CS499 Homework 9 (First Draft)

Intersteller

Exercise 9.1

We define $f_1 : \mathbb{N} \to \mathbb{N}$

$f_1(0) = 0, f_1(1) = 1, \cdots, f_1(n) = n.$

We define $f_2 : \mathbb{N} \to \mathbb{N}^2$ based on this graph:

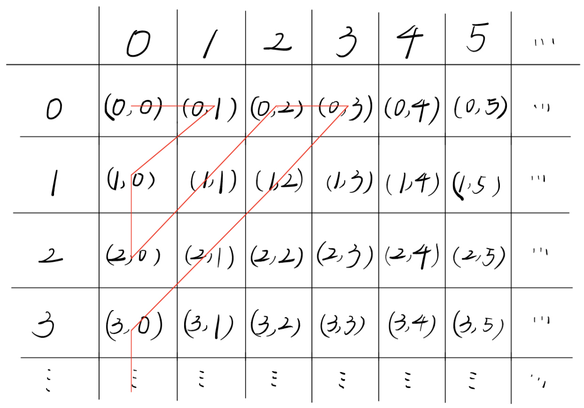

Figure 1:

$f_2(0) = (0,0), f_2(1) = (0,1), f_2(2) = (1,0) \cdots$

We define $f_3 : \mathbb{N} \to \mathbb{N}^3$ based on this graph:

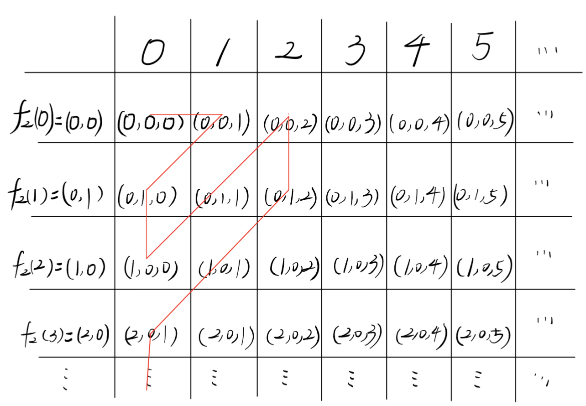

Figure 2:

$f_2(0) = (0,0,0), f_2(1) = (0,0,1), f_2(2) = (0,1,0) \cdots$

And so on, we can define $f_k, k \in \mathbb{N}$. Now we can define a bijection $\mathbb{N} \to \mathbb{N}^*$ base on this graph:

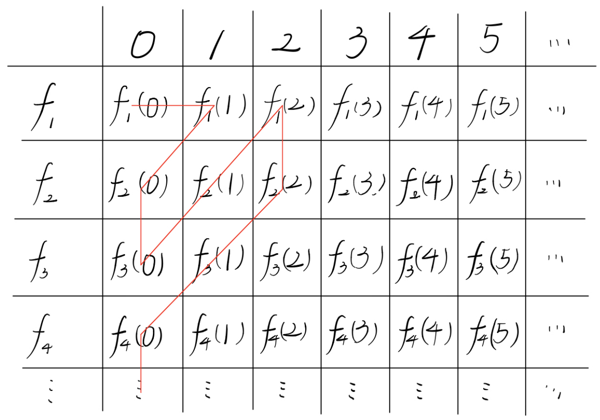

Figure 3:

We have $0 \to f_1(0), 1 \to f_1(1), 2 \to f_2(0) \cdots$. This is a bijection $\mathbb{N} \to \mathbb{N}^*$.

Exercise 9.2

We can define a bijection from $\{0,1\}^{\mathbb{N}}$ to $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ as follows.
Given $A = (a_1a_2a_3a_4\cdots, b_1b_2b_3b_4\cdots)$, we define $f(A) = a_1b_1a_2b_2a_3b_3a_4b_4\cdots$. To be more precisely,

$$f(A)[i] = \begin{cases} A[1][\frac{i+1}{2}],\ i\ is\ odd\ number \\ A[2][\frac{i}{2}],\ i\ is\ even\ number \end{cases}$$

Obviously, for each $A \in \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$, there is only one $f(A) \in \{0,1\}^{\mathbb{N}}$. For each $B \in \{0,1\}^{\mathbb{N}}$, there is only one $B = f^{-1}(B) \in \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. Therefore, f is a bijection and $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.
Using the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$, we can get $\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$.

Exercies 9.3

We use the Cantor's method to proove that. For any $A \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$, we define that $f(A)$ is the $\{0,1\}^{\mathbb{N}}$ sequence we get by following the blue line as follows.

