CS499 Homework 5 (First Draft)

Intersteller

Exercise 5.1

(1) The degree of a vertex is defined as the number of edges linked to this vertex. And the score of a graph is a sequence ranking degree of all vertices from small to big.

(2) Graph score theorem states that, if we can find a graph for graph score $(d_1, \cdots, d_{n-1}, d_n)$, then we can find a graph for graph score $(d_1, \cdots, d_{n-d_n-1}, d_{n-d_n} - 1, \cdots d_{n-1} - 1)$, and vice versa. If we finally get graph score (ϕ), the graph exists.

(3) Graph score algorithm:
First, we get a graph score $(d_1, \cdots, d_{n-1}, d_n)$.
If $d_N > n - 1$, we cannot find a graph. Otherwise, we add an edge from d_n to $d_{n-d_n}, \cdots d_{n-1}$, and check the graph score $(d_1, \cdots, d_{n-d_n-1}, d_{n-d_n} - 1, \cdots d_{n-1} - 1)$ after it is sorted.
We repeat the previous step. If the graph score finally comes to (ϕ), the graph exists.

(4) The most difficult part is to prove if we can find a graph for graph score $(d_1, \cdots, d_{n-1}, d_n)$, then we can find a graph for graph score $(d_1, \cdots, d_{n-d_n-1}, d_{n-d_n} - 1, \cdots d_{n-1} - 1)$. We can suppose there is a solution without edge between n and k $(n - d_n \leq k \leq n - 1)$, so n must have another link with j $(j \leq n - d_{n-1} < k)$. As $j < k$, we know $d_j \leq d_k$, so k must have edge with some point l and $l \neq k$. We change the edges (n, j) (k, l) to (n, k) (j, l), and we add an edge between n and k without changing the score. In this way, we can transform the answer to make sure there is an edge from d_n to $d_{n-d_n}, \cdots, d_{n-1}$. Then we delete these edges, we get a graph for score $(d_1, \cdots, d_{n-d_n-1}, d_{n-d_n} - 1, \cdots d_{n-1} - 1)$.

Exercise 5.2

Let's define an operation of a sequence:subtract 1 from the two largest number.

Theorem 5.3 can be written as: Let $(a_1, ..., a_n) \in N_0^n$. There is a multigraph with this score if and only if after $\frac{\sum_{n=1}^{N} a_n}{2}$ operations,all the numbers in the sequence are 0 (we name a sequence consists of 0 a "zero sequence").

Exercise 5.4

proof

(1) If there is a multigraph with score $(a_1, ..., a_n)$ satisfying$a_1 \leq a_2 \leq ... \leq a_n$:

(i) if $a_{n-1} = 1$, there must be an even number of 1s, it's obvious that we can change its score sequence to a zero sequence through several operations.

(ii) if $a_{n-1} = 1$ and there is an edge(v_{n-1}, v_n), delete this edge and we get a multigraph with score $(a_1, ...a_{n-1} - 1, a_n - 1)$. Repeat this operation and we would get a zero sequence eventually.

(iii) else there must be an edge (v_k, v_n) and an edge(v_l, v_{n-1}) such that $k \neq l$. Delte these two edges and add an edge(v_k, v_l) and we get a multigraph with score $(a_1, ...a_{n-1} - 1, a_n - 1)$.Repeat this operation and we would get a zero sequence eventually.

(2)

If a sequence can transform into a zero sequence, we can get a multigraph with the score sequence by undoing an operation and add an edge accordingly $\frac{\sum_{n=1}^{N} a_n}{2}$ times.
Theorem 5.3 proves ture based on **1** and **2**.

Exercise 5.5&5.6&5.7

Let $(a_1, \ldots, a_n) \in \mathbb{R}_0^n$.There is a weighted graph with this score if and only if suppose $a_1, \ldots, a_n$ are arranged from small to large,(a_n)≥$\sum_{i=1}^{n-1} a_i$

proof

(1)When $n = 2$,it is obvious that there is a weighted graph if and only if (a_1)=(a_2).

