CS499 Homework 9 (First Draft)

Intersteller

Exercise 9.1

We define $f_1 : \mathbb{N} \to \mathbb{N}$

$f_1(0) = 0, f_1(1) = 1, \cdots, f_1(n) = n.$

We define $f_2 : \mathbb{N} \to \mathbb{N}^2$ based on this graph:

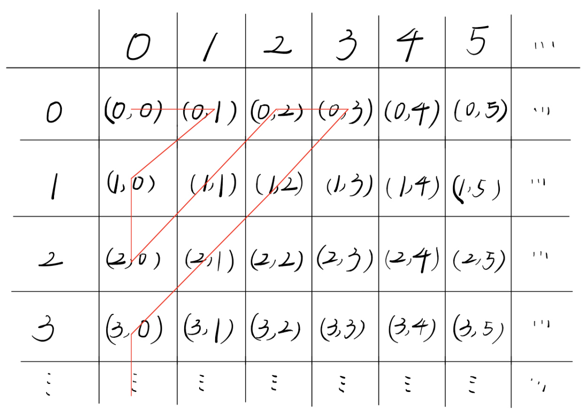

Figure 1:

$f_2(0) = (0,0), f_2(1) = (0,1), f_2(2) = (1,0) \cdots$

We define $f_3 : \mathbb{N} \to \mathbb{N}^3$ based on this graph:

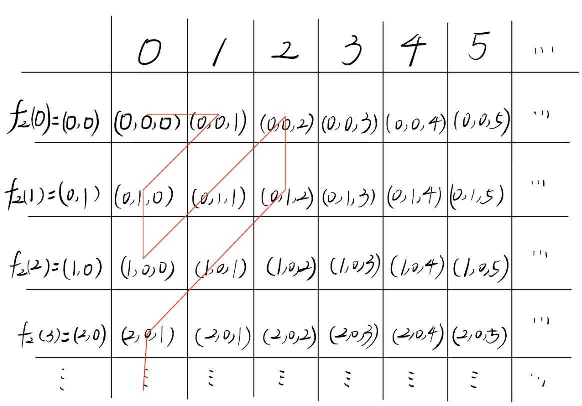

Figure 2:

$f_2\left(0\right)=\left(0,0,0\right), f_2\left(1\right)=\left(0,0,1\right), f_2\left(2\right)=\left(0,1,0\right)\cdots$

And so on, we can define $f_k, k\in\mathbb{N}$. Now we can define a bijection $\mathbb{N}\to\mathbb{N}^*$ base on this graph:

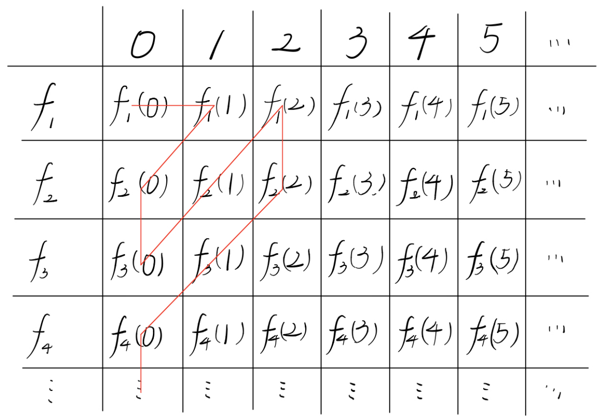

Figure 3:

We have $0\to f_1\left(0\right), 1\to f_1\left(1\right), 2\to f_2\left(0\right)\cdots$. This is a bijection $\mathbb{N}\to\mathbb{N}^*$.

Exercise 9.2

We can define a bijection from $\{0,1\}^{\mathbb{N}}$ to $\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$ as follows.
Given $A=(a_1a_2a_3a_4\cdots, b_1b_2b_3b_4\cdots)$, we define $f(A)=a_1b_1a_2b_2a_3b_3a_4b_4\cdots$. To be more precisely,

$$f(A)[i]=\begin{cases} A[1][\dfrac{i+1}{2}],\ i\ is\ odd\ number \\ A[2][\dfrac{i}{2}],\ i\ is\ even\ number \end{cases}$$

Obviously, for each $A\in\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$, there is only one $f(A)\in\{0,1\}^{\mathbb{N}}$. For each $B\in\{0,1\}^{\mathbb{N}}$, there is only one $B=f^{-1}(B)\in\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$. Therefore, f is a bijection and $\{0,1\}^{\mathbb{N}}\cong\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}$.
Using the fact that $\mathbb{R}\cong\{0,1\}^{\mathbb{N}}$, we can get $\mathbb{R}\cong\mathbb{R}\times\mathbb{R}$.

