CS499 Homework 9 (First Draft)

Intersteller

Exercise 9.1

We define $f_1 : N \rightarrow N$

$f_1(0) = 0, f_1(1) = 1, \cdots, f_1(n) = n.$

We define $f_2 : N \rightarrow N^2$ based on this graph:

	0	1	2	3	4	5	...
0	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	...
1	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	...
2	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	...
3	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Figure 1:

$f_2(0) = (0,0), f_2(1) = (0,1), f_2(2) = (1,0) \cdots$

We define $f_3 : N \rightarrow N^3$ based on this graph:

	0	1	2	3	4	5	...
$f_2(0)=(0,0)$	(0,0,0)	(0,0,1)	(0,0,2)	(0,0,3)	(0,0,4)	(0,0,5)	...
$f_2(1)=(0,1)$	(0,1,0)	(0,1,1)	(0,1,2)	(0,1,3)	(0,1,4)	(0,1,5)	...
$f_2(2)=(1,0)$	(1,0,0)	(1,0,1)	(1,0,2)	(1,0,3)	(1,0,4)	(1,0,5)	...
$f_2(3)=(2,0)$	(2,0,1)	(2,0,1)	(2,0,2)	(2,0,3)	(2,0,4)	(2,0,5)	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Figure 2:

$f_2\left(0\right)=\left(0,0,0\right), f_2\left(1\right)=\left(0,0,1\right), f_2\left(2\right)=\left(0,1,0\right)\cdots$

And so on, we can define $f_k, k \in N$. Now we can define a bijection $N \to N^*$ base on this graph:

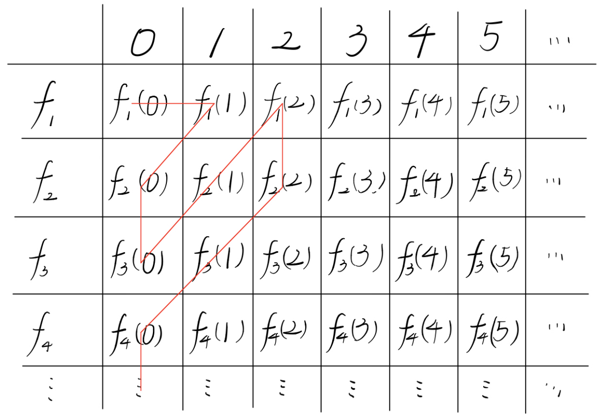

Figure 3:

We have $0 \to f_1\left(0\right), 1 \to f_1\left(1\right), 2 \to f_2\left(0\right)\cdots$. This is a bijection $N \to N^*$.

Exercise 9.2

We can define a bijection from $\{0,1\}^N$ to N as follows.
Given $A=(a_1,a_2,\cdots,a_n)$, $a_i \in \{0,1\}$ for $1 \leq i \leq n$, let $f(A)=2^n+a_1 2^{n-1}+a_2 2^{n-2}+\cdots+a_n 2^0-1$. For every element A in $\{0,1\}^N$, there is one $f(A)$ in N. For every element $a \in N$, according to the binary code, ther is one $f^{-1}(a) \in \{0,1\}^N$. So, f is a bijection from $\{0,1\}^N$ to N. Therefore, $\{0,1\}^N \cong N$.
Using the given fact that $N \times N \cong N$, it is clear that $\{0,1\}^N \cong \{0,1\}^N \times \{0,1\}^N$.
Using the given fact that $R \cong \{0,1\}^N$, it is clear that $R \cong R \times R$.

Exercies 9.3

Using the given fact that $R \cong \{0,1\}^N$ and the fact we proved in **9.2** that $\{0,1\}^N \cong N$, we only need to prove that $N \cong N^N$. N^N is actually $N^* - \{\in\}$. Using the method of Hilbert's Hotel, it is obvious that $N^N \cong N^*$. We have proved in **9.1** that $N \cong N^*$. Therefore, $R \cong R^N$.

Exercise 9.5

$000\cdots, 100\cdots, 1100\cdots, 11100\cdots$ According to this rule, the first n bits of the n_{th} sequence are 1, and the remaining bits are 0. Obviously, these sequences constitute a countably and infinite chain.

Exercise 9.6

$100\cdots, 0100\cdots, 00100\cdots, 000100\cdots$ According to this rule, the n_{th} bit of the n_{th} sequence is 1, and the remaining bits are 0. Obviously, these sequences constitute a countably and infinite antichain. **Exercise 9.8**

We can form a bijection f from $0,1^N$ to set A, which is a subset of $0,1^N$ as follow: Assume a string s is an element of $0,1^N$ and $t=f(s)$, the first k digit of s_1(we call it s_k)determines the $(2^k+1)-th$ to $2^{k+1}-th$ digits of t(we call them $t_{2^k+1}tot_{2^{k+1}}$ as the following rule:

Consider the first k digits as a binary number a_k, then t_{2^k+1} to $t_{2^k+a_k}$ are 1 and the $t_{2^k+a_k+1}$ to $t_{2^{k+1}}$ are 0. Specially, we define that the first 2 digits of t are always $'00'$.

Here is an example:

$$s=1011......$$

$$f(s)=00,10,1100,11111000,1111111111100000,......$$

Obviously, f is a bijection and A is uncountable. Also, any two elements t_1,t_2 of A is comparable. Assume two elements s_1,s_2 is different and their first different digit is the k-th digit. The k-th element of s_1 is 1. Then for any m such that $m \geq k$,the binary number of the first m digit of s_1 is greater than that of s_2, which leads to the conclusion that the according string of t_1 is ”greater” than t_2. Thus A is the chain required.