(2)When $n = 3$, suppose $a = a_3 = wdge(A_3), b = a_2 = wdge(A_2), c = a_1 = wdge(A_1)$,

$x = W_{\{A_3,A_2\}}, y = W_{\{A_3,A_1\}}, z = W_{\{A_1,A_2\}}, a \geq b \geq c$,then we have

$$\begin{cases} x + y = a \\ x + z = b \\ y + z = c \end{cases} \tag{1}$$

$$\Rightarrow$$

$$\begin{cases} x = \frac{a+b-c}{2} \\ y = \frac{a+c-b}{2} \\ z = \frac{b+c-a}{2} \end{cases} \tag{2}$$

So

if there is a weighted graph,then

$$x \geq 0 \Rightarrow a + b - c \geq 0$$
$$y \geq 0 \Rightarrow a + c - b \geq 0$$
$$z \geq 0 \Rightarrow b + c - a \geq 0 \Rightarrow a \leq b + c \Rightarrow a_3 \leq a_1 + a_2.$$

if $a \geq b \geq c$ and $a \leq b + c$,then

$x \geq 0, y \geq 0, z \geq 0 \Rightarrow$ there is a weighted graph.

(3) Suppose when $|V| = n$ the theorem is right,we talk about the situation of $|V| = n + 1$.

If $a_{n+1} \leq \sum_{i=1}^{n} a_i$, then we have

if $a_{n+1} \geq a_n + a_1$,we have$(a_2, \cdots, a_n, a_{n+1} - a_1)$and $a_{n+1} - a_1 \leq \sum_{i=2}^{n} a_i$, so there is a weighted graph G whose score is$(a_2, \cdots, a_n, a_{n+1} - a_1)$.Now we add vertex u and $edge(u, v)$ whose weight is $a_1(wdge(v) = a_{n+1} - a_1$ in $G)$ to get $G\prime$.Obviously the score of $G\prime$ is $(a_1, \cdots, a_n, a_{n+1})$.

if $a_{n+1} < a_n + a_1$,we have$(a_2, \cdots, a_n - \frac{a_1}{2}, a_{n+1} - \frac{a_1}{2})$,we have two situations:

1.

$a_{n+1} - \frac{a_1}{2}$ is the $max\{a_2, \cdots, a_n - \frac{a_1}{2}, a_{n+1} - \frac{a_1}{2}\}$.

Obviously $a_{n+1} - \frac{a1}{2} < a_n - \frac{a1}{2} < a2 + \cdots + a_n - \frac{a_1}{2}$

So we have a weighted graph G whose score is $(a_2, \cdots, a_n - \frac{a_1}{2}, a_{n+1} - \frac{a_1}{2})$.

Now we add vertex u and $edge(u, v)$ whose weight is $\frac{a_1}{2}(wdge(v) = a_{n+1} - \frac{a_1}{2}$ in $G)$and $edge(u, v\prime)(wdge(v\prime) = a_{n+1} - \frac{a_1}{2}$ in $G)$ to get $G\prime$.

Obviously the score of $G\prime$ is $(a_1, \cdots, a_n, a_{n+1})$.

2.

a_{n-1} is the $max\{a_2, \cdots, a_n - \frac{a_1}{2}, a_{n+1} - \frac{a_1}{2}\}$.

Obviously $a_{n-1} < a_n < a_n + a_{n+1} - a_1 = a_n - \frac{a_1}{2} + a_{n+1} - \frac{a_1}{2} \leq a_2 + \cdots + a_n - \frac{a_1}{2} + a_{n+1} - \frac{a_1}{2}$.

So we have a weighted graph G whose score is $(a_2, \cdots, a_n - \frac{a_1}{2}, a_{n+1} - \frac{a_1}{2})$.

In a similar way we get a weighted graph $G\prime$ whose score is $(a_1, \cdots, a_n, a_{n+1})$.

