# CS499 Homework 9

**Intersteller**

### Exercise 9.1

We define $f_1 : \mathbb{N} \to \mathbb{N}$

$f_1(0) = 0, f_1(1) = 1, \cdots, f_1(n) = n.$

We define $f_2 : \mathbb{N} \to \mathbb{N}^2$ based on this graph:

| | 0 | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|---|
| 0 | (0,0) | (0,1) | (0,2) | (0,3) | (0,4) | (0,5) | ... |
| 1 | (1,0) | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | ... |
| 2 | (2,0) | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | ... |
| 3 | (3,0) | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | ... |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Figure 1:

$f_2(0) = (0,0), f_2(1) = (0,1), f_2(2) = (1,0) \cdots$

We define $f_3 : \mathbb{N} \to \mathbb{N}^3$ based on this graph:

| | 0 | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|---|
| $f_2(0)=(0,0)$ | (0,0,0) | (0,0,1) | (0,0,2) | (0,0,3) | (0,0,4) | (0,0,5) | ... |
| $f_2(1)=(0,1)$ | (0,1,0) | (0,1,1) | (0,1,2) | (0,1,3) | (0,1,4) | (0,1,5) | ... |
| $f_2(2)=(1,0)$ | (1,0,0) | (1,0,1) | (1,0,2) | (1,0,3) | (1,0,4) | (1,0,5) | ... |
| $f_2(3)=(2,0)$ | (2,0,1) | (2,0,1) | (2,0,2) | (2,0,3) | (2,0,4) | (2,0,5) | ... |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Figure 2:

$f_2\left(0\right)=\left(0,0,0\right), f_2\left(1\right)=\left(0,0,1\right), f_2\left(2\right)=\left(0,1,0\right)\cdots$

And so on, we can define $f_k, k\in\mathbb{N}$. Now we can define a bijection $\mathbb{N}\to\mathbb{N}^*$ base on this graph:

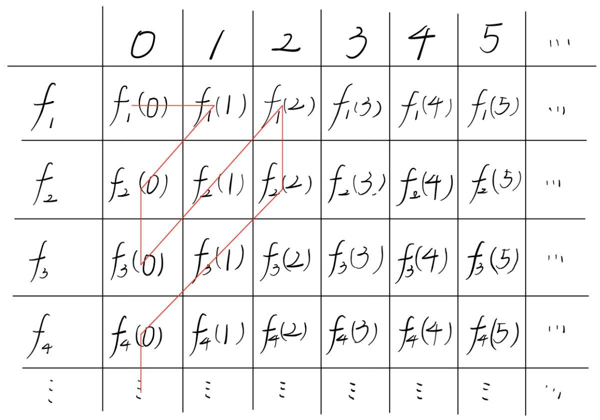


Figure 3:

We have $0\to\{\in\}, 1\to f_1\left(0\right), 2\to f_1\left(1\right), 3\to f_2\left(0\right)\cdots$. This is a bijection $N\to N*$.

**Exercise 9.2**

We can define a bijection from $\{0,1\}^{\mathbb{N}}$ to $\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$ as follows.
Given $A=(a_1a_2a_3a_4\cdots, b_1b_2b_3b_4\cdots)$, we define $f(A)=a_1b_1a_2b_2a_3b_3a_4b_4\cdots$. To be more precisely,

$$f(A)[i]=\begin{cases} A[1][\frac{i+1}{2}],\ i\ is\ odd\ number \\ A[2][\frac{i}{2}],\ i\ is\ even\ number \end{cases}$$

Obviously, for each $A\in\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$, there is only one $f(A)\in\{0,1\}^{\mathbb{N}}$. For each $B\in\{0,1\}^{\mathbb{N}}$, there is only one $B=f^{-1}(B)\in\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$. Therefore, $f$ is a bijection and $\{0,1\}^{\mathbb{N}}\cong\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$.
Using the fact that $\mathbb{R}\cong\{0,1\}^{\mathbb{N}}$, we can get $\mathbb{R}\cong\mathbb{R}\times\mathbb{R}$.

**Exercies 9.3**

We use the Cantor's method to proove that. For any $A\in(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$, we define that $f(A)$ is the $\{0,1\}^{\mathbb{N}}$ sequence we get by following the blue line as follows.

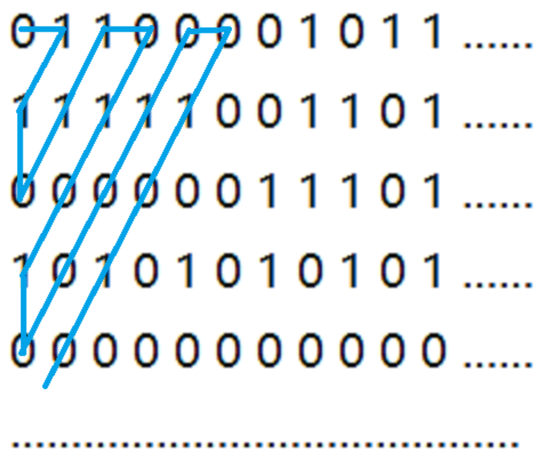


Figure 4:

It is obvious that for any $B \in \{0,1\}^{\mathbb{N}}$, we can get to $f^{-1} \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ by writing it down following the blue line. Therefore, $f$ is a bijection and $\{0,1\}^{\mathbb{N}} \cong (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$. Using the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$, we can get $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$.

