CS499 Homework 8 (First Draft)

Intersteller

Exercise 8.1

1.

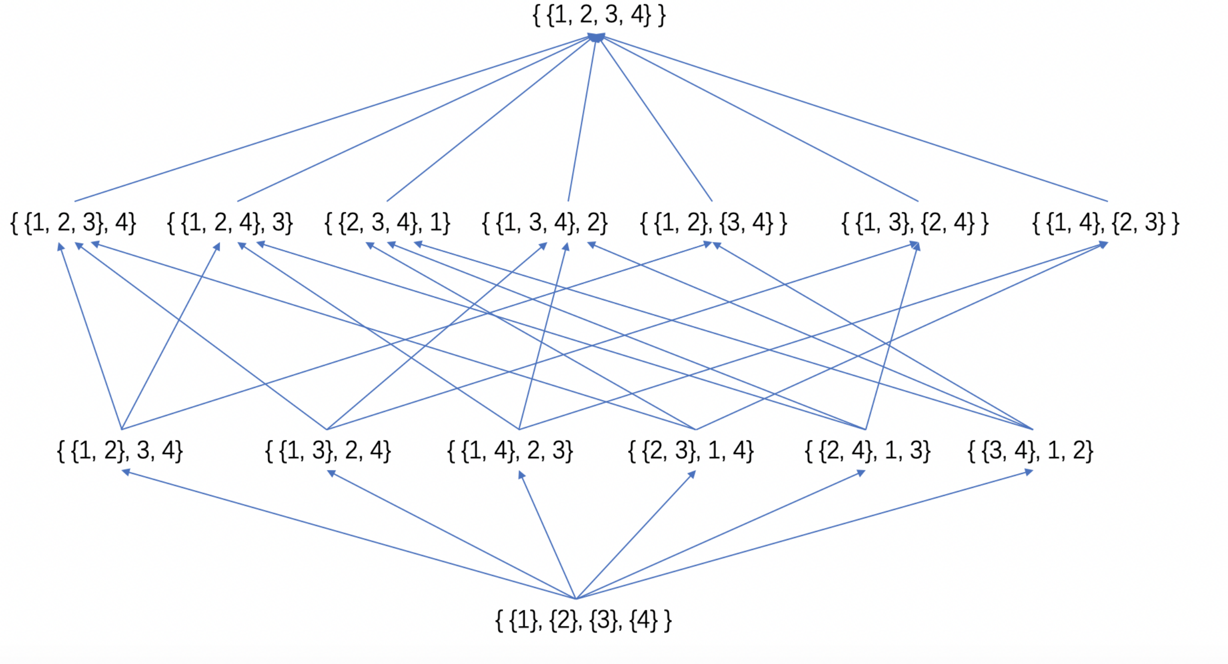

Figure 1:

2. The size of the largest chain is 4.
3. The size of the largest antichain is 7.

Exercise 8.2

1.$(0, 0, 0, 0, 0, \cdots)$ are minimal. There is not a maximum.

2.There is a minimum, but there is not a maximum.

3.Yes. For example,

$$(1, 1, 1, 1, \cdots)$$
$$(2, 1, 1, 1, \cdots)$$
$$(3, 1, 1, 1, \cdots)$$

$$(4, 1, 1, 1, \cdots)$$

$$\vdots$$

4.

When $n = 1$, we can't find an antichain.

When $n \geq 2$, we can find an antichain. For example,

$$(0, k, 0, \cdots, 0)$$

$$(1, k-1, 0, \cdots, 0)$$

$$(2, k-2, 0, \cdots, 0)$$

$$\vdots$$

$$(k-1, 1, 0, \cdots, 0)$$

Exercise 8.3

Yes. We prove it by mathematical induction.

(1)When $n = 1$, obviously, we can sort elements from small to large to get an infinite chain.

(2)We suppose every infinite subset $S \subseteq N_0^n$ contain an infinite chain, then when $n = N + 1$, we can take the first element of each set to constitute a sequence and sort the sequence from small to large. There are two situations:

1.The sequence is bounded.

Obviously at least one element we call it a has appeared countless times. Based on the inductive assumption, we can find an infinite chain in all sets whose first element is a and whose a is removed. Then we add a back to these sets which constitute the infinite chain to get the final infinite chain.

2.The sequence is unbounded.

We take the set A whose first element is smallest in this sequence as the first set in the chain. Based on the inductive assumption, we can find an infinite chain contained this set which remove the first element in all sets removing their first elements. Obviously the successor of the set A removing its first element in this chain will be the successor of A in the final chain if it adds its first element and we call it B. Removing the first element of B, according to hypothesis, we have an infinite chain of N elements. So successors of every element are infinite. However, in the sequence of the first element, there are limited numbers less than B, so we can find an element which is larger than B. So we can find the successor of B in the final chain. And so on, we get an infinite chain when $n = N + 1$.