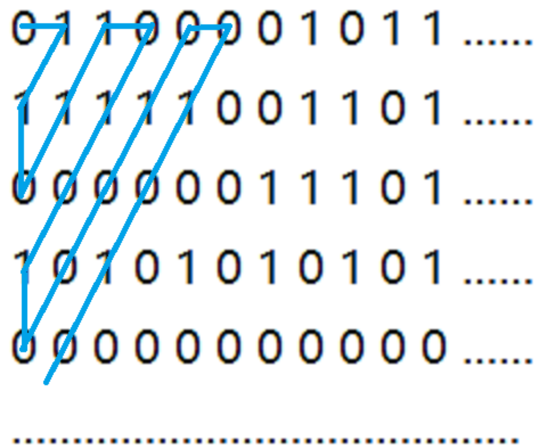

Figure 4:

It is obvious that for any $B \in \{0,1\}^{\mathbb{N}}$, we can get to $f^{-1} \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ by writing it down following the blue line. Therefore, f is a bijection and $\{0,1\} \cong (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$. Using the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$, we can get $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$.

Exercise 9.4

For one continuous function f, if $f(q)$ is fixed for $\forall q \in \mathbb{Q}$, then f is fixed. Thus we can define a bijection g from $\mathbb{Q}$. Since $\mathbb{Q}$ can be listed, $\forall q_i \in \mathbb{Q}$ we have $g(q_i)$ which is a real number. For example:

$$\mathbb{Q} : q_1, q_2, q_3, q_4, q_5, \ldots\ldots$$

$$\mathbb{R} : g(q_1), g(q_2), g(q_3), g(q_4), g(q_5), \ldots\ldots.$$

Obviously g is a bijection. So, one continuous function can be expressed by an infinite sequence of real numbers. Thus, we have $\mathcal{F} \cong \mathbb{R}^{\mathbb{N}}$. According to **Exercise 9.3**, $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. Therefore, we have $\mathcal{F} \cong \mathbb{R}^{\mathbb{N}} \cong \mathbb{R}$.

Exercise 9.5

$000\cdots, 100\cdots, 1100\cdots, 11100\cdots$ According to this rule, the first n bits of the n_{th} sequence are 1, and the remaining bits are 0. Obviously, these sequences constitute a countably infinite chain.

Exercise 9.6

$100\cdots, 0100\cdots, 00100\cdots, 000100\cdots$ According to this rule, the n_{th} bit of the n_{th} sequence is 1, and the remaining bits are 0. Obviously, these sequences constitute a countably infinite antichain.

In the following part, we denote the i_{th} digit of a sequence X as x_i.

Exercise 9.7

$\forall S \in \{0,1\}^{\mathbb{N}}$, define $f(S) = T$ as follows.

If $s_k = 0$, then $t_{2k-1} := 0$, $t_{2k} := 1$.
If $s_k = 1$, then $t_{2k-1} := 1$, $t_{2k} := 0$.
For example:

$$S = 1011010........$$

$$f(S) = 10, 01, 10, 10, 01, 10, 01, \ldots\ldots$$

Obviously, f is a bijection. Let $A = \{f(S) | \forall S \in \{0,1\}^{\mathbb{N}}\}$. $A \cong \{0,1\}^{\mathbb{N}}$ is uncountable. Also, $\forall T_a, T_b \in A$ is not comparable since "01" and "10" is not comparable. Thus, A is an required antichain.

Exercise 9.8

$\forall S \in \{0,1\}^{\mathbb{N}}$, define $f(S) = T$ as follows.

Consider the sequence consisting of the first k digits in S $s_{1:k}$ as a binary number a_k. Let $t_{(2^k+1):(2^k+a_k)} := 1$, $t_{(2^k+a_k+1):(2^{k+1})} := 0$. Specially, we define that the first 2 digits of T are always 0.

For example:

$$s = 1011......$$
$$f(s) = 00, 10, 1100, 11111000, 1111111111100000, ...$$

Obviously, f is a bijection. Let $A = \{f(S) | \forall S \in \{0,1\}^{\mathbb{N}}\}$. $A \cong \{0,1\}^{\mathbb{N}}$ is uncountable. Also, $\forall T_a, T_b \in A$ is comparable for the following reason.
$\forall T_a, T_b \in A$, $T_a \neq T_b$, assume $f^{-1}(T_a) = S_a$, $f^{-1}(T_b) = S_b$. Assume that the first different digit of S_a and S_b is the k_{th} digit and the k_{th} digit of S_a is 1. Then, $\forall m \geq k$, the binary number consisting of the first m digit of S_a is larger than that of S_b, which leads to the fact that T_a is "greater" than T_b. Thus, A is the required chain.

Exercise 9.9

$\forall S \in \{0,1\}^{\mathbb{N}}$, define $f(S) = T$ as follows.

$$t_n = \sum_{i=1}^{n} (s_i + 1) 3^{i-1}$$

. Obviously, f is a bijection. Let $X = \{f(S) | \forall S \in \{0,1\}^{\mathbb{N}}\}$. $X \cong \{0,1\}^{\mathbb{N}}$ is uncountable. And $f(S)$ is infinite. Whenever distinct $x, y \in X (x = \{x_1, x_2, ...\})$, suppose $m = f^{-1}(x), n = f^{-1}(y)$ and assume the first different digit between m and n is the k_{th} digit. Then $x \cap y = \{x_1, x_2, x_3, \cdots, x_{k-1}\}$, which is finite.

Question

How to prove that $\mathbb{R}$ is smaller than $2^{\mathbb{R}}$.