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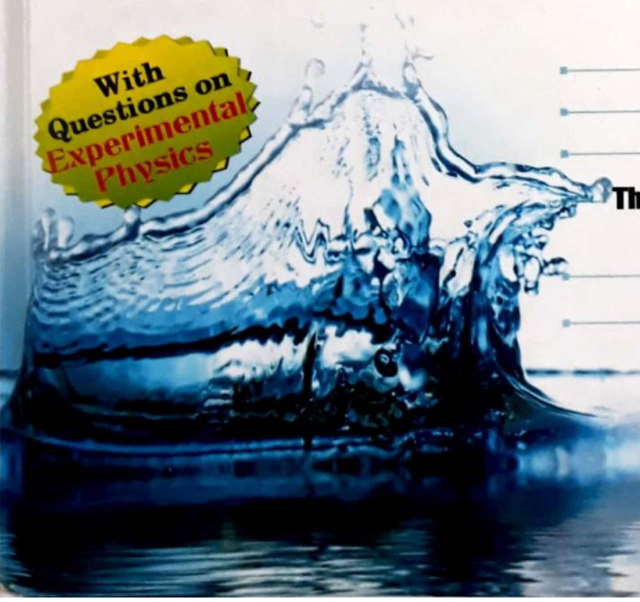

- **Wave Motion**
- **Superposition of Waves**
- **Sound Waves**
- **Thermometry, Thermal Expansion & Kinetic Theory of Gases**
- **Law of Thermodynamics**
- **Calorimetry & Heat Transfer**

DC Pandey

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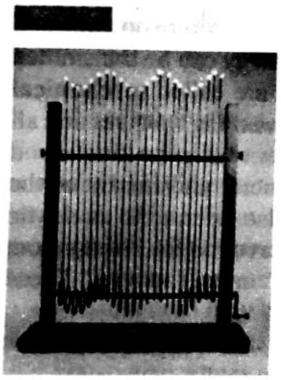


14

WAVE MOTION

Chapter Contents

14.1 Introduction

"A wave is any disturbance from a normal or equilibrium condition that propagates without the transport of matter. In general, a wave transports both energy and momentum."

The disturbance created by a wave is represented by a wave function $y(x, t)$. For a string the wave function is a (vector) displacement; whereas for sound waves it is a (scalar) pressure or density fluctuation. In the case of light or radio waves, the wave function is either an electric or a magnetic field vector.

Wave motion appears in almost every branch of physics. We are all familiar with water waves, sound waves and light waves. Waves occur when a system is disturbed from its equilibrium position and this disturbance travels or propagates from one region of the system to other. Energy can be transmitted over considerable distances by wave motion. The waves requiring a medium are called **mechanical waves** and those which do not require a medium are called **non-mechanical waves**. Light waves and all other electromagnetic waves are nonmechanical. The energy in the mechanical waves is the kinetic and potential energy of the matter. In the propagation of mechanical waves elasticity and inertia of the medium play an important role. This is why mechanical waves sometimes are also referred to as **elastic waves.** Note that the medium itself does not move as a whole along with the wave motion. Apart from mechanical and nonmechanical waves there is also another kind of waves called **"matter waves"**. These represent wave like properties of particles.

14.2 Transverse and Longitudinal Waves

There are two distinct classes of wave motion :

(i) transverse and
(ii) longitudinal.

In a **transverse wave motion** the particles of the medium oscillate about their mean or equilibrium position at right angles to the direction of propagation of wave motion itself.

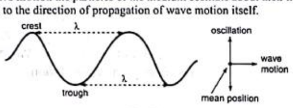


Fig. 14.1

This form of wave motion travels in the form of **crests and troughs**, as for example, waves travelling along a stretched string. This type of waves are possible in media which possess elasticity of shape or rigidity, *i.e.*, in solids. These are also possible on the surface of liquids also, even though they do not possess the property of rigidity. This is because they possess another equally effective property (surface tension) of resisting any vertical displacement of their particles (or keeping their level). Gases, however, possess neither rigidity nor do they resist any vertical displacement of particles (or keep their level). A transverse wave motion is therefore not possible in a gaseous medium. An electromagnetic wave is necessarily a transverse wave because of the electric and magnetic fields being perpendicular to its direction of propagation. The distance between two successive crests or troughs is known as the **wavelength** (λ) of the wave.

In a **longitudinal wave motion** the particles of the medium oscillate about their mean or equilibrium position along the direction of propagation of the wave motion itself. This type of wave motion travels in the form of **compressions** and **rarefactions** and is possible in media possessing elasticity of volume, *i.e.*, in solids, liquids and gases.

Fig. 14.2

The distance between two successive compressions or rarefactions constitute one wavelength. Sound waves in a gas are longitudinal in nature. In some cases, the waves are neither purely transverse nor purely longitudinal as, for example **ripples** on water surface (produced by dropping a stone in water), in which the particles of the medium (here water) oscillate across as well as along the direction of propagation of the wave motion describing elliptical paths. We need not however, bother ourselves with any such type of waves here.

Again waves may be **one dimensional, two dimensional or three dimensional** according as they propagate energy in just one, two or three dimensions.

Transverse waves along a string are one dimensional, ripples on water surface are two dimensional and sound waves proceeding radially from a point source are three dimensional.

14.3 The General Equation of Wave Motion

As we have already read, in a wave motion, some physical quantity (say y) is made to oscillate at one place and these oscillations of y propagate to other places. The y may be,

(i) displacement of particles from their mean position in case of transverse wave in a rope or longitudinal sound wave in a gas.
(ii) pressure difference (dP) or density difference ($d\rho$) in case of sound wave or
(iii) electric and magnetic fields in case of electromagnetic waves.

The oscillations of y may or may not be simple harmonic in nature. Now let us consider a one dimensional wave travelling along x-axis. In this case y is a function of position (x) and time (t). The reason is that one may be interested in knowing the value of y at a general point x at any time t. Thus, we can say that,

$$y = y(x, t)$$

But only those functions of x and t, represent a wave motion which satisfy the differential equation,

$$\frac{\partial^2 y}{\partial t^2} = k\frac{\partial^2 y}{\partial x^2}$$

Here k is a constant, which is equal to square of the wave speed, or

$$k = v^2$$

Thus, the above equation can be written as,

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2} \qquad \text{...(i)}$$

The general solution of this equation is of the form

$$y(x, t) = f(ax \pm bt) \qquad \text{...(ii)}$$

Thus, any function of x and t which satisfies Eq. (i) or which can be written as Eq. (ii) represents a wave. The only condition is that it should be finite everywhere and at all times. Further, if these conditions are satisfied, then speed of wave (v) is given by,

$$v = \frac{\text{coefficient of } t}{\text{coefficient of } x} = \frac{b}{a}$$

The plus (+) sign between ax and bt implies that the wave is travelling along negative x-direction and minus (–) sign shows that it is travelling along positive x-direction.

Sample Example 14.1 *Show that the equation,* $y = a\sin(\omega t - kx)$ *satisfies the wave equation* $\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$. *Find speed of wave and the direction in which it is travelling.*

Solution
$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 a\sin(\omega t - kx)$$

and
$$\frac{\partial^2 y}{\partial x^2} = -k^2 a\sin(\omega t - kx)$$

We can write these two equations as,

$$\frac{\partial^2 y}{\partial t^2} = \frac{\omega^2}{k^2}\cdot\frac{\partial^2 y}{\partial x^2}$$

Comparing this with,
$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$$

We get,
$$\text{wave speed } v = \frac{\omega}{k} \qquad \textbf{Ans.}$$

The negative sign between ωt and kx implies that wave is travelling along positive x-direction.

Sample Example 14.2 *Which of the following functions represent a wave*

(a) $(x - vt)^2$ *(b)* $\ln(x + vt)$ *(c)* $e^{-(x-vt)^2}$ *(d)* $\frac{1}{x+vt}$

Solution Although all the four functions are written in the form $f(ax \pm bt)$, only (c) among the four functions is finite everywhere at all times. Hence only (c) represents a wave.

Sample Example 14.3 $y(x, t) = \frac{0.8}{[(4x+5t)^2+5]}$ *represents a moving pulse where x and y are in metre and t in second. Then choose the correct alternative(s):* **(JEE 1999)**

(a) pulse is moving in positive x-direction

(b) in 2 s it will travel a distance of 2.5 m
(c) its maximum displacement is 0.16 m
(d) it is a symmetric pulse

Solution (b), (c) and (d) are correct options.

The shape of pulse at $x = 0$ and $t = 0$ would be as shown in Fig. 14.3.

$$y(0, 0) = \frac{0.8}{5} = 0.16\text{ m}$$

From the figure it is clear that $y_{max} = 0.16$ m

Pulse will be symmetric (symmetry is checked about y_{max}) if

At $t = 0$; $\quad y(x) = y(-x)$

From the given equation

$$\left.\begin{aligned} y(x) &= \frac{0.8}{16x^2+5} \\ \text{and} \quad y(-x) &= \frac{0.8}{16x^2+5} \end{aligned}\right\} \text{at } t = 0$$

or
$$y(x) = y(-x)$$

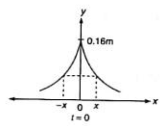


Fig. 14.3

Therefore, pulse is symmetric.

Speed of pulse : At $t = 1$ s and $x = -1.25$ m

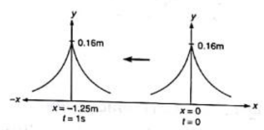


Fig. 14.4

value of y is again 0.16 m, *i.e.*, pulse has travelled a distance of 1.25 m in 1 second in negative x-direction or we can say that the speed of pulse is 1.25 m/s and it is travelling in negative x-direction. Therefore, it will travel a distance of 2.5 m in 2 seconds. The above statement can be better understood from Fig. 14.4.

Alternate method :

If equation of a wave pulse is

$$y = f(ax \pm bt)$$

the speed of wave is $\frac{b}{a}$ in negative x-direction for $y = f(ax + bt)$ and positive x-direction for $y = f(ax - bt)$. Comparing this from given equation we can find that speed of wave is $\frac{5}{4} = 1.25$ m/s and it is travelling in negative x-direction.

Sample Example 14.4 *In a wave motion $y = a\sin(kx - \omega t)$, y can represent:* (JEE 1999)
(a) *electric field* (b) *magnetic field* (c) *displacement* (d) *pressure*

Solution (a, b, c, d)

In case of sound wave, y can represent pressure and displacement, while in case of an electromagnetic wave it represents electric and magnetic fields.

- *In general, y is any general physical quantity which is made to oscillate at one place and these oscillations are propagated to other places.*

Introductory Exercise 14.1

1. Prove that the equation $y = a\sin\omega t$ does not satisfy the wave equation and hence it does not represent a wave.
2. A wave pulse is described by $y(x,t) = ae^{-(bx-ct)^2}$, where a, b and c are positive constants. What is the speed of this wave?
3. The displacement of a wave disturbance propagating in the positive x-direction is given by
$$y = \frac{1}{1+x^2} \text{ at } t = 0 \quad \text{and} \quad y = \frac{1}{1+(x-1)^2} \quad \text{at} \quad t = 2\text{ s}$$
where x and y are in metre. The shape of the wave disturbance does not change during the propagation. What is the velocity of the wave?
4. A travelling wave pulse is given by, $y = \dfrac{10}{5+(x+2t)^2}$
Here x and y are in metre and t in second. In which direction and with what velocity is the pulse propagating? What is the amplitude of pulse?
5. If at $t = 0$, a travelling wave pulse on a string is described by the function,
$$y = \frac{10}{(x^2+2)}$$
Here x and y are in metre and t in second. What will be the wave function representing the pulse at time t, if the pulse is propagating along positive x-axis with speed 2 m/s?

14.4 Plane Progressive Harmonic Wave

Consider a function $y = f(x)$, represented graphically by the solid curve shown in Fig. 14.5.

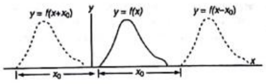


Fig. 14.5

If we replace x by $x - x_0$, we get the function,
$$y = f(x - x_0)$$

It is clear that shape of the curve has not changed, the same value of y occurs for values of x increased by the amount x_0. In other words, assuming that x_0 is positive, we see that the curve has been displaced to

the right an amount x_0 without deformation. Similarly, $y = f(x + x_0)$ corresponds to a displacement of the curve to the left by an amount x_0.

For example if we have two functions:

$y_1 = x^2$ and $y_2 = (x-5)^2$ and $y_1 = 16$ at $x = 4$ then y_2 has the same value, *i.e.*, 16 at $x = 4+5$ or $x = 9$.

Now, if $x_0 = vt$, where t is the time, we get a travelling curve. That is $y = f(x - vt)$ represents a curve moving to the right with a velocity v, called the **wave velocity or phase velocity**.

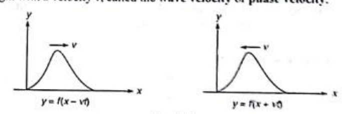


Fig. 14.6

Similarly $y = f(x + vt)$ represents a curve moving to the left with velocity v. Therefore, we conclude that a mathematical expression of the form,
$$y(x,t) = f(x \pm vt)$$
is adequate for describing a physical situation that 'travels' or 'propagates' without deformation along negative or positive x-axis. The quantity $y(x,t)$ may represent the deformation in a solid, the pressure in a gas, an electric or magnetic field, etc.

When $y(x,t)$ is a sine or cosine function such as,
$$y(x,t) = A\sin k(x - vt)$$
or
$$y(x,t) = A\cos k(x - vt)$$
it is called **plane progressive harmonic wave**. In plane progressive harmonic wave oscillations of y are simple harmonic in nature.

The quantity k has a special meaning. Replacing the value of x by $x + \dfrac{2\pi}{k}$, we get the same value of y, *i.e.*,
$$y\left(x + \frac{2\pi}{k}, t\right) = A\sin k\left(x + \frac{2\pi}{k} - vt\right) = A\sin[k(x - vt) + 2\pi]$$
$$= A\sin k(x - vt) = y(x,t)$$

The quantity,
$$\frac{2\pi}{k} = \lambda$$
designated as **wavelength**, is the **space period** of the curve, that is the curve repeats itself every length λ. The quantity $k = \dfrac{2\pi}{\lambda}$ represents the number of wavelengths in the distance 2π and is called the **wave number**.

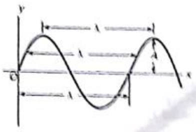


Fig. 14.7

Therefore,
$$y(x,t) = A\sin k(x - vt) = A\sin\frac{2\pi}{\lambda}(x - vt)$$

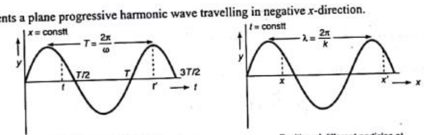
represents a plane progressive harmonic wave of wavelength λ propagating towards the positive x-axis with speed v. The above equation can also be written as,
$$y(x,t) = A\sin(kx - \omega t)$$
where
$$\omega = kv = \frac{2\pi v}{\lambda}$$
gives the **angular f-equency** of the wave. Further, $\omega = 2\pi f$, where f is the **frequency** with which y oscillates at every point x. We have the important relation,
$$v = \lambda f$$

Also if T is the **period** of oscillation then,
$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

We may also write,
$$y = A\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

Thus, the equation of a plane progressive harmonic wave moving along positive x-direction can be written as,
$$y = A\sin k(x - vt) = A\sin(kx - \omega t) = A\sin\frac{2\pi}{\lambda}(x - vt) = A\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

Similarly, the expressions
$$y = A\sin k(x + vt) = A\sin(kx + \omega t) = A\sin 2\pi\left(\frac{x}{\lambda} + \frac{t}{T}\right)$$
represents a plane progressive harmonic wave travelling in negative x-direction.



Displacement of a particle at different instants

Position of different particles at same instant in a string.

Fig. 14.8

Note that as y propagates in the medium (or space), it repeats itself in space after one period, because
$$\lambda = vT$$
which shows that, **the wavelength is the distance advanced by the wave motion in one period.**

Therefore, in plane progressive harmonic wave we have two periodicities, one in time given by the period T, and one in space given by the wavelength λ, with the two related by
$$\lambda = vT$$

• *Important points in* Wave motion Read so far

1. As we have read in Art. 14.3, any function of x and t which satisfies equation number (i) of the same article or which can be written in the form of equation number (ii) represents a wave provided it is finite everywhere at all times. What we have read in Art. 14.4 is about plane progressive harmonic wave. If $f(ax \pm bt)$ is a sine or cosine function, it is called plane progressive harmonic wave. The only special characteristic of this wave is that oscillations of y are simple harmonic in nature.
2. The general expression of a plane progressive harmonic wave is,
$$y = A\sin(kx \pm \omega t \pm \phi)$$
or
$$y = A\cos(kx \pm \omega t \pm \phi)$$
Here ϕ represents the initial phase.
3. I have seen students often confused whether the equation of a plane progressive wave should be,
$$y = A\sin(kx - \omega t)$$
or
$$y = A\sin(\omega t - kx)$$
Because some books write the first while the others write the second. It hardly matters whether you write the first or the second. Both the equations represent a travelling wave travelling in positive x-direction with speed $v = \dfrac{\omega}{k}$. The difference between them is that they are out of phase, *i.e.*, phase difference between them is π. It means, if a particle in position $x = 0$ at time $t = 0$ is in its mean position and moving upwards (represented by first wave) then the same particle will be in its mean position but moving downwards (represented by the second wave). Similarly the waves, $y = A\sin(kx - \omega t)$ and $y = -A\sin(kx - \omega t)$ are also out of phase.
4. **Particle velocity (v_p) and acceleration (a_p) in a sinusoidal wave :** In plane progressive harmonic wave particles of the medium oscillate simple harmonically about their mean position. Therefore, all the formulae what we have read in SHM apply to the particles here also. For example, maximum particle velocity is $\pm A\omega$ at mean position and it is zero at extreme positions etc. Similarly maximum particle acceleration is $\pm\omega^2 A$ at extreme positions and zero at mean position. However the wave velocity is different from the particle velocity. This depends on certain characteristics of the medium. Unlike the particle velocity which oscillates simple harmonically (between $+A\omega$ and $-A\omega$) the wave velocity is constant for given characteristics of the medium.
Suppose the wave function is,
$$y(x,t) = A\sin(kx - \omega t) \qquad \text{...(i)}$$
Let us differentiate this function partially with respect to t and x.
$$\frac{\partial y(x,t)}{\partial t} = -A\omega\cos(kx - \omega t) \qquad \text{...(ii)}$$
$$\frac{\partial y(x,t)}{\partial x} = Ak\cos(kx - \omega t) \qquad \text{...(iii)}$$
Now, these can be written as,
$$\frac{\partial y(x,t)}{\partial t} = -\left(\frac{\omega}{k}\right)\frac{\partial y(x,t)}{\partial x}$$
Here,
$$\frac{\partial y(x,t)}{\partial t} = \text{particle velocity } v_P$$
$$\frac{\omega}{k} = \text{wave velocity } v$$

and $$\frac{\partial y(x,t)}{\partial x} = \text{slope of the wave}$$

Thus, $$v_P = -v\,(\text{slope}) \qquad \text{...(iv)}$$

i.e., particle velocity at a given position and time is equal to negative of the product of wave velocity with slope of the wave at that point at that instant.

The acceleration of the particle is the second partial derivative of $y(x,t)$ with respect to t.

$\therefore$ $$a_P = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A \sin(kx-\omega t) = -\omega^2 y(x,t)$$

i.e., the acceleration of the particle equals $-\omega^2$ times its displacement, which is the result we obtained for SHM. Thus,

$$a_P = -\omega^2\,(\text{displacement}) \qquad \text{...(v)}$$

We can also show that,

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \left(\frac{\omega^2}{k^2}\right)\frac{\partial^2 y(x,t)}{\partial x^2}$$

or $$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2\frac{\partial^2 y(x,t)}{\partial x^2} \qquad \text{...(vi)}$$

which is also the wave equation.

Figure shows the velocity (v_P) and acceleration (a_P) given by Eqs. (iv) and (v) for two points 1 and 2 on a string as a sinusoidal wave is travelling in it along positive x-direction.

At 1 : Slope of the curve is positive. Hence from Eq. (iv) particle velocity (v_P) is negative or downwards. Similarly displacement of the particle is positive, so from Eq. (v) acceleration will be negative or downwards.

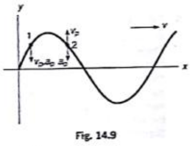


Fig. 14.9

At 2 : Slope is negative while displacement is positive. Hence v_P will be positive (upwards) and a_P is negative (downwards).

Note *The direction of v_P will change if the wave travels along negative x-direction.*

Sample Example 14.5 *The equation of a wave is,*

$$y(x,t) = 0.05 \sin\left[\frac{\pi}{2}(10x - 40t) - \frac{\pi}{4}\right] m$$

Find : (a) the wavelength, the frequency and the wave velocity
(b) the particle velocity and acceleration at $x = 0.5$ m and $t = 0.05$ s

Solution (a) The equation may be rewritten as,

$$y(x,t) = 0.05\sin\left(5\pi x - 20\pi t - \frac{\pi}{4}\right)\text{m}$$

Comparing this with equation of plane progressive harmonic wave,

$$y(x,t) = A\sin(kx - \omega t + \phi) \text{ we have,}$$

$$\text{wave number } k = \frac{2\pi}{\lambda} = 5\pi \text{ rad/m}$$

$\therefore$ $$\lambda = 0.4 \text{ m} \qquad \textbf{Ans.}$$

The angular frequency is,

$$\omega = 2\pi f = 20\pi \text{ rad/s}$$

$\therefore$ $$f = 10 \text{ Hz} \qquad \textbf{Ans.}$$

The wave velocity is,

$$v = f\lambda = \frac{\omega}{k} = 4 \text{ m/s in } +x \text{ direction} \qquad \textbf{Ans.}$$

(b) The particle velocity and acceleration are,

$$\frac{\partial y}{\partial t} = -(20\pi)(0.05)\cos\left(\frac{5\pi}{2} - \pi - \frac{\pi}{4}\right)$$

$$= 2.22 \text{ m/s} \qquad \textbf{Ans.}$$

$$\frac{\partial^2 y}{\partial t^2} = -(20\pi)^2(0.05)\sin\left(\frac{5\pi}{2} - \pi - \frac{\pi}{4}\right)$$

$$= 140 \text{ m/s}^2 \qquad \textbf{Ans.}$$

Sample Example 14.6 *Figure shows a snapshot of a sinusoidal travelling wave taken at $t = 0.3$ s. The wavelength is 7.5 cm and the amplitude is 2 cm. If the crest P was at $x = 0$ at $t = 0$, write the equation of travelling wave.*

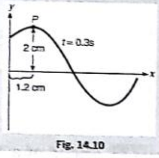


Fig. 14.10

Solution Given, $A = 2$ cm, $\lambda = 7.5$ cm

$\therefore$ $$k = \frac{2\pi}{\lambda} = 0.84 \text{ cm}^{-1}$$

The wave has travelled a distance of 1.2 cm in 0.3 s. Hence, speed of the wave,

$$v = \frac{1.2}{0.3} = 4 \text{ cm/s}$$

$\therefore$ Angular frequency $\omega = (v)(k) = 3.36$ rad/s

Since the wave is travelling along positive x-direction and crest (maximum displacement) is at $x = 0$ at $t = 0$, we can write the wave equation as,

$$y(x,t) = A\cos(kx - \omega t)$$

or $$y(x,t) = A\cos(\omega t - kx) \quad \text{as} \quad \cos(-\theta) = \cos\theta$$

Therefore, the desired equation is,

$$y(x,t) = (2\text{ cm})\cos[(0.84\text{ cm}^{-1})x - (3.36\text{ rad/s})t]\text{cm} \qquad \textbf{Ans.}$$

Introductory Exercise 14.2

1. The equation of a travelling wave is,
$$y(x,t) = 0.02\sin\left(\frac{x}{0.05} + \frac{t}{0.01}\right)\text{m}$$
Find :
(a) The wave velocity and
(b) the particle velocity at $x = 0.2$ m and $t = 0.3$ s.
Given $\cos\theta = -0.85$
where $\theta = 34$ rad
2. Is there any relationship between wave speed and the maximum particle speed for a wave travelling on a string? If so, what is it?
3. Consider a sinusoidal travelling wave shown in figure. The wave velocity is $+40$ cm/s.
Find :
(a) the frequency
(b) the phase difference between points 2.5 cm apart
(c) how long it takes for the phase at a given position to change by 60°
(d) the velocity of a particle at point P at the instant shown.

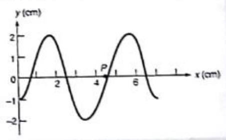


Fig. 14.11

4. Transverse waves on a string have wave speed 12.0 m/s, amplitude 0.05 m and wavelength 0.4 m. The waves travel in the $+x$ direction and at $t = 0$ the $x = 0$ end of the string has zero displacement and is moving upwards.
(a) Write a wave function describing the wave.
(b) Find the transverse displacement of a point at $x = 0.25$ m at time $t = 0.15$ s.
(c) How much time must elapse from the instant in part (b) until the point at $x = 0.25$ m has zero displacement?

14.5 Speed of a Transverse Wave on a String

One of the key properties of any wave is the wave speed. In this section we'll see what determines the speed of propagation of transverse waves on a string. The physical quantities that determine the speed of transverse waves on a string are the **tension** in the string and its **mass per unit length** (also called linear mass density).

We might guess that increasing the tension showed increase the restoring forces that tend to straighten the string when it is disturbed, thus increasing the wave speed. We might also guess increasing the mass should make the motithat on more sluggish and decrease the speed. Both these guesses turn out to be right. We will develop the exact relationship between wave speed, tension and mass per unit length.

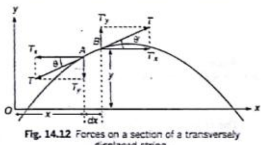


Fig. 14.12 Forces on a section of a transversely displaced string.

Under equilibrium conditions a string subject to a tension T is straight. Suppose that we now displace the string sidewise, or perpendicular to its length, by a small amount as shown in figure. Consider a small section AB of the string of length dx, that has been displaced a distance y from the equilibrium position. On each end a tangential force T is acting. Due to the curvature of the string, the two forces are not directly opposed but make angles θ and θ' with the x-axis. The resultant upward force on the section AB of the string is,

$$F_y = T_y' - T_y \qquad \text{...(i)}$$

Under the acting of this force, the section AB of the string moves up and down.

Rewriting Eq. (i) we have,

$$F_y = T(\sin\theta' - \sin\theta)$$

Since θ and θ' are almost equal, we may write

$$F_y = Td(\sin\theta)$$

If the curvature of the string is not very large, the angles θ and θ' are small, and the sines can be replaced by their tangents. So the upward force is,

$$F_y = Td(\tan\theta) = T.\left\{\frac{d}{dx}(\tan\theta)\right\}.dx$$

But $\tan\theta$ is the slope of the curve adopted by the string, which is equal to $\frac{dy}{dx}$. Hence

$$F_y = T\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\}dx = T\left(\frac{d^2y}{dx^2}\right)dx$$

This force must be equal to the mass of the section AB multiplied by its upward acceleration $\frac{d^2y}{dt^2}$. If μ is the linear density of the string, the mass of the section AB is μdx. We use the relation $F = ma$ and write the equation of motion of this section of the string as,

$$(\mu dx)\frac{d^2y}{dt^2} = T\left(\frac{d^2y}{dx^2}\right)dx$$

or $$\frac{d^2y}{dt^2} = \frac{T}{\mu}\cdot\frac{d^2y}{dx^2}$$

Comparing this with wave equation,

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$$

the wave speed, $$v = \sqrt{\frac{T}{\mu}}$$

Alternate Method

Consider a pulse travelling along a string with a speed v to the right. If the amplitude of the pulse is small compared to the length of the string, the tension T will be approximately constant along the string. In the reference frame moving with speed v to the right, the pulse in stationary and the string moves with a speed v to the left. Figure shows a small segment of the string of length Δl. This segment forms part of a

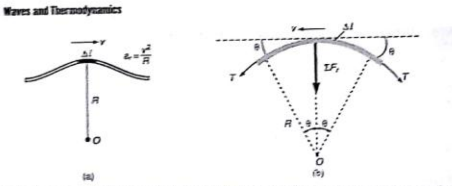


Fig. 14.13 (a) To obtain the speed v of a wave on a stretched string, it is convenient to describe the motion of a small segment of the string in a moving frame of reference.

(b) In the moving frame of reference, the small segment of length Δl moves to the left with speed v. The net force on the segment is in the radial direction because the horizontal components of the tension force cancel.

circular arc of radius R. Instantaneously the segment is moving with speed v in a circular path, so it has a centripetal acceleration $\frac{v^2}{R}$. The forces acting on the segment are the tension T at each end. The horizontal components of these forces are equal and opposite and thus cancel. The vertical components of these forces point radially inward toward the centre of the circular arc. These radial forces provide the centripetal acceleration. Let the angle subtended by the segment at centre be 2θ. The net radial force acting on the segment is

$$\Sigma F_r = 2T \sin\theta = 2T\theta$$

Where we have used the approximation $\sin\theta = \theta$ for small θ.

If μ is the mass per unit length of the string, the mass of the segment of length Δl is

$$m = \mu \Delta l = 2\mu R\theta \qquad (\text{as } \Delta l = 2R\theta)$$

From Newton's second law

$$\Sigma F_r = ma = \frac{mv^2}{R}$$

or

$$2T\theta = (2\mu R\theta)\left(\frac{v^2}{R}\right) \quad \therefore \quad v = \sqrt{\frac{T}{\mu}}$$

• Speed of Wave Motion

1. Speed of transverse wave on a string is given by,

$$v = \sqrt{\frac{T}{\mu}}$$

Here,
μ = mass per unit length of the string

$$= \frac{m}{l}$$

$$= \frac{mA}{lA} \qquad (A = \text{area of cross-section of the string})$$

$$= \left(\frac{m}{V}\right)A \qquad (V = \text{volume of string})$$

$$= \rho A \qquad (\rho = \text{density of string})$$

Hence, the above expression can also be written as,

$$v = \sqrt{\frac{T}{\rho A}}$$

2. Speed of longitudinal wave through a gas (or a liquid) is given by,

$$v = \sqrt{\frac{B}{\rho}}$$

Here, B = Bulk modulus of the gas (or liquid)
and ρ = density of the gas (or liquid)

Now, Newton who first deduced this relation for v assumed that during the passage of a sound wave through a gas (or air), the temperature of the gas remains constant i.e., sound wave travels under isothermal conditions and hence took B to be the isothermal elasticity of the gas and which is equal to its pressure P. So, **Newton's** formula for the velocity of a sound wave (or a longitudinal wave) in a gaseous medium becomes,

$$v = \sqrt{\frac{P}{\rho}}$$

If, however, we calculate the velocity of sound in air at NTP with the help of this formula by substituting:

$$P = 1.01 \times 10^5 \text{ N/m}^2 \quad \text{and} \quad \rho = 1.29 \times 10^{-3} \text{ kg/m}^3$$

then v comes out to be nearly 280 m/s. Actually the velocity of sound in air at NTP as measured by Newton himself, is found to be 332 m/s. Newton could not explain this large discrepancy between his theoretical and experimental results.

La'place after 140 years correctly argued that a sound wave passes through a gas (or air) very rapidly. So adiabatic conditions are developed. So, he took B to be the adiabatic elasticity of the gas, which is equal to γP where γ is the ratio of C_p (molar heat capacity at constant pressure) and C_V (molar heat capacity at constant volume). Thus, Newton's formula as corrected by La'place becomes,

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

For air, $\gamma = 1.41$ So that in air,

$$v = \sqrt{\frac{1.41\, P}{\rho}}$$

which gives 331.6 m/s as the velocity of sound (in air) at NTP which is in agreement with Newton's experimental result.

Note *We will carry out the derivation of formula,*

$$v = \sqrt{\frac{B}{\rho}}$$

in the chapter of sound (Chapter-16).

3. Speed of longitudinal wave in a thin rod or wire is given by,

$$v = \sqrt{\frac{Y}{\rho}}$$

Here, Y is the Young's modulus of elasticity.

Sample Example 14.7 *One end of 12.0 m long rubber tube with a total mass of 0.9 kg is fastened to a fixed support. A cord attached to the other and passes over a pulley and supports an object with a mass of 5.0 kg. The tube is struck a transverse blow at one end. Find the time required for the pulse to reach the other end. ($g = 9.8$ m/s²)*

Solution Tension in the rubber tube AB, $T = mg$

or
$$T = (5.0)(9.8) = 49 \text{ N}$$

Mass per unit length of rubber tube,

$$\mu = \frac{0.9}{12} = 0.075 \text{ kg/m}$$

$\therefore$ Speed of wave on the tube,

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{49}{0.075}} = 25.56 \text{ m/s}$$

$\therefore$ The required time is,

$$t = \frac{AB}{v} = \frac{12}{25.56} = 0.47 \text{ s} \qquad \textbf{Ans.}$$

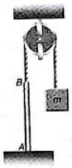


Fig. 14.14

Sample Example 14.8 *A uniform rope of mass 0.1 kg and length 2.45 m hangs from a ceiling.*
(a) Find the speed of transverse wave in the rope at a point 0.5 m distant from the lower end.
(b) Calculate the time taken by a transverse wave to travel the full length of the rope.

Solution (a) As the string has mass and it is suspended vertically, tension in it will be different at different points. For a point at a distance x from the free end, tension will be due to the weight of the string below it. So, if m is the mass of string of length l, the mass of length x of the string will be, $\left(\frac{m}{l}\right)x$.

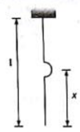


Fig. 14.15

$\therefore$
$$T = \left(\frac{m}{l}\right)xg = \mu xg \qquad \left(\frac{m}{l} = \mu\right)$$

$\therefore$
$$\frac{T}{\mu} = xg$$

or
$$v = \sqrt{\frac{T}{\mu}} = \sqrt{xg} \qquad \ldots\text{(i)}$$

At $x = 0.5$ m,
$$v = \sqrt{0.5 \times 9.8} = 2.21 \text{ m/s}$$

(b) From Eq. (i) we see that velocity of the wave is different at different points. So, if at point x the wave travels a distance dx in time dt, then

$$dt = \frac{dx}{v} = \frac{dx}{\sqrt{gx}}$$

$\therefore$
$$\int_0^t dt = \int_0^l \frac{dx}{\sqrt{gx}}$$

or
$$t = 2\sqrt{\frac{l}{g}} = 2\sqrt{\frac{2.45}{9.8}}$$

$$= 1.0 \text{ s} \qquad \textbf{Ans.}$$

Introductory Exercise 14.3

1. Calculate the velocity of a transverse wave along a string of length 2 m and mass 0.06 kg under a tension of 500 N.
2. Calculate the speed of a transverse wave in a wire of 1.0 mm² cross-section under a tension of 0.98 N. Density of the material of wire is 9.8×10^3 kg/m³.

14.6 Energy in Wave Motion

Every wave motion has energy associated with it. In wave motion, energy and momentum are transferred or propagated.

To produce any of the wave motions, we have to apply a force to a portion of the wave medium. The point where the force is applied moves, so we do work on the system. As the wave propagates, each portion of the medium exerts a force and does work on the adjoining portion. In this way a wave can transport energy from one region of space to other.

Regarding the energy in wave motion, we come across three terms namely, energy density (u), power (P) and intensity (I). Now let us take them one by one.

Energy density (u)

By the energy density of a plane progressive wave we mean **the total mechanical energy (kinetic + potential) per unit volume** of the medium through which the wave is passing. Let us proceed to obtain an expression for it.

Consider a string attached to a tuning fork. As the fork vibrates, it imparts energy to the segment of the string attached to it. For example, as the fork moves through its equilibrium position, it stretches the segment, increasing its potential energy, and the fork imparts transverse speed to the segment, increasing its kinetic energy. As a wave moves along the string, energy is imparted to the other segments of the string.

Kinetic energy per unit volume

We can calculate the kinetic energy of unit volume of the string from the wave function. Mass of unit volume is the density ρ. Its displacement from equilibrium is the wave function $y = A \sin(kx - \omega t)$. Its speed is $\frac{dy}{dt}$ where x is considered to be fixed. The kinetic energy of unit volume ΔK is then

$$\Delta K = \frac{1}{2}(\Delta m)v_y^2 = \frac{1}{2}\rho\left(\frac{dy}{dt}\right)^2$$

Using $y = A\sin(kx - \omega t)$, we obtain

$$\frac{dy}{dt} = -\omega A\cos(kx - \omega t)$$

So the kinetic energy per unit volume is

$$\Delta K = \frac{1}{2}\rho^2\omega^2A^2\cos^2(kx - \omega t) \qquad \text{...(i)}$$

Potential energy per unit volume

The potential energy of a segment is the work done in stretching the string and depends on the slope $\frac{dy}{dx}$. The potential energy per unit volume of the string is related to the slope and tension T by (for small slopes)

$$\Delta U = \frac{1}{2}\rho v^2\left(\frac{dy}{dx}\right)^2 \qquad \text{...(ii)}$$

where v = wave speed $= \frac{\omega}{k}$

Using $\frac{dy}{dx} = kA\cos(kx - \omega t)$, we obtain for the potential energy

$$\Delta U = \frac{1}{2}\rho\omega^2A^2\cos^2(kx - \omega t) \qquad \text{...(iii)}$$

Which is the same as the kinetic energy. The total energy per unit volume is

$$\Delta E = \Delta K + \Delta U = \rho\omega^2A^2\cos^2(kx - \omega t) \qquad \text{...(iv)}$$

We see that ΔE varies with time. Since the average value of $\cos^2(kx - \omega t)$ at any point is $\frac{1}{2}$, the average energy per unit volume (called the energy density u) is

$$u = \frac{1}{2}\rho\omega^2A^2 \qquad \text{...(v)}$$

Note *(i) Equation (v) is the same result as for a mass ρ attached to a spring and oscillating simple harmonic wave. However for a mass attached to a spring the potential energy is maximum when the displacement is maximum. For a string segment, the potential energy depends on the slope of the string and is maximum when the slope is maximum, which is at the equilibrium position of the segment, the same position for which the kinetic energy is maximum.*

At A : Kinetic energy and potential energy both are zero.

At B : Kinetic energy and potential energy both are maximum.

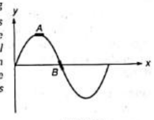


Fig. 14.16

(ii) Equation (ii) has been derived in miscellaneous example number 13.

Power (P)

Power is the instantaneous rate at which energy is transferred along the string (if we consider a transverse wave on a string).

Its value depends on the position x on the string at time t. Note that energy is being transferred only at points where the string has a nonzero slope ($\partial y/\partial x \neq 0$), so that there is a transverse component of the tension force, and where the string has a nonzero velocity ($\partial y/\partial t \neq 0$) so that the transverse force can do work. Let us obtain an expression for power transmitted through a string.

In unit time, the wave will travel a distance v. If s be the area of cross-section of the string, then volume of this length would be sv and energy transmitted per unit time (called power) would be,

$$P = (\text{energy density})\,(\text{volume})$$

$$\therefore \qquad P = \frac{1}{2}\rho\omega^2A^2sv$$

Note *This power is really the average power. The instantaneous power is given by,*

$$P(x,t) = F_y(x,t)\cdot v_y(x,t)$$

which comes out to be, $\rho\omega^2A^2sv\sin^2(kx - \omega t)$ or $\rho\omega^2A^2sv\cos^2(kx - \omega t)$ depending on the displacement equation $y(x,t)$. The average value of $\sin^2$ or $\cos^2$ function, averaged over any whole number of cycles is $\frac{1}{2}$. Hence the average power will be $\frac{1}{2}\rho\omega^2A^2sv$.

Intensity (I)

Flow of energy per unit area of cross-section of the string in unit time is known as the intensity of the wave. Thus,

$$I = \frac{\text{power}}{\text{area of cross-section}} = \frac{P}{s}$$

or

$$I = \frac{1}{2}\rho\omega^2A^2v$$

Again this is the average intensity transmitted through the string. The instantaneous intensity $\rho\omega^2A^2v\sin^2(kx - \omega t)$ or $\rho\omega^2A^2v\cos^2(kx - \omega t)$ depends on x and t.

Note *(i) Although the above relations for power and intensity have been discussed for a transverse wave on a string, they hold good for other waves also.*

*(ii) **Intensity due to a point source :** If a point source emits wave uniformly in all directions, the energy at a distance r from the source is distributed uniformly on a spherical surface of radius r and area $S = 4\pi r^2$. If P is the power emitted by the source, the power per unit area at a distance r from the source is $\frac{P}{4\pi r^2}$. The average power per unit area that is incident perpendicular to the direction of propagation is called the intensity. Therefore:*

$$I = \frac{P}{4\pi r^2}$$

or

$$I \propto \frac{1}{r^2}$$

Now, as amplitude $A \propto \sqrt{I}$, a spherical harmonic wave emanating from a point source can therefore, be written as

$$y(r,t) = \frac{A}{r}\sin(kr - \omega t)$$

Sample Example 14.9 *A stretched string is forced to transmit transverse waves by means of an oscillator coupled to one end. The string has a diameter of 4 mm. The amplitude of the oscillation is 10^{-4} m and the frequency is 10 Hz. Tension in the string is 100 N and mass density of wire is $4.2\times10^3\ kg/m^3$. Find :*

(a) the equation of the waves along the string
(b) the energy per unit volume of the wave
(c) the average energy flow per unit time across any section of the string and
(d) power required to drive the oscillator.

Solution (a) Speed of transverse wave on the string is,

$$v = \sqrt{\frac{T}{\rho S}} \qquad (\text{as } \mu = \rho S)$$

Substituting the values, we have

$$v = \sqrt{\frac{100}{(4.2\times10^3)\left(\frac{\pi}{4}\right)(4.0\times10^{-3})^2}}$$

$$= 43.53\ \text{m/s}$$

$$\omega = 2\pi f = 20\pi\ \frac{\text{rad}}{\text{s}} = 62.83\ \frac{\text{rad}}{\text{s}}$$

$$k = \frac{\omega}{v} = 1.44\ \text{m}^{-1}$$

$\therefore$ Equation of the waves along the string,

$$y(x,t) = A\sin(kx - \omega t)$$

$$= (10^{-4}\ \text{m})\sin\left[(1.44\ \text{m}^{-1})x - \left(62.83\ \frac{\text{rad}}{\text{s}}\right)t\right] \qquad \textbf{Ans.}$$

(b) Energy per unit volume of the string,

$$u = \text{energy density} = \frac{1}{2}\rho\omega^2A^2$$

Substituting the values, we have

$$u = \left(\frac{1}{2}\right)(4.2\times10^3)(62.83)^2(10^{-4})^2$$

$$= 8.29\times10^{-2}\ \text{J/m}^3 \qquad \textbf{Ans.}$$

(c) Average energy flow per unit time,

$$P = \text{power}$$

$$= \left(\frac{1}{2}\rho\omega^2A^2\right)(sv) = (u)(sv)$$

Substituting the values, we have

$$P = (8.29\times10^{-2})\left(\frac{\pi}{4}\right)(4.0\times10^{-3})^2(43.53)$$

$$= 4.53\times10^{-5}\ \text{J/s} \qquad \textbf{Ans.}$$

(d) Power required to drive the oscillator is obviously 4.53×10^{-5} W. **Ans.**

Introductory Exercise 14.4

1. Spherical waves are emitted from a 1.0 W source in an isotropic non-absorbing medium. What is the wave intensity 1.0 m from the source?
2. A line source emits a cylindrical expanding wave. Assuming the medium absorbs no energy, find how the amplitude and intensity of the wave depend on the distance from the source?

Solved Examples

For JEE Main

Example 1 *Equation of a transverse wave travelling in a rope is given by* $y = 5\sin(4.0t - 0.02x)$ *where* y *and* x *are expressed in cm and time in seconds. Calculate*

(a) *the amplitude, frequency, velocity and wavelength of the wave.*

(b) *the maximum transverse speed and acceleration of a particle in the rope.*

Solution (a) Comparing this with the standard equation of wave motion

$$y = A\sin\left(2\pi ft - \frac{2\pi}{\lambda}x\right)$$

where A, f and λ are amplitude, frequency and wavelength respectively.

Thus, amplitude $$A = 5\text{ cm}$$

$$2\pi f = 4$$

$\Rightarrow$ $$\text{Frequency } f = \frac{4}{2\pi} = 0.637\text{ Hz}$$

$$\text{Again } \frac{2\pi}{\lambda} = 0.02$$

or $$\text{Wavelength } \lambda = \frac{2\pi}{0.02} = 100\pi\text{ cm}$$

Velocity of the wave

$$v = f\lambda = \frac{4}{2\pi}\frac{2\pi}{0.02} = 200\text{ cm/s}$$ **Ans.**

(b) Transverse velocity of the particle

$$u = \frac{dy}{dt} = 5\times 4\cos(4.0t - 0.02x)$$

$$= 20\cos(4.0t - 0.02x)$$

Maximum velocity of the particle $= 20\text{ cm/s}$

Particle acceleration $a = \dfrac{d^2y}{dt^2} = -20\times 4\sin(4.0t - 0.02x)$

Maximum particle acceleration $= 80\text{ cm/s}^2$

Ans.

Example 2 *A wire of uniform cross-section is stretched between two points 100 cm apart. The wire is fixed at one end and a weight is hung over a pulley at the other end. A weight of 9 kg produces a fundamental frequency of 750 Hz.*

(a) *What is the velocity of the wave in wire ?*

(b) *If the weight is reduced to 4 kg, what is the velocity of wave ? What is the wavelength and frequency?*

Solution (a) $L = 100\,\text{cm}$, $f_1 = 750\,\text{Hz}$

$$v_1 = 2Lf_1 = 2 \times 100 \times 750$$
$$= 150000\,\text{cms}^{-1} = 1500\,\text{ms}^{-1}$$

(b)
$$v_1 = \sqrt{\frac{T_1}{m}} \text{ and } v_2 = \sqrt{\frac{T_2}{m}}$$
$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}} \Rightarrow \frac{v_2}{1500} = \sqrt{\frac{4}{9}}$$

$\therefore$
$$v_2 = 1000\,\text{ms}^{-1}$$
$$\lambda_2 = \text{wavelength} = 2L = 200\,\text{cm} = 2\,\text{m}$$
$$f_2 = \frac{v_2}{\lambda_2} = \frac{1000}{2} = 500\,\text{Hz}$$

Ans.

Example 3 *Determine the speed of sound waves in water, and find the wavelength of a wave having a frequency of 242 Hz. Take $B_{water} = 2 \times 10^9$ Pa.*

Solution Speed of sound wave,

$$v = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{(2 \times 10^9)}{10^3}} = 1414\,\text{m/s}$$

Wavelength
$$\lambda = \frac{v}{f} = 5.84\,\text{m}$$

Ans.

Example 4 *The faintest sound the human ear can detect at frequency 1 kHz corresponds to an intensity of about 10^{-12} W/m^2. Determine the pressure amplitude and the maximum displacement associated with this sound assuming the density of the air $= 1.3$ kg/m^3 and velocity of sound in air $= 332$ m/s.*

Solution
$$I = \frac{P^2}{2\rho v}$$

$\Rightarrow$
$$P = \sqrt{I \times 2\rho v}$$
$$= \sqrt{10^{-12} \times 2 \times 1.3 \times 332}$$
$$= 2.94 \times 10^{-5}\,\text{N/m}^2$$

Again,
$$P = \rho v \omega A$$

$\Rightarrow$
$$A = \frac{P}{\rho v \omega} = \frac{2.94 \times 10^{-5}}{1.3 \times 332 \times 2\pi \times 10^3}$$
$$= 1.1 \times 10^{-11}\,\text{m}$$

Ans.

Example 5 *For the speed of a transverse wave on a string, we write the formula*

$$v=\sqrt{\frac{T}{\mu}}$$

and sometimes we also write $v=f\lambda$. Are these two same or different? Explain.

Solution Speed of a wave depends on some special characteristics of the medium. For Problem, in case of a string it depends on T and μ. Frequency f depends on the source. Wavelength is self adjusted $\left(=\frac{v}{f}\right)$.

Let us take an Problem.

Suppose tension in a string is 100 N, linear mass density (μ) of this wire is 1 kg/m. Then in such a string speed of the transverse wave is,

$$v=\sqrt{\frac{T}{\mu}}=\sqrt{\frac{100}{1}}=10\ \text{m/s}$$

This value of 10 m/s is now fixed under the given conditions ($T=100$ N, $\mu=1$ kg/m). Now suppose we keep the frequency of a wave 10 Hz, then wavelength of such a wave will become 1 m $\left(=\frac{v}{f}\right)$. Now if frequency is reduced to 5 Hz (half), the wavelength will become two times or 2 m, so that product of λ and f remains constant ($=10$ m/s).

Example 6 *A wave moves with speed 300 m/s on a wire which is under a tension of 500 N. Find how much the tension must be changed to increase the speed to 312 m/s?*

Solution Speed of a transverse wave on a wire is,

$$v=\sqrt{\frac{T}{\mu}} \qquad \text{...(i)}$$

Differentiating with respect to tension, we have

$$\frac{dv}{dT}=\frac{1}{2\sqrt{\mu T}} \qquad \text{...(ii)}$$

Dividing Eq. (ii) by Eq. (i), we have

$$\frac{dv}{v}=\frac{1}{2}\frac{dT}{T}$$

or

$$dT=(2T)\frac{dv}{v}$$

Substituting the proper values, we have

$$dT=\frac{(2)(500)(312-300)}{300}=6.67\ \text{N} \qquad \textbf{Ans.}$$

i.e., tension should be increased by 6.67 N.

Example 7 *For the wave shown in figure, write the equation of this wave if its position is shown at $t=0$. Speed of wave is $v=300$ m/s*

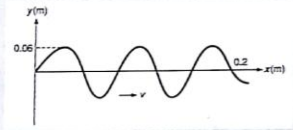


Fig. 14.17

Solution The amplitude,

$$A=0.06\ \text{m}$$

$$\frac{5}{2}\lambda=0.2\ \text{m}$$

$\therefore$

$$\lambda=0.08\ \text{m}$$

$$f=\frac{v}{\lambda}=\frac{300}{0.08}=3750\ \text{Hz}$$

$$k=\frac{2\pi}{\lambda}=78.5\ \text{m}^{-1}$$

and

$$\omega=2\pi f=23562\ \text{rad/s}$$

At $t=0$, $x=0$,

$$\frac{dy}{dx}=\text{positive}$$

and the given curve is a sine curve.

Hence, equation of wave travelling in positive x-direction should have the form,

$$y(x,t)=A\sin(kx-\omega t)$$

Substituting the values, we have

$$y(x,t)=(0.06\ \text{m})\sin[(78.5\ \text{m}^{-1})x-(23562\ \text{s}^{-1})t]\ \text{m} \qquad \textbf{Ans.}$$

Example 8 *For a wave described by, $y=A\sin(\omega t-kx)$, consider the following points*

(i) $x=0$, (ii) $x=\frac{\pi}{4k}$, (iii) $x=\frac{\pi}{2k}$ and (iv) $x=\frac{3\pi}{4k}$.

For a particle at each of these points at $t=0$, describe whether the particle is moving or not and in what direction and describe whether the particle is speeding up, slowing down or instantaneously not accelerating?

Solution

$$y=A\sin(\omega t-kx)$$

Particle velocity $v_P(x,t)=\frac{\partial y}{\partial t}=\omega A\cos(\omega t-kx)$ and particle acceleration

$$a_P(x,t)=\frac{\partial^2 y}{\partial t^2}=-\omega^2 A\sin(\omega t-kx)$$

(i) $t=0$, $x=0$:

$$v_P=+\omega A \quad \text{and} \quad a_P=0$$

i.e., particle is moving **upwards** but its **acceleration is zero.**

Note *Direction of velocity can be obtained in a different manner as under.*
At $t=0$,

$$y=A\sin(-kx)=-A\sin kx$$

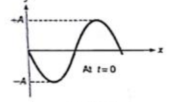


Fig. 14.18

i.e., y-x graph is as shown in figure. At $x=0$, slope is negative. Therefore, particle velocity is positive ($v_P=-v\times$ slope), as the wave is travelling along positive x-direction.

(ii) $t=0$, $x=\frac{\pi}{4k}$:

$$x=\frac{\pi}{4k} \quad \therefore \quad kx=\frac{\pi}{4}$$

$$v_P=\omega A\cos\left(-\frac{\pi}{4}\right)=+\frac{\omega A}{\sqrt{2}}$$

and

$$a_P=-\omega^2 A\sin\left(-\frac{\pi}{4}\right)=+\frac{\omega^2 A}{\sqrt{2}}$$

Velocity of particle is positive, *i.e.*, the particle is moving **upwards** (along positive y-direction). Further v_P and a_P are in the same direction (both are positive). Hence, the particle is **speeding up.**

(iii) $t=0$, $x=\frac{\pi}{2k}$: $x=\frac{\pi}{2k}$ $\therefore$ $kx=\frac{\pi}{2}$

$$v_P=\omega A\cos(-\pi/2)=0$$

$$a_P=-\omega^2 A\sin(-\pi/2)=\omega^2 A$$

i.e., particle is **stationary** or at its extreme position ($y=-A$). So, it is **speeding up** at this instant.

(iv) $t=0$, $x=\frac{3\pi}{4k}$:

$$x=\frac{3\pi}{4k} \quad \therefore \quad kx=\frac{3\pi}{4}$$

$\therefore$

$$v_P=\omega A\cos\left(-\frac{3\pi}{4}\right)=-\frac{\omega A}{\sqrt{2}}$$

$$a_P=-\omega^2 A\sin\left(-\frac{3\pi}{4}\right)=+\frac{\omega^2 A}{\sqrt{2}}$$

Velocity of particle is negative, *i.e.*, the particle is moving **downwards.** Further v_P and a_P are in opposite directions, *i.e.*, the particle is **slowing down.**

For JEE Advanced

Example 1 *A thin string is held at one end and oscillates vertically so that,*

$$y(x=0,t)=8\sin 4t\ \text{(cm)}$$

Neglect the gravitational force. The string's linear mass density is 0.2 kg/m and its tension is 1 N. The string passes through a bath filled with 1 kg water. Due to friction heat is transferred to the bath. The heat transfer efficiency is 50%. Calculate how much time passes before the temperature of the bath rises one degree kelvin?

Solution Comparing the given equation with equation of a travelling wave,

$$y=A\sin(kx\pm\omega t) \quad \text{at } x=0 \quad \text{we find,}$$

$$A=8\ \text{cm}=8\times10^{-2}\ \text{m}$$

$$\omega=4\ \text{rad/s}$$

Speed of travelling wave,

$$v=\sqrt{\frac{T}{\mu}}=\sqrt{\frac{1}{0.2}}=2.236\ \text{m/s}$$

Further, $\rho S=\mu=0.2$ kg/m

The average power over a period is,

$$P=\frac{1}{2}(\rho S)\omega^2 A^2 v$$

Substituting the values, we have

$$P=\frac{1}{2}(0.2)(4)^2(8\times10^{-2})^2(2.236)$$

$$=2.29\times10^{-2}\ \frac{\text{J}}{\text{s}}$$

The power transferred to the bath is,

$$P'=0.5P=1.145\times10^{-2}\ \frac{\text{J}}{\text{s}}$$

Now let, it takes t second to raise the temperature of 1 kg water by 1 degree kelvin. Then

$$P't=ms\Delta t$$

Here, $s=$ specific heat of water $=4.2\times10^3$ J/kg-°K

$\therefore$

$$t=\frac{ms\Delta t}{P'}=\frac{(1)(4.2\times10^3)(1)}{1.145\times10^{-2}}$$

$$=3.6\times10^5\ \text{s}\approx4.2\ \text{day} \qquad \textbf{Ans.}$$

Example 2 *Consider a wave propagating in the negative x-direction whose frequency is 100 Hz. At $t=5$ s, the displacement associated with the wave is given by,*

$$y=0.5\cos(0.1x)$$

where x and y are measured in centimetres and t in seconds. Obtain the displacement (as a function of x) at $t=10$ s. What is the wavelength and velocity associated with the wave?

Solution A wave travelling in negative x-direction can be represented as,

$$y(x, t) = A\cos(kx + \omega t + \phi)$$

At $t = 5$ s,

$$y(x, t = 5) = A\cos(kx + 5\omega + \phi)$$

Comparing this with the given equation,

We have, $\quad A = 0.5\,\text{cm}, \quad k = 0.1\,\text{cm}^{-1}$

and

$$5\omega + \phi = 0 \qquad \text{...(i)}$$

Now,

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.1} = 20\pi\ \text{cm} \qquad \textbf{Ans.}$$

$$\omega = 2\pi f = 200\pi\ \frac{\text{rad}}{\text{s}}$$

$\therefore$

$$v = \frac{\omega}{k} = \frac{200\pi}{0.1} = 2000\,\pi\ \frac{\text{cm}}{\text{s}} \qquad \textbf{Ans.}$$

From Eq. (i),

$$\phi = -5\omega$$

At $t = 10$ s,

$$y(x, t = 10) = 0.5\cos(0.1\,x + 10\,\omega - 5\omega)$$

$$= 0.5\cos(0.1\,x + 5\,\omega)$$

Substituting $\omega = 200\,\pi$,

$$y(x, t = 10) = 0.5\cos(0.1x + 1000\pi)$$

$$= 0.5\cos(0.1x) \qquad \textbf{Ans.}$$

Example 3 *A simple harmonic wave of amplitude* 8 *units travels along positive x-axis. At any given instant of time, for a particle at a distance of* 10 *cm from the origin, the displacement is* +6 *units, and for a particle at a distance of* 25 *cm from the origin, the displacement is* + 4 *units. Calculate the wavelength.*

Solution

$$y = A\sin\frac{2\pi}{\lambda}(vt - x)$$

or

$$\frac{y}{A} = \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

In the first case,

$$\frac{y_1}{A} = \sin 2\pi\left(\frac{t}{T} - \frac{x_1}{\lambda}\right)$$

Here, $\quad y_1 = +6, \quad A = 8, \quad x_1 = 10\,\text{cm}$

$\therefore$

$$\frac{6}{8} = \sin 2\pi\left(\frac{t}{T} - \frac{10}{\lambda}\right) \qquad \text{...(i)}$$

Similarly in the second case,

$$\frac{4}{8} = \sin 2\pi\left(\frac{t}{T} - \frac{25}{\lambda}\right) \qquad \text{...(ii)}$$

From Eq. (i),

$$2\pi\left(\frac{t}{T}-\frac{10}{\lambda}\right)=\sin^{-1}\left(\frac{6}{8}\right)=0.85 \text{ rad}$$

or $$\frac{t}{T}-\frac{10}{\lambda}=0.14 \quad \text{...(iii)}$$

Similarly from Eq. (ii), $$2\pi\left(\frac{t}{T}-\frac{25}{\lambda}\right)=\sin^{-1}\left(\frac{4}{8}\right)=\frac{\pi}{6} \text{ rad}$$

or $$\frac{t}{T}-\frac{25}{\lambda}=0.08 \quad \text{...(iv)}$$

Subtracting Eq. (iv) from Eq. (iii), we get

$$\frac{15}{\lambda}=0.06$$

$\therefore$ $$\lambda = 250 \text{ cm}$$ **Ans.**

Example 4 *A wave pulse on a horizontal string is represented by the function*

$$y(x, t)\frac{5.0}{1.0+(x-2t)^2} \quad (CGS\ units)$$

Plot this function at $t = 0$, 2.5 and 5.0 s.

Solution At the given times, the function representing the wave pulse is

$$y(x, 0)=\frac{5.0}{1.0+x^2}$$

$$y(x, 2.5\text{ s})=\frac{5.0}{1.0+(x-5.0)^2}$$

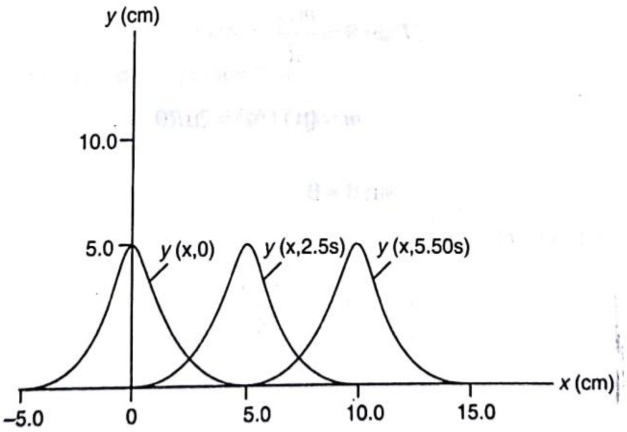


Fig. 14.19

$$y(x, 5.0\,s) = \frac{5.0}{1.0 + (x - 10.0)^2}$$

The maximum of $y(x, 0)$ is 5.0 cm; it is located at $x = 0$. Notice in the plot of figure that the pulse is also centred at this position. At $t = 2.5$ and 5.0 s, the centre of the pulse has moved to $x = 5.0$ and 10.0 cm, respectively. So, in each 2.5 s time interval, the pulse moves 5.0 cm in the positive x-direction. Its velocity is therefore +2.0 cm/s a value that is also evident from the given function.

