Mathematical Foundations of Computer Science

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- Homework assignment published on Monday, 2018-03-11
- Submit questions and first solutions by Sunday, 2018-03-17 by email to dominik.scheder@gmail.com and the TAs.
- You will receive feedback by Wednesday, 2018-03-20
- Revise your solution and submit your final solution by Sunday, 2018-03-24 by email to dominik.scheder@gmail.com and the TAs.

3 Tossing Coins

Let us toss a coin infinitely often, producing a bit string $x = x_1 x_2 x_3 \dots$. For a finite bit string $z \in \{0,1\}^*$, let T_z denote the number of tosses until z appears the first time. For example, if we toss 0010110, then $T_{10} = 4$ and $T_{110} = 7$.

Exercise 3.1. Consider an unbiased coin, i.e., 0 and 1 come up with probability $p = 1/2$ each.

1. Give an explicit[1] formula for $\Pr[T_{10} = n]$.
2. Give an explicit formula for $\Pr[T_{11} = n]$.

[1] By *explicit* in this context I mean something not containing a recurrence; not containing $\sum$ or $\prod$. You may, however, use stuff we have used before, like $\binom{n}{k}$, Fibonacci numbers F_n, Catalan numbers C_n.

3. Compute $\mathbb{E}[T_{11}]$ and $\mathbb{E}[T_{10}]$. You can choose any of the three proof methods above (but two of them won't be fun; in particular, using Point 1 and Point 2 of this exercise will not help you so much).

You might have noticed that $\mathbb{E}[T_{10}] < \mathbb{E}[T_{11}]$, i.e., 10 appears earlier than 11, on average.

Exercise 3.2. Let $\mathcal{E}$ be the event that 10 appears earlier than 11. What is $\Pr[\mathcal{E}]$?

Exercise 3.3. Toss two coins repeatedly, producing two sequences $x_1x_2\dots$ and $y_1y_2\dots$. We stop once we see 10 in the first sequence or 11 in the second sequence. Formally, we toss the two coins

$$T := \min(T_{10}(x), T_{11}(y))$$

times.

1. What is $\mathbb{E}[T]$?
2. What are the probabilities that 10 appears in x (a) before 11 appears in y, (b) at the same time as 11, (c) later than 11?

Hint. I think the only way to solve this exercise without going crazy is by applying proof method 3 from above, drawing a small automaton; or in this case, not so extremely small anymore.

Exercise 3.4. Let us toss a biased coin, that is 1 comes up with some probability $p \in [0, 1]$. Let T be the number of tosses until the first 1 appears.

1. Give an explicit formula for $\mathbb{E}[T^2]$ in terms of p.
2. Give an explicit formula for $\mathbb{E}\left[\frac{1}{T}\right]$ in terms of p.