WORK, Energy, Power

Work Done by a Force :

Let a constant force $\vec{F}$ be applied on the body such that it makes an angle θ with the horizontal and body is displaced through a distance S. Since the body is being displaced in the direction of $F\cos\theta$, there fore workdone by the force in displacing the body through a distance s is given by

$$W = (F\cos\theta)\, s = FS\cos\theta.$$

$$\therefore \quad W = \vec{F}\cdot\vec{S} = FS\cos\theta$$

= Magnitude of the force × Component of the displacement along the direction of force.

Nature of workdone

- Positive work $(0^\circ \leq \theta < 90^\circ)$

  If force and displacement is along the same direction. It signifies that the external force favors the motion of the body.

  $$W = FS\cos 0^\circ = FS.$$

- Negative work $(90^\circ < \theta \leq 180^\circ)$

  If the force is opposite to displacement. It signifies that external force opposes the motion of the body.

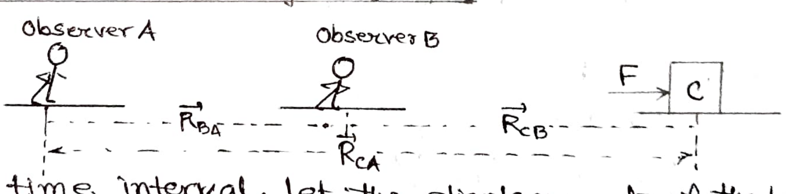
  $$W = FS\cos 180^\circ = -FS$$

- Zero work

  If the force is perpendicular to the displacement $(\vec{F} \perp \vec{S})$

  $$W = FS\cos 90^\circ = 0$$

- Work depends on the frame of reference

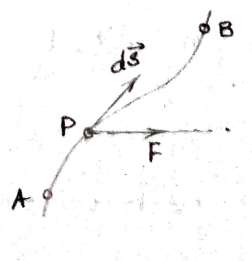


During certain time interval, let the displacement of the block relative to the observer A and B be $\Delta\vec{r}_{CA}$ and $\Delta\vec{r}_{CB}$, respectively. Then the corresponding work done can be given as

$$W_1 = \vec{F}\cdot\Delta\vec{r}_{CA} \quad \text{and} \quad W_2 = \vec{F}\cdot\Delta\vec{r}_{CB}$$

The above expression tells us that same force (F) may perform different works (W_1 and W_2) relative to different observers (reference frames) A and B. Hence the work done by a force depends on the reference frames.

- Workdone by a variable force



$$dW = \vec{F}\cdot d\vec{s}$$

∴ total work done in going from A to B is

$$W_{AB} = \int_A^B \vec{F}\cdot d\vec{s} = \int_A^B F\,ds\cos\theta$$

In terms of rectangular Components

$$d\vec{s} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\therefore \quad W = \int \vec{F} \cdot d\vec{s}$$

$$W_{AB} = \int (F_x \hat{i} + F_y \hat{j} + F_z \hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_{x_A}^{x_B} F_x\,dx + \int_{y_A}^{y_B} F_y\,dy + \int_{z_A}^{z_B} F_z\,dz.$$

Dimension:

$$[W] = [MLT^{-2}] \times [L] = [ML^2T^{-2}]$$

Units (Absolute):

joule: One joule of work is said to be done when a force of one newton displaces a body through a distance of one meter in its own direction.

$$1\,J = 1\,N \times 1\,m = 1\,Nm.$$

erg: One erg of work is said to be done if a force of One dyne displaces a body through a distance of one centemeter in its own direction.

$$1\,erg = 1\,dyne \times 1\,cm$$

Relation between joule and erg

$$1\,joule = 1\,newton \times 1\,meter.$$

$$= 10^5\,dyn \times 10^2\,cm = 10^7\,dyne\,cm$$

$$1\,joule = 10^7\,erg.$$

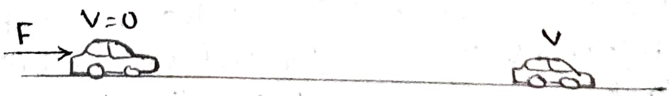
Energy: Energy of a body is defined as its capacity or ability to do work.

