CS499 Homework 4 (First Draft)

Intersteller

Exercise 4.1

This proof goes wrong when it calculates $E[X_T]$ in a second way. According to this proof, T is a certain number. Thus, after walking T steps, actually we can reach i with probability $Pr[T,i](i = 0,1,2,\cdots,k)$. And obviously, none of $Pr[T,i]$ equals 0 for a certain T. So we cannot derive the equation $E[X_T] = k \cdot p_j + 0 \cdot (1-p_j)$. If we slightly change the proof, the equation can be derived correctly. We let $T \to \infty$, of course,

$$\lim_{T\to\infty} Pr[T,i] = 0(i = 1,2,\cdots,k-1)$$

$$\lim_{T\to\infty} Pr[T,k] = P_j$$

$$\lim_{T\to\infty} Pr[T,0] = 1 - P_j$$

In this way the equation works.

Exercise 4.2

1. Since

$$E(T) = \sum_{n=0}^{\infty}(2n+1)C_n p^{n+1}(1-p)^n$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty}(2n+1)C_n p^{n+1}(1-p)^n &= p\sum_{n=0}^{\infty}\frac{2n+1}{n+1}\binom{2n}{n}[p(1-p)]^n \\
&< 2p\sum_{n=0}^{\infty}\binom{2n}{n}[p(1-p)]^n \\
&< 2p\sum_{n=0}^{\infty}2^{2n}[p(1-p)]^n \\
&= 2p\sum_{n=0}^{\infty}[4p(1-p)]^n
\end{aligned}$$

we have

$$E(T) < 2p\sum_{n=0}^{\infty}[4p(1-p)]^n$$

Since $p > \frac{1}{2}$, then $4p(1-p) < 1$.
Thus,

$$E(T) < \frac{2p}{1-4p(1-p)}$$

$E(T)$ is finite.

2. We denote
$$g(x) = C_0 + C_1x + C_2x^2 + ... + C_nx^n + ...$$
where C_n is Calatan number.
We can compute that
$$[g(x)]^2 = C_0^2+(C_0C_1+C_1C_0)x+(C_0C_2+C_1^2+C_2C_0)x^2+...+(C_0C_n+C_1C_{n-1}+...+C_nC_0)x^n+...$$
Since
$$C_n = C_0C_{n-1} + C_1C_{n-2} + ... + C_{n-1}C_0$$
we have
$$[g(x)]^2 = C_0^2 + C_2x + C_3x^2 + C_4x^3 + ... + C_{n+1}x^n + ...$$
Since $C_0 = C_1 = 1$, we have $x[g(x)]^2 = g(x) - 1$.
We solve the equation and get
$$g(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$
Since $g(0) = 1$, thus
$$g(x) = \frac{1 - \sqrt{1-4x}}{2x}$$
$$g'(x) = \frac{1}{x\sqrt{1-4x}} - \frac{1-\sqrt{1-4x}}{2x^2}$$
So,
$$\begin{aligned}
E(T) &= \sum_{n=0}^{\infty}(2n+1)C_np^{n+1}(1-p)^n \\
&= 2p\sum_{n=0}^{\infty} nC_n[p(1-p)]^n + p\sum_{n=0}^{\infty} C_n[p(1-p)]^n \\
&= 2p^2(1-p) \times (g[p(1-p)])' + p \times g[p(1-p)] \\
&= \frac{2p^2(1-p)}{p(1-p)\sqrt{1-4p(1-p)}} - \frac{1-\sqrt{1-4p(1-p)}}{1-p} + p \times \frac{1-\sqrt{1-4p(1-p)}}{2p(1-p)} \\
&= \frac{2p}{\sqrt{1-4p(1-p)}} - \frac{1-\sqrt{1-4p(1-p)}}{2(1-p)} \\
&= \frac{2p}{2p-1} - \frac{2(1-p)}{2(1-p)} \\
&= \frac{1}{2p-1}
\end{aligned}$$

Exercise 4.3

This proof went wrong because when $p < \frac{1}{2}$, T does not have a distribution sequence. As we have already known, $\sum_{i=0}^{\infty} Pr[T = i] = \sum_{n=0}^{\infty} C_np^{n+1}(1-p)^n < 1$, and it fails to satisfy the normalization of a distribution sequence that $\sum_{i=0}^{\infty} Pr[T = i] = 1$, so T can never have an expectation.

