

## CS499 Homework 6 (First Draft)

### Interstellar

#### Exercise 6.1

(1) As the picture shows:

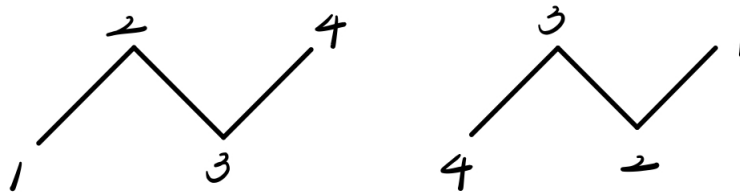


Figure 1:

The number of automorphisms is 2. (2) Suppose the original picture is

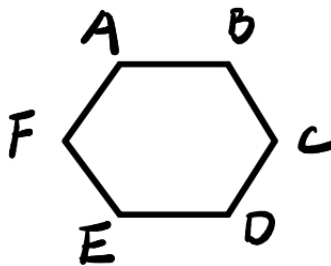


Figure 2:

Original  $A$  position now is  $A'$ , so original  $B$  position now is the point connected with  $A'$  before and there are two possibilities.

If  $B'$  is determined, the original  $F$  position is the other point connected with  $A'$  apart from  $B'$ .

Therefore other positions is also determined for the same reason, as in the following figure.

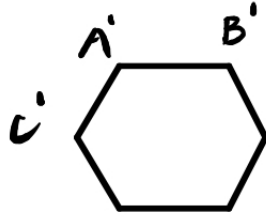


Figure 3:

So the number of automorphisms is  $6 \times 2 = 12$ .

(3) For the same reason of (2), if there are 3 points determined, the graph is determined.

So the number of automorphisms is  $8 \times 3! = 48$ .

### Exercise 6.2

The complete graph on 999 vertices has  $C_{999}^2$  edges. The  $|E|$  is odd, so the  $|E|$  of original graph and the  $|E|$  of complement graph is not equal. So there is no self-complementary graph on 999 vertices.

### Exercise 6.3

**Theorem:** There is a self-complementary graph on  $n$  vertices if and only if  $n = 4k$  or  $n = 4k + 1$ . (here and in the following,  $k = 1, 2, 3, \dots$ )

**proof:**

(1) If  $n = 4k + 2$  or  $n = 4k + 3$ , since a complete graph has  $\frac{(|V|-1)(|V|-2)}{2}$  edges,  $|E|$  will be an odd number. It is obvious that if  $|E|$  is an odd number, the graph can not be self-complementary.

(2) If  $n = 4k$ , we can show the graph is self-complementary in the following method:

We divide the vertices into 4 sequences A, B, C and D, each sequence has  $k$  vertices. Then we add  $\frac{(k-1)(k-2)}{2}$  edges into sequence A and sequence B respectively. After that we add  $\frac{(k)(k-1)}{2}$  edges between sequence A and sequence B, sequence A and C, sequence B and sequence D respectively, as in the following figure.

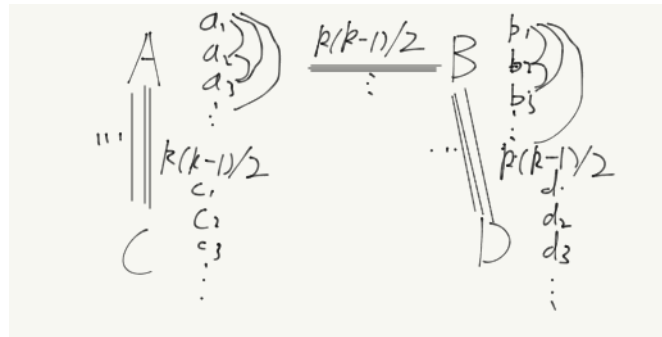


Figure 4:

In this way we get a graph  $G$ , whose complementary graph is  $G'$ , which has  $\frac{(k-1)(k-2)}{2}$  edges in sequence C and in sequence D respectively, and  $\frac{(k)(k-1)}{2}$  edges between sequence A and sequence D, sequence B and sequence C, sequence C and sequence D respectively. Through the bijection  $C \rightarrow A$  ( $c_1 \rightarrow a_1, c_2 \rightarrow a_2, \dots$  (and the same in the following)),  $A \rightarrow D$ ,  $B \rightarrow C$  and  $D \rightarrow B$ , we can show that  $G$  and  $G'$  are automorphism, which means  $G$  or  $G'$  is a self-complementary graph.

(3) If  $n = 4k + 1$ , we divide the vertices into 4 sequences same as above and 1 special dot. Then for the sequences we use the same method to add edges, except in graph  $G$  we add  $k$  edges between  $A$  and the dot,  $B$  and the dot respectively. Then we will have  $k$  edges between  $C$  and the dot,  $D$  and the dot respectively in  $G'$ . In the bijection we add  $dot \rightarrow dot$ . In this way,  $G$  or  $G'$  is still self-complementary graph.

#### Exercise 6.4

For  $k = 3$  and  $k = 4$ , the corresponding graphs are in the following figure.

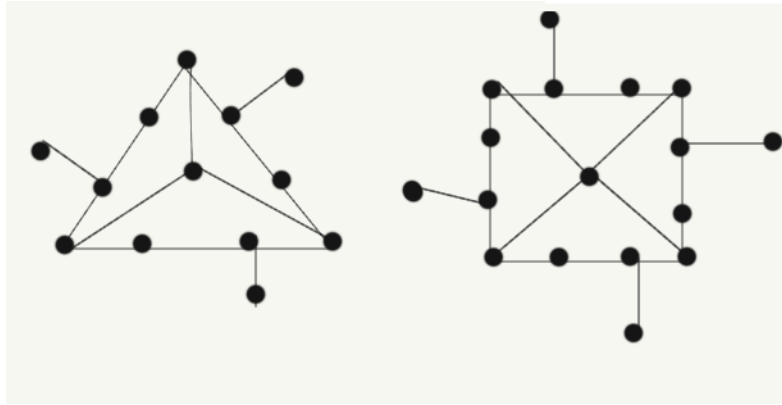


Figure 5:

**One general construction method:** For  $k = n$ , first we draw a polygon that has  $n$  edges. Then we add one vertex in the middle of the polygon and link it with  $n$  vertices respectively. After that we add 2 vertices on each edge. Finally, we add one vertex and one edge in every 3 vertices clockwise.

#### Question:

In Exercise 6.7, if we do not include the constraint that  $P$  is a prime number, what conclusion can we get? First, we make a prime factorization on  $P$ , which is  $P = P_1^{A_1} \cdot P_2^{A_2} \cdot \dots \cdot P_n^{A_n}$ . For every  $i$ ,  $P_i$  is prime number and  $A_i$  is positive integer. We have a hypothesis that the number of vertices might be  $O(P_{max} + A_{max})$ . But we don't know how to prove it. Or do we have another answer?