Mechanical Energy:

- Kinetic Energy
- Potential Energy.

Kinetic Energy: The energy possessed by a body by virtue of its motion is called its Kinetic energy.

$$dW = \vec{F} \cdot d\vec{s} = F\,ds\cos 0^\circ$$



$\therefore$ Total workdone

$$W = \int dW = \int F\,ds = \int m \cdot \frac{dv}{dt} \cdot ds \qquad \left[\because F = m \cdot a = m\frac{dv}{dt} = m\frac{dv}{ds} \cdot \frac{ds}{dt}\right]$$

$$W = \int m \frac{dv}{ds} \cdot \frac{ds}{dt}\,ds = \int_0^v m\,v\,dv$$

$$W = m \int_0^v v\,dv = m\left[\frac{v^2}{2}\right]_0^v = \frac{1}{2}mv^2$$

This workdone appears as Kinetic energy $K = \frac{1}{2}mv^2$

Work Energy Theorem

Suppose a variable force $\vec{F}$ acts on a body of mass m and produces displacement $d\vec{s}$ in its own direction ($\theta = 0^\circ$)

$\therefore$ $$dw = \vec{F}\cdot d\vec{s} = F\,ds\cos 0^\circ = F\,ds$$

According to Newton's second law

$$F = ma = m\frac{dv}{dt}$$

$$dw = m\frac{dv}{dt}\cdot ds = mv\,dv$$

$\therefore$ $$W = \int dw = \int_u^v mv\,dv = m\int_u^v v\,dv$$

$$= m\left[\frac{v^2}{2}\right]_u^v = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

$$W = K_f - K_i$$

$\therefore$ Workdone = change in Kinetic energy.

Potential Energy.

Energy stored in a body or a system by virtue of its position in a field of force or by its configuration.

Suppose F is the force needed to lift the body up with zero acceleration.

$\therefore$ F = weight of the body = mg

$\therefore$ Workdone on the body in raising it through ds height

$$dw = \vec{F}\cdot d\vec{s} = F\,ds\cos 180^\circ \quad [\text{As F is opposite to ds}]$$

$\therefore$ Total workdone to lift it from ground to h height

$$W = \int_0^h -F\,ds = -F\int_0^h ds = -mgh$$

This workdone against the gravity is stored as potential energy (U) of the body.

$$U = mgh.$$

Conservative and Non-Conservative Forces

Conservative Force :

- The workdone by a conservative force on a particle moving between any two points is independent of the path taken by the particle.
- The workdone by a conservative force on a particle moving through any closed path is zero.

Non-Conservative Force :

- The workdone by a non-conservative force not depends only on the initial and final positions but also on the path followed.
- The workdone by a non-conservative force along a closed path is not zero.

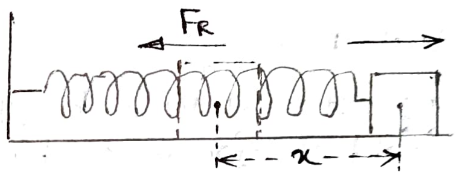
Properties of Conservative forces:

- A force F is conservative, if it can be defined from the scalar potential energy function $U(x)$ by the relation
$$F(x) = -\frac{dU(x)}{dx}$$
- The workdone is path independent
$$W = \int_{x_i}^{x_f} F(x)\,dx = K_f - K_i = U_i - U_f.$$
- The workdone along a closed path
$$W_{\text{closed path}} = \oint F(x)\,dx = 0$$
- If only the conservative forces are acting on body, then its total mechanical energy is conserved.

Potential Energy of a Spring

According to Hooke's Law

$$F = -Kx \quad \text{---(i)}$$



$\therefore$ Workdone for dx extension

$$dW = \vec{F}\cdot d\vec{x} = Kx\,dx$$

$\therefore$ Total workdone for the displacement $x=0$ to $x=x$.