Exercise 4.4

$$E[\frac{1}{T+1}] = \sum_{n=0}^{\infty} \frac{1}{2(n+1)} C_np^{n+1}(1-p)^n$$
We denote
$$g(x) = C_0 + C_1x + C_2x^2 + ... + C_nx^n + ...$$
where C_n is Calatan number.

We can compute that

$[g(x)]^2 = C_0^2 + (C_0C_1 + C_1C_0)x + (C_0C_2 + C_1^2 + C_2C_0)x^2 + ... + (C_0C_n + C_1C_{n-1} + ... + C_nC_0)x^n + ...$

Since

$$C_n = C_0C_{n-1} + C_1C_{n-2} + ... + C_{n-1}C_0$$

we have

$$[g(x)]^2 = C_0^2 + C_2x + C_3x^2 + C_4x^3 + ... + C_{n+1}x^n + ...$$

Since $C_0 = C_1 = 1$, we have $x[g(x)]^2 = g(x) - 1$.

We solve the equation and get

$$g(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Since $g(0) = 1$, thus

$$g(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} C_n x^{n+1} = \int_0^x g(t)dt = \ln(\sqrt{1-4x} + 1) - \sqrt{1-4x} - \ln 2 + 1$$

Since $p = \frac{1}{2}$, then

$$\begin{aligned} E[\frac{1}{T+1}] &= \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{n+1} C_n (\frac{1}{4})^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} C_n (\frac{1}{4})^{n+1} \\ &= G(\frac{1}{4}) \\ &= 1 - \ln 2 \end{aligned}$$

Exercise 4.5

Proof. Suppose A_i whose strength is a_i and B_j whose strength is b_j fight.After fighting,for A_i we have

$$E(strength_{A_i}) = \frac{a_i}{a_i + b_j} \cdot (a_i + b_j) + \frac{b_j}{a_i + b_j} \cdot 0 = a_i$$

So after $\forall k$ wars,

$$E(\sum strength_{A_i}) = \sum_{i=1}^{m} a_i$$

$$E(\sum strength_{B_j}) = \sum_{j=1}^{n} b_j$$

Now let's compute $E(\sum strength_{A_i})$ in a different way:

Wars has only two results:Alice'team wins or Bob's team wins.

The former happens with probability p, the latter with probability $1 - p$.

Thus

$$E(\sum strength_{A_i}) = (\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j) \cdot p + 0 \cdot (1 - p)$$

Therefore,

$$p = \frac{\sum_{i=1}^{m} a_i}{\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j}$$

So the probability of Alice's team winning does not depend on the order in which Alice and Bob send their monsters into the arena.

Exercise 4.6

proof Define Alice's winning probability of an order $\{a_{i_1}, a_{i_2}, \cdot, a_{i_n}\}$ is $Pr_{i_1 i_2 \cdot i_n : m}$,in which m is the number of robots in team B, $p_{ij} = \frac{ai}{ai+bj}$,$q_{ij} = 1 - pij$ We use the induction to prove.

lemma Consider there are only two robot in Alice's team(we call it team A) and m in Bob's team(we call it team B), the probability of team A is

$$Pr_{12:m} = \frac{q_{12}p_{21} \ldots p_{2m} - q_{21}p_{11} \ldots p_{1m}}{p_{21} - p_{11}} \tag{1}$$

We use induction to prove.When $m = 2$,we can easily get

$$Pr_{12} = Pr_{21} = \frac{a_1 a_2 + a_1 b_2 + a_2 b_2}{(a_1 + b_2)(a_2 + b_2)} = \frac{q_{12}p_{21}p_{22} - q_{21}p_{11}p_{12}}{p_{21} - p_{11}} \tag{2}$$

Assume the lemma is true when m=k-1,then

$$\begin{aligned} Pr_{12:k} &= p_{11}Pr_{12:k-1} + q_{11}p_{21} \cdots p_{2k} \\ &= p_{11} \cdot \frac{q_{12}p_{22} \ldots p_{2k} - q_{22}p_{12} \ldots p_{1k}}{p_{22} - p_{12}} + q_{11}p_{21} \cdots p_{2k} \end{aligned} \tag{3}$$

Consider that

$$\frac{p_{22}q_{12}}{p_{22} - p_{12}} = \frac{q_{11}p_{21}}{p_{21} - p_{11}} = \frac{a_1}{a_2 - a_1}$$
$$\frac{p_{12}q_{22}}{p_{22} - p_{12}} = \frac{q_{21}p_{11}}{p_{21} - p_{11}} = \frac{a_2}{a_2 - a_1}$$

Plug it into the formula of Pr

$$Pr_{12:k} = \frac{q_{12}p_{21} \ldots p_{2k} - q_{21}p_{11} \ldots p_{1k}}{p_{21} - p_{11}}$$

So the lemma proved true.

