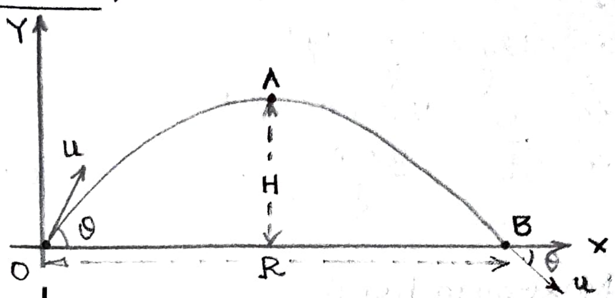
MOTION IN TWO DIMENSION
(PROJECTILE MOTION)

The motion of bodies under gravity is generally known as projectile motion.
Example: Motion of cricket or footballs in air.



The assumptions of projectile motion are as follows:

- There is no air resistance.
- The effect due to curvature of earth is negligible
- The effect due to rotation of earth is negligible
- For all points of the trajectory, the acceleration due to gravity g is constant in magnitude and direction.

O → Point of projection
u → Initial velocity or velocity of projection
θ → Angle of projection
R → Range
H → Maximum height reached.

Calculation of various parameters in projectile motion

Let a particle is projected with velocity u at an angle θ with the horizontal from point O as shown in figure.

Along x-axis

Initial velocity $u_x = u\cos\theta$

Acceleration $a_x = 0$ [As No force exists]

Displacement $x = u_x t + \frac{1}{2}a_x t^2$

$\therefore x = (u\cos\theta)\,t$

$\therefore \boxed{t = \frac{x}{u\cos\theta}}$ --- (i)

Along Y-axis

Initial velocity $u_y = u\sin\theta$

Acceleration $a_y = -g$

Displacement $y = u_y t + \frac{1}{2}a_y t^2$

$\therefore y = (u\sin\theta)t - \frac{1}{2}gt^2$ --- (ii)

velocity at any instance

$v_y = u_y + a_y t$

$\therefore v_y = u\sin\theta - gt$ --- (iii)

Equation of projectile

From eqn (i) & (ii)

$$y = (u\sin\theta)\frac{x}{u\cos\theta} - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}$$

$$\therefore y = x\tan\theta - \frac{g x^2}{2u^2\cos^2\theta}$$

∴ This eqn is in the form

$y = ax + bx^2$ where $a = \tan\theta$ and $b = -\frac{g}{2u^2\cos^2\theta}$

∴ This is the eqn of a parabola. Thus the eqn. of the trajectory of the projectile is a parabola.

$$\therefore \boxed{y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2} \text{ ------ (iv)}$$

Half time of flight

At maximum height $v_y = 0$

$\therefore$ $v_y = u_y + a_y t$

$\therefore$ $0 = u\sin\theta - gt$

$\therefore$ $\boxed{t = \dfrac{u\sin\theta}{g}}$ ------- (v)

Maximum height

At maximum height $v_y = 0$

$\therefore$ $v_y^2 = u_y^2 + 2a_y s$

$\therefore$ $0 = u^2\sin^2\theta - 2gH$

$\therefore$ $\boxed{H = \dfrac{u^2\sin^2\theta}{2g}}$ ------ (vi)

Total time of flight

At point B, the y displacement becomes zero.

$\therefore$ $y = u_y t + \frac{1}{2} a_y t^2$

$0 = (u\sin\theta)T + \frac{1}{2} g T^2$

$\therefore$ $\boxed{T = \dfrac{2u\sin\theta}{g}}$ ----- (vii)

Range of the projectile

$R = u_x \times T$

$\therefore$ $R = u\cos\theta \cdot \dfrac{2u\sin\theta}{g}$

$\boxed{R = \dfrac{u^2\sin 2\theta}{g}}$ ---- (viii)

For range to be maximum $\dfrac{dR}{d\theta} = 0$

$\therefore$ $\dfrac{d}{d\theta}\left[\dfrac{u^2\sin 2\theta}{g}\right] = 0$

$\therefore$ $\dfrac{2u^2}{g}\cos 2\theta = 0$; $u \neq 0$ $\therefore \cos 2\theta = 0 = \cos 90°$ $\therefore \theta = 45°$

$\therefore$ A projectile will have maximum range when it is projected at an angle $\pi/4$ (45°) to the horizontal and the maximum range will be (u^2/g).