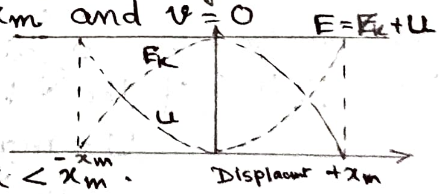
$$W = \int dW = \int_0^x F\cdot dx = K\int_0^x x\,dx = K\frac{x^2}{2} = \frac{1}{2}Kx^2$$

The workdone by the external force F will be stored as elastic potential energy of the spring.

Conservation of energy in an elastic spring

At the extreme position, $x = \pm x_m$ and $v = 0$

$$K = \frac{1}{2}mv^2 = 0 \quad \text{(Minimum)},$$
$$U = \frac{1}{2}Kx_m^2 \quad \text{(Maximum)}$$



At any intermediate position $-x_m < x < x_m$.

Total energy = Kinetic energy + Potential energy

$$\frac{1}{2}\cdot x_m^2 = \frac{1}{2}mv^2 + \frac{1}{2}\ x^2$$

$$\therefore E_k = \frac{1}{2}mv^2 = \frac{1}{2}Kx_m^2 - \frac{1}{2}Kx^2 = \frac{1}{2}K(x_m^2 - x^2)$$

$$\therefore \text{velocity } (v) = \sqrt{\frac{K}{m}(x_m^2 - x^2)}.$$

At equilibrium $x = 0$

$$\therefore v = \sqrt{\frac{K}{m}}\,x_m. \quad \text{Maximum velocity.}$$

POWER: Power is defined as the rate of workdone.

$$P = \frac{W}{t} = \frac{\text{workdone}}{\text{time}}$$

It is a scalar quantity.

Dimension: $[P] = \frac{[W]}{[t]} = \frac{[ML^2T^{-2}]}{[T]} = [ML^2T^{-3}]$

Units: The S.I unit is watt (W)

The power of an agent is one watt if it does work at the rate of 1 joule per second.

$$1 \text{ watt} = \frac{1 \text{ joule}}{1 \text{ second}} \qquad 1W = 1 J s^{-1}$$

Instantaneous power:

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}.$$

As $dW = \vec{F} \cdot d\vec{r}$

$$\therefore P = \frac{dW}{dt} = F \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v} \quad \left[\because \vec{v} = \frac{d\vec{r}}{dt}\right]$$

Relation between kWh and joule:

$$1 kWh = 1 kW \times 1 h = 1000 W \times 3600 s = 3.6 \times 10^6 Ws$$
$$= 3.6 \times 10^6 J.$$

COLLISION

A collision means any short interaction of particles or bodies either through direct contact or otherwise.

Types of Collision

(a) Perfectly elastic collision: A collision in which the kinetic energy and momentum are both conserved is said to be perfectly elastic.

Ex: Collision between two billiard balls (approximately)

(b) Inelastic collision: A collision in which the kinetic energy is not conserved only the momentum is conserved is said to be inelastic collision.

(c) Perfectly inelastic collision: A perfectly inelastic collision is one in which the two colliding bodies stick together after collision. In such collision the loss of kinetic energy is maximum.

Ex: collision between two clay spheres or collision electron with a proton (electron capture).

Coefficient of Restitution (e) :

It measures the degree of elasticity of a collision.

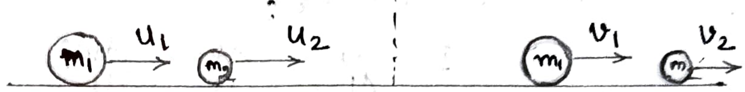
$$e = \frac{\text{relative velocity of separation after collision}}{\text{relative velocity of approach before collision}}$$

- If $e = 1$, the collision is perfectly elastic
- If $e < 1$, the collision is inelastic
- If $e = 0$, the collision is perfectly inelastic
- If $e > 1$, the collision is superelastic.

Perfectly Elastic Collision In One Dimension

Let m_1, m_2 = Masses of the particle A and B

u_1, u_2 = Initial velocity of A and B.



Before Collision

After Collision.

