

CS499 Homework 9 (First Draft)

Interstellar

Exercise 9.1

We define $f_1 : \mathbb{N} \rightarrow \mathbb{N}$

$f_1(0) = 0, f_1(1) = 1, \dots, f_1(n) = n.$

We define $f_2 : \mathbb{N} \rightarrow \mathbb{N}^2$ based on this graph:

	0	1	2	3	4	5	...
0	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	...
1	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	...
2	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	...
3	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	...
...

Figure 1:

$f_2(0) = (0,0), f_2(1) = (0,1), f_2(2) = (1,0) \dots$

We define $f_3 : \mathbb{N} \rightarrow \mathbb{N}^3$ based on this graph:

	0	1	2	3	4	5	...
$f_2(0)=(0,0)$	(0,0,0)	(0,0,1)	(0,0,2)	(0,0,3)	(0,0,4)	(0,0,5)	...
$f_2(1)=(0,1)$	(0,1,0)	(0,1,1)	(0,1,2)	(0,1,3)	(0,1,4)	(0,1,5)	...
$f_2(2)=(1,0)$	(1,0,0)	(1,0,1)	(1,0,2)	(1,0,3)	(1,0,4)	(1,0,5)	...
$f_2(3)=(2,0)$	(2,0,1)	(2,0,2)	(2,0,3)	(2,0,4)	(2,0,5)
...

Figure 2:

$f_2(0) = (0, 0, 0), f_2(1) = (0, 0, 1), f_2(2) = (0, 1, 0) \dots$

And so on, we can define $f_k, k \in \mathbb{N}$. Now we can define a bijection $\mathbb{N} \rightarrow \mathbb{N}^*$ base on this graph:

	0	1	2	3	4	5	...
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	$f_1(4)$	$f_1(5)$...
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	$f_2(4)$	$f_2(5)$...
f_3	$f_3(0)$	$f_3(1)$	$f_3(2)$	$f_3(3)$	$f_3(4)$	$f_3(5)$...
f_4	$f_4(0)$	$f_4(1)$	$f_4(2)$	$f_4(3)$	$f_4(4)$	$f_4(5)$...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots

Figure 3:

We have $0 \rightarrow f_1(0), 1 \rightarrow f_1(1), 2 \rightarrow f_2(0) \dots$. This is a bijection $\mathbb{N} \rightarrow \mathbb{N}^*$.

Exercise 9.2

We can define a bijection from $\{0, 1\}^{\mathbb{N}}$ to $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ as follows.

Given $A = (a_1 a_2 a_3 a_4 \dots, b_1 b_2 b_3 b_4 \dots)$, we define $f(A) = a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 \dots$. To be more precisely,

$$f(A)[i] = \begin{cases} A[1][\frac{i+1}{2}], & i \text{ is odd number} \\ A[2][\frac{i}{2}], & i \text{ is even number} \end{cases}$$

Obviously, for each $A \in \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$, there is only one $f(A) \in \{0, 1\}^{\mathbb{N}}$. For each $B \in \{0, 1\}^{\mathbb{N}}$, there is only one $B = f^{-1}(B) \in \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$. Therefore, f is a bijection and $\{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$.

Using the fact that $\mathbb{R} \cong \{0, 1\}^{\mathbb{N}}$, we can get $\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$.

Exercises 9.3

We use the Cantor's method to prove that. For any $A \in (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$, we define that $f(A)$ is the $\{0, 1\}^{\mathbb{N}}$ sequence we get by following the blue line as follows.

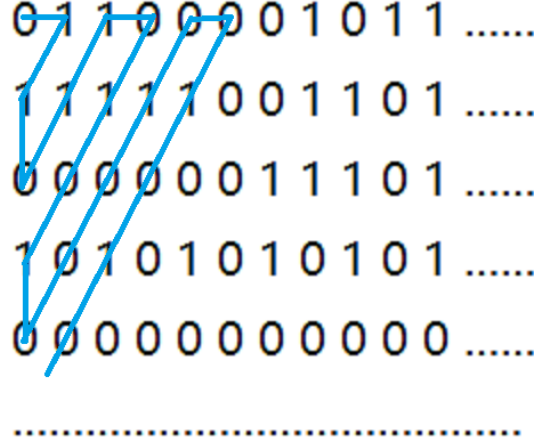


Figure 4:

It is obvious that for any $B \in \{0, 1\}^{\mathbb{N}}$, we can get to $f^{-1} \in (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$ by writing it down following the blue line. Therefore, f is a bijection and $\{0, 1\} \cong (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$. Using the fact that $\mathbb{R} \cong \{0, 1\}^{\mathbb{N}}$, we can get $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$.