- When the range is maximum, the maximum height H reached by the projectile is

$$H = \frac{u^2\sin^2\theta}{2g} = \frac{u^2\sin^2 45}{2g} = \frac{u^2}{4g} = \frac{R_{max}}{4}$$

- Relation between horizontal range and maximum height:

$R = \dfrac{u^2\sin 2\theta}{g}$ and $H = \dfrac{u^2\sin^2\theta}{2g}$ $\therefore \dfrac{R}{H} = \dfrac{u^2\sin 2\theta/g}{u^2\sin^2\theta/2g} = 4\cot\theta$

$\therefore$ $R = 4H\cot\theta$.

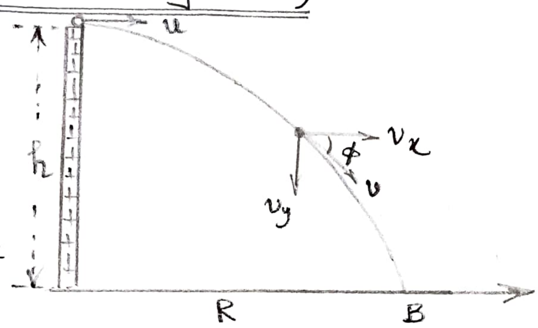
For two complimentary angle of projection, the range is same

$$R_\theta = \frac{u^2 \sin 2\theta}{g} \quad ; \quad R_{90-\theta} = \frac{u^2 \sin 2(90^\circ - \theta)}{g} = \frac{u^2 \sin(180^\circ - 2\theta)}{g}$$

$$\therefore R_{90-\theta} = \frac{u^2 \sin 2\theta}{g} = R_\theta .$$

Horizontal Projection (Horizontal Projectile)

A body is projected horizontally from a certain height y vertically above the ground with initial velocity u.



- Trajectory of horizonal projectile

$$x = ut \quad \therefore t = \frac{x}{u} \quad \text{------ (i)}$$

The verticle displacement in t time

$$y = -\frac{1}{2} g t^2 \quad (\text{Initial velocity} = 0) \quad \text{------ (ii)}$$

$\therefore$ From eqn (i) & (ii)

$$y = -\frac{1}{2} \frac{g x^2}{u^2} \quad \text{------ (iii)}$$

- Displacement of the projectile

$$r = \sqrt{(ut)^2 + \left(\frac{gt^2}{2}\right)^2} = ut\sqrt{1 + \left(\frac{gt}{2u}\right)^2} \quad \text{------ (iv)}$$

$$\tan\alpha = \frac{\frac{1}{2} g t^2}{ut} = \frac{1}{u}\sqrt{\frac{gy}{2}} \quad \left[\text{As } t = \sqrt{\frac{2y}{g}}\right] \quad \text{------ (v)}$$

- Instantaneous velocity:

$$v_y = 0 - gt = -gt$$

So, $\vec{v} = v_x \hat{i} - v_y \hat{j} = u\hat{i} - gt\hat{j}$

$$v = \sqrt{u^2 + (gt)^2} = u\sqrt{1 + (gt/u)^2}$$

Again $\vec{v} = u\hat{i} - \sqrt{2gy}\,\hat{j}$

$$v = \sqrt{u^2 + 2gy} \quad \text{------ (vi)}$$

- Direction of instantaneous velocity:

$$\tan\phi = \frac{v_y}{v_x} \quad \therefore \phi = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{\sqrt{2gy}}{u}\right) = \tan^{-1}\left(\frac{gt}{u}\right) \quad \text{------ (vii)}$$

- Time of flight

$$-h = 0 - \frac{1}{2} g T^2 \quad \therefore T = \sqrt{\frac{2h}{g}} \quad \text{------ (viii)}$$

