which are maximal?
2. Is there a minimum? A maximum?
3. Does it have an infinite chain?
4. Does it have arbitrarily large antichains? That is, can you find an antichain A of size $|A| = k$ for every $k \in \mathbb{N}$?

***Exercise 8.3.** Does every infinite subset $S \subseteq \mathbb{N}_0^n$ contain an infinite chain?

Exercise 8.4. Show that $(\mathbb{N}_0^n, \leq)$ has no infinite antichain. **Hint.** Use the previous exercise.

Consider the induced ordering on $\{0,1\}^n$. That is, for $x, y \in \{0,1\}^n$ we have $x \leq y$ if $x_i \leq y_i$ for every coordinate $i \in [n]$.

Exercise 8.5. Draw the Hasse diagrams of $(\{0,1\}^n, \leq)$ for $n = 2, 3$.

Exercise 8.6. Determine the maximum, minimum, maximal, and minimal elements of $\{0,1\}^n$.

Exercise 8.7. What is the longest chain of $\{0,1\}^n$?

****Exercise 8.8.** *What is the largest antichain of* $\{0,1\}^n$*?*