Exercise 9.4

For one continuous function f , if $f(q)$ is fixed for $\forall q \in \mathbb{Q}$, then f is fixed. Thus we can define a bijection g from \mathbb{Q} to \mathbb{R} . Since \mathbb{Q} can be listed, $\forall q_i \in \mathbb{Q}$, $\exists g(q_i) \in \mathbb{R}$. For example:

$$\begin{aligned} \mathbb{Q} &: q_1, q_2, q_3, q_4, q_5, \dots \\ \mathbb{R} &: g(q_1), g(q_2), g(q_3), g(q_4), g(q_5), \dots \end{aligned}$$

Obviously g is a bijection. So, one continuous function can be expressed by an infinite sequence of real numbers. Thus, we have $\mathcal{F} \cong \mathbb{R}^{\mathbb{N}}$. According to **Exercise 9.3**, $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. Therefore, we have $\mathcal{F} \cong \mathbb{R}^{\mathbb{N}} \cong \mathbb{R}$.

Exercise 9.5

$000 \dots, 100 \dots, 1100 \dots, 11100 \dots$ According to this rule, the first n bits of the n_{th} sequence are 1, and the remaining bits are 0. Obviously, these sequences constitute a countably infinite chain.

Exercise 9.6

$100 \dots, 0100 \dots, 00100 \dots, 000100 \dots$ According to this rule, the n_{th} bit of the n_{th} sequence is 1, and the remaining bits are 0. Obviously, these sequences constitute a countably infinite antichain.

In the following part, we denote the i_{th} digit of a sequence X as x_i .

Exercise 9.7

$\forall S \in \{0, 1\}^{\mathbb{N}}$, define $f(S) = T$ as follows.

If $s_k = 0$, then $t_{2k-1} := 0, t_{2k} := 1$.

If $s_k = 1$, then $t_{2k-1} := 1, t_{2k} := 0$.

For example:

$$\begin{aligned} S &= 1011010 \dots \\ f(S) &= 10, 01, 10, 10, 01, 10, 01, \dots \end{aligned}$$

Obviously, f is a bijection. Let $A = \{f(S) | S \in \{0, 1\}^{\mathbb{N}}\}$. $A \cong \{0, 1\}^{\mathbb{N}}$ is uncountable. Also, $\forall T_a, T_b \in A$ is not comparable since "01" and "10" is not comparable. Thus, A is an required antichain.

Exercise 9.8

$\forall S \in \{0, 1\}^{\mathbb{N}}$, define $f(S) = T$ as follows.

Consider the sequence consisting of the first k digits in S $s_{1:k}$ as a binary number a_k . Let $t_{(2^k+1):(2^k+a_k)} := 1$, $t_{(2^k+a_k+1):(2^{k+1})} := 0$. Specially, we define that the first 2 digits of T are always 0.

For example:

$$s = 1011.....$$

$$f(s) = 00, 10, 1100, 11111000, 1111111111100000,$$

Obviously, f is a bijection. Let $A = \{f(S) | \forall S \in \{0, 1\}^{\mathbb{N}}\}$. $A \cong \{0, 1\}^{\mathbb{N}}$ is uncountable. Also, $\forall T_a, T_b \in A$ is comparable for the following reason.

$\forall T_a, T_b \in A$, $T_a \neq T_b$, assume $f^{-1}(T_a) = S_a$, $f^{-1}(T_b) = S_b$. Assume that the first different digit of S_a and S_b is the k_{th} digit and the k_{th} digit of S_a is 1. Then, $\forall m \geq k$, the binary number consisting of the first m digit of S_a is larger than that of S_b , which leads to the fact that T_a is "greater" than T_b . Thus, A is the required chain.

Exercise 9.9

$\forall S \in \{0, 1\}^{\mathbb{N}}$, define $f(S) = T$ as follows.

$$t_n = \sum_{i=1}^n (s_i + 1) 3^{i-1}$$

. Obviously, f is a bijection. Let $X = \{f(S) | \forall S \in \{0, 1\}^{\mathbb{N}}\}$. $X \cong \{0, 1\}^{\mathbb{N}}$ is uncountable. And $f(S)$ is infinite. Whenever distinct $x, y \in X$ ($x = \{x_1, x_2, \dots\}$), suppose $m = f^{-1}(x)$, $n = f^{-1}(y)$ and assume the first different digit between m and n is the k_{th} digit. Then $x \cap y = \{x_1, x_2, x_3, \dots, x_{k-1}\}$, which is finite.

Question

How to prove that \mathbb{R} is smaller than $2^{\mathbb{R}}$.