CS499 Homework 4 (First Draft)

Intersteller

Exercise 4.2

1. Since
$$E(T) = \sum_{n=0}^{\infty} (2n+1) C_n p^{n+1} (1-p)^n$$
and
$$\begin{aligned}\sum_{n=0}^{\infty} (2n+1) C_n p^{n+1} (1-p)^n &= p \sum_{n=0}^{\infty} \frac{2n+1}{n+1} \binom{2n}{n} [p(1-p)]^n \\ &< 2p \sum_{n=0}^{\infty} \binom{2n}{n} [p(1-p)]^n \\ &< 2p \sum_{n=0}^{\infty} 2^{2n} [p(1-p)]^n \\ &= 2p \sum_{n=0}^{\infty} [4p(1-p)]^n\end{aligned}$$
we have
$$E(T) < 2p \sum_{n=0}^{\infty} [4p(1-p)]^n$$
Since $p > \frac{1}{2}$, then $4p(1-p) < 1$.
Thus,
$$E(T) < \frac{2p}{1-4p(1-p)}$$
$E(T)$ is finite.
2. We denote
$$g(x) = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n + ...$$
where C_n is Calatan number.
We can compute that
$$[g(x)]^2 = C_0^2 + (C_0C_1 + C_1C_0)x + (C_0C_2 + C_1^2 + C_2C_0)x^2 + ... + (C_0C_n + C_1C_{n-1} + ... + C_nC_0)x^n + ...$$
Since
$$C_n = C_0C_{n-1} + C_1C_{n-2} + ... + C_{n-1}C_0$$
we have
$$[g(x)]^2 = C_0^2 + C_2 x + C_3 x^2 + C_4 x^3 + ... + C_{n+1} x^n + ...$$
Since $C_0 = C_1 = 1$, we have $x[g(x)]^2 = g(x) - 1$.
We solve the equation and get
$$g(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Since $g(0) = 1$, thus

$$g(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

$$g'(x) = \frac{1}{x\sqrt{1 - 4x}} - \frac{1 - \sqrt{1 - 4x}}{2x^2}$$

So,

$$\begin{aligned}
E(T) &= \sum_{n=0}^{\infty}(2n+1)C_n p^{n+1}(1-p)^n \\
&= 2p\sum_{n=0}^{\infty} nC_n[p(1-p)]^n + p\sum_{n=0}^{\infty} C_n[p(1-p)]^n \\
&= 2p^2(1-p) \times (g[p(1-p)])' + p \times g[p(1-p)] \\
&= \frac{2p^2(1-p)}{p(1-p)\sqrt{1-4p(1-p)}} - \frac{1-\sqrt{1-4p(1-p)}}{1-p} + p \times \frac{1-\sqrt{1-4p(1-p)}}{2p(1-p)} \\
&= \frac{2p}{\sqrt{1-4p(1-p)}} - \frac{1-\sqrt{1-4p(1-p)}}{2(1-p)} \\
&= \frac{2p}{2p-1} - \frac{2(1-p)}{2(1-p)} \\
&= \frac{1}{2p-1}
\end{aligned}$$

Exercise 4.4

$$E[\frac{1}{T+1}] = \sum_{n=0}^{\infty} \frac{1}{2(n+1)} C_n p^{n+1}(1-p)^n$$

We denote

$$g(x) = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n + ...$$

where C_n is Calatan number.

We can compute that

$$[g(x)]^2 = C_0^2 + (C_0C_1 + C_1C_0)x + (C_0C_2 + C_1^2 + C_2C_0)x^2 + ... + (C_0C_n + C_1C_{n-1} + ... + C_nC_0)x^n + ...$$

Since

$$C_n = C_0C_{n-1} + C_1C_{n-2} + ... + C_{n-1}C_0$$

we have

$$[g(x)]^2 = C_0^2 + C_2 x + C_3 x^2 + C_4 x^3 + ... + C_{n+1}x^n + ...$$

Since $C_0 = C_1 = 1$, we have $x[g(x)]^2 = g(x) - 1$.

We solve the equation and get

$$g(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}$$

Since $g(0) = 1$, thus

$$g(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} C_n x^{n+1} = \int_0^x g(t)dt = \ln(\sqrt{1-4x} + 1) - \sqrt{1-4x} - \ln 2 + 1$$

Since $p = \frac{1}{2}$, then

$$\begin{aligned} E[\frac{1}{T+1}] &= \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{n+1}C_n(\frac{1}{4})^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n+1}C_n(\frac{1}{4})^{n+1} \\ &= G(\frac{1}{4}) \\ &= 1 - \ln 2 \end{aligned}$$

Exercise 4.5

Proof. Suppose A_i whose strength is a_i and B_j whose strength is b_j fight.After fighting,for A_i we have

$$E(strength_{A_i}) = \frac{a_i}{a_i + b_j} \cdot (a_i + b_j) + \frac{b_j}{a_i + b_j} \cdot 0 = a_i$$

So after $\forall k$ wars,

$$E(\sum strength_{A_i}) = \sum_{i=1}^{m} a_i$$

$$E(\sum strength_{B_j}) = \sum_{j=1}^{n} b_j$$

Now let's compute $E(\sum strength_{A_i})$ in a different way:

Wars has only two results:Alice'team wins or Bob's team wins.

The former happens with probability p, the latter with probability $1 - p$.

Thus

$$E(\sum strength_{A_i}) = (\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j) \cdot p + 0 \cdot (1 - p)$$

Therefore,

$$p = \frac{\sum_{i=1}^{m} a_i}{\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j}$$

So the probability of Alice's team winning does not depend on the order in which Alice and Bob send their monsters into the arena.