By the Conservation of momentum

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

or $$m_1(u_1 - v_1) = m_2(v_2 - u_2) \quad \text{-----(1)}$$

Since the collision is perfectly elastic so, the kinetic energy is also conserved. Thus

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

or $$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \quad \text{--(2)}$$

Dividing eqn (2) by (1), we get

$$\frac{m_1(u_1^2 - v_1^2)}{m_1(u_1 - v_1)} = \frac{m_2(v_2^2 - u_2^2)}{m_2(v_2 - u_2)}$$

or $$(u_1 + v_1) = (v_2 + u_2)$$

$$\therefore (u_1 - u_2) = (v_2 - v_1) \quad \text{-----(3)}$$

$$\therefore u_{12} = v_{21}$$

$\therefore$ Relative velocity of body A w.r.t B = Relative velocity of body B w.r.t A.

or Relative velocity of approach = Relative velocity of separation.

$$\therefore \quad e = \frac{v_2 - v_1}{u_1 - u_2} = 1$$

$\therefore$ It is a perfetly elastic collision.

From eqn (3) $v_2 = u_1 - u_2 + v_1$

substituting the value of v_2 in eqn (1), we get

$$m_1(u_1 - v_1) = m_2(u_1 - u_2 + v_1 - u_2)$$

or $m_1 u_1 - m_1 v_1 = m_2 u_1 - 2 m_2 u_2 + m_2 v_1$

or $(m_1 + m_2) v_1 = (m_1 - m_2) u_1 + 2 m_2 u_2$

or $$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 + \left(\frac{2m_2}{m_1 + m_2}\right) u_2 \quad \text{----(5)}$$

Similarly $$v_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) u_2 + \left(\frac{2m_1}{m_1 + m_2}\right) u_1 \quad \text{----(6)}$$

Special Cases :

(a) When colliding particles are of equal masses i.e $m_1 = m_2$

putting m_1 for m_2 in eqn (5), we get

$$v_1 = \left(\frac{m_1 - m_1}{m_1 + m_1}\right) u_1 + \left(\frac{2m_1}{m_1 + m_2}\right) u_2 = u_2$$

Similarly, putting m_1 for m_2 in eqn (6), we get

$$v_2 = \left(\frac{m_1 - m_1}{m_1 + m_1}\right) u_1 + \left(\frac{2m_1}{m_1 + m_2}\right) u_1 = u_1$$

∴ The particles exchange their velocities during collision.

(b) When one of the colliding particles (i.e target) is at rest.

Let the target particle B be a rest so that $u_2 = 0$

putting $u_2 = 0$ in eqn (5) and (6), we obtain

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 \quad \text{-------(7)}$$

and $$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right) u_1 \quad \text{-------(8)}$$

- When both particles are of the same mass i.e $m_1 = m_2$

$$v_1 = \left(\frac{m_1 - m_1}{m_1 + m_1}\right) u_1 = 0$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_1}\right) u_1 = u_1$$

Therefore, after impact the incident particle A stops and the target particle B moves with the initial velocity A i.e the velocities of A and B are interchanged on collision.

- When $m_1 \ll m_2$ mass of the incident particle is negligible as compared to m_2. Therefrom eqn (7) and (8)

$$v_1 \approx -u_1 \quad \text{and} \quad v_2 \simeq 0$$

Velocity of the incident particle A after impact is equal and opposite to its velocity before impact. The velocity of target particle B after impact is approximately zero.

- When $m_2 << m_1$; Mass of the target particle is negligible as compared to m_1 (mass of incident particle) From eqn (7) and (8)

$$v_1 \approx u_1 \quad \text{and} \quad v_2 \simeq 2u_1.$$

It means that the velocity of the incident particle A, after impact, is nearly equal to its velocity before impact and velocity of the particle B, after impact, is nearly double the velocity of the incident particle A before impact.

Show that the fractional loss in kinetic energy of an incident neutron in a head-on collision with an atomic nucleus, initially at rest, is $4A/(A+1)^2$ where A is the mass number of the nucleus.