Assume that for team B with $m-1$ robots,the winning probability of team A is regardless of the order. **i** Consider team B with m robots, the winning prabability with the order$\{1, 2, 3, \cdots, n\}$ is

$$\begin{aligned} Pr_{123\cdots(n-1)n:m} &= p_{11}Pr_{123\cdots n:(m-1)} + q_{11}Pr_{123\cdots n-1:m} \\ &= p_{11}Pr_{123\cdots n:(m-1)} + q_{11}p_{21}Pr_{123\cdots n-1:m} + \cdots \\ &+ q_{11}q_{12} \cdots q_{(n-2)1}p_{(n-1)1}Pr_{(n-1)n:k-1} + q_{11}q12 \cdots q_{(n-2)1}q_{(n-1)1}Pr_{n:k} \end{aligned} \tag{4}$$

exchange the order of last two robots, the winning prabability is

$$\begin{aligned} Pr_{123\cdots n(n-1):m} &= p_{11}Pr_{123\cdots n:(m-1)} + q_{11}Pr_{123\cdots n-1:m} \\ &= p_{11}Pr_{123\cdots n:(m-1)} + q_{11}p_{21}Pr_{123\cdots n-1:m} + \cdots \\ &+ q_{11}q_{12} \cdots q_{(n-2)1}p_{n1}Pr_{(n(n-1):k-1} + q_{11}q_{12} \cdots q_{(n-2)1}q_{n1}Pr_{n-1:k} \end{aligned} \tag{5}$$

It's observed that only the last 2 terms are different,compare it by subtraction

$$Pr_{123\cdots(n-1)n:m} - Pr_{123\cdots n(n-1):m} = q_{11}q_{12} \cdots q_{(n-2)1}(p_{(n-1)1}Pr_{(n-1)n:k-1} + q_{(n-1)1}Pr_{n:k} - p_{n1}Pr_{2n(n-1)k-1} - q_{n1}Pr_{n-}$$

Consider that

$$Pr_{n:k} = p_{n1}p_{n2} \cdots p_{nk}$$
$$Prn - 1 : k = p_{(n-1)1}p_{(n-1)2} \cdots p_{(n-1)k}$$

and plug the lemma to rewrite $Pr_{n(n-1):k-1} and Pr_{(n-1)n:k-1}$

$$Pr_{n(n-1):k-1} = Pr_{(n-1)n:k-1} = \frac{q_{n1}p_{(n-1)1} \cdots p_{(n-1)k} - q_{(n-1)1}p_{n1} \cdots p_{nk}}{p_{(n-1)1} - pn1}$$

then we can get the result

$$Pr_{123\cdots(n-1)n:m} - Pr_{123\cdots n(n-1):m} = 0$$

ii Similarly, when exchanging the order of any 2 robots, we can use the method showed in **i** to prove they have the same winning probability. Thus,the proposition proved true.

Questions

1.

When we work on Exercise 4.5 & 4.6, we find an interesting phenomenon. In Exercise 4.5, we have proved that the probability of victory only depends on the sum of robots strengths. But in 4.6, it does not work. For example, if two teams both have 1 robot with strength 1, obviously probability of victory is 50%. Then if one team separates the robot into n parts, each part with strength $\frac{1}{n}$, its probability of loss is $(\frac{n}{n+1})^n$. We can calculate that

$$\lim_{n\to\infty}(\frac{n}{n+1})^n = \frac{1}{e}$$

which means one can raise its winning probability by dividing its robot. So in 4.6, we cannot use the sum of robots strength exclusively as the strength of a team. Then how can we evaluate a teams strength?

2.

How can we deal with problems like Exercise 4.6, with lots of variables and circumstances?