Example 5 *A uniform circular hoop of string is rotating clockwise in the absence of gravity. The tangential speed is v_0. Find the speed of the wave travelling on this string.*

Solution

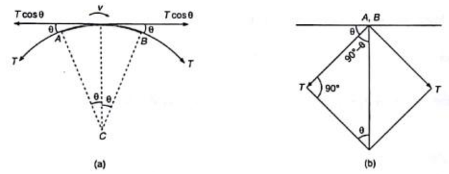


Fig. 14.20

Let T be the tension in the string. Consider a small circular element AB of the string of length,

$$\Delta l = R\,(2\theta) \qquad (R = \text{radius of hoop})$$

The components of tension $T\cos\theta$ are equal and opposite and thus cancel out. The components towards centre C (*i.e.*, $T\sin\theta$) provides the necessary centripetal force to element AB.

$$\therefore \qquad 2T\sin\theta = \frac{mv_0^2}{R} \qquad \text{...(i)}$$

Here $\qquad m = (\mu)(\Delta l) = 2\mu R\theta \qquad \left(\mu = \frac{\text{mass}}{\text{length}}\right)$

As θ is small, $\qquad \sin\theta \approx \theta$

Substituting in Eq. (i), we get

$$2T\theta = \frac{(2\mu R\theta)\,v_0^2}{R}$$

or

$$\frac{T}{\mu} = v_0^2$$

or

$$\sqrt{\frac{T}{\mu}} = v_0 \qquad \text{...(ii)}$$

Speed of wave travelling on this string,

$$v = \sqrt{\frac{T}{\mu}} = v_0 \qquad \text{[from Eq. (ii)]}$$

i.e., the velocity of the transverse wave along the hoop of string is the same as the velocity of rotation of the hoop, *viz.* v_0. **Ans.**

Example 6 *A sinusoidal wave travelling in the positive direction on a stretched string has amplitude 2.0 cm, wavelength 1.0 m and wave velocity 5.0 m/s. At $x = 0$ and $t = 0$, it is given that $y = 0$ and $\frac{\partial y}{\partial t} < 0$. Find the wave function $y(x, t)$.*

Solution We start with a general form for a rightward moving wave,

$$y(x, t) = A\sin(kx - \omega t + \phi)$$

The amplitude given is $\qquad A = 2.0\text{ cm} = 0.02\text{ m}$

The wavelength is given as, $\qquad \lambda = 1.0\text{ m}$

$\therefore$ Wave number $\qquad k = \frac{2\pi}{\lambda} = 2\pi\text{ m}^{-1}$

Angular frequency $\qquad \omega = vk = 10\pi\text{ rad/s}$

$$\therefore \qquad y(x, t) = (0.02)\sin[2\pi(x - 5.0t) + \phi]$$

We are told that for $x = 0$, $t = 0$,

$$y = 0 \quad\text{and}\quad \frac{\partial y}{\partial t} < 0$$

i.e., $\qquad 0.02\sin\phi = 0 \qquad (\text{as } y = 0)$

and $\qquad -0.2\pi\cos\phi < 0$

From these conditions, we may conclude that

$$\phi = 2n\pi \qquad \text{where } n = 0, 2, 4, 6\ldots$$

Therefore,

$$y(x, t) = (0.02\text{ m})\sin[(2\pi\text{ m}^{-1})x - (10\pi\text{ s}^{-1})t]\text{ m} \qquad \textbf{Ans.}$$

Example 7 *A pulse is propagating on a long stretched string along its length taken as positive x-axis. Shape of the string at $t = 0$ is given by,*

$$y = \sqrt{a^2 - x^2} \qquad \textit{when } |x| \ge a$$
$$= 0 \qquad \textit{when } |x| \le 0$$

Study the propagation of this pulse if it is travelling in positive x-direction with speed v.

Solution The pulse is semicircular in shape as shown in figure. For the wave travelling in positive x-direction,

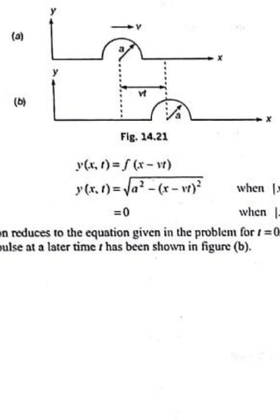


Fig. 14.21

$$y(x, t) = f(x - vt)$$

Hence, $\qquad y(x, t) = \sqrt{a^2 - (x - vt)^2} \qquad \text{when } |x - vt| \le a$

$$= 0 \qquad \text{when } |x - vt| \ge 0$$

The above equation reduces to the equation given in the problem for $t = 0$.
The shape of the pulse at a later time t has been shown in figure (b).

EXERCISES

For JEE Main

Subjective Question

Wave Equation and Plane Progressive Harmonic Wave

1. A certain transverse wave is described by

$$y(x, t) = (6.50\text{ mm})\cos 2\pi\left(\frac{x}{28.0\text{cm}} - \frac{t}{0.0360\text{s}}\right)$$

Determine the wave's (a) amplitude, (b) wavelength, (c) frequency, (d) speed of propagation and (e) direction of propagation.

2. For the wave $y = 5\sin 30\pi[t - (x/240)]$, where x and y are in cm and t is in seconds, find the
   (a) displacement when $t = 0$ and $x = 2$ cm, (b) wavelength,
   (c) velocity of the wave and (d) frequency of the wave.

3. The equation for a wave travelling in x-direction on a string is

$$y = (3.0\text{cm})\sin[(3.14\text{cm}^{-1})x - (314\text{s}^{-1})t]$$

   (a) Find the maximum velocity of a particle of the string.
   (b) Find the acceleration of a particle at $x = 6.0$ cm at time $t = 0.11$ s.

4. A wave of frequency 500 Hz has a wave velocity of 350 m/s.
   (a) Find the distance between two points which are 60° out of phase.
   (b) Find the phase difference between two displacements at a certain point at times 10^{-3} s apart.

5. At $t = 0$, a transverse wave pulse in wire is described by the function;

$$y = \frac{6}{x^2 + 3}$$

where x and y are in metres. Write the function $y(x, t)$ that describes this wave if it is travelling in the positive x-direction with a speed of 4.50 m/s.

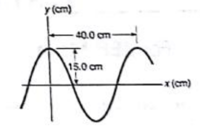
6. A wave is described by the equation;

$$y = (1.0\text{ mm})\sin\pi\left(\frac{x}{2.0\text{cm}} - \frac{t}{0.01\text{s}}\right)$$

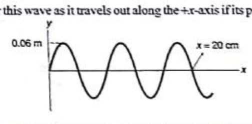
   (a) Find the time period and the wavelength.
   (b) Write the equation for the velocity of the particles. Find the speed of the particle at $x = 1.0$ cm at time $t = 0.01$ s.
   (c) What are the speeds of the particles at $x = 3.0$ cm, 5.0 cm and 7.0 cm at $t = 0.01$ s?
   (d) What are the speeds of the particles at $x = 1.0$ cm at $t = 0.011$, 0.012 and 0.013 s?

7. A sinusoidal wave travelling in the positive x-direction has an amplitude of 15.0 cm, a wavelength of 40.0 cm and a frequency of 8.00 Hz. The vertical displacement of the medium at $t = 0$ and $x = 0$ is also 15.0 cm, as shown in figure.
   (a) Find the angular wave number k, period T, angular frequency ω and speed v of the wave.

(b) Write a general expression for the wave function.



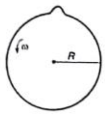
8. For the wave shown in figure, find its amplitude, frequency and wavelength if its speed is 300 m/s. Write the equation for this wave as it travels out along the $+x$-axis if its position at $t=0$ is as shown.



9. Transverse waves on a string have wave speed 8.00 m/s, amplitude 0.0700 m and wavelength 0.32 m. The waves travel in the negative x-direction and at $t=0$ the $x=0$ end of the string has its maximum upward displacement.
   (a) Find the frequency, period and wave number of these waves.
   (b) Write a wave function describing the wave.
   (c) The transverse displacement of a particle at $x=0.360$ m at time $t=0.150$ s.
   (d) How much time must elapse from the instant in part (c) until the particle at $x=0.360$ m next has maximum upward displacement ?

Speed of a Transverse Wave on a String

10. A copper wire 2.4 mm in diameter is 3 m long and is used to suspend a 2 kg mass from a beam. If a transverse disturbance is sent along the wire by striking it lightly with a pencil, how fast will the disturbance travel ? The density of copper is $8920\ kg/m^3$.
11. A uniform rope of length 12 m and mass 6 kg hangs vertically from a rigid support. A block of mass 2 kg is attached to the free end of the rope. A transverse pulse of wavelength 0.06 m is produced at the lower end of the rope. What is the wavelength of the pulse when it reaches the top of the rope?
12. A flexible steel cable of total length L and mass per unit length μ hangs vertically from a support at one end. (a) Show that the speed of a transverse wave down the cable is $v=\sqrt{g(L-x)}$, where x is measured from the support. (b) How long will it take for a wave to travel down the cable?
13. A loop of rope is whirled at a high angular velocity ω, so that it becomes a taut circle of radius R. A kink develops in the whirling rope. (a) Show that the speed of the kink in the rope is $v=\omega R$. (b) Under what conditions does the kink remain stationary relative to an observer on the ground?



14. A wire of variable mass per unit length $\mu=\mu_0 x$, is hanging from the ceiling as shown in figure. The length of wire is l_0. A small transverse disturbance is produced at its lower end. Find the time after which the disturbance will reach to the other end.

15. A non-uniform wire of length L and mass M has a variable linear mass density given by $\mu=kx$, where x is distance from one end of wire and k is a constant. Find the time taken by a pulse starting at one end to reach the other end when the tension in wire is T.
16. One end of a horizontal rope is attached to a prong of an electrically driven tuning fork that vibrates at 120 Hz. The other end passes over a pulley and supports a 1.50 kg mass. The linear mass density of the rope is 0.0550 kg/m.
   (a) What is the speed of a transverse wave on the rope ?
   (b) What is the wavelength ?
   (c) How would your answers to parts (a) and (b) change if the mass were increased to 3.00 kg?

Energy in Wave Motion

17. A certain 120 Hz wave on a string has an amplitude of 0.160 mm. How much energy exists in an 80 g length of the string?
18. A taut string for which $\mu=5.00\times10^{-2}$ kg/m is under a tension of 80.0 N. How much power must be supplied to the string to generate sinusoidal waves at a frequency of 60.0 Hz and an amplitude of 6.00 cm?
19. A 200 Hz wave with amplitude 1 mm travels on a long string of linear mass density 6 g/m kept under a tension of 60 N.
   (a) Find the average power transmitted across a given point on the string.
   (b) Find the total energy associated with the wave in a 2.0 m long portion of the string.
20. A transverse wave of amplitude 0.50 mm and frequency 100 Hz is produced on a wire stretched to a tension of 100 N . If the wave speed is 100 m/s. What average power is the source transmitting to the wire ?

Objective Questions

Single Correct Option

1. A harmonic oscillator vibrates with amplitude of 4 cm and performs 150 oscillations in one minute. If the initial phase is 45° and it starts moving away from the origin, then the equation of motion is
   (a) $0.04\sin\left(5\pi t+\frac{\pi}{4}\right)$ (b) $0.04\sin\left(5\pi t-\frac{\pi}{4}\right)$ (c) $0.04\sin\left(4\pi t+\frac{\pi}{4}\right)$ (d) $0.04\sin\left(4\pi t-\frac{\pi}{4}\right)$
2. The speed of propagation of a wave in a medium is $300\ m/s^{-1}$. The equation of motion of point at $x=0$ is given by $y=0.04\sin 600\pi t$ (metre). The displacement of a point $x=75$ cm at $t=0.01$ s is
   (a) 0.02 m (b) 0.04 m (c) zero (d) 0.028 m
3. The displacement function of a wave travelling along positive x-direction is $y=\frac{1}{2+3x^2}$ at $t=0$ and by $y=\frac{1}{2+3(x-2)^2}$ at $t=2$ s, where y and x are in metre. The velocity of the wave is
   (a) $2\ m/s^{-1}$ (b) 0.5 m/s (c) 1 m/s (d) 3 m/s
4. The displacement from the position of equilibrium of a point 4 cm from a source of oscillation is half the amplitude at the moment $t=\frac{T}{6}$, where T is the time period. The wavelength of the travelling wave is
   (a) 0.24 m (b) 0.48 m (c) 0.96 m (d) None of these
5. A source oscillates with a frequency 25 Hz and the wave propagates with 300 m/s. Two points A and B are located at distances 10 m and 16 m away from the source. The phase difference between A and B is
   (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π
6. The equation of a transverse wave propagating in a string is given by
$$y=0.02\sin(x+30t)$$
   where x and y are in metre and t is in second.
   If linear density of the string is 1.3×10^{-4} kg/m, then the tension in the string is
   (a) 0.12 N (b) 1.2 N (c) 12 N (d) 120 N
7. The angle between wave velocity and particle velocity in a travelling wave may be
   (a) zero (b) $\frac{\pi}{2}$ (c) π (d) All the these
8. A 100 Hz sinusoidal wave is travelling in the positive x-direction along a string with a linear mass density of 3.5×10^{-3} kg/m and a tension of 35 N. At time $t=0$, the point $x=0$, has maximum displacement in the positive y direction. Next when this point has zero displacement the slope of the string is $\pi/20$. Which of the following expression represent (s) the displacement of string as a function of x (in metre) and t (in second).
   (a) $y=0.025\cos(200\pi t-2\pi x)$ (b) $y=0.5\cos(200\pi t-2\pi x)$
   (c) $y=0.025\cos(100\pi t-10\pi x)$ (d) $y=0.5\cos(100\pi t-10\pi x)$

9. Vibrations of period 0.25 s propagate along a straight line at a velocity of 48 cm/s. One second after the emergence of vibrations at the initial point, displacement of the point, 47 cm from it is found to be 3 cm. [Assume that at initial point particle is in its mean position at $t=0$ and moving upwards]. Then,
   (a) amplitude of vibrations is 6 cm (b) amplitude of vibrations is $3\sqrt{2}$ cm
   (c) amplitude of vibrations is 3 cm (d) None of the above
10. Transverse waves are generated in two uniform steel wires A and B by attaching their free ends to a fork of frequency 500 Hz. The diameter of wire A is half that of B and tension in wire A is half the tension in wire B. What is the ratio of velocities of waves in A and B?
   (a) 1 : 2 (b) $\sqrt{2}:1$ (c) 2 : 1 (d) $1:\sqrt{2}$
11. The frequency of A note is 4 times that of B note. The energies of two notes are equal. The amplitude of B note as compared to that of A note will be
   (a) double (b) equal (c) four times (d) eight times
12. If at $t=0$, a travelling wave pulse on a string is described by the function.
$$y=\frac{6}{x^2+3}$$
   What will be the wave function representing the pulse at time t, if the pulse is propagating along positive x-axis with speed 4 m/s?
   (a) $y=\frac{6}{(x+4t)^2+3}$ (b) $y=\frac{6}{(x-4t)^2+3}$ (c) $y=\frac{6}{(x-t)^2}$ (d) $y=\frac{6}{(x-t)^2+12}$
13. If the speed of longitudinal waves equals 10 times the speed of the transverse waves in a stretched wire of material which has modulus of elasticity E, then the stress in the wire is
   (a) $10E$ (b) $100E$ (c) $E/10$ (d) $E/100$
14. Equation of progressive wave is given by, $y=4\sin\left[\pi\left(\frac{t}{5}-\frac{x}{9}\right)+\frac{\pi}{6}\right]$, where x and y are in metre. Then
   (a) $v=5$ m/s (b) $\lambda=18$ m (c) $A=0.04$ m (d) $f=50$ Hz
15. The equation of a wave is given by $Y=5\sin 10\pi(t-0.01x)$ along the x-axis. (All the quantities are expressed in SI units). The phase difference between the points separated by a distance of 10 m along x-axis is
   (a) $\frac{\pi}{2}$ (b) π (c) 2π (d) $\frac{\pi}{4}$
16. A transverse wave travelling on a stretched string is represented by the equation $y=\frac{2}{(2x-6.2t)^2+20}$. Then,
   (a) velocity of the wave is 3.1 m/s (b) amplitude of the wave is 0.1 m
   (c) frequency of the wave is 20 Hz (d) wavelength of the wave is 1 m

17. For energy density, power and intensity of any wave choose the correct options.

(a) u = energy density $= \frac{1}{2}\rho\omega^2 A^2$

(b) P = power $= \frac{1}{2}\rho\,\omega^2 A^2 v$

(c) I = intensity $= \frac{1}{2}\rho\,\omega^2 A^2 Sv$

(d) $I = \frac{P}{S}$

18. For the transverse wave equation $y = A\sin(\pi x + \pi t)$, choose the correct options at $t = 0$

(a) points at $x = 0$ and $x = 1$ are at mean positions
(b) points at $x = 0.5$ and $x = 1.5$ have maximum accelerations
(c) points at $x = 0.5$ and $x = 1.5$ are at rest
(d) All of the above

19. In the wave equation,

$$y = A\sin\frac{2\pi}{a}(x - bt)$$

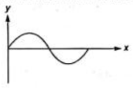
(a) speed of wave is a
(b) speed of wave is b
(c) wavelength of wave is a/b
(d) wavelength of wave is a

20. In the wave equation,

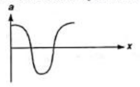
$$y = A\sin 2\pi\left(\frac{x}{a} - \frac{t}{b}\right)$$

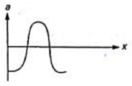
(a) speed of wave is a/b
(b) speed of wave is b/a
(c) wavelength of wave is a
(d) time period of wave is b

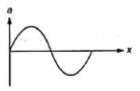
21. Corresponding to y-x graph of a transverse harmonic wave shown in figure

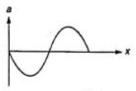


Choose the correct options at same time.

(a)


(b)


(c)


(d)


For **JEE Advanced**

Assertion and Reason

Directions : Choose the correct option.

(a) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*
(b) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*
(c) *If **Assertion** is true, but the **Reason** is false.*
(d) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** Transverse waves can't travel in gaseous medium.
Reason : They do not possess modulus of rigidity.

2. **Assertion :** Transverse waves can travel on the surface of water.
Reason : Surface tension property of water plays the role of modulus of rigidity.

3. **Assertion :** Two wave equations are $y_1 = A\sin(\omega t - kx)$ and $y_2 = A\sin(kx - \omega t)$. These two waves have a phase difference of π.
Reason : They are travelling in opposite directions.

4. **Assertion :** Wave speed is given by $v = f\lambda$
If frequency f is doubled, v will become two times.
Reason : For given conditions of medium wave speed remains constant.

5. **Assertion :** On moon you cannot hear your friend standing at some distance from you.
Reason : There is vacuum on moon.

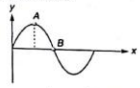
6. **Assertion :** Wave number is the number of waves per unit length.
Reason : Wave number $= \frac{1}{\lambda}$.

7. **Assertion :** Electromagnetic waves don't require medium for their propagation.
Reason : They can't travel in a medium.

8. **Assertion :** Two strings shown in figure have the same tension. Speed of transverse waves in string-1 will be more.

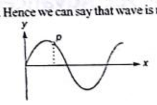
Reason : $v \propto \frac{1}{\sqrt{\mu}}$, Here μ is mass per unit length of string.

9. **Assertion :** y-x graph of a transverse wave on a string is as shown in figure. At point A potential energy and kinetic energy both are minimum.



Reason : At point B kinetic energy and potential energy both are maximum.

10. **Assertion :** y-x graph of a transverse wave on a string is as shown in figure. At the given instant point P is moving downwards. Hence we can say that wave is moving towards positive x-direction.



Reason : Particle velocity is given by :

$$v_P = -\frac{v \cdot \partial y}{\partial x}$$

11. **Assertion :** If two waves of same amplitude produce a resultant wave of same amplitude, then phase difference between them will be 120°.
Reason : The resultant amplitude of two waves is equal to sum of amplitude of two waves.

Match the Columns

1. For the wave equation,

$$y = a\sin(bt - cx)$$

match the following two columns.

Column I	Column II
(a) wave speed	(p) $\frac{b}{2\pi}$
(b) maximum particle speed	(q) $\frac{c}{2\pi}$
(c) wave frequency	(r) $\frac{b}{c}$
(d) wavelength	(s) None

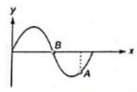
2. For the wave equation,

$$y = (4\,\text{cm})\sin[\pi t + 2\pi x]$$

Here t is in second and x in meters.

Column I	Column II
(a) at $x = 0$, particle velocity is maximum at t =	(p) 0.5 s
(b) at $x = 0$, particle acceleration is maximum at t =	(q) 1.0 s
(c) at $x = 0.5$ m, particle velocity is maximum at t =	(r) zero
(d) at $x = 0.5$ m, particle acceleration is maximum at t =	(s) 1.5 s

3. y-x graph of a transverse wave at a given instant is shown in figure. Match the following two columns.



Column I	Column II
(a) velocity of particle A	(p) positive
(b) acceleration of particle A	(q) negative
(c) velocity of particle B	(r) zero
(d) acceleration of particle B	(s) can't tell

4. For a travelling wave match the following two columns.

Column I	Column II
(a) energy density	(p) $[ML^2T^{-3}]$
(b) power	(q) $\frac{1}{2}\rho\omega^2 A^2 Sv$
(c) intensity	(r) $[M^0L^{-1}T]$
(d) wave number	(s) None

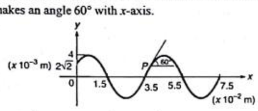
5. Match following two columns.

Column I	Column II
(a) $y = A\sin(\omega t - kx)$	(p) travelling in positive x-direction
(b) $y = A\sin(kx - \omega t)$	(q) travelling in negative x-direction
(c) $y = -A\cos(\omega t + kx)$	(r) at $t = 0$, velocity of particle is positive at $x = 0$
(d) $y = -A\cos(kx - \omega t)$	(s) at $t = 0$ acceleration of particle is positive at $x = 0$

Subjective Questions

1. The figure shows a snap photograph of a vibrating string at $t = 0$. The particle P is observed moving up with velocity $20\sqrt{3}$ cm/s. The tangent at P makes an angle 60° with x-axis.

(a) Find the direction in which the wave is moving.
(b) Write the equation of the wave.
(c) The total energy carried by the wave per cycle of the string. Assuming that the mass per unit length of the string = 50 g/m.

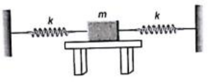


2. A long string having a cross-sectional area 0.80 mm^2 and density 12.5 g/cm^3, is subjected to a tension of 64 N along the positive x-axis. One end of this string is attached to a vibrator at $x = 0$ moving in transverse direction at a frequency of 20 Hz. At $t = 0$, the source is at a maximum displacement $y = 1.0$ cm.

(a) Find the speed of the wave travelling on the string.
(b) Write the equation for the wave.

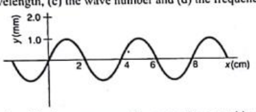
(c) What is the displacement of the particle of the string at $x = 50$ cm at time $t = 0.05$ s ?
(d) What is the velocity of this particle at this instant?

3. One end of each of two identical springs, each of force-constant 0.5 N/m are attached on the opposite sides of a wooden block of mass 0.01 kg. The other ends of the springs are connected to separate rigid supports such that the springs are unstretched and are collinear in a horizontal plane. To the wooden piece is fixed a pointer which touches a vertically moving plane paper. The wooden piece, kept on a smooth horizontal table is now displaced by 0.02 m along the line of springs and released. If the speed of paper is 0.1 m/s, find the equation of the path traced by the pointer on the paper and the distance between two consecutive maximas on this path.



4. A wave pulse is travelling on a string with a speed v towards the positive x-axis. The shape of the string at $t = 0$ is given by $y(x) = A\sin(x/a)$, where A and a are constants.
(a) What are the dimensions of A and a?
(b) Write the equation of the wave for a general time t, if the wave speed is v.

5. Figure shows a plot of the transverse displacement of the particle of a string at $t = 0$ through which a travelling wave is passing in the positive x-direction. The wave speed is 20 cm/s. Find (a) the amplitude, (b) the wavelength, (c) the wave number and (d) the frequency of the wave.



6. Two wires of different densities but same area of cross-section are soldered together at one end and are stretched to a tension T. The velocity of a transverse wave in the first wire is double of that in the second wire. Find the ratio of the density of the first wire to that of the second wire.

7. Two long strings A and B, each having linear mass density 1.2×10^{-2} kg/m are stretched by different tensions 4.8 N and 7.5 N respectively and are kept parallel to each other with their left ends at $x = 0$. Wave pulses are produced on the strings at the left ends at $t = 0$ on string A and at $t = 20$ ms on string B. When and where will the pulse on B overtake that on A ?

8. A sinusoidal transverse wave travels on a string. The string has length 8.00 m and mass 6.00 g. The wave speed is 30.0 m/s and the wavelength is 0.200 m. (a) If the wave is to have an average power of 50.0 W, what must be the amplitude of the wave ? (b) For this same string, if the amplitude and wavelength are the same as in part (a) what is the average power for the wave if the tension is increased such that the wave speed is doubled ?

9. A uniform rope with length L and mass m is held at one end and whirled in a horizontal circle with angular velocity ω. You can ignore the force of gravity on the rope. Find the time required for a transverse wave to travel from one end of the rope to the other.

Hint : $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

ANSWERS

Introductory Exercise 14.1

2. c/b 3. 0.5 m/s
4. The pulse is travelling along negative x-axis with velocity 2 m/s. The amplitude of the pulse is 2 m.
5. $y = \frac{10}{(x - 2t)^2 + 2}$

Introductory Exercise 14.2

1. (a) −5 m/s (b) −1.7 m/s 2. Yes, $(v_p)_{max} = (kA)v$
3. (a) 10 Hz (b) $\frac{5\pi}{4}$ rad (c) $\frac{1}{60}$ s (d) −1.26 m/s
4. (a) $y(x, t) = (0.05\ \text{m}) \sin[(60\pi\ \text{s}^{-1})t - (5\pi\ \text{m}^{-1})x]$ (b) − 3.54 cm (c) 4.2 ms

Introductory Exercise 14.3

1. 129.1 m/s 2. 10 m/s

Introductory Exercise 14.4

1. $\frac{1}{4\pi}$ W/m^2 2. $A \propto \frac{1}{\sqrt{r}}$, $I \propto \frac{1}{r}$

For JEE Main

Subjective Questions

1. (a) 6.50 mm (b) 28.0 cm (c) 27.8 Hz (d) 7.8 m/s (e) positive x
2. (a) −3.535 cm (b) 16 cm (c) 240 cm/s (d) 15 Hz
3. (a) 9.4 m/s (b) zero 4. (a) 0.116 m (b) 180° 5. $y = \frac{5}{[(x - 4.5t)^2 + 3]}$
6. (a) 20 ms, 4.0 cm (b) $v = -(\pi/10\ \text{m/s}) \cos \pi\left[\frac{x}{2.0\ \text{cm}} - \frac{t}{0.01\ \text{s}}\right]$, zero
(c) 0 m/s, 0 m/s, 0 m/s (d) 9.7 cm/s, 18 cm/s, 25 cm/s
7. (a) 0.157 rad/cm, 0.125 s, 50.3 rad/s, 320 cm/s (b) $y = (15.0\ \text{cm}) \cos(0.157x - 50.3t)$
8. Amplitude = 0.06 m, Frequency = 3750 Hz, Wavelength = 8 cm
$y = (0.06\ \text{m}) \sin[(78.5\ \text{m}^{-1})x - (23562\ \text{s}^{-1})t]$
9. (a) 25.0 Hz, 0.04 s, 19.6 rad/m (b) $y(x, t) = (0.07\ \text{m}) \cos 2\pi\left(\frac{x}{0.32\ \text{m}} + \frac{t}{0.04\ \text{s}}\right)$
(c) 0.0495 m (d) 0.015 s 10. 22 m/s 11. 0.12 m 12. (b) $t = 2\sqrt{\frac{l}{g}}$
13. (b) The kink will be stationary with respect to the ground if it moves clockwise with respect to the rope.
14. $\sqrt{\frac{8l_0}{g}}$ 15. $\frac{2}{3}\sqrt{\frac{2ML}{T}}$ 16. (a) 16.3 m/s (b) 0.136 m (c) both increase by $\sqrt{2}$ times.
17. 0.58 mJ 18. 512 W 19. (a) 0.47 W (b) 9.4 mJ 20. 49 mW

Objective Questions

1.(a) 2.(b) 3.(c) 4.(b) 5.(c) 6.(a) 7.(d) 8.(a) 9.(a) 10.(b)
11.(c) 12.(b) 13.(d) 14.(b) 15.(b) 16.(a,b) 17.(a,d) 18.(d) 19.(b,d) 20.(a,c,d)
21.(d)

For JEE Advanced

Assertion and Reason

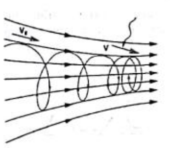
1. (a) 2. (a) 3. (c) 4. (d) 5. (a) 6. (a) 7. (c) 8. (d) 9. (b) 10. (d)
11. (c)

Match the Columns

1. (a) → r (b) → s (c) → p (d) → s
2. (a) → q,r (b) → p,s (c) → q,r (d) → p,s
3. (a) → s (b) → p (c) → s (d) → r
4. (a) → s (b) → p,q (c) → s (d) → s
5. (a) → p,r (b) → p (c) → q,s (d) → p,s .

Subjective Questions

1. (a) Negative x (b) $y = (0.4\ \text{cm}) \sin\left(10\pi t + \frac{\pi}{2}x + \frac{\pi}{4}\right)$s (c) 1.6×10^{-5} J
2. (a) 80 m/s (b) $y = (1.0\ \text{cm}) \cos\left[(40\pi\ \text{s}^{-1})t - \left(\frac{\pi}{2}\ \text{m}^{-1}\right)x\right]$ (c) $\frac{1}{\sqrt{2}}$ cm (d) 89 cm/s
3. $y = 0.02 \cos(10t - 100x)$ m, 0.0628 m 4. (a) L, L (b) $y(x, t) = A\sin\left(\frac{x - vt}{a}\right)$
5. (a) 1.0 mm (b) 4 cm (c) 1.6 cm^{-1} (d) 5 Hz 6. 0.25 7. at $t = 0.1$ s, at $x = 2.0$ m
8. (a) 7.07 cm (b) 400.0 W 9. $\frac{\pi}{\sqrt{2}\omega}$



15

SUPERPOSITION OF WAVES

Chapter Contents

15.1 Introduction

When a wave strikes the boundaries of its medium, some part of it is reflected and some is transmitted. The incident and the reflected waves overlap in the same region of the medium. This overlapping of two or more waves gives rise to some special characteristics of waves. The overlapping of waves is based on the principle of superposition discussed in Art. 15.2. The phenomenon like interference of waves and beats in sound are based on superposition principle. Young's double slit experiment is an example of interference in light. Standing wave is an another example of interference.

When there are two boundary points or surfaces such as a guitar string tied at both ends, we get repeated reflections. In such situations we find that sinusoidal waves can occur only for certain special frequencies, which are determined by the properties and the dimensions of the medium. These special frequencies and their associated wave patterns are called **normal modes**.

In this chapter our focus will be on the interference of mechanical waves on a stretched string. But interference is also important in light (electromagnetic) waves. It explains the colours seen in soap bubbles.

15.2 Principle of Superposition

Two or more waves can travel simultaneously in a medium without affecting the motion of one another. Therefore, the resultant displacement of each particle of the medium at any instant is equal to the vector sum of the displacements produced by the two waves separately. This principle is called 'principle of superposition'. It holds for all types of waves, provided the waves are not of very large amplitude. We can express the superposition principle in the form,

$$y(x, t) = y_1(x, t) + y_2(x, t) + \ldots + y_n(x, t)$$

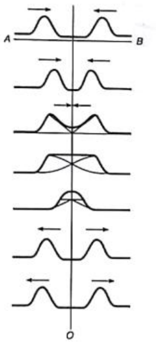
or

$$y(x, t) = \sum_{j=1}^{n} y_j(x, t)$$

Here, the y_j are the individual wave functions, and their sum, the wave function $y(x, t)$ describes the resultant behaviour of the medium as a function of position and time.

The principle of superposition is of central importance in all types of waves. When a friend talks to you while you are listening to music, you can distinguish the sound of speech and the sound of music from each other. This is precisely because according to principle of superposition two different waves can travel in a medium simultaneously without disturbing the other. Superposition also applies to electromagnetic waves (such as light) and many other types of waves.

As an example, consider a long stretched string AB. At the end A, a pulse is generated which propagates towards B with some speed (say v). At B an identical pulse is generated which travels towards A with the same speed v. The snapshot of the



Overlap of two wave pulses travelling in opposite directions. Time increases from top to bottom.

Fig. 15.1

string at different times are shown in figure. From this figure you might have understood how the two waves superimpose and what is the meaning of principle of superposition.

• *Important Points in* SUPERPOSITION

1. Consider the superposition of two sinusoidal waves of same frequency at a point. Let us assume that the two waves are travelling in the same direction with same velocity. The equation of the two waves reaching at a point can be written as,

$$y_1 = A_1 \sin(kx - \omega t)$$

and

$$y_2 = A_2 \sin(kx - \omega t + \phi)$$

The resultant displacement of the point where the waves meet is

$$\begin{aligned} y &= y_1 + y_2 \\ &= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t + \phi) \\ &= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t)\cos\phi + A_2 \cos(kx - \omega t)\sin\phi \\ &= (A_1 + A_2\cos\phi)\sin(kx - \omega t) + A_2 \sin\phi \cos(kx - \omega t) \\ &= A\cos\theta \sin(kx - \omega t) + A\sin\theta\cos(kx - \omega t) \end{aligned}$$

or

$$y = A\sin(kx - \omega t + \theta)$$

Here,

$$A_1 + A_2\cos\phi = A\cos\theta$$

and

$$A_2 \sin\phi = A\sin\theta$$

or

$$A^2 = (A_1 + A_2\cos\phi)^2 + (A_2\sin\phi)^2$$

or

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

and

$$\tan\theta = \frac{A\sin\theta}{A\cos\theta} = \frac{A_2\sin\phi}{A_1 + A_2\cos\phi}$$

2. The above result can be obtained by graphical method as well. Assume a vector $\vec{A}_1$ of length A_1 to represent the amplitude of first wave.

Another vector $\vec{A}_2$ of length A_2, making an angle ϕ with $\vec{A}_1$ represent the amplitude of second wave. The resultant of $\vec{A}_1$ and $\vec{A}_2$ represent the amplitude of resulting function y. The angle θ represents the phase difference between the resulting function and the first wave.

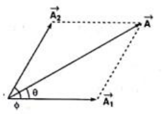


Fig. 15.2

Sample Example 15.1 *Two harmonic waves are represented in SI units by,*

$y_1(x, t) = 0.2\sin(x - 3.0t)$ *and* $y_2(x, t) = 0.2\sin(x - 3.0t + \phi)$

(a) Write the expression for the sum $y = y_1 + y_2$ *for* $\phi = \frac{\pi}{2}$ *rad.*

(b) Suppose the phase difference ϕ *between the waves is unknown and the amplitude of their sum is* 0.32 *m, what is* ϕ*?*

Solution (a)

$$\begin{aligned} y &= y_1 + y_2 \\ &= 0.2\sin(x - 3.0t) + 0.2\sin\left(x - 3.0t + \frac{\pi}{2}\right) \\ &= A\sin(x - 3.0t + \theta) \end{aligned}$$

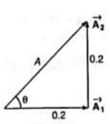


Fig. 15.3

Here,

$$A = \sqrt{(0.2)^2 + (0.2)^2} = 0.28 \text{ m}$$

and

$$\theta = \frac{\pi}{4}$$

$\therefore$

$$y = 0.28\sin\left(x - 3.0t + \frac{\pi}{4}\right) \quad \textbf{Ans.}$$

(b) Since the amplitude of the resulting wave is 0.32 m and $A = 0.2$ m, we have

$$0.32 = \sqrt{(0.2)^2 + (0.2)^2 + (2)(0.2)(0.2)\cos\phi}$$

Solving this, we get

$$\phi = \pm 1.29 \text{ rad} \quad \textbf{Ans.}$$

EXERCISE *What is the resultant wave obtained when* $y_1(x, t) = A\sin(kx - \omega t + \phi_1)$ *and* $y_2 = A\sin(kx - \omega t + \phi_2)$ *are added? Use* $\phi_1 = \frac{\pi}{6}$ *rad and* $\phi_2 = \frac{\pi}{2}$ *rad.*

Ans $y(x, t) = \sqrt{3}A\sin(kx - \omega t + \pi/3)$

15.3 Interference of Waves

We have seen earlier that if two sinusoidal waves of same angular frequency ω meet at a point where phase difference between them is ϕ, the resulting amplitude is given by,

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\phi \quad \ldots\text{(i)}$$

Further in Art. 14.6, we have also read that intensity of a wave is given by,

$$I = \frac{1}{2}\rho A^2\omega^2 v$$

i.e.,

$$I \propto A^2$$

So, if ρ, ω and v are same for both interfering waves, Eq. (i) can also be written as,

$$I = I_1 + I_2 + 2\sqrt{I_1I_2}\cos\phi \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we see that the resulting amplitude A and intensity I depends on the phase difference ϕ between the interfering waves. Where $\cos\phi = +1$, $A = A_{max} = A_1 + A_2$

or

$$I = I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2$$

and the waves are said to be interfering **constructively**.

Similarly where $\cos\phi = -1$,

$$A = A_{min} = A_1 \sim A_2$$

or

$$I = I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2$$

and the waves are said to be interfering **destructively**.

Note *Detailed discussion on interference will be done in the chapter of wave optics.*

Sample Example 15.2 *Two waves of equal frequencies have their amplitudes in the ratio of 3 : 5. They are superimposed on each other. Calculate the ratio of maximum and minimum intensities of the resultant wave.*

Solution Given,

$$\frac{A_1}{A_2} = \frac{3}{5}$$

$\therefore$

$$\sqrt{\frac{I_1}{I_2}} = \frac{3}{5} \quad (\text{as } I \propto A^2)$$

Maximum intensity is obtained, where

$$\cos\phi = 1 \text{ and } I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2$$

Minimum intensity is found, where

$$\cos\phi = -1 \text{ and } I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2$$

Hence,

$$\frac{I_{max}}{I_{min}} = \left(\frac{\sqrt{I_1} + \sqrt{I_2}}{\sqrt{I_1} - \sqrt{I_2}}\right)^2 = \left(\frac{\sqrt{\frac{I_1}{I_2}} + 1}{\sqrt{\frac{I_1}{I_2}} - 1}\right)^2$$

$$= \left(\frac{3/5 + 1}{3/5 - 1}\right)^2 = \frac{64}{4} = \frac{16}{1} \quad \textbf{Ans.}$$

EXERCISE *The ratio of intensities of two waves is 9 : 16. If these two waves interfere, then determine the ratio of the maximum and minimum possible intensities.*

Ans 49 : 1.

15.4 Reflection and Transmission of a Wave

Before starting the reflection and transmission of a wave from a boundary where two media separate each other, let us talk about a denser and a rare medium in reference to its behaviour towards a wave.

A medium is said to be denser (relative to the other) if the speed of wave in this medium is less than the speed of the wave in the other medium. Rather we can say speed of a wave in a denser medium is less than its speed in the rare medium. Thus, it is the speed of wave which decides whether the medium is denser or rare for that particular wave and

$$v_{denser} < v_{rare}$$

If a medium is denser for one type of wave then at the same time the same medium can be rare for the other type of wave. For example, water is denser for electromagnetic (light) waves compared to air, because the speed of electromagnetic waves is less in water than in air. At the same time, for sound wave water is a rare medium because speed of sound wave in water is more. The laws of refraction (or transmission from one medium to the other) and reflection remain the same *i.e.*,

Angle of incidence = angle of reflection (law of reflection) and a ray bends towards the normal it it travels from a rare medium to a denser medium and *vice-versa* (law of refraction).

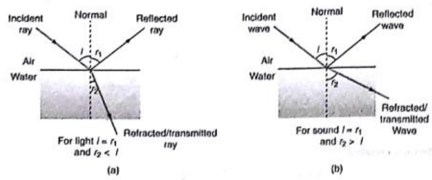


Fig. 15.4

Any type of wave is associated with the following physical quantities :

(i) speed of wave (v)
(ii) frequency (f), time period (T) and angular frequency (ω)
(iii) wavelength (λ) and wave number (k)
(iv) amplitude (A) and intensity (I) and
(v) phase (ϕ)

Now, let us see what happens to these physical quantities when they are either reflected or transmitted.

(i) Speed of wave (v) : Speed of a wave depends on the medium and some of its characteristics. For example, speed of a transverse wave on a stretched string depends on the tension T and its mass per unit length μ. In reflection medium and hence, its characteristics do not change. So, in reflection speed of wave does not change. On the other hand, in transmission, medium and hence the speed of wave do change.

(ii) Frequency (f), time period (T) and angular frequency (ω) : These three are related to each other by the relation

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Or we can say, if any one of them is known, other two can easily be obtained.

Frequency of the wave depends on the source from where wave originates. In reflection and transmission, since source does not change. Hence, none of the three change.

(iii) Wavelength (λ) and wave number (k) : These two are related to each other by the simple relation,

$$k = \frac{2\pi}{\lambda}$$

Further, $$\lambda = \frac{v}{f}$$

Here, v depends on medium and f on source.

So, if any one of them is changed, the wavelength will change. Therefore, in reflection λ and k do not change as medium and source both remain unchanged, while in transmission, as the medium is changed so λ and k do change.

(iv) Amplitude (A) and Intensity (I) : Expression for the intensity of a wave is,

$$I = \frac{1}{2}\rho\omega^2 A^2 v$$

$$\therefore \quad I \propto A^2$$

Intensity is the energy transmitted per unit area per unit time. When a wave is incident on a boundary (separating two media) part of it is reflected and part is transmitted. Hence, intensity and amplitude both change in reflection as well as transmission. Unless 100% reflection or 100% transmission is there.

(v) Phase : In transmission no phase change takes place. While in reflection phase change is zero if the wave is reflected from a rare medium and it is π if it is reflected from a denser medium. For example, in Fig. 15.4 (a), the reflected ray suffers a phase difference of π with the incident ray, because for electromagnetic wave water is denser, while in Fig. 15.4 (b) the reflected wave is in phase with the incident wave. Because for sound wave water is a rare medium.

The above results in tabular form are given below.

Table 15.1

Wave property	Reflection	Transmission (Refraction)
v	does not change	changes
f, T, ω	do not change	do not change
λ, k	do not change	change
A, I	change	change
ϕ	$\Delta\phi = 0$, from a rarer medium $\Delta\phi = \pi$, from a denser medium	does not change

Sample Example 15.3 *Two strings 1 and 2 are taut between two fixed supports (as shown in figure) such that the tension in both strings is same. Mass per unit length of 2 is more than that of 1. Explain which string is denser for a transverse travelling wave.*

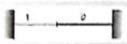


Fig. 15.5

Solution Speed of a transverse wave on a string

$$v = \sqrt{\frac{T}{\mu}}$$

or $$v \propto \frac{1}{\sqrt{\mu}}$$

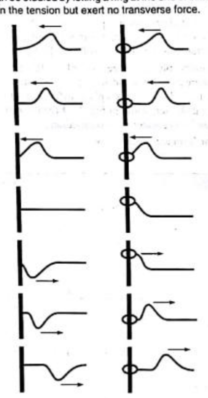
Now, $$\mu_2 > \mu_1 \quad \text{(given)}$$

$$\therefore \quad v_2 < v_1$$

i.e., medium 2 is denser and medium 1 is rarer.

• *Important Points in* REFLECTION AND TRANSMISSION

1. When a pulse travelling along a string reaches the end, it is reflected. If the end is fixed as in figure (a), the pulse returns inverted. This is because as the leading edge reaches the wall, the string pulls up the wall. According to Newton's third law, the wall will exert an equal and opposite force on the string at all instants. This force is therefore, directed first down and then up. It produces a pulse that is inverted but otherwise identical to the original.
   The motion of free end can be studied by letting a ring at the end of string sliding smoothly on the rod. The ring and rod maintain the tension but exert no transverse force.



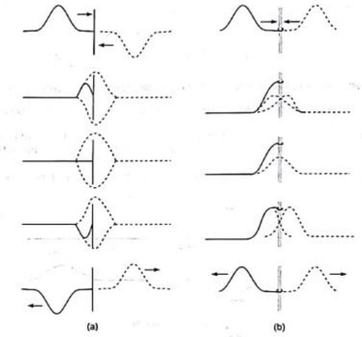
(a) (b)

Reflection of a wave pulse (a) at a fixed end of a string and (b) at a free end. Time increases from top to bottom in each figure.

Fig. 15.6

When a wave arrives at this free end, the ring slides along the rod. The ring reaches a maximum displacement. At this position the ring and the string come momentarily to rest as in the fourth drawing from the top in figure (b). But the string is stretched in this position, giving increased tension, so the free end of the string is pulled back down, and again a reflected pulse is produced, but now the direction of the displacement is the same as for the initial pulse.

2. The formation of the reflected pulse is similar to the overlap of two pulses travelling in opposite directions. The net displacement at any point is given by the principle of superposition.



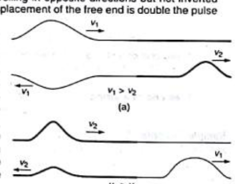
(a) (b)

Fig. 15.7

Fig. (a) shows two pulses with the same shape, one inverted with respect to the other, travelling in opposite directions. Because these two pulses have the same shape the net displacement of the point where the string is attached to the wall is zero at all times.

Fig. (b) shows two pulses with the same shape, travelling in opposite directions but not inverted relative to each other. Note that at one instant, the displacement of the free end is double the pulse height.

3. As a general case, a pulse may encounter the boundary between a light string and a heavy string. This results in partial reflection and partial transmission. Since the tensions are the same, the relative magnitudes of the wave velocities are determined by the mass densities. In figure (a), the pulse approaches from the light string. The heavy string behaves somewhat like a wall but it can move, and so part of the original pulse is transmitted to the heavy string. In figure (b) the pulse approaches from a heavy string. The light string offers little resistance and now approximates a free end. Consequently, the reflected pulse is not inverted.



(b)

Fig. 15.8

4. Suppose the equations of incident wave, reflected wave and transmitted wave are,

$$y_i = A_i \sin(kx - \omega t)$$

$$y_r = A_r \sin(kx + \omega t)$$

and

$$y_t = A_t \sin(kx - \omega t)$$

Then, $$A_r = \frac{v_2 - v_1}{v_1 + v_2} A_i \quad \text{and} \quad A_t = \frac{2v_2}{v_1 + v_2} A_i$$

Now let's take a look at what happens under certain special conditions.

Note *The above relations can be proved from energy conservation principle. For more details see solved example 4.*

$v_1 > v_2$: *When* $v_1 > v_2$, *i.e., medium 1 is rare and 2 is denser,* A_r = *negative and* $|A_r|$ *and* $|A_t|$ *both are individually less than* $|A_i|$. *Negative value of* A_r *indicates that the reflected wave suffers a phase change of* π.

$v_1 < v_2$: *If opposite is the case, i.e.,* $v_1 < v_2$, *both* A_r *and* A_t *are positive. Also,* $A_t > A_i$.

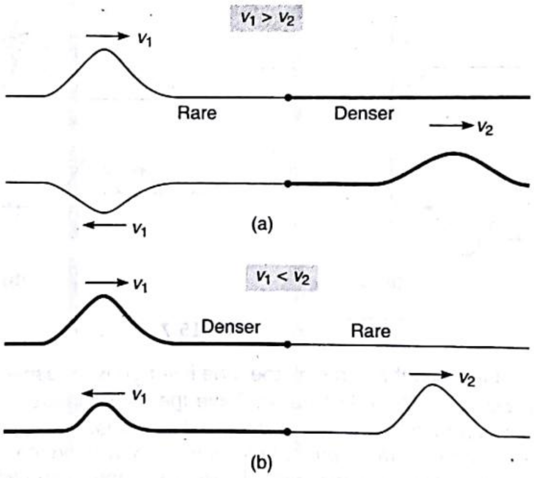


Fig. 15.9

The situations for $v_1 > v_2$ and for $v_1 < v_2$ are depicted in figures (a) and (b)

Fixed end of a string : Since the fixed end is equivalent to a string of infinite linear mass density, $v_2 = \sqrt{T/\mu_2} = 0$, and we obtain,

$$A_t = 0 \quad \text{and} \quad A_r = -A_i$$

Free end of a string : In this case $\mu_2 \to 0$, so $v_2 \to \infty$ and we can show that,

$$A_t = 0 \quad \text{and} \quad A_r = A_i$$

Sample Example 15.4 *A harmonic wave is travelling on string 1. At a junction with string 2 it is partly reflected and partly transmitted. The linear mass density of the second string is four times that of the first string, and that the boundary between the two strings is at* $x = 0$. *If the expression for the incident wave is,*

$$y_i = A_i \cos(k_1 x - \omega_1 t)$$

(a) What are the expressions for the transmitted and the reflected waves in terms of A_i, k_1 and ω_1?
(b) Show that the average power carried by the incident wave is equal to the sum of the average power carried by the transmitted and reflected waves.

Solution (a) Since $v = \sqrt{T/\mu}$, $T_2 = T_1$ and $\mu_2 = 4\mu_1$

we have,
$$v_2 = \frac{v_1}{2} \quad \ldots(i)$$

From Table 17.1, we can see that the frequency does not change, that is

$$\omega_1 = \omega_2 \quad \ldots(ii)$$

Also, because $k = \frac{\omega}{v}$, the wave numbers of the harmonic waves in the two strings are related by,

$$k_2 = \frac{\omega_2}{v_2} = \frac{\omega_1}{v_1/2} = 2\frac{\omega_1}{v_1} = 2k_1 \quad \ldots(iii)$$

The amplitudes are,
$$A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i = \left[\frac{2\,(v_1/2)}{v_1 + (v_1/2)}\right] = \frac{2}{3} A_i \quad \ldots(iv)$$

and
$$A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_i = \left[\frac{(v_1/2) - v_1}{v_1 + (v_1/2)}\right] A_i = -\frac{A_i}{3} \quad \ldots(v)$$

Now with Eqs. (ii), (iii) and (iv), the transmitted wave can be written as,

$$y_t = \frac{2}{3} A_i \cos\,(2k_1 x - \omega_1 t)$$ **Ans.**

Similarly the reflected wave can be expressed as,

$$y_r = -\frac{A_i}{3} \cos\,(k_1 x + \omega_1 t)$$
$$= \frac{A_i}{3} \cos\,(k_1 x + \omega_1 t + \pi)$$ **Ans.**

(b) The average power of a harmonic wave on a string is given by,

$$P = \frac{1}{2} \rho A^2 \omega^2 s v$$
$$= \frac{1}{2} A^2 \omega^2 \mu v \quad (\text{as } \rho s = \mu)$$

Now,
$$P_i = \frac{1}{2} \omega_1^2 A_i^2 \mu_1 v_1 \quad \ldots(vi)$$

$$P_t = \frac{1}{2}\omega_1^2 \left(\frac{2}{3} A_i\right)^2 (4\mu_1)\left(\frac{v_1}{2}\right) = \frac{4}{9}\omega_1^2 A_i^2 \mu_1 v_1 \quad \ldots(vii)$$

and
$$P_r = \frac{1}{2}\omega_2^2\left(-\frac{A_i}{3}\right)^2 (\mu_1)(v_1) = \frac{1}{18}\omega_1^2 A_i^2 \mu_1 v_1 \quad \ldots(viii)$$

From Eqs. (vi), (vii) and (viii), we can show that

$$P_i = P_t + P_r$$ **Hence proved.**

Sample Example 15.5 *A triangular pulse moving at 2 cm/s on a rope approaches an end at which it is free to slide on a vertical pole.*

(a) *Draw the pulse at $\frac{1}{2}$ s interval until it is completely reflected.*

(b) *What is the particle speed on the trailing edge at the instant depicted?*

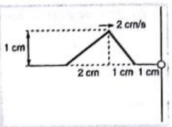
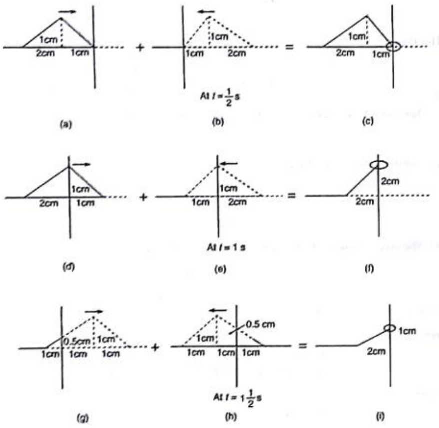


Fig. 15.10

Solution (a) Reflection of a pulse from a free boundary is really the superposition of two identical waves travelling in opposite direction. This can be shown as under.



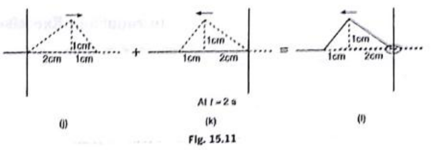


Fig. 15.11

In every $\frac{1}{2}$ s, each pulse (one real moving towards right and one imaginary moving towards left) travels a distance of 1 cm, as the wave speed is 2 cm/s.

(b) Particle speed, $v_P = |-v \text{ (slope)}|$

Here, v = wave speed = 2 cm/s and slope $= \frac{1}{2}$

$\therefore$ Particle speed = 1 cm/s **Ans.**

Sample Example 15.6 *Figure shows a rectangular pulse and triangular pulse approaching each other. The pulse speed is 0.5 cm/s. Sketch the resultant pulse at $t = 2$ s*

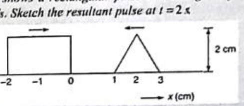


Fig. 15.12

Solution In 2 s each pulse will travel a distance of 1 cm.

The two pulses overlap between 0 and 1 cm as shown in figure. So, A_1 and A_2 can be added as shown in figure (c).

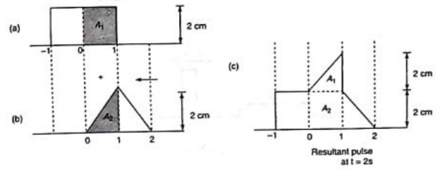


Fig. 15.13

Introductory Exercise 15.1

1. Two pulses of identical shape overlap such that the displacement of the rope is momentarily zero at all points, what happens to the energy at this time?
2. The pulse shown in figure has a speed of 10 cm/s.

Fig. 15.14

   (a) If the linear mass density of the right string is 0.25 that of the left string, at what speed does the transmitted pulse travel?

   (b) Compare the heights of the transmitted pulse and the reflected pulse to that of the incident pulse.

3. A wave $y_i = 0.3 \cos(2.0x - 40t)$ is travelling along a string toward a boundary at $x = 0$. Write expressions for the reflected waves if

   (a) the string has a fixed end at $x = 0$ and

   (b) the string has a free end at $x = 0$.

   Assume SI units.

4. The harmonic wave $y_i = (2.0 \times 10^{-3}) \cos \pi (2.0x - 50t)$ travels along a string toward a boundary at $x = 0$ with a second string. The wave speed on the second string is 50 m/s. Write expressions for reflected and transmitted waves. Assume SI units.
5. A string that is 10 cm long is fixed at both ends. At $t = 0$, a pulse travelling from left to right at 1 cm/s is 4.0 cm from the right end as shown in figure. Determine the next two times when the pulse will be at that point again. State in each case whether the pulse is upright or inverted.

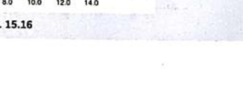


Fig. 15.15

6. Two pulses travelling in opposite directions along a string are shown for $t = 0$ in the figure. Plot the shape of the string at $t = 1.0, 2.0, 3.0, 4.0$ and 5.0 s respectively.

Fig. 15.16

15.5 Standing Waves

We now consider what happens when two harmonic waves of equal frequency and amplitude travel through a medium (say string) in opposite directions. Suppose the two waves are,

$$y_1 = A \sin(kx - \omega t)$$

and

$$y_2 = A \sin(kx + \omega t)$$

By the principle of superposition their sum is

$$y = y_1 + y_2$$

or

$$y = A[\sin(kx - \omega t) + \sin(kx + \omega t)]$$

By using the identity,

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right), \quad \text{we obtain}$$

$$y = 2A \sin(kx)\cos(\omega t) \qquad \text{...(i)}$$

This expression is different from wave representations that we have encountered upto now. It doesn't have the form $f(x \pm vt)$ or $f(ax \pm bt)$ and therefore, does not describe a travelling wave. Instead Eq. (i) represents what is known as a **standing wave.**

Eq. (i) can also be written as,

$$y = A(x)\cos \omega t \qquad \text{...(ii)}$$

where,

$$A(x) = 2A \sin kx \qquad \text{...(iii)}$$

This equation of a standing wave [Eq. (ii)] is really an equation of simple harmonic motion, whose amplitude [Eq. (iii)] is a function of x.

$$A(x) = 0, \quad \text{where} \quad \sin kx = 0$$

or

$$kx = 0, \pi, 2\pi, \ldots, n\pi \qquad (n = 0, 1, 2, \ldots)$$

Substituting $k = \frac{2\pi}{\lambda}$, we have

$$A(x) = 0 \quad \text{where,} \quad x = 0, \frac{\lambda}{2}, \lambda, \ldots, \frac{n\lambda}{2}$$

These are the points who never displace from their mean position. These are known as the **nodes** of the standing wave. The distance between two adjacent nodes is $\frac{\lambda}{2}$.

Further, from Eq. (iii), we can see that maximum value of $|A(x)|$ is $2A$, where

$$\sin kx = \pm 1$$

or

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, (2n-1)\frac{\pi}{2} \qquad (n = 0, 1, 2, \ldots)$$

or $$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \ldots, (2n-1)\frac{\lambda}{4}$$

These are the points of maximum displacement called **antinodes**. The distance between two adjacent antinodes is also $\frac{\lambda}{2}$, while that between a node and an antinode is $\frac{\lambda}{4}$.

In Figs. (a), (b), (c) and (d), the two travelling waves and their resultant standing waves are shown for four different times over one period T of the travelling waves. At $t = 0$, the two travelling waves have the same displacement everywhere and add to produce the standing wave shown. At $t = T/4$, each wave

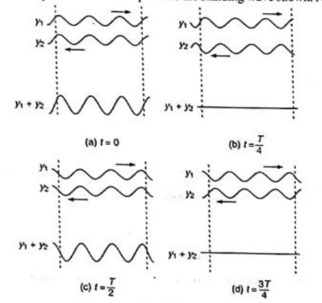


Fig. 15.17

has moved a distance of $\lambda/4$ in opposite directions, so they differ in phase by π rad and completely cancel. At $t = T/2$ they are again in phase but the positions of the crests and the troughs of the standing wave at $t = 0$ and $T/2$ have been interchanged. At $t = \frac{3T}{4}$, the travelling waves completely cancel once more. Finally at $t = T$, or after one period, the standing wave reassumes the shape it had at $t = 0$.

Note that in a travelling wave each particle vibrates with the same amplitude. However, in case of a standing wave it is not the same for different particles but varies with the location x of the particle. Energy is not transported in stationary waves it remains standing, although it alternates between vibrational kinetic energy and the elastic potential energy. We call the motion a wave motion because we can think of it as a superposition of waves travelling in opposite directions. We can equally regard the motion as an oscillation of the string as a whole, each particle oscillating with SHM of angular frequency ω and with an amplitude that depends on its location.

In the figure, we have shown how the energy associated with the oscillating string shifts back and forth between kinetic energy K and potential energy U during one cycle.

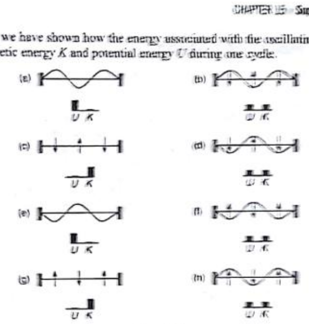


Fig. 15.18

Table 15.2 Comparison between travelling and stationary waves

S. No.	Travelling waves	Stationary waves
1.	These waves advance in a medium with definite velocity.	These waves remain stationary between two boundaries in the medium.
2.	In these waves, all particles of the medium oscillate with same frequency and amplitude.	In these waves, all particles except nodes oscillate with same frequency but different amplitudes. Amplitude is zero at nodes and maximum at antinodes.
3.	At any instant phase of vibration varies continuously from one particle to the other, i.e., phase difference between two particles can have any value between 0 and 2π.	At any instant the phase of all particles between two successive nodes is the same, but phase of particles on one side of a node is opposite to the phase of particles on the other side of the node, i.e., phase difference between any two particles can be either zero or π.
4.	In these waves, at no instant all the particles of the medium pass through their mean positions simultaneously.	In these waves all particles of the medium pass through their mean positions simultaneously twice in each time period.
5.	These waves transmit energy in the medium.	These waves do not transmit energy in the medium.

15.6 Normal Modes of a String

In an unbounded continuous medium, there is no restriction on the frequencies or wavelengths of the standing waves. However, if the waves are confined in space—for example, when a string is tied at both ends—standing waves can be set up for a discrete set of frequencies or wavelengths. Consider a string of definite length L rigidly held at both ends. When we set up a sinusoidal wave on such a string, it gets reflected from the fixed ends. By the superposition of two identical waves travelling in opposite directions, standing waves are established on the string. The only requirement we have to satisfy is that the end points be nodes as these points can not oscillate. They are permanently at rest. There may be any number of nodes in between or none at all, so that the wavelength associated with the standing waves can take on many different values. The distance between adjacent nodes is $\lambda/2$, so that in a string of length L there must be exactly an integral number n of half wavelengths, $\lambda/2$. That is,

$$\frac{n\lambda}{2} = L$$

or $$\lambda = \frac{2L}{n} \qquad (n = 1, 2, 3 \ldots)$$

But $\lambda = \frac{v}{f}$ and $v = \sqrt{\frac{T}{\mu}}$, so that the natural frequencies of oscillation of the system are:

$$f = n\left(\frac{v}{2L}\right) = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \qquad (n = 1, 2, 3 \ldots)$$

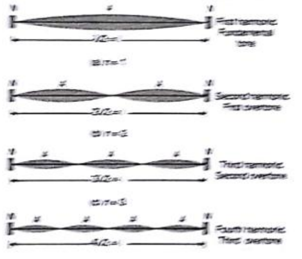


Fig. 15.19

The smallest frequency f_1 corresponds to the largest wavelength $(n = 1)$, $\lambda_1 = 2L$.

$$f_1 = \frac{v}{2L}$$

This is called the **fundamental frequency**. The other standing wave frequencies are,

$$f_2 = \frac{2v}{2L} = 2f_1$$

$$f_3 = \frac{3v}{2L} = 3f_1 \quad \text{and so on.}$$

These frequencies are called **harmonics**. Musicians sometimes call them **overtone**. Students are advised to remember these frequencies by name.