Let m_1 = Mass of incident neutron
u_1 = initial velocity of incident neutron.
v_1, v_2 = velocities of the neutron and the nucleus after collision.
K_i, K_f = initial and final kinetic energy of neutron.

clearly,
$$\frac{K_f}{K_i} = \frac{\frac{1}{2} m_1 v_1^2}{\frac{1}{2} m_1 u_1^2} = \left(\frac{v_1}{u_1}\right)^2$$

from eqn (7) and (8)

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1, \quad \frac{v_1}{u_1} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right).$$

Thus,
$$\frac{K_f}{K_i} = \frac{(m_1 - m_2)^2}{(m_1 + m_2)^2}$$

Fractional loss in kinetic energy of neutron.

$$f = \frac{K_i - K_f}{K_i} = 1 - \frac{K_f}{K_i} = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 = \frac{4 m_1 m_2}{(m_1 + m_2)^2}$$

As $m_1 = 1$ and $m_2 = A$.

$$f = \frac{4A}{(A+1)^2}$$

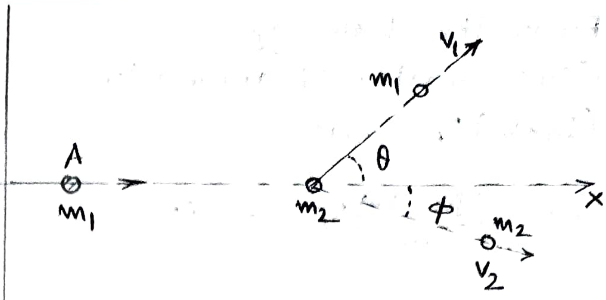
Note:
$$\frac{K_f}{K_i} = \left(\frac{1-A}{1+A}\right)^2 = \left(\frac{A-1}{A+1}\right)^2$$

Elastic Collision in Two Dimension :

Let m_1, m_2 = Masses of the particles A and B

u_1 = Velocity of the particle A before Collision

v_1, v_2 = Velocities of the particles A and B after collision.



Applying the principle of Conservation of linear momentum along x-axis

$$m_1 u_1 = m_1 v_1 \cos\theta + m_2 v_2 \cos\phi$$

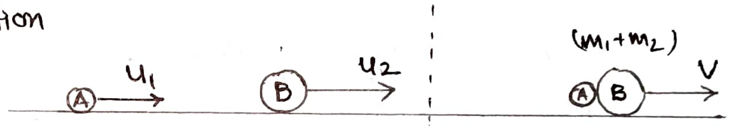
Along Y-axis $\quad m_1 v_1 \sin\theta - m_2 v_2 \sin\phi = 0$

Again by conservation of K.E

$$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Perfectly Inelastic collisions :

From Law of Conservation of energy



$$m_1 u_1 + m_2 u_2 = (m_1+m_2) v$$

$$v = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} \quad \text{---} (1)$$

Initial Kinetic energy $\quad E_{ki} = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$

Final Kinetic energy $\quad E_{kf} = \frac{1}{2}(m_1+m_2) v^2$

$\therefore$ change in kinetic energy $\Delta E_k = (E_{ki} - E_{kf})$

$$\therefore \Delta E_k = \frac{1}{2}\left[m_1 u_1^2 + m_2 u_2^2 - (m_1+m_2) v^2\right] \quad \text{---} (ii)$$

Now Substitute $V = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$ in eqn (ii)

$$\Delta E_k = \frac{1}{2}\left[m_1 u_1^2 + m_2 u_2^2 - (m_1+m_2)\left(\frac{m_1 u_1 + m_2 u_2}{m_1+m_2}\right)^2\right]$$

$$= \frac{1}{2}\left[m_1 u_1^2 + m_2 u_2^2 - \frac{(m_1 u_1 + m_2 u_2)^2}{m_1+m_2}\right]$$

$$= \frac{1}{2} \frac{m_1^2 u_1^2 + m_2^2 u_2^2 + m_1 m_2 u_1^2 + m_1 m_2 u_2^2 - m_1^2 u_1^2 - m_2^2 u^2 + 2 m_1 m_2 u_1 u_2}{(m_1+m_2)}$$

