Mathematical Foundations of Computer Science

CS 499, Shanghai Jiaotong University, Dominik Scheder

Spring 2019

- Homework assignment published on Thursday, 2019-05-02.
- Work on it and submit a first solution or questions by Wednesday, 2019-05-08, 12:00 by email to me and the TAs.
- You will receive feedback by Sunday, 2019-05-12
- Submit your final solution by Wednesday, 2019-05-15 to me and the TAs.

9 Infinite Sets

In the lecture (and the lecture notes) we have showed that $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$, i.e., there is a bijection $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. From this, and by induction, it follows quite easily that $\mathbb{N}^k \cong \mathbb{N}$ for every k.

Exercise 9.1. Consider $\mathbb{N}^*$, the set of all finite sequences of natural numbers, that is, $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \cup \dots$. Here, ϵ is the empty sequence. Show that $\mathbb{N} \cong \mathbb{N}^*$ by defining a bijection $\mathbb{N} \rightarrow \mathbb{N}^*$.

Exercise 9.2. Show that $R \cong R \times R$. **Hint:** Use the fact that $R \cong \{0,1\}^{\mathbb{N}}$ and thus show that $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

Exercise 9.3. Consider $\mathbb{R}^{\mathbb{N}}$, the set of all infinite sequences $(r_1, r_2, r_3, \dots)$ of real numbers. Show that $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. **Hint:** Again, use the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$.

Exercise 9.4. Let $\mathcal{F}$ be the set of all *continuous* functions $f : \mathbb{R} \to \mathbb{R}$. Show that $\mathcal{F} \cong \mathbb{R}$.

Next, let us view $\{0,1\}^{\mathbb{N}}$ as a partial ordering: given two elements $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathbb{N}}$, that is, sequences $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$, we define $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all $i \in \mathbb{N}$. Clearly, $(0, 0, \dots)$ is the minimum element in this ordering and $(1, 1, \dots)$ the maximum.

Exercise 9.5. Give a countably infinite chain in $\{0,1\}^{\mathbb{N}}$. Remember that a set A is countably infinite if $A \cong \mathbb{N}$.

Exercise 9.6. Find a countably infinite antichain in $\{0,1\}^{\mathbb{N}}$.

Exercise 9.7. Find an uncountable antichain in $\{0,1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.

****Exercise 9.8.** *Find an uncountable chain in* $\{0,1\}^{\mathbb{N}}$. *That is, a chain* A *with* $A \cong \mathbb{R}$.

****Exercise 9.9.** *Find a set* $X \subseteq 2^{\mathbb{N}}$ *(that is,* X *is a set of subsets of* $\mathbb{N}$*) such that (1) every* $x \in X$ *is an infinite subset of* $\mathbb{N}$*, (2)* $x \cap y$ *is finite whenever* $x, y \in X$ *are distinct, (3)* X *is uncountable.*