For example,

f_1 = fundamental tone or first harmonic

$f_2 = 2f_1$ = first overtone or second harmonic

$f_3 = 3f_1$ = second overtone or third harmonic and so on.

A **normal mode** of an oscillating system is a motion in which all particles of the system move sinusoidally with the same frequency. A harmonic oscillator which has only one oscillating particle, has only one normal mode and one characteristic frequency. By contrast, a string fixed at both ends has infinitely many normal modes, because it is made up of a very large (effectively infinite) number of particles. Fig. 15.19 shows the first four normal mode patterns and their associated frequencies and wavelengths. If we could displace a string so that its shape is the same as one of the normal-mode patterns and then release it, it would vibrate with the frequency of that mode.

15.7 Resonance

If the string (discussed in Art. 15.6) is set vibrating in any one of the normal modes and left to itself, the oscillations gradually die out. The motion is damped by dissipation of energy through the elastic supports at the ends and by the resistance of the air to the motion. We can pump energy into the system by applying a driving force. If the driving frequency is equal to any natural frequency of the string, the string will vibrate at that frequency with a larger amplitude. This phenomenon is called **resonance**. Because the string has a large number of natural frequencies, resonance can occur at many different frequencies.

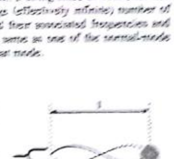


Fig. 15.20

A simple way to set up resonant waves is shown in figure. One prong of a tuning fork is attached to one end of the string. The string hangs over a pulley and a weight determines the tension in it.

The waves produced by the tuning fork travel down the string are inverted by the reflection at the pulley (fixed end) and travel back to the fork. Since the fork is vibrating with a small amplitude, it acts essentially as a fixed end as far as reflection is concerned.

Let us look at a particular wave crest generated at the left end of the string by the fork. It travels to the right end of the string, is reflected back, and is reflected again at the fork. Since it has been reflected twice, it has been inverted twice and now differs from the next wave crest coming from the fork only in that the first has already travelled a distance $2L$ where L is the length of the string. If this distance is exactly equal to wavelength λ, the twice-reflected wave will be in phase with the second wave and the two will interfere, constructively. The resultant wave has an amplitude twice that of either wave (assuming no loss by reflection). When this resulting wave travels the distance $2l$ to the pulley and back and has been reflected twice, it will exactly overlap the third wave generated by the tuning fork. Thus, each new wave is in phase with the waves reflected at the tuning fork and the amplitude continues to increase as the string absorbs energy from the tuning fork. Various damping effects put a limit on the maximum amplitude that can be reached. This maximum amplitude is much larger than that of the tuning fork. Such waves exist in both the directions and they interfere to give standing waves of large amplitude in the string. Thus, the tuning fork is in resonance with the string when the tuning fork frequency is such that the wavelength in the string equals twice the length of the string.

Resonance will occur if the distance $2l$ is any integer times the wavelength. Thus, the condition for resonance is,

$$2l = n\lambda$$

or

$$\lambda = \frac{2l}{n} \qquad (n = 1, 2, 3, \ldots)$$

In terms of the frequency of the waves, the condition for resonance is

$$f = \frac{v}{\lambda} = \frac{nv}{2l}$$

Substituting $v = \sqrt{\frac{T}{\mu}}$, we have

$$f = \frac{n}{2l}\sqrt{\frac{T}{\mu}}$$

This is the same what we obtained in Art 15.6. It is sometimes convenient to think of the resonance condition in terms of the time necessary for the first wave to travel to the end and back. Since this distance is $2l$, the time will be $\frac{2l}{v}$, where v is the wave speed. If this time equals the period of vibration of the fork, the first wave will add constructively to the second wave. Resonance will also result if this time equals any integer number of periods. Thus, we can write the resonance condition,

$$\frac{2l}{v} = nT = \frac{n}{f}$$

or

$$f = n\frac{v}{2l}$$

which is the same as that found by fitting an integral number of wavelengths into the distance $2l$.

Note 1. *By varying the tension in the string by changing the hanging weight we can adjust the wave speed such that the condition, $\frac{2l}{v} = nT$ is satisfied and resonance is obtained.*

2. *What happens when the frequency of the tuning fork is not equal to one of the frequencies discussed above? When the first wave has travelled a distance $2l$ and is reflected from the fork, it*

differs in phase from the wave being generated at the fork. It combines with this wave to produce a resultant new wave whose amplitude may be greater or smaller than that of the original wave, depending on the phase difference. When this resultant wave travels a distance $2l$ and is again reflected at the fork, it will differ in phase from the next wave generated. The phase difference of the incoming waves depends on the number of reflections. The interference will be constructive, in which case the amplitude of the wave on the string will increase or destructive, in which case the amplitude of the wave will decrease. On the average, the amplitude will not increase but remain that of the first wave generated, i.e., the amplitude of the resultant wave will equal to that of the tuning fork, which is small. The string does not absorb energy on the average. Only when the frequency of the tuning fork equals one of the frequencies given in Art. 15.7 or 15.6 will the waves add in phase and the amplitude builds up.

Sample Example 15.7 *The displacement of a standing wave on a string is given by*

$$y(x, t) = 0.4 \sin (0.5x) \cos (30t)$$

where x and y are in centimetres.

(a) Find the frequency, amplitude and wave speed of the component waves.

(b) What is the particle velocity at $x = 2.4$ cm at $t = 0.8$ s?

Solution (a) The given wave can be written as the sum of two component waves as

$$y(x, t) = 0.2 \sin (0.5x - 30t) + 0.2 \sin (0.5x + 30t)$$

That is the two component waves are

$$y_1(x, t) = 0.2 \sin (0.5x - 30t)$$

travelling in positive x-direction and

$$y_2(x, t) = 0.2 \sin (0.5x + 30t)$$

travelling in negative x-direction.

Now, $\omega = 30 \frac{\text{rad}}{\text{s}}$ and $k = 0.5 \text{ cm}^{-1}$

$\therefore$ frequency, $f = \frac{\omega}{2\pi} = \frac{15}{\pi}$ Hz **Ans.**

amplitude, $A = 0.2$ cm **Ans.**

and wave speed, $v = \frac{\omega}{k} = \frac{30}{0.5} = 60$ cm/s **Ans.**

(b) Particle velocity

$$v_P(x, t) = \frac{\partial y}{\partial t} = -12 \sin (0.5x) \sin (30t)$$

$$\therefore \quad v_P(x = 2.4 \text{ cm}, t = 0.8 \text{ s}) = -12 \sin (1.2) \sin (24)$$

$$= 10.12 \text{ cm/s}$$ **Ans.**

Sample Example 15.8 *A string fixed at both ends has consecutive standing wave modes for which the distances between adjacent nodes are 18 cm and 16 cm respectively.*

(a) What is the minimum possible length of the string?

(b) If the tension is 10 N and the linear mass density is 4 g/m, what is the fundamental frequency?

Solution (a)

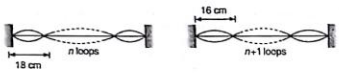


Fig. 15.21

Let l be the length of the string. Then

$$18n = l \qquad \ldots(i)$$

$$16(n+1) = l \qquad \ldots(ii)$$

From Eqs. (i) and (ii), wet get

$$n = 8$$

and $l = 144$ cm

Therefore, the minimum possible length of the string can be 144 cm. **Ans.**

(b) For fundamental frequency, $l = \lambda/2$

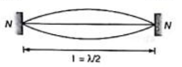


Fig. 15.22

or $\lambda = 2l = 288 \text{ cm} = 2.88 \text{ m}$

Speed of wave on the string

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{10}{4 \times 10^{-3}}} = 50 \text{ m/s}$$

$\therefore$ Fundamental frequency,

$$f = \frac{v}{\lambda} = \frac{50}{2.88} = 17.36 \text{ Hz}$$ **Ans.**

Introductory Exercise 15.2

1. A string vibrate according to the equation $y = 5 \sin \frac{\pi x}{3} \cos 40\pi t$
where x and y are in centimetres and t is in seconds.
(a) What is the speed of the component wave?
(b) What is the distance between the adjacent nodes?
(c) What is the velocity of the particle of the string at the position $x = 1.5$ cm when $t = \frac{9}{8}$ s?
2. If two waves differ only in amplitude and are propagated in opposite directions through a medium, will they produce standing waves. Is energy transported?
3. Figure shows different standing wave patterns on a string of linear mass density 4.0×10^{-2} kg/m under a tension of 100 N. The amplitude of antinodes is indicated in each figure. The length of the string is 2.0 m.

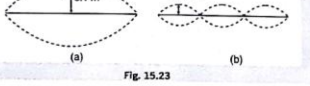


Fig. 15.23

(i) obtain the frequencies of the modes shown in figures (a) and (b).
(ii) Write down the transverse displacement y as a function of x and t for each mode. (Take the initial configuration of the wire in each mode to be as shown by the dark lines in the figure).

4. A 160 g rope 4 m long is fixed at one end and tied to a light string of the same length at the other end. Its tension is 400 N.
(a) What are the wavelengths of the fundamental and the first two overtones?
(b) What are the frequencies of these standing waves?
[**Hint :** In this case, fixed end is a node and the end tied with the light string is antinode.]
5. A string fastened at both ends has successive resonances with wavelengths of 0.54 m for the nth harmonic and 0.48 m for the $(n+1)$th harmonic.
(a) Which harmonics are these?
(b) What is the length of the string?
(c) What is the wavelength of the fundamental frequency?
6. If the frequencies of the second and fifth harmonics of a string differ by 54 Hz. What is the fundamental frequency of the string?
7. A wire is attached to a pan of mass 200 g that contains a 2.0 kg mass, as shown in the figure. When plucked, the wire vibrates at a fundamental frequency of 220 Hz. An additional unknown mass M is then added to the pan and a fundamental frequency of 260 Hz is detected. What is the value of M?

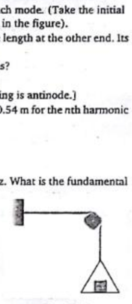


Fig. 15.24

8. A wire fixed at both ends is 1.0 m long and has a mass of 36 g. One of its oscillation frequencies is 250 Hz and the next higher one is 300 Hz.
(a) Which harmonics do these frequencies represent?
(b) What is the tension in the wire?

Extra Points

- Till now we have come across following three sets of equations

$$\left.\begin{array}{l} y = A \sin (\omega t \pm \phi) \\ y = A \cos (\omega t \pm \phi) \end{array}\right\} \text{SHM}$$

$$\left.\begin{array}{l} y = A \sin (kx \pm \omega t \pm \phi) \\ y = A \cos (kx \pm \omega t \pm \phi) \end{array}\right\} \text{Travelling wave}$$

$$\left.\begin{array}{lll} y = A \sin kx \cos \omega t & \text{or} & 2A \sin kx \cos \omega t \\ y = 2A \sin \omega t \cos kx & \text{or} & A \sin \omega t \cos kx \\ y = A \sin kx \sin \omega t & \text{or} & 2A \sin kx \sin \omega t \\ y = 2A \cos kx \cos \omega t & \text{or} & A \cos kx \cos \omega t \end{array}\right\} \text{Standing wave}$$

Note *In standing waves we have given four set of equations. The equation of standing wave basically depends on the component waves. Further if the maximum amplitude is $2A$, it means amplitude of travelling waves is A and if it is A, then amplitude of travelling waves is $\frac{A}{2}$.*

- Standing wave is an example of interference. Nodes means destructive interference and antinodes means constructive interference.
- Two identical waves moving in opposite directions along the string will still produce standing waves even if their amplitudes are unequal. This is the case when an incident travelling wave is only partially reflected from a boundary, the resulting superposition of two waves having different amplitudes and travelling in opposite directions gives a standing wave pattern of waves whose envelope is shown in figure.

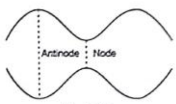


Fig. 15.25

The standing wave ratio (SWR) is defined as

$$\frac{A_{max}}{A_{min}} = \frac{A_i + A_r}{A_i - A_r}$$

For 100% reflection, SWR = ∞ and for no reflection, SWR = 1.

- The intensity of a travelling wave is given by

$$I = \frac{1}{2}\rho A^2\omega^2 v$$

i.e., $\quad I \propto A^2$

So, we can write, $\quad \frac{I_1}{I_2} = \left(\frac{A_1}{A_2}\right)^2 \quad$ if ρ, ω and v are same for two waves.

For example, when an incident travelling wave is partly reflected and partly transmitted from a boundary, we can write

$$\frac{I_i}{I_r} = \left(\frac{A_i}{A_r}\right)^2$$

as incident and reflected waves are in the same medium hence, they have same values of ρ and v. But we can not write

$$\frac{I_i}{I_t} = \left(\frac{A_i}{A_t}\right)^2$$

as they have different values of ρ and v.

Solved Examples

For JEE Main

Example 1 *Two wires are fixed on a sonometer. Their tensions are in the ratio 8 : 1, their lengths are in the ratio 36:35, the diameters are in the ratio 4:1 and densities are in the ratio 1:2. Find the frequencies of the beats produced if the note of the higher pitch has a frequency of 360 s.*

Solution Given $\quad \frac{T_1}{T_2} = \frac{8}{1}, \quad \frac{L_1}{L_2} = \frac{36}{35}$

$$\frac{D_1}{D_2} = \frac{4}{1}, \quad \frac{\rho_1}{\rho_2} = \frac{1}{2}$$

Let μ_1 and μ_2 be the linear densities.

$\therefore \quad \mu_1 = \pi \times \frac{D_1^2}{4} \times \rho_1 \quad$ and $\quad \mu_2 = \pi \times \frac{D_2^2}{4} \times \rho_2$

$\therefore \quad \frac{\mu_1}{\mu_2} = \left(\frac{D_1}{D_2}\right)^2 \times \frac{\rho_1}{\rho_2} = \left(\frac{4}{1}\right)^2 \times \frac{1}{2} = \frac{8}{1}$

$\therefore \quad \frac{f_1}{f_2} = \frac{L_2}{L_1} \times \sqrt{\frac{T_1}{T_2} \times \frac{\mu_2}{\mu_1}} = \frac{35}{36}\sqrt{\frac{8}{1} \times \frac{1}{8}} = \frac{35}{36}$

$$f_2 > f_1.$$

we have $\quad f_2 = 360 \quad \therefore \quad f_1 = 350$

$$\text{Beats} = f_2 - f_1 = 10/\text{s}.$$

Example 2 *The vibrations of a string of length 60 cm fixed at both ends are represented by the equation $y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi t)$ where x and y are in cm and t in seconds.*

(a) What is the maximum displacement of a point at x = 5 cm?
(b) Where are the nodes located along the string?
(c) What is the velocity of the particle at x = 7.5 cm and at t = 0.25 s?
(d) Write down the equations of the component wave whose superposition give the above wave.

Solution (a) At $x = 5$ cm the standing wave equation gives

$$y = 4\sin\left(\frac{5\pi}{15}\right)\cos(96\pi t)$$

$$= 4\sin\frac{\pi}{3}\cos(96\pi t) = 4 \times \frac{\sqrt{3}}{2}\cos(96\pi t)$$

$\therefore$ Maximum displacement $= 2\sqrt{3}$ cm

(b) The nodes are the points of permanent rest. Thus, they are those points for which

$$\sin\left(\frac{\pi x}{15}\right) = 0$$

i.e., $\quad \frac{\pi x}{15} = n\pi, \quad n = 0, 1, 2, 3, 4 \ldots$

$x = 15n$, *i.e.*, at $x = 0, 15, 30, 45$ and 60 cm

(c) The particle velocity is equal to

$$\left(\frac{\partial y}{\partial t}\right) = 4\sin\left(\frac{\pi x}{15}\right)(96\pi)(-\sin 96\pi t)$$

$$= -384\pi\sin\left(\frac{\pi x}{15}\right)\sin(96\pi t)$$

at $x = 7.5$ cm and $t = 0.25$ s, we get

$$\left(\frac{\partial y}{\partial t}\right) = -384\pi\sin\left(\frac{\pi x}{15}\right)\sin(96\pi t)$$

$$= -384\pi\sin\left(\frac{\pi}{2}\right)\sin(24\pi) = 0$$

(d) The equations of the component waves are :

$$y_1 = 2\sin\left(\frac{\pi x}{15} + 96\pi t\right) \quad \text{and} \quad y_2 = 2\sin\left(\frac{\pi x}{15} - 96\pi t\right)$$

as we can see that $y = y_1 + y_2$

Example 3 *In a stationary wave pattern that forms as a result of reflection of waves from an obstacle the ratio of the amplitude at an antinode and a node is $\beta = 1.5$. What percentage of the energy passes across the obstacle?*

Solution As we have studied in earlier when incident wave and reflected wave superimpose to produce stationary wave, the ratio of amplitudes at antinode and at node is given by,

$$\frac{A_{max}}{A_{min}} = \frac{A_i + A_r}{A_i - A_r}$$

This ratio is given as 1.5 or $\frac{3}{2}$.

$\therefore \quad \frac{A_i + A_r}{A_i - A_r} = \frac{3}{2}$

or $\quad \frac{1 + \frac{A_r}{A_i}}{1 - \frac{A_r}{A_i}} = \frac{3}{2}$

Solving this equation, we get $\quad \frac{A_r}{A_i} = \frac{1}{5}$

$\therefore \quad \frac{I_r}{I_i} = \left(\frac{A_r}{A_i}\right)^2 = \frac{1}{25}$

or $\quad I_r = 0.04\, I_i$

i.e., 4% of the incident energy is reflected or 96% energy passes across the obstacle. **Ans.**

Example 4 *From energy conservation principle prove the relations.*

$$A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)A_i \quad \text{and} \quad A_t = \left(\frac{2v_2}{v_1 + v_2}\right)A_i$$

Here symbols have their usual meanings.

Solution By conservation of energy, the average incident power equals the average reflected power plus the average transmitted power

or $\quad P_i = P_r + P_t$

$\therefore \quad \frac{1}{2}\mu_1\omega^2 A_i^2 v_1 = \frac{1}{2}\mu_1\omega^2 A_r^2 v_1 + \frac{1}{2}\mu_2\omega^2 A_t^2 v_2$

or $\quad \left(\frac{T}{v_1^2}\right)\omega^2 A_i^2 v_1 = \left(\frac{T}{v_1^2}\right)\omega^2 A_r^2 v_1 + \left(\frac{T}{v_2^2}\right)\omega^2 A_t^2 v_2$

or $\quad \frac{A_i^2}{v_1} = \frac{A_r^2}{v_1} + \frac{A_t^2}{v_2} \quad \ldots(i)$

Further $\quad A_i + A_r = A_t \quad \ldots(ii)$

Solving these two equations for A_r and A_t, we get

$$A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)A_i \quad \text{and} \quad A_t = \left(\frac{2v_2}{v_1 + v_2}\right)A_i$$

Hence proved.

Example 5 *A standing wave is formed by two harmonic waves, $y_1 = A\sin(kx - \omega t)$ and $y_2 = A\sin(kx + \omega t)$ travelling on a string in opposite directions. Mass density of the string is ρ and area of cross-section is s. Find the total mechanical energy between two adjacent nodes on the string.*

Solution The distance between two adjacent nodes is $\frac{\lambda}{2}$ or $\frac{\pi}{k}$.

$\therefore$ Volume of string between two nodes will be

$$V = (\text{area of cross-section})\,(\text{distance between two nodes})$$

$$= (s)\left(\frac{\pi}{k}\right)$$

Energy density (energy per unit volume) of a travelling wave is given by

$$u = \frac{1}{2}\rho A^2\omega^2$$

A standing wave is formed by two identical waves travelling in opposite directions. Therefore, the energy stored between two nodes in a standing wave

$$E = 2\,[\text{energy stored in a distance of } \frac{\pi}{k} \text{ of a travelling wave}]$$

$$= 2\,(\text{energy density})\,(\text{volume})$$

$$= 2\left(\frac{1}{2}\rho A^2\omega^2\right)\left(\frac{\pi s}{k}\right)$$

or $$E = \frac{\rho A^2\omega^2 \pi s}{k}$$ **Ans.**

Alternate Method : The equation of the standing wave

$$y = y_1 + y_2 = 2A \sin kx \cos \omega t = A(x) \cos \omega t$$

Here, $A(x) = 2A \sin kx$

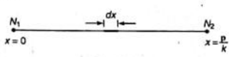


Fig. 15.26

i.e., first node is at $x = 0$ and the next node is at $x = \frac{\pi}{k}$. Let us take an element of length dx at distance x from N_1. Mass of this element is $dm\ (= \rho s dx)$. This can be treated as point mass. This element oscillates simple harmonically with angular frequency ω and amplitude $2A \sin kx$.

Hence, energy of this element

$$dE = \frac{1}{2}(dm)(2A \sin kx)^2(\omega^2)$$

$$dE = \frac{1}{2}(\rho s dx)(2A \sin kx)^2\omega^2$$

Integrating this with the limits from $x = 0$ to $x = \frac{\pi}{k}$, we get the same result.

Example 6 *A string of linear mass density 5.0×10^{-3} kg/m is stretched under a tension of 65 N between two rigid supports 60 cm apart.*

(a) If the string is vibrating in its second overtone so that the amplitude at one of its antinodes is 0.25 cm, what are the maximum transverse speed and acceleration of the string at antinodes?

(b) What are these quantities at a distance 5.0 cm from an node?

Solution (a) In second overtone

$$l = \frac{3\lambda}{2}$$

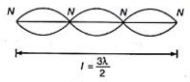


Fig. 15.27

or $$\lambda = \frac{2l}{3} = \frac{2 \times 60}{3} = 40\ \text{cm} = 0.4\ \text{m}$$

$\therefore$ $$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.4} = 5\pi\ \text{m}^{-1}$$

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{65}{5.0 \times 10^{-3}}} = 114\ \text{m/s}$$

$\therefore$ $$\omega = kv = 570\,\pi\ \text{rad/s}$$

Maximum transverse speed at antinode $= A_0\omega$

Here, A_0 = amplitude of antinode $= 0.25\ \text{cm} = 2.5 \times 10^{-3}$ m

$\therefore$ $$\text{Maximum speed} = (2.5 \times 10^{-3})(570\,\pi)\ \text{m/s}$$

$$= 4.48\ \text{m/s}$$ **Ans.**

$$\text{Maximum acceleration} = \omega^2 A_0$$

$$= (570\,\pi)^2(2.5 \times 10^{-3})\ \text{m/s}^2$$

$$= 8.0 \times 10^3\ \text{m/s}^2$$ **Ans.**

(b) At a distance x from the node the amplitude can be written as,

$$A = A_0 \sin kx = (2.5 \times 10^{-3}) \sin (5\pi x)\ \text{metre}$$

Here x is in metres.

Therefore, at $$x = 5.0\ \text{cm} = 5.0 \times 10^{-2}\ \text{m}$$

$$A = (2.5 \times 10^{-3}) \sin (5\pi \times 5.0 \times 10^{-2})$$

$$= 1.8 \times 10^{-3}\ \text{m}$$

$$\text{maximum speed} = A\omega$$

$$= (1.8 \times 10^{-3})(570\,\pi)\ \text{m/s}$$

$$= 3.22\ \text{m/s}$$ **Ans.**

and $$\text{maximum acceleration} = \omega^2 A$$

$$= (570\,\pi)^2(1.8 \times 10^{-3})\ \text{m/s}^2$$

$$= 5.8 \times 10^3\ \text{m/s}^2$$ **Ans.**

Example 7 *An aluminium wire of cross-sectional area 10^{-6} m^2 is joined to a steel wire of the same cross-sectional area. This compound wire is stretched on a sonometer pulled by a weight of 10 kg. The total length of the compound wire between the bridges is 1.5 m of which the aluminium wire is 0.6 m and the rest is steel wire. Transverse vibrations are set-up in the wire by using an external source of variable frequency. Find the lowest frequency of excitation for which the standing waves are formed such that the joint in the wire is a* **node**. *What is the total number of nodes at this frequency? The density of aluminium is 2.6×10^3 kg/m^3 and that of steel is 1.04×10^4 kg/m^3 ($g = 10$ m/s^2).*

Solution Let n_a loops are formed in aluminium wire and n_s in steel. Then

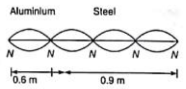


Fig. 15.28

$$f_a = f_s$$

or $$n_a\left(\frac{v_a}{2l_a}\right) = n_s\left(\frac{v_s}{2l_s}\right)$$

or $$\frac{n_a}{n_s} = \left(\frac{v_s}{v_a}\right)\left(\frac{l_a}{l_s}\right)$$

But $$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\rho s}} \propto \frac{1}{\sqrt{\rho}}$$

Therefore, $$\frac{v_s}{v_a} = \sqrt{\frac{\rho_a}{\rho_s}}$$

$\therefore$ $$\frac{n_a}{n_s} = \sqrt{\frac{\rho_a}{\rho_s}} \cdot \frac{l_a}{l_s}$$

Substituting the values, we have

$$\frac{n_a}{n_s} = \sqrt{\frac{2.6 \times 10^3}{1.04 \times 10^4}} \cdot \frac{0.6}{0.9} = \frac{1}{3}$$

i.e., at lowest frequency, one loop is formed in aluminium wire and three loops are formed in steel wire as shown in figure.

$\therefore$ $$f_{\min} = n_a\left(\frac{v_a}{2l_a}\right) = \frac{n_a\sqrt{T/\rho_a s}}{2l_a}$$

$$= \frac{(1)\sqrt{100/(2.6 \times 10^3 \times 10^{-6})}}{2 \times 0.6} \qquad (T = mg = 100\ \text{N})$$

or $$f_{\min} = 163.4\ \text{Hz}$$ **Ans.**

Total number of nodes are **five** as shown in figure. **Ans.**

Note *In such type of problems nature of junction will be known to us. Then we have to equate frequencies on the two sides. By equating the frequencies, we find $\frac{n_1}{n_2}$. Suppose this comes out to be 0.4. Write it, $\frac{n_1}{n_2} = \frac{2}{5}$.*

At lowest oscillation frequency 2 loops are formed on side 1 and 5 on side 2. At next higher frequency 4 loops will be formed on side 1 and 10 on side 2 and so on.

Example 8 *Find the resultant amplitude and phase of a point at which N sinusoidal waves interfere. All the waves have same amplitude A and their phases increase in arithmetic progression of common difference ϕ.*

Solution The diagram for their sum is shown in figure for $N = 6$. The resultant amplitude is A_R. The apex angle of every isosceles triangle is ϕ. So, the angle subtended by the resultant is $N\phi$. Since the heads of the vectors are all at the same distance r from the apex of the diagram.

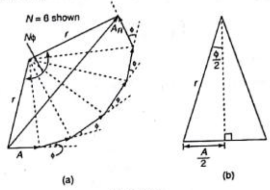


Fig. 15.29

From figure (a), $$\frac{A_R}{2} = r \sin \frac{N\phi}{2} \qquad \text{...(i)}$$

and from figure (b), $$\frac{A}{2} = r \sin \frac{\phi}{2} \qquad \text{...(ii)}$$

Dividing Eq. (i) by (ii), we get $$\frac{A_R}{A} = \frac{\sin \frac{N\phi}{2}}{\sin \frac{\phi}{2}}$$

or $$A_R = A\left(\frac{\sin \frac{N\phi}{2}}{\sin \frac{\phi}{2}}\right)$$ **Ans.**

Note *Suppose $N = 2$ and $\phi = 90°$, then*

$$A_R = \sqrt{2}A$$

Similarly we can also check the above result for other special cases.

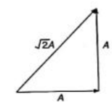


Fig. 15.30

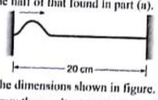
EXERCISES

For JEE Main

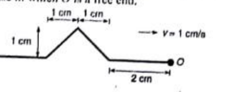
Subjective Questions

Reflection and Transmission of a Wave: Principle of Superposition and Interference

1. Two waves are travelling in the same direction along a stretched string. The waves are 90° out of phase. Each wave has an amplitude of 4.0 cm. Find the amplitude of the resultant wave.
2. Two wires of different densities are soldered together end to end then stretched under tension T. The wave speed in the first wire is twice that in the second wire.
   (a) If the amplitude of incident wave is A, what are amplitudes of reflected and transmitted waves?
   (b) Assuming no energy loss in the wire, find the fraction of the incident power that is reflected at the junction and fraction of the same that is transmitted.
3. A wave is represented by
$$y_1 = 10\cos(5x + 25t)$$
where x is measured in metres and t in seconds. A second wave for which
$$y_2 = 20\cos\left(5x + 25t + \frac{\pi}{3}\right)$$
interferes with the first wave. Deduce the amplitude and phase of the resultant wave.
4. Two waves passing through a region are represented by
$$y_1 = (1.0\,\text{cm})\sin[(3.14\,\text{cm}^{-1})x - (157\,\text{s}^{-1})t]$$
and
$$y_2 = (1.5\,\text{cm})\sin[(1.57\,\text{cm}^{-1})x - (314\,\text{s}^{-1})t]$$
Find the displacement of the particle at $x = 4.5$ cm at time $t = 5.0$ ms.
5. A string of length 20 cm and linear mass density 0.4 g/cm is fixed at both ends and is kept under a tension of 16 N. A wave pulse is produced at $t = 0$ near an end as shown in figure which travels towards the other end. (a) When will the string have the shape shown in the figure again ? (b) Sketch the shape of the string at a time half of that found in part (a).



6. A wave pulse on a string has the dimensions shown in figure. The wave speed is $v = 1$ cm/s.
   (a) If point O is a fixed end, draw the resultant wave on the string at $t = 3$ s and $t = 4$ s.
   (b) Repeat part (a) for the case in which O is a free end.



Stationary Waves and Normal Modes of a String

7. Two sinusoidal waves travelling in opposite directions interfere to produce a standing wave described by the equation
$$y = (1.5\,\text{m})\sin(0.400x)\cos(200t)$$
where x is in metres and t is in seconds. Determine the wavelength, frequency and speed of the interfering waves.
8. Two sinusoidal waves combining in a medium are described by the equations
$$y_1 = (3.0\,\text{cm})\sin\pi(x + 0.60t)$$
and
$$y_2 = (3.0\,\text{cm})\sin\pi(x - 0.60t)$$
where x is in centimetres and t is in seconds. Determine the maximum displacement of the medium at
   (a) $x = 0.250$ cm,
   (b) $x = 0.500$ cm and
   (c) $x = 1.50$ cm.
   (d) Find the three smallest values of x corresponding to antinodes.
9. A standing wave is formed by the interference of two travelling waves, each of which has an amplitude $A = \pi$ cm, angular wave number $k = (\pi/2)$ per centimetre.
   (a) Calculate the distance between two successive antinodes.
   (b) What is the amplitude of the standing wave at $x = 0.50$ cm from a node ?
10. Find the fundamental frequency and the next three frequencies that could cause a standing-wave pattern on a string that is 30.0 m long, has a mass per unit length of 9.00×10^{-3} kg/m and is stretched to a tension of 20.0 N.
11. A string vibrates in its first normal mode with a frequency of 220 vibrations/s. The vibrating segment is 70.0 cm long and has a mass of 1.20 g.
   (a) Find the tension in the string.
   (b) Determine the frequency of vibration when the string vibrates in three segments.
12. A 60.0 cm guitar string under a tension of 50.0 N has a mass per unit length of 0.100 g/cm. What is the highest resonance frequency of the string that can be heard by a person able to hear frequencies upto 20,000 Hz ?
13. A wire having a linear density of 0.05 g/cm is stretched between two rigid supports with a tension of 450 N. It is observed that the wire resonates at a frequency of 420 Hz. The next higher frequency at which the same wire resonates is 490 Hz. Find the length of the wire.
14. The vibrations from an 800 Hz tuning fork set up standing waves in a string clamped at both ends. The wave speed in the string is known to be 400 m/s for the tension used. The standing wave is observed to have four antinodes. How long is the string ?
15. A string vibrates in 4 segments to a frequency of 400 Hz.
   (a) What is its fundamental frequency ?
   (b) What frequency will cause it to vibrate into 7 segments?
16. A sonometer wire has a total length of 1 m between the fixed ends. Where should the two bridges be placed below the wire so that the three segments of the wire have their fundamental frequencies in the ratio 1 : 2 : 3 ?

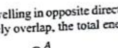
17. A guitar string is 90 cm long and has a fundamental frequency of 124 Hz. Where should it be pressed to produce a fundamental frequency of 186 Hz?
18. Adjacent antinodes of a standing wave on a string are 15.0 cm apart. A particle at an antinode oscillates in simple harmonic motion with amplitude 0.850 cm and period 0.0750 s. The string lies along the $+x$-axis and is fixed at $x = 0$.
   (a) Find the displacement of a point on the string as a function of position and time.
   (b) Find the speed of propagation of a transverse wave in the string.
   (c) Find the amplitude at a point 3.0 cm to the right of an antinode.
19. A 1.50 m long rope is stretched between two supports with a tension that makes the speed of transverse waves 48.0 m/s. What are the wavelength and frequency of
   (a) the fundamental ? (b) the second overtone ? (c) the fourth harmonic ?
20. A thin, taut string tied at both ends and oscillating in its third harmonic has its shape described by the equation $y(x,t) = (5.60\,\text{cm})\sin[(0.0340\,\text{rad/cm})x]\sin[(50.0\,\text{rad/s})t]$, where the origin is at the left end of the string, the x-axis is along the string and the y-axis is perpendicular to the string.
   (a) Draw a sketch that shows the standing wave pattern.
   (b) Find the amplitude of the two travelling waves that make up this standing wave.
   (c) What is the length of the string ?
   (d) Find the wavelength, frequency, period and speed of the travelling wave.
   (e) Find the maximum transverse speed of a point on the string.
   (f) What would be the equation $y(x, t)$ for this string if it were vibrating in its eighth harmonic ?
21. A wire with mass 40.0 g is stretched so that its ends are tied down at points 80.0 cm apart. The wire vibrates in its fundamental mode with frequency 60.0 Hz and with an amplitude at the antinodes of 0.300 cm.
   (a) What is the speed of propagation of transverse wave in the wire ?
   (b) Compute the tension in the wire.
   (c) Find the maximum transverse velocity and acceleration of particles in the wire.

Objective Questions

Single Correct Option

1. When tension of a string is increased by 2.5 N, the initial frequency is altered in the ratio of 3 : 2. The initial tension in the string is
   (a) 6 N (b) 5 N (c) 4 N (d) 2 N
2. The lengths of two wires are in the ratio 1 : 2, their tensions are in the ratio 1 : 2 and their diameters are in the ratio 1 : 3. The ratio of the notes they emit when sounded together by the same source is
   (a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $2\sqrt{3}$ (d) $3\sqrt{2}$
3. If f_1, f_2 and f_3 are the fundamental frequencies of three segments into which a string is divided, then the original fundamental frequency f_0 of the whole string is
   (a) $f_0 = f_1 + f_2 + f_3$
   (b) $\frac{1}{f_0} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$
   (c) $\frac{1}{\sqrt{f_0}} = \frac{1}{\sqrt{f_1}} + \frac{1}{\sqrt{f_2}} + \frac{1}{\sqrt{f_3}}$
   (d) None of these

4. Two identical harmonic pulses travelling in opposite directions in a taut string approach each other. At the instant when they completely overlap, the total energy of the string will be



   (a) zero
   (b) partly kinetic and partly potential
   (c) purely kinetic
   (d) purely potential
5. Two transverse waves A and B superimpose to produce a node at $x = 0$. If the equation of wave A is $y = a\cos(kx + \omega t)$, then the equation of wave B is
   (a) $+a\cos(kx - \omega t)$ (b) $-a\cos(kx + \omega t)$ (c) $-a\cos(kx - \omega t)$ (d) $+a\cos(\omega t - kx)$
6. In a standing wave, node is a point of
   (a) maximum strain (b) maximum pressure (c) maximum density (d) All of these
7. In a stationary wave system, all the particles of the medium
   (a) have zero displacement simultaneously at some instant
   (b) have maximum displacement simultaneously at some instant
   (c) are at rest simultaneously at some instant
   (d) All the above
8. Three one dimensional mechanical waves in an elastic medium is given as :
$$y_1 = 3A\sin(\omega t - kx),$$
$$y_2 = A\sin(\omega t - kx + \pi)$$
and
$$y_3 = 2A\sin(\omega t + kx)$$
are superimposed with each other. The maximum displacement amplitude of the medium particle would be
   (a) $4A$ (b) $3A$ (c) $2A$ (d) A
9. A string is stretched so that its length is increased by $\frac{1}{\eta}$ of its original length. The ratio of fundamental frequency of transverse vibration to that of fundamental frequency of longitudinal vibration will be
   (a) $\eta : 1$ (b) $1 : \eta$ (c) $\sqrt{\eta} : 1$ (d) $1 : \sqrt{\eta}$
10. A string of length 1 m and linear mass density 0.01 kg/m is stretched to a tension of 100 N. When both ends of the string are fixed, the three lowest frequencies for standing wave are f_1, f_2 and f_3. When only one end of the string is fixed, the three lowest frequencies for standing wave are n_1, n_2 and n_3. Then
   (a) $n_3 = 5n_1 = f_3 = 125$ Hz
   (b) $f_3 = 5f_1 = n_2 = 125$ Hz
   (c) $f_3 = n_2 = 3f_1 = 150$ Hz
   (d) $n_2 = \frac{f_1 + f_2}{2} = 75$ Hz
11. In a standing wave on a string
   (a) In one time period all the particles are simultaneously at rest twice
   (b) All the particles must be at their positive extremes simultaneously once in a time period
   (c) All the particles may be at their positive extremes simultaneously once in a time period
   (d) All the particles are never at rest simultaneously

12. Three resonant frequencies of string with both rigid ends are 90, 150 and 210 Hz. If the length of the string is 80 cm, what is the speed of the transverse wave in the string?
(a) 45 m/s (b) 75 m/s (c) 48 m/s (d) 80 m/s

13. Three coherent waves having amplitudes 12 mm, 6 mm and 4 mm arrive at a given point with successive phase difference of $\pi/2$. Then the amplitude of the resultant wave is
(a) 7 mm (b) 10 mm (c) 5 mm (d) 4.8 mm

14. For a certain stretched string, three consecutive resonance frequencies are observed as 105, 175 and 245 Hz respectively. Then the fundamental frequency is
(a) 30 Hz (b) 45 Hz (c) 35 Hz (d) None of these

15. A sonometer wire has a length 114 cm between two fixed ends. Where should two bridges be placed so as to divide the wire into three segments whose fundamental frequencies are in the ratio 1 : 3 : 4
(a) $l_1 = 72$ cm, $l_2 = 24$ cm, $l_3 = 18$ cm
(b) $l_1 = 60$ cm, $l_2 = 40$ cm, $l_3 = 14$ cm
(c) $l_1 = 52$ cm, $l_2 = 30$ cm, $l_3 = 32$ cm
(d) $l_1 = 65$ cm, $l_2 = 30$ cm, $l_3 = 19$ cm

16. The frequency of a sonometer wire is f. The frequency becomes $f/2$ when the mass producing the tension is completely immersed in water and on immersing the mass in a certain liquid, frequency becomes $f/3$. The relative density of the liquid is
(a) 1.32 (b) 1.67 (c) 1.41 (d) 1.18

17. A string of length 1.5 m with its two ends clamped is vibrating in fundamental mode. Amplitude at the centre of the string is 4 mm. Distance between the two points having amplitude 2 mm is
(a) 1 m (b) 75 cm (c) 60 cm (d) 50 cm

18. A man generates a symmetrical pulse in a string by moving his hand up and down. At $t = 0$, the point in his hand moves downwards from mean position. The pulse travels with speed 3 m/s on the string and his hand passes 6 times in each second from the mean position. Then the point on the string at a distance 3 m will reach its upper extreme first time at $t =$
(a) 1.25 s (b) 1 s (c) 11/12 s (d) $\frac{23}{24}$ s

19. Among two interfering sources, let S_1 be ahead of the phase by 90° relative to S_2. If an observation point P is such that $PS_1 - PS_2 = 1.5\lambda$, the phase difference between the waves from S_1 and S_2 reaching P is
(a) 3π (b) $\frac{5\pi}{2}$ (c) $\frac{7\pi}{2}$ (d) 4π

20. A wire having a linear density 0.1 kg/m is kept under a tension of 490 N. It is observed that it resonates at a frequency of 400 Hz. The next higher frequency is 450 Hz. Find the length of the wire
(a) 0.4 m (b) 0.7 m
(c) 0.6 m (d) 0.49 m

21. A string 1 m long is drawn by a 300 Hz vibrator attached to its end. The string vibrates in three segments. The speed of transverse waves in the string is equal to
(a) 100 m/s (b) 200 m/s
(c) 300 m/s (d) 400 m/s

22. If $\lambda_1, \lambda_2, \lambda_3$ are the wavelength of the waves giving resonance to the fundamental, first and second overtone modes respectively in a string fixed at both ends. The ratio of the wavelengths $\lambda_1 : \lambda_2 : \lambda_3$ is
(a) 1 : 2 : 3 (b) 1 : 3 : 5 (c) $1 : \frac{1}{2} : \frac{1}{3}$ (d) $1 : \frac{1}{3} : \frac{1}{5}$

23. The period of oscillations of a point is 0.04 s and the velocity of propagation of oscillation is 300 m/s. The difference of phases between the oscillations of two points at distances 10 m and 16 m respectively from the source of oscillations is
(a) 2π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) π

24. A transverse wave described by an equation $y = 0.02 \sin (x + 30t)$ where x and t are in metre and second, is traveling along a wire of area of cross-section 1 mm² and density 8000 kg/m³. What is the tension in the string?
(a) 20 N (b) 7.2 N (c) 30 N (d) 14.4 N

25. A string vibrates in 5 segments to a frequency of 480 Hz. The frequency that will cause it to vibrate in 2 segments will be
(a) 96 Hz (b) 192 Hz (c) 1200 Hz (d) 2400 Hz

26. A wave travels on a light string. The equation of the waves is $Y = A \sin (kx - \omega t + 30°)$. It is reflected from a heavy string tied to an end of the light string at $x = 0$. If 64% of the incident energy is reflected then the equation of the reflected wave is
(a) $Y = 0.8 A \sin (kx - \omega t + 30° + 180°)$ (b) $Y = 0.8 A \sin (kx + \omega t + 30° + 180°)$
(c) $Y = 0.8 A \sin (kx - \omega t - 30°)$ (d) $Y = 0.8 A \sin (kx + \omega t + 30°)$

27. The tension, length, diameter and density of a string B are double than that of another string A. Which of the following overtones of B is same as the fundamental frequency of A?
(a) 1st (b) 2nd (c) 3rd (d) 4th

Passage : (Q. 28 - Q. 30)

Incident wave $y = A \sin\left(ax + bt + \frac{\pi}{2}\right)$ is reflected by an obstacle at $x = 0$ which reduces intensity of reflected wave by 36%. Due to superposition a resulting wave consist of standing wave and travelling wave given by

$$y = -1.6 \sin ax \sin bt + cA \cos (bt + ax)$$

where A, a, b and c are positive constants.

28. Amplitude of reflected wave is
(a) $0.6A$ (b) $0.8A$ (c) $0.4A$ (d) $0.2A$

29. Value of c is
(a) 0.2 (b) 0.4 (c) 0.6 (d) 0.8

30. Position of second antinode is
(a) $x = \frac{\pi}{3a}$ (b) $x = \frac{3\pi}{a}$ (c) $x = \frac{3\pi}{2a}$ (d) $x = \frac{2\pi}{3a}$

31. If the tension in a stretched string fixed at both ends is increased by 21% the fundamental frequency is found to change by 15 Hz. Then the
(a) original frequency is 150 Hz
(b) velocity of propagation of the transverse wave along the string increases by 5%
(c) velocity of propagation of the transverse wave along the string increases by 10%
(d) fundamental wavelength on the string does not change

32. For interference to take place
(a) sources must be coherent (b) sources must have same amplitude
(c) waves should travel in opposite directions (d) sources must have same frequency

33. Regarding stationary waves choose the correct options.
(a) This is an example of interference (b) Amplitudes of waves may be different
(c) Particles at nodes are always at rest (d) Energy is conserved

34. When a wave travels from a denser to rarer medium
(a) speed of wave increases (b) wavelength of wave decreases
(c) amplitude of wave increases (d) there is no change in phase angle

35. A wire is stretched and fixed at two ends. Transverse stationary waves are formed in it. It oscillates in its third overtone mode. The equation of stationary wave is

$$Y = A \sin kx \cos \omega t$$

Choose the correct options.
(a) Amplitude of constituent waves is $\frac{A}{2}$ (b) The wire oscillates in three loops
(c) The length of the wire is $\frac{4\pi}{k}$ (d) Speed of stationary wave is $\frac{\omega}{k}$

36. Which of the following equations can form stationary waves?
(i) $Y = A \sin (\omega t - kx)$ (ii) $Y = A \cos (\omega t - kx)$
(iii) $Y = A \sin (\omega t + kx)$ (iv) $Y = A \cos (\omega t + kx)$
(a) (i) and (ii) (b) (i) and (iii) (c) (iii) and (iv) (d) (ii) and (iv)

37. Two waves

$$y_1 = A \sin (\omega t - kx) \text{ and } y_2 = A \sin (\omega t + kx)$$

superimpose to produce a stationary wave then,
(a) $x = 0$ is a node (b) $x = 0$ is an antinode
(c) $x = \frac{\pi}{k}$ is a node (d) $\pi = \frac{2\pi}{k}$ is an antinode

For JEE Advanced

Assertion and Reason

Directions : *Choose the correct option.*
(a) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*
(b) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*
(c) *If **Assertion** is true, but the **Reason** is false.*
(d) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** Two waves $y_1 = A \sin (\omega t + kx)$ and $y_2 = A \cos (\omega t - kx)$ are superimposed, then $x = 0$, becomes a node.
Reason : At node net displacement due to two waves should be zero.

2. **Assertion :** Stationary waves are so called because particles are at rest in stationary waves.
Reason : They are formed by the superposition of two identical waves travelling in opposite directions.

3. **Assertion :** When a wave travels from a denser medium to rarer medium amplitude of oscillation increases.
Reason : In denser medium speed of wave is less compared to a rarer medium.

4. **Assertion :** A wire is stretched and then fixed at two ends. It oscillates in its second overtone mode. There are total four nodes and three antinodes.
Reason : In second overtone mode, length of wire should be $l = \frac{3\lambda}{2}$, where λ is wavelength.

5. **Assertion :** If we see the oscillations of a stretched wire at higher overtone mode, frequency of oscillation increases but wavelength decreases.
Reason : From $v = f\lambda$, $\lambda \propto \frac{1}{f}$ as $v =$ constant.

6. **Assertion :** Standing waves are formed when amplitudes of two constituent waves are equal.
Reason : At any point net displacement at a given time is resultant of displacement of constituent waves.

7. **Assertion :** In a standing wave $x = 0$ is a node. Then total mechanical energy lying between $x = 0$ and $x = \frac{\lambda}{4}$ is not equal to the energy lying between $x = \frac{\lambda}{4}$ and $x = \frac{\lambda}{2}$.
Reason : In standing waves different particles oscillate with different amplitudes.

8. **Assertion :** Ratio of maximum intensity and minimum intensity in interference is 25 : 1. Hence amplitude ratio of two waves should be 3 : 2.
Reason : $\frac{I_{max}}{I_{min}} = \left(\frac{A_1 + A_2}{A_1 - A_2}\right)^2$

9. **Assertion :** Three waves of equal amplitudes interfere at a point. Phase difference between two successive waves is $\frac{\pi}{2}$. Then resultant intensity is same as the intensity due to individual wave.
Reason : For interference to take place sources must be coherent.

10. Assertion : For two sources to be coherent phase difference between two waves at all points should be same.

Reason : Two different light sources are never coherent.

Match the Columns

1. In the figure shown mass per unit length of string-2 is nine times that of string-1. Tension in both the strings is same. A transverse wave is incident at the boundary as shown. Part of wave is reflected in medium-1 and part is transmitted to medium-2. Match the following two columns.

Column I	Column II
(a) $\lvert A_1/A_2 \rvert$	(p) 9
(b) v_1/v_2	(q) 1
(c) I_1/I_2	(r) 3
(d) P_1/P_2	(s) data insufficient

Here : A is amplitude, v the speed of wave, I the intensity and P the power. Abbreviation 1 is used for reflected wave and 2 for transmitted wave.

2. Transverse waves are produced in a stretched wire. Both ends of the string are fixed. Let us compare between second overtone mode (in numerator) and fifth harmonic mode (in denominator). Match the following two columns.

Column I	Column II
(a) Frequency ratio	(p) 2/3
(b) Number of nodes ratio	(q) 4/5
(c) Number of antinodes ratio	(r) 3/5
(d) Wavelength ratio	(s) 5/3

3. A wave travels from a denser medium to rarer medium, then match the following two columns.

Column I	Column II
(a) speed of wave	(p) will increase
(b) wavelength of wave	(q) will decrease
(c) amplitude of wave	(r) will remain constant
(d) frequency of wave	(s) may increase or decrease

4. Two waves of same amplitude and same intensity interfere at one point. Phase difference between them is θ. Match the following two columns.

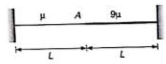
Column I	Column II
(a) Resultant amplitude for $\theta = 60°$	(p) 2 times
(b) Resultant amplitude for $\theta = 120°$	(q) 3 times
(c) Resultant intensity for $\theta = 90°$	(r) 4 times
(d) Resultant intensity for $\theta = 0°$	(s) None

5. A wire is stretched and fixed at its two ends. Its second overtone frequency is 210 Hz. Then match the following two columns.

Column I	Column II
(a) Fundamental frequency	(p) 210 Hz
(b) Third harmonic frequency	(q) 350 Hz
(c) Third overtone frequency	(r) 280 Hz
(d) Second harmonic frequency	(s) None

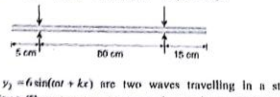
Subjective Questions

1. Three pieces of string, each of length L, are joined together end-to-end, to make a combined string of length $3L$. The first piece of string has mass per unit length μ_1, the second piece has mass per unit length $\mu_2 = 4\mu_1$ and the third piece has mass per unit length $\mu_3 = \mu_1/4$.
   (a) If the combined string is under tension F, how much time does it take a transverse wave to travel the entire length $3L$? Give your answer in terms of L, F and μ_1.
   (b) Does your answer to part (a) depend on the order in which the three piece are joined together? Explain.
2. In a stationary wave that forms as a result of reflection of waves from an obstacle, the ratio of the amplitude at an antinode to the amplitude at node is 6. What percentage of energy is transmitted?
3. A standing wave $y = a \sin kx \cos \omega t$ is maintained in a homogeneous rod with cross-sectional area S and density ρ. Find the total mechanical energy confined between the sections corresponding to the adjacent nodes.
4. A string of mass per unit length μ is clamped at both ends such that one end of the string is at $x = 0$ and the other is at $x = l$. When string vibrates in fundamental mode amplitude of the mid-point of the string is a and tension in the string is T. Find the total oscillation energy stored in the string.
5. A long wire PQR is made by joining two wires PQ and QR of equal radii. PQ has a length 4.8 m and mass 0.06 kg. QR has length 2.56 m and mass 0.2 kg. The wire PQR is under a tension of 80 N. A sinusoidal wave pulse of amplutde 3.5 cm is sent along the wire PQ from the end P. No power is dissipated during the propagation of the wave pulse. Calculate :
   (a) The time taken by the wave pulse to reach the other end R.
   (b) The amplitude of the reflected and transmitted wave pulse after the incident wave pulse crosses the joint Q.
6. A light string is tied at one end to fixed support and to a heavy string of equal length L at the other end as shown in figure. Mass per unit length of the strings are μ and 9μ and the tension is T. Find the possible values of frequencies such that point A is a node/antinode.



7. A string fixed at both ends is vibrating in the lowest possible mode of vibration for which a point at quarter of its length from one end is a point of maximum displacement. The frequency of vibration in this mode is 100 Hz. What will be the frequency emitted when it vibrates in the next mode such that this point is again a point of maximum displacement ?

8. Sources separated by 20 m vibrate according to the equation $y_1 = 0.06 \sin \pi t$ and $y_2 = 0.02 \sin \pi t$. They send out waves along a rod with speed 3 m/s. What is the equation of motion of a particle 12 m from the first source and 8 m from the second. y_1, y_2 are in m?
9. Three component sinusoidal waves progressing in the same direction along the same path have the same period but their amplitudes are A, $\frac{A}{2}$ and $\frac{A}{3}$. The phases of the variation at any position x on their path at time $t = 0$ are 0, $-\frac{\pi}{2}$ and $-\pi$ respectively. Find the amplitude and phase of the resultant wave.
10. A metal rod of length 1 m is clamped at two points as shown in the figure. Distance of the clamp from the two ends are 5 cm and 15 cm respectively. Find the minimum and next higher frequency of natural longitudinal oscillation of the rod. Given that Young's modulus of elasticity and density of aluminium are $Y = 1.6 \times 10^{11}$ Nm^{-2} and $\rho = 2500$ kgm^{-3} respectively.



11. $y_1 = 8 \sin(\omega t - kx)$ and $y_2 = 6 \sin(\omega t + kx)$ are two waves travelling in a string of area of cross-section s and density ρ. These two waves are superimposed to produce a standing wave.
   (a) Find the energy of the standing wave between two consecutive nodes.
   (b) Find the total amount of energy crossing through a node per second.

ANSWERS

Introductory Exercise 15.1

1. All is kinetic 2. (a) 20 cm/s (b) $\frac{A_t}{A_i} = \frac{4}{3}$, $\frac{A_r}{A_i} = \frac{1}{3}$
3. (a) $y(x, t) = 0.3 \cos(2.0x + 40t + \pi)$ (b) $y(x, t) = 0.3 \cos(2.0x + 40t)$ SI Units
4. (a) $(6.67 \times 10^{-4}) \cos \pi(2.0x + 50t)$ (b) $(2.67 \times 10^{-3}) \cos \pi(1.0x - 50t)$ SI Units
5. 8 s, inverted, 20 s upright

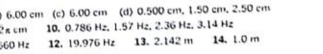
Introductory Exercise 15.2

1. (a) 120 cm/s (b) 3 cm (c) zero 2. Yes, Yes
3. (i) Fundamental 12.5 Hz, third harmonic 37.5 Hz
   (ii) (a) $y = 0.1 \sin \frac{\pi x}{2} \sin 25\pi t$ (b) $y = 0.04 \sin \frac{3\pi x}{2} \sin 75\pi t$
4. (a) 16 m, 5.33 m, 3.2 m (b) 6.25 Hz, 18.75 Hz, 31.25 Hz
5. (a) 8th and 9th (b) 2.16 m (c) 4.32 m
6. 18 Hz 7. 0.873 kg 8. (a) 5th and 6th (b) 360 N

For JEE Main

Subjective Questions

1. 5.66 cm 2. (a) $-\frac{A}{3}, \frac{2}{3}A$ (b) $\frac{1}{9}, \frac{8}{9}$ 3. 24.46 cm, $(5x + 25t + 0.714)$ rad 4. $\frac{-1}{2\sqrt{2}}$ cm
5. (a) 0.02 s 6. (a) (i) (ii) (b) (i) (ii)



7. (a) 15.7 m, 31.8 Hz, 500 m/s
8. (a) 4.24 cm (b) 6.00 cm (c) 6.00 cm (d) 0.500 cm, 1.50 cm, 2.50 cm
9. (a) 2 cm (b) $\sqrt{2}\pi$ cm 10. 0.786 Hz, 1.57 Hz, 2.36 Hz, 3.14 Hz
11. (a) 163 N (b) 660 Hz 12. 19.976 Hz 13. 2.142 m 14. 1.0 m
15. (a) 100 Hz (b) 700 Hz
16. One bridge at 6/11 m from one end and the other at 2/11 m from the other end.
17. The string should be pressed at 60 cm from one end.
18. (a) $y(x, t) = (0.85 \text{ cm}) \sin\left[\frac{2\pi x}{0.3 \text{ m}}\right] \sin\left[\frac{2\pi t}{0.075 \text{ s}}\right]$ (b) 4.00 m/s (c) 0.688 cm
19. (a) 3.0 m, 16.0 Hz (b) 1.0 m, 48.0 Hz (c) 0.75m, 64.0 Hz
20. (b) 2.80 cm (c) 277 cm (d) 185 cm, 7.9 Hz 0.126 sec, 1470 cm/s (e) 280 cm/s
    (f) $y(x, t) = (5.6 \text{ cm}) \sin[(0.0907 \text{ rad/cm})x] \sin[(133 \text{ rad/s})t]$
21. (a) 96.0 m/s (b) 461 N (c) 1.13 m/s, 426.4 m/s^2

Objective Questions

1.(d)	2.(d)	3.(b)	4.(c)	5.(c)	6.(d)	7.(d)	8.(a)	9.(d)	10.(d)
11.(a)	12.(c)	13.(b)	14.(c)	15.(a)	16.(d)	17.(a)	18.(c)	19.(b)	20.(b)
21.(b)	22.(c)	23.(d)	24.(b)	25.(b)	26.(b)	27.(c)	28.(b)	29.(a)	30.(c)
31.(a,c,d)	32.(a,d)	33.(a,b,d)	34.(a,c,d)	35.(a,c)	36.(b,d)	37.(b,d)			

For JEE Advanced

Assertion and Reason

1. (d) 2. (d) 3. (b) 4. (b) 5. (a) 6. (d) 7. (a) 8. (a) 9. (b) 10. (d)

Match the Columns

1. (a) → q (b) → r (c) → s (d) → r
2. (a) → r (b) → p (c) → r (d) → s
3. (a) → p (b) → p (c) → p (d) → r
4. (a) → s (b) → s (c) → p (d) → r
5. (a) → s (b) → p (c) → r (d) → s

Subjective Questions

1. (a) $\frac{7L}{2}\sqrt{\frac{\mu_1}{F}}$ (b) No 2. 49% 3. $\frac{\pi S\rho\omega^2 a^2}{4k}$ 4. $\frac{\pi^2 a^2 T}{4l}$ 5. (a) 0.14 s (b) −1.5 cm, 2.0 cm
6. $\frac{f}{2}, f, \frac{3f}{2}$... etc., when A is a node $\frac{3}{4}f, \frac{5}{4}f, \frac{7}{4}f$... etc., when A is an antinode. Here, $f = \frac{1}{L}\sqrt{\frac{T}{\mu}}$ 7. 300 Hz 8. $0.05 \sin \pi t - 0.0173 \cos \pi t$
9. $\frac{5}{6}A, -\tan^{-1}\left(\frac{3}{4}\right)$ 10. 40 kHz, 120 kHz 11. (a) $\frac{50\pi}{k}\rho\omega^2 s$ (b) $\frac{2\rho\omega^3 s}{k}$

16

SOUND WAVES

Chapter Contents

16.1 Introduction

Of all the mechanical waves that occur in nature, the most important in our everyday lives are longitudinal waves in a medium, usually air, called **sound waves.** The reason is that the human ear is tremendously sensitive and can detect sound waves even of very low intensity. The human ear is sensitive to waves in the frequency range from about 20 to 20,000 Hz called the **audible range**, but we also use the term sound for similar waves with frequencies above **(ultrasonic)** and below **(infrasonic)** the range of human hearing. Our main concern in this chapter is with sound waves in air, but sound can travel through any gas, liquid or solid.

In this chapter we will discuss several important properties of sound waves, including **pressure wave** and **density wave.** We will find that superposition of two sound waves differing slightly in frequency causes a phenomenon called **beats.** When a source of sound or a listener moves through the air, the listener may hear a different frequency than the one emitted by the source. This is the **Doppler effect.**

16.2 Displacement Wave, Pressure Wave and Density Wave

Upto this point we have described mechanical waves primarily in terms of displacement, however, a description of sound waves in terms of pressure fluctuations is often more appropriate, largely because the ear is primarily sensitive to changes in pressure.

Harmonic sound waves can be generated by a tuning fork or loudspeaker that is vibrating with simple harmonic motion. The vibrating source causes the air molecules next to it to oscillate with simple harmonic motion about their equilibrium position. These molecules collide with neighbouring molecules, causing them to oscillate, thereby, propagating the sound wave. As we have discussed in chapter-16, a sinusoidal wave equation $y(x, t)$ travelling in positive x-direction can be written as,

$$y(x, t) = A \sin (kx - \omega t)$$

which gives the instantaneous displacement y of a particle in the medium at position x at time t.

Note that in a longitudinal wave the displacements are parallel to the direction of travel of the wave. So, x and y are measured parallel to each other, not perpendicular as in a transverse wave. The amplitude A is the maximum displacement of a particle in a medium from its equilibrium position.

These displacements are along the direction of the motion of the wave lead to **variations in the density and pressure** of air.

Hence, a sound wave can also be described in terms of variations of pressure or density at various points. The pressure fluctuations are of the order of 1 Pa $(= 1\ N/m^2)$, whereas atmospheric pressure is about 10^5 Pa. In a sinusoidal sound wave in air, the pressure fluctuates above and below atmospheric pressure P_0 in a sinusoidal variation with the same frequency as the motions of the air particles. Similarly the density of air also vibrates sinusoidally above and below its normal level. So, we can express a sound wave either in terms of $y(x, t)$ or $\Delta P(x, t)$ or $\Delta \rho(x, t)$. All the three equations are related to one another. For example, amplitude of pressure variation $(\Delta P)_m$ is related to amplitude of displacement A by the equation,

$$\Delta P_m = BAk$$

where B is the bulk modulus of the medium. Moreover pressure or density wave is 90° out of phase with the displacement wave, *i.e.*, when the displacement is zero, the pressure and density changes are either

maximum or minimum and when the displacement is a maximum or minimum, the pressure and density changes are zero. Now, let us find the relation between them.