$$= \frac{m_1 m_2}{2(m_1+m_2)}\left[u_1^2 + u_2^2 - 2 u_1 u_2\right]$$

$$\therefore \Delta E_k = \frac{m_1 m_2}{2(m_1+m_2)} (u_1 - u_2)^2$$

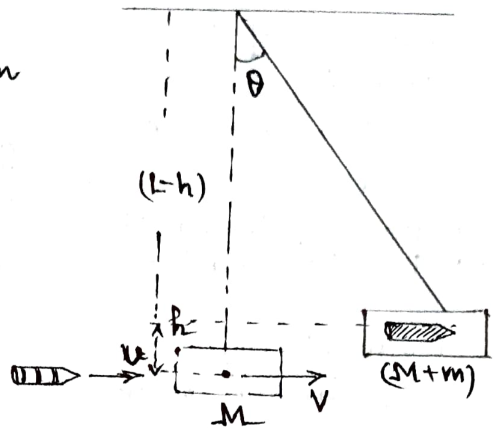
As m_1 & m_2 can not be negative and $(u_1 - u_2)^2$ is also positive therefor ΔE_k is also positive.

Balastic Pendulum

From the Law of Conservation of momentum

$(M+m)V = mv$

$$v = \left(\frac{M+m}{m}\right)V \quad \text{- - - (1)}$$



After the collision, the total Kinetic energy of the system is $\frac{1}{2}(M+m)V^2$. All this Kinetic energy is converted into potential energy $(M+m)gh$. Thus

$$\frac{1}{2}(M+m)V^2 = (M+m)gh$$

$$V = \sqrt{2gh} \quad \text{- - - - - (2)}$$

$\therefore$ From eqn (1) and (2)

$$v = \frac{M+m}{m}\sqrt{2gh} \quad \text{- - - - - - (3)}$$

Again from figure $\cos\theta = \frac{L-h}{L} = 1 - \frac{h}{L}$

$\therefore$ From eqn (3)

$$v^2 = \left(\frac{M+m}{m}\right)^2 2gh$$

$$\therefore \quad h = \frac{1}{2g}\left(\frac{mv}{M+m}\right)^2$$

$$\therefore \quad \cos\theta = 1 - \frac{h}{L} = 1 - \frac{1}{2gL}\left(\frac{mv}{M+m}\right)^2$$

$$\therefore \quad \theta = \cos^{-1}\left[1 - \frac{1}{2gL}\left(\frac{mv}{M+m}\right)^2\right]$$

WORK, Energy, POWER

WORKSHEET (DOSE-I)

1. A Constant force $\vec{F} = (3\hat{i} + 2\hat{j} + 2\hat{k})$ N acts on a particle displacing it from a position $\vec{r_1} = (-\hat{i} + \hat{j} - 2\hat{k})$ m to a new position $\vec{r_2} = (\hat{i} - \hat{j} + 3\hat{k})$ m. Find the workdone by the force.

   Ans: 12J

2. A block of mass 10 kg is slowly slid up on a smooth inclined plane of inclination 37° by a person. Calculate the work done by the person in moving the block through a distance of 2.0 m, if the driving force is applied
   a. parallel to the inclined plane
   b. in the horizontal direction

   Ans: 120J, 120J

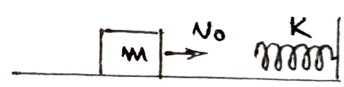
3. The displacement of a particle of mass 1kg on a horizontal smooth surface is a function of time given by $x = \frac{1}{3}t^3$. Find out the workdone by the external agent for the first one second.

   Ans: 0.5J

4. A block of mass m is moving with an initial velocity V_0 towards a stationary spring of stiffness K attached to the wall as shown in fig.

   Ans: $x_0 = V_0\sqrt{m/k}$.

   a. Find the maximum compression in the spring



   b. Is the work done by the spring negative or positive?