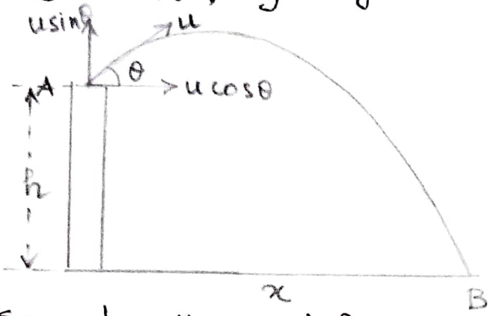
- Horizontal range

$$R = uT + \frac{1}{2} 0 T^2 \quad \therefore R = u\sqrt{\frac{2h}{g}} \quad \text{------ (ix)}$$

Projectile from hight at certain angle with horizontal

Case I : Projectile at an angle θ above horizontal

$u_x = u\cos\theta$, $a_y = -g$



$S_y = -h$, $u_y = u\sin\theta$, $a_y = -g$, $t = T$

$S_y = u_y t + \frac{1}{2} a_y t^2$

$-h = (u\sin\theta)T - \frac{1}{2} g T^2$

Range $R = u_x T = (u\cos\theta)\, T$

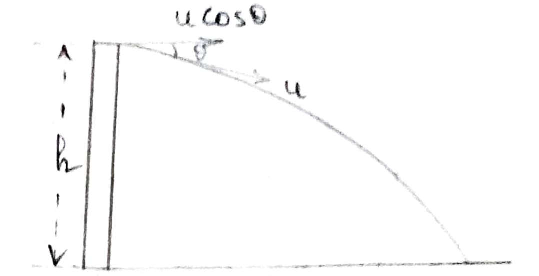
also $v_y^2 = u_y^2 + 2a_y S_y = u^2\sin^2\theta + 2gh$

$v_x = u\cos\theta$

$v_B = \sqrt{v_y^2 + v_x^2} = \sqrt{u^2 + 2gh}$.

Case II : Projectile at an angle θ below horizontal

$u_x = u\cos\theta$, $u_y = -u\sin\theta$, $a_y = -g$



$S_y = u_y t + \frac{1}{2} a_y t^2$

$S_y = -h$, $u_y = -u\sin\theta$

$t = T$, $a_y = -g$

$\therefore -h = -(u\sin\theta)T - \frac{1}{2} g T^2$

$h = (u\sin\theta)T + \frac{1}{2} g T^2$

Range $R = u_x T = (u\cos\theta)\, T$

$v_x = u\cos\theta$

$\therefore v_y^2 = u_y^2 + 2a_y S_y$

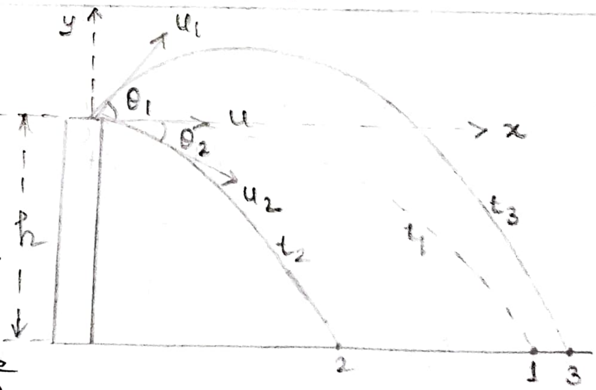
$\therefore v_y^2 = u^2\sin^2\theta + 2(-g)(-h)$

$v_y^2 = u^2\sin^2\theta + 2gh$

$v_B = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + 2gh}$.

Important observation:

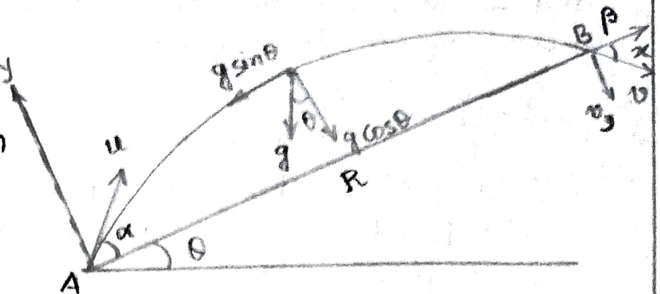
If three objects are thrown from the same height in different directions as shown in figure then



- If $u_1 = u_2 = u$ and $\theta_1 = \theta_2$
  All the three particle will strike the ground with the same final speed.
- If $u_1 \neq u_2 \neq u$, then all the three particles will strike the ground with the different final speed.
- If $|u_1\sin\theta_1| = |u_2\sin\theta_2|$, then the time of flights of the particles are related as $t_3 = \sqrt{t_1 t_2}$.