If there is a weighted graph whose score is $(a_1, \cdots, a_n)$,then we have

$a_n \leq \sum i = 1n - 1a_i$.Otherwise this graph is unable to satisfy a_n.

Exercise 5.8&5.9

(Score Theorem for Graphs with Real Edge Weights). Let $(a_1, \cdots, a_n) \in \mathbb{R}^n$. There is a graph with real edge weights with this score if and only if $n = 2$ and $a_1 = a_2$ or $n \geq 3$.

Exercise 5.10

Proof: If $n = 2$ and $a_1 = a_2$, it is obviously true.

Consider $n \geq 3$, we let the graph be a n polygon. For each edge between adjacent vertexes, the edge weight is x_i. Thus we have $(x_1, \cdots, x_n) \in \mathbb{R}^n$ and n equations:

$$
\begin{aligned}
x_n + x_1 &= a_1 \\
x_1 + x_2 &= a_2 \\
x_2 + x_3 &= a_3 \\
&\cdots \\
x_{n-1} + x_n &= a_n
\end{aligned}
$$

Obviously there are n variables and n equations, so there must exist real solutions for x_i.

Thus, for any $(a_1, \cdots, a_n) \in \mathbb{R}^n$ and $n \geq 3$, there must exist a graph with real edge weights with this score.

Exercise 5.11

(For convenience, we ignore the 0 (a dot) in the following)

ID:517030910250

(1)(2) It is neither a graph score nor a multigtaph score because the sum of the ID is an odd number.
(3) It is a weighted graph score, as is shown in the following figure.

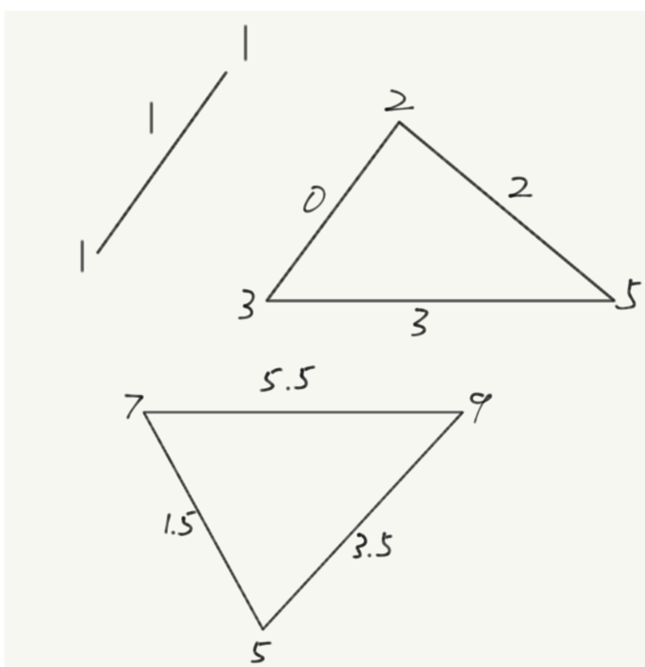

Figure 1:

4Since it is a weighted graph score, it is the score of a graph with real edge weights.

ID:517030910258

(1)(2) It is neither a graph score nor a multigtaph score because the sum of the ID is an odd number.
(3) It is a weighted graph score, as is shown in the following figure.

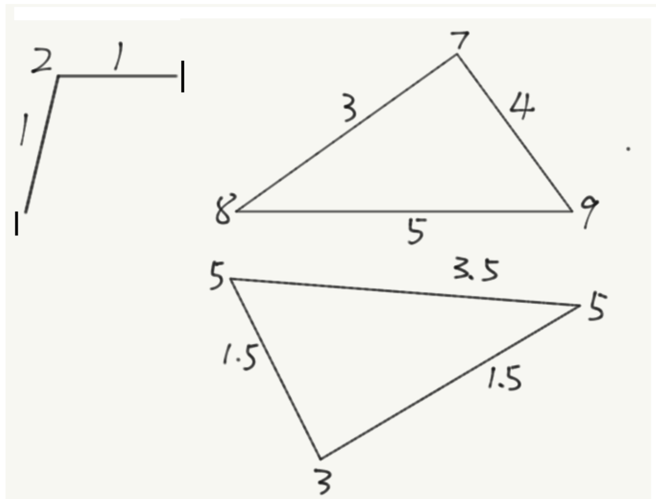

Figure 2:

4Since it is a weighted graph score, it is the score of a graph with real edge weights.