Relation between displacement wave and pressure wave

Figure shows a harmonic displacement wave moving through air contained in a long tube of cross sectional area S.

The volume of gas that has a thickness Δx in the horizontal direction is $V_i = S \cdot \Delta x$. The change in volume ΔV is $S \cdot \Delta y$, where Δy is the difference between the value of y at $x + \Delta x$ and the value of y at x. From the definition of bulk modulus, the pressure variation in the gas is

$$\Delta P = -B \frac{\Delta V}{V_i}$$

$$= -B \left(\frac{S \cdot \Delta y}{S \cdot \Delta x} \right)$$

$$= -B \left(\frac{\Delta y}{\Delta x} \right)$$

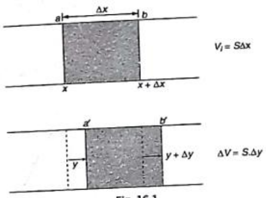


Fig. 16.1

As Δx approaches zero, the ratio $\frac{\Delta y}{\Delta x}$ becomes $\frac{\partial y}{\partial x}$ (The partial derivative indicates that we are interested in the variation of y with position at a fixed time).

Therefore,
$$\Delta P = -B \frac{\partial y}{\partial x} \quad \text{...(i)}$$

So, this is the equation which relates the displacement equation with the pressure equation. Suppose the displacement equation is

$$y = A \cos (kx - \omega t) \quad \text{...(ii)}$$

Then,
$$\frac{\partial y}{\partial x} = -kA \sin (kx - \omega t) \quad \text{...(iii)}$$

and from Eqs. (i) and (iii), we find that

$$\Delta P = BAk \sin (kx - \omega t) = (\Delta P)_m \sin (kx - \omega t) \quad \text{...(iv)}$$

Here,
$$(\Delta P)_m = BAk$$

is the amplitude of pressure variation. From Eqs. (ii) and (iv), we see that pressure equation is 90° out of phase with displacement equation. When the displacement is zero, the pressure variation is either maximum or minimum and *vice-versa*. Fig. 16.2 (a) shows displacement from equilibrium of air molecules in a harmonic sound wave versus position at some instants. Points x_1 and x_3 are points of zero displacement. Now, refer Fig. 16.2 (b) : Just to the left of x_1, the displacement.

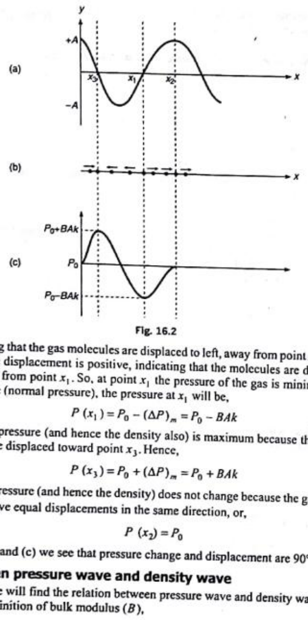


Fig. 16.2

is negative, indicating that the gas molecules are displaced to left, away from point x_1 at this instant. Just to the right of x_1, the displacement is positive, indicating that the molecules are displaced to the right, which is again away from point x_1. So, at point x_1 the pressure of the gas is minimum. So, if P_0 is the atmospheric pressure (normal pressure), the pressure at x_1 will be,

$$P(x_1) = P_0 - (\Delta P)_m = P_0 - BAk$$

At point x_3, the pressure (and hence the density also) is maximum because the molecules on both sides of that point are displaced toward point x_3. Hence,

$$P(x_3) = P_0 + (\Delta P)_m = P_0 + BAk$$

At point x_2 the pressure (and hence the density) does not change because the gas molecules on both sides of that point have equal displacements in the same direction, or,

$$P(x_2) = P_0$$

From figures (a) and (c) we see that pressure change and displacement are 90° out of phase.

Relation between pressure wave and density wave

In this section we will find the relation between pressure wave and density wave.
According to definition of bulk modulus (B),

$$B = \left(-\frac{dP}{dV/V} \right) \quad \text{...(i)}$$

Further,
$$\text{volume} = \frac{\text{mass}}{\text{density}}$$

or
$$V = \frac{m}{\rho}$$

or
$$dV = -\frac{m}{\rho^2} \cdot d\rho = -\frac{V}{\rho} \cdot d\rho$$

or
$$\frac{dV}{V} = -\frac{d\rho}{\rho}$$

Substituting in Eq. (i), we get

$$d\rho = \frac{\rho\,(dP)}{B} = \frac{dP}{v^2} \qquad \left(\frac{\rho}{B} = \frac{1}{v^2} \right)$$

Or this can be written as,

$$\Delta \rho = \frac{\rho}{B} \cdot \Delta P = \frac{1}{v^2} \cdot \Delta P$$

So, this relation relates the pressure equation with the density equation. For example, if

$$\Delta P = (\Delta P)_m \sin (kx - \omega t)$$

then
$$\Delta \rho = (\Delta \rho)_m \sin (kx - \omega t)$$

where,
$$(\Delta \rho)_m = \frac{\rho}{B} (\Delta P)_m = \frac{(\Delta P)_m}{v^2}$$

Thus, density equation is in phase with the pressure equation and this is 90° out of phase with the displacement equation.

Sample Example 16.1 *(a) What is the displacement amplitude for a sound wave having a frequency of 100 Hz and a pressure amplitude of 10 Pa?*
(b) The displacement amplitude of a sound wave of frequency 300 Hz is 10^{-7} m. What is the pressure amplitude of this wave? Speed of sound in air is 340 m/s and density of air is 1.29 kg/m^3.

Solution (a)
$$(\Delta P)_m = BAk$$

Here,
$$k = \frac{\omega}{v} = \frac{2\pi f}{v}$$

and
$$B = \rho v^2 \qquad \left(\text{as } v = \sqrt{\frac{B}{\rho}} \right)$$

$\therefore$
$$(\Delta P)_m = (\rho v^2)(A)\left(\frac{2\pi f}{v} \right)$$

$\therefore$
$$A = \frac{(\Delta P)_m}{2\pi v \rho f} \quad \text{...(i)}$$

Substituting the values, we have

$$A = \frac{(10)}{2 \times 3.14 \times 100 \times 1.29 \times 340}$$

$$= 3.63 \times 10^{-5}\ \text{m} \qquad \textbf{Ans.}$$

(b) From Eq. (i)

$$(\Delta P)_m = 2\pi f \rho v A$$

Substituting the values, we have

$$(\Delta P)_m = 2 \times 3.14 \times 300 \times 1.29 \times 340 \times 10^{-7}$$

$$= 8.26 \times 10^{-2}\ \text{N/m}^2 \qquad \textbf{Ans.}$$

Introductory Exercise 16.1

1. Calculate the bulk modulus of air from the following data for a sound wave of wavelength 35 cm travelling in air. The pressure at a point varies between $(10^5 \pm 14)$ Pa and the particles of the air vibrate in SHM of amplitude 5.5×10^{-6} m.
2. Find the minimum and maximum wavelengths of sound in water that is in the audible range for an average human ear. Speed of sound in water is 1450 m/s.
3. A typical loud sound wave with a frequency of 1 kHz has a pressure amplitude of about 10 Pa
   (a) At $t = 0$, the pressure is a maximum at some point x_1. What is the displacement at that point at $t = 0$?
   (b) What is the maximum value of the displacement at any time and place? Take the density of air to be $1.29\ \text{kg/m}^3$ and speed of sound in air is 340 m/s.
4. The pressure variation in a sound wave in air is given by
   $$\Delta P = 12 \sin(8.18x - 2700t + \pi/4)\ \text{N/m}^2$$
   Find the displacement amplitude. Density of air $= 1.29\ \text{kg/m}^3$.

16.3 Speed of a Longitudinal Wave

First we calculate the speed at which a longitudinal pulse propagates through a fluid. We will apply Newton's second law to the motion of an element of the fluid and from this we derive the wave equation.

Consider a fluid element 'ab' confined to a tube of cross sectional area S as shown in figure. The element has a thickness Δx. We assume that the equilibrium pressure of the fluid is P_0. Because of the disturbance, the section 'a' of the element moves a distance y from its mean position and section 'b' moves a distance $y + \Delta y$ to a new position b'. The pressure on the left side of the element becomes $P_0 + \Delta P_1$ and on the right side it becomes $P_0 + \Delta P_2$. If ρ is the equilibrium density, the mass of the element is $\rho S \Delta x$. (When the element moves its mass does not change, even though its volume and density do change).

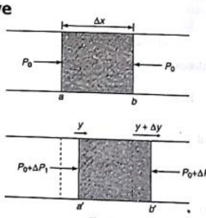


Fig. 16.3

The net force acting on the element is,

$$F = (\Delta P_1 - \Delta P_2) S$$

and its acceleration is

$$a = \frac{\partial^2 y}{\partial t^2}$$

Thus, Newton's second law applied to the motion of the element is

$$(\Delta P_1 - \Delta P_2) S = \rho S \Delta x \frac{\partial^2 y}{\partial t^2} \qquad \text{...(i)}$$

Next we divide both sides by Δx and note that in the limit as $\Delta x \to 0$, we have $(\Delta P_1 - \Delta P_2)/\Delta x \to \partial P/\partial x$, Eq. (i) then takes the form

$$-\frac{\partial P}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2} \qquad \text{...(ii)}$$

The excess pressure ΔP may be written as

$$\Delta P = -B \frac{\partial y}{\partial x}$$

When this is used in Eq. (ii), we obtain the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{B} \cdot \frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial t^2} = \frac{B}{\rho} \frac{\partial^2 y}{\partial x^2}$$

Comparing this equation with the wave equation

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

We have

$$v = \sqrt{\frac{B}{\rho}} \qquad \text{(speed of longitudinal wave in a fluid)}$$

This is the speed of longitudinal waves within a gas or a **liquid.**

When a longitudinal wave propagates in a solid rod or bar, the rod expands sideways slightly when it is compressed longitudinally and the speed of a longitudinal wave in a rod is given by

$$v = \sqrt{\frac{Y}{\rho}} \qquad \text{(speed of a longitudinal wave in a solid rod)}$$

16.4 Sound Waves in Gases

In the preceding section we derived equation

$$v = \sqrt{B/\rho}$$

for the speed of longitudinal waves in a fluid of bulk modulus B and density ρ. We use this to find the speed of sound in an ideal gas. The bulk modulus of the gas, however depends on the process. When a wave travels through a gas, are the compressions and expansions adiabatic, or is there enough heat conduction between adjacent layers of gas to maintain a nearly constant temperature throughout?

Because thermal conductivities of gases are very small, it turns out that for ordinary sound frequencies (20 Hz to 20,000 Hz) propagation of sound is very nearly adiabatic. Thus, in the above equation, we use the **adiabatic bulk modulus (B_s)**, which is given by

$$B_s = \gamma P$$

Here, γ is the ratio of molar heat capacity C_P/C_V. Thus,

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

We can get a useful alternative form of the above equation by substituting density ρ of an ideal gas

$$\rho = \frac{PM}{RT}$$

where R is the gas constant, M the molecular mass and T is the absolute temperature combining all these equations, we can write

$$v = \sqrt{\frac{\gamma RT}{M}} \qquad \text{(speed of sound in an ideal gas)}$$

Effect of temperature, pressure and humidity on the speed of sound in air

(i) **Effect of temperature :** From the equation,

$$v = \sqrt{\frac{\gamma RT}{M}}$$

We can see that,

$$v \propto \sqrt{T}$$

or

$$\frac{v_1}{v_2} = \sqrt{\frac{T_1}{T_2}}$$

At STP, the temperature is 0°C or 273 K. If the speed of sound at 0°C is v_0, its value at t°C will satisfy

$$\frac{v_t}{v_0} = \sqrt{\frac{273 + t}{273}} = \left(1 + \frac{t}{273}\right)^{\frac{1}{2}} = 1 + \frac{t}{546}$$

$\therefore$

$$v_t = v_0 \left(1 + \frac{t}{546}\right)$$

If the speed of sound in air at 0°C (v_0) be taken 332 m/s, then

$$v_t = 332\left(1 + \frac{t}{546}\right)$$

or

$$v_t = 332 + 0.61t$$

Thus, the velocity of sound in air increases roughly by 0.61 m/s per degree centigrade rise in temperature.

(ii) **Effect of pressure :** From the formula for the speed of sound in a gas $v = \sqrt{\frac{\gamma P}{\rho}}$, it appears that $v \propto \sqrt{P}$. But actually it is not so. Because

$$\frac{P}{\rho} = \frac{RT}{M} = \text{constant at constant temperature.}$$

That is, at constant temperature if P changes then ρ also changes in such a way that P/ρ remains constant. Hence, in the formula $v = \sqrt{\frac{\gamma P}{\rho}}$, the value of P/ρ does not change when P changes. From this it is clear that if the temperature of the gas remains constant, then there is no effect of the pressure change on the speed of sound.

(iii) **Effect of humidity :** The density of moist air (*i.e.*, air mixed with water-vapour) is less than the density of dry air. This is because in moist air heavy dust particles settle down due to condensation. Hence, density of air gets decreased thus increasing the speed of sound. Therefore, assuming the value of γ for moist air same as for dry air (which is actually slightly less than that for dry air) it is clear from the formula $v = \sqrt{\frac{\gamma P}{\rho}}$ that the speed of sound in moist air is slightly greater than in dry air.

Sample Example 16.2 *Calculate the speed of longitudinal waves in the following gases at 0° C and 1 atm ($= 10^5$ Pa):*

(a) oxygen for which the bulk modulus is 1.41×10^5 Pa and density is $1.43\ \text{kg/m}^3$.

(b) helium for which the bulk modulus is 1.7×10^5 Pa and the density is $0.18\ \text{kg/m}^3$.

Solution (a)

$$v_{O_2} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{1.41 \times 10^5}{1.43}} = 314\ \text{m/s} \qquad \textbf{Ans.}$$

(b)

$$v_{He} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{1.7 \times 10^5}{0.18}} = 972\ \text{m/s} \qquad \textbf{Ans.}$$

Sample Example 16.3 *Find speed of sound in hydrogen gas at 27° C. Ratio C_P/C_V for H_2 is 1.4. Gas constant $R = 8.31$ J/mol-K.*

Solution

$$v = \sqrt{\frac{\gamma RT}{M}}$$

Here, $T = 27 + 273 = 300$ K, $\gamma = 1.4$, $R = 8.31$ J/mol-K, $M = 2 \times 10^{-3}$ kg/mol

$\therefore$

$$v = \sqrt{\frac{1.4 \times 8.31 \times 300}{2 \times 10^{-3}}} = 1321\ \text{m/s} \qquad \textbf{Ans.}$$

Note *In the above formula, put $M = 2 \times 10^{-3}$ kg/mol. Don't put it 2 kg.*

Sample Example 16.4 *At what temperature will the speed of sound in hydrogen be the same as in oxygen at 100°C? Molar masses of oxygen and hydrogen are in the ratio 16 : 1.*

Solution

$$v=\sqrt{\frac{\gamma RT}{M}}$$

$$v_{H_2}=v_{O_2}$$

$$\therefore \quad \sqrt{\frac{\gamma_{H_2}RT_{H_2}}{M_{H_2}}}=\sqrt{\frac{\gamma_{O_2}RT_{O_2}}{M_{O_2}}}$$

$$\gamma_{H_2}=\gamma_{O_2} \qquad \text{(as both are diatomic)}$$

$$\therefore \quad T_{H_2}=\left(\frac{M_{H_2}}{M_{O_2}}\right)(T_{O_2})=\left(\frac{1}{16}\right)(100+273)$$

$$=23.31\text{ K}$$

$$\approx -249.7°\text{C}$$

Ans.

Introductory Exercise 16.2

1. Calculate the temperature at which the velocity of sound in air is double its velocity at 0°C.
2. Calculate the difference in the speeds of sound in air at − 3° C, 60 cm pressure of mercury and 30° C, 75 cm pressure of mercury. The speed of sound in air at 0°C is 332 m/s.
3. In a liquid with density 900 kg/m³, longitudinal waves with frequency 250 Hz are found to have wavelength 8.0 m. Calculate the bulk modulus of the liquid.
4. Calculate the speed of sound in oxygen at 273 K.

16.5 Sound Intensity

Travelling sound waves, like all other travelling waves, transfer energy from one region of space to another. We define the **intensity** of a wave (denoted by I) to be the time average rate at which energy is transported by the wave, per unit area across a surface perpendicular to the direction of propagation. We have already derived an expression for the intensity of a mechanical wave in Art. 14.6, which is

$$I=\frac{1}{2}\rho A^2\omega^2 v \qquad \text{...(i)}$$

For a sound wave,

$$(\Delta P)_m=BAk=BA\left(\frac{\omega}{v}\right)$$

or

$$\omega=\frac{(\Delta P)_m v}{BA}$$

Substituting this value in Eq. (i), we have

$$I=\frac{1}{2}\rho A^2\frac{(\Delta P)_m^2 v^2}{B^2A^2}v \qquad (\rho v^2=B)$$

or

$$I=\frac{v(\Delta P)_m^2}{2B} \qquad \text{...(ii)}$$

Thus, intensity of a sound wave can be calculated by either of the Eqs. (i) or (ii).

Sound Intensity in decibels

The physiological sensation of loudness is closely related to the intensity of wave producing the sound. At a frequency of 1 kHz people are able to detect sounds with intensities as low as 10^{-12} W/m². On the other hand an intensity of 1 W/m² can cause pain, and prolonged exposure to sound at this level will damage a person's ears. Because the range in intensities over which people hear is so large, it is convenient to use a logarithmic scale to specify intensities. This scale is defined as follows.

If the intensity of sound in watts per square meter is I, then the intensity level β in decibels (dB) is given by,

$$\beta=10\log\frac{I}{I_0}$$

where the base of the logarithm is 10, and $I_0=10^{-2}$ W/m²
(roughly the minimum intensity that can be heard).

On the decibel scale, the pain threshold of 1 W/m² is then

$$\beta=10\log\frac{1}{10^{-12}}=120\text{ dB}$$

Table 16.1 gives typical values for the intensity levels of some of the common sounds.

Table 16.1 Sound Intensity levels in decibels
(Threshold of hearing = 0 dB; threshold of pain = 120 dB)

Source of sound	dB
Rustling leaves	10
Whisper	20
Quiet room	30
Normal level of speech (inside)	65
Street traffic (inside car)	80
Riveting tool	100
Thunder	110
Indoor rock concert	120

Sample Example 16.5 *A point source of sound emits a constant power with intensity inversely proportional to the square of the distance from the source. By how many decibels does the sound intensity level drop when you move from point P_1 to P_2? Distance of P_2 from the source is two times the distance of source from P_1.*

Solution We label the two points 1 and 2, and we use the equation $\beta=10\log\frac{I}{I_0}$ (dB) twice. The difference in sound intensity level $\beta_2-\beta_1$ is given by

$$\beta_2-\beta_1=(10\text{ dB})\left(\log\frac{I_2}{I_0}-\log\frac{I_1}{I_0}\right)$$

$$=(10\text{ dB})[(\log I_2-\log I_0)-(\log I_1-\log I_0)]$$

$$=(10\text{ dB})\log\frac{I_2}{I_1}$$

Now,

$$I\propto\frac{1}{r^2}$$

$$\therefore \quad \frac{I_2}{I_1}=\left(\frac{r_1}{r_2}\right)^2=\frac{1}{4} \quad \text{as} \quad r_2=2r_1$$

$$\therefore \quad \beta_2-\beta_1=(10\text{ dB})\log\left(\frac{1}{4}\right)=-6.0\text{ dB}$$

Ans.

Sample Example 16.6 *For a person with normal hearing, the faintest sound that can be heard at a frequency of 400 Hz has a pressure amplitude of about 6.0×10^{-5} Pa. Calculate the corresponding intensity in W/m². Take speed of sound in air as 344 m/s and density of air 1.2 kg/m³.*

Solution

$$I=\frac{v(\Delta P)_m^2}{2B}$$

Substituting $B=\rho v^2$, the above equation reduces to,

$$I=\frac{(\Delta P)_m^2}{2\rho v}=\frac{(6.0\times10^{-5})^2}{2\times1.2\times344}$$

$$=4.4\times10^{-12}\text{ W/m}^2$$

Ans.

Introductory Exercise 16.3

1. A sound wave in air has a frequency of 300 Hz and a displacement amplitude of 6.0×10^{-3} mm. For this sound wave calculate the
   (a) pressure amplitude
   (b) intensity
   (c) sound intensity level (in dB)
   Speed of sound = 344 m/s and density of air = 1.2 kg/m³.

2. Most people interpret a 9.0 dB increase in sound intensity level as a doubling in loudness. By what factor must the sound intensity be increased to double the loudness?
3. A baby's mouth is 30 cm from her father's ear and 3.0 m from her mother's ear. What is the difference between the sound intensity levels heard by the father and by the mother.
4. The faintest sound that can be heard has a pressure amplitude of about 2×10^{-5} N/m² and the loudest that can be heard without pain has a pressure amplitude of about 28 N/m². Determine in each case
   (a) the intensity of the sound both in W/m² and in dB and
   (b) the amplitude of the oscillations if the frequency is 500 Hz. Assume an air density of 1.29 kg/m³ and a velocity of sound is 345 m/s.

16.6 Interference of Sound Waves

The principle of superposition introduced in Art. 15.2 is valid for sound waves as well. When two or more waves meet at some point on a medium, the resultant disturbance is equal to the sum of the disturbances produced by individual waves. Depending on the phase difference, the waves can interfere constructively or destructively leading to a corresponding increase or decrease in the resultant intensity. Before studying the interference of sound waves let us first discuss the two terms :

(i) phase difference and
(ii) coherent sources which will be frequently used later.

Phase Difference $\Delta\phi$

(i) Phase difference between two different points P_1 and P_2 on the path of a travelling wave separated by a distance Δx is

$$\Delta\phi=\left(\frac{2\pi}{\lambda}\right)\Delta x$$

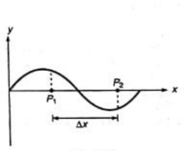


Fig. 16.4

(ii) A phase difference may also arise between two waves generated by the same source when they travel along paths of unequal lengths. Suppose the difference in lengths is Δx then the phase difference will be same i.e., $\frac{2\pi}{\lambda}\Delta x$.

(iii) The phase difference at a point P on the path of a travelling wave at two different times t_1 and t_2 ($>t_1$) is

$$\Delta\phi=\frac{2\pi}{T}\Delta t$$

Here, $\Delta t=t_2-t_1$

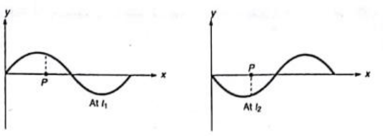


Fig. 16.5

Coherent Sources

Two sources which are in phase or have a constant phase difference are called coherent sources. For the two sources to be coherent their frequencies must be same. But the converse is not always true, *i.e.*, the two different sources having the same frequency are not always coherent. If the phase difference of the sources changes erratically with time, even if they have the same frequency the sources are said to be incoherent. This is what happens with light sources composed of a large number of same kind of atoms, which emit light of the same frequency. Since there are many atoms involved in each source and they do not oscillate in phase, they are incoherent. Thus, two different light sources cann't be coherent.

Now, let us come to the point.

Consider two coherent point sources of sound S_1 and S_2 which oscillate in phase with the same angular frequency ω. A point P is situated at a distance x from S_1 and $x + \Delta x$ from S_2, so that the path difference between the two waves reaching P from S_1 and S_2 is Δx. The displacement equations at P due to the two waves are described by

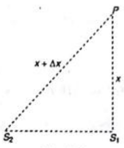


Fig. 16.6

$$y_1 = A_1 \sin (kx - \omega t)$$

and $$y_2 = A_2 \sin [(kx - \omega t) + \phi]$$

where $$\phi = \left(\frac{2\pi}{\lambda}\right)\Delta x$$

is the phase difference between the two waves reaching P. The resultant wave at P is given by

$$y = A \sin [(kx - \omega t) + \theta]$$

where $$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos \phi \qquad \text{...(i)}$$

and $$\tan \theta = \frac{A_2 \sin \phi}{A_1 + A_2 \cos \phi} \qquad \text{...(ii)}$$

Now, as $I \propto A^2$, Eq. (i) can be written as

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \phi$$

The resultant amplitude (or intensity) is maximum when $\phi = 2n\pi$ and minimum when $\phi = (2n + 1)\pi$ where n is an integer. These are the conditions for constructive and destructive interference. Thus

$$\phi = 2n\pi \quad \text{(condition for constructive interference)}$$
$$\phi = (2n + 1)\pi \quad \text{(condition for destructive interference)}$$

Using $\phi = \frac{2\pi}{\lambda}\Delta x$, the above conditions may be written in terms of the path difference as

$$\Delta x = n\lambda \quad \text{(constructive interference)}$$
$$\Delta x = \left(n + \frac{1}{2}\right)\lambda \quad \text{(destructive interference)}$$

At constructive interference

$$A = A_1 + A_2$$

or $$I = (\sqrt{I_1} + \sqrt{I_2})^2$$

and at destructive interference

$$A = A_1 - A_2$$

or $$I = (\sqrt{I_1} - \sqrt{I_2})^2$$

If $A_1 = A_2$ then A or $I = 0$ at destructive interference (where $\Delta x = \lambda/2, 3\lambda/2 \dots$ etc.) or no sound is detected at such a point. If the sources have an initial phase difference ϕ_0 between them, then it is either added or subtracted with ϕ to get the net phase difference at P, depending on the condition whether S_1 leads or lags behind in phase with S_2.

If the sources are incoherent, the phase difference between the sources keep on changing. At any point P, sometimes constructive and sometimes destructive interference takes place. If the intensity due to each source is I, the resultant intensity rapidly and randomly changes between $4I$ and zero, so that the average observable intensity is $2I$. If intensities due to individual sources is I_1 and I_2 the resultant intensity is

$$I = I_1 + I_2 \quad \text{(incoherent sources)}$$

No interference effect is therefore observed. For observable interference, the sources must be coherent. One way to obtain a pair of coherent sources is to obtain two sound waves from the same source by dividing the original wave along two different paths and then combining them. The two waves then differ in phase only because of different paths travelled.

A wave enters from the left and splits equally at point A. At point B, where the waves combine, there is a phase difference because of the path difference. If λ is the wavelength and Δx the path difference, the phase difference is $\phi = 2\pi\Delta x/\lambda$.

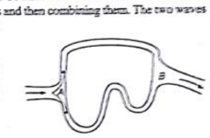


Fig. 16.7

• Important Points in INTERFERENCE

1. If the two waves emitted from S_1 and S_2 have already a phase difference of π, the conditions of maximas and minimas are interchanged, *i.e.*, path difference

$$\Delta x = \frac{\lambda}{2}, \frac{3\lambda}{2} \dots \quad \text{(for constructive interference)}$$

and $$\Delta x = \lambda, 2\lambda, \dots \quad \text{(for destructive interference)}$$

2. Interference of pressure variations can also be obtained in the similar manner.
3. Most of the problems of interference can be solved by calculating the path difference Δx and then by putting

$$\Delta x = 0, \lambda, 2\lambda, \dots \quad \text{(constructive interference)}$$
$$\Delta x = \frac{\lambda}{2}, \frac{3\lambda}{2} \dots \quad \text{(destructive interference)}$$

provided the waves emitted from S_1 and S_2 are in phase. If they have a phase difference of π, the conditions are interchanged.

Sample Example 16.7 *Two sound sources S_1 and S_2 emit pure sinusoidal waves in phase. If the speed of sound is 350 m/s,*

(a) for what frequencies does constructive interference occur at P?

(b) for what frequencies does destructive interference occur at point P?

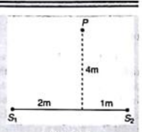


Fig. 16.8

Solution Path difference $\Delta x = S_1P - S_2P$

$$= \sqrt{(2)^2 + (4)^2} - \sqrt{(1)^2 + (4)^2}$$
$$= 4.47 - 4.12 = 0.35 \text{ m}$$

(a) Constructive interference occurs when the path difference is an integer number of wavelength.

or $$\Delta x = n\lambda = \frac{nv}{f}$$

or $$f = \frac{n(v)}{\Delta x} \qquad \text{where } n = 1, 2, 3, \dots$$

$\therefore$ $$f = \frac{350}{0.35}, \frac{2 \times 350}{0.35}, \frac{3 \times 350}{0.35}, \dots$$

$$f = 1000 \text{ Hz}, 2000 \text{ Hz}, 3000 \text{ Hz}, \dots, \text{etc.}$$ **Ans.**

(b) Destructive interference occurs when the path difference is a half-integer number of wavelengths

or $$\Delta x = (2n + 1)\frac{\lambda}{2} \qquad n = 0, 1, 2, \dots$$

or $$\Delta x = (2n + 1)\frac{v}{2f}$$

$\therefore$ $$f = \frac{(2n + 1)v}{2\Delta x}$$

$$= \frac{350}{2 \times 0.35}, \frac{3 \times 350}{2 \times 0.35}, \frac{5 \times 350}{2 \times 0.35}, \dots$$

$$= 500 \text{ Hz}, 1500 \text{ Hz}, 2500 \text{ Hz}, \dots$$ **Ans.**

Introductory Exercise 16.4

Note *Students are advised to solve this exercise after studying Young's double slit experiment in class 12th.*

1. Two sound waves emerging from a source reach a point simultaneously along two paths. When the path difference is 12 cm or 36 cm then there is silence at that point. If the speed of sound in air be 330 m/s then calculate least possible frequency of the source.
2. A wave of frequency 500 cycles/sec has a phase velocity of 350 m/s.
   (a) How far apart are two points 60° out of phase?
   (b) What is the phase difference between two displacements at a certain point at times 10^{-3} s apart?
3. A source S and a detector D of high frequency waves are a distance d apart on the ground. The direct wave from S is found to be in phase at D with the wave from S that is reflected from a horizontal layer at an altitude H. The incident and reflected rays make the same angle with the reflecting layer. When the layer rises a distance h, no signal is detected at D. Neglect absorption in the atmosphere and find the relation between d, h, H and the wavelength λ of the waves.

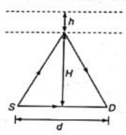


Fig. 16.9

4. Two sound speakers are driven in phase by an audio amplifier at frequency 600 Hz. The speed of sound is 340 m/s. The speakers are on the y-axis, one at $y = +1.0$ m and the other at $y = -1.0$ m. A listener begins at $y = 0$ and walks along a line parallel to the y-axis at a very large distance x away.
   (a) At what angle θ (between the line from the origin to the listener at the x-axis) will she first hear a minimum sound intensity?
   (b) At what angle will she first hear a maximum (after $\theta = 0°$) sound intensity?
   (c) How many maxima can she possibly hear if she keeps walking in the same direction?
5. Two speakers separated by some distance emit sound of the same frequency. At some point P the intensity due to each speaker separately is I_0. The path difference from P to one of the speakers is $\frac{1}{2}\lambda$ greater than that from P to the other speaker. What is the intensity at P if
   (a) the speakers are coherent and in phase;
   (b) the speakers are incoherent; and
   (c) the speakers are coherent but have a phase difference of 180°?
6. Two loudspeakers radiate in phase at 170 Hz. An observer sits at 8 m from one speaker and 11 m from the other. The intensity level from either speaker acting alone is 60 dB. The speed of sound is 340 m/s.

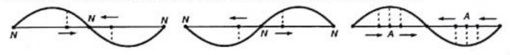
(a) Find the observed intensity level when both speakers are on together.
(b) Find the observed intensity level when both speakers are on together but one has its leads reversed so that the speakers are 180° out of phase.
(c) Find the observed intensity level when both speakers are on and in phase but the frequency is 85 Hz.

7. Two identical speakers emit sound waves of frequency 680 Hz uniformly in all directions with a total audio output of 1 mW each. The speed of sound in air is 340 m/s. A point P is a distance 2.00 m from one speaker and 3.00 m from the other.
(a) Find the intensities I_1 and I_2 from each speaker at point P separately.
(b) If the speakers are driven coherently and in phase, what is the intensity at point P?
(c) If they are driven coherently but out of phase by 180°, what is the intensity at point P?
(d) If the speakers are incoherent, what is the intensity at point P?

16.7 Standing Longitudinal Waves in Organ Pipes

When longitudinal waves propagate in a fluid in a pipe with finite length, the waves are reflected from the ends in a same way the transverse waves on a string are reflected at its ends. The superposition of the waves travelling in opposite directions forms a longitudinal standing wave.

Transverse waves on a string, including standing waves, are usually described only in terms of the displacement of the string. But longitudinal standing waves in a fluid may be described either in terms of the displacement of the fluid or in terms of the pressure variation in the fluid. To avoid confusion we will use the terms **displacement node** and **displacement antinode** to refer to points where particles of the fluid have zero displacement and maximum displacement, respectively. Similarly, **pressure node** and **pressure antinode** refer to the points where pressure and density variation in the fluid is zero or maximum, respectively. Because the pressure wave is 90° out of phase with the displacement wave. Consequently, the displacement node behaves as a pressure antinode and *vice-versa*.



(a) (b) (c)

Fig. 16.10

This can be understood physically by realizing that two small volume elements of fluid on opposite sides of a displacement node are vibrating in opposite phase. Hence, when they approach each other [see figure (a)] the pressure at this node is a maximum, and when they recede from each other, [see figure (b)] the pressure at this node is a minimum. Similarly, two small elements of fluid which are on opposite sides of a displacement antinode vibrate in phase and therefore give rise to no pressure variations at the antinode [see figure (c)].

Pressure equation of standing longitudinal wave

If two identical (same frequency) longitudinal waves travel in opposite directions, standing waves are produced by their superposition. If the equations of the two waves are written as

$$\Delta P_1 = (\Delta P)_m \sin (kx - \omega t)$$

and

$$\Delta P_2 = (\Delta P)_m \sin (kx + \omega t)$$

From the principle of superposition, the resultant wave is

$$\Delta P = \Delta P_1 + \Delta P_2 = 2(\Delta P)_m \sin kx \cos \omega t \quad \text{...(i)}$$

This equation is similar to the equation obtained in chapter 17 for standing waves on a string. From Eq. (i), we can see that

$$\Delta P = 0 \quad \text{at } x = 0, \lambda/2, \lambda, \ldots, \text{etc.} \qquad \text{(pressure nodes)}$$

and

$$\Delta P = \text{maximum at } x = \lambda/4, 3\lambda/4, \ldots, \text{etc.} \qquad \text{(pressure antinodes)}$$

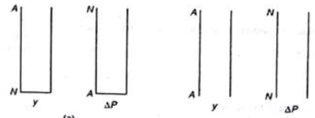
The distance between two adjacent nodes or between two adjacent antinodes is $\lambda/2$. Longitudinal standing waves can be produced in air columns trapped in tubes of cylindrical shape. Organ pipes are such vibrating air columns.

Conditions at the boundary of an organ pipe

Let us take an example of a closed organ pipe. In a closed pipe one end is closed and the other is open. when a longitudinal wave encounters the closed end of the pipe it gets reflected from this end. But the reflected wave is 180° out of phase with the incident wave, *i.e.*, a compression is reflected as a compression and a rarefaction is reflected as a rarefaction. This is a necessary condition because the displacement of the small volume elements at the closed end must always be zero. Hence, a closed end is a displacement node.

A sound wave is also reflected from an open end. You may wonder how a sound wave can reflect from an open end, since there may not appear to be a change in the medium at this point. At the open end pressure is the same as atmospheric pressure and does not vary. Thus, there is a pressure node (or displacement antinode) at this end. A compression is therefore reflected as a rarefaction and rarefaction as a compression. Now let us see how this reflection takes place. When a rarefaction reaches an open end, the surrounding air rushes towards this region and creates a compression that travels back along the pipe. Similarly, when a compression reaches an open end, the air expands to form a rarefaction. This can be said in a different way as : at the open end of the tube fluid elements are free to move, so there is a displacement antinode. Thus, in a nutshell we can say that closed end of an organ pipe is a displacement node or pressure antinode and open end of the pipe is displacement antinode or pressure node.

Similarly, both ends of an open organ pipe (open at both ends) are displacement antinodes or pressure nodes.



(a) Closed organ pipe

(b) Open organ pipe

Fig. 16.11

Note *In practice, the pressure nodes lie slightly beyond the ends of the tube. A compression reaching an open end does not reflect until it passes beyond the end. For a thin walled tube of circular cross section, this* ***end correction*** *is approximately 0.6R where R is the tube's radius. Hence, the effective length of the tube is longer than the true length l. However, we neglect this small correction if the length of the tube is much larger than its diameter.*

Standing waves in a closed organ pipe

Before reading this article, students are advised to read Art. 15.7 for the condition of resonance.

To get resonance in a closed organ pipe sound waves are sent in by a source (normally a tuning fork) near the open end. Resonance corresponds to a pressure antinode at the closed end and a pressure node at the open end. The standing wave patterns for the three lowest harmonics in this situation are shown in figure. Since the node-antinode separation is $\frac{\lambda}{4}$, the resonance condition for the first harmonic is, $l = \frac{\lambda_1}{4}$.

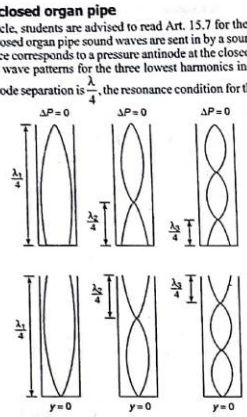


Fig. 16.12

Similarly, the resonance conditions for the higher harmonics are $l = \frac{3\lambda_2}{4}, \frac{5\lambda_3}{4}, \ldots$, etc. The natural frequencies of oscillation of the air in the tube closed at one end and open at the other are, therefore,

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{4l} = nf_1 \quad (n = 1, 3, 5, \ldots)$$

Here, v = speed of sound in the tube

$$f_1 = \frac{v}{4l} \qquad \text{(fundamental frequency or the first harmonic)}$$

$$f_3 = 3\frac{v}{4l} = 3f_1 \qquad \text{(first overtone or the third harmonic)}$$

and

$$f_5 = 5\frac{v}{4l} = 5f_1 \qquad \text{(second overtone or the fifth harmonic)}$$

and so on.

Thus, in a pipe closed at one end and open at the other, the natural frequencies of oscillation form a harmonic series that includes only odd integer multiples of the fundamental frequency.

Standing waves in an open organ pipe

Since both ends of the tube are open, there are pressure nodes (or displacement antinodes) at both ends. Figure shows the resulting standing waves for the three lowest resonant frequencies since the distance between pressure nodes is $\lambda/2$, the resonance condition is $l = \frac{n\lambda n}{2}$ where $n = 1, 2, 3, \ldots$ and l is the length of the tube. The resonant frequencies for a tube open at both ends are then

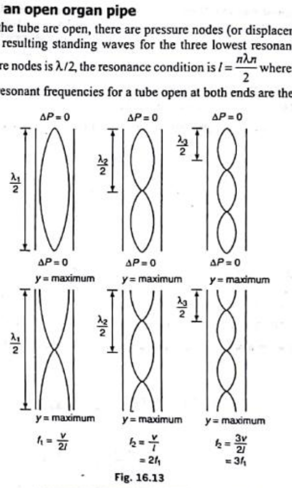


Fig. 16.13

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2l} = nf_1 \quad (n = 1, 2, 3, \ldots)$$

Here,

$$f_1 = \frac{v}{2l} \qquad \text{(fundamental frequency or first harmonic)}$$

$$f_2 = \frac{v}{l} = 2f_1 \qquad \text{(first overtone o second harmonic)}$$

$$f_3 = 3\frac{v}{2l} = 3f_1 \qquad \text{(second overtone or third harmonic)}$$

and so on.

Thus, in a pipe open at both ends, the natural frequencies of oscillation form a harmonic series that includes all integral multiples of the fundamental frequency.

Note *It is interesting to investigate what happens to the frequencies of instruments based on air columns and strings as the temperature rises. The sound emitted by a flute, becomes sharp (increases in frequency) because the speed of sound increases inside the flute. The sound produced by a violin becomes flat (decreases in frequency) as the strings expand thermally because the expansion causes their tension to decrease.*

Sample Example 16.8 *Third overtone of a closed organ pipe is in unison with fourth harmonic of an open organ pipe. Find the ratio of the lengths of the pipes.*

Solution Third overtone of closed organ pipe means seventh harmonic. Given

$$(f_7)_{\text{closed}} = (f_4)_{\text{open}}$$

$$7\left(\frac{v}{4l_c}\right) = 4\left(\frac{v}{2l_o}\right)$$

$\therefore$
$$\frac{l_c}{l_o} = \frac{7}{8}$$ **Ans.**

Sample Example 16.9 *The water level in a vertical glass tube 1.0 m long can be adjusted to any position in the tube. A tuning fork vibrating at 660 Hz is held just over the open top end of the tube. At what positions of the water level will there be resonance? Speed of sound is 330 m/s.*

Solution Resonance corresponds to a pressure antinode at closed end and pressure node at open end. Further, the distance between a pressure node and a pressure antinode is $\frac{\lambda}{4}$, the condition of resonance would be,

$$\text{Length of air column } l = n\frac{\lambda}{4} = n\left(\frac{v}{4f}\right)$$

Here, $n = 1, 3, 5, \ldots$

$$l_1 = (1)\left(\frac{330}{4 \times 660}\right) = 0.125 \text{ m}$$

$$l_2 = 3l_1 = 0.375 \text{ m}$$

$$l_3 = 5l_1 = 0.625 \text{ m}$$

$$l_4 = 7l_1 = 0.875 \text{ m}$$

$$l_5 = 9l_1 = 1.125 \text{ m}$$

Since, $l_5 > 1$ m (the length of tube), the length of air columns can have the values from l_1 to l_4 only. Therefore, level of water at resonance will be

$$(1.0 - 0.125) \text{ m} = 0.875 \text{ m}$$

$$(1.0 - 0.375) \text{ m} = 0.625 \text{ m}$$

$$(1.0 - 0.625) \text{ m} = 0.375 \text{ m}$$

and
$$(1.0 - 0.875) \text{ m} = 0.125 \text{ m}$$ **Ans.**

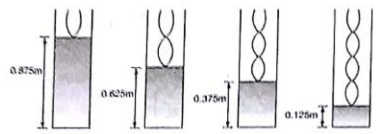


Fig. 16.14

In all the four cases shown in figure, the resonance frequency is 660 Hz but first one is the fundamental tone or first harmonic. Second is first overtone or third harmonic and so on.

Sample Example 16.10 *A tube 1.0 m long is closed at one end. A stretched wire is placed near the open end. The wire is 0.3 m long and has a mass of 0.01 kg. It is held fixed at both ends and vibrates in its fundamental mode. It sets the air column in the tube into vibration at its fundamental frequency by resonance. Find*

(a) the frequency of oscillation of the air column and
(b) the tension in the wire.
Speed of sound in air = 330 m/s.

Solution (a) Fundamental frequency of closed pipe $= \frac{v}{4l}$

$$= \frac{330}{4 \times 1} = 82.5 \text{ Hz}$$ **Ans.**

(b) At resonance, given :

fundamental frequency of stretched wire (at both ends) = fundamental frequency of air column

$\therefore$
$$\frac{v}{2l} = 82.5 \text{ Hz}$$

$\therefore$
$$\frac{\sqrt{T/\mu}}{2l} = 82.5$$

or
$$T = \mu\,(2 \times 0.3 \times 82.5)^2$$

$$= \left(\frac{0.01}{0.3}\right)(2 \times 0.3 \times 82.5)^2$$

$$= 81.675 \text{ N}$$ **Ans.**

Sample Example 16.11 *An open organ pipe has a fundamental frequency of 300 Hz. The first overtone of a closed organ pipe has the same frequency as the first overtone of the open pipe. How long is each pipe? (Speed of sound in air = 330 m/s)*

Solution Fundamental frequency of an open pipe,

$$f_1 = \frac{v}{2l_o}$$

$\therefore$
$$l_o = \frac{v}{2f_1} = \frac{330}{2 \times 300} = 0.55 \text{ m}$$

$$= 55 \text{ cm}$$ **Ans.**

Given, first overtone of closed pipe = first overtone of open pipe

Hence,
$$3\left(\frac{v}{4l_c}\right) = 2\left(\frac{v}{2l_o}\right)$$

$\therefore$
$$l_c = \frac{3}{4}l_o = \left(\frac{3}{4}\right)(0.55)$$

$$= 0.4125 \text{ m}$$

$$= 41.25 \text{ cm}$$ **Ans.**

Introductory Exercise 16.5

1. The fundamental frequency of a closed organ pipe is 220 Hz.
   (a) Find the length of this pipe.
   (b) The second overtone of this pipe has the same wavelength as the third harmonic of an open pipe. Find the length of this open pipe. Take speed of sound in air 345 m/s.
2. Standing sound waves are produced in a pipe that is 0.8 m long, open at one end, and closed at the other. For the fundamental and first two overtones, where along the pipe (measured from the closed end) are
   (a) the displacement antinodes
   (b) the pressure antinodes.
3. An organ pipe has two successive harmonics with frequencies 400 and 560 Hz. The speed of sound in air is 344 m/s.
   (a) Is this an open or a closed pipe?
   (b) What two harmonics are these?
   (c) What is the length of the pipe?
4. The atomic mass of iodine is 127 g/mol. A standing wave in iodine vapour at 400 K has nodes that are 6.77 cm apart when the frequency is 1000 Hz. At this temperature, is iodine vapour monoatomic or diatomic.
5. A tuning fork whose natural frequency is 440 Hz is placed just above the open end of a tube that contains water. The water is slowly drained from the tube while the tuning fork remains in place and is kept vibrating. The sound is found to be enhanced when the air column is 60 cm long and when it is 100 cm long. Find the speed of sound in air.

16.8 Beats

When two wavetrains of the same frequency travel along the same line in opposite directions, standing waves are formed in accordance with the principle of superposition. In standing waves amplitude is a function of distance. This illustrates a type of interference that we can call **interference in space.** The same principle of superposition leads us to another type of interference, which we can call **interference in time.** It occurs when two wavetrains of slightly different frequency travel through the same region.

If the waves are in phase at some time (say $t = 0$) the interference will be constructive and the resultant amplitude at this moment will be $A_1 + A_2$, where A_1 and A_2 are the amplitudes of individual

wavetrains. But at some later time (say $t = t_0$), because the frequencies are different, the waves will be out of phase or the interference will be destructive and the resultant amplitude will be $A_1 - A_2$ (if $A_1 > A_2$). Later, we will see that the time t_0 is $\frac{2}{f_1 - f_2}$.

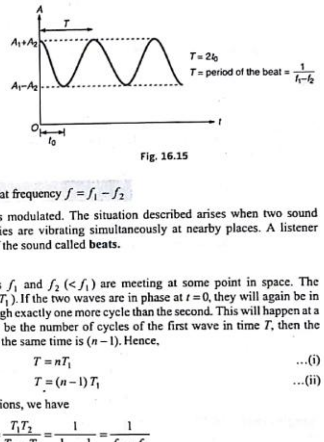


Fig. 16.15

Thus, the resultant amplitude oscillates between $A_1 + A_2$ and $A_1 - A_2$ with a time period $T = 2t_0 = \frac{1}{f_1 - f_2}$ or with a frequency $f = f_1 - f_2$ known as **beat frequency.** Thus

$$\text{Beat frequency } f = f_1 - f_2$$

It is then said that the amplitude is modulated. The situation described arises when two sound sources of close but different frequencies are vibrating simultaneously at nearby places. A listener observes a fluctuation in the intensity of the sound called **beats.**

Calculation of beat frequency

Suppose two waves of frequencies f_1 and f_2 ($< f_1$) are meeting at some point in space. The corresponding periods are T_1 and T_2 ($> T_1$). If the two waves are in phase at $t = 0$, they will again be in phase when the first wave has gone through exactly one more cycle than the second. This will happen at a time $t = T$, the period of the beat. Let n be the number of cycles of the first wave in time T, then the number of cycles of the second wave in the same time is $(n - 1)$. Hence,

$$T = nT_1 \quad \ldots\text{(i)}$$

and
$$T = (n-1)\,T_1 \quad \ldots\text{(ii)}$$

Eliminating n from these two equations, we have

$$T = \frac{T_1T_2}{T_2 - T_1} = \frac{1}{\frac{1}{T_1} - \frac{1}{T_2}} = \frac{1}{f_1 - f_2}$$

The reciprocal of the beat period is the beat frequency

$$f = \frac{1}{T} = f_1 - f_2.$$

Alternate Method

Let the oscillations at some point in space (say $x = 0$) due to two waves be,

$$y_1 = A_1 \sin 2\pi f_1 t \quad \text{and} \quad y_2 = A_2 \sin 2\pi f_2 t$$

If they are in phase at some time t, then

$$2\pi f_1 t = 2\pi f_2 t$$

or
$$f_1 t = f_2 t \quad \ldots\text{(i)}$$

They will be again in phase at time $(t+T)$ if,

$$2\pi f_1(t+T) = 2\pi f_2(t+T) + 2\pi$$

or

$$f_1(t+T) = f_2(t+T) + 1 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get $\quad T = \dfrac{1}{f_1 - f_2}$

Note *(i) When $A_1 = A_2 = A$ (say) that is, when the two amplitudes are equal, we obtain an amplitude which oscillates between zero and $2A$.*

(ii) At frequency difference greater than about 6 or 7 Hz, we no longer hear individual beats. For example, if you listen to a whistle that produces sounds at 2000 Hz and 2100 Hz, you will hear not only these tones but also a much lower 100 Hz tone.

(iii) If the frequency of a tuning fork is f and it produces Δf beats per second with a standard fork of frequency f_0, then

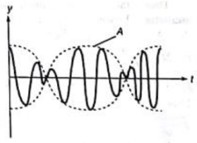


Fig. 16.16

$$f = f_0 \pm \Delta f$$

If on filing the arms of an unknown fork the beat frequency decreases then,

$$f = f_0 - \Delta f$$

This is because filing of an arm of a tuning fork increases its frequency.
Similarly, if on loading/waxing of the unknown fork, the beat frequency decreases then the frequency of the unknown fork is, $f = f_0 + \Delta f$. This is because loading/waxing decreases the frequency of tuning fork.
Similarly, $f = f_0 + \Delta f$ if on filing beat frequency increases and $f = f_0 - \Delta f$ if on loading/waxing beat frequency increases.

Thus, $f = f_0 - \Delta f$ *if on filing beat frequency is decreased*
$= f_0 - \Delta f$ *if on filing beat frequency is increased*
$= f_0 + \Delta f$ *if on loading/waxing beat frequency is increased*
$= f_0 - \Delta f$ *if on loading/waxing beat frequency is decreased*

Sample Example 16.12 *The string of a violin emits a note of 400 Hz at its correct tension. The string is bit taut and produces 5 beats per second with a tuning fork of frequency 400 Hz. Find frequency of the note emitted by this taut string.*

Solution The frequency of vibration of a string increases with increase in the tension. Thus, the note emitted by the string will be a little more than 400 Hz. As it produces 5 beats per second with the 440 Hz tuning fork, the frequency will be 405 Hz.

Introductory Exercise 16.6

1. A tuning fork produces 4 beats per second with another tuning fork of frequency 256 Hz. The first one is now loaded with a little wax and the beat frequency is found to increase to 6 per second. What was the original frequency of the first tuning fork?
2. A tuning fork of unknown frequency makes three beats per second with a standard fork of frequency 384 Hz. The beat frequency decreases when a small piece of wax is put on a prong of the first fork. What is the frequency of this fork?

3. A piano wire A vibrates at a fundamental frequency of 600 Hz. A second identical wire B produces 6 beats per second with it when the tension in A is slightly increased. Find the ratio of the tension in A to the tension in B.
4. A tuning fork of frequency 256 Hz produces 4 beats per second with a wire of length 25 cm vibrating in its fundamental mode. The beat frequency decreases when the length is slightly shortened. What could be the minimum length by which the wire be shortened so that it produces no beats with the tuning fork?

16.9 The Doppler Effect

If a wave source and a receiver are moving relative to each other, the frequency observed by the receiver (f') is different from the actual source frequency (f). This phenomenon is called the **Doppler effect**, named after the Austrian physicist Christian Johann Doppler (1803–1853), who discovered it in light waves.

Perhaps you might have noticed how the sound of a vehicle's horn changes as the vehicle moves past you. The frequency (pitch) of the sound you hear as the vehicle approaches you is higher than the frequency you hear as it moves away from you. The Doppler effect, applies to waves in general. Let us apply it to sound waves. We consider the special case in which the source and observer move along the line joining them. We will use the following symbols,

v = speed of sound
v_s = speed of source and
v_o = speed of observer.

First we take two special cases. Then the results obtained there can be generalised.

(a) Source at rest, observer moves

Suppose that the observer O moves towards the source S at speed v_o. The speed of the sound waves relative to O is $v_r = v + v_o$, but wavelength has its normal value $\lambda = \dfrac{v}{f}$. Thus, the frequency heard by O is,

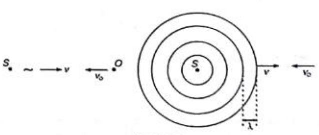


Fig. 16.17

$$f' = \frac{v_r}{\lambda} = \left(\frac{v + v_o}{v}\right) f$$

If O were moving away from S, the frequency heard by O would be

$$f' = \left(\frac{v - v_o}{v}\right) f$$

Combining these two expressions, we find

$$f' = \left(\frac{v \pm v_o}{v}\right) f \qquad \text{...(i)}$$

(b) Source moves, observer at rest

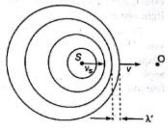
Suppose that the source S moves towards O as shown in figure. If S were at rest, the distance between two consecutive wave pulses emitted by sound would be $\lambda = \dfrac{v}{f} = vT$. However, in one time period S moves a distance v_sT before it emits the next pulse. As a result the wavelength is modified. Directly ahead of S the effective wavelength (for both S and O) is

Fig. 16.18

$$\lambda' = vT - v_sT = \left(\frac{v - v_s}{f}\right)$$

The speed of sound waves relative to O is simply v. Thus, the frequency observed by O is

$$f' = \frac{v}{\lambda'} = \left(\frac{v}{v - v_s}\right) f$$



The wavelength in front of the source is less than the normal whereas in the rear it is larger than normal.

Fig. 16.19

If S were moving away from O, the effective wavelength would be $\lambda' = \left(\dfrac{v + v_s}{f}\right)$ and the apparent frequency would be

$$f' = \left(\frac{v}{v + v_s}\right) f$$

Combining these two results, we have

$$f' = \left(\frac{v}{v \pm v_s}\right) f \qquad \text{...(ii)}$$

All four possibilities can be combined into one equation

$$f' = \left(\frac{v \pm v_o}{v \mp v_s}\right) f \qquad \text{...(iii)}$$

where the upper signs (+ numerator, − denominator) correspond to the source and observer along the line joining the two in the direction toward the other, and the lower signs in the direction away from the other.

Alternate Method

The above formulae can be derived alternately as follows.

Assume that the source and observer are moving along the same line and that the observer O is to the right of the source S. Suppose that at time $t = 0$, when the source and the observer are separated by a distance

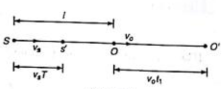


Fig. 16.20

$SO = l$, the source emits a wave pulse (say p_1) that reaches the observer at a later time t_1. In that time the observer has moved a distance v_ot_1 and the total distance travelled by p_1 in the time t_1 has been $l + v_ot_1$. If v is the speed of sound, this distance is also vt_1. Then

$$vt_1 = l + v_ot_1 \quad \text{or} \quad t_1 = \frac{l}{v - v_0} \qquad \text{...(iv)}$$

At time $t = T$, the source is at S' and the wave pulse (say p_2) emitted at this time will reach the observer at a time say t_2, measured from the same time origin as before. The total distance travelled by p_2 till it is received by the observer (measured from S') is $(l - v_sT) + v_ot_2$. The actual travel time for p_2 is $(t_2 - T)$ and the distance travelled is $v(t_2 - T)$. Therefore,

$$v(t_2 - T) = (l - v_sT) + v_0t_2 \quad \text{or} \quad t_2 = \frac{l + (v - v_s)T}{v - v_o} \qquad \text{...(v)}$$

The time interval reckoned by the observer between the two pulses emitted by the source at S and at S' is

$$T' = t_2 - t_1 = \left(\frac{v - v_s}{v - v_o}\right) T$$

This is really the changed time period as observed by the observer. Hence, the new frequency is

$$f' = \frac{1}{T'} = \left(\frac{v - v_o}{v - v_s}\right)\frac{1}{T}$$

or

$$f' = \left(\frac{v - v_o}{v - v_s}\right) f$$

This is a result which we can expect by using equation $f' = \left(\dfrac{v \pm v_o}{v \mp v_s}\right) f$ in the case when source is moving towards observer and observer is moving away from the source.

• *Important Points in* DOPPLER EFFECT

1. If the medium (air) itself starts moving, appropriate changes must be made in Eq. (iii). If wind blows at a speed v_w from the source to the observer, take $v \to v + v_w$ (both in numerator and denominator) and if in opposite direction (*i.e.*, from observer to source), take $v \to v - v_w$. Thus, the modified formula is

$$f' = \left(\frac{v \pm v_w \pm v_o}{v \pm v_w \mp v_s}\right) f \qquad \text{...(vi)}$$

2. We have derived equation number (iii) by assuming that v_o and v_s are along the line joining source and observer. If the motion is along some other direction, the components of velocities along the line joining source and observer are considered.

For example in the figure shown,

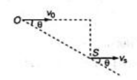


Fig. 16.21

$$f' = \left(\frac{v + v_o \cos\theta}{v + v_o \sin\theta}\right) f$$

3. Change in frequency depends on the fact that whether the source is moved towards the observer or the observer is moved towards the source. But when the speed of source and observer are much lesser than that of sound, the change in frequency becomes independent of the fact whether the source is moving or the observer. This can be shown as under.

Suppose a source is moving towards a stationary observer, with speed u and the speed of sound is v, then

$$f' = \left(\frac{v}{v-u}\right) f = \left(\frac{1}{1-\frac{u}{v}}\right) f$$
$$= \left(1-\frac{u}{v}\right)^{-1} f$$

Using the binomial expansion, we have

$$\left(1-\frac{u}{v}\right)^{-1} \approx 1+\frac{u}{v}+\frac{u^2}{v^2}$$

$\therefore$
$$f' \approx \left(1+\frac{u}{v}+\frac{u^2}{v^2}\right) f$$
$$f' \approx \left(1+\frac{u}{v}\right) f \quad \text{if} \quad u << v$$

On the other hand, if an observer moves towards a stationary source with same speed u, then

$$f' = \left(\frac{v+u}{v}\right) f = \left(1+\frac{u}{v}\right) f$$

which is same as above.

4. The Doppler effect is important in light. But the speed of light is so great that only astronomical or atomic sources, which have high velocities compared to speed of light, show pronounced Doppler effect.

There are differences, however, in the Doppler effect formula for light and for sound. In sound it is not just the relative motion of source and observer that determines the frequency change. Even when the relative motion is the same, we obtain different results, depending on whether the source or the observer is moving. For light however, it is the relative motion between the source and the observer which matters.

Sample Example 16.13 *A siren emitting a sound of frequency 1000 Hz moves away from you toward a cliff at a speed of 10 m/s.*

(a) What is the frequency of the sound you hear coming directly from the siren?
(b) What is the frequency of the sound you hear reflected off the cliff?
(c) What beat frequency would you hear? Take the speed of sound in air as, 330 m/s.

Solution The situation is as shown in figure.

Fig. 16.22

(a) Frequency of sound reaching directly to us (by S)

$$f_1 = \left(\frac{v}{v+v_s}\right) f = \left(\frac{330}{330+10}\right)(1000)$$
$$= 970.6 \text{ Hz}$$ **Ans.**

(b) Frequency of sound which is reflected of from the cliff (from S')

$$f_2 = \left(\frac{v}{v-v_s}\right) f = \left(\frac{330}{330-10}\right)(1000)$$
$$= 1031.3 \text{ Hz}$$ **Ans.**

(c) Beat frequency $= f_2 - f_1$
$= 60.7$ Hz **Ans.**

Note *Numerically the beat frequency comes out to be 60.7 Hz. But, beats between two tones can be detected by ear upto a frequency of about 7 per second. At higher frequencies beats cannot be distinguished in the sound produced. Hence, the correct answer of part (c) should be zero.*

Sample Example 16.14 *A car approaching a crossing at a speed of 20 m/s sounds a horn of frequency 500 Hz when 80 m from the crossing. Speed of sound in air is 330 m/s. What frequency is heard by an observer 60 m from the crossing on the straight road which crosses car road at right angles?*

Solution The situation is as shown in figure.

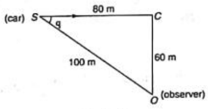


Fig. 16.23

$$\cos\theta = \frac{80}{100} = \frac{4}{5}$$

$\therefore$ Apparent frequency is

$$f_{app} = \left(\frac{v}{v - v_s\cos\theta}\right) f$$
$$= \left(\frac{330}{330 - 20 \times \frac{4}{5}}\right)(500)$$
$$= 525.5 \text{ Hz}$$ **Ans.**

Introductory Exercise 16.7

1. Show that when the speed of the source and the observer are small compared to the speed of sound in the medium, the change in frequency becomes independent of the fact whether the source is moving or the observer.
2. A sound source moves with a speed of 80 m/s relative to still air toward a stationary listener. The frequency of sound is 200 Hz and speed of sound in air is 340 m/s.
   (a) Find the wavelength of the sound between the source and the listener.
   (b) Find the frequency heard by the listener.
3. If the source and receiver are at rest relative to each other but the wave medium is moving relative to them, will the receiver detect any wavelength or frequency shift.
4. A railroad train is travelling at 30 m/s in still air. The frequency of the note emitted by the locomotive whistle is 500 Hz. What is the wavelength of the sound waves :
   (a) in front of the locomotive? (b) behind the locomotive?
   What is the frequency of the sound heard by a stationary listener
   (c) in front of the locomotive? (d) behind the locomotive?
   Speed of sound in air is 344 m/s.