Projectile Motion on an Inclined Plane :

Consider a plane is inclined at an angle θ with horizontal. Let a projectile is projected from the foot of the inclined plane with velocity u at an angle α with the inclined plane as shown in figure.



Take x-axis parallel to the inclined plane and y-axis perpendicular to the inclined plane.

Now, $u_x = u\cos\alpha$, $u_y = u\sin\alpha$, $a_x = -g\sin\theta$, $a_y = -g\cos\theta$. From A to B : $S_x = R$, $S_y = 0$ and let time taken from A to B is T.

$\therefore$

$$S_y = u_y t + \frac{1}{2} a_y t^2$$

$$0 = (u\sin\alpha)T - \frac{1}{2} g\cos\theta\, T^2 \quad \therefore T = \frac{2u\sin\alpha}{g\cos\theta} \quad \text{----(i)}$$

Now, by applying $S_x = u_x t + \frac{1}{2} a_x t^2$ we get

$$R = (u\cos\alpha)T - \frac{1}{2} g\sin\theta\, T^2 \quad \text{----(ii)}$$

Now, put the value of T from (i) in (ii), we get

$$R = u\cos\theta \,\frac{2u\sin\theta}{g\cos\theta} - \frac{1}{2} g\sin\theta \left(\frac{2u\sin\alpha}{g\cos\theta}\right)^2$$

$$= \frac{2u^2\sin\alpha\cos\alpha}{g\cos\theta} - \frac{2u^2\sin^2\alpha\sin\theta}{g\cos^2\theta}$$

$$= \frac{u^2}{g\cos^2\theta}\left[\sin 2\alpha\cos\theta - \sin\theta\, 2\sin^2\alpha\right]$$

$$= \frac{u^2}{g\cos^2\theta}\left[\sin 2\alpha\cos\theta - \sin\theta(1-\cos 2\alpha)\right]$$

$$R = \frac{u^2}{g\cos^2\theta}\left[\sin(2\alpha+\theta) - \sin\theta\right] \quad \text{------(iii)}$$

Maximum Range : Range will be maximum if $\sin(2\alpha+\theta) = 1$

$\therefore \quad 2\alpha + \theta = \pi/2 \quad$ or $\quad \alpha = \left(\pi/4 - \frac{\theta}{2}\right) \quad$ ------(iv)

$$\therefore \quad R_{max} = \frac{u^2}{g\cos^2\theta}\left[1-\sin\theta\right] = \frac{u^2(1-\sin\theta)}{g(1-\sin^2\theta)} = \frac{u^2}{g(1+\sin\theta)}.$$

Velocity at point B

Applying $v_x = u_x + a_x t$

$$v\cos\beta = u\cos\alpha - g\sin\theta . T$$

$$= u\cos\alpha - g\sin\theta \cdot \frac{2u\sin\alpha}{g\cos\theta}$$

$$= u\cos\alpha(1 - 2\tan\alpha\tan\theta)$$

Applying $v_y = u_y + a_y t$

$$-v\sin\beta = u\sin\alpha - g\cos\theta \cdot \frac{2u\sin\alpha}{g\cos\theta}$$

$$v\sin\beta = u\sin\alpha .$$

Note:

- Magnitude of velocity along y-axis is same at both the points A and B. This can also be concluded from the equation $v_y^2 = u_y^2 + 2a_y s$. Here $S_y = 0$ from A to B. So $v_y^2 = u_y^2$ $\therefore |v_y| = |u_y|$ $\therefore v\sin\beta = u\sin\alpha$.