ID:517030910029

(1)(2) It is neither a graph score nor a multigtaph score because the sum of the ID is an odd number.
(3) It is a weighted graph score, as is shown in the following figure.

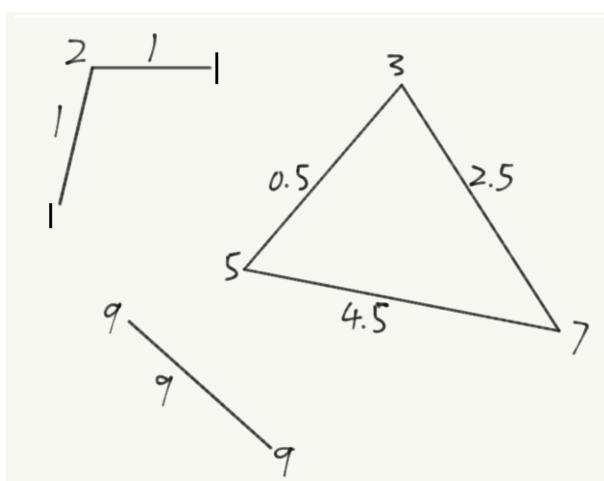

Figure 3:

4Since it is a weighted graph score, it is the score of a graph with real edge weights.

ID:517030910227

(1)(2) It is neither a graph score nor a multigtaph score because the sum of the ID is an odd number.
(3) It is a weighted graph score, as is shown in the following figure.

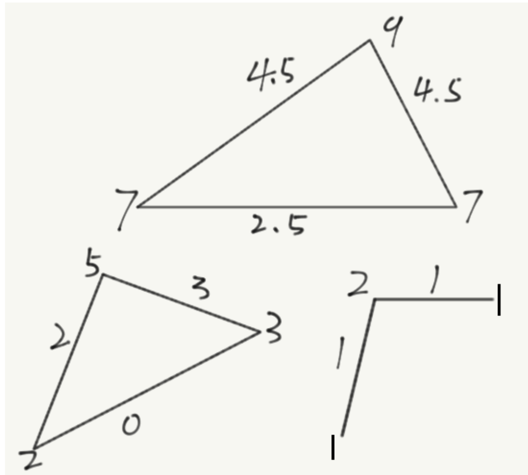

Figure 4:

4Since it is a weighted graph score, it is the score of a graph with real edge weights.

ID:517030910263

(1)(2) It is neither a graph score nor a multigtaph score because the sum of the ID is an odd number.
(3) It is a weighted graph score, as is shown in the following figure.

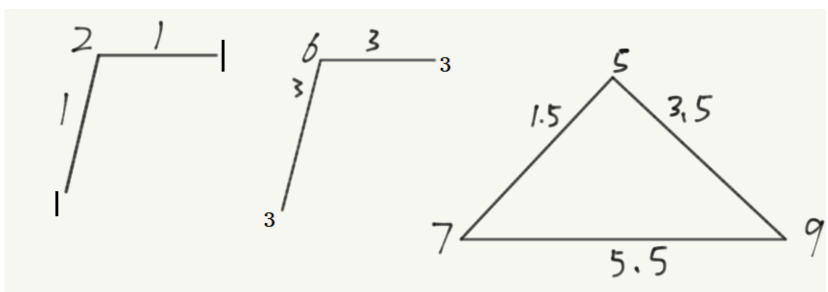

Figure 5:

4Since it is a weighted graph score, it is the score of a graph with real edge weights.

Questions