5. A 300 Hz source, an observer and a wind are moving as shown in the figure with respect to the ground. What frequency is heard by the observer? Take speed of sound in air = 340 m/s

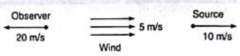


Fig. 16.24

6. Prove that at a distance r straight in front of a point source with power P, moving with constant speed v_s, the wave intensity is $\frac{P}{4\pi r^2}\left(\frac{v-v_s}{v}\right)$. Here v = speed of sound..

EXTRA POINTS

CHARACTERISTICS OF SOUND

Following are the characteristics:

(i) Intensity of wave

The amount of energy flowing per unit area and per unit time is called the intensity of wave. It is represented by I. Its units are J/m^{2}s or w/m^2.

$$I = 2\pi^2 f^2 A^2 \rho v$$

i.e., $\quad I \propto f^2$ and $I \propto A^2$

If P is the power of an isotropic point source, intensity at a distance r is given by,

$$I = \frac{P}{4\pi r^2}$$

or $\quad I \propto \frac{1}{r^2}$ (for a point source)

If P is the power of a line source, then intensity at a distance r is given by,

$$I = \frac{P}{2\pi r l}$$

or $\quad I \propto \frac{1}{r}$ (for a line source)

As, $\quad I \propto A^2$

Therefore, $\quad A \propto \frac{1}{r}$ (for a point source)

and $\quad A \propto \frac{1}{\sqrt{r}}$ (for a line source)

(ii) Intensity level (loudness)

Depends on **intensity** of sound.

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

Here, I_0 is the intensity of minimum audible sound which is 10^{-12} w/m^2.

Note *For more details of intensity and loudness read article 16.5.*

(iii) Quality

Quality is that characteristics of sound by which we can differentiate between the sounds coming from different sources.

(iv) Pitch

Pitch is the characteristics of sound that depends on **frequency**. It determines the **shrillness** or **graveness** of sound.

Smaller the frequency smaller is the pitch, greater the frequency greater is the pitch. Frequency of ladies' voice is usually higher than that of gents. Therefore, lidies' voice has higher pitch (sharper) than gents.

Humming of mosquito has high pitch (high frequency) but low intensity (low loudness) while the roar of a lion has high intensity (loudness) but low pitch.

Ultrasonic, Infrasonic and Audible Sound

Sound waves can be classified in three groups according to their range of frequencies.

Infrasonic Waves

Longitudinal waves having frequencies below 20 Hz are called infrasonic waves. They cannot be heard by human beings. They are produced during earthquakes. Infrasonic waves can be heard by snakes.

Audible Waves

Longitudinal waves having frequencies lying between 20–20,000 Hz are called audible waves.

Ultrasonic Waves

Longitudinal waves having frequencies above 20,000 Hz are called ultrasonic waves. They are produced and heard by bats. They have a large energy content.

Applications of Ultrasonic Waves

Ultrasonic waves have a large range of application. Some of them are as follows :

(i) The fine internal cracks in a metal can be detected by ultrasonic waves.
(ii) Ultrasonic waves can be used for determining the depth of the sea, lakes etc.
(iii) Ultrasonic waves can be used to detect submarines, icebergs etc.
(iv) Ultrasonic waves can be used to clean clothes, fine machinery parts etc.
(v) Ultrasonic waves can be used to kill smaller animals like rats, fish and frogs etc.

Shock Waves

If the speed of the body in air is greater than the speed of the sound, then it is called supersonic speed. Such a body leaves behind a conical region of disturbance which spreads continuously. Such a disturbance is called a **'Shock Waves'**.

This wave carries huge energy. If it strikes a building, then the building may be damaged.

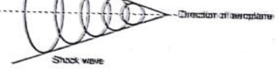


Fig. 16.25

Musical Scale

The arrangement of notes having definite ratio in respective fundamental frequencies is called a musical scale.

Musical scales are of two types.

(a) Diatonic scale

It is known as 'Sargam' in Indian system. It contains eight notes with definite ratios in their frequencies. The note of lowest frequency is called key note and the highest (which is double of first) is called an octave. Harmonium, piano etc. are based on this scale.

(b) Tempered Scale

It contains 13 notes. The ratio of frequencies of successive notes is $2^{1/12}$.

Musical Interval

The ratio between the frequencies of two notes is called the musical interval.

Following are the names of some musical intervals.

(i) Unison $\frac{n_2}{n_1}=1$ (ii) Octave $\frac{n_2}{n_1}=2$

(iii) Major tone $\frac{n_2}{n_1}=\frac{9}{8}$ (iv) Minor tone $\frac{n_2}{n_1}=\frac{10}{9}$

(v) Semi tone $\frac{n_2}{n_1}=\frac{16}{15}$ (vi) Fifth tone $\frac{n_2}{n_1}=\frac{3}{2}$

Echo

Multiple reflection of sound is called an echo. If the distance of reflector from the source is d then,

$$2d = vt$$

Hence, v = speed of sound and t, the time of echo.

$$\therefore \qquad d=\frac{vt}{2}$$

Since, the effect of ordinary sound remains on our ear for 1/10 second, therefore, if the sound returns to the starting point before 1/10 second, then it will not be distinguished from the original sound and no echo will be heard. Therefore, the minimum distance of the reflector is,

$$d=\frac{v\times t}{2}=\left(\frac{330}{2}\right)\left(\frac{1}{10}\right)=16.5\text{ m}$$

Reverberation Time

A listener receives a series of sound waves due to a large number of reflections from the walls, ceiling and floor of the enclosed space which give him the impression of persistance or prolongation of the sound, which we call reverberation. The sound continues to be heard till the intensity falls below the zero level of intensity or threshold of hearing. The time for which sound continues to be heard after the

source has stopped producing sound is called **reverberation time.** The standard reverberation time of a room is defined as the interval of time taken by a sustained note to fall in intensity to 10^{-6} of its original value.

Sabine formula for reverberation time (T) of a hall is,

$$T=\frac{0.16V}{\Sigma aS}$$

where V is the volume of the hall in m^3 and $\Sigma aS = a_1S_1 + a_2S_2 + \dots$ is the absorption of the hall.

Here, $S_1, S_2, \dots$ are the area of the surfaces (in m^2) which absorb sound and $a_1, a_2, \dots$ are their respective absorption coefficients.

It has been established that for speech, optimum value of reverberation time is 0.5 second and for music, the optimum time of reverberation time may lie between 1 to 2 second. A room with zero reverberation time is called a **dead room.**

Doppler's Effect in Light

Light waves also show Doppler's effect. If a light source is moving away from a stationary observer then the frequency of light waves appears decreased and wavelength appears increased and *vice-versa.*

If the light source or the observer is moving with a velocity v such that the distance between them is decreasing, then the apparent frequency of the source will be given by,

$$f'=f\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

If the distance between the light source and the observer is increasing, then the apparent frequency of the source is given by,

$$f'=f\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

The change in wavelength can be determined by,

$$\Delta\lambda=\frac{v}{c}\cdot\lambda$$

If the light source is moving away from the observer, the shift in the spectrum is towards red and if it is moving towards the observer the shift is towards the violet.

Note *That Doppler's effect in light depends only on the relative motion between the source and the observer while the Doppler's effect in sound also depends upon whether the source is moving or the observer is moving.*

Solved Examples

For JEE Main

Example 1 *Determine the speed of sound waves in water, and find the wavelength of a wave having a frequency of 242 Hz. Take $B_{water}=2\times10^9$ Pa.*

Solution Speed of sound wave,

$$v=\sqrt{\frac{B}{\rho}}=\sqrt{\frac{(2\times10^9)}{10^3}}=1414\text{ m/s}$$

Wavelength $\qquad \lambda=\frac{v}{f}=5.84\text{ m}$ **Ans.**

Example 2 *A flute which we treat as a pipe open at both ends is 60 cm long.*

(a) What is the fundamental frequency when all the holes are covered?

(b) How far from the mouthpiece should a hole be uncovered for the fundamental frequency to be 330 Hz? Take speed of sound in air as 340 m/s.

Solution (a) Fundamental frequency when the pipe is open at both ends is

$$f_1=\frac{v}{2l}=\frac{340}{2\times0.6}=283.33\text{ Hz}$$ **Ans.**

(b) Suppose the hole is uncovered at a length l from the mouthpiece, the fundamental frequency will be

$$f_1=\frac{v}{2l}$$

$$\therefore \qquad l=\frac{v}{2f_1}=\frac{340}{2\times330}$$

$$=0.515\text{ m}=51.5\text{ cm}$$ **Ans.**

Note *Opening holes in the side effectively shortens the length of the resonance column, thus increasing the frequency.*

Example 3 *A window whose area is 2 m^2 opens on a street where the street noise results at the window an intensity level of 60 dB. How much acoustic power enters the window through sound waves ? Now, if a sound absorber is fitted at the window, how much energy from the street will it collect in a day ?*

Solution By definition sound level $=10\log\frac{I}{I_0}=60$

or $\qquad \frac{I}{I_0}=10^6$

$$I=10^{-12}\times10^6=1\,\mu\text{W/m}^2$$

Power entering the room

$$1\times10^{-6}\times2=2\,\mu\text{W}$$

Energy collected in a day $=2\times10^{-6}\times86400$

$=0.173$ J **Ans.**

Example 4 *For a certain organ pipe, three successive resonance frequencies are observed at 425, 595 and 765 Hz respectively. Taking the speed of sound in air to be 340 m/s*

(a) explain whether the pipe is closed at one end or open at both ends.

(b) determine the fundamental frequency and length of the pipe.

Solution (a) The given frequencies are in the ratio 5 : 7 : 9. As the frequencies are odd multiple of 85 Hz the pipe must be closed at one end.

(b) Now, the fundamental frequency is the lowest *i.e.*, 85 Hz.

$$\therefore \qquad 85=\frac{v}{4l} \Rightarrow l=\frac{340}{4\times85}=1\text{ m}$$ **Ans.**

Example 5 *Two tuning forks P and Q when set vibrating, give 4 beats per second. If a prong of the fork P is filed, the beats are reduced to 2 per second. Determine the frequency of P, if that of Q is 250 Hz.*

Solution There are four beats between P and Q, therefore the possible frequencies of P are 246 or 254 (that is 250 ± 4) Hz.

When the prong of P is filed, its frequency becomes greater than the original frequency.

If we assume that the original frequency of P is 254, then on filing its frequency will be greater than 254. The beats between P and Q will be more than 4. But it is given that the beats are reduced to 2, therefore, 254 is not possible.

Therefore, the required frequency must be 246 Hz.

(This is true, because on filing the frequency may increase to 248, giving 2 beats with Q of frequency 250 Hz) **Ans.**

Example 6 *Two tuning forks A and B sounded together give 8 beats per second. With an air resonance tube closed at one end, the two forks give resonances when the two air columns are 32 cm and 33 cm respectively. Calculate the frequencies of forks.*

Solution Let the frequency of the first fork be f_1 and that of second be f_2.

We then have,

$$f_1=\frac{v}{4\times32} \quad\text{and}\quad f_2=\frac{v}{4\times33}$$

We also see that $\qquad f_1>f_2$

$$\therefore \qquad f_1-f_2=8 \qquad \dots(i)$$

and $$\frac{f_1}{f_2}=\frac{33}{32} \qquad \dots(ii)$$

Solving Eqs. (i) and (ii), we get

$$f_1=264\text{ Hz} \quad\text{and}\quad f_2=256\text{ Hz}$$ **Ans.**

Example 7 *A siren emitting a sound of frequency 1000 Hz moves away from you towards a cliff at a speed of 10 m/s.*
(a) What is the frequency of the sound you hear coming directly from the siren?
(b) What is the frequency of sound you hear reflected off the cliff. Speed of sound in air is 330 m/s?

Solution (a) Sound heard directly.

$$f_1 = f_0\left(\frac{v}{v+v_s}\right)$$

$$v_s = 10 \text{ m/s}$$

$\therefore$
$$f_1 = \left(\frac{330}{330+10}\right) \times 1000$$

$$= \frac{33}{34} \times 1000 \text{ Hz}$$ **Ans.**

(b) The frequency of the reflected sound is given by

$$f_2 = f_0\left(\frac{v}{v-v_s}\right)$$

$$f_2 = \left(\frac{330}{330-10}\right) \times 1000$$

$$= \frac{33}{32} \times 1000 \text{ Hz}$$ **Ans.**

Example 8 *A whistle of frequency 540 Hz rotates in a circle of radius 2 m at a linear speed of 30 m/s. What is the lowest and highest frequency heard by an observer a long distance away at rest with respect to the centre of circle. Take speed of sound in air as 330 m/s. Can the apparent frequency be ever equal to actual?*

Solution Apparent frequency will be minimum when the source is at N and moving away from the observer

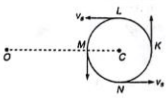


Fig. 16.28

$$f_{min} = \left(\frac{v}{v+v_s}\right) f$$

$$= \left(\frac{330}{330+30}\right)(540)$$

$$= 495 \text{ Hz}$$

Frequency will be maximum when source is at L and approaching the observer. **Ans.**

$$f_{max} = \left(\frac{v}{v-v_s}\right) f$$

$$= \left(\frac{330}{330-30}\right)(540) = 594 \text{ Hz}$$ **Ans.**

Further when source is at M and K, angle between velocity of source and line joining source and observer is 90° or $v_s \cos\theta = v_s \cos 90° = 0$. So, there will be no change in the apparent frequency.

For JEE Advanced

Example 1 *Twocoherent narrow slits emitting sound of wavelength λ in the same phase are placed parallel to each other at a small separation of 2λ. The sound is detected by moving a detector on the screen at a distance D ($>> \lambda$) from the slit S_1 as shown in figure. Find the distance y such that the intensity at P is equal to intensity at O.*

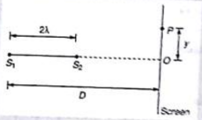


Fig. 16.29

Solution At point O on the screen the path difference between the sound waves reaching from S_1 and S_2 is 2λ, *i.e.*, constructive interference is obtained at O. At a very large distance from point O on the screen the path difference is zero.

Thus, we can conclude that as we move away from point O on the screen path difference decreases from 2λ to zero. At O constructive interference is obtained (where $\Delta x = 2\lambda$). So, next constructive interference will be obtained where $\Delta x = \lambda$. Hence,

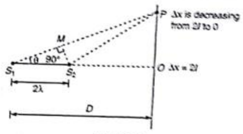


Fig. 16.30

$$S_1P - S_2P = \lambda$$

or
$$\sqrt{D^2+y^2} - \sqrt{y^2+(D-2\lambda)^2} = \lambda$$

$\therefore$
$$\sqrt{D^2+y^2} - \lambda = \sqrt{y^2+(D-2\lambda)^2}$$

Squaring both sides, we get

$$D^2+y^2+\lambda^2-2\lambda\sqrt{D^2+y^2} = y^2+D^2+4\lambda^2-4\lambda D$$

or
$$2\sqrt{D^2+y^2} = 4D-3\lambda$$

as $D >> \lambda$,
$$4D-3\lambda \approx 4D$$

$\therefore$
$$2\sqrt{D^2+y^2} = 4D \quad \text{or} \quad \sqrt{D^2+y^2} = 2D$$

Again squaring both sides, we get

$$D^2+y^2 = 4D^2 \quad \text{or} \quad y = \sqrt{3}\,D$$ **Ans.**

Alternate method : Let $\Delta x = \lambda$ at angle θ as shown. Path difference between the waves is $S_1M = 2\lambda\cos\theta$

$\therefore$
$$2\lambda\cos\theta = \lambda \qquad (\Delta x = \lambda)$$

or
$$\theta = 60°$$

Now,
$$PO = S_1O\cot 30°$$

or
$$y = \sqrt{3}\,D$$ **Ans.**

Example 2 *Two tuning forks with natural frequencies 340 Hz each move relative to a stationary observer. One fork moves away from the observer while the other moves towards him at the same speed. The observer hears beats of frequency 3 Hz. Find the speed of the tuning fork (velocity of sound in air is 340 m/s).*

Note *The difference in apparent frequencies is very small (3 Hz). So, we may conclude that the speed of source (v_s) << speed of sound (v). Therefore, we can neglect the higher terms of $\frac{v_s}{v}$.*

Solution Given $f_1 - f_2 = 3$

or
$$\left(\frac{v}{v-v_s}\right) f - \left(\frac{v}{v+v_s}\right) f = 3$$

or
$$\left[\frac{1}{\left(1-\frac{v_s}{v}\right)} - \frac{1}{\left(1+\frac{v_s}{v}\right)}\right] f = 3$$

or
$$\left[\left(1-\frac{v_s}{v}\right)^{-1} - \left(1+\frac{v_s}{v}\right)^{-1}\right] f = 3$$

or
$$\left[\left(1+\frac{v_s}{v}\right) - \left(1-\frac{v_s}{v}\right)\right] f = 3$$

or
$$\frac{2v_s f}{v} = 3$$

or
$$\text{Speed of tuning fork } v_s = \frac{3v}{2f}$$

Substituting the values, we get

$$v_s = \frac{(3)(340)}{(2)(340)} = 1.5 \text{ m/s}$$ **Ans.**

Example 3 *A fighter plane moving in a vertical loop with constant speed of radius R. The centre of the loop is at a height h directly overhead of an observer standing on the ground. The observer receives maximum frequency of the sound produced by the plane when it is nearest to him. Find the speed of the plane. Velocity of sound in air is v.*

Solution Let the speed of the plane (source) be v_s. Maximum frequency will be observed by the observer when v_s is along SO. The observer receives maximum frequency when the plane is nearest to him. That is as soon as the wave pulse reaches from S to O with speed v the plane reaches from S to S' with speed v_s. Hence,

$$t = \frac{SO}{v} = \frac{SS'}{v_s}$$

or
$$v_s = \left(\frac{SS'}{SO}\right) v$$

$$= \frac{R\theta}{\sqrt{h^2-R^2}}\, v$$ **Ans.**

Here,
$$\cos\theta = \frac{R}{h}$$

or
$$\theta = \cos^{-1}\left(\frac{R}{h}\right)$$

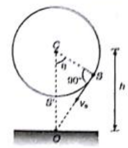


Fig. 16.31

Example 4 *A source of sound of frequency 1000 Hz moves uniformly along a straight line with velocity 0.8 times velocity of sound. An observer is located at a distance l = 250 m from the line. Find :*
(a) the frequency of the sound at the instant when the source is closest to the observer.
(b) the distance of the source when he observes no change in the frequency.

Solution (a) Suppose the pulse which is emitted when the source is at S reaches the observer O in the same time in which the source reaches from S to S', then

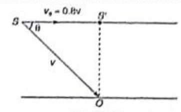


Fig. 16.32

$$\cos\theta = \frac{SS'}{SO} = \frac{v_s t}{vt} = \frac{v_s}{v} = 0.8$$

Now,
$$f' = \left(\frac{v}{v - v_s\cos\theta}\right) f$$

$$= \left\{\frac{v}{v-(0.8v)(0.8)}\right\}(1000)$$

$$= \left(\frac{1}{1-0.64}\right)(1000)$$

$$= 2777.7 \text{ Hz}$$ **Ans.**

(b) The observer will observe no change in the frequency when the source is at S as shown in figure. In the time when the wave pulse reaches from S to O, the source will reach from S to S'. Hence

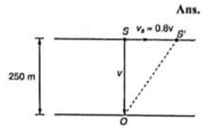


Fig. 16.33

$$t = \frac{SO}{v} = \frac{SS'}{v_s}$$

$\therefore$
$$SS' = \left(\frac{v_s}{v}\right) SO = (0.8)(250) = 200 \text{ m}$$

Therefore, distance of observer from source at this instant is

$$S'O = \sqrt{(SO)^2+(SS')^2}$$

$$= \sqrt{(250)^2+(200)^2} = 320 \text{ m}$$ **Ans.**

Example 5 *The air column in a pipe closed at one end is made to vibrate in its second overtone by tuning fork of frequency 440 Hz. The speed of sound in air is 330 m/s. End corrections may be neglected. Let P_0 denote the mean pressure at any point in the pipe, and ΔP_0 the maximum amplitude of pressure variation.*

(a) *Find the length L of the air column.*
(b) *What is the amplitude of pressure variation at the middle of the column?*
(c) *What are the maximum and minimum pressures at the open end of the pipe?*
(d) *What are the maximum and minimum pressures at the closed end of the pipe?*

Solution (a) Frequency of second overtone of the closed pipe $= 5\left(\frac{v}{4L}\right) = 440$ Hz (Given)

$$\therefore \quad L = \frac{5v}{4 \times 440}\ \text{m}$$

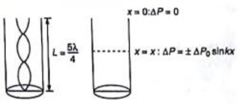


Fig. 16.34

Substituting v = speed of sound in air = 330 m/s

$$L = \frac{5 \times 330}{4 \times 440} = \frac{15}{16}\ \text{m}$$

$$\lambda = \frac{4L}{5} = \frac{4\,(15/16)}{5} = \frac{3}{4}\ \text{m}$$ **Ans.**

(b) Open end is displacement antinode. Therefore, it would be a pressure node or at $x = 0$; $\Delta P = 0$

Pressure amplitude at $x = x$ can be written as

$$\Delta P = \pm \Delta P \sin kx$$

where $$k = \frac{2\pi}{\lambda} = \frac{2\pi}{3/4} = \frac{8\pi}{3}\ \text{m}^{-1}$$

Therefore, pressure amplitude at $x = \frac{L}{2} = \frac{15/16}{2}$ m or (15/32) m will be

$$\Delta P = \pm \Delta P_0 \sin\left(\frac{8\pi}{3}\right)\left(\frac{15}{32}\right) = \pm \Delta P_0 \sin\left(\frac{5\pi}{4}\right)$$

$$\Delta P = \pm \frac{\Delta P_0}{\sqrt{2}}$$ **Ans.**

(c) Open end is a pressure node, *i.e.*, $\Delta P = 0$

Hence, $$P_{max} = P_{min} = \text{Mean pressure } (P_0)$$

(d) Closed end is a displacement node or pressure antinode.

Therefore, $$P_{max} = P_0 + \Delta P_0$$

and $$P_{min} = P_0 - \Delta P_0$$

Example 6 *At a distance 20 m from a point source of sound the loudness level is 30 dB. Neglecting the damping, find*

(a) *the loudness at 10 m from the source*
(b) *the distance from the source at which sound is not heard.*

Solution (a) Intensity due to a point source varies with distance r from it as

$$I \propto \frac{1}{r^2}$$

or $$\frac{I_1}{I_2} = \left(\frac{r_2}{r_1^2}\right)^2$$

Now, $$L_1 = 10 \log \frac{I_1}{I_0}$$

and $$L_2 = 10 \log \frac{I_2}{I_0}$$

$$\therefore \quad L_1 - L_2 = 10\left[\log \frac{I_1}{I_0} - \log \frac{I_2}{I_0}\right]$$

$$= 10 \log \frac{I_1}{I_2} = 10 \log \left(\frac{r_2}{r_1}\right)^2$$

Substituting, $L_1 = 30$ dB, $r_1 = 20$ m and $r_2 = 10$ m

we have $$30 - L_2 = 10 \log \left(\frac{10}{20}\right)^2 = -6.0$$

or $$L_2 = 36\ \text{dB}$$ **Ans.**

(b) $$L_1 - L_2 = 10 \log \left(\frac{r_2}{r_1}\right)^2$$

Sound is not heard at a point where $L_2 = 0$

or $$30 = 10 \log \left(\frac{r_2}{r_1}\right)^2$$

$$\therefore \quad \left(\frac{r_2}{r_1}\right)^2 = 1000 \quad \text{or} \quad \frac{r_2}{r_1} = 31.62$$

$$\therefore \quad r_2 = (31.62)(20) \approx 632\ \text{m}$$ **Ans.**

Example 7 *The first overtone of an open organ pipe beats with the first overtone of a closed organ pipe with a beat frequency of 2.2 Hz. The fundamental frequency of the closed organ pipe is 110 Hz. Find the lengths of the pipes. Speed of sound in air v = 330 m/s.*

Solution Let l_1 and l_2 be the lengths of closed and open pipes respectively.

Fundamental frequency of closed organ pipe is given by

$$f_1 = \frac{v}{4l}$$

v = speed of sound in air = 330 m/s

But $$f_1 = 110\ \text{Hz is given}$$

Therefore, $$\frac{v}{4l_1} = 110\ \text{Hz}$$

$$\therefore \quad l_1 = \frac{v}{4 \times 110} = \frac{330}{4 \times 110}\ \text{m} = 0.75\ \text{m}$$

First overtone of closed organ pipe will be

$$f_3 = 3f_1 = 3\,(110)\ \text{Hz} = 330\ \text{Hz}$$

This produces a beat frequency of 2.2 Hz with first overtone of open organ pipe. Therefore, first overtone frequency of open organ pipe is either

$$(330 + 2.2)\ \text{Hz} = 332.2\ \text{Hz}$$

or $$(330 - 2.2)\ \text{Hz} = 327.8\ \text{Hz}$$

If it is 332.2 Hz, then

$$2\left(\frac{v}{2l_2}\right) = 332.2\ \text{Hz}$$

or $$l_2 = \frac{v}{332.2} = \frac{330}{332.2}\ \text{m} = 0.99\ \text{m}$$

and if it is 327.8 Hz, then

$$2\left(\frac{v}{2l_2}\right) = 327.8\ \text{Hz}$$

or $$l_2 = \frac{v}{327.8}\ \text{m} = \frac{330}{327.8}\ \text{m} = 1.0067\ \text{m}$$

Therefore, length of the closed organ pipe is $l_1 = 0.75$ m while length of open pipe is either $l_2 = 0.99$ m or 1.0067 m. **Ans.**

Example 8 *A point A is located at a distance r = 1.5 m from a point source of sound of frequency 600 Hz. The power of the source is 0.8 W. Speed of sound in air is 340 m/s and density of air is 1.29 kg/m³. Find at the point A,*

(a) *the pressure oscillation amplitude $(\Delta P)_m$*
(b) *the displacement oscillation amplitude A.*

Solution (a) At a distance r from a point source of power P, the intensity of the sound is

$$I = \frac{P}{4\pi r^2} = \frac{0.8}{(4\pi)(1.5)^2}$$

or $$I = 2.83 \times 10^{-2}\ \text{W/m}^2 \quad \text{...(i)}$$

Further, the intensity of sound in terms of $(\Delta P)_m$, ρ and v is given by

$$I = \frac{(\Delta P)_m^2}{2\rho v} \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$(\Delta P)_m = \sqrt{2 \times 2.83 \times 10^{-2} \times 1.29 \times 340}$$

$$= 4.98\ \text{N/m}^2$$ **Ans.**

(b) Pressure oscillation amplitude $(\Delta P)_m$ and displacement oscillation amplitude A are related by the equation

$$(\Delta P)_m = BAk$$

Substituting $B = \rho v^2$, $k = \frac{\omega}{v}$ and $\omega = 2\pi f$

we get $$(\Delta P)_m = 2\pi A \rho v f$$

$$\therefore \quad A = \frac{(\Delta P)_m}{2\pi \rho v f} = \frac{4.98}{(2\pi)(1.29)(340)(600)}$$

$$= 3.0 \times 10^{-6}\ \text{m}$$ **Ans.**

Example 9 *A boat is travelling in a river with a speed 10 m/s along the stream flowing with a speed 2 m/s. From this boat, a sound transmitter is lowered into the river through a rigid support. The wavelength of the sound emitted from the transmitter inside the water is 14.45 mm. Assume that attenuation of sound in water and air is negligible.*

(a) *What will be the frequency detected by a receiver kept inside the river downstream?*
(b) *The transmitter and the receiver are now pulled up into air. The air is blowing with a speed 5 m/s in the direction opposite the river stream. Determine the frequency of the sound detected by the receiver.*

(Temperature of the air and water = 20° C; Density of river water = 10^3 kg/m³; Bulk modulus of the water = 2.088×10^9 Pa; Gas constant, R = 8.31 J/mol-K; Mean molecular mass of air = 28.8×10^{-3} kg/mol; C_P/C_V for air = 1.4)

Solution Velocity of sound in water is

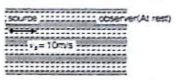


Fig. 16.35

$$v_w = \sqrt{\frac{B}{\rho}}$$

$$= \sqrt{\frac{2.088 \times 10^9}{10^3}} = 1445\ \text{m/s}$$

Frequency of sound in water will be

$$f_0 = \frac{v_w}{\lambda_w} = \frac{1445}{14.45 \times 10^{-3}}\ \text{Hz}$$

$$f_0 = 10^5\ \text{Hz}$$

(a) Frequency of sound detected by receiver (observer) at rest would be

$$f_1 = f_0\left(\frac{v_w + v_r}{v_w + v_r - v_s}\right)$$

$$= (10^5)\left(\frac{1445+2}{1445+2-10}\right) \text{Hz}$$

$$f_1 = 1.0069 \times 10^5 \text{ Hz}$$ **Ans.**

(b) Velocity of sound in air is

$$v_a = \sqrt{\frac{\gamma RT}{M}}$$

$$= \sqrt{\frac{(1.4)(8.31)(20+273)}{28.8\times10^{-3}}}$$

$$= 344 \text{ m/s}$$

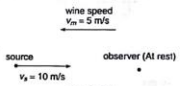


Fig. 16.36

Frequency does not depend on the medium. Therefore, frequency in air is also $f_0 = 10^5$ Hz.

∴ Frequency of sound detected by receiver (observer) in air would be

$$f_2 = f_0\left(\frac{v_a - v_w}{v_a - v_w - v_s}\right) = 10^5\left[\frac{344-5}{344-5-10}\right] \text{Hz}$$

$$f_2 = 1.0304 \times 10^5 \text{ Hz}$$ **Ans.**

Example 10 *Three sound sources A, B and C have frequencies 400, 401 and 402 Hz respectively. Calculate the number of beats noted per second.*

Solution Let us make the following table :

Group	Beat frequency $(f_1 - f_2)$ Hz	Beat time period $\left(\frac{1}{f_1 - f_2}\right)$ sec
A and B	1	1
B and C	1	1
A and C	2	0.5

Beat time period for A and B is 1 s. It implies that if A and B are in phase at time $t = 0$, they are again in phase after 1 s. Same is the case with B and C. But beat time period for A and C is 0.5 s.

Therefore, beat time period for all together A, B and C will be 1 s. Because if, at $t = 0$, A, B and C all are in phase then after 1 s. (A and B) and (B and C) will again be in phase for the first time while (A and C) will be in phase for the second time. Or we can say all A, B and C are again in phase after 1 s.

∴ Beat time period $T_b = 1$ s

or Beat frequency $f_b = \dfrac{1}{T_b} = 1$ Hz **Ans.**

Alternate method Suppose at time t, the equations of waves are

$$y_1 = A_1 \sin 2\pi f_A t$$

$$y_2 = A_2 \sin 2\pi f_B t$$

and $$y_3 = A_3 \sin 2\pi f_C t$$

If they are in phase at the given instant then.

$$2\pi f_A t = 2\pi f_B t = 2\pi f_C t \qquad \text{...(i)}$$

Let T_b be the beat time period, *i.e.*, after time T_b they all are again in phase. As $f_C > f_B > f_A$, so

$$2\pi f_C (t + T_b) = 2\pi f_A (t + T_b) + 2m\pi \qquad \text{...(ii)}$$

and $$2\pi f_B (t + T_b) = 2\pi f_A (t + T_b) + 2n\pi \qquad \text{...(iii)}$$

Here, m and n ($< m$) are positive integers.

From Eqs. (i) and (ii)

$$(f_C - f_A) T_b = m \qquad \text{...(iv)}$$

Similarly, from Eqs. (i) and (iii)

$$(f_B - f_A) T_b = n \qquad \text{...(v)}$$

Dividing (iv) by (v)

$$\frac{m}{n} = \frac{f_C - f_A}{f_B - f_A} = \frac{402 - 400}{401 - 400} = \frac{2}{1}$$

Thus, letting $m = 2$ and $n = 1$

$$T_b = \frac{m}{f_C - f_A} \qquad \text{[from Eq. (iv)]}$$

$$= \frac{2}{2} = 1 \text{ Hz}$$

∴ Beat frequency $f_b = \dfrac{1}{T_b} = 1$ Hz **Ans.**

EXERCISES

For JEE Main

Subjective Questions

Speed of Longitudinal Waves

1. A person standing between two parallel hills fires a gun. He hears the first echo after 3/2 s. and a second echo after 5/2 s. If speed of sound is 332 m/s, calculate the distance between the hills. When will he hear the third echo ?
2. Using the fact that hydrogen gas consists of diatomic molecules with $M = 2$ kg/K-mol. Find the speed of sound in hydrogen at 27°C.
3. Helium is a monoatomic gas that has a density of 0.179 kg/m^3 at a pressure of 76 cm of mercury and a temperature of 0°C. Find the speed of compressional waves (sound) in helium at this temperature and pressure.
4. (a) In a liquid with density 1300 kg/m^3, longitudinal waves with frequency 400 Hz are found to have wavelength 8.00 m. Calculate the bulk modulus of the liquid.
   (b) A metal bar with a length of 1.50 m has density 6400 kg/m^3. Longitudinal sound waves take 3.90×10^{-4} s to travel from one end of the bar to the other. What is Young's modulus for this metal ?
5. What must be the stress (F/A) in a stretched wire of a material whose Young's modulus is Y for the speed of longitudinal waves equal to 30 times the speed of transverse waves?
6. A gas is a mixture of two parts by volume of hydrogen and one part by volume of nitrogen. If the velocity of sound in hydrogen at 0°C is 1300 m/s. Find the velocity of sound in the gaseous mixture at 27°C.

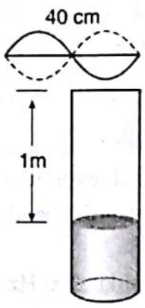
Intensity and Sound Level

7. About how many times more intense will the normal ear perceive a sound of 10^{-6} W/m^2 than one of 10^{-9} W/m^2?
8. The explosion of a fire cracker in the air at a height of 40 m produces a 100 dB sound level at ground below. What is the instantaneous total radiated power? Assuming that it radiates as a point source.
9. (a) What is the intensity of a 60 dB sound ?
   (b) If the sound level is 60 dB close to a speaker that has an area of 120 cm^2. What is the acoustic power output of the speaker?
10. (a) By what factor must the sound intensity be increased to increase the sound intensity level by 13.0 dB ?
    (b) Explain why you do not need to know the original sound intensity?
11. The speed of a certain compressional wave in air at standard temperature and pressure is 330 m/s. A point source of frequency 300 Hz radiates energy uniformly in all directions at the rate of 5 watt.

    (a) What is the intensity of the wave at a distance of 20 m from the source?
    (b) What is the amplitude of the wave there? [Density of air at STP $= 1.29 \text{ kg/m}^3$]
12. What is the amplitude of motion for the air in the path of a 60 dB, 800 Hz sound wave ? Assume that $\rho_{air} = 1.29 \text{ kg/m}^3$ and $v = 330$ m/s.
13. A rock band gives rise to an average sound level of 102 dB at a distance of 20 m from the centre of the band. As an approximation, assume that the band radiates sound equally into a sphere. What is the sound power output of the band ?
14. If it were possible to generate a sinusoidal 300 Hz sound wave in air that has a displacement amplitude of 0.200 mm. What would be the sound level of the wave ? (Assume $v = 330$ m/s and $\rho_{air} = 1.29 \text{ kg/m}^3$)
15. (a) A longitudinal wave propagating in a water-filled pipe has intensity $3.00 \times 10^{-6} \text{ W/m}^2$ and frequency 3400 Hz. Find the amplitude A and wavelength λ of the wave. Water has density 1000 kg/m^3 and bulk modulus 2.18×10^9 Pa. (b) If the pipe is filled with air at pressure 1.00×10^5 Pa and density 1.20 kg/m^3, what will be the amplitude A and wavelength λ of a longitudinal wave with the same intensity and frequency as in part (a)? (c) In which fluid is the amplitude larger, water or air? What is the ratio of the two amplitudes? Why is this ratio so different from one ?
16. For a person with normal hearing, the faintest sound that can be heard at a frequency of 400 Hz has a pressure amplitude of about 6.0×10^{-5} Pa. Calculate the corresponding intensity and sound intensity level at 20°C.

Organ Pipes

17. The fundamental frequency of an open pipe is 594 Hz. What is the fundamental frequency if one end is closed ?
18. Find the fundamental frequency and the frequency of the first three overtones of a pipe 45.0 cm long. (a) If the pipe is open at both ends. (b) If the pipe is closed at one end. Use $v = 344$ m/s.
19. A uniform tube of length 60 cm stands vertically with its lower end dipping into water. First two air column lengths above water are 15 cm and 45 cm, when the tube responds to a vibrating fork of frequency 500 Hz. Find the lowest frequency to which the tube will respond when it is open at both ends.
20. Write the equation for the fundamental standing sound waves in a tube that is open at both ends. If the tube is 80 cm long and speed of the wave is 330 m / s. Represent the amplitude of the wave at an antinode by A.
21. A long glass tube is held vertically, dipping into water, while a tuning fork of frequency 512 Hz is repeatedly struck and held over the open end. Strong resonance is obtained, when the length of the tube above the surface of water is 50 cm and again 84 cm, but not at any intermediate point. Find the speed of sound in air and next length of the air column for resonance.
22. A wire of length 40 cm which has a mass of 4 g oscillates in its second harmonic and sets the air column in the tube to vibrations in its fundamental mode as shown in figure. Assuming the speed of sound in air as 340 m/s, find the tension in the wire.



23. In a resonance tube experiment to determine the speed of sound in air, a pipe of diameter 5 cm is used. The column in pipe resonates with a tuning fork of frequency 480 Hz when the minimum length of the air column is 16 cm. Find the speed of sound in air column at room temperature.
24. The air column in a pipe closed at one end is made to vibrate in its second overtone by tuning fork of frequency 440 Hz. The speed of sound in air is 330 m / s. End corrections may be neglected. Let P_0 denote the mean pressure at any point in the pipe, and ΔP_0 the maximum amplitude of pressure variation.
(a) Find the length L of the air column.
(b) What is the amplitude of pressure variation at the middle of the column ?
(c) What are the maximum and minimum pressure at the open end of the pipe ?
(d) What are the maximum and minimum pressure at the closed end of the pipe ?
25. On a day when the speed of sound is 345 m/s, the fundamental frequency of a closed organ pipe is 220 Hz. (a) How long is this closed pipe ? (b) The second overtone of this pipe has the same wavelength as the third harmonic of an open pipe. How long is the open pipe ?
26. A closed organ pipe is sounded near a guitar, causing one of the strings to vibrate with large amplitude. We vary the tension of the string until we find the maximum amplitude. The string is 80% as long as the closed pipe. If both the pipe and the string vibrate at their fundamental frequency, calculate the ratio of the wave speed on the string to the speed of sound in air.

Beats and Doppler Effect

27. A police siren emits a sinusoidal wave with frequency $f_S = 300$ Hz. The speed of sound is 340 m/s. (a) Find the wavelength of the waves if the siren is at rest in the air. (b) If the siren is moving at 30 m/s, find the wavelength of the waves ahead of and behind the source.
28. Two identical violin strings, when in tune and stretched with the same tension, have a fundamental frequency of 440.0 Hz. One of the string is retuned by adjusting its tension. When this is done, 1.5 beats per second are heard when both strings are plucked simultaneously. (a) What are the possible fundamental frequencies of the retuned string? (b) By what fractional amount was the string tension changed if it was (i) increased (ii) decreased ?
29. A swimming duck paddles the water with its feet once every 1.6 s, producing surface waves with this period. The duck is moving at constant speed in a pond where the speed of surface waves is 0.32 m/s, and the crests of the waves ahead of the duck are spaced 0.12 m apart. (a) What is the duck's speed? (b) How far apart are the crests behind the duck?

30. A railroad train is travelling at 30.0 m/s in still air. The frequency of the note emitted by the train whistle is 262 Hz. What frequency is heard by a passenger on a train moving in the opposite direction to the first at 18.0 m/s and (a) approaching the first? (b) receding from the first? Speed of sound in air = 340 m/s.

31. A boy is walking away from a wall at a speed of 1.0 m/s in a direction at right angles to the wall. As he walks, he blows a whistle steadily. An observer towards whom the boy is walking hears 4.0 beats per second. If the speed of sound is 340 m/s, what is the frequency of the whistle?

32. The first overtone of an open organ pipe beats with the first overtone of a closed organ pipe with a beat frequency of 2.2 Hz. The fundamental frequency of the closed organ pipe is 110 Hz. Find the lengths of the pipes. Speed of sound in air $v = 330$ m/s.

33. A tuning fork P of unknown frequency gives 7 beats in 2 seconds with another tuning fork Q. When Q runs towards a wall with a speed of 5 m/s it gives 5 beats per second with its echo. On loading P with wax, it gives 5 beats per second with Q. What is the frequency of P ? Assume speed of sound = 332 m/s.

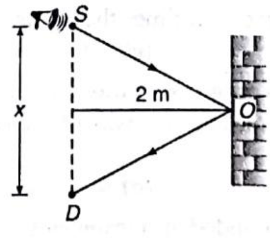
34. A stationary observer receives sonic oscillations from two tuning forks one of which approaches and the other recedes with the same velocity. As this takes place, the observer hears the beats of frequency $f = 2.0$ Hz. Find the velocity of each tuning fork if their oscillation frequency is $f_0 = 680$ Hz and the velocity of sound in air is $v = 340$ m/s.

Interference in Sound

35. Sound waves from a tuning fork A reach a point P by two separate paths ABP and ACP. When ACP is greater than ABP by 11.5 cm, there is silence at P. When the difference is 23 cm the sound becomes loudest at P and when 34.5 cm there is silence again and so on. Calculate the minimum frequency of the fork if the velocity of sound is taken to be 331.2 m/s.

36. Two loudspeakers S_1 and S_2 each emit sounds of frequency 220 Hz uniformly in all directions. S_1 has an acoustic output of 1.2×10^{-3} W and S_2 has 1.8×10^{-3} W. S_1 and S_2 vibrate in phase. Consider a point P such that $S_1P = 0.75$ m and $S_2P = 3$ m. How are the phases arriving at P related? What is the intensity at P when both S_1 and S_2 are on? Speed of sound in air is 330 m/s.

37. A source of sound emitting waves at 360 Hz is placed in front of a vertical wall, at a distance 2 m from it. A detector is also placed in front of the wall at the same distance from it. Find the minimum distance between the source and the detector for which the detector detects a maximum of sound. Take speed of sound in air = 360 m/s.



Objective Questions

Single Correct Option

1. Velocity of sound in vacuum is
(a) equal to 330 m/s (b) greater than 330 m/s
(c) less than 330 m/s (d) None of these

2. Longitudinal waves are possible in
(a) solids (b) liquids (c) gases (d) All of these

3. The temperature at which the velocity of sound in oxygen will be the same as that of nitrogen at 15°C is
(a) 112°C (b) 72°C (c) 56°C (d) 17°C

4. A closed organ pipe is excited to vibrate in the third overtone. It is observed that there are
(a) three nodes and three antinodes (b) three nodes and four antinodes
(c) four nodes and three antinodes (d) four nodes and four antinodes

5. When temperature is increased, the frequency of organ pipe
(a) increases (b) decreases (c) remains same (d) Nothing can be said

6. When a sound wave travels from water to air, it
(a) bends towards normal (b) bends away from normal
(c) may bend in any direction (d) data insufficient

7. A closed organ pipe and an open organ pipe are tuned to the same fundamental frequency. The ratio of their lengths is
(a) 1 : 2 (b) 2 : 1 (c) 1 : 4 (d) 4 : 1

8. A sonometer wire under a tension of 10 kg weight is in unison with a tuning fork of frequency 320 Hz. To make the wire vibrate in unison with a tuning fork of frequency 256 Hz, the tension should be altered by
(a) 3.6 kg decreased (b) 3.6 kg increased (c) 6.4 kg decreased (d) 6.4 kg increased

9. A tuning fork of frequency 256 Hz is moving towards a wall with a velocity of 5 m/s. If the speed of sound is 330 m/s, then the number of beats heard per second by a stationary observer lying between tuning fork and the wall is
(a) 2 (b) 4 (c) zero (d) 8

10. Two sound waves of wavelength 1 m and 1.01 m in a gas produce 10 beats in 3 s. The velocity of sound in the gas is
(a) 330 m/s (b) 337 m/s (c) 360 m/s (d) 300 m/s

11. When a source is going away from a stationary observer with the velocity equal to that of sound in air, then the frequency heard by observer is n times the original frequency. The value of n is
(a) 0.5 (b) 0.25 (c) 1.0 (d) no sound is heard

12. When interference is produced by two progressive waves of equal frequencies, then the maximum intensity of the resulting sound are N times the intensity of each of the component waves. The value of N is
(a) 1 (b) 2 (c) 4 (d) 8

13. A tuning fork of frequency 500 Hz is sounded on a resonance tube. The first and second resonances are obtained at 17 cm and 52 cm. The velocity of sound is
(a) 170 m/s (b) 350 m/s (c) 520 m/s (d) 850 m/s

14. A vehicle, with a horn of frequency n is moving with a velocity of 30 m/s in a direction perpendicular to the straight line joining the observer and the vehicle. The observer perceives the sound to have a frequency $(n + n_1)$. If the sound velocity in air is 300 m/s, then
(a) $n_1 = 10n$ (b) $n_1 = 0$ (c) $n_1 = 0.1n$ (d) $n_1 = -0.1n$

15. How many frequencies below 1 kHz of natural oscillations of air column will be produced if a pipe of length 1 m is closed at one end ? [velocity of sound in air is 340 m/s]
(a) 5 (b) 6 (c) 4 (d) 8

16. A sound source emits frequency of 180 Hz when moving towards a rigid wall with speed 5 m/s and an observer is moving away from wall with speed 5 m/s. Both source and observer moves on a straight line which is perpendicular to the wall. The number of beats per second heard by the observer will be [speed of sound = 355 m/s]
(a) 5 beats/s (b) 10 beats/s (c) 6 beats/s (d) 8 beats/s

17. Two sound waves of wavelengths λ_1 and λ_2 $(\lambda_2 > \lambda_1)$ produce n beats/s, the speed of sound is
(a) $\dfrac{n\lambda_1\lambda_2}{\lambda_2 - \lambda_1}$ (b) $n\left(\dfrac{1}{\lambda_1} - \dfrac{1}{\lambda_2}\right)$ (c) $n(\lambda_2 - \lambda_1)$ (d) $n(\lambda_2 + \lambda_1)$

18. A, B and C are three tuning forks. Frequency of A is 350 Hz. Beats produced by A and B are 5/s and by B and C are 4/s. When a wax is put on A beat frequency between A and B is 2 Hz and between A and C is 6 Hz. Then frequency of B and C respectively are
(a) 355 Hz, 349 Hz (b) 345 Hz, 341 Hz (c) 355 Hz, 341 Hz (d) 345 Hz, 349 Hz

19. The first resonance length of a resonance tube is 40 cm and the second resonance length is 122 cm. The third resonance length of the tube will be
(a) 200 cm (b) 202 cm (c) 203 cm (d) 204 cm

20. Two identical wires are stretched by the same tension of 100 N and each emits a note of frequency 200 Hz. If the tension in one wire is increased by 1 N, then the beat frequency is
(a) 2 Hz (b) $\dfrac{1}{2}$ Hz (c) 1 Hz (d) None

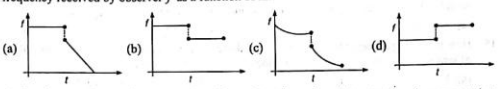
21. A tuning fork of frequency 340 Hz is sounded above an organ pipe of length 120 cm. Water is now slowly poured in it. The minimum height of water column required for resonance is (speed of sound in air = 340 m/s)
(a) 25 cm (b) 95 cm (c) 75 cm (d) 45 cm

22. In a closed end pipe of length 105 cm, standing waves are set up corresponding to the third overtone. What distance from the closed end, amongst the following is a pressure node ?
(a) 20 cm (b) 60 cm (c) 85 cm (d) 45 cm

23. If the fundamental frequency of a pipe closed at one end is 512 Hz. The frequency of a pipe of the same dimension but open at both ends will be
(a) 1024 Hz (b) 512 Hz (c) 256 Hz (d) 128 Hz

24. Oxygen is 16 times heavier than hydrogen. At NTP equal volume of hydrogen and oxygen are mixed. The ratio of speed of sound in the mixture to that in hydrogen is
(a) $\sqrt{8}$ (b) $\sqrt{\dfrac{1}{8}}$ (c) $\sqrt{\dfrac{2}{17}}$ (d) $\sqrt{\dfrac{32}{17}}$

25. A train is moving towards a stationary observer. Which of the following curve best represents the frequency received by observer f as a function of time?



26. A closed organ pipe and an open pipe of same length produce 4 beats when they are set into vibrations simultaneously. If the length of each of them were twice their initial lengths, the number of beats produced will be
(a) 2 (b) 4 (c) 1 (d) 8

27. One train is approaching an observer at rest and another train is receding from him with the same velocity 4 m/s. Both trains blow whistles of same frequency of 243 Hz. The beat frequency in Hz as heard by the observer is (speed of sound in air = 320 m/s)
(a) 10 (b) 6 (c) 4 (d) 1

28. Speed of sound in air is 320 m/s. A pipe closed at one end has a length of 1 m and there is another pipe open at both ends having a length of 1.6 m. Neglecting end corrections, both the air columns in the pipes can resonate for sound of frequency
(a) 80 Hz (b) 240 Hz (c) 320 Hz (d) 400 Hz

29. Four sources of sound each of sound level 10 dB are sounded together in phase, the resultant intensity level will be $(\log_{10} 2 = 0.3)$
(a) 40 dB (b) 26 dB (c) 22 dB (d) 13 dB

30. A longitudinal sound wave given by $P = 2.5 \sin \dfrac{\pi}{2}(x - 600t)$ (P is in N/m^2 and x is in metre and t is in second) is sent down a closed organ pipe. If the pipe vibrates in its second overtone, the length of the pipe is
(a) 6 m (b) 8 m (c) 5 m (d) 10 m

31. Sound waves of frequency 600 Hz fall normally on perfectly reflecting wall. The distance from the wall at which the air particles have the maximum amplitude of vibration is
(speed of sound in air = 330 m/s)
(a) 13.75 cm (b) 40.25 cm (c) 70.5 cm (d) 60.75 cm

32. The wavelength of two sound waves are 49 cm and 50 cm respectively. If the room temperature is 30°C then the number of beats produced by them is approximately
(velocity of sound in air at 30°C = 332 m/s)
(a) 6 (b) 10 (c) 13 (d) 18

33. Two persons A and B, each carrying a source of frequency 300 Hz, are standing a few metre apart. A starts moving towards B with velocity 30 m/s. If the speed of sound is 300 m/s, which of the following is true?
(a) Number of beats heard by A is higher than that heard by B
(b) The number of beats heard by B are 30 Hz
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

34. A fixed source of sound emitting a certain frequency appears as f_a when the observer is approaching the source with speed v_0 and f_r when the observer recedes from the source with the same speed. The frequency of the source is
(a) $\dfrac{f_r + f_a}{2}$ (b) $\dfrac{f_r - f_a}{2}$ (c) $\sqrt{f_a f_r}$ (d) $\dfrac{2f_r f_a}{f_r + f_a}$

For JEE Advanced

Assertion and Reason

Directions : *Choose the correct option.*
(a) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the Assertion.*
(b) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of Assertion.*
(c) *If **Assertion** is true, but the **Reason** is false.*
(d) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** A closed pipe and an open organ pipe are of same length. Then neither of their frequencies can be same.
Reason : In the above case fundamental frequency of closed pipe will be two times the fundamental frequency of open pipe.

2. **Assertion :** A sound source is approaching towards a stationary observer along the line joining them. Then apparent frequency to the observer will go on increasing.
Reason : If there is no relative motion between source and observer apparent frequency is equal to the actual frequency.

3. **Assertion :** In longitudinal wave pressure is maximum at a point where displacement is zero.
Reason : There is a phase difference of $\dfrac{\pi}{2}$ between $y(x, t)$ and $\Delta P(x, t)$ equation in case of longitudinal wave.

4. **Assertion :** A train is approaching towards a hill. The driver of the train will hear beats.
Reason : Apparent frequency of reflected sound observed by driver will be more than the frequency of direct sound observed by him.

5. **Assertion :** Sound level increases linearly with intensity of sound.
Reason : If intensity of sound is doubled sound level increases approximately 3 dB.

6. **Assertion :** Speed of sound in gases is independent of pressure of gas.
Reason : With increase in temperature of gas speed of sound will increase.

7. **Assertion :** Beat frequency between two tuning forks A and B is 4 Hz. Frequency of A is greater than the frequency of B. When A is loaded with wax, beat frequency may increase or decrease.
Reason : When a tuning fork is loaded with wax its frequency decreases.

8. **Assertion :** Two successive frequencies of an organ pipe are 450 Hz and 750 Hz. Then this pipe is a closed pipe.
Reason : Fundamental frequency of this pipe is 150 Hz.

9. **Assertion :** Fundamental frequency of a narrow pipe is more.
Reason : According to Laplace end correction if radius of pipe is less, frequency should be more.

10. **Assertion :** In the experiment of finding speed of sound by resonance tube method, as the level of water is lowered, wavelength increases.

Reason : By lowering the water level number of loops increases.

Objective Questions

Single Correct Option

1. A plane wave of sound travelling in air is incident upon a plane water surface. The angle of incidence is 60°. If velocity of sound in air and water are 330 m/s and 1400 m/s, then the wave undergoes
   (a) refraction only (b) reflection only
   (c) Both reflection and refraction (d) neither reflection nor refraction

2. An organ pipe of 3.9 π m long, open at both ends is driven to third harmonic standing wave. If the amplitude of pressure oscillation is 1% of mean atmospheric pressure [$p_0 = 10^5 \text{ N/m}^2$]. The maximum displacement of particle from mean position will be
   [Given velocity of sound = 200 m/s and density of air = 1.3 kg/m^3]
   (a) 2.5 cm (b) 5 cm (c) 1 cm (d) 2 cm

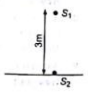
3. A plane sound wave passes from medium 1 into medium 2. The speed of sound in medium 1 is 200 m/s and in medium 2 is 100 m/s. The ratio of amplitude of the transmitted wave to that of incident wave is
   (a) $\frac{3}{4}$ (b) $\frac{4}{5}$ (c) $\frac{5}{6}$ (d) $\frac{2}{3}$

4. Two sources of sound are moving in opposite directions with velocities v_1 and v_2 ($v_1 > v_2$). Both are moving away from a stationary observer. The frequency of both the sources is 1700 Hz. What is the value of $(v_1 - v_2)$ so that the beat frequency observed by the observer is 10 Hz? $v_{sound} = 340$ m/s and assume that v_1 and v_2 both are very much less than v_{sound}.
   (a) 1 m/s (b) 2 m/s (c) 3 m/s (d) 4 m/s

5. A sounding body emitting a frequency of 150 Hz is dropped from a height. During its fall under gravity it crosses a balloon moving upwards with a constant velocity of 2 m/s one second after it started to fall. The difference in the frequency observed by the man in balloon just before and just after crossing the body will be (velocity of sound = 300 m/s, $g = 10 \text{ m/s}^2$)
   (a) 12 (b) 6 (c) 8 (d) 4

6. A closed organ pipe has length L. The air in it is vibrating in third overtone with maximum amplitude a. The amplitude at distance $\frac{L}{7}$ from closed end of the pipe is
   (a) 0 (b) a (c) $\frac{a}{2}$ (d) Data insufficient

7. S_1 and S_2 are two coherent sources of sound having no initial phase difference. The velocity of sound is 330 m/s. No maxima will be formed on the line passing through S_2 and perpendicular to the line joining S_1 and S_2. If the frequency of both the sources is



   (a) 330 Hz (b) 120 Hz
   (c) 100 Hz (d) 220 Hz

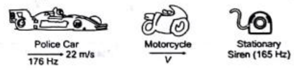
8. A source is moving with constant speed $v_s = 20$ m/s towards a stationary observer due east of the source. Wind is blowing at the speed of 20 m/s at 60° north of east. The source has frequency 500 Hz. Speed of sound = 300 m/s. The frequency registered by the observer is approximately
   (a) 541 Hz (b) 552 Hz (c) 534 Hz (d) 517 Hz

9. A car travelling towards a hill at 10 m/s sounds its horn which has a frequency 500 Hz. This is heard in a second car travelling behind the first car in the same direction with speed 20 m/s. The sound can also be heard in the second car by reflection of sound from the hill. The beat frequency heard by the driver of the second car will be (speed of sound in air = 340 m/s)
   (a) 31 Hz (b) 24 Hz (c) 21 Hz (d) 34 Hz

10. Two sounding bodies are producing progressive waves given by $y_1 = 2\sin(400\pi t)$ and $y_2 = \sin(404\pi t)$ where t is in second, which superpose near the ears of a person. The person will hear
    (a) 2 beats/s with intensity ratio 9/4 between maxima and minima
    (b) 2 beats/s with intensity ratio 9 between maxima and minima
    (c) 4 beats/s with intensity ratio 16 between maxima and minima
    (d) 4 beats/s with intensity ratio 16/9 between maxima and minima

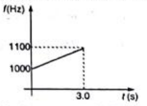
11. The air in a closed tube 34 cm long is vibrating with two nodes and two antinodes and its temperature is 51°C. What is the wavelength of the waves produced in air outside the tube, when the temperature of air is 16°C?
    (a) 42.8 cm (b) 68 cm (c) 17 cm (d) 102 cm

12. A police car moving at 22 m/s, chase a motorcyclist. The police man sounds his horn at 176 Hz, while both of them move towards a stationary siren of frequency 165 Hz. Calculate the speed of the motorcyclist, if he does not observe any beats. (velocity of sound in air = 330 m/s)
    (a) 33 m/s (b) 22 m/s (c) zero (d) 11 m/s



13. A closed organ pipe resonates in its fundamental mode at a frequency of 200 Hz with O_2 in the pipe at a certain temperature. If the pipe now contains 2 moles of O_2 and 3 moles of ozone, then what will be the fundamental frequency of same pipe at same temperature?
    (a) 268.23 Hz (b) 175.4 Hz (c) 149.45 Hz (d) None of these

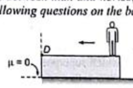
14. A detector is released from rest over a source of sound of frequency $f_0 = 10^3$ Hz. The frequency observed by the detector at time t is plotted in the graph. The speed of sound in air is ($g = 10 \text{ m/s}^2$)



    (a) 330 m/s (b) 350 m/s (c) 300 m/s (d) 310 m/s

Passage : (Q. 15 - Q. 17)

A man of mass 50 kg is running on a plank of mass 150 kg with speed of 8 m/s relative to plank as shown in the figure (both were initially at rest and the velocity of man with respect to ground any how remains constant). Plank is placed on smooth horizontal surface. The man, while running whistles with frequency f_0. A detector (D) placed on plank detects frequency. The man jumps off with same velocity (w.r.t. to ground) from point D and slides on the smooth horizontal surface [Assume coefficient of friction between man and horizontal is zero]. The speed of sound in still medium is 330 m/s. Answer following questions on the basis of above situations.

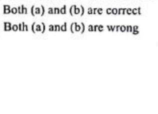


15. The frequency of sound detected by detector D, before man jumps off the plank is
    (a) $\frac{332}{324} f_0$ (b) $\frac{330}{322} f_0$ (c) $\frac{328}{336} f_0$ (d) $\frac{330}{338} f_0$

16. The frequency of sound detected by detector D, after man jumps off the plank is
    (a) $\frac{332}{324} f_0$ (b) $\frac{330}{322} f_0$ (c) $\frac{328}{336} f_0$ (d) $\frac{330}{338} f_0$

17. Choose the correct plot between the frequency detected by detector versus position of the man relative to detector

(a)



(b)

(c)

(d)

18. Sound waves are travelling along positive x-direction. Displacement of particle at any time t is as shown in figure. Select the wrong statement.

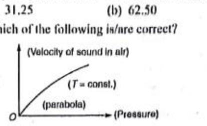
    (a) Particle located at E has its velocity in negative x-direction
    (b) Particle located at D has zero velocity
    (c) Both (a) and (b) are correct
    (d) Both (a) and (b) are wrong

More than One Correct Options

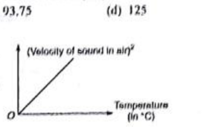
19. An air column in a pipe, which is closed at one end, is in resonance with a vibrating tuning fork of frequency 264 Hz. If $v = 330$ m/s, the length of the column in cm is (are)
    (a) 31.25 (b) 62.50 (c) 93.75 (d) 125

20. Which of the following is/are correct?

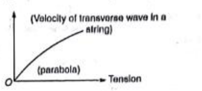
(a)



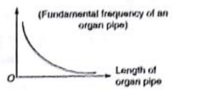
(b)



(c)



(d)



21. Choose the correct options for longitudinal wave
    (a) maximum pressure variation is BAk
    (b) maximum density variation is ρAk
    (c) pressure equation and density equation are in phase
    (d) pressure equation and displacement equation are out of phase

22. Second overtone frequency of a closed pipe and fourth harmonic frequency of an open pipe are same. Then choose the correct options.
    (a) Fundamental frequency of closed pipe is more than the fundamental frequency of open pipe
    (b) First overtone frequency of closed pipe is more than the first overtone frequency of open pipe
    (c) Fifteenth harmonic frequency of closed pipe is equal to twelfth harmonic frequency of open pipe
    (d) Tenth harmonic frequency of closed pipe is equal to eighth harmonic frequency of open pipe

23. For fundamental frequency f of a closed pipe, choose the correct options.
    (a) If radius of pipe is increased f will decrease
    (b) If temperature is increased f will increase
    (c) If molecular mass of the gas filled in the pipe is increased f will decrease.
    (d) If pressure of gas (filled in the pipe) is increased without change in temperature, f will remain unchanged

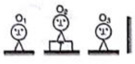
24. A source is approaching towards an observer with constant speed along the line joining them. After crossing the observer, source recedes from observer with same speed. Let f is apparent frequency heard by observer. Then
    (a) f will increase during approaching
    (b) f will decrease during receding
    (c) f will remain constant during approaching
    (d) f will remain constant during receding

Match the Columns

1. Fundamental frequency of an open organ pipe is f. Match the following two columns for a closed pipe of double the length.