Exercies 9.3

We use the Cantor's method to proove that. For any $A\in(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$, we define that $f(A)$ is the $\{0,1\}^{\mathbb{N}}$ sequence we get by following the blue line as follows.

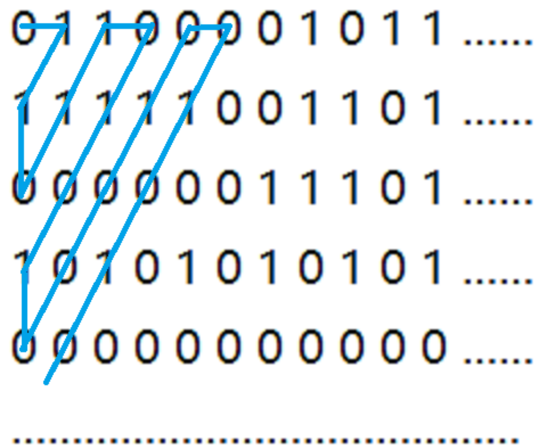

Figure 4:

It is obvious that for any $B \in \{0,1\}^{\mathbb{N}}$, we can get to $f^{-1} \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ by writing it down following the blue line. Therefore, f is a bijection and $\{0,1\} \cong (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$. Using the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$, we can get $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$.

Exercise 9.4

We know that one continuous function can be expressed by an infinite sequence of real numbers. Thus we have $\mathcal{F} \cong \mathbb{R}^{\mathbb{N}}$. According to **Exercise 9.3**, $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. So we have $\mathcal{F} \cong \mathbb{R}^{\mathbb{N}} \cong \mathbb{R}$.

Exercise 9.5

$000\cdots, 100\cdots, 1100\cdots, 11100\cdots$ According to this rule, the first n bits of the n_{th} sequence are 1, and the remaining bits are 0. Obviously, these sequences constitute a countably infinite chain.

Exercise 9.6

$100\cdots, 0100\cdots, 00100\cdots, 000100\cdots$ According to this rule, the n_{th} bit of the n_{th} sequence is 1, and the remaining bits are 0. Obviously, these sequences constitute a countably infinite antichain.

Exercise 9.7

We can define A using a bijection f from $\{0,1\}^{\mathbb{N}}$ to A, which is a subset of $\{0,1\}^{\mathbb{N}}$ as follows.
Assuming a string s is an element of $\{0,1\}^{\mathbb{N}}$ and $f(s) = t$, let s_k determines t_{2k-1} and t_{2k} by the following rule.
If $s_k = 0$, then $t_{2k-1} := 0$, $t_{2k} := 1$. If $s_k = 1$, then $t_{2k-1} := 1$, $t_{2k} := 0$.
For example:

$$s = 1011010........$$
$$f(s) = 10, 01, 10, 10, 01, 10, 01, ...$$

Obviously, f is a bijection and A is uncountable. Also, any two elements t_a, t_b of A is not comparable since "01" and "10" is not comparable. Thus, A is an required antichain.

Exercise 9.8

We can define A using a bijection f from $\{0,1\}^{\mathbb{N}}$ to A, which is a subset of $\{0,1\}^{\mathbb{N}}$ as follows.
Assuming $s \in \{0,1\}^{\mathbb{N}}$ and $f(s) = t$, the first k digit of s (we call it $s_{1:k}$) determines $t_{(2^k+1):(2^{k+1})}$ by the following rule.

Consider the $s_{1:k}$ as a binary number a_k, then $t_{(2^k+1):(2^k+a_k)} := 1$ and the $t_{(2^k+a_k+1):(2^{k+1})} := 0$.
Specially, we define that the first 2 digits of t are always 0.

For example:

$$s = 1011......$$
$$f(s) = 00, 10, 1100, 11111000, 1111111111100000, ...$$

Obviously, f is a bijection and A is uncountable. Also, any two elements $t_a, t_b \in A$ is comparable. Assuming that two elements s_a, s_b is different, their first different digit is the k_{th} digit and the k_{th} digit of s_a is 1, then for any m such that $m \geq k$, the binary number of the first m digit of s_a is greater than that of s_b, which leads to the conclusion that string t_a is "greater" than t_b. Thus, A is the required chain.