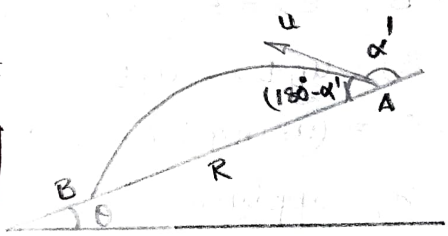
- $\tan\beta = \dfrac{\tan\alpha}{1 - 2\tan\alpha\tan\theta}$.

  → If $\beta < 90^\circ$, then $1 - 2\tan\alpha\tan\theta > 0$ and in this condition $\tan\beta > \tan\alpha$ or $\beta > \alpha$.

  → If $\beta > 90^\circ$, then $1 - 2\tan\alpha\tan\theta < 0$ and in this condition also $\beta > \alpha$.

If the projectile were projected down the inclined plane:

Replacing R with (−R) and α with $(180 - \alpha')$ in the range formula, we get



$$-R = \frac{u^2}{g\cos^2\theta}\left[\sin\{2(180^\circ - \alpha') + \theta\} + \sin\theta\right]$$

$$= \frac{u^2}{g\cos^2\theta}\left[\sin\{360^\circ - (2\alpha' - \theta)\} - \sin\theta\right]$$

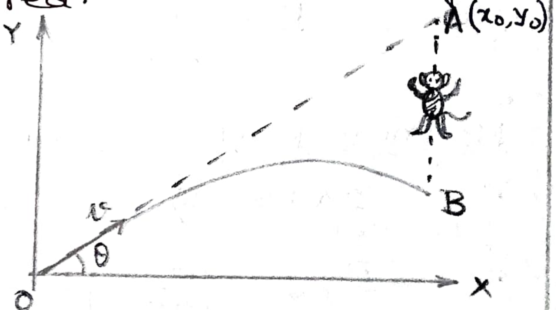
$$= \frac{u^2}{g\cos^2\theta}\left[-\sin(2\alpha' - \theta) - \sin\theta\right]$$

$$\therefore R = \frac{u^2}{g\cos^2\theta}\left[\sin(2\alpha' - \theta) + \sin\theta\right]$$

For maximum range $2\alpha' - \theta = \pi/2$ or $\alpha' = \dfrac{\pi}{4} + \dfrac{\theta}{2}$

and $R_{max} = \dfrac{u^2}{g\cos^2\theta}(1 + \sin\theta) = \dfrac{u^2}{g(1 - \sin\theta)}$.

- Show that if a monkey sitting on a tree starts falling freely at the instant a gun aimed at it is fired, the monkey is bound to hit by the bullet so fired.



In absence of gravity ($g = 0$)

$$\left.\begin{aligned} x_0 &= (v\cos\theta)t \\ y_0 &= (v\sin\theta)t \end{aligned}\right\} \quad \text{------ (i)}$$

$\therefore$ Co-ordinate of the monkey at $t = 0$ (x_0, y_0).

In presence of gravity, at any instance t, the co-ordinate of the bullet

$$\left.\begin{aligned} x &= (v\cos\theta)t = x_0 \\ y &= (v_0\sin\theta)t - \tfrac{1}{2}gt^2 = y_0 - \tfrac{1}{2}gt^2 \end{aligned}\right\} \quad \text{------ (ii)}$$

At any instance the co-ordinate of freely falling monkey

$x_M = x_0 = x$, $y_m = y_0 - \frac{1}{2}gt^2 = y$

$\therefore$ At any instance t the co-ordinate of the bullet and the monkey is same, thus the bullet hits the monkey.

MOTION IN TWO DIMENSION
(PROJECTILE MOTION)

WORKSHEET (DOSE-I)

1. The velocity of a projectile when it is at its heighest point is $\sqrt{2/5}$ of its velocity when it is at half of its gratest height. What is the angle of projection?