Column I	Column II
(a) Fundamental frequency	(p) $1.25f$
(b) Second overtone frequency	(q) f
(c) Third harmonic frequency	(r) $0.75f$
(d) First overtone frequency	(s) None

2. A train T horns a sound of frequency f. It is moving towards a wall with speed $\frac{1}{4}$ th the speed of sound. There are three observers O_1, O_2 and O_3 as shown. Match the following two columns :



Column I	Column II
(a) Beat frequency observed to O_1	(p) $\frac{2}{3}f$
(b) Beat frequency observed to O_2	(q) $\frac{8}{15}f$
(c) Beat frequency observed to O_3	(r) None
(d) If train moves in opposite direction with same speed then beat frequency observed to O_3	(s) Zero

3. A tuning fork is placed near a vibrating stretched wire. A boy standing near the two hears a beat frequency f. It is known that frequency of tuning fork is greater than frequency of stretched wire. Match the following two columns.

Column I	Column II
(a) If tuning fork is loaded with wax.	(p) beat frequency must increase.
(b) If prongs of tuning fork are filed.	(q) beat frequency must decrease.
(c) If tension in stretched wire is increased.	(r) beat frequency may increase.
(d) If tension in stretched wire is decreased	(s) beat frequency may decrease.

4. I represents intensity of sound wave, A the amplitude and r the distance from the source. Then match the following two columns.

Column I	Column II
(a) Intensity due to a point source.	(p) Proportional to $r^{-1/2}$
(b) Amplitude due to a point source.	(q) Proportional to r^{-1}
(c) Intensity due to a line source.	(r) Proportional to r^{-2}
(d) Amplitude due to a line source.	(s) Proportional to r^{-4}

5. Equation of longitudinal stationary wave in second overtone mode in a closed organ pipe is

$$y = (4 \text{ mm}) \sin \pi x \cos \pi t$$

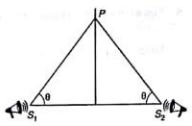
Here x is in metre and t in second. Then match the following two columns ;

Column I	Column II
(a) Length of pipe	(p) 1 m
(b) Wavelength	(q) 1.5 m
(c) Distance of displacement node from the closed end	(r) 2.0 m
(d) Distance of pressure node from the closed end	(s) None

Subjective Questions

1. A train of length l is moving with a constant speed v along a circular track of radius R. The engine of the train emits a sound of frequency f. Find the frequency heard by a guard at the rear end of the train.
2. A 3 m long organ pipe open at both ends is driven to third harmonic standing wave. If the amplitude of pressure oscillations is 1 per cent of mean atmospheric pressure ($P_0 = 10^5 \text{ N/m}^2$). Find the amplitude of particle displacement and density oscillations. Speed of sound $v = 332$ m/s and density of air $\rho = 1.03 \text{ kg/m}^3$.
3. A siren creates a sound level of 60 dB at a location 500 m from the speaker. The siren is powered by a battery that delivers a total energy of 1.0 kJ. Assuming that the efficiency of siren is 30%, determine the total time the siren can sound.
4. A cylinder of length 1 m is divided by a thin perfectly flexible diaphragm in the middle. It is closed by similar flexible diaphragms at the ends. The two chambers into which it is divided contain hydrogen and oxygen. The two diaphragms are set in vibrations of same frequency. What is the minimum frequency of these diaphragms for which the middle diaphragm will be motionless ? Velocity of sound in hydrogen is 1100 m/s and that in oxygen is 300 m/s.
5. A conveyor belt moves to the right with speed $v = 300$ m/min. A very fast pieman puts pies on the belt at a rate of 20 per minute and they are received at the other end by a pieeater.
   (a) If the pieman is stationary find the spacing x between the pies and the frequency with which they are received by the stationary pieeater.
   (b) The pieman now walks with speed 30 m/min towards the receiver while continuing to put pies on the belt at 20 per minute. Find the spacing of the pies and the frequency with which they are received by the stationary pieeater.
6. A point sound source is situated in a medium of bulk modulus $1.6 \times 10^5 \text{ N/m}^2$. An observer standing at a distance 10 m from the source writes down the equation for the wave as $y = A \sin (15\pi x - 6000 \pi t)$. Here y and x are in metres and t is in second. The maximum pressure amplitude received to the observer's ear is 24π Pa, then find :
   (a) the density of the medium,
   (b) the displacement amplitude A of the waves received by the observer and
   (c) the power of the sound source.

7. Two sources of sound S_1 and S_2 vibrate at the same frequency and are in phase. The intensity of sound detected at a point P (as shown in figure) is I_0.
   (a) If $\theta = 45°$ what will be the intensity of sound detected at this point if one of the sources is switched off ?
   (b) What will be intensity of sound detected at P if $\theta = 60°$ and both the sources are now switched on?



8. Two narrow cylindrical pipes A and B have the same length. Pipe A is open at both ends and is filled with a monoatomic gas of molar mass M_A. Pipe B is open at one end and closed at the other end and is filled with a diatomic gas of molar mass M_B. Both gases are at the same temperature.
   (a) If the frequency of the second harmonic of the fundamental mode in pipe A is equal to the frequency of the third harmonic of the fundamental mode in pipe B, determine the value of $\frac{M_A}{M_B}$.
   (b) Now, the open end of pipe B is also closed (so that the pipe is closed at both ends.) Find the ratio of the fundamental frequency in pipe A to that in pipe B.
9. A boat is travelling in a river with a speed 10 m/s along the stream flowing with a speed 2 m/s. From this boat a sound transmitter is lowered into the river through a rigid support. The wavelength of the sound emitted from the transmitter inside the water is 14.45 mm. Assume that attenuation of sound in water and air is negligible.
   (a) What will be the frequency detected by a receiver kept inside the river downstream?
   (b) The transmitter and the receiver are now pulled up into the air. The air is blowing with a speed 5 m/s in the direction opposite the river stream. Determine the frequency of the sound detected by the receiver.

   (Temperature of the air and water = 20°C; Density of river water = 10^3 kg/m^3; Bulk modulus of the water $= 2.088 \times 10^9$ Pa; Gas constant $R = 8.31$ J/mol-K; Mean molecular mass of air $= 28.8 \times 10^{-3}$ kg per mol and C_P/C_V for air = 1.4)
10. A string 25 cm long and having a mass of 2.5 g is under tension. A pipe closed at one end is 40 cm long. When the string is set vibrating in its first overtone and the air in the pipe in its fundamental frequency, 8 beats per second are heard. It is observed that decreasing the tension in the string decrease the beat frequency. If the speed of sound in air is 320 m/s, find the tension in the string.
11. A source emits sound waves of frequency 1000 Hz. The source moves to the right with a speed of 32 m/s relative to ground. On the right a reflecting surface moves towards left with a speed of 64 m/s relative to the ground. The speed of sound in air is 332 m/s. Find :
    (a) the wavelength of sound in ahead of the source,
    (b) the number of waves arriving per second which meets the reflecting surface,
    (c) the speed of reflected waves and
    (d) the wavelength of reflected waves.

ANSWERS

Introductory Exercise 16.1
1. $1.4 \times 10^5 \text{ N/m}^2$ 2. 7.25 cm, 72.5 m 3. (a) Zero (b) 3.63×10^{-6} m
4. 1.04×10^{-5} m

Introductory Exercise 16.2
1. 819°C 2. 20.06 m/s 3. 3.6×10^9 Pa 4. 315 m/s

Introductory Exercise 16.3
1. (a) 4.67 Pa (b) $2.64 \times 10^{-2} \text{ W/m}^2$ (c) 104 dB 2. 7.9 3. 20 dB
4. Faintest (a) $4.49 \times 10^{-13} \text{ W/m}^2$, −3.48 dB (b) 1.43×10^{-11} m
   Loudest (a) 0.881 W/m^2, +119 dB (b) 2.01×10^{-5} m

Introductory Exercise 16.4
1. 1375 Hz 2. (a) 11.7 cm (b) 180° 3. $\lambda = 2\sqrt{4(H+h)^2 + d^2} - 2\sqrt{4H^2 + d^2}$
4. (a) 8.14° (b) 16.5° (c) 3 Maxima beyond the $\theta = 0°$ maximum
5. (a) 0 (b) $2I_0$ (c) $4I_0$ 6. (a) 0 (b) 66 dB (c) 63 dB
7. (a) $I_1 = 19.9 \,\mu\text{W/m}^2$, $I_2 = 8.84 \,\mu\text{W/m}^2$ (b) $55.3 \,\mu\text{W/m}^2$ (c) $2.2 \,\mu\text{W/m}^2$ (d) $28.7 \,\mu\text{W/m}^2$

Introductory Exercise 16.5
1. (a) 0.392 m (b) 0.470 m
2. (a) Fundamental 0.8 m, first overtone 0.267 m, 0.8 m, second overtone 0.16 m, 0.48 m, 0.8 m
   (b) Fundamental 0, first overtone 0, 0.533 m. Second overtone 0, 0.32 m, 0.64 m
3. (a) closed (b) 5, 7 (c) 1.075 m 4. Diatomic 5. 352 m/s

Introductory Exercise 16.6
1. 252 Hz 2. 387 Hz 3. 1.02 4. 0.4 cm

Introductory Exercise 16.7
2. (a) 1.3 m (b) 262 Hz 3. No 4. (a) 0.628 m (b) 0.748 m (c) 548 Hz (d) 460 Hz
5. 274 Hz

For JEE Main

Subjective Questions
1. 10.664 m, 4 s 2. 1321 m/s 3. 972 m/s
4. (a) 1.33×10^{10} Pa (b) 9.47×10^{10} Pa 5. $Y/900$ 6. 591 m/s 7. Two times
8. 201 W 9. (a) 10^{-6} W/m^2 (b) 1.2×10^{-8} W 10. (a) 20
11. (a) $9.95 \times 10^{-4} \text{ W/m}^2$ (b) 1.15×10^{-6} m 12. 13.6 nm 13. 80 W 14. 134.4 dB
15. (a) 9.44×10^{-11} m, 0.43 m (b) 5.66×10^{-9} m, 0.434 m (d) air, $A_a/A_\omega = 60$
16. $4.5 \times 10^{-12} \text{ W/m}^2$, 6.53 dB 17. 297 Hz
18. (a) 382.2Hz, 764.4 Hz, 1146.7 Hz (b) 191.1Hz, 573.3 Hz, 955.5 Hz
19. 250 Hz 20. $y = A\cos(3.93x)\sin(1297t)$ 21. 348.16 m/s, 118 cm 22. 11.56 N
23. 336 m/s 24. (a) $\frac{15}{16}$ m (b) $\pm \frac{\Delta P_0}{\sqrt{2}}$ (c) $P_{max} = P_{min} = P_0$ (d) $P_{max} = P_0 + \Delta P_0$, $P_{min} = P_0 - \Delta P_0$

25. (a) 0.392 m (b) 0.470 m 26. 0.40 27. (a) 1.13 m (b) 1.03 m, 1.23 m
28. (a) 441.5 Hz, 438.5 Hz (b) (i) + 0.68% (ii) –0.68% 29. (a) 0.245 m/s (b) 0.904 m
30. (a) 302 Hz (b) 228 Hz 31. 680 Hz
32. Length of closed organ pipe is l_1 = 0.75 m while length of open pipe is either l_2 = 0.99 m or 1.0067 m.
33. 160 Hz 34. 0.5 m/s 35. 1440 Hz 36. $\phi_1 = \pi, \phi_2 = 4\pi$, Resultant power = 6.0×10^{-5} W
37. 7.5 m

Objective Questions

1.(d)	2.(d)	3.(c)	4.(d)	5.(a)	6.(a)	7.(a)	8.(a)	9.(c)	10.(b)
11.(a)	12.(c)	13.(b)	14.(b)	15.(a)	16.(a)	17.(a)	18.(b)	19.(d)	20.(c)
21.(d)	22.(d)	23.(a)	24.(c)	25.(b)	26.(a)	27.(b)	28.(d)	29.(c)	30.(c)
31.(a)	32.(c)	33.(d)	34.(a)						

For JEE Advanced

Assertion and Reason

1. (c)	2. (d)	3. (d)	4. (a)	5. (d)	6. (d)	7. (b)	8. (b)	9. (a)	10. (d)

Objective Questions

1.(b)	2.(a)	3.(d)	4.(b)	5.(a)	6.(b)	7.(c)	8.(c)	9.(a)	10.(b)
11.(a)	12.(b)	13.(b)	14.(c)	15.(a)	16.(c)	17.(a)	18.(c)		

More than One Correct Options

19. (a,c) 20. (c,d) 21. (a,b,c) 22. (b,c,d) 23. (a,b,c,d) 24. (c,d)

Match the Columns

1. (a) → s (b) → p (c) → r (d) → r
2. (a) → q (b) → p (c) → s (d) → s
3. (a) → r,s (b) → p (c) → r, s (d) → p
4. (a) → r (b) → q (c) → q (d) → p
5. (a) → s (b) → r (c) → p,r (d) → q

Subjective Questions

1. f 2. 0.28 cm, 9.0×10^{-3} kg/m³ 3. 95.5 s 4. 1650 Hz
5. (a) 15 m, 20 min⁻¹ (b) 13.5 m, 22.22 min⁻¹ 6. (a) 1 kg/m³ (b) 10μm (c) 288 π^3 W
7. (a) $\frac{l_0}{4}$ (b) l_0 8. (a) $\frac{400}{189}$ (b) $\frac{3}{4}$ 9. (a) 1.0069×10^5 Hz (b) 1.0304×10^5 Hz
10. 27.04 N 11. (a) 0.3 m (b) 1320 (c) 332 m/ss (d) 0.2 m

17 THERMOMETRY, THERMAL EXPANSION & KINETIC THEORY OF GASES

Chapter Contents

17.1 Thermometers and The Celsius Temperature Scale

Thermometers are devices that are used to measure temperatures. All thermometers are based on the principle that some physical properties of a system change as the system's temperature changes. Some physical properties that change with temperature are

(1) the volume of a liquid
(2) the length of a solid
(3) the pressure of a gas at constant volume
(4) the volume of a gas at constant pressure and
(5) the electric resistance of a conductor.

A common thermometer in everyday use consists of a mass of liquid, usually mercury or alcohol that expands in a glass capillary tube when heated. In this case the physical property is the **change in volume of the liquid.** Any temperature change is proportional to the change in length of the liquid column. The thermometer can be calibrated accordingly. On the celcius temperature scale, a thermometer is usually calibrated between 0°C (called the ice point of water) and 100°C (called the steam point of water). Once the liquid levels in the thermometer have been established at these two points, the distance between the two points is divided into 100 equal segments to create the celsius scale.

Thus, each segment denotes a change in temperature of one celsius degree (1°C). A practical problem in this type of thermometer is that readings may vary for two different liquids. When one thermometer reads a temperature, for example 40°C the other may indicate a slightly different value. This discrepancies between thermometers are especially large at temperatures far from the calibration points. To surmount this problem we need a universal thermometer whose readings are independent of the substance used in it. The gas thermometer used in the next article meets this requirement.

17.2 The Constant Volume Gas Thermometer and The Absolute Temperature Scale

The physical property used by the constant volume gas thermometer is the **change in pressure of a gas at constant volume.**

The pressure versus temperature graph for a typical gas taken with a constant volume is shown in figure. The two dots represent the two reference temperatures namely, the ice and steam points of water. The line connecting them serves as a calibration curve for unknown temperatures. Experiments show that the thermometer readings are nearly independent of the type of gas used as long as the gas pressure is low and the temperature is well above the point at which the gas liquifies.

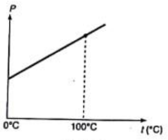


Fig. 17.1

If you extend the curves shown in figure toward negative temperatures, you find, in every case, that the pressure is zero when the temperature is –273.15°C. This significant temperature is used as the basis for the **absolute temperature scale,** which sets –273.15°C as its zero point.

This temperature is often referred to as **absolute zero.** The size of a degree on the absolute temperature scale is identical to the size of a degree on the Celsius scale. Thus, the conversion between these temperatures is

$$T_C = T - 273.15 \qquad ...(i)$$

In 1954, by the International committee on weights and measures, the **triple point of water** was chosen as the reference temperature for this new scale. The triple point of water is the single combination of temperature and pressure at which liquid water, gaseous water and ice (solid water) coexist in equilibrium. This triple point occurs at a temperature of approximately 0.01°C and a pressure of 4.58 mm of mercury. On the new scale, which uses the unit **kelvin,** the temperature of water at the triple point was set at 273.16 kelvin, abbreviated as 273.16 K. (No degree sign is used with the unit kelvin).

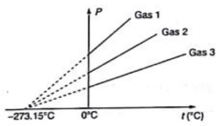


Fig. 17.2

This new absolute temperature scale (also called the kelvin scale) employs the SI unit of absolute temperature, the kelvin which is defined to be "$\frac{1}{273.16}$ of the difference between absolute zero and the temperature of the triple point of water".

The Celsius, Fahrenheit and Kelvin Temperature Scales

Equation (i) shows the relation between the temperatures in celsius scales and kelvin scale. Because the size of a degree is the same on the two scales, a temperature difference of 10°C is equal to a temperature difference of 10 K. The two scales differ only in the choice of the zero point. The ice point temperature on the kelvin scale, 273.15 K, corresponds to 0.00°C and the kelvin steam point 373.15 K, is equivalent to 100.00°C.

A common temperature scale in everyday use in US is the **Fahrenheit scale.** The ice point in this scale is 32°F and the steam point is 212°F. The distance between these two points are divided in 180 equal parts. The relation between celsius scale and fahrenheit scale is as derived below :

Fig. 17.3

100 parts of celsius scale = 180 parts of fahrenheit scale

$\therefore$ 1 part of celsius scale = $\frac{9}{5}$ parts of fahrenheit scale

Relation among Kelvin, Celsius and Fahrenheit temperature scales

Fig. 17.4

Hence, $$T_F = 32 + \frac{9}{5} T_C \quad \text{...(ii)}$$

Further, $$\Delta T_C = \Delta T = \frac{5}{9} \Delta T_F \quad \text{...(iii)}$$

Sample Example 17.1 *Express a temperature of 60°F in degrees celsius and in kelvins.*

Solution Substituting $T_F = 60° \text{ F}$ in Eq. (ii)

$$T_C = \frac{5}{9}(T_F - 32) = \frac{5}{9}(60° - 32°) = 15.55°\text{C} \quad \textbf{Ans.}$$

From Eq. (i)

$$T = T_C + 273.15 = 15.55°\text{C} + 273.15 = 288.7 \text{ K} \quad \textbf{Ans.}$$

Sample Example 17.2 *The temperature of an iron piece is heated from 30°C to 90°C. What is the change in its temperature on the fahrenheit scale and on the kelvin scale?*

Solution $$\Delta T_C = 90°\text{C} - 30°\text{C} = 60°\text{C}$$

Using Eq. (iii),

$$\Delta T_F = \frac{9}{5} \Delta T_C = \frac{9}{5}(60°\text{C}) = 108° \text{ F} \quad \textbf{Ans.}$$

and $$\Delta T = \Delta T_C = 60 \text{ K} \quad \textbf{Ans.}$$

• *Important points in* THERMOMETERS

1. Different Thermometers

Thermometric property : It is the property that can be used to measure the temperature. It is represented by any physical quantity such as length, volume, pressure and resistance etc., which varies linearly with a certain range of temperature. Let X denote the thermometric physical quantity and X_0, X_{100} and X_t be its values at 0°C, 100°C and t °C respectively. Then,

$$t = \left(\frac{X_t - X_0}{X_{100} - X_0} \right) \times 100° \text{C}$$

(i) Constant volume gas thermometer : The pressure of a gas at constant volume is the thermometric property. Therefore,

$$t = \left(\frac{P_t - P_0}{P_{100} - P_0} \right) \times 100° \text{C}$$

(ii) Platinum resistance thermometer : The resistance of a platinum wire is the thermometric property. Hence,

$$t = \left(\frac{R_t - R_0}{R_{100} - R_0} \right) \times 100° \text{C}$$

(iii) Mercury thermometer : In this thermometer the length of a mercury column from some fixed point is taken as thermometric property. Thus,

$$t = \left(\frac{l_t - l_0}{l_{100} - l_0} \right) \times 100° \text{C}$$

2. Two other thermometers, commonly used are **thermocouple thermometer** and **total radiation pyrometer.**
3. **Total radiation pyrometer** is used to measure very high temperatures. When a body is at a high temperature, it glows brightly and the radiation emitted per second from unit area of the surface of the body is proportional to the fourth power of the absolute temperature of the body. If this radiation is measured by some device, the temperature of the body is calculated. This is the principle of a total radiation pyrometer. The main advantage of this thermometer is that the experimental body is not kept in contact with it. Hence, there is no definite higher limit of its temperature-range. It can measure temperature from 800°C to 3000°C–4000°C. However it cannot be used to measure temperatures below 800°C because at low temperatures the emission of radiation is so poor that it cannot be measured directly.
4. **Ranges of different thermometers :**

Thermometer	Lower Limit	Upper Limit
Mercury Thermometer	–30°C	300°C
Gas Thermometer	–268°C	1500°C
Platinum Resistance Thermometer	–200°C	1200°C
Thermocouple Thermometer	–200°C	1600°C
Radiation Thermometer	800°C	No limit

5. **Reaumer's Scale :** Other than Celsius, Fahrenheit and Kelvin temperature scales **Reaumer's Scale** was designed by Reaumer in 1730. The lower fixed point is 0°R representing melting point of ice. The upper fixed point is 80°R, which represents boiling point of water. The distance between the two fixed points is divided into 80 equal parts. Each part represents 1°R. If T_C, T_F and T_R are temperature values of a body on Celsius scale, Fahrenheit scale and Reaumer scale respectively, then,

$$\frac{T_C - 0}{100} = \frac{T_F - 32}{180} = \frac{T_R - 0}{80}$$

6. A substance is found to exist in three states solid, liquid and gas. For each substance there is a set of temperature and pressure at which all the three states may coexist. This is called **triple point** of that substance. For water, the values of pressure and temperature corresponding to triple point are 4.58 mm of Hg and 273.16°K.

17.3 Quantity of Heat

When a cold body is brought in contact with a hot body, the cold body warms up and the hot body cools down as they approach thermal equilibrium. Fundamentally a transfer of energy takes place from one substance to the other. This type of energy transfer that takes place solely because of a temperature difference is called **heat flow or heat transfer** and energy transfer in this way is called **heat.**

Water can be warmed up by vigorous stirring with a paddle wheel. The paddle wheel adds energy to the water by doing work on it. The same temperature change can also be caused by putting the water in contact with some hotter body. Hence, this interaction must also involve an energy exchange. Before exploring the relation between heat and mechanical energy let us define a unit of quantity of heat.

"One **calorie** (1 cal) is defined as the amount of heat required to raise the temperature of one gram of water from 14.5°C to 15.5°C."

Experiments have shown that

$$1\ \text{cal} = 4.186\ \text{J}$$

Similarly $\qquad 1\ \text{kcal} = 1000\ \text{cal} = 4186\ \text{J}$

The calorie is not a fundamental SI unit.

17.4 Thermal Expansion

Most substances expand when they are heated. Thermal expansion is a consequence of the change in average separation between the constituent atoms of an object. Atoms of an object can be imagined to be connected to one another by stiff springs as shown in figure. At ordinary temperatures, the atoms in a solid oscillate about their equilibrium positions with an amplitude of approximately 10^{-11} m. The average spacing between the atoms is about 10^{-10} m. As the temperature of solid increases, the atoms oscillate with greater amplitudes, as a result the average separation between them increases, consequently the object expands.

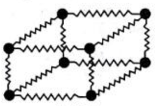
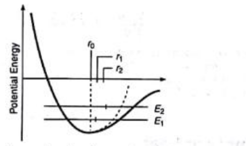


Fig. 17.5

More precisely, thermal expansion arises from the asymmetrical nature of the potential energy curve.

At the atomic level, thermal expansion may be understood by considering how the potential energy of the atoms varies with distance. The equilibrium position of an atom will be at the minimum of the potential energy well if the well is symmetric. At a given temperature each atom vibrates about its equilibrium position and its average position remains at the minimum point. If the shape of the well is not symmetrical, as shown in figure, the average position of an atom will not be at the minimum point. When the temperature is raised the amplitude of the vibrations increases and the average position is located at a greater interatomic separation. This increased separation is manifested as expansion of the material.



The potential energy of an atom. Thermal expansion arises because the "wall" is not symmetrical about the equilibrium position r_0. As the temperature rises, the energy of the atom changes. The average position r when the energy is E_2 is not the same as that when the E_1.

Fig. 17.6

Linear expansion

Suppose that the temperature of a thin rod of length l is changed from T to $T + \Delta T$. It is found experimentally that, if ΔT is not too large, the corresponding change in length Δl of the rod is directly proportional to ΔT and l. Thus,

$$\Delta l \propto \Delta T \quad \text{and} \quad \Delta l \propto l$$

Introducing a proportionality constant α (which is different for different materials) we may write Δl as

$$\Delta l = l\alpha\Delta T \qquad \text{...(i)}$$

Here, the constant α is called the **coefficient of linear expansion** of the material of the rod and its units are K^{-1} or $[(°C)^{-1}]$. Remember that $\Delta T = \Delta T_C$.

Actually, α does depend slightly on the temperature, but its variation is usually small enough to be negligible, even over a temperature range of 100°C. We will always assume that α is a constant.

Volume expansion

Because the linear dimensions of an object change with temperature, it follows that surface area and volume change as well. Just as with linear expansion, experiments show that if the temperature change ΔT is not too great (less than 100°C or so), the increase in volume ΔV is proportional to both the temperature change ΔT and the initial volume V. Thus,

$$\Delta V \propto \Delta T \quad \text{and} \quad \Delta V \propto V$$

Introducing a proportionality constant γ, we may write ΔV as,

$$\Delta V = V \times \gamma \times \Delta T \qquad \text{...(ii)}$$

Here, γ is called the **coefficient of volume expansion**. The units of γ are K^{-1} or $(°C)^{-1}$.

Relation between γ and α

For an isotropic solid (which has the same value of α in all directions) $\gamma = 3\alpha$. To see that $\gamma = 3\alpha$ for a solid, consider a cube of length l and volume $V = l^3$.

When the temperature of the cube is increased by dT, the side length increases by dl and the volume increases by an amount dV given by

$$dV = \left(\frac{dV}{dl}\right) \cdot dl = 3l^2 \cdot dl$$

Now, $\qquad dl = l\alpha dT$

$\therefore \qquad dV = 3l^3\alpha dT = (3\alpha)VdT$

This is consistent with Eq. (ii),

$$dV = \gamma V dT, \text{ only if}$$

$$\gamma = 3\alpha \qquad \text{...(iii)}$$

Average values of α and γ for some materials are listed in Table 17.1. You can check the relation $\gamma = 3\alpha$, for the materials given in the table.

Table 17.1

Material	α [K^{-1} or $(°C)^{-1}$]	γ [K^{-1} or $(°C)^{-1}$]
Steel	1.2×10^{-5}	3.6×10^{-5}
Copper	1.7×10^{-5}	5.1×10^{-5}
Brass	2.0×10^{-5}	6.0×10^{-5}
Aluminium	2.4×10^{-5}	7.2×10^{-5}

The anomalous expansion of water

Most liquids also expand when their temperatures increase. Their expansion can also be described by Eq. (ii). The volume expansion coefficients for liquids are about 100 times larger than those for solids.

Some substances contract when heated over a certain temperature range. The most common example is water.

Figure shows how the volume of 1 gm of water varies with temperature at atmospheric pressure. The volume decreases as the temperature is raised from 0°C to about 4°C, at which point the volume is a minimum and the density is a maximum ($1000\ kg/m^3$). Above 4°C, water expands with increasing temperature like most substances.

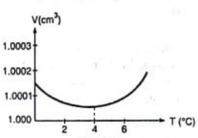


Fig. 17.7

This anomalous behaviour of water causes ice to form first at the surface of a lake in cold weather. As winter approaches, the water temperature increases initially at the surface. The water there sinks because of its increased density. Consequently, the surface reaches 0°C first and the lake becomes covered with ice. Aquatic life is able to survive the cold winter as the lake bottom remains unfrozen at a temperature of about 4°C.

• *Important points in* THERMAL EXPANSION

1. If a solid object has a hole in it, what happens to the size of the hole, when the temperature of the object increases? A common misconception is that if the object expands, the hole will shrink because material expands into the hole. But the truth is that if the object expands, the hole will expand too, because every linear dimension of an object changes in the same way when the temperature changes.

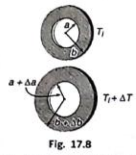


Fig. 17.8

2. **Expansion of a bimetallic strip :** As table 17.1 indicates, each substance has its own characteristic average coefficient of expansion. For example, when the temperatures of a brass rod and a steel rod of equal length are raised by the same amount from some common initial value, the brass rod expands more than the steel rod because brass has a greater average coefficient of expansion than steel. Such type of bimetallic strip is found in practical devices such as thermostats to break or make electrical contact.

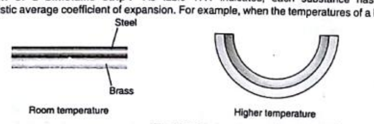


Fig. 17.9

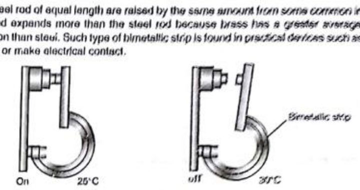


Fig. 17.10

3. **Variation of density with temperature :** Most substances expand when they are heated, *i.e.*, volume of a given mass of a substance increases on heating, so the density should decrease $\left(\text{as } \rho \propto \frac{1}{V}\right)$. Let us see how the density (ρ) varies with increase in temperature.

$$\rho = \frac{m}{V}$$

or $\qquad \rho \propto \frac{1}{V} \qquad$ (for a given mass)

$\therefore \qquad \frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V + \Delta V} = \frac{V}{V + \gamma V\Delta T} = \frac{1}{1 + \gamma\Delta T}$

$\therefore \qquad \rho' = \frac{\rho}{1 + \gamma\Delta T}$

This expression can also be written as,

$$\rho' = \rho(1 + \gamma\Delta T)^{-1}$$

As γ is small, $\qquad (1 + \gamma\Delta T)^{-1} \approx 1 - \gamma\Delta T$

$\therefore \qquad \rho' = \rho(1 - \gamma\Delta T)$

4. **Effect of temperature on upthrust :** When a solid body is completely immersed in a liquid its apparent weight gets decreased due to an upthrust acting on it by the liquid. The apparent weight is given by,

$$w_{app} = w - F$$

Here F = upthrust = $V_S \rho_L g$

where V_S = volume of solid and ρ_L = density of liquid

Now, as the temperature is increased V_S increases while ρ_L decreases. So, F may increase or decrease (or may remain constant also) depending upon the condition that which factor dominates on the other. We can write

$$F \propto V_S \rho_L$$

or $$\frac{F'}{F} = \frac{V_S'}{V_S} \cdot \frac{\rho_L'}{\rho_L} = \frac{(V_S + \Delta V_S)}{V_S} \cdot \left(\frac{1}{1+\gamma_L \Delta T}\right)$$

$$= \left(\frac{V_S + \gamma_S V_S \Delta T}{V_S}\right)\left(\frac{1}{1+\gamma_L \Delta T}\right)$$

or $$F' = F\left(\frac{1+\gamma_S \Delta T}{1+\gamma_L \Delta T}\right)$$

Now, if $\gamma_S > \gamma_L, F' > F$ or $w' < w$ and *vice-versa*.

And if $\gamma_S = \gamma_L$, $F' = F$ or $W' = W$

5. **Effect of temperature on the time period of a pendulum :** The time period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

or $$T \propto \sqrt{l}$$

As the temperature is increased length of the pendulum and hence, time period gets increased or a pendulum clock becomes slow and it loses the time.

$$\frac{T'}{T} = \sqrt{\frac{l'}{l}} = \sqrt{\frac{l + \Delta l}{l}}$$

Here, we put $\Delta l = l\alpha\Delta\theta$ in place of $l\alpha\Delta T$ so as to avoid the confusion with change in time period. Thus,

$$\frac{T'}{T} = \sqrt{\frac{l + l\alpha\Delta\theta}{l}} = (1+\alpha\Delta\theta)^{1/2}$$

or $$T' = T\left(1 + \frac{1}{2}\alpha\Delta\theta\right)$$

or $$\Delta T = T' - T = \frac{1}{2}T\alpha\Delta\theta$$

Time lost in time t (by a pendulum clock whose actual time period is T and the changed time period at some higher temperature is T') is

$$\Delta t = \left(\frac{\Delta T}{T'}\right)t$$

Similarly, if the temperature is decreased the length and hence, the time period gets decreased. A pendulum clock in this case runs fast and it gains the time.

$$\frac{T'}{T} = \sqrt{\frac{l'}{l}} = \sqrt{\frac{l - l\alpha\Delta\theta}{l}} = 1 - \frac{1}{2}\alpha\Delta\theta$$

or $$T' = T\left(1 - \frac{1}{2}\alpha\Delta\theta\right)$$

$$\Delta T = T - T' = \frac{1}{2}T\alpha\Delta\theta$$

and time gained in time t is the same, *i.e.*,

$$\Delta t = \left(\frac{\Delta T}{T'}\right)t$$

6. At some higher temperature a scale will expand and scale reading will be lesser than true values, so that

$$\text{true value} = \text{scale reading } (1 + \alpha\Delta T)$$

Here ΔT is the temperature difference.
However, at lower temperatures scale reading will be more or true value will be less.

7. When a rod whose ends are rigidly fixed such as to prevent from expansion or contraction undergoes a change in temperature, thermal stresses are developed in the rod. This is because, if the temperature is increased, the rod has a tendency to expand but since, it is fixed at two ends, the rod exerts a force on supports.

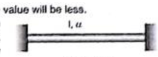


Fig. 17.11

$$\text{Thermal strain} = \frac{\Delta l}{l} = \alpha \cdot \Delta T$$

So $$\text{thermal stress} = (Y)\,(\text{thermal strain}) = Y\alpha\Delta T$$

or $$\text{force on supports } F = A\,(\text{stress}) = YA\alpha\Delta T$$

Here, Y = Young's modulus of elasticity of the rod.

$$F = YA\alpha\Delta T$$

8. **Expansion of liquids :** For heating a liquid it has to be put in some container. When the liquid is heated, the container will also expand. We define coefficient of apparent expansion of a liquid as the apparent increase in volume per unit original volume per °C rise in temperature. It is represented by γ_a. Thus,

$$\gamma_a = \gamma_r - \gamma_g$$

Here, γ_r = coefficient of real expansion of a liquid
and γ_g = coefficient of cubical expansion of the container

Sample Example 17.3 *A steel ruler exactly 20 cm long is graduated to give correct measurements at 20°C.*
(a) Will it give readings that are too long or too short at lower temperatures?
(b) What will be the actual length of the ruler be when it is used in the desert at a temperature of 40°C ? $\alpha_{steel} = 1.2 \times 10^{-5}\ (°C)^{-1}$.

Solution (a) If the temperature decreases, the length of the ruler also decreases through thermal contraction. Below 20°C, each centimetre division is actually somewhat shorter than 1.0 cm, so the steel ruler gives readings that are too long.

(b) At 40°C, the increase in length of the ruler is

$$\Delta l = l\alpha\Delta T$$
$$= (20)\,(1.2 \times 10^{-5})\,(40° - 20°)$$
$$= 0.48 \times 10^{-2} \text{ cm}$$

∴ The actual length of the ruler is,

$$l' = l + \Delta l = 20.0048 \text{ cm}$$ **Ans.**

Sample Example 17.4 *Find the coefficient of volume expansion for an ideal gas at constant pressure.*

Solution For an ideal gas

$$PV = nRT$$

As P is constant, we have

$$P \cdot dV = nRdT$$

∴ $$\frac{dV}{dT} = \frac{nR}{P}$$

or $$\gamma = \frac{1}{V} \cdot \frac{dV}{dT} = \frac{nR}{PV} = \frac{nR}{nRT} = \frac{1}{T}$$

∴ $$\gamma = \frac{1}{T}$$ **Ans.**

Sample Example 17.5 *The scale on a steel metre stick is calibrated at 15°C. What is the error in the reading of 60 cm at 27°C?* $\alpha_{steel} = 1.2 \times 10^{-5}\ (°C)^{-1}$.

Solution At higher temperatures actual reading is more than the scale reading. The error in the reading will be

$$\Delta l = (\text{scale reading})\,(\alpha)\,(\Delta T)$$
$$= (60)\,(1.2 \times 10^{-5})\,(27° - 15°)$$
$$= 0.00864 \text{ cm}$$ **Ans.**

Sample Example 17.6 *A second's pendulum clock has a steel wire. The clock is calibrated at 20°C. How much time does the clock lose or gain in one week when the temperature is increased to 30°C?* $\alpha_{steel} = 1.2 \times 10^{-5} (°C)^{-1}$.

Solution The time period of second's pendulum is 2 second. As the temperature increases length and hence, time period increases. Clock becomes slow and it loses the time. The change in time period is

$$\Delta T = \frac{1}{2}T\alpha\Delta\theta$$
$$= \left(\frac{1}{2}\right)(2)\,(1.2 \times 10^{-5})\,(30° - 20°)$$
$$= 1.2 \times 10^{-4} \text{ s}$$

∴ New time period is,

$$T' = T + \Delta T = (2 + 1.2 \times 10^{-4})$$
$$= 2.00012 \text{ s}$$

∴ Time lost in one week

$$\Delta t = \left(\frac{\Delta T}{T'}\right)t = \frac{(1.2 \times 10^{-4})}{(2.00012)}(7 \times 24 \times 3600)$$
$$= 36.28 \text{ s}$$ **Ans.**

Introductory Exercise 17.1

1. What is the value of
   (a) 0°F in celsius scale?
   (b) 0 K on Fahrenheit scale?
2. At what temperature is the Fahrenheit scale reading equal to
   (a) twice
   (b) half of Celsius?
3. A Faulty thermometer reads 5°C in melting ice and 99°C in steam. Find the correct temperature in °F when this faulty thermometer reads 52°C.
4. At what temperature the Fahrenheit and Kelvin scales of temperature give the same reading?
5. At what temperature the Fahrenheit and Celsius scales of temperature give the same reading?
6. A pendulum clock of time period 2 sec gives the correct time at 30°C. The pendulum is made of iron. How many seconds will it lose or gain per day when the temperature falls to 0°C? $\alpha_{Fe} = 1.2 \times 10^{-5} (°C)^{-1}$.
7. A block of wood is floating in water at 0°C. The temperature of water is slowly raised from 0°C to 10°C. How will the percentage of volume of block above water level change with rise in temperature?
8. A piece of metal floats on mercury. The coefficient of volume expansion of metal and mercury are γ_1 and γ_2 respectively. If the temperature of both mercury and metal are increased by an amount ΔT, by what factor does the fraction of the volume of the metal submerged in mercury changes?
9. A brass disc fits snugly in a hole in a steel plate. Should you heat or cool the system to loosen the disc from the hole? Given that $\alpha_B > \alpha_{Fe}$.
10. Show that the volume thermal expansion coefficient for an ideal gas at constant pressure is $\frac{1}{T}$.

17.5 Concept of an Ideal Gas

A gas has no shape and size and can be contained in a vessel of any size or shape. It expands indefinitely and uniformly to fill the available space. It exerts pressure on its surroundings.

The gases whose molecules are point masses (mass without volume) and do not attract each other are called **ideal** or **perfect** gases. It is a hypothetical concept which can't exist in reality. The gases such as hydrogen, oxygen or helium which cannot be liquified easily are called **permanent gases**. An actual gas behaves as ideal gas most closely at low pressure and high temperature.

17.6 Gas Laws

Assuming permanent gases to be ideal, through experiments, it was established that gases irrespective of their nature obey the following laws :

(a) Boyle's law

According to this law, for a given mass of a gas the volume of a gas at constant temperature (called **isothermal** process) is inversely proportional to its pressure, *i.e.*,

$$V \propto \frac{1}{P} \qquad (T = \text{constant})$$

or $$PV = \text{constant}$$

or $$P_iV_i = P_fV_f$$

Thus, P-V graph in an isothermal process is a rectangular hyperbola. Or PV *versus* P or V graph is a straight line parallel to P or V axis.

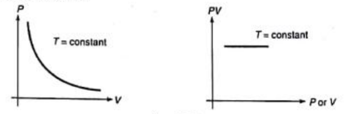


Fig. 17.12

(b) Charle's law

According to this law, for a given mass of a gas the volume of a gas at constant pressure (called **isobaric** process) is directly proportional to its absolute temperature, *i.e.,*

$$V \propto T \qquad (P = \text{constant})$$

or $$\frac{V}{T} = \text{constant}$$

or $$\frac{V_i}{T_i} = \frac{V_f}{T_f}$$

Thus, V-T graph in an isobaric process is a straight line passing through origin. Or V/T versus V or T graph is a straight line parallel to V or T axis.

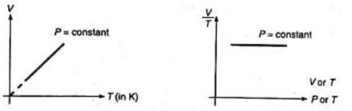


Fig. 17.13

(c) Gay Lussac's law or Pressure law

According to this law, for a given mass of a gas the pressure of a gas at constant volume (called **isochoric** process) is directly proportional to its absolute temperature, *i.e.,*

$$P \propto T \qquad (V = \text{constant})$$

or $$\frac{P}{T} = \text{constant}$$

or $$\frac{P_i}{T_i} = \frac{P_f}{T_f}$$

Thus, P-T graph in an isochoric process is a straight line passing through origin or P/T versus P or T graph is a straight line parallel to P or T axis.

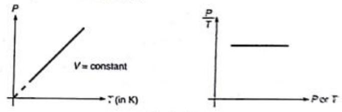


Fig. 17.14

(d) Avogadro's law

According to this law, at same temperature and pressure equal volumes of all gases contain equal number of molecules.

17.7 Ideal Gas Equation

All the above four laws can be written in one single equation known as ideal gas equation. According to this equation,

$$PV = nRT = \frac{m}{M}RT$$

In this equation n = number of moles of the gas

$$= \frac{m}{M}$$

m = total mass of the gas,

M = molecular mass of the gas

and R = Universal gas constant

$= 8.31$ J/mol-K

$= 2.0$ cal/mol-K

The above four laws can be derived from this single equation. For example, for a given mass of a gas (m = constant)

PV = constant at constant temperature (Boyle's law)

$\frac{P}{T}$ = constant at constant volume (Pressure law)

$\frac{V}{T}$ = constant at constant pressure (Charle's law)

and if P, V and T are constants then

n = constant for all gases.

And since, equal number of moles contain equal number of molecules. So, at constant pressure, volume and temperature all gases will contain equal number of molecules. Which is nothing but Avogadro's law.

• *Important Points in* GAS LAWS

1. In our previous discussion we have read Charle's law and pressure law in absolute temperature scale. In centigrade scale these laws are as under :

Charle's law : When a given mass of a gas is heated at constant pressure then for each 1°C rise in temperature the volume of the gas increases by a fraction α of its volume at 0°C. Thus, if the volume of a given mass of a gas at 0° is V_0, then on heating at constant pressure to t°C its volume will increase by $V_0\alpha t$. Therefore, if its volume at t° be V_t, then

$$V_t = V_0 + V_0\alpha t$$

or $$V_t = V_0(1 + \alpha t)$$

Here α is called the 'volume coefficient' of the gas. For all gases the experimental value of α is nearly $\frac{1}{273}$ per °C.

$$\therefore \quad V_t = V_0\left(1 + \frac{t}{273}\right)$$

Thus, V_t *versus* t graph is a straight line with slope $\frac{V_0}{273}$ and positive intercept V_0. Further $V_t = 0$ at $t = -273°$.

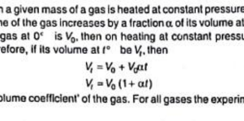


Fig. 17.15

Pressure law : According to this law, when a given mass of a gas is heated at constant volume then for each 1°C rise in temperature, the pressure of the gas increases by a fraction β of its pressure at 0°C. Thus, if the pressure of a given mass of a gas at 0°C be P_0, then on heating at constant volume to t° , its pressure will increase by $P_0\beta t$. Therefore, if its pressure at t° be P_t, then

$$P_t = P_0 + P_0\beta t$$

or $$P_t = P_0(1 + \beta t)$$

Here β is called the 'pressure coefficient' of the gas. For all gases the experimental value of β is also $\frac{1}{273}$ per°C.

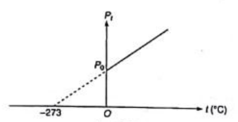


Fig. 17.16

$$\therefore \quad P_t = P_0\left(1 + \frac{t}{273}\right)$$

The P_t versus t graph is as shown in figure.

2. The above forms of Charle's law and pressure law can be simply expressed in terms of absolute temperature.

Let at constant pressure, the volume of a given mass of a gas at 0°C, t_1° and t_2° be V_0, V_1 and V_2 respectively. Then,

$$V_1 = V_0\left(1 + \frac{t_1}{273}\right) = V_0\left(\frac{273 + t_1}{273}\right)$$

$$V_2 = V_0\left(1 + \frac{t_2}{273}\right) = V_0\left(\frac{273 + t_2}{273}\right)$$

$$\therefore \quad \frac{V_1}{V_2} = \frac{273 + t_1}{273 + t_2} = \frac{T_1}{T_2}$$

where T_1 and T_2 are the absolute temperatures corresponding to t_1° and t_2° . Hence,

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}$$

or $$\frac{V}{T} = \text{constant}$$

or $$V \propto T$$

This is the form of Charle's law which we have already studied in article 17.6. In the similar manner we can prove the pressure law.

3. Under isobaric conditions (P = constant), V-T graph is a straight line passing through origin (where T is in kelvin). The slope of this line is $\left(\frac{nR}{P}\right)$ as $V = \left(\frac{nR}{P}\right)T$ or slope of the line is directly proportional to $\frac{n}{P}$.

$$\text{slope} = \frac{nR}{P} \quad \text{or} \quad \text{slope} \propto \frac{n}{P}$$

Similarly, under isochoric conditions (V = constant), P-T graph is a straight line passing through origin whose slope is $\frac{nR}{V}$ or slope is directly proportional to $\frac{n}{V}$.

4. **Density of a gas :** The ideal gas equation is,

$$PV = nRT = \frac{m}{M}RT$$

$$\therefore \quad \frac{m}{V} = \rho = \frac{PM}{RT} \qquad (\rho = \text{density})$$

Fig. 17.17

$$\therefore \qquad \rho = \frac{PM}{RT}$$

From this equation we can see that ρ-P graph is straight line passing through origin at constant temperature ($\rho \propto P$) for a given gas and ρ-T graph is rectangular hyperbola at constant pressure $\left(\rho \propto \frac{1}{T}\right)$. Similarly for a given mass of a gas ρ-V graph is a rectangular hyperbola $\left(\rho \propto \frac{1}{V}\right)$.

Sample Example 17.7 *P-V diagrams of same mass of a gas are drawn at two different temperatures T_1 and T_2. Explain whether $T_1 > T_2$ or $T_2 > T_1$.*

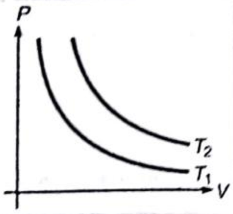


Fig. 17.18

Solution The ideal gas equation is,

$$PV = nRT$$

or

$$T = \frac{PV}{nR}$$

$T \propto PV$ if number of moles of the gas are kept constant. Here mass of the gas is constant, which implies that number of moles are constant, *i.e.*, $T \propto PV$. In the given diagram product of P and V for T_2 is more than T_1 at all points (keeping either P or V same for both graphs). Hence,

$$T_2 > T_1 \qquad \textbf{Ans.}$$

Sample Example 17.8 *The P-V diagram of two different masses m_1 and m_2 are drawn (as shown) at constant temperature T. State whether $m_1 > m_2$ or $m_2 > m_1$?*

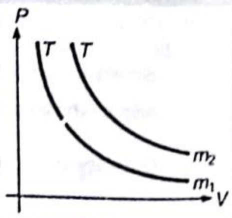


Fig. 17.19

Solution

$$PV = nRT = \frac{m}{M}RT$$

$$\therefore \qquad m = (PV)\left(\frac{M}{RT}\right)$$

or $\qquad m \propto PV \qquad$ if T = constant

From the graph we can see that $P_2V_2 > P_1V_1$ (for same P or V). Therefore,

$$m_2 > m_1 \qquad \textbf{Ans.}$$

Sample Example 17.9 *The P-T graph for the given mass of an ideal gas is shown in figure. What inference can be drawn regarding the change in volume (whether it is constant, increasing or decreasing)?*

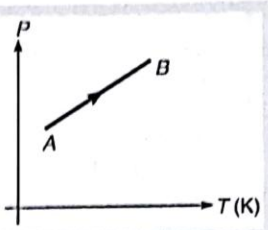


Fig. 17.20

HOW TO PROCEED Definitely it is not constant. Because when volume is constant, P-T graph is a straight line passing through origin. The given line does not pass through origin, hence, volume is not constant.

$$V = (nR)\left(\frac{T}{P}\right)$$

Now, to see the volume of the gas we will have to see whether $\frac{T}{P}$ is increasing or decreasing.

Solution From the given graph we can write the P-T equation as,

$$P = aT + b \qquad (y = mx + c)$$

here a and b are positive constants. Further,

$$\frac{P}{T} = a + \frac{b}{T}$$

Now, $T_B > T_A$ $\therefore$ $\frac{b}{T_B} < \frac{b}{T_A}$ or $\left(\frac{P}{T}\right)_B < \left(\frac{P}{T}\right)_A$

or $$\left(\frac{T}{P}\right)_B > \left(\frac{P}{T}\right)_A$$

or $$V_B > V_A$$ **Ans.**

Thus, as we move from A to B, volume of the gas is increasing.

Introductory Exercise 17.2

1. From the graph for an ideal gas state whether m_1 or m_2 is greater?

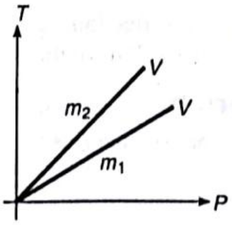


Fig. 17.21

2. A vessel is filled with an ideal gas at a pressure of 20 atm and is at a temperature of 27°C. One-half of the mass is removed from the vessel and the temperature of the remaining gas is increased to 87°C. At this temperature find the pressure of the gas.
3. A vessel contains a mixture of 7 g of nitrogen and 11 g of carbondioxide at temperature $T = 290$ K. If pressure of the mixture is 1 atm ($= 1.01 \times 10^5$ N / m^2), calculate its density ($R = 8.31$ J / mol - K).
4. An electric bulb of volume 250 cm^3 was sealed off during manufacture at a pressure of 10^{-3} mm of mercury at 27°C. Compute the number of air molecules contained in the bulb. Given that $R = 8.31$ J / mol · K and $N_A = 6.02 \times 10^{23}$ per mol.
5. State whether $P_1 > P_2$ or $P_2 > P_1$ for given mass of a gas?

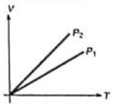


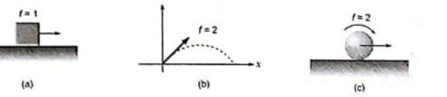
Fig. 17.22

6. For a given mass of a gas what is the shape of P versus $\frac{1}{V}$ graph at constant temperature?

17.8 Degree of Freedom (f)

The term degree of freedom refers to the number of possible independent ways in which a system can have energy.

For example : In figure (a) block has one degree of freedom, because it is confined to move in a



(a) (b) (c)

Fig. 17.23

straight line and has only one translational degree of freedom.

In figure (b), the projectile has two degrees of freedom because it is confined to move in a plane and so it has two translational degrees of freedom.

In figure (c), the sphere has two degrees of freedom one rotational and another translational.

Similarly a particle free to move in space will have three translational degrees of freedom.

Degree of freedom of gas molecules

A gas molecule can have following types of energies :

(i) translational kinetic energy
(ii) rotational kinetic energy
(iii) vibrational energy (potential + kinetic)

Vibrational energy

The forces between different atoms of a gas molecule may be visualized by imagining every atom as being connected to its neighbours by springs. Each atom can vibrate along the line joining the atoms. Energy associated with this is called vibrational energy.

Degree of freedom of monoatomic gas

A monoatomic gas molecule (like He) consists of a single atom. It can have translational motion in any direction in space. Thus, it has 3 translational degrees of freedom.

$$f = 3 \quad \text{(all translational)}$$

It can also rotate but due to its small moment of inertia, rotational kinetic energy is neglected.

Degree of freedom of a diatomic and linear polyatomic gas

The molecules of a diatomic and linear polyatomic gas (like O_2, CO_2 and H_2) cannot only move bodily but also rotate about any one of the three co-ordinate axes as shown in figure. However, its moment of inertia about the axis joining the two atoms (x-axis) is negligible. Hence, it can have only two rotational degrees of freedom. Thus, a diatomic molecule has 5 degrees of freedom : 3 translational and 2 rotational. At sufficiently high temperatures it has vibrational energy as well providing it two more degrees of freedom (one vibrational kinetic energy and another vibrational potential energy). Thus, at high temperatures a diatomic molecule has 7 degrees of freedom, 3 translational, 2 rotational and 2 vibrational. Thus,

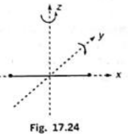


Fig. 17.24

$$f = 5$$

(3 translational + 2 rotational) at room temperatures

and $$f = 7$$

(3 translational + 2 rotational + 2 vibrational) at high temperatures.

Degree of freedom of nonlinear polyatomic gas

A nonlinear polyatomic molecule (such as NH_3) can rotate about any of three co-ordinate axes. Hence, it has 6 degrees of freedom 3 translational and 3 rotational. At room temperatures a polyatomic gas molecule has vibrational energy greater than that of a diatomic gas. But at high enough temperatures it is also significant. So it has 8 degrees of freedom 3 rotational, 3 translational and 2 vibrational. Thus,

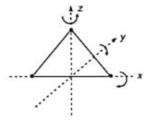


Fig. 17.25

$$f = 6$$

(3 translational + 3 rotational) at room temperatures

and $$f = 8$$

(3 translational + 3 rotational + 2 vibrational) at high temperatures.

Degree of freedom of a solid

An atom in a solid has no degrees of freedom for translational and rotational motion. At high temperatures due to vibration along 3 axes it has $3 \times 2 = 6$ degrees of freedom.

$$f = 6 \quad \text{(all vibrational) at high temperatures}$$

Note (i) *Degrees of freedom of a diatomic and polyatomic gas depends on temperature and since there is no clear cut demarcation line above which vibrational energy become significant. Moreover, this temperature varies from gas to gas. On the other hand for a monoatomic gas there is no such confusion. Degree of freedom here is 3 at all temperatures. Unless and until stated in the question you can take $f = 3$ for monoatomic gas, $f = 5$ for a diatomic gas and $f = 6$ for a non-linear polyatomic gas.*

(ii) *When a diatomic or polyatomic gas dissociates into atoms it behaves as a monoatomic gas. Whose degrees of freedom are changed accordingly.*

17.9 Internal Energy of an Ideal Gas

Suppose a gas is contained in a closed vessel as shown in figure. If the container as a whole is moving with some speed then this motion is called the **ordered motion** of the gas. Source of this motion is some external force. The zig zag motion of gas molecules within the vessel is known as the **disordered motion**. This motion is directly related to the temperature of the gas. As the temperature is increased, the disordered motion of the gas molecules gets fast. The internal energy (U) of the gas is concerned only with its disordered motion. It is in no way concerned with its ordered motion. When the temperature of the gas is increased, its disordered motion and hence its internal energy is increased.

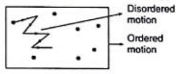
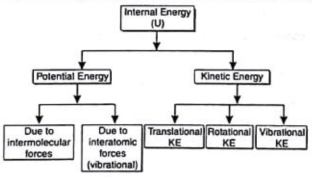


Fig. 17.26

Intermolecular forces in an ideal gas is zero. Thus, P.E. due to intermolecular forces of an ideal gas is zero. A monoatomic gas is having a single atom. Hence its vibrational energy is zero. For dia



and polyatomic gases vibrational energy is significant only at high temperatures. So, they also have only translational and rotational K.E. We may thus conclude that at room temperature the internal energy of an ideal gas (whether it is mono, dia or poly) consists of only translational and rotational K.E. Thus,

$$U \text{ (of an ideal gas)} = K_T + K_R \qquad \text{at room temperatures.}$$

Later in the next article we will see that K_T (translational KE) and K_R (rotational KE) depends on T only. They are directly proportional to the absolute temperature of the gas. Thus, **internal energy of an ideal gas depends only on its absolute temperature (T) and is directly proportional to T.**

Or, $$U \propto T$$

17.10 Law of Equipartition of Energy

An ideal gas is just like an ideal father. As an ideal father distributes whole of its assets equally among his children. Same is the case with an ideal gas. It distributes its internal energy equally in all degrees of freedom. In each degree of freedom energy of one mole of an ideal gas is $\frac{1}{2}RT$ where T is the absolute temperature of the gas. Thus, if f be the number of degrees of freedom, the internal energy of 1 mole of the gas will be $\frac{f}{2}RT$ or internal energy of n moles of the gas will be $\frac{n}{2}fRT$. Thus,

$$U = \frac{n}{2}fRT \qquad \text{...(i)}$$

For a monoatomic gas, $f = 3$.

Therefore, $$U = \frac{3}{2}RT \qquad \text{(for 1 mole of a monoatomic gas)}$$

For a dia and linear polyatomic gas at low temperatures, $f = 5$, so,

$$U = \frac{5}{2}RT \qquad \text{(for 1 mole)}$$

and for nonlinear polyatomic gas at low temperatures, $f = 6$, so

$$U = \frac{6}{2}RT = 3RT \qquad \text{(for 1 mole)}$$

Note *From Eq. (i) we can see that internal energy of an ideal gas depends only on its temperature and which is directly proportional to its absolute temperature T. In an isothermal process T = constant. Therefore, the internal energy of the gas does not change or $dU = 0$.*

17.11 Molar Heat Capacity

"Molar heat capacity C is the heat required to raise the temperature of 1 mole of a gas by 1°C (or 1 K)." Thus,

$$C = \frac{\Delta Q}{n\Delta T} \quad \text{or} \quad \Delta Q = nC\Delta T$$

For a gas the value of C depends on the process through which its temperature is raised.

For example, in an isothermal process $\Delta T = 0$ or $C_{iso} = \infty$. In an adiabatic process (we will discuss it later) $\Delta Q = 0$. Hence, $C_{adi} = 0$. Thus, molar heat capacity of a gas varies from 0 to ∞ depending on the process. In general experiments are made either at constant volume or at constant pressure. In case of solids and liquids, due to small thermal expansion, the difference in measured values of molar heat capacities is very small and is usually neglected. However, in case of gases molar heat capacity at constant volume C_V is quite different from that at constant pressure C_P. Later in the next chapter we will derive the following relations, for an ideal gas

$$C_V = \frac{dU}{dT} = \frac{f}{2}R = \frac{R}{\gamma - 1}$$

$$C_P = C_V + R$$

$$\gamma = \frac{C_P}{C_V} = 1 + \frac{2}{f}$$

Here U is the internal energy of one mole of the gas. The most general expression for C in the process $PV^x =$ constant is,

$$C = \frac{R}{\gamma - 1} + \frac{R}{1 - x} \qquad \text{(we will derive it later)}$$

For example : For isobaric process $P =$ constant or $x = 0$ and

$$C = C_P = \frac{R}{\gamma - 1} + R = C_V + R$$

For isothermal process, $PV =$ constant or $x = 1$

$\therefore$ $C = \infty$ and

for adiabatic process $PV^{\gamma} =$ constant or $x = \gamma$

$\therefore$ $C = 0$

Values of f, U, C_V, C_P and γ for different gases are shown in table 17.2.

Table 17.2

Nature of gas	f	$U = \frac{f}{2}RT$	$C_V = dU/dT = \frac{f}{2}R$	$C_P = C_V + R$	$\gamma = C_P/C_V = 1 + \frac{2}{f}$
Monoatomic	3	$\frac{3}{2}RT$	$\frac{3}{2}R$	$\frac{5}{2}R$	1.67
Dia and linear polyatomic	5	$\frac{5}{2}RT$	$\frac{5}{2}R$	$\frac{7}{2}R$	1.4
Non-linear polyatomic	6	$3RT$	$3R$	$4R$	1.33

17.12 Kinetic Theory of Gases

We have studied the mechanics of single particles. When we approach the mechanics associated with the many particles in systems such as gases, liquids and solids, we are faced with analyzing the dynamics of a huge number of particles. The dynamics of such many particle systems is called statistical mechanics.

The game involved in studying a system with a large number of particles is similar to what happens after every physics test. Of course we are interested in our individual marks, but we also want to know the class average.

The kinetic theory that we study in this article is a special aspect of the statistical mechanics of large number of particles. We begin with the simplest model for a monoatomic ideal gas, a dilute gas whose particles are single atoms rather than molecules.

