

CS499 Homework 9 (First Draft)

Interstellar

Exercise 9.1

We define $f_1 : N \rightarrow N$

$$f_1(0) = 0, f_1(1) = 1, \dots, f_1(n) = n.$$

We define $f_2 : N \rightarrow N^2$ based on this graph:

	0	1	2	3	4	5	...
0	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	...
1	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	...
2	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	...
3	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	...
...

Figure 1:

$$f_2(0) = (0,0), f_2(1) = (0,1), f_2(2) = (1,0) \dots$$

We define $f_3 : N \rightarrow N^3$ based on this graph:

	0	1	2	3	4	5	...
$f_2(0)=(0,0)$	(0,0,0)	(0,0,1)	(0,0,2)	(0,0,3)	(0,0,4)	(0,0,5)	...
$f_2(1)=(0,1)$	(0,1,0)	(0,1,1)	(0,1,2)	(0,1,3)	(0,1,4)	(0,1,5)	...
$f_2(2)=(1,0)$	(1,0,0)	(1,0,1)	(1,0,2)	(1,0,3)	(1,0,4)	(1,0,5)	...
$f_2(3)=(2,0)$	(2,0,0)	(2,0,1)	(2,0,2)	(2,0,3)	(2,0,4)	(2,0,5)	...
...

Figure 2:

$f_2(0) = (0, 0, 0), f_2(1) = (0, 0, 1), f_2(2) = (0, 1, 0) \dots$

And so on, we can define $f_k, k \in N$. Now we can define a bijection $N \rightarrow N^*$ base on this graph:

	0	1	2	3	4	5	...
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	$f_1(4)$	$f_1(5)$...
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	$f_2(4)$	$f_2(5)$...
f_3	$f_3(0)$	$f_3(1)$	$f_3(2)$	$f_3(3)$	$f_3(4)$	$f_3(5)$...
f_4	$f_4(0)$	$f_4(1)$	$f_4(2)$	$f_4(3)$	$f_4(4)$	$f_4(5)$...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots

Figure 3:

We have $0 \rightarrow f_1(0), 1 \rightarrow f_1(1), 2 \rightarrow f_2(0) \dots$. This is a bijection $N \rightarrow N^*$.

Exercise 9.5

$000 \dots, 100 \dots, 1100 \dots, 11100 \dots$ According to this rule, the first n bits of the n_{th} sequence are 1, and the remaining bits are 0. Obviously, these sequences constitute a countably and infinite chain.

Exercise 9.6

$100 \dots, 0100 \dots, 00100 \dots, 000100 \dots$ According to this rule, the n_{th} bit of the n_{th} sequence is 1, and the remaining bits are 0. Obviously, these sequences constitute a countably and infinite antichain. **Exercise 9.8**

We can form a bijection f from $0, 1^N$ to set A , which is a subset of $0, 1^N$ as follow: Assume a string s is an element of $0, 1^N$ and $t = f(s)$, the first k digit of s_1 (we call it s_k) determines the $(2^k + 1) - th$ to $2^{k+1} - th$ digits of t (we call them $t_{2^k+1} to t_{2^{k+1}}$ as the following rule:

Consider the first k digits as a binary number a_k , then t_{2^k+1} to $t_{2^k+a_k}$ are 1 and the $t_{2^k+a_k+1}$ to $t_{2^{k+1}}$ are 0. Specially, we define that the first 2 digits of t are always '00'.

Here is an example:

$$s = 1011 \dots$$

$$f(s) = 00, 10, 1100, 11111000, 1111111111100000, \dots$$

Obviously, f is a bijection and A is uncountable. Also, any two elements t_1, t_2 of A is comparable. Assume two elements s_1, s_2 is different and their first different digit is the k -th digit. The k -th element of s_1 is 1. Then for any m such that $m \geq k$, the binary number of the first m digit of s_1 is greater than that of s_2 , which leads to the conclusion that the according string of t_1 is "greater" than t_2 . Thus A is the chain required.