CS499 Homework 10 (First Draft)

Intersteller

Exercise 10.1

Since
$$\sum_{v\in V} f(s,v) = \sum_{v\in V\setminus S} f(s,v) + \sum_{v\in S} f(s,v)$$
we only need to prove that
$$\sum_{v\in S} f(s,v) = \sum_{u\in S-s,\ v\in V\setminus S} f(u,v)$$
Since
$$\sum_{v\in S} f(s,v) = -\sum_{v\in S-s} f(v,s)$$
we only need to prove that
$$\sum_{u\in S-s,\ v\in s+V\setminus S} f(u,v) = 0$$
It is obvious to see that
$$\sum_{u\in S-s,\ v\in S-s} f(u,v) = 0$$
So, we only need to prove that
$$\sum_{u\in S-s,\ v\in s+V\setminus S} f(u,v) + \sum_{u\in S-s,\ v\in S-s} f(u,v) = \sum_{u\in S-s,\ v\in V} f(u,v) = 0$$
According to the defination,
$$\sum_{u\in S-s,\ v\in V} f(u,v) = 0$$
Done.

Exercise 10.2

Define the minimum cut between i and j as minCut(i,j). According to the Max Flow Min Cut Theorem, minCut(s,r)≥k, minCut(r,t)≥k. Obviously, minCut(s,t)≥min{minCut(s,r),minCut(r,t)}≥k, which means there is a flow from s to r of value k.

Exercise 10.4

Suppose there is a s-t-path in G that has less that k edges. Then, at least one edge in the path moves more than one level forward, which contradicts Definition 10.3. So, $dist(s,t) \geq k$.

Exercise 10.5

Since dist$(s,t) = k$, we have a path $s \to u_1 \to u_2 \cdots u_{k-1} \to t$.For convenience, we call $s : u_0, t : u_k$. We construct $V_0, V_1 \cdots V_K$ as follows:

Step1:$u_0 \in V_0, u_1 \in V_1, u_2 \in V_2 \cdots u_{k-1} \in V_{K-1}, u_k \in V_K$.

Step2:$\forall v \in V\setminus V_k$, if there is an edge $(v,t), v \in V_{k-1}$. Obviously, $\forall n, 0 \leq n \leq k-2, u_n$ can't link to t. Otherwise, we can skip u_{k-1} and get a shorter path from s to t, which contradicts $dist(s,t) = k$.

Step3:$\forall v \in V \backslash (V_k \cup V_{k-1})$, if $\exists w \in V_{k-1}, \exists edge(v, w), v \in V_{k-2}$. Similarly, $\forall n, 0 \leq n \leq k-3$, u_n can't link to any vertex in V_{k-1}.

$\vdots$

Step$k$:$\forall v \in V \backslash \bigcup_{i=2}^{k} V_i$, if $\exists w \in V_2, \exists edge(v, w), v \in V_1$. Similarly, u_0 can't link to any vertex in V_2.

Step$k+1$:$\forall v \in V$ and $v \notin V_i, i \neq 1, 2, 3 \cdots k, v \in V_0$.

Therefore, if dist$(s, t) = k$, (G, s, t, c) has a k-layering.

Exercise 10.6

Obviously, $(1) s \in V_0$ and $(2) t \in V_k$ are satisfied, we consider condition (3). Since (G, s, t, c) is a flow network and $V_0, V_1, ..., V_k$ is an optimal layering, every edge in G moves at most one level forward and $dist_G(s, t) = k$. And we denote the path p as $s \rightarrow v_1 \rightarrow v_2 \rightarrow ... \rightarrow v_{k-1} \rightarrow t$. Then we can get $s \in V_0$, $v_1 \in V_1$, ... ,$v_i \in V_i$, ... ,$v_{k-1} \in V_{k-1}$ and $t \in V_k$. Each edge is between two adjacent layerings. Since residual network G_f only add a reverse edge in each edge in p, these additional edge is also between two adjacent layerings. So $V_0, V_1, ..., V_k$ satisfy condition 3 and it is a layering of (G_f, s, t, c_f).

Exercise 10.7

Because there is a path from s to t, we suppose $dist_G(s, t) = k$. Based on the construction method in 10.5, we can find a k-layering and it is an optimal layering. Therefore, every network(G, s, t, c) has an optimal layering, provided there is a path from s to t.

Exercise 10.8

We consider each while-loop of EK algorithm. In every iteration EK algorithm choose p to be a shortest s-t-path in G_f. And we denote the path p as $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow ... \rightarrow v_{k-1} \rightarrow v_k$ and $v_0 = s \in V_0$, $v_1 \in V_1$, ... ,$v_i \in V_i$, ... ,$v_{k-1} \in V_{k-1}$ and $v_k = t \in V_k$. Then EK algorithm routes c_{min} flow along p. So in G there $\exists (v_i, v_{i+1}) \in p$ where $i \in 0, 1, 2..., k$, $c_f(v_i, v_{i+1}) = c_{min}$ and in G_f, $c_f(v_i, v_{i+1}) = 0$ and $c_f(v_{i+1}, v_i) = c_{min}$. Obviously, after that $c_f(v_i, v_{i+1})$ is always 0 if $V_0, V_1, ..., V_k$ is still an optimal layering, otherwise there is a feasible s-t-path from v_{i+1} to v_i which is impossible. Therefore in every iteration, the total number of edges, which are from V_i to $V_{i+1}(\forall i \in 0, 1, 2, .., k)$ and in feasible s-t-path$(dist(s, t) = k)$, will minus at least one. Obviously these edges are less than or equal to m. So after m iteration, there no feasible s-t-path which $dist(s, t) = k$ and $dist(s, t)$ will be large than k. Therefore after at most m iterations of the while-loop, $V_0, V_1, ..., V_k$ ceases to be an optimal layering.

Exercise 10.9

proof According to **Exercise 10.8**, a particular layering is no more optimal after at moat m iterations. Since a layering is at least 1-layering and at most n-latering, after at most $m * n$ iterations, there is no optimal layering, which means there is no s-t-path, the algorithm terminates.

Exercise 10.10

proof According to **Exercise 10.9**, the Edmonds-Karp algorithm terminates after n·m iterations of the while-loop, which is to say, we can get the max flow f after finite steps by Edmonds-Karp algorithm.

Question