Macroscopic variables of a gas are pressure, volume and temperature and microscopic properties are speed of gas molecules, momentum of molecules, etc. Kinetic theory of gases relates the microscopic properties to macroscopic properties. Further more, the kinetic theory provides us with a physical basis for our understanding of the concept of pressure and temperature.

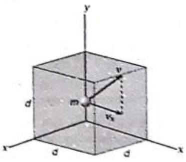
The ideal gas approximation

We make the following assumptions while describing an ideal gas :

1. The number of particles in the gas is very large.
2. The volume V containing the gas is much larger than the total volume actually occupied by the gas particles themselves.
3. The dynamics of the particles is governed by Newton's laws of motion.
4. The particles are equally likely to be moving in any direction.
5. The gas particles interact with each other and with the walls of the container only via elastic collisions.
6. The particles of the gas are identical and indistinguishable.

The pressure of an ideal gas

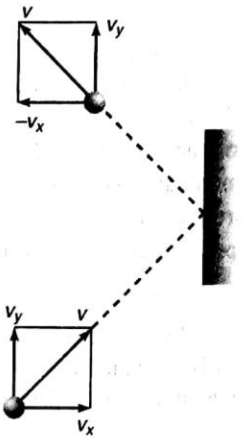
Consider an ideal gas consisting of N molecules in a container of volume V. The container is a cube with edges of length d. Consider the collision of one molecule moving with a velocity $\vec{v}$ toward the right hand face of the cube. The molecule has velocity components v_x, v_y and v_z. Previously we used m to represent the mass of a sample, but in this article we shall use m to represent the mass of one molecule. As the molecule collides with the wall elastically its x-component of velocity is reversed, while its y and z components of velocity remain unaltered. Because the



A cubical box with sides of length d containing an ideal gas. The molecule shown moves with velocity v.

Fig. 17.27

x-component of the momentum of the molecule is mv_x before the collision and $-mv_x$ after the collision, the change in momentum of the molecule is



A molecule makes an elastic collision with the wall of the container. Its x component of momentum is reversed, while its y component remains unchanged. In this construction, we assume that the molecule moves in the xy plane.

Fig. 17.28

$$\Delta p_x = -mv_x - (mv_x) = -2mv_x$$

Applying impulse = change in momentum to the molecule

$$F\Delta t = \Delta p_x = -2mv_x$$

where F is the magnitude of the average force exerted by the wall on the molecule in time Δt. For the molecules to collide twice with the same wall, it must travel a distance $2d$ in the x-direction. Therefore, the time interval between two collisions with the same wall is $\Delta t = \frac{2d}{v_x}$. Over a time interval that is long compared with Δt, the average force exerted on the molecules for each collision is

$$F = \frac{-2mv_x}{\Delta t} = \frac{-2mv_x}{2d/v_x} = \frac{-mv_x^2}{d}$$

According to Newton's third law, the average force exerted by the molecule on the wall is, $\frac{mv_x^2}{d}$.

Each molecule of the gas exerts a force on the wall. We find the total force exerted by all the molecules on the wall by adding the forces exerted by the individual molecules.

$$\therefore \qquad F_{\text{wall}} = \frac{m}{d}(v_{x_1}^2 + v_{x_2}^2 + \ldots + v_{x_N}^2)$$

This can also be written as,

$$F_{\text{wall}} = \frac{Nm}{d}\bar{v}_x^2$$

where
$$\bar{v}_x^2 = \frac{v_{x_1}^2 + v_{x_2}^2 + \ldots + v_{x_N}^2}{N}$$

Since the velocity has three components v_x, v_y and v_z, we can have

$$\bar{v}^2 = \bar{v}_x^2 + \bar{v}_y^2 + \bar{v}_z^2 \qquad (\text{as } v^2 = v_x^2 + v_y^2 + v_z^2)$$

Because the motion is completely random, the average values $\bar{v}_x^2$, $\bar{v}_y^2$ and $\bar{v}_z^2$ are equal to each other. So,

$$\bar{v}^2 = 3\bar{v}_x^2 \quad \text{or} \quad \bar{v}_x^2 = \frac{1}{3}\bar{v}^2$$

Therefore,
$$F_{\text{wall}} = \frac{N}{3}\left(\frac{m\bar{v}^2}{d}\right)$$

$\therefore$ Pressure on the wall

$$P = \frac{F_{\text{wall}}}{A} = \frac{F_{\text{wall}}}{d^2} = \frac{1}{3}\left(\frac{N}{d^3}m\bar{v}^2\right)$$

$$= \frac{1}{3}\left(\frac{N}{V}\right)m\bar{v}^2 = \frac{2}{3}\left(\frac{N}{V}\right)\left(\frac{1}{2}m\bar{v}^2\right)$$

$\therefore$
$$P = \frac{1}{3}\frac{mN}{V}\bar{v}^2 = \frac{2}{3}\left(\frac{N}{V}\right)\left(\frac{1}{2}m\bar{v}^2\right) \qquad \ldots\text{(i)}$$

This result indicates that the pressure is proportional to the number of molecules per unit volume (N/V) and to the average translational kinetic energy of the molecules $\frac{1}{2}m\bar{v}^2$. This result relates the large scale quantity (macroscopic) of pressure to an atomic quantity (microscopic)—the average value of the square of the molecular speed. The above equation varifies some features of pressure with which you are probably familiar. One way to increase the pressure inside a container is to increase the number of molecules per unit volume in the container.

The meaning of the absolute temperature

Rewriting Eq. (i) in the more familiar form

$$PV = \frac{2}{3}N\left(\frac{1}{2}m\bar{v}^2\right)$$

Let us now compare it with the ideal gas equation

$$PV = nRT$$

$$nRT = \frac{2}{3}N\left(\frac{1}{2}m\bar{v}^2\right)$$

Here $$n = \frac{N}{N_A} \qquad (N_A = \text{Avogadro number})$$

$\therefore$ $$T = \frac{2}{3}\left(\frac{N_A}{R}\right)\left(\frac{1}{2}m\bar{v}^2\right)$$

or $$T = \frac{2}{3k}\left(\frac{1}{2}m\bar{v}^2\right) \qquad \text{...(ii)}$$

where k is **Boltzmann's constant** which has the value

$$k = \frac{R}{N_A} = 1.38 \times 10^{-23} \text{ J/K}$$

By rearranging Eq. (ii) we can relate the translational molecular kinetic energy to the temperature

$$\frac{1}{2}mv^2 = \frac{3}{2}kT$$

That is, the average translational kinetic energy per molecule is $\frac{3}{2}kT$. Because

$$\bar{v}_x^2 = \frac{1}{3}\bar{v}^2, \text{ it follows that}$$

$$\frac{1}{2}m\bar{v}_x^2 = \frac{1}{2}kT$$

In the similar manner it follows that

$$\frac{1}{2}m\bar{v}_y^2 = \frac{1}{2}kT \quad \text{and} \quad \frac{1}{2}m\bar{v}_z^2 = \frac{1}{2}kT$$

Thus, in each translational degree of freedom one gas molecule has an energy $\frac{1}{2}kT$. One mole of a gas has N_A number of molecules. Thus, one mole of the gas has an energy $\frac{1}{2}(kN_A)T = \frac{1}{2}RT$ in each degree of freedom. Which is nothing but the law of equipartition of energy. The total translational kinetic energy of one mole of an ideal gas is therefore, $\frac{3}{2}RT$.

$$(\text{KE})_{\text{Trans}} = \frac{3}{2}RT \quad \text{(of one mole)}$$

Root mean square speed

The square root of $\bar{v}^2$ is called the root mean square (rms) speed of the molecules. From Eq. (ii) we obtain, for the rms speed

$$v_{\text{rms}} = \sqrt{\bar{v}^2} = \sqrt{\frac{3kT}{m}}$$

Using $$k = \frac{R}{N_A}, \quad mN_A = M \quad \text{and} \quad \frac{RT}{M} = \frac{P}{\rho}$$

we can write,

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3P}{\rho}}$$

Mean speed or average speed

The particles of a gas have a range of speeds. The average speed is found by taking the average of the speeds of all the particles at a given instant. Remember that the speed is a positive scalar since it is the magnitude of the velocity.

$$v_{\text{av}} = \frac{v_1 + v_2 + \ldots + v_N}{N}$$

From Maxwellian speed distribution law, we can show that

$$v_{\text{av}} = \sqrt{\frac{8kT}{\pi m}} = \sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{8P}{\pi\rho}}$$

Most probable speed

This is defined as the speed which is possessed by maximum fraction of total number of molecules of the gas. For example, if speeds of 10 molecules of a gas are, 1, 2, 2, 3, 3, 3, 4, 5, 6, 6 km/s, then the most probable speed is 3 km/s, as maximum fraction of total molecules possess this speed. Again from Maxwellian speed distribution law (out of JEE syllabus)

$$v_{\text{mp}} = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2RT}{M}} = \sqrt{\frac{2P}{\rho}}$$

Note
1. *In the above expressions of v_{rms}, v_{av} and v_{mp}, M is the molar mass in kilogram per mole. For example, molar mass of hydrogen is 2×10^{-3} kg/mol.*
2. $v_{\text{rms}} > v_{\text{av}} > v_{\text{mp}}$ *(RAM)*
3. $v_{\text{rms}} : v_{\text{av}} : v_{\text{mp}} = \sqrt{3} : \sqrt{\frac{8}{\pi}} : \sqrt{2}$

   and since, $\frac{8}{\pi} \approx 2.5$, *we have* $v_{\text{rms}} : v_{\text{av}} : v_{\text{mp}} = \sqrt{3} : \sqrt{2.5} : \sqrt{2}$

Sample Example 17.10 *A tank used for filling helium balloons has a volume of 0.3 m^3 and contains 2.0 mol of helium gas at 20.0°C. Assuming that the helium behaves like an ideal gas.*
(a) What is the total translational kinetic energy of the molecules of the gas?
(b) What is the average kinetic energy per molecule?

Solution (a) Using $(\text{KE})_{\text{Trans}} = \frac{3}{2}nRT$

with $n = 2.0$ mol and $T = 293$ K, we find that

$$(\text{KE})_{\text{Trans}} = \frac{3}{2}(2.0)(8.31)(293)$$
$$= 7.3 \times 10^3 \text{ J}$$ **Ans.**

(b) The average kinetic energy per molecule is $\frac{3}{2}kT$.

or $$\frac{1}{2}m\bar{v}^2 = \frac{1}{2}m\bar{v}_{\text{rms}}^2 = \frac{3}{2}kT$$
$$= \frac{3}{2}(1.38 \times 10^{-23})(293)$$
$$= 6.07 \times 10^{-21} \text{ J}$$ **Ans.**

Sample Example 17.11 *Consider an 1100 particles gas system with speeds distribution as follows :*
1000 particles each with speed 100 m/s
2000 particles each with speed 200 m/s
4000 particles each with speed 300 m/s
3000 particles each with speed 400 m/s and 1000 particles each with speed 500 m/s
Find the average speed, and rms speed.

Solution The average speed is:

$$v_{\text{av}} = \frac{(1000)(100) + (2000)(200) + (4000)(300) + (3000)(400) + (1000)(500)}{1100}$$
$$= 309 \text{ m/s}$$ **Ans.**

The rms speed is :

$$v_{\text{rms}} = \sqrt{\frac{(1000)(100)^2 + (2000)(200)^2 + (4000)(300)^2 + (3000)(400)^2 + (1000)(500)^2}{1100}}$$
$$= 328 \text{ m/s}$$ **Ans.**

Note *Here* $\frac{v_{\text{rms}}}{v_{\text{av}}} = \sqrt{\frac{3}{8/\pi}}$ *as values and gas molecules are arbitrarily taken.*

Sample Example 17.12 *Calculate the change in internal energy of 3.0 mol of helium gas when its temperature is increased by 2.0 K.*

Solution Helium is a monoatomic gas. Internal energy of n moles of the gas is,

$$U = \frac{3}{2}nRT$$

$\therefore$ $$\Delta U = \frac{3}{2}nR(\Delta T)$$

Substituting the values,

$$\Delta U = \left(\frac{3}{2}\right)(3)(8.31)(2.0) = 74.8 \text{ J}$$ **Ans.**

Sample Example 17.13 *In a crude model of a rotating diatomic molecule of chlorine (Cl_2), the two Cl atoms are 2.0×10^{-10} m apart and rotate about their centre of mass with angular speed $\omega = 2.0 \times 10^{12}$ rad/s. What is the rotational kinetic energy of one molecules of Cl_2, which has a molar mass of 70.0 g/mol?*

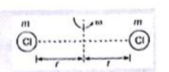


Fig. 17.29

Solution Moment of inertia,

$$I = 2(mr^2) = 2mr^2$$

Here $$m = \frac{70 \times 10^{-3}}{2 \times 6.02 \times 10^{23}} = 5.81 \times 10^{-26} \text{ kg}$$

and $$r = \frac{2.0 \times 10^{-10}}{2} = 1.0 \times 10^{-10} \text{ m}$$

$\therefore$ $$I = 2(5.81 \times 10^{-26})(1.0 \times 10^{-10})^2$$
$$= 1.16 \times 10^{-45} \text{ kg-m}^2$$

$\therefore$ $$K_R = \frac{1}{2}I\omega^2$$
$$= \frac{1}{2} \times (1.16 \times 10^{-45}) \times (2.0 \times 10^{12})^2$$
$$= 2.32 \times 10^{-21} \text{ J}$$ **Ans.**

Note *At $T = 300$ K, rotational K.E. should be equal to* $\frac{1}{2}kT = \frac{1}{2} \times (1.38 \times 10^{-23}) \times (300) = 2.07 \times 10^{-21}$ J

Sample Example 17.14 *Prove that the pressure of an ideal gas is numerically equal to two third of the mean translational kinetic energy per unit volume of the gas.*

Solution Translational KE per unit volume

$$E = \frac{1}{2}(\text{mass per unit volume})(\bar{v}^2)$$
$$= \frac{1}{2}(\rho)\left(\frac{3P}{\rho}\right) = \frac{3}{2}P$$

or $$P = \frac{2}{3}E$$ **Hence Proved.**

Note *Students are advised to remember this result. In this expression E is the translational KE per unit volume.*

Introductory Exercise 17.3

1. The average speed of all the molecules in a gas at a given instant is not zero, whereas the average velocity of all the molecules is zero. Explain why?
2. A sample of helium gas is at a temperature of 300 K and a pressure of 0.5 atm. What is the average kinetic energy of a molecule of a gas?
3. A sample of helium and neon gases has a temperature of 300 K and pressure of 1.0 atm. The molar mass of helium is 4.0 g/mol and that of neon is 20.2 g/mol.
   (a) Find the rms speed of the helium atoms and of the neon atoms.
   (b) What is the average kinetic energy per atom of each gas?
4. At what temperature will the particles in a sample of helium gas have an rms speed of 1.0 km/s?
5. At 0°C and 1.0 atm ($= 1.01 \times 10^5\ N/m^2$) pressure the densities of air, oxygen and nitrogen are $1.293\ kg/m^3$, $1.429\ kg/m^3$ and $1.251\ kg/m^3$ respectively. Calculate the percentage of nitrogen in the air from these data, assuming only these two gases to be present.
6. An air bubble of $20\ cm^3$ volume is at the bottom of a lake 40 meters deep where the temperature is 4°C. The bubble rises to the surface which is at a temperature of 20°C. Take the temperature to be the same as that of the surrounding water and find its volume just before it reaches the surface.
7. If the water molecules in 1.0 g of water were distributed uniformly over the surface of earth, how many such molecules would there be in $1.0\ cm^2$ of earth's surface?
8. For a certain gas the heat capacity at constant pressure is greater than that at constant volume by 29.1 J/K.
   (a) How many moles of the gas are there?
   (b) If the gas is monoatomic, what are heat capacities at constant volume and pressure ?
   (c) If the gas molecules are diatomic which rotate but do not vibrate, what are heat capacities at constant volume and at constant pressure.
9. The heat capacity at constant voiume of a sample of a monoatomic gas is 35 J/K. Find :
   (a) the number of moles
   (b) the internal energy at 0°C
   (c) the molor heat capacity at constant pressure.
10. For any distribution of speeds $v_{rms} \ge v_{av}$. Is this statement true or false?

Extra Points

- Pressure exerted by an ideal gas is numerically equal to two-third of the mean kinetic energy of translation per unit volume of the gas. Thus,

$$P = \frac{2}{3}E$$

- **Mean Free Path :** Every gas consists of a very large number of molecules. These molecules are in a state of continuous rapid and random motion. They undergo perfectly elastic collisions against one another. Therefore, path of a single gas molecule consists of a series of short zig-zag paths of different lengths.

  The mean free path of a gas molecule is the average distance between two successive collisions. It is represented by λ.

$$\lambda = \frac{kT}{\sqrt{2}\pi\sigma^2 p}$$

Here, σ = diameter of the molecule

k = Boltzmann's constant

- **Avagadro's Hypothesis :** At constant temperature and pressure equal volumes of different gases contain equal number of molecules. In 1 gm-mole of any gas there are 6.02×10^{23} molecules of that gas. This is called Avagadro's number. Thus,

$$N = 6.02 \times 10^{23} \text{ per gm-mole}$$

Therefore, the number of molecules in mass m of the substance :

$$\text{Number of molecules} = nN = \frac{m}{M} \times N$$

- **Dalton' Law of Partial Pressure :** According to this law if the gases filled in a vessel do not react chemically, then the combined pressure of all the gases is due to the partial pressure of the molecules of the individual gases. If $P_1, P_2, \dots$ represent the partial pressures of the different gases, then the total pressure is,

$$P = P_1 + P_2 \dots$$

- **Van der Waal's Equation :** Experiments have proved that real gases deviate largely from ideal behaviour. The reason of this deviation is two wrong assumptions in the kinetic theory of gases.
  (i) The size of the molecules is much smaller in comparison to the volume of the gas, hence, it may be neglected.
  (ii) Molecules do not exert intermolecular force on each other.
  Van der Waal made corrections for these assumptions and gave a new equation. This equation is known as Van der Waal's equation for real gases.
  (i) Correction for the finite size of molecules : Molecules occupy some volume. Therefore, the volume in which they perform thermal motion is less than the observed volume of the gas. It is represented by $(V - b)$. Here, b is a constant which depends on the effective size and number of molecules of the gas. Therefore, we should use $(V - b)$ in place of V in gas equation.
  (ii) Correction for intermolecular attraction : Due to the intermolecular force between gas molecules the molecules which are very near to the wall experiences a net inward force. Due to this inward force there is a decrease in momentum of the particles of a gas. Thus, the pressure exerted by real gas molecules is less than the pressure exerted by the molecules of an ideal gas. So, we use $\left(P + \frac{a}{V^2}\right)$ in place of P in gas equation. Here, again a is a constant.
  Van der Waal's equation of state for real gases thus becomes,

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

- **Critical Temperature, Pressure and Volume :** Gases can't be liquified above a temperature called critical temperature (T_C) however large the pressure may be. The pressure required to liquify the gas at critical temperature is called critical pressure (P_C) and the volume of the gas at critical temperature and pressure is called critical volume (V_C). Value of critical constants in terms of Van der Waal's constants 'a' and 'b' are as under :

$$V_C = 3b, \quad P_C = \frac{a}{27b^2} \quad \text{and} \quad T_C = \frac{8a}{27Rb}$$

Further, $\frac{RT_C}{P_C V_C} = \frac{8}{3}$ is called critical coefficient and is same for all gases.

Detailed Discussion on molar heat capacity (can be read only for personal interest):

- For monoatomic gases value of C_V is $\frac{3}{2}R$. No variation is observed in this. So, value of C_V, C_P, $C_P - C_V$ and γ comes out to be same for different monoatomic gases (Table 17.3).

Table 17.3

Gas	C_P	C_V	$C_P - C_V$	$\gamma = C_P/C_V$
Monoatomic Gases				
He	20.8	12.5	8.33	1.67
Ar	20.8	12.5	8.33	1.67
Diatomic Gases				
H_2	28.8	20.4	8.33	1.41
N_2	29.1	20.8	8.33	1.40
O_2	29.4	21.1	8.33	1.40
CO	29.3	21.0	8.33	1.40
Cl_2	34.7	25.7	8.96	1.35
Polyatomic Gases				
CO_2	37.0	28.5	8.50	1.30
SO_2	40.4	31.4	9.00	1.29
H_2O	35.4	27.0	8.37	1.30
CH_4	35.5	27.1	8.41	1.31

*All values except that for water were obtained at 300 K. SI units are used for C_P and C_V.

For dia and polyatomic gases these values are not equal for different gases. These values vary from gas to gas. Even for one gas values are different at different temperatures.

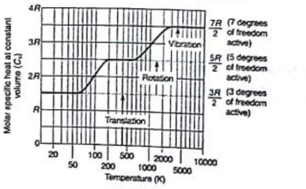


Fig. 17.30

Figure illustrates the variation in the molar specific heat (at constant volume) for H_2 over a wide range in temperatures. (Note that T is drawn on a logrithmic scale). Below about 100 K, C_V is $\frac{3R}{2}$ which is characteristic of three translational degrees of freedom. At room temperature (300 K) it is $\frac{5R}{2}$ which includes the two rotational degrees of freedom. It seems, therefore, that at low temperatures, rotation is not allowed. At high temperatures, C_V starts to rise toward the value $\frac{7R}{2}$. Thus, the vibrational degrees of freedom contribute only at these high temperatures. In table 17.3 the large values of C_V for some polyatomic molecules show the contributions of vibrational energy. In addition, a molecule with three or more atoms that are not in a straight line has three, not two, rotational degrees of freedom.

From this discussion we expect heat capacities to be temperature-dependent, generally increasing with increasing temperature.

- **Solids :** In crystalline solids (monoatomic), the atoms are arranged in a three dimensional array, called a lattice. Each atom in a lattice can vibrate along three mutually perpendicular directions, each of which has two degrees of freedom. One corresponding to vibrations KE and the other vibrational PE Thus, each atom has a total of six degrees of freedom. The volume of a solid does not change significantly with temperature, and so there is little difference between C_V and C_P for a solid. The molar heat capacity is expected to be,

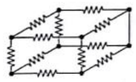


Fig. 17.31

$$C = \frac{f}{2}R = \frac{6}{2}R$$

or $$C = 3R \quad \text{(ideal monoatomic solid)}$$

Its numerical value is $C = 25$ J/mol-K $= 6$ cal/mol-K. This result was first found experimentally by **Dulong and Petit.**

Figure shows that the Dulong and Petit law is obeyed quite well at high (> 250 K) temperatures. At low temperatures, the heat capacities decreases.

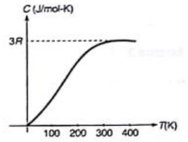


Fig. 17.32

Solved Examples

For JEE Main

Example 1 *A platinum resistance thermometer reads 0°C when its resistance is 80 Ω and 100°C when its resistance is 90 Ω. Find the temperature at which the resistance is 86 Ω.*

Solution The temperature on the platinum scale is

$$t=\frac{R_t-R_0}{R_{100}-R_0}\times 100°=\frac{86-80}{90-80}\times 100°\text{C}=60°\text{C}$$

Example 2 *The steam point and the ice point of a mercury thermometer are marked as 80° and 10°. At what temperature on centigrade scale the reading of this thermometer will be 59°?*

Solution Let the relation between the thermometer reading and centigrade be $y=ax+b$

given at $x=100$, $y=80$ and at $x=0$, $y=10$

$\therefore$ $$80=100a+b, 10=b \Rightarrow a=0.7$$

Now, we have to find x when $y=59$

$\therefore$ $$59=0.7x+b \Rightarrow x=70$$

$\therefore$ The answer is 70°C

Example 3 *Find the rms speed of hydrogen molecules at room temperature (= 300 K).*

Solution Mass of 1 mole of hydrogen gas

$$=2\text{ g}=2\times10^{-3}\text{ kg}$$

$\Rightarrow$ $$v_{rms}=\sqrt{\frac{3RT}{M}}=\sqrt{\frac{3\times8.31\times300}{2\times10^{-3}}}$$

$$=1.93\times10^{3}\text{ m/s}$$

Ans.

Example 4 *Find the temperature at which oxygen molecules would have the same rms speed as of hydrogen molecules at 300 K.*

Solution If T be the corresponding temperature,

$$\sqrt{\frac{3RT}{M_O}}=\sqrt{\frac{3R(300)}{M_H}} \Rightarrow T=(300)\left(\frac{M_O}{M_H}\right)$$

$$=4800\text{ K}$$

Example 5 *A sphere of diameter 7 cm and mass 266.5 g floats in a bath of liquid. As the temperature is raised, the sphere just sinks at a temperature of 35° C. If the density of the liquid at 0° C is 1.527 g/cm³ find the coefficient of cubical expansion of the liquid.*

Solution The sphere will sink in the liquid at 35°C, when its density becomes equal to the density of liquid at 35° C.

The density of sphere,

$$\rho_{35}=\frac{266.5}{\frac{4}{3}\times\left(\frac{22}{7}\right)\times\left(\frac{7}{2}\right)^3}=1.483\text{ g/cm}^3$$

Now $$\rho_0=\rho_{35}[1+\gamma\Delta T)$$

$$1.527=1.483[1+\gamma\times35]$$

$$1.029=1+\gamma\times35$$

$$\gamma=\frac{1.029-1}{35}=0.00083/°\text{C}$$

Example 6 *4 g hydrogen is mixed with 11.2 litre of He at STP in a container of volume 20 litre. If the final temperature is 300 K find the pressure.*

Solution 4 g hydrogen = 2 moles hydrogen.

$$11.2\text{ litre He at STP}=\frac{1}{2}\text{ mole of He}$$

$$P=P_H+P_{He}=(n_H+n_{He})\frac{RT}{V}$$

$$=\left(2+\frac{1}{2}\right)\frac{8.31\times(300\text{ K})}{(20\times10^{-3})\text{ m}^3}$$

$$=3.12\times10^{5}\text{ N/m}^2$$

Ans.

Example 7 *One mole of an ideal monoatomic gas is taken at a temperature of 300 K. Its volume is doubled keeping its pressure constant. Find the change in internal energy.*

Solution Since, pressure is constant

$\therefore$ $$V\propto T$$

$\therefore$ $$\frac{V_i}{T_i}=\frac{V_f}{T_f}$$

$\therefore$ $$T_f=\frac{V_f}{V_i}T_i \Rightarrow T_f=2T_i=600\text{K}$$

$\therefore$ $$\Delta U=\frac{f}{2}n.R\Delta T=\frac{3}{2}R(600-300)=450R$$

Example 8 *A beam of particles each of mass m_0 and speed v is directed along the x-axis. The beam strikes an area 1 mm square with 10^{15} particles striking per second. Find the pressure on the area due to the beam if the particles stick to the area when they hit. Evaluate for an electron beam in a television tube where $m_0=9.1\times10^{-31}$ kg and $v=8\times10^{7}$ m/s.*

Solution Each electron exerts an impulse of m_0v when it strikes and sticks to the surface. This times the number striking in unit time divided by area is pressure. Hence,

$$P=\frac{10^{15}}{(10^{-3})^2}m_0v=10^{21}\,m_0v$$

Substituting the given values, we have

$$P=(9.1\times10^{-31})(8\times10^{7})(10^{21})=0.073\text{ N/m}^2$$

Ans.

Example 9 *Two perfect monoatomic gases at absolute temperatures T_1 and T_2 are mixed. There is no loss of energy. Find the temperature of the mixture if the number of moles in the gases are n_1 and n_2.*

Solution From energy conservation principle,

$$E_i=E_f$$

or $$E_1+E_2=E$$

$\therefore$ $$n_1\left(\frac{3}{2}RT_1\right)+n_2\left(\frac{3}{2}RT_2\right)=(n_1+n_2)\left(\frac{3}{2}RT\right)$$

$\therefore$ $$T=\frac{n_1T_1+n_2T_2}{n_1+n_2}$$

Ans.

Example 10 *A glass beaker holds exactly 1 lt at 0°C*

(a) What is its volume at 50°C?

(b) If the beaker is filled with mercury at 0°C, what volume of mercury overflows when the temperature is 50°C? $\alpha_g=8.3\times10^{-6}$ per °C and $\gamma_{Hg}=1.82\times10^{-4}$ per °C.

Solution (a) The volume of beaker after the temperature change is,

$$V_{beaker}=V_0(1+3\alpha_g\Delta\theta)$$

$$=(1)[1+3\times8.3\times10^{-6}\times50]$$

$$=1.001\text{ lt}$$

Ans.

(b) Volume of mercury at 50°C is

$$V_{mercury}=V_0(1+\gamma_{Hg}\Delta\theta)$$

$$=(1)[1+1.82\times10^{-4}\times50]=1.009\text{ lt}$$

The overflow is thus $1.009-1.001=0.008$ lt or 8 ml

Ans.

For JEE Advanced

Example 1 *A cubical box of side 1 m contains helium gas (atomic weight 4) at a pressure of 100 N/m². During an observation time of 1 second, an atom travelling with the root-mean-square speed parallel to one of the edges of the cube, was found to make 500 hits with a particular wall, without any collision with other atoms. Take $R=\frac{25}{3}$ J/mol-K and $k=1.38\times10^{-23}$ J/K.*

(a) Evaluate the temperature of the gas.

(b) Evaluate the average kinetic energy per atom.

(c) Evaluate the total mass of helium gas in the box.

Solution Volume of the box = 1 m³, pressure of the gas = 100 N/m². Let T be the temperature of the gas.

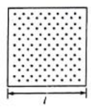


Fig. 17.33

(a) Time between two consecutive collisions with one wall $=\frac{1}{500}$ sec

This time should be equal to $\frac{2l}{v_{rms}}$, where l is the side of the cube.

$$\frac{2l}{v_{rms}}=\frac{1}{500}$$

or $$v_{rms}=1000\text{ m/s}$$ (as $l=1$ m)

$\therefore$ $$\sqrt{\frac{3RT}{M}}=1000$$

or $$T=\frac{(1000)^2M}{3R}=\frac{(10^6)(4\times10^{-3})}{3\left(\frac{25}{3}\right)}=160\text{ K}$$

Ans.

(b) Average kinetic energy per atom $=\frac{3}{2}kT$

$$=\frac{3}{2}(1.38\times10^{-23})(160)\text{ J}$$

$$=3.312\times10^{-21}\text{ J}$$

Ans.

(c) From $PV=nRT=\frac{m}{M}RT$

we get mass of helium gas in the box,

$$m=\frac{PVM}{RT}$$

Substituting the values, we get

$$m=\frac{(100)(1)(4\times10^{-3})}{\left(\frac{25}{3}\right)(160)}$$

$$=3.0\times10^{-4}\text{ kg}$$

Ans.

Example 2 *An ideal diatomic gas with $C_V = \frac{5R}{2}$ occupies a volume V_i at a pressure P_i. The gas undergoes a process in which the pressure is proportional to the volume. At the end of the process, it is found that the rms speed of the gas molecules has doubled from its initial value. Determine the amount of energy transferred to the gas by heat.*

Solution Given that, $P \propto V$ or PV^{-1} = constant

As we know, molar heat capacity in the process PV^x = constant is,

$$C = \frac{R}{\gamma - 1} + \frac{R}{1-x} = C_V + \frac{R}{1-x}$$

In the given problem, $C_V = \frac{5R}{2}$ and $x = -1$

$$\therefore \quad C = \frac{5R}{2} + \frac{R}{2} = 3R \qquad \text{...(i)}$$

At the end of the process rms speed is doubled, *i.e.*, temperature has become four times ($v_{rms} \propto \sqrt{T}$).

Now,

$$\begin{aligned} \Delta Q &= nC\Delta T \\ &= nC(T_f - T_i) \\ &= nC(4T_i - T_i) \\ &= 3T_i nC \\ &= (3T_i)(n)(3R) \\ &= 9(nRT_i) \end{aligned}$$

or $\Delta Q = 9P_iV_i$ **Ans.**

Example 3 *1 g mole of oxygen at 27°C and 1 atmospheric pressure is enclosed in a vessel.*

(a) Assuming the molecules to be moving with v_{rms}, find the number of collisions per second which the molecules make with one square metre area of the vessel wall.

(b) The vessel is next thermally insulated and moved with a constant speed v_0. It is then suddenly stopped. The process results in a rise of temperature of the gas by 1°C. Calculate the speed v_0. [$k = 1.38 \times 10^{-23}$ J/K and $N_A = 6.02 \times 10^{23}$/mol]

Solution (a) Mass of one oxygen molecule

$$\begin{aligned} m &= \frac{M}{N_A} \\ &= \frac{32}{6.02 \times 10^{23}} \text{ g} \\ &= 5.316 \times 10^{-23} \text{ g} \\ &= 5.316 \times 10^{-26} \text{ kg} \end{aligned}$$

$$v_{rms} = \sqrt{\frac{3kT}{m}}$$

$$= \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 300}{5.316 \times 10^{-26}}}$$

$$= 483.35 \text{ m/s}$$

Change in momentum per collision

$$\begin{aligned} \Delta p &= mv_{rms} - (-mv_{rms}) = 2mv_{rms} \\ &= (2)(5.316 \times 10^{-26})(483.35) \\ &= 5.14 \times 10^{-23} \text{ kg-m/s} \end{aligned}$$

Now, suppose n particles strike per second

$$F = n\Delta p = (n)(5.14 \times 10^{-23}) \text{ N} \qquad \left(F_{ext} = \frac{dP}{dt}\right)$$

Now, as $P = \frac{F}{A}$, for unit area $F = P$

$\therefore$ $(n)(5.14 \times 10^{-23}) = 1.01 \times 10^5$

or $n = 1.965 \times 10^{27}$ per second **Ans.**

(b) When the vessel is stopped the ordered motion of the vessel converts into disordered motion and temperature of the gas is increased.

$$\therefore \quad \frac{1}{2}mv_0^2 = \Delta U \qquad \text{...(i)}$$

$$U = \frac{5}{2}RT \qquad (\text{for } O_2)$$

$$\therefore \quad \Delta U = \frac{5}{2}R\Delta T$$

Here m is not the mass of one gas molecule but it is the mass of the whole gas.

m = mass of 1 mol $= 32 \times 10^{-3}$ kg

Substituting these values in Eq. (i)

$$v_0 = \sqrt{\frac{5R\Delta T}{m}}$$

$$= \sqrt{\frac{5 \times 8.31 \times 1}{32 \times 10^{-3}}}$$

$$= 36 \text{ m/s}$$ **Ans.**

Example 4 *Given: Avogadro's number $N = 6.02 \times 10^{23}$ and Boltzmann's constant $k = 1.38 \times 10^{-23}$ J/K. Calculate*

(a) the average kinetic energy of translation of the molecules of an ideal gas at 0°C and at 100°C.

(b) also calculate the corresponding energies per mole of the gas.

Solution (a) According to the kinetic theory, the average kinetic energy of translation per molecule of an ideal gas at kelvin temperature T is $\left(\frac{3}{2}\right)kT$, where k is Boltzmann's constant.

At 0°C ($T = 273$ K), the kinetic energy of translation $= \frac{3}{2}kT$

$$= \frac{3}{2} \times (1.38 \times 10^{-23}) \times 273 = 5.65 \times 10^{-21} \text{ J/molecule}$$

At 100°C ($T = 373$ K), the energy is

$$\frac{3}{2} \times (1.38 \times 10^{-23}) \times 373 = 7.72 \times 10^{-21} \text{ J/molecule}$$

(b) 1 mole of gas contains N ($= 6.02 \times 10^{23}$) molecules. Therefore, at 0°C, the kinetic energy of translation of 1 mole of the gas is

$$= (5.65 \times 10^{-21})(6.02 \times 10^{23}) \approx 3401 \text{ J/mol and at } 100°\text{C}.$$

the kinetic energy of translation of 1 mole of gas is

$$= (7.72 \times 10^{-21})(6.02 \times 10^{23}) \approx 4647 \text{ J/mol}$$

Example 5 *An air bubble starts rising from the bottom of a lake. Its diameter is 3.6 mm at the bottom and 4 mm at the surface. The depth of the lake is 250 cm and the temperature at the surface is 40°C. What is the temperature at the bottom of the lake? Given atmospheric pressure = 76 cm of Hg and $g = 980$ cm/s^2.*

Solution At the bottom of the lake, volume of the bubble

$$V_1 = \frac{4}{3}\pi r_1^3 = \frac{4}{3}\pi (0.18)^3 \text{ cm}^3$$

Pressure on the bubble P_1 = Atmospheric pressure + Pressure due to a column of 250 cm of water

$$\begin{aligned} &= 76 \times 13.6 \times 980 + 250 \times 1 \times 980 \\ &= (76 \times 13.6 + 250)\, 980 \text{ dyne/cm}^2 \end{aligned}$$

At the surface of the lake, volume of the bubble

$$V_2 = \frac{4}{3}\pi r_2^3 = \frac{4}{3}\pi (0.2)^3 \text{ cm}^3$$

Pressure on the bubble

$$\begin{aligned} P_2 &= \text{atmospheric pressure} \\ &= (76 \times 13.6 \times 980) \text{ dyne/cm}^2 \\ T_2 &= 273 + 40°\text{C} = 313° \text{ K} \end{aligned}$$

Now

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$$

or

$$\frac{(76 \times 13.6 + 250) 980 \times \left(\frac{4}{3}\right)\pi (0.18)^3}{T_1}$$

$$= \frac{(76 \times 13.6) \times 980 \left(\frac{4}{3}\right)\pi (0.2)^3}{313}$$

or $T_1 = 283.37$ K

$\therefore$ $T_1 = 283.37 - 273 = 10.37$°C

Example 6 *P-V diagram of n moles of an ideal gas is as shown in figure. Find the maximum temperature between A and B.*

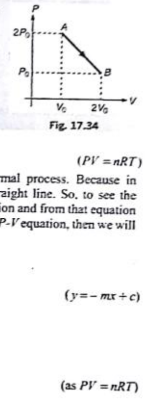


Fig. 17.34

HOW TO PROCEED For given number of moles of a gas,

$$T \propto PV \qquad (PV = nRT)$$

Although $(PV)_A = (PV)_B$ or $T_A = T_B$, still it is not an isothermal process. Because in isothermal process P-V graph is a rectangular hyperbola while it is a straight line. So, to see the behaviour of temperature first we will find either T-V equation or T-P equation and from that equation we can judge how the temperature varies. From the graph first we will write P-V equation, then we will convert it either in T-V equation or in T-P equation.

Solution From the graph the P-V equation can be written as,

$$P = -\left(\frac{P_0}{V_0}\right)V + 3P_0 \qquad (y = -mx + c)$$

or

$$PV = -\left(\frac{P_0}{V_0}\right)V^2 + 3P_0V$$

or

$$nRT = 3P_0V - \left(\frac{P_0}{V_0}\right)V^2 \qquad (\text{as } PV = nRT)$$

or

$$T = \frac{1}{nR}\left[3P_0V - \left(\frac{P_0}{V_0}\right)V^2\right]$$

This is the required T-V equation. This is quadratic in V. Hence, T-V graph is a parabola. Now, to find maximum or minimum value of T we can substitute.

$$\frac{dT}{dV} = 0$$

or

$$3P_0 - \left(\frac{2P_0}{V_0}\right)V = 0$$

or
$$V = \frac{3}{2}V_0$$

Further $\frac{d^2T}{dV^2}$ is negative at $V = \frac{3}{2}V_0$

Hence, T is maximum at $V = \frac{3}{2}V_0$ and this maximum value is,

$$T_{max} = \frac{1}{nR}\left[(3P_0)\left(\frac{3V_0}{2}\right) - \left(\frac{P_0}{V_0}\right)\left(\frac{3V_0}{2}\right)^2\right]$$

or
$$T_{max} = \frac{9P_0V_0}{4nR} \qquad \textbf{Ans.}$$

Thus, T-V graph is as shown in figure.

$$T_A = T_B = \frac{2P_0V_0}{nR}$$

and
$$T_{max} = \frac{9P_0V_0}{4nR} = 2.25\frac{P_0V_0}{nR}$$

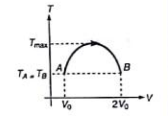


Fig. 17.35

Note *Most of the problems of T_{max}, P_{max} and V_{max} are solved by differentiation. Sometimes graph will be given and sometimes direct equation will be given. For Problem, for P_{max} you will require either P-V or P-T equation.*

EXERCISES

For JEE Main

Subjective Questions

Temperature Scales

1. Change each of the given temperatures to the Celsius and Kelvin scales: 68° F, 5° F and 176° F.
2. Change each of the given temperatures to the Fahrenheit and Rankine scales: 30°C, 5°C and −20°C.
3. At what temperature do the Celsius and Fahrenheit readings have the same numerical value?
4. You work in a materials testing lab and your boss tells you to increase the temperature of a sample by 40.0°C. The only thermometer you can find at your workbench reads in °F. If the initial temperature of the sample is 68.2° F. What is its temperature in ° F when the desired temperature increase has been achieved?
5. The steam point and the ice point of a mercury thermometer are marked as 80° and 20°. What will be the temperature in centigrade mercury scale when this thermometer reads 32° ?

Thermometers

6. The pressure of the gas in a constant volume gas thermometer is 80 cm of mercury in melting ice at 1 atm. When the bulb is placed in a liquid, the pressure becomes 160 cm of mercury. Find the temperature of the liquid.
7. The resistances of a platinum resistance thermometer at the ice point, the steam point and the boiling point of sulphur are 2.50, 3.50 and 6.50 Ω respectively. Find the boiling point of sulphur on the platinum scale. The ice point and the steam point measure 0° and 100° respectively.
8. In a constant volume gas thermometer, the pressure of the working gas is measured by the difference in the levels of mercury in the two arms of a U-tube connected to the gas at one end. When the bulb is placed at the room temperature 27.0°C, the mercury column in the arm open to atmosphere stands 5.00 cm above the level of mercury in the other arm. When the bulb is placed in a hot liquid, the difference of mercury levels becomes 45.0 cm. Calculate the temperature of the liquid. (Atmospheric pressure = 75.0 cm of mercury.)

Thermal Expansion

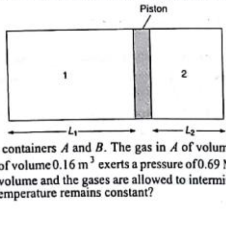
9. An iron ball has a diameter of 6 cm and is 0.010 mm too large to pass through a hole in a brass plate when the ball and plate are at a temperature of 30°C. At what temperature, the same for ball and plate, will the ball just pass through the hole ?
10. (a) An aluminum measuring rod which is correct at 5°C, measures a certain distance as 88.42 cm at 35 °C. Determine the error in measuring the distance due to the expansion of the rod. (b) If this aluminum rod measures a length of steel as 88.42 cm at 35°C, what is the correct length of the steel at 35°C?
11. A steel tape is calibrated at 20°C. On a cold day when the temperature is −15°C, what will be the percentage error in the tape ?

12. A steel wire of 2.0 mm² cross-section is held straight (but under no tension) by attaching it firmly to two points a distance 1.50 m apart at 30 °C. If the temperature now decreases to −10°C and if the two points remain fixed, what will be the tension in the wire ? For steel, Y = 20,0000 MPa.
13. A metallic bob weighs 50 g in air. If it is immersed in a liquid at a temperature of 25°C, it weighs 45 g. When the temperature of the liquid is raised to 100°C, it weighs 45.1 g. Calculate the coefficient of cubical expansion of the liquid. Given that coefficient of cubical expansion of the metal is 12×10^{-6} °C^{-1}.

Gas Laws and Ideal Gas Equation

14. Find the mass (in kilogram) of an ammonia molecule NH_3.
15. An ideal gas exerts a pressure of 1.52 MPa when its temperature is 298.15 K and its volume is 10^{-2} m³. (a) How many moles of gas are there? (b) What is the mass density if the gas is molecular hydrogen? (c) What is the mass density if the gas is oxygen?
16. A compressor pumps 70 L of air into a 6 L tank with the temperature remaining unchanged. If all the air is originally at 1 atm. What is the final absolute pressure of the air in the tank?
17. A partially inflated balloon contains 500 m³ of helium at 27°C and 1 atm pressure. What is the volume of the helium at an altitude of 18000 ft, where the pressure is 0.5 atm and the temperature is −3°C?
18. A cylinder whose inside diameter is 4.00 cm contains air compressed by a piston of mass m = 13.0 kg which can slide freely in the cylinder. The entire arrangement is immersed in a water bath whose temperature can be controlled. The system is initially in equilibrium at temperature t_i = 20°C. The initial height of the piston above the bottom of the cylinder is h_i = 4.00 cm.
   The temperature of the water bath is gradually increased to a final temperature t_f = 100°C. Calculate the final height h_f of the piston.

19. The closed cylinder shown in figure has a freely moving piston separating chambers 1 and 2. Chamber 1 contains 25 mg of N_2 gas and chamber 2 contains 40 mg of helium gas. When equilibrium is established what will be the ratio L_1 / L_2? What is the ratio of the number of moles of N_2 to the number of moles of He? (Molecular weights of N_2 and He are 28 and 4).



20. Two gases occupy two containers A and B. The gas in A of volume 0.11 m³ exerts a pressure of 1.38 MPa. The gas in B of volume 0.16 m³ exerts a pressure of 0.69 MPa. Two containers are united by a tube of negligible volume and the gases are allowed to intermingle. What is the final pressure in the container if the temperature remains constant?

21. A glass bulb of volume 400 cm³ is connected to another of volume 200 cm³ by means of a tube of negligible volume. The bulbs contain dry air and are both at a common temperature and pressure of 20°C and 1.00 atm. The larger bulb is immersed in steam at 100°C and the smaller in melting ice at 0°C. Find the final common pressure.
22. The condition called standard temperature and pressure (STP) for a gas is defined as temperature of 0° C = 273.15 K and a pressure of 1 atm = 1.013×10^5 Pa. If you want to keep a mole of an ideal gas in your room at STP, how big a container do you need?
23. A large cylindrical tank contains 0.750 m³ of nitrogen gas at 27°C and 1.50×10^5 Pa (absolute pressure). The tank has a tightfitting piston that allows the volume to be changed. What will be the pressure if the volume is decreased to 0.480 m³ and the temperature is increased to 157°C.
24. A vessel of volume 5 litres contains 1.4 g of N_2 and 0.4 g of He at 1500 K. If 30% of the nitrogen molecules are dissociated into atoms then find the gas pressure.

Degree of Freedom, Internal Energy and Molar Heat Capacity

25. Temperature of diatomic gas is 300 K. If moment of inertia of its molecules is 8.28×10^{-38} g-cm². Calculate their root mean square angular velocity.
26. How many degrees of freedom have the gas molecules, if under standard conditions the gas density is $\rho = 1.3$ kg/m³ and velocity of sound propagation on it is v = 330 m/s?
27. Three moles of an ideal gas having $\gamma = 1.67$ are mixed with 2 moles of another ideal gas having $\gamma = 1.4$. Find the equivalent value of γ for the mixture.
28. If the kinetic energy of the molecules in 5 litres of helium at 2 atm is E. What is the kinetic energy of molecules in 15 litres of oxygen at 3 atm in terms of E?
29. Find the number of degrees of freedom of molecules in a gas. Whose molar heat capacity
   (a) at constant pressure $C_P = 29$ J mol^{-1}K^{-1}
   (b) $C = 29$ J mol^{-1} K^{-1} in the process PT = constant.
30. In a certain gas $\frac{2}{5}$ th of the energy of molecules is associated with the rotation of molecules and the rest of it is associated with the motion of the centre of mass.
   (a) What is the average translational energy of one such molecule when the temperature is 27°C?
   (b) How much energy must be supplied to one mole of this gas at constant volume to raise the temperature by 1°C?
31. A mixture contains 1 mole of helium ($C_P = 2.5R, C_V = 1.5R$) and 1 mole of hydrogen ($C_P = 3.5R, C_V = 2.5R$). Calculate the values of C_P, C_V and γ for the mixture.
32. Two ideal gases have the same value of $C_P/C_V = \gamma$. What will be the value of this ratio for a mixture of the two such gases in the ratio 1 : 2?
33. An ideal gas ($C_P / C_V = \gamma$) is taken through a process in which the pressure and the volume vary as $P = aV^b$. Find the value of b for which the specific heat capacity in the process is zero.
34. An ideal gas is taken through a process in which the pressure and the volume are changed according to the equation $P = kV$. Show that the molar heat capacity of the gas for the process is given by $C = C_V + \frac{R}{2}$.

Kinetic Theory of Gases

35. Calculate the root mean square speed of hydrogen molecules at 373.15 K.

36. Five gas molecules chosen at random are found to have speed of 500, 600, 700, 800 and 900 m/s. Find the rms speed. Is it the same as the average speed?

37. The pressure of a gas in a 100 mL container is 200 kPa and the average translational kinetic energy of each gas particle is 6×10^{-26} J. Find the number of gas particles in the container. How many moles are there in the container?

38. One gram mole NO_2 at 57°C and 2 atm pressure is kept in a vessel. Assuming the molecules to be moving with rms velocity. Find the number of collisions per second which the molecules make with one square meter area of the vessel wall.

39. A 2.00 mL volume container contains 50 mg of gas at a pressure of 100 kPa. The mass of each gas particle is 8.0×10^{-26} kg. Find the average translational kinetic energy of each particle.

40. Call the rms speed of the molecules in an ideal gas v_0 at temperature T_0 and pressure p_0. Find the speed if (a) the temperature is raised from $T_0 = 293$ K to 573 K (b) the pressure is doubled and $T = T_0$ (c) the molecular weight of each of the gas molecules is tripled.

41. At what temperature is the "effective" speed of gaseous hydrogen molecules (molecular weight = 2) equal to that of oxygen molecules (molecular weight = 32) at 47°C?

42. At what temperature is v_{rms} of H_2 molecules equal to the escape speed from earth's surface. What is the corresponding temperature for escape of hydrogen from moon's surface? Given $g_m = 1.6\ m/s^2$, $R_e = 6367$ km and $R_m = 1750$ km.

43. (a) What is the average translational kinetic energy of a molecule of an ideal gas at temperature of 27°C?
    (b) What is the total random translational kinetic energy of the molecules in one mole of this gas?
    (c) What is the rms speed of oxygen molecules at this temperature?

Objective Questions

Single Correct Option

1. The average velocity of molecules of a gas of molecular weight M at temperature T is
   (a) $\sqrt{\frac{3RT}{M}}$ (b) $\sqrt{\frac{8RT}{\pi M}}$ (c) $\sqrt{\frac{2RT}{M}}$ (d) zero

2. Four particles have velocities 1, 0, 2 and 3 m/s. The root mean square velocity of the particles (definition wise) is
   (a) 3.5 m/s (b) $\sqrt{3.5}$ m/s (c) 1.5 m/s (d) $\sqrt{\frac{14}{3}}$ m/s

3. A steel rod of length 1 m is heated from 25° to 75°C keeping its length constant. The longitudinal strain developed in the rod is (Given coefficient of linear expansion of steel $= 12 \times 10^{-6}/°C$)
   (a) 6×10^{-4} (b) -6×10^{-5} (c) -6×10^{-4} (d) zero

4. In a process the pressure of a gas remains constant. If the temperature is doubled, then the change in the volume will be
   (a) 100% (b) 200% (c) 50% (d) 25%

5. The average kinetic energy of the molecules of an ideal gas at 10°C has the value E. The temperature at which the kinetic energy of the same gas becomes $2E$ is
   (a) 5°C (b) 10°C (c) 40°C (d) None of these

6. A polyatomic gas with n degrees of freedom has a mean energy per molecule given by
   (a) $\frac{n}{2}RT$ (b) $\frac{1}{2}RT$ (c) $\frac{n}{2}kT$ (d) $\frac{1}{2}kT$

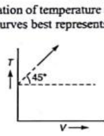
7. The temperature of an ideal gas is increased from 27°C to 927°C. The rms speed of its molecules becomes
   (a) twice (b) half (c) four times (d) one-fourth

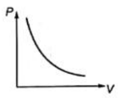
8. In case of hydrogen and oxygen at NTP, which of the following is the same for both?
   (a) Average linear momentum per molecule (b) Average KE per molecule
   (c) KE per unit volume (d) KE per unit mass

9. Two thermally insulated vessels 1 and 2 are filled with air at temperatures (T_1, T_2), volumes (V_1, V_2) and pressures (P_1, P_2) respectively. If the valve joining the two vessels is opened, the temperature inside the vessel at equilibrium will be (P = common pressure)
   (a) $T_1 + T_2$ (b) $(T_1 + T_2)/2$
   (c) $\frac{T_1 T_2 P(V_1 + V_2)}{P_1 V_1 T_2 + P_2 V_2 T_1}$ (d) $\frac{T_1 T_2 (P_1 V_1 + P_2 V_2)}{P_1 V_1 T_2 + P_2 V_2 T_1}$

10. Two marks on a glass rod 10 cm apart are found to increase their distance by 0.08 mm when the rod is heated from 0°C to 100°C. A flask made of the same glass as that of rod measures a volume of 100 cc at 0°C. The volume it measures at 100°C in cc is
    (a) 100.24 (b) 100.12 (c) 100.36 (d) 100.48

11. The given curve represents the variation of temperature as a function of volume for one mole of an ideal gas. Which of the following curves best represents the variation of pressure as a function of volume?



(a)


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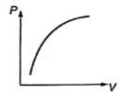

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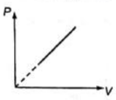

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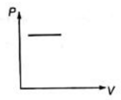


12. A gas is found to obey the law P^2V = constant. The initial temperature and volume are T_0 and V_0. If the gas expands to a volume $3V_0$, then the final temperature becomes
    (a) $\sqrt{3}\,T_0$ (b) $\sqrt{2}\,T_0$ (c) $\frac{T_0}{\sqrt{3}}$ (d) $\frac{T_0}{\sqrt{2}}$

13. Air fills a room in winter at 7°C and in summer at 37°C. If the pressure is the same in winter and summer, the ratio of the weight of the air filled in winter and that in summer is
    (a) 2.2 (b) 1.75 (c) 1.1 (d) 3.3

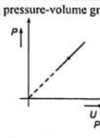
14. Three closed vessels A, B and C are at the same temperature T and contain gases which obey Maxwell distribution law of velocities. Vessel A contains O_2, B only N_2 and C mixture of equal quantities of O_2 and N_2. If the average speed of the O_2 molecules in vessel A is v_1 that of N_2 molecules in vessel B is v_2, then the average speed of the O_2 molecules in vessel C is
    (a) $\frac{(v_1 + v_2)}{2}$ (b) v_1 (c) $\sqrt{v_1 v_2}$ (d) None of these

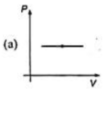
15. In a very good vacuum system in the laboratory, the vacuum attained was 10^{-13} atm. If the temperature of the system was 300 K, the number of molecules present in a volume of 1 cm^3 is
    (a) 2.4×10^6 (b) 24 (c) 2.4×10^9 (d) zero

16. If nitrogen gas molecule goes straight up with its rms speed at 0°C from the surface of the earth and there are no collisions with other molecules, then it will rise to an approximate height of
    (a) 8 km (b) 12 km (c) 12 m (d) 8 m

17. The coefficient of linear expansion of steel and brass are $11 \times 10^{-6}/°C$ and $19 \times 10^{-6}/°C$ respectively. If their difference in lengths at all temperatures has to kept constant at 30 cm, their lengths at 0°C should be
    (a) 71.25 cm and 41.25 cm (b) 82 cm and 52 cm
    (c) 92 cm and 62 cm (d) 62.25 cm and 32.25 cm

18. The given P-U graph shows the variation of internal energy of an ideal gas with increase in pressure. Which of the following pressure-volume graph is equivalent to this graph?



(a)


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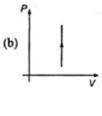

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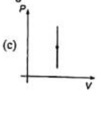

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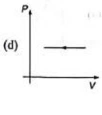
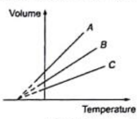


19. The expansion of an ideal gas of mass m at a constant pressure P is given by the straight line B. Then, the expansion of the same ideal gas of mass $2m$ at a pressure $2P$ is given by the straight line
    (a) C
    (b) A
    (c) B
    (d) data insufficient



20. 28 g of N_2 gas is contained in a flask at a pressure of 10 atm and at a temperature of 57°C. It is found that due to leakage in the flask, the pressure is reduced to half and the temperature to 27°C. The quantity of N_2 gas that leaked out is
    (a) 11/20 g (b) 20/11 g (c) 5/63 g (d) 63/5 g

21. A mixture of 4g of hydrogen and 8 g of helium at NTP has a density about
    (a) 0.22 kg/m^3 (b) 0.62 kg/m^3 (c) 1.12 kg/m^3 (d) 0.13 kg/m^3

22. The pressure (P) and the density (ρ) of given mass of a gas expressed by Boyle's law, $P = K\rho$ holds true
    (a) for any gas under any condition (b) for some gases under any conditions
    (c) only if the temperature is kept constant (d) none of these

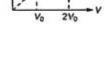
More than One Correct Options

23. During an experiment, an ideal gas is found to obey a condition $\frac{P^2}{\rho}$ = constant. (ρ = density of the gas). The gas is initially at temperature T, pressure P and density ρ. The gas expands such that density changes to $\rho/2$.
    (a) The pressure of the gas changes to $\sqrt{2}\,P$
    (b) The temperature of the gas changes to $\sqrt{2}\,T$
    (c) The graph of the above process on P-T diagram is parabola
    (d) The graph of the above process on P-T diagram is hyperbola

24. During an experiment, an ideal gas is found to obey a condition VP^2 = constant. The gas is initially at a temperature T, pressure P and volume V. The gas expands to volume $4V$.
    (a) The pressure of gas changes to $\frac{P}{2}$
    (b) The temperature of gas changes to $4T$
    (c) The graph of the above process on P-T diagram is parabola
    (d) The graph of the above process on P-T diagram is hyperbola

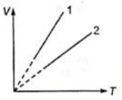
25. Find the correct options
    (a) Ice point in Fahrenheit scale is 32°F (b) Ice point in Fahrenheit scale is 98.8°F
    (c) Steam point in Fahrenheit scale is 212°F (d) Steam point in Fahrenheit scale is 252°F

26. In the P-V diagram shown in figure, choose the correct options for the process a-b:
    (a) density of gas has reduced to half
    (b) temperature of gas has increased to two times
    (c) internal energy of gas has increased to four times
    (d) T-V graph is a parabola passing through origin



27. Choose the **wrong** options
(a) Translational kinetic energy of all ideal gases at same temperature is same
(b) In one degree of freedom all ideal gases has internal energy $= \frac{1}{2} RT$
(c) Translational kinetic energy of all ideal gases is three
(d) Translational kinetic energy of all ideal gases is $\frac{3}{2} RT$

28. Along the line-1, mass of gas is m_1 and pressure is P_1. Along the line-2 mass of same gas is m_2 and pressure is P_2. Choose the correct options.



(a) m_1 may be less than m_2
(b) m_2 may be less than m_1
(c) P_1 may be less than P_2
(d) P_2 may be less than P_1

29. Choose the correct options
(a) In $P = \frac{m}{M} RT$, m is mass of gas per unit volume
(b) In $PV = \frac{m}{M} RT$, m is mass of one molecule of gas
(c) In $P = \frac{1}{3} \frac{mN}{V} v_{rms}^2$, m is total mass of gas.
(d) In $v_{rms} = \sqrt{\frac{3kT}{m}}$, m is mass of one molecule of gas

For **JEE Advanced**

Assertion and Reason

Directions : *Choose the correct option.*

(*a*) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*
(*b*) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*
(*c*) *If **Assertion** is true, but the **Reason** is false.*
(*d*) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** Straight line on *P*-*T* graph for an ideal gas represents isochoric process.
Reason : If $P \propto T$, V = constant.

2. **Assertion :** Vibrational kinetic energy is insignificant at low temperatures.
Reason : Interatomic forces are responsible for vibrational kinetic energy.

3. **Assertion :** In the formula $P = \frac{2}{3} E$, the term E represents translational kinetic energy per unit volume of gas.
Reason : In case of monoatomic gas translational kinetic energy and total kinetic energy are equal.

4. **Assertion :** If a gas container is placed in a mooving train, the temperature of gas will increase.
Reason : Kinetic energy of gas molecules will increase.

5. **Assertion :** According to the law of equipartition of energy, internal energy of an ideal gas at a given temperature, is equally distributed in translational and rotational kinetic energies.
Reason : Rotational kinetic energy of a monoatomic gas is zero.

6. **Assertion :** Real gases behave as ideal gases most closely at low pressure and high temperature.
Reason : Intermolecular force between ideal gas molecules is assumed to be zero.

7. **Assertion :** A glass of water is filled at 4°C. Water will overflow, if temperature is increased or decreased. (Ignore expansion of glass).
Reason : Density of water is minimum at 4°C.

8. **Assertion :** If pressure of an ideal gas is doubled and volume is halved, then its internal energy will remain unchanged.
Reason : Internal energy of an ideal gas is a function of temperature only.

9. **Assertion :** In equation $P = \frac{1}{3} \alpha v_{rms}^2$, the term α represents density of gas.
Reason : $v_{rms} = \sqrt{\frac{3RT}{M}}$.

10. **Assertion :** In isobaric process, *V*-*T* graph is a straight line passing through origin. Slope of this line is directly proportional to mass of the gas. V is taken on y-axis.
Reason : $V = \left(\frac{nR}{P}\right) T$
$\therefore$ slope $\propto n$ or slope $\propto m$

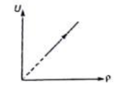
Match the Columns

1. Match the following two columns for 2 moles of a diatomic gas at room temperature T.

Column I	Column II
(a) translational kinetic energy	(p) $2RT$
(b) rotational kinetic energy	(q) $4RT$
(c) potential energy	(r) $3RT$
(d) total internal energy	(s) None

2. In the graph shown, U is the internal energy of gas and ρ the density. Corresponding to given graph match the following two columns.