2. If the time of flight of a bullet over the horizonal range R is T, find the angle of projection. Ans: $\theta = \tan^{-1}\left(\frac{T^2 g}{2R}\right)$

3. A car is moving horizontally with velocity v. A shell is fired with velocity u inclined at an angle θ with the horizontal. Find the horizontal range of the shell relative to the ground. Ans: $\frac{2u\sin\theta\,(v+u\cos\theta)}{g}$

4. A stone is thrown from the ground towards a wall 6 m high at a distance 4 m such that it just clears the top of the wall at its heighest point. Find the speed and angle of projection of the stone. Ans: 11.43 m/s, 71.6°

5. A ball is projected from a point on the floor with a speed of 15 m/s at an angle of 60° with the horizontal. Will it hit a vertical wall 5 m away from the point of projection and perpendicular to the plane of projection without hitting the floor? Will the answer differ if the wall is 22 m away?

6. A boy standing on a long railroad car throws a ball straight upwards. The car is moving on the horizontal road with an acceleration of $1\ m/s^2$ and the projection velocity in the vertical direction is 9.8 m/s. How far behind the boy will the ball fall on the car? Ans: 2 m.

7. A particle is projected at an angle θ from the horizontal with kinetic energy, K. Show that the kinetic energy of the particle at the maximum height is $K\cos^2\theta$.

8. The range of the particle when launched at an angle of 15° with the horizontal is 1.5 km. What is the range of the projectile when launched at an angle of 45° to the horizontal. Ans: 3 km

9. A football is kicked with a velocity 20m/s at a projection angle of 45°. A receiver on the goal line 25 m away in the direction of the kick runs at the same instance to meet the ball. What must be his speed if he is to catch the ball before it hits the ground? Ans: 5.5 m/s

10. A projectile can have the same range R for two angles of projection. If t_1 and t_2 be the time of flight in the two cases, then prove that $t_1 t_2 = \frac{2R}{g}$.

Multiple choice Questions

1. A projectile is fired at an angle of 30° to the horizontal such that the vertical component of its initial velocity is $80\,m/s$. Its time of flight is T. Its velocity at $t = T/4$ has a magnitude of nearly

(a) 200 m/s (b) 300 m/s (c) 100 m/s (d) None of these

2. Two particles A and B are projected simultaneously from a fixed point of the ground. Particle A is projected on a smooth horizontal surface with speed V, while particle B is projected in air with speed $\frac{2V}{\sqrt{3}}$. If particle B hits the particle A, the angle of projection of B with the vartical is.

(a) 30° (b) 60° (c) 45° (d) Both (a) and (b)

3. Two second after projection, a projectile is travelling in a direction inclined at 30° to the horizontal. After 1 more second it is travelling horizontally. Then ($g = 10\,m/s^2$)

(a) the velocity of projection is $20\sqrt{3}$ m/s
(b) the angle of projection is 30° with horizontal
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong.

4. A projectile is launched with a speed of 10 m/s at an angle 60° with the horizontal from a inclined plane of inclination 30°. The range R is (Take $g = 10\,ms^{-2}$)

(a) 4.9 m (b) 13.3 m (c) 9.1 m (d) 12.6 m

5. A ball is thrown from the ground to clear a wall 3 m high at a distance of 6 m and falls 18 m away from the wall, the angle of projection of ball is

(a) $\tan^{-1}(3/2)$ (b) $\tan^{-1}(2/3)$ (c) $\tan^{-1}(1/2)$ (d) $\tan^{-1}(3/4)$

6. A particle is projected from horizontal making an angle of 53° with initial velocity 100 m/s. The time taken by the particle to make angle 45° from horizontal is

(a) 14 s (b) 2.0 s (c) 12 s (d) 4 s.

7. A projectile is thrown with some initial velocity at an angle α to the horizontal. Its velocity when it is at the highest point is $\sqrt{2/5}$ times the velocity when it is at height half of the maximum height. Find the angle of projection α with the horizontal

(a) 30° (b) 45° (c) 60° (d) 37°

8. A projectile is thrown with velocity u making an angle θ with vertical. If just crosses the top of two poles each of height h after 1 s and 3 s respectively. The maximum hight of the projectile is

(a) 9.8 m (b) 19.6 m (c) 39.2 m (d) 4.9 m.

9. A body is projected with a speed u m/s at angle β with the horizontal. The kinetic energy at the heighest point is 3/4 th of the initial kinetic energy. The value of β as

(a) 30° (b) 45° (c) 60° (d) 120°