**Exercise 9.4**

We can form a bijection from $\mathcal{F}$ to $\mathbb{R}^{\mathbb{N}}$ as follows. For $\forall f \in \mathcal{F}$, we can determine it solely if and only if we know $f(q)$ for $\forall q \in \mathbb{Q}$. Since $\mathbb{Q}$ is countable, we can list $\mathbb{Q}$ as a sequence of $q_0, q_1, q_2, ..., q_n, ...$ Thus we can determine f if and only if we know the sequence $f(q_0), f(q_1), ..., f(q_n), ...$ And $f(q_i) \in \mathbb{R}$ where $i \in \mathbb{N}$, so we can form a bijection from $\mathcal{F}$ to $\mathbb{R}^{\mathbb{N}}$. Thus, we have $\mathcal{F} \cong \mathbb{R}^{\mathbb{N}}$. According to **Exercise 9.3**, $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. Therefore, we have $\mathcal{F} \cong \mathbb{R}^{\mathbb{N}} \cong \mathbb{R}$.

**Exercise 9.5**

$000\cdots, 100\cdots, 1100\cdots, 11100\cdots$ According to this rule, the first $n$ bits of the $n^{th}$ sequence are 1, and the remaining bits are 0. Obviously, these sequences constitute a countably infinite chain.

**Exercise 9.6**

$100\cdots, 0100\cdots, 00100\cdots, 000100\cdots$ According to this rule, the $n^{th}$ bit of the $n^{th}$ sequence is 1, and the remaining bits are 0. Obviously, these sequences constitute a countably infinite antichain.

**Notion: In the following part, we denote the $i^{th}$ digit of a sequence $X$ as $x_i$.**

**Exercise 9.7**

$\forall S \in \{0,1\}^{\mathbb{N}}$, define $f(S) = T$ as follows.

If $s_k = 0$, then $t_{2k-1} := 0$, $t_{2k} := 1$.
If $s_k = 1$, then $t_{2k-1} := 1$, $t_{2k} := 0$.
For example:

$$S = 1011010........$$

$$f(S) = 10, 01, 10, 10, 01, 10, 01, ......$$

Let $A = \{f(S) | \forall S \in \{0,1\}^{\mathbb{N}}\}$. Obviously, $f$ is a bijection from $\{0,1\}^{\mathbb{N}}$ to $A$. $A \cong \{0,1\}^{\mathbb{N}}$ is uncountable. Also, $\forall T_a, T_b \in A$ is not comparable since "01" and "10" is not comparable. Thus, $A$ is an required antichain.

**Exercise 9.8**

$\forall S \in \{0,1\}^{\mathbb{N}}$, define $f(S) = T$ as follows.

Consider the sequence consisting of the first k digits in S $s_{1:k}$ as a binary number $a_k$. Let $t_{(2^k+1):(2^k+a_k)} := 1$, $t_{(2^k+a_k+1):(2^{k+1})} := 0$. Specially, we define that the first 2 digits of $T$ are always 0.

For example:

$$s = 1011......$$

$$f(s) = 00, 10, 1100, 11111000, 1111111111100000, ......$$

Let $A = \{f(S) | \forall S \in \{0,1\}^{\mathbb{N}}\}$. Obviously, $f$ is a bijection from $\{0,1\}^{\mathbb{N}}$ to $A$. $A \cong \{0,1\}^{\mathbb{N}}$ is uncountable. Also, $\forall T_a, T_b \in A$ is comparable for the following reason.
$\forall T_a, T_b \in A$, $T_a \neq T_b$, assume $f^{-1}(T_a) = S_a$, $f^{-1}(T_b) = S_b$. Assume that the first different digit of $S_a$ and $S_b$ is the $k^{th}$ digit and the $k^{th}$ digit of $S_a$ is 1. Then, $\forall m \geq k$, the binary number consisting of the first $m$ digit of $S_a$ is larger than that of $S_b$, which leads to the fact that $T_a$ is "greater" than $T_b$. Thus, $A$ is the required chain.

**Exercise 9.9**

$\forall S \in \{0,1\}^{\mathbb{N}}$, define $f(S) = T$ as follows.

$$t_n = \sum_{i=1}^{n} (s_i + 1) 3^{i-1},\ n \in \mathbb{N}^*$$

Let $X = \{f(S) | \forall S \in \{0,1\}^{\mathbb{N}}\}$.

Therefore, $f$ is a bijection from $\{0,1\}^{\mathbb{N}}$ to $X$. $X \cong \{0,1\}^{\mathbb{N}}$ is uncountable. And $f(S)$ is an infinite subset of $\mathbb{N}$. Whenever distinct $x, y \in X (x = \{x_1, x_2, ...\})$, suppose $m = f^{-1}(x), n = f^{-1}(y)$ and assume the first different digit between $m$ and $n$ is the $k^{th}$ digit. Then $x \cap y = \{x_1, x_2, x_3, \cdots, x_{k-1}\}$, which is finite.

**Question**

How to prove that $\mathbb{R} \ncong 2^{\mathbb{R}}$.