So every infinite subset $S \subseteq N_0^n$ contain an infinite chain.

Exercise 8.4

We assume that $(\mathbb{N}_0^n, \leq)$ has an infinite antichain T. Obviously, $T \subseteq (\mathbb{N}_0^n, \leq)$. According to **exercise 8.3**, we get T contains an infinite chain, which contradicts T is an infinite antichain. Thus $(\mathbb{N}_0^n, \leq)$ has no infinite antichain.

Exercise 8.5

$n = 2$

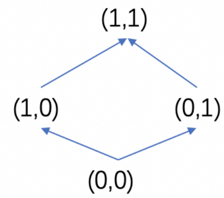

Figure 2:

$n = 3$

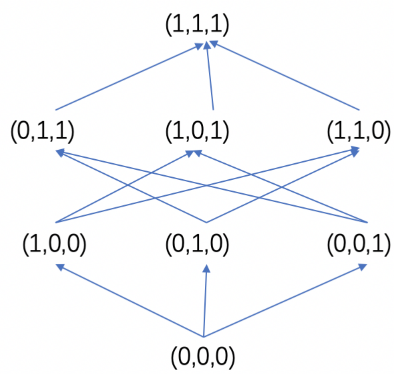

Figure 3:

Exercise 8.6

Maximum is $(\underbrace{1, 1, 1, \cdots, 1}_{n})$

Minimum is $(\underbrace{0, 0, 0, \cdots, 0}_{n})$

Maximal is $(\underbrace{1, 1, 1, \cdots, 1}_{n})$

Minimal is $(\underbrace{0, 0, 0, \cdots, 0}_{n})$

Exercise 8.7

We construct the Hasse diagrams like **Exercise 8.5**. The first layer is an element with n one. The second layer is n elements with $n-1$ one. The third layer is $\binom{n}{2}$ elements with $n-2$ one......The k layer is $\binom{n}{k-1}$ elements with $n+1-k$ one, where $1 \leq k \leq n+1$.

Obviously elements in the same layer is antichain, since elements in chain must have different numbers of one and in the same layer the number of one is equivalent. Thus it has $n+1$ antichain partition.

According to Mirsky's Theorem, max size of chain=min size of antichain partition. It means max size of chain must less than or equal to the size of each antichain partition. Thus the longest chain of $\{0,1\}^n \leq n+1$. Since we get $n+1$ antichain partition in above Hasse diagrams, then the longest chain of $\{0,1\}^n$ is $n+1$. For example, $(0,0,...,0), (0,0,...,1), (0,0,...,0,1), ..., (1,1,...,1)$ is a chain with $n+1$ elements.

Exercise 8.8

The largest antichain of $\{0,1\}^n$ is $\binom{n}{[n/2]}$.According to the Dilworth Theorem,the largest antichain equals to the minimum size of chain partition.

We define a layer as a set of strings containing same number of $'1'$ and is sorted by how many $'1'$ a string in this layer contains.

1.There are $\binom{n}{[n/2]}$ strings in the middle layer, which has the most strings. Since any two strings from the same layer are not comparable, there are at least $\binom{n}{[n/2]}$ chain partitions.

2. All strings in any layer except the middle one can form chains with unique strings in its adjacent layer with the following method:

Assuming there are more $'1'$ than $'0'$ in this layer, we calculate a strings score by the following rules: scan the string from the beginning and the initial score is 0, add one if current digit is 1,minus one otherwise. Find the digit where the first highest score appears (which must be a $'1'$),change it to 0. Then we get a string belongs to its adjacent layer and these two strings can form a chain(they are comparable).Now we prove that this string is unique:

Assume that there are two different strings that transform into a same string. Assume that the first string changes the i-th digit, and the other changes the j-th digit (with no loss of generality, assume $i < j$). Then the i-th digit of the second string and the j-th digit of the first string are 0, whereas other digits are the same. Assume that the score of the $(i-1)$-th digit is k.Then the score of the i-th digit is $k+1$ for the first string and $(k-1)$ for the second. Assume that the score of the $(j-1)$-th digit for the first string is $k+1+p$,then the score of the $(j-1)$-th digit for the second string is $k-1+p$.The score of the j-th digit for the second string is $k+p$. Since the changing digit is where the first largest score occurs, we have

$$k+1 \geq k+1+p$$

$$k < k+p$$

where we get $p \geq 0$ and $p > 0$ which contradict each other. So different strings cannot transform into a same string by the method. Similarly, if there are more $'0'$ than $'1'$ in the layer,we can use a similar method to form chains with unique strings in its adjacent layer.

Question

1How to prove the proposition that the set of integer has the smallest cardinality among all the infinite sets?

2How to handle the Russell's paradox?