Column I	Column II
(a) Pressure	(p) is constant
(b) Volume	(q) is increasing
(c) Temperature	(r) is decreasing
(d) Ratio T/V	(s) data insufficient



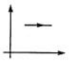
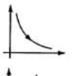
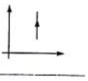
3. At a given temperature T
$v_1 = \sqrt{\frac{x_1 RT}{M}}$ = rms speed of gas molecules, $v_2 = \sqrt{\frac{x_2 RT}{M}}$ = average speed of gas molecules
$v_3 = \sqrt{\frac{x_3 RT}{M}}$ = most probable speed of gas molecules
$v_4 = \sqrt{\frac{x_4 RT}{M}}$ = speed of sound

Column I	Column II
(a) x_1	(p) 1.5
(b) x_2	(q) 2.0
(c) x_3	(r) 3.0
(d) x_4	(s) None of these/data insufficient

4. With increase in temperature, match the following two columns.

Column I	Column II
(a) Density of water	(p) will increase
(b) Fraction of a solid floating in a liquid	(q) will decrease
(c) Apparent weight of a solid immersed in water	(r) will remain unchanged
(d) Time period of pendulum	(s) may increase or decrease

5. Corresponding to isobaric process match the following two columns.

Column I	Column II
(a) *P*-*T* graph	(p) 
(b) *U*-ρ graph	(q) 
(c) *T*-*V* graph	(r) 
(d) *T*-ρ graph	(s) 

Note First physical quantity is along y-axis.

Subjective Questions

1. A cubical vessel contains one gram molecule of nitrogen at a pressure of 2 atm and temperature 300 K. If the molecules are assumed to move with the rms velocity.
(a) Find the number of collisions per unit area per second which the molecules may make with the wall of the vessel.
(b) Further if the vessel now thermally insulated moved with a constant speed v and then suddenly stopped results in a rise of temperature of 2°C, find v.

2. The volume of a diatomic gas ($\gamma = 7/5$) is increased two times in a polytropic process with molar heat capacity $C = R$. How many times will the rate of collision of molecules against the wall of the vessel be reduced as a result of this process?

3. A perfectly conducting vessel of volume $V = 0.4\ m^3$ contains an ideal gas at constant temperature $T = 273$ K. A portion of the gas is let out and the pressure of the gas falls by $\Delta P = 0.24$ atm. (Density of the gas at STP is $\rho = 1.2\ kg/m^3$). Find the mass of the gas which escapes from the vessel.

4. A thin-walled cylinder of mass m, height h and cross-sectional area A is filled with a gas and floats on the surface of water. As a result of leakage from the lower part of the cylinder, the depth of its submergence has increased by Δh. Find the initial pressure P_1 of the gas in the cylinder if the atmospheric pressure is P_0 and the temperature remains constant.

5. A cubical box of side 1 m contains helium gas (atomic weight 4) at a pressure of $100\ N/m^2$. During an observation time of 1 s an atom travelling with the rms speed parallel to one of the edges of the cube , was found to make 500 hits with a particular wall, without any collision with other atoms. Take $R = 8.31$ J/mol-K and $k = 1.38 \times 10^{-23}$ J/K.
(a) Evaluate the temperature of the gas.
(b) Evaluate the average kinetic energy per atom.
(c) Evaluate the total mass of helium gas in the box.

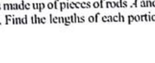
6. An air bubble of volume V is released from the bottom of a lake which has a depth of 11 m. Find the volume of the bubble at the surface of the lake. The water has density $1000\ kg/m^3$. At the release point of the bubble and at the surface, temperature is 4°C. The atmospheric pressure is $1.01 \times 10^5\ N/m^2$.

7. Find the minimum attainable pressure of an ideal gas in the process $T = T_0 + \alpha V^2$ where T_0 and α are positive constants and V is the volume of one mole of gas.

8. A solid body floats in a liquid at a temperature $t = 50°C$ being completely submerged in it. What percentage of the volume of the body is submerged in the liquid after it is cooled to $t_0 = 0°C$, if the coefficient of cubic expansion for the solid is $\gamma_s = 0.3 \times 10^{-5}\ °C^{-1}$ and of the liquid $\gamma_l = 8 \times 10^{-5}\ °C^{-1}$.

9. Two vessels connected by a pipe with a sliding plug contain mercury. In one vessel, the height of mercury column is 39.2 cm and its temperature is 0°C, while in the other, the height of mercury column is 40 cm and its temperature is 100°C. Find the coefficient of cubical expansion for mercury. The volume of the connecting pipe should be neglected.

10. Two steel rods and an aluminium rod of equal length l_0 and equal cross-section are joined rigidly at their ends as shown in the figure below. All the rods are in a state of zero tension at 0°C. Find the length of the system when the temperature is raised to θ. Coefficient of linear expansion of aluminium and steel are α_a and α_s respectively. Young's modulus of aluminium is Y_a and of steel is Y_s.



11. A metal rod A of 25 cm length expands by 0.050 cm when its temperature is raised from 0°C to 100°C. Another rod B of a different metal of length 40 cm expands by 0.040 cm for the same rise in temperature. A third rod C of 50 cm length is made up of pieces of rods A and B placed end to end expands by 0.03 cm on heating from 0° C to 50° C. Find the lengths of each portion of the composite rod.

ANSWERS

Introductory Exercise 17.1

1. (a) −17.8°C (b) − 459.67°F 2. (a) 160°C (b) − 24.6°C 3. 122°F
4. 574.25 5. −40°C 6. Gains, 15.55 s 7. It will first increase and then decrease
8. $(\gamma_2 - \gamma_1)\Delta T$ 9. Cool the system

Introductory Exercise 17.2

1. $m_1 > m_2$ 2. 12 atm 3. 1.5 kg/m^3 4. 8×10^{15} 5. $P_1 > P_2$
6. Straight line passing through origin

Introductory Exercise 17.3

1. Speed is a scalar quantity while velocity is a vector quantity. 2. 6.21×10^{-21} J
3. (a) 1368 m/s, 609 m/s (b) 6.21×10^{-21} J 4. 160 K 5. 76.5% by mass
6. 103 cm^3 7. 6.5×10^3
8. (a) 3.5 mol (b) 43.65 J/mol-K, 72.75 J/mol-K (c) 72.75 J/mol-K, 101.85 J/mol-K
9. (a) 2.81 mol (b) 9.56 kJ (c) 20.8 J/mol-K 10. True

For JEE Main

Subjective Questions

1. 20°C, −15°C, 80°C, 293K, 258K, 353K 2. 86°F, 41°F, − 4°F, 546°R, 501°R, 456°R
3. − 40°F = − 40°C 4. 140.2°F 5. 20°C 6. 546.30 K 7. 400° 8. 177.07°C
9. 53.8°C 10. (a) 0.068 m (b) 88.49 cm 11. − 0.042% 12. 192 N
13. 3.1×10^{-4} per °C 14. 2.82×10^{-26} kg 15. (a) 6.135 mol (b) 1.24 kg/m^3 (c) 19.6 kg/m^3
16. 12.7 atm absolute pressure 17. 900 m^3 18. 5.09 cm 19. 0.089 20. 0.97 MPa
21. 1.13 atm 22. 22.4 L 23. 3.36×10^5 Pa 24. 4.1×10^5 N/m^2 25. 10^{12} rad/s
26. 5 27. 1.54 28. 7.5 E 29. (a) 5 (b) 3 30. (a) 6.225×10^{-21} J (b) 20.8 J
31. $3R, 2R, 1.5$ 32. γ 33. $-\gamma$ 35. 2.15 km/s 36. $v_{rms} = 714$ m/s, $v_{av} = 700$ m/s
37. 5×10^{21}, 8.3×10^{-3} mol 38. 3.1×10^{27} 39. 4.8×10^{-22} J
40. (a) 1.40 v_0 (b) v_0 (c) 0.58 v_0 or $v_0/\sqrt{3}$ 41. −253°C 42. $T_E = 10059$ K, $T_M = 449$ K
43. (a) 6.21×10^{-21} J (b) 3740 J (c) 484 m/s

Objective Questions

1.(b) 2.(b) 3.(c) 4.(a) 5.(d) 6.(c) 7.(a) 8.(b) 9.(c) 10.(a)
11.(a) 12.(a) 13.(c) 14.(b) 15.(a) 16.(b) 17.(a) 18.(b) 19.(c) 20.(d)
21.(d) 22.(c)

More than One Correct Options

23. (b,d) 24. (a,d) 25. (a,c) 26. (a,c,d) 27. (a,b) 28. (a,b,c,d) 29. (a,d)

For JEE Advanced

Assertion and Reason

1. (d) 2. (b) 3. (b) 4. (d) 5. (d) 6. (b) 7. (c) 8. (b) 9. (b) 10. [illegible]

Match the Columns

1. (a) → r (b) → p (c) → s (d) → s
2. (a) → p (b) → q (c) → q (d) → p
3. (a) → r (b) → s (c) → q (d) → s
4. (a) → s (b) → s (c) → s (d) → p
5. (a) → q (b) → r (c) → p (d) → r

Subjective Questions

1. (a) 4.2×10^{27} (b) 54.5 m/s 2. $(2)^{4/3}$ times 3. 115.2 g
4. $P_1 = \left(P_0 + \frac{mg}{A}\right)\left(1 - \frac{\Delta h}{h}\right)$ 5. (a) 160 K (b) 3.31×10^{-21} J (c) 3.0×10^{-4} kg
6. 2.089 V 7. $2R\sqrt{\alpha T_0}$ 8. 99.99% 9. 2.0×10^{-4} per °C
10. $l_0\left[1 + \left(\frac{\alpha_b Y_b + 2\alpha_s Y_s}{Y_b + 2Y_s}\right)\theta\right]$ 11. 10 cm, 40 cm

18

LAWS OF THERMODYNAMICS

Chapter Contents

18.1 The First Law of Thermodynamics

The first law of thermodynamics, is an extension of the principle of conservation of energy. To state energy relationships precisely, we need the concept of a **thermodynamic system.** A thermodynamic system is a system that can interact (and exchange energy) with its surroundings, or environment, in at least one ways, one of which is heat transfer. In this chapter the thermodynamic system will be an ideal gas contained in a vessel in most of the cases. A process in which there are changes in the state of a thermodynamic system (P, V and T in case of a gas) is called a **thermodynamic process.**

We now come to the first law.

Let a system changes from an initial equilibrium state i to a final equilibrium state f in a definite way, the heat absorbed by the system being Q and the work done by the system being W. Then we compute the $Q - W$. While Q and W both depend on the thermodynamic path taken between two equilibrium states, their difference $Q - W$ does not. We do this over and over again, using different paths each time. We find that in every case the quantity $Q - W$ is the same. The students may recall from mechanics that when an object is moved from an initial point i to a final point f in a gravitational field in the absence of friction, the work done depends only on the positions of the two points and not at all on the path through which the body is moved.

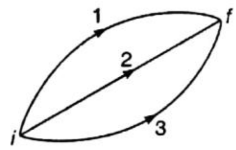


Fig. 18.1

From this we concluded that there is a function of the space coordinates of the body whose final value minus its initial value equals the work done in displacing the body. We called it the potential energy function. In thermodynamics there is a function of the thermodynamic coordinates (P, V and T) whose final value minus its initial value equals the change $Q - W$ in the process. We call this function the **internal energy** function. We have,

$$\Delta U = U_f - U_i = Q - W \qquad \ldots(i)$$

This equation is known as the first law of thermodynamics.

We describe the energy relations in any thermodynamic process in terms of the quantity of heat Q added to the system and the work W done by the system. Both Q and W may be positive, negative or zero. A positive value of Q represents heat flow into the system, negative Q represents heat flow out of the system. A positive value of W represents work done by the system against its surroundings, such as work done by an expanding gas. Negative W represents work done on the system by its surroundings such as work done during compression of a gas.

Table 18.1 Thermodynamic sign conventions for heat and work.

Process	Convention
Heat added to the system	$Q > 0$
Heat removed from the system	$Q < 0$
Work done by the system	$W > 0$
Work done on the system	$W < 0$

Eq. (i) can be written as

$$Q_1 - W_1 = Q_2 - W_2 = \ldots\ldots\ldots$$

or

$$\Delta U_1 = \Delta U_2 = \ldots\ldots\ldots$$

that is the change in the internal energy of the system between two points is path independent. It depends on thermodynamic co-ordinates of the two points. For example, in case of an ideal gas it depends only on the initial and final temperatures.

Often the first law must be used in its differential form, which is

$$dU = dQ - dW \qquad \text{...(ii)}$$

This can also be written as

$$dQ = dU + dW \qquad \text{...(iii)}$$

or

$$Q = \Delta U + W \qquad \text{...(iv)}$$

The first law can be expressed in other words as under.

Suppose a heat Q is given to a system. This heat is partly used by the system in doing work against its surroundings and partly its internal energy gets increased and from energy conservation principle $Q = \Delta U + W$. An another analogous example is from our daily life. Consider a person X. Suppose his monthly income is Rs. 50,000 (Q). He spends Rs. 30,000 (W) as his monthly expenditure. Then obviously the remaining Rs. 20,000 goes to his savings (ΔU). In some month it is also possible that he spends more than his income. In that case he will withdraw it from his bank or his savings will get reduced ($\Delta U < 0$). In the similar manner other combinations can be made.

Sample Example 18.1 *When a system goes from state A to state B, it is supplied with 400 J of heat and it does 100 J of work.*

(a) For this transition, what is the system's change in internal energy?

(b) If the system moves from B to A, what is the change in internal energy?

(c) If in moving from A to B along a different path in which $W'_{AB} = 400\ J$ of work is done on the system, how much heat does it absorb?

Solution (a) From the first law,

$$\Delta U_{AB} = Q_{AB} - W_{AB} = (400 - 100)\ \text{J}$$
$$= 300\ \text{J}$$

(b) Consider a closed path that passes through the state A and B. **Internal energy is a state function so ΔU is zero for a closed path.**

Thus,

$$\Delta U = \Delta U_{AB} + \Delta U_{BA} = 0$$

or

$$\Delta U_{BA} = -\ \Delta U_{AB}$$
$$= -\ 300\ \text{J}$$

(c) The change in internal energy is the same for any path, so

$$\Delta U_{AB} = \Delta U'_{AB} = Q'_{AB} - W'_{AB}$$
$$300\ \text{J} = Q'_{AB} - (-\ 400\ \text{J})$$

and the heat exchanged is

$$Q'_{AB} = -\ 100\ \text{J}$$

The negative sign indicates that the system loses heat in this transition.

Exercise *The quantities in the following table represent four different paths for the same initial and final state. Find a, b, c, d, e, f and g.*

Table 18.2

Q (J)	W (J)	ΔU (J)
-80	-120	d
90	c	e
a	40	f
b	-40	g

Ans. $a = 80$ J $b = 0$ $c = 50$ J $d = 40$ J $e = 40$ J $f = 40$ J $g = 40$ J

18.2 Further Explanation of The Three Terms Used in First Law

First law of thermodynamics basically revolves round the three terms Q, ΔU and W. If you substitute these three terms correctly with proper signs in the equation $Q = \Delta U + W$, then you are able to solve most of the problems of first law. Let us take each term one by one. Here, we are taking the system an ideal gas.

(i) Heat transfer (Q or dQ)

For heat transfer, apply

$$Q = nC\Delta T$$

or in differential form

$$dQ = nCdT$$

where C is the molar heat capacity of the gas and n is the number of moles of the gas. Always take,

$$\Delta T = T_f - T_i$$

where T_f is the final temperature and T_i the initial temperature of the gas. Further, we have discussed in chapter 20 , that molar heat capacity of an ideal gas in the process PV^x = constant is,

$$C = \frac{R}{\gamma - 1} + \frac{R}{1-x} = C_V + \frac{R}{1-x}$$

$C = C_V = \frac{R}{\gamma - 1}$ in isochoric process and

$C = C_P = C_V + R$ in isobaric process

Mostly C_P and C_V are used.

(ii) Change in internal energy (U or dU)

For change in internal energy of the gas apply,

$$\Delta U = nC_V \Delta T$$

or in differential form,

$$dU = nC_V dT$$

Students are often confused that the result $\Delta U = nC_V \Delta T$ can be applied only in case of an isochoric process (as C_V is here used). However it is not so. It can be applied in any process, whether it is isobaric, isothermal adiabatic or else.

(iii) Work done (W or dW)

This is the most important of the three.

Work done during volume changes

A gas in a cylinder with a movable piston is a simple example of a thermodynamic system.

Figure shows a gas confined to a cylinder that has a movable piston at one end. If the gas expands against the piston, it exerts a force through a distance and does work on the piston. If the piston compresses the gas as it is moved inward, work is also done—in this case on the gas. The work associated with such volume changes can be determined as follows.

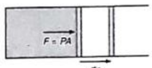


Fig. 18.2

Let the gas pressure on the piston face be P. Then the force on the piston due to gas is PA, where A is the area of the face.

When the piston is pushed outward an infinitesimal distance dx, the work done by the gas is

$$dW = F \cdot dx = PA\,dx$$

which, since the change in volume of the gas is $dV = Adx$, becomes

$$dW = PdV$$

For a finite change in volume from V_i to V_f, this equation is then integrated between V_i to V_f to find the net work

$$W = \int dW = \int_{V_i}^{V_f} PdV$$

Now, there are two methods of finding work done by a gas.

Method 1. This is used when P-V equation is known to us. Suppose P as a function of V is known to us.

$$P = f(V)$$

then work done can be found by,

$$W = \int_{V_i}^{V_f} f(V)\,dV$$

Method 2. The work done by a gas is also equal to the area under P-V graph. Following different cases are possible.

Case 1. When volume is constant

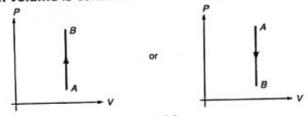


Fig. 18.3

V = constant

$\therefore$ $W_{AB} = 0$

Case 2. When volume is increasing

V is increasing

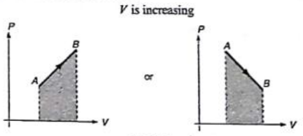


Fig. 18.4

$\therefore$ $W_{AB} > 0$

W_{AB} = Shaded area

Case 3. When volume is decreasing

V is decreasing

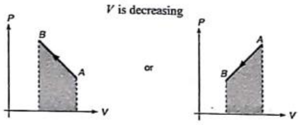
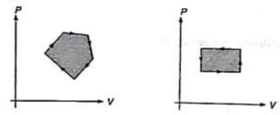


Fig. 18.5

$\therefore$ $W_{AB} < 0$

W_{AB} = – Shaded area

Case 4. Cyclic process



(a) (b)

Fig. 18.6

$W_{\text{clockwise cycle}}$ = + Shaded area [in figure (a)]

$W_{\text{anticlockwise cycle}}$ = – Shaded area [in figure (b)]

Case 5. Incomplete cycle

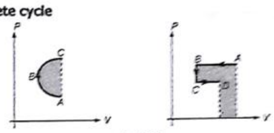


Fig. 18.7

W_{ABC} = + Shaded area

W_{ABCD} = – Shaded area

Sample Example 18.2 *A certain amount of an ideal gas passes from state A to B first by means of process 1, then by means of process 2. In which of the process is the amount of heat absorbed by the gas greater?*

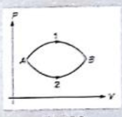


Fig. 18.8

Solution

$$Q_1 = W_1 + \Delta U_1$$

and

$$Q_2 = W_2 + \Delta U_2$$

U is a state function. Hence, ΔU depends only on the initial and final positions. Therefore,

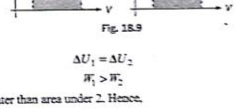


Fig. 18.9

$$\Delta U_1 = \Delta U_2$$

But

$$W_1 > W_2$$

as the area under 1 is greater than area under 2. Hence,

$$Q_1 > Q_2$$

Sample Example 18.3 *Find the ratio of $\frac{\Delta Q}{\Delta U}$ and $\frac{\Delta Q}{\Delta W}$ in an isobaric process. The ratio of molar heat capacities $\frac{C_P}{C_V} = \gamma$.*

Solution In an isobaric process P = constant. Therefore, $C = C_P$.

Now,
$$\frac{\Delta Q}{\Delta U} = \frac{nC_P \Delta T}{nC_V \Delta T} = \frac{C_P}{C_V} = \gamma$$

and
$$\frac{\Delta Q}{\Delta W} = \frac{\Delta Q}{\Delta Q - \Delta U}$$
$$= \frac{nC_P \Delta T}{nC_P \Delta T - nC_V \Delta T}$$
$$= \frac{C_P}{C_P - C_V} = \frac{C_P / C_V}{C_P / C_V - 1}$$
$$= \frac{\gamma}{\gamma - 1}$$

Sample Example 18.4 *Boiling water: Suppose* 1.0 g *of water vaporizes isobarically at atmospheric pressure* $(1.01 \times 10^5 \ Pa)$. *Its volume in the liquid state is* $V_i = V_{liquid} = 1.0 \ cm^3$ *and its volume in vapour state is* $V_f = V_{vapour} = 1671 \ cm^3$. *Find the work done in the expansion and the change in internal energy of the system. Ignore any mixing of the steam and the surrounding air. Take latent heat of vaporization* $L_v = 2.26 \times 10^6 \ J/kg$.

Solution Because the expansion takes place at constant pressure, the work done is

$$W = \int_{V_i}^{V_f} P_0 dV = P_0 \int_{V_i}^{V_f} dV = P_0 (V_f - V_i)$$
$$= (1.01 \times 10^5)(1671 \times 10^{-6} - 1.0 \times 10^{-6})$$
$$= 169 \text{ J}$$
$$Q = mL_v = (1.0 \times 10^{-3})(2.26 \times 10^6) = 2260 \text{ J}$$

Hence, from the first law, the change in internal energy

$$\Delta U = Q - W = 2260 - 169 = 2091 \text{ J}$$

Note *The positive value of ΔU indicates that the internal energy of the system increases. We see that most $\left(\frac{2091 \text{ J}}{2260 \text{ J}} = 93\%\right)$ of the energy transferred to the liquid goes into increasing the internal energy of the system only $\frac{169 \text{ J}}{2260 \text{ J}} = 7\%$ leaves the system by work done by the steam on the surrounding atmosphere.*

Introdcutory Exercise 18.1

1. A gas in a cylinder is held at a constant pressure of 1.7×10^5 Pa and is cooled and compressed from $1.20 \ m^3$ to $0.8 \ m^3$. The internal energy of the gas decreases by 1.1×10^5 J.
   (a) Find the work done by the gas.
   (b) Find the magnitude of the heat flow into or out of the gas, and state the direction of heat flow.
   (c) Does it matter whether or not the gas is ideal?

2. A thermodynamic system undergoes a cyclic process as shown in figure.
   (a) over one complete cycle, does the system do positive or negative work.
   (b) over one complete cycle, does heat flow into or out of the system.
   (c) In each of the loops 1 and 2, does heat flow into or out of the system.

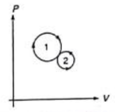


Fig. 18.10

3. A well insulated box contains a partition dividing the box into two equal volumes as shown in figure. Initially the left hand side contains an ideal monoatomic gas and the other half is a vacuum. The partition is suddenly removed so that the gas now is contained throughout the entire box.
   (a) Does the temperature of the gas change?
   (b) Does the internal energy of the system change?
   (c) Does the gas work?

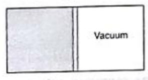


Fig. 18.11

4. How many moles of helium at temperature 300 K and 1.00 atm pressure are needed to make the internal energy of the gas 100 J?
5. A 1.0 kg bar of copper is heated at atmospheric pressure $(1.01 \times 10^5 \ N/m^2)$. If its temperature increases from 20°C to 50°C, calculate the change in its internal energy. $\alpha = 7.0 \times 10^{-6}$ per°C, $\rho = 8.92 \times 10^3 \ kg/m^3$ and $c = 387 \ J/kg\text{-}°C$.

18.3 Different Thermodynamic Processes

Among the thermodynamic processes we will consider are the following :
(i) An isothermal process during which the system's temperature remains constant.
(ii) An adiabatic process during which no heat is transferred to or from the system.
(iii) An isobaric process during which the pressure of the system is constant.
(iv) An isochoric process during which the system's volume does not change.

There are, of course, many other processes that do not fit into any of these four categories. We will mostly consider an ideal gas.

(i) Isothermal process

An isothermal process is a constant temperature process. In an isothermal process :
(i) T = constant or $\Delta T = 0$
(ii) $P \propto \frac{1}{V}$ or PV = constant *i.e.*, P-V graph is a rectangular hyperbola with $P_i V_i = P_f V_f$.
(iii) As T = constant, hence U = constant for an ideal gas, because U is a function of T only.
(iv) $\Delta U = 0 \ \therefore \ Q = W$

Work done in isothermal process :

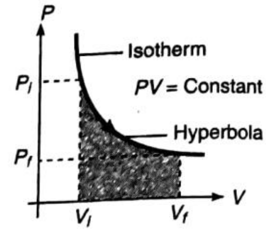
$$W = \int_{V_i}^{V_f} P\, dV$$

$$= \int_{V_i}^{V_f} \left(\frac{nRT}{V}\right) dV \quad \left(\text{as } P = \frac{nRT}{V}\right)$$

$$= nRT \int_{V_i}^{V_f} \frac{dV}{V} \quad (\text{as } T = \text{constant})$$

$$= nRT \ln\left(\frac{V_f}{V_i}\right)$$

$$= nRT \ln\left(\frac{P_i}{P_f}\right) \quad (\text{as } P_i V_i = P_f V_f)$$



Isothermal expansion of an ideal gas

Fig. 18.12

Thus in an isothermal process :

$$\Delta U = 0 \quad \text{and} \quad Q = W = nRT \ln\left(\frac{V_f}{V_i}\right) = nRT \ln\left(\frac{P_i}{P_f}\right)$$

Note *For a process to be isothermal, any heat flow into or out of the system must occur* ***slowly*** *enough, so that thermal equilibrium is maintained.*

(ii) Adiabatic Process

An adiabatic process is defined as one with no heat transfer into or out of a system : $Q = 0$. We can prevent heat flow either by surrounding the system with thermally insulating material or by carrying out the process so **quickly** that there is not enough time for appreciable heat flow. From the first law we find that for every adiabatic process,

$$W = -\Delta U \quad (\text{as } Q = 0)$$

$$= -nC_V \Delta T$$

$$= -nC_V (T_f - T_i)$$

$$= n\left(\frac{R}{\gamma - 1}\right)(T_i - T_f) \quad \left(\text{as } C_V = \frac{R}{\gamma - 1}\right)$$

$$= \frac{P_i V_i - P_f V_f}{\gamma - 1} \quad (\text{as } nRT = PV)$$

Thus, in an adiabatic process,

$$Q = 0 \quad \text{and} \quad W = -\Delta U = \frac{P_i V_i - P_f V_f}{\gamma - 1}$$

Note *In adiabatic process $W = -\Delta U$. Therefore, if the work done by a gas is positive (i.e., volume of the gas is increasing), then ΔU will be negative. That is U and hence T will decrease. The cooling of air can be experienced practically during bursting of a tyre. The process is so fast that it can be assumed as adiabatic. As the gas expands. Therefore, it cools. On the other hand the compression stroke in an internal combustion engine is an approximately adiabatic process. The temperature rises as the air fuel mixture in the cylinder is compressed.*

P-V relation : In adiabatic process $dQ = 0$

and $$dW = -dU$$

$\therefore$ $$PdV = -C_V dT \qquad \text{(for } n = 1)$$

$\therefore$ $$dT = -\frac{PdV}{C_V} \qquad \ldots(i)$$

Also, for 1 mole of an ideal gas,

$$d(PV) = d(RT)$$

or $$PdV + VdP = RdT$$

or $$dT = \frac{PdV + VdP}{R} \qquad \ldots(ii)$$

From Eqs. (i) and (ii)

$$C_V VdP + (C_V + R)\,PdV = 0$$

or $$C_V VdP + C_P PdV = 0$$

Dividing this equation by PV, we are left with

$$C_V \frac{dP}{P} + C_P \frac{dV}{V} = 0$$

or $$\frac{dP}{P} + \gamma \frac{dV}{V} = 0$$

or $$\int \frac{dP}{P} + \gamma \int \frac{dV}{V} = 0$$

or $$\ln(P) + \gamma \ln(V) = \text{constant}$$

We can write this in the form

$$PV^{\gamma} = \text{constant}$$

This equation is the condition that must be obeyed by an ideal gas in an adiabatic process. For example, if an ideal gas makes an adiabatic transition from a state with pressure and volume P_i and V_i to a state with P_f and V_f, then

$$P_i V_i^{\gamma} = P_f V_f^{\gamma}$$

The equation $PV^{\gamma} = \text{constant}$ can be written in terms of other pairs of thermodynamic variables by combining it with the ideal gas law ($PV = nRT$). In doing so we will find that,

$$TV^{\gamma - 1} = \text{constant} \quad \text{and} \quad T^{\gamma} P^{1-\gamma} = \text{constant}$$

Slope of *P-V* graph

In an adiabatic process ($PV^{\gamma} = \text{constant}$), the slope of PV diagram at any point is

$$\frac{dP}{dV} = \frac{d}{dV}\left(\frac{\text{constant}}{V^{\gamma}}\right) = -\gamma\left(\frac{P}{V}\right)$$

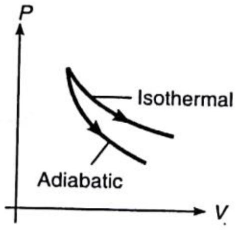
Thus, $$(\text{Slope})_{\text{adiabatic}} = -\gamma\left(\frac{P}{V}\right)$$

Similarly in an isothermal process (PV = constant), the slope of PV diagram at any point is,

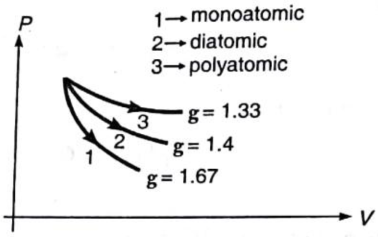
$$\frac{dP}{dV} = \frac{d}{dV}\left(\frac{\text{constant}}{V}\right) = -\frac{P}{V}$$

or $$(\text{Slope})_{\text{isothermal}} = -\frac{P}{V}$$

Because $\gamma > 1$, the isothermal curve is not as steep as that for the adiabatic expansion.



Adiabatic and isothermal expansion of an ideal gas



Adiabatic expansion of mono, dia and polyatomic gases.

Fig. 18.13

(iii) Isobaric process

An isobaric process is a constant pressure process. In an isobaric process :

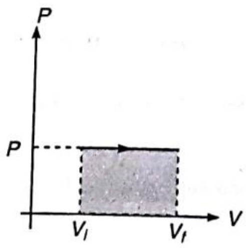
(i) P = constant or $\Delta P = 0$

(ii) $V \propto T$ or $\frac{V}{T}$ = constant

i.e., V-T graph is a straight line passing through origin.

(iii) $$Q = nC_P\Delta T, \quad \Delta U = nC_V\Delta T \quad \text{and therefore}$$
$$W = Q - \Delta U = n(C_P - C_V)\Delta T = nR\Delta T$$
$$= nR(T_f - T_i)$$
$$= P(V_f - V_i) \qquad (\text{as } nRT = PV)$$

Thus, in an isobaric process



Isobaric expansion of a gas

Fig. 18.14

$$Q = nC_P\Delta T, \quad \Delta U = nC_V\Delta T \quad \text{and} \quad W = P(V_f - V_i)$$

(iv) Isochoric process

An isochoric process is a constant volume process. In an isochoric process :

(i) $V =$ constant or $\Delta V = 0$

(ii) $P \propto T$ or $\dfrac{P}{T} =$ constant

i.e., *P-T* graph is a straight line passing through origin.

(iii)

As $V =$ constant, hence $W = 0$ and from first law of thermodynamics

$$Q = \Delta U = nC_V \Delta T$$

Thus in an isochoric process :

$$W = 0 \quad \text{and} \quad Q = \Delta U = nC_V \Delta T$$

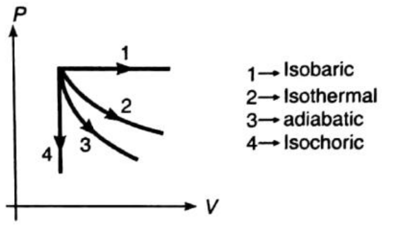


Fig. 18.15

P-V diagram of different processes is shown in one graph.

Table 18.3 shows Q, ΔU and W for different processes discussed above.

Table 18.3

Name of the process	Q	ΔU	W
Isothermal	$Q = W$	0	$nRT \ln\left(\dfrac{V_f}{V_i}\right) = nRT \ln\left(\dfrac{P_i}{P_f}\right)$
Adiabatic	0	$nC_V \Delta T$	$\dfrac{P_i V_i - P_f V_f}{\gamma - 1} = -\Delta U$
Isobaric	$nC_P\ \Delta T$	$nC_V \Delta T$	$P(V_f - V_i)$
Isochoric	$Q = \Delta U = nC_V \Delta T$	$nC_V \Delta T$	0

Sample Example 18.5 *What is the heat input needed to raise the temperature of 2 moles of helium gas from 0°C to 100°C*

(a) at constant volume,

(b) at constant pressure?

(c) What is the work done by the gas in part (b)?

Give your answer in terms of R.

Solution Helium is monoatomic gas. Therefore,

$$C_V = \frac{3R}{2} \quad \text{and} \quad C_P = \frac{5R}{2}$$

(a) At constant volume,

$$\begin{aligned} Q &= nC_V \Delta T \\ &= (2)\left(\frac{3R}{2}\right)(100) \\ &= 300R \end{aligned}$$

(b) At constant pressure,

$$\begin{aligned} Q &= nC_P \Delta T \\ &= (2)\left(\frac{5R}{2}\right)(100) \\ &= 500R \end{aligned}$$

(c) At constant pressure,

$$\begin{aligned} W &= Q - \Delta U \\ &= nC_P \Delta T - nC_V \Delta T \\ &= nR\Delta T = (2)(R)(100) \\ &= 200R \end{aligned}$$

Sample Example 18.6 *An ideal monoatomic gas at 300 K expands adiabatically to twice its volume. What is the final temperature?*

Solution For an ideal monoatomic gas,

$$\gamma = \frac{5}{3}$$

In an adiabatic process,

$$TV^{\gamma-1} = \text{constant}$$

$\therefore$

$$T_f V_f^{\gamma-1} = T_i V_i^{\gamma-1}$$

or

$$T_f = T_i \left(\frac{V_i}{V_f}\right)^{\gamma-1}$$

$$= (300)\left(\frac{1}{2}\right)^{\frac{5}{3}-1} = 189\ \text{K}$$

Sample Example 18.7 *An ideal gas expands isothermally along AB and does 700 J of work.*

(a) How much heat does the gas exchange along AB?

(b) The gas then expands adiabatically along BC and does 400 J of work. When the gas returns to A along CA, it exhausts 100 J of heat to its surroundings. How much work is done on the gas along this path?

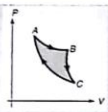


Fig. 18.16

Solution (a) AB is an isothermal process. Hence,

$$\Delta U_{AB} = 0$$

and

$$Q_{AB} = W_{AB} = 700\ \text{J}$$

(b) BC is an adiabatic process. Hence,

$$Q_{BC} = 0$$

$$W_{BC} = 400\ \text{J} \qquad \text{(given)}$$

$\therefore$

$$\Delta U_{BC} = -W_{BC} = -400\ \text{J}$$

ABC is a cyclic process and internal energy is a state function. Therefore,

$$(\Delta U)_{\text{whole cycle}} = 0 = \Delta U_{AB} + \Delta U_{BC} + \Delta U_{CA}$$

and from first law of thermodynamics,

$$Q_{AB} + Q_{BC} + Q_{CA} = W_{AB} + W_{BC} + W_{CA}$$

Substituting the values,

$$700 + 0 - 100 = 700 + 400 + \Delta W_{CA}$$

$\therefore$

$$\Delta W_{CA} = -500\ \text{J}$$

Negative sign implies that work is done on the gas.

Table 18.4 shows different values in different processes.

Table 18.4

Process	Q (J)	W (J)	ΔU (J)
AB	700	700	0
BC	0	400	−400
CA	−100	−500	400
For complete cycle	600	600	0

Note *Total work done is 600 J, which implies that area of the closed curve is also 600 J.*

Introductory Exercise 18.2

1. One mole of an ideal monoatomic gas is initially at 300 K. Find the final temperature if 200 J of heat are added as follows :
   (a) at constant volume
   (b) at constant pressure
2. Prove that work done by an ideal gas in an adiabatic process is $W = \dfrac{P_i V_i - P_f V_f}{\gamma - 1}$, using the integraion $\int PdV$.

3. When a gas expands along AB it does 500 J of work and absorbs 250 J of heat. When the gas expands along AC, it does 700 J of work and absorbs 300 J of heat.
(a) How much heat does the gas exchange along BC?
(b) When the gas makes the transition from C to A along CDA, 800 J of work are done on it from C to D. How much heat does it exchange along CDA?

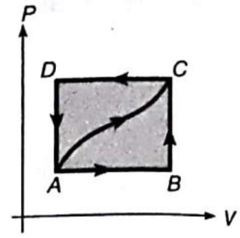


Fig. 18.17

4. One mole of an ideal monoatomic gas occupies a volume of 1.0×10^{-2} m^3 at a pressure of 2.0×10^5 N/m^2.
(a) What is the temperature of the gas?
(b) The gas undergoes a quasi-static adiabatic compression until its volume is decreased to 5.0×10^{-3} m^3. What is the new gas temperature?
(c) How much work is done on the gas during the compression?
(d) What is the change in the internal energy of the gas?

5. A bullet of mass 10 g travelling horizontally at 200 m/s strikes and embeds in a pendulum bob of mass 2.0 kg.
(a) How much mechanical energy is dissipated in the collision?
(b) Assuming that C_v for the bob plus bullet is $3R$, calculate the temperature increase of the system due to the collision. Take the molecular mass of the system to be 200 g/mol.

6. An ideal gas is carried through a thermodynamic cycle consisting of two isobaric and two isothermal processes, as shown in figure. Show that the net work done in the entire cycle is given by the equation.
$$W_{net} = P_1 (V_2 - V_1)\ln \frac{P_2}{P_1}$$

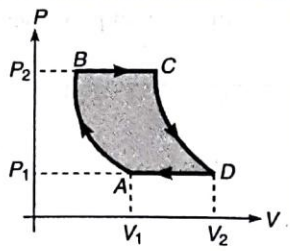


Fig. 18.18

7. Consider the cyclic process depicted in figure. If Q is negative for the process BC, and if ΔU is negative for the process CA, what are the signs of Q, W and ΔU that are associated with each process?

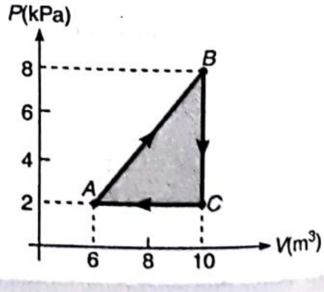


Fig. 18.19

8. An ideal gas is enclosed in a cylinder with a movable piston on top. The piston has mass of 8000 g and an area of 5.00 cm^2 and is free to slide up and down, keeping the pressure of the gas constant. How much work is done as the temperature of 0.200 mol of the gas is raised from 200°C to 300°C?

9. A sample of ideal gas is expanded to twice its original volume of 1.00 m^3 in a quasi-static process for which $P = \alpha V^2$, with $\alpha = 5.00$ atm/m^6, as shown in figure. How much work is done by the expanding gas?

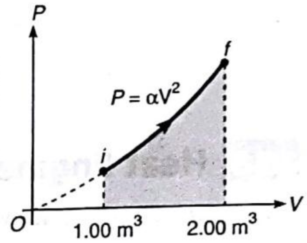


Fig. 18.20

18.4 Efficiency of a Cycle

In a cyclic process.

$$\Delta U = 0 \quad \text{and} \quad Q_{\text{net}} = W_{\text{net}}$$

(from first law of thermodynamics)

First we see what is the meaning of efficiency of a cycle. Suppose 100 J of heat is supplied to a system (in our case it is an ideal gas) and the system does 60 J of work. Then efficiency of the cycle is 60%. Thus, efficiency (η) of a cycle can be defined as

$$\eta = \left(\frac{\begin{array}{c}\text{Work done by the working substance}\\\text{(an ideal gas in our case) during a cycle}\end{array}}{\text{Heat supplied to the gas during the cycle}}\right)\times 100$$

$$= \frac{W_{\text{Total}}}{|Q_{+\text{ve}}|}\times 100$$

$$= \frac{|Q_{+\text{ve}}| - |Q_{-\text{ve}}|}{|Q_{+\text{ve}}|}\times 100$$

$$= \left\{1 - \left|\frac{Q_{-\text{ve}}}{Q_{+\text{ve}}}\right|\right\}\times 100$$

Thus,
$$\eta = \frac{W_{\text{Total}}}{|Q_{+\text{ve}}|}\times 100 = \left\{1 - \left|\frac{Q_{-\text{ve}}}{Q_{+\text{ve}}}\right|\right\}\times 100$$

Note *(i) There cann't be a cycle whose efficiency is 100%. Hence, η is always less than 100%.*

Thus, $$W_{\text{Total}} \neq Q_{+\text{ve}}$$

(ii) It is just like a shopkeeper. He takes some money from you. (Suppose he takes Rs. 100/- from you). In lieu of this he provides services to you (suppose he provides services of worth Rs. 80/-). Then the efficiency of the shopkeeper is 80%. There cann't be a shopkeeper whose efficiency is 100%. Otherwise what will he save?

Sample Example 18.8 *In Sample Example 18.7 find the efficiency of the given cycle.*

Solution From table 18.4 we can see that Q_{+ve} during the cycle is 700 J, while the total work done in the cycle is 600 J.

$\therefore$

$$\eta = \frac{W_{\text{Total}}}{|Q_{+ve}|} \times 100 = \left(\frac{600}{700}\right) \times 100$$

$$= 85.71\%$$

18.5 Heat Engines

A heat engine is a device which converts heat energy into mechanical energy.

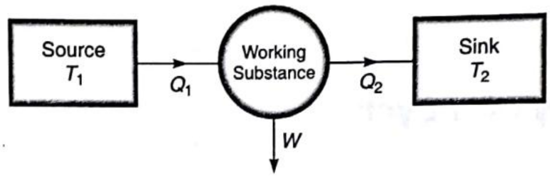


Fig. 18.21

Any heat engine has some working substance (normally a gas in a chamber). In every heat engine the working substance absorbs some heat (Q_1) from a source (at temperature T_1), converts a part of it into work (W) and the rest (Q_2) is rejected to the sink (at temperature T_2).

From conservation of energy.

$$Q_1 = W + Q_2$$

As discussed in article 18.4 **thermal efficiency** of a heat engine is defined as the ratio of net work done per cycle by the engine to the total amount of heat absorbed per cycle by the working substance from the source. It is denoted by η. Thus,

$$\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

As some heat is always rejected to the sink $Q_2 \neq 0$. Therefore, η is always less than 1 *i.e.*, thermal efficiency of a heat engine is always less than 100%.

Types of Heat Engines

In practice, heat engine are of two types :

(a) External combustion engine

In which heat is produced by burning the fuel in a chamber outside the main body (working substance) of the engine. Steam engine is an external combustion engine. The thermal efficiency of a steam engine varies from 10 to 20%.

(b) Internal combustion engine

In which heat is produced by burning the fuel inside the main body of the engine. Petrol engine and diesel engines are internal combustion engine.

Carnot Engine

Carnot cycle consists of the following four stages :

(i) Isothermal expansion (process AB)g

(ii) Adiabatic expansion (process BC)

(iii) Isothermal compression (process CD) and

(iv) Adiabatic compression (process DA)

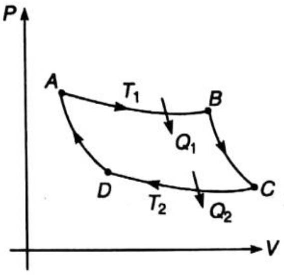


Fig. 18.22

The P-V diagram of the cycle is shown in the figure.

In process AB heat Q_1 is taken by the working substance at constant temperature T_1 and in process CD heat Q_2 is liberated from the working substance at constant temperature T_2. The net work done is area of graph $ABCD$. After doing the calculations for different processes we can show that :

$$\frac{Q_2}{Q_1} = \frac{T_2}{T_1}$$

Therefore, efficiency of the cycle is,

$$\eta = 1 - \frac{T_2}{T_1}$$

Note *That efficiency of Carnot engine is maximum (not 100%) for given temperatures T_1 and T_2. But still Carnot engine is not a practical engine because many ideal situations have been assumed while designing this engine which can practically not be obtained.*

Otto or Petrol Engine

This engine was made by Otto. This is also a four stroke engine. Four stroke means, in a cycle there are four processes. The working substance in it is 2% petrol and 98% air. The four processes are, charging stroke, compression stroke, working stroke and exhaust stroke. The efficiency of a petrol engine is about 52%.

Diesel Engine

The Diesel engine was made by a German Engineer Diesel. The efficiency of Diesel engine is about 64%.

18.6 Refrigerator

Refrigerator is an apparatus which takes heat from a cold body, work is done on it and the work done together with the heat absorbed is rejected to the source.

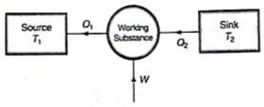


Fig. 18.23

An ideal refrigerator can be regarded as Carnot's ideal heat engine working in the reverse direction.

Coefficient of Performance

(β) of a refrigerator is defined as the ratio of quantity of heat removed per cycle (Q_2) to the work done on the working substance per cycle to remove this heat. Thus,

$$\beta = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2}$$

By doing the calculations we can show that,

$$\beta = \frac{T_2}{T_1 - T_2} = \frac{1-\eta}{\eta}$$

Here, η is the efficiency of Carnot's cycle.

Sample Example 18.9 *The P-V diagram of 0.2 mol of a diatomic ideal gas is shown in figure. Process BC is adiabatic. The value of γ for this gas is 1.4.*

(a) Find the pressure and volume at points A, B and C.

(b) Calculate ΔQ, ΔW and ΔU for each of the three processes.

(c) Find the thermal efficiency of the cycle.

Take $1\ atm = 1.0 \times 10^5\ N/m^2$.

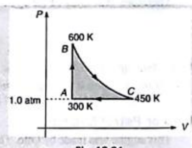


Fig. 18.24

Solution (a) $P_A = P_C = 1\ \text{atm} = 1.01 \times 10^5\ \text{N/m}^2$

Process AB is an isochoric process.

$$\therefore \quad P \propto T \quad \text{or} \quad \frac{P_B}{P_A} = \frac{T_B}{T_A}$$

$$\therefore \quad P_B = \left(\frac{T_B}{T_A}\right) P_A = \left(\frac{600}{300}\right)(1\ \text{atm}) = 2\ \text{atm}$$

$$= 2.02 \times 10^5\ \text{N/m}^2$$

From ideal gas equation $\quad V = \frac{nRT}{P}$

$$\therefore \quad V_A = V_B = \frac{nRT_A}{P_A}$$

$$= \frac{(0.2)(8.31)(300)}{(1.01 \times 10^5)} = 5.0 \times 10^{-3}\ m^3$$

$$= 5\ \text{litre}$$

and $\quad V_C = \frac{nRT_C}{P_C} = \frac{(0.2)(8.31)(455)}{(1.01 \times 10^5)}$

$$= 7.6 \times 10^{-3}\ m^3$$

$$= 7.6\ \text{litre}$$

State	P	V
A	1 atm	5 lt
B	2 atm	5 lt
C	1 atm	7.6 lt

(b) **Process AB** is an isochoric process. Hence,

$$\Delta W_{AB} = 0$$

$$\Delta Q_{AB} = \Delta U_{AB} = nC_V \Delta T = n\left(\frac{5}{2}R\right)(T_B - T_A)$$

$$= (0.2)\left(\frac{5}{2}\right)(8.31)(600 - 300) = 1246\ J$$

Process BC is an adiabatic process. Hence,

$$\Delta Q_{BC} = 0$$

$$\therefore \quad \Delta W_{BC} = -\Delta U_{BC}$$

$$\Delta U_{BC} = nC_V \Delta T = nC_V (T_C - T_B)$$

$$= (0.2)\left(\frac{5}{2}R\right)(455 - 600)$$

$$= (0.2)\left(\frac{5}{2}\right)(8.31)(-145)\ J$$

$$= -602\ J$$

$$\therefore \quad \Delta W_{BC} = -\Delta U_{BC} = 602\ J$$

Process CA is an isobaric process. Hence,

$$\Delta Q_{CA} = nC_P \Delta T = n\left(\frac{7}{2}R\right)(T_A - T_C)$$

$$= (0.2)\left(\frac{7}{2}\right)(8.31)(300 - 455)$$

$$= -902\ J$$

$$\Delta U_{CA} = nC_V \Delta T$$

$$= \frac{\Delta Q_{CA}}{\gamma} \qquad \left(\text{as } \gamma = \frac{C_P}{C_V}\right)$$

$$= -\frac{902}{1.4} = -644\ J$$

$$\therefore \quad \Delta W_{CA} = \Delta Q_{CA} - \Delta U_{CA} = -258\ J$$

Process	ΔQ (in J)	ΔW (in J)	ΔU (in J)
AB	1246	0	1246
BC	0	602	−602
CA	−902	−258	−644
Total	344	344	0

(c) Efficiency of the cycle

$$\eta = \frac{W_{Total}}{|Q_{+ve}|} \times 100$$

$$= \frac{344}{1246} \times 100$$

$$= 27.6\%$$

Introductory Exercise 18.3

1. Three moles of an ideal gas being initially at a temperature $T_i = 273$ K were isothermally expanded 5 times its initial volume and then isochorically heated so that the pressure in the final state become equal to that in the initial state. The total heat supplied in the process is 80 kJ. Find $\gamma \left(= \frac{C_P}{C_V}\right)$ of the gas.

2. As a result of the isobaric heating by $\Delta T = 72$ K, one mole of a certain ideal gas obtains an amount of heat $Q = 1.6$ kJ. Find the work performed by the gas, the increment of its internal energy and γ.

3. A gas undergoes the cycle shown in figure. The cycle is repeated 100 times per minute. Determine the power generated.

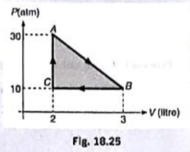


Fig. 18.25

Sample Example 18.10 *Carnot's engine takes in a thousand kilo calories of heat from a reservoir at 827°C and exhausts it to a sink at 27°C. How, much work does it perform ? What is the efficiency of the engine?*

Solution Given, $Q_1 = 10^6$ cal

$$T_1 = (827 + 273) = 1100\ K$$

and $\quad T_2 = (27 + 273) = 300\ K$

as, $\quad \frac{Q_2}{Q_1} = \frac{T_2}{T_1}$

$$\therefore \quad Q_2 = \frac{T_2}{T_1} \cdot Q_1 = \left(\frac{300}{1100}\right)(10^6)$$

$$= 2.72 \times 10^5\ cal$$

Efficiency of the cycle,

$$\eta = \left(1 - \frac{T_2}{T_1}\right) \times 100$$

or $\quad \eta = \left(1 - \frac{300}{1100}\right) \times 100$

$$= 72.72\%$$

Sample Example 18.11 *Calculate the least amount of work that must be done to freeze one gram of water at 0°C by means of a refrigerator. Temperature of surroundings is 27°C. How much heat is passed on the surroundings in this process? Latent heat of fusion L = 80 cal/g.*

Solution $\quad Q_2 = mL = 1 \times 80 = 80\ cal$

$$T_2 = 0°C = 273\ K$$

and $\quad T_1 = 27°C = 300\ K$

$$\frac{Q_2}{W} = \frac{T_2}{T_1 - T_2}$$

$$\therefore \quad W = \frac{Q_2(T_1 - T_2)}{T_2}$$

$$= \frac{80\,(300 - 273)}{273} = 7.91\ cal \qquad \textbf{Ans.}$$

$$Q_1 = Q_2 + W$$

$$= (80 + 7.91) = 87.91\ cal \qquad \textbf{Ans.}$$

18.7 Zeroth Law of Thermodynamics

The zeroth law of thermodynamics states that if two systems, A and B, are in thermal equilibrium with a third system, C, then A and B are in thermal equilibrium with each other. It is analogous to the transitive property in math (if $A = C$ and $B = C$, then $A = B$). Another way of stating the zeroth law is that every object has a certain temperature, and when two objects are in thermal equilibrium, their temperatures are equal. It is called the zeroth law because it came to light after the first and second laws of thermodynamics had already been established and named, but was considered more fundamental and thus was given a lower number zero.

18.8 Second Law of Thermodynamics

Before studying the second law of thermodynamics we will have to understand the concept of entropy.

Entropy

Entropy is an extensive thermodynamic property that is the measure of a system's thermal energy per unit temperature that is unavailable for doing useful work. In thermodynamics, entropy has the dimension of energy divided by temperature, which has a unit of joules per kelvin (J/K) in the SI units. As a physical system becomes more disordered, and its energy becomes more evenly distributed, that energy becomes less able to do work. For example, a car rolling along a road has kinetic energy that could do work (by carrying or colliding with something). As friction slows it down and its energy is distributed to its surroundings as heat, it loses this ability. The amount of entropy is often thought of as the amount of disorder in a system.

Second law of thermodynamics

The First Law of Thermodynamics, commonly known as the Law of Conservation of Matter, states that matter/energy cannot be created nor can it be destroyed. The quantity of matter/energy remains the same. It can change from solid to liquid to gas to plasma and back again, but the total amount of matter/energy in the universe remains constant.

The second law is a statement that all processes go only in one direction, which is the direction of greater and greater degradation of energy. In other words, to a state of higher and higher entropy. This is also known as the Law of Increased Entropy. While quantity remains the same (First Law), the quality of matter/energy deteriorates gradually over time. How so? Usable energy is inevitably used for productivity, growth and repair. In the process, usable energy is converted into unusable energy. Thus, usable energy is irretrievably lost in the form of unusable energy. The implications of the Second Law of Thermodynamics are considerable. The universe is constantly losing usable energy and never gaining. We logically conclude that the universe is not eternal. The universe had a finite beginning — the moment at which it was at zero entropy (its most ordered possible state). Like a wind-up clock, the universe is winding down, as if at one point it was fully wound up and has been winding down ever since.

There are three alternative statements of second law of thermodynamics.

Clausius statement of the second law It is impossible to transfer heat from a cooler to a hotter body.

Kelvin-Planck statement of the second law It is impossible to build a heat engine that has 100% efficiency.

Entropy statement of the second law It is impossible for any system in a way that entropy is destroyed. Therefore in any process change in entropy $(S_f - S_i)$ may be positive or zero. It cannot be negative.

or $\quad \Delta S \geq 0 \quad$ Always

Note *Actual statements may be different. We have given them in simple form for better understanding.*

18.9 Reversible And Irreversible Processes

Irreversible Process

Most thermodynamic processes proceed naturally from one direction to another. In other words, they have a preferred direction. Heat flows from a hotter object to a colder one. Gases expand to fill a room, but will not spontaneously contract to fill a smaller space.

The process is said to be an irreversible process if it cannot return the system and the surroundings to their original conditions when the process is reversed. The irreversible process is not at equilibrium throughout the process.

For example, when we are driving the car uphill, it consumes a lot of fuel and this fuel is not returned when we are driving down the hill. The basic concept is that most of the thermodynamic processes have a preferred direction just as heat always flows from hotter object to colder object. Once a gas is released in a room, it expands in room and never contracts without indulgence of any external force etc.

Many factors contribute in making any process irreversible. *The most common of these are:*

(i) Friction

(ii) Unrestrained expansion of a fluid

(iii) Heat transfer through a finite temperature difference

(iv) Mixing of two different substances

Reversible Process

In some systems, the reverse occurs. Normally it happens when that system is close to thermal equilibrium. This equilibrium has to be inside the system itself and also within the system and its surroundings. When this stage is reached, even a small change can change the direction of the process and therefore such a reversible process is also known as an equilibrium process.

For Example

A very simple example can be of two metal jars A and B which are at a thermal equilibrium and are in contact with each other. Now when we heat jar A slightly, heat starts to flow from Jar A to Jar B. This is the direction of this process. Now this process can be reversed just by cooling Jar A slightly. When Jar A is cooled, heat flows from Jar B to Jar A till thermal equilibrium is reached.

Note *One of the important uses of second law of thermodynamics is to determine whether a given process is reversible or irreversible. Any process which is in agreement with second law of thermodynamics is irreversible.*

Extra Points

- In the above discussion we have seen that:

$$W = \int_{V_i}^{V_f} P\,dV$$

From this equation it seems as if work done can be calculated only when P-V equation is known and the limits V_i and V_f are known to us. But it is not so. We can calculate work done even if we know the limits of temperature.

For example, the temperature of n moles of an ideal gas is increased from T_0 to $2T_0$ through a process $P = \frac{\alpha}{T}$ and we are interested in finding the work done by the gas. Then

$$PV = nRT \qquad \text{(ideal gas equation)...(i)}$$

and $$P = \frac{\alpha}{T} \qquad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get $V = \frac{nRT^2}{\alpha}$

or $$dV = \frac{2nRT}{\alpha}\,dT$$

$\therefore$ $$W = \int_{V_i}^{V_f} P\,dV = \int_{T_0}^{2T_0}\left(\frac{\alpha}{T}\right)\left(\frac{2nRT}{\alpha}\right)dT = 2nRT_0$$

So, we have found the work done without putting the limits of volume.

- Sometimes the piston (which is massless) is attached to a spring of force constant k and a mass m is placed over the piston. The area of the piston is A. The gas expands. To make the calculation easy we assume that initially the spring was in its natural length. We are required to find the work done by the gas. As the piston is massless, net force on it at every instant is zero.

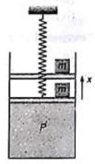


Fig. 18.26

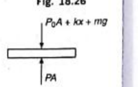


Fig. 18.27

$\therefore$ $$PA = kx + mg + P_0A$$

or $$P = P_0 + \frac{kx}{A} + \frac{mg}{A}$$

$$dW = PdV = P\,(Adx)$$
$$= (AP_0 + kx + mg)\,dx$$

$\therefore$ $$W = \int_0^x P\,dV$$

or $$W = \int_0^x (AP_0 + kx + mg)\,dx$$
$$= P_0Ax + \frac{1}{2}kx^2 + mgx \qquad (\text{as } Ax = \Delta V)$$

or $$W = P_0\Delta V + \frac{1}{2}kx^2 + mgx$$

The result can be stated in a different manner as under.

The gas does work against the atmospheric pressure P_0 (which is constant), the spring force kx (which varies linearly with x) and the gravity force mg (which is again constant).

$\therefore$ W_1 = Work done against $P_0 = P_0\Delta V$

W_2 = Work done against $kx = \frac{1}{2}kx^2$

and W_3 = Work done against $mg = mgx$

So that $W_{Total} = W_1 + W_2 + W_3 = P_0\Delta V + \frac{1}{2}kx^2 + mgx$

- From point number (2). We may conclude that work done by a gas is zero if the other side of the piston is vacuum.

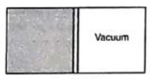


Fig. 18.28

- $C_V = \frac{dU}{dT}$: Let us derive the relation $C_V = \frac{dU}{dT}$. Where U = internal energy of 1 mole of the gas.

Consider 1 mole ($n = 1$) of an ideal monoatomic gas which undergoes an isochoric process (V = constant). From the first law of thermodynamics.

$$dQ = dW + dU \qquad \text{...(i)}$$

Here, $dW = 0$ as $V =$

$dQ = CdT = C_VdT$ (In $dQ = nC_VdT$, $n = 1$ and $C = C_V$)

Substituting in Eq. (i), we have

$$C_VdT = dU$$

or $$C_V = \frac{dU}{dT}$$ **Hence Proved**

- $C_P - C_V = R$: To prove this relation (also known as **Mayor's formula**) let us consider 1 mole of an ideal gas which undergoes an isobaric (P = constant) process. From first law of thermodynamics,

$$dQ = dW + dU$$

Here, $dQ = C_PdT$ (as $n = 1$ and $C = C_P$)

$dU = C_VdT$

and $dW = PdV = Pd\left(\frac{RT}{P}\right)$ $\quad \left(V = \frac{RT}{P}\right)$

$= d(RT)$ (as $P =$)

$= R \cdot dT$

Substituting these values in equation (i)

We have $C_PdT = R \cdot dT + C_VdT$

or $C_P - C_V = R$ **Hence proved**

- $C_V = \frac{R}{\gamma - 1}$: We have already derived,

$$C_P - C_V = R$$

dividing this equation by C_V, we have

$$\frac{C_P}{C_V} - 1 = \frac{R}{C_V}$$

or $\quad \gamma - 1 = \frac{R}{C_V} \qquad \left(\text{as } \frac{C_P}{C_V} = \gamma\right)$

$\therefore \quad C_V = \frac{R}{\gamma - 1}$ **Hence Proved**

- **Thermodynamic parameters for a mixture of gases :**

(i) Equivalent molar mass : When n_1 moles of a gas with molar mass M_1 are mixed with n_2 moles of a gas with molar mass M_2, the equivalent molar mass of the mixture is given by

$$M = \frac{n_1 M_1 + n_2 M_2}{n_1 + n_2}$$

(ii) Internal energy of the mixture : The total energy of the mixture is

$$U = U_1 + U_2$$

(iii) C_V of the mixture :

$$U = U_1 + U_2$$

$\therefore \quad dU = dU_1 + dU_2$

or $\quad nC_V dT = n_1 C_{V_1} dT + n_2 C_{V_2} dT \qquad \text{...(ii)}$

or $\quad (n_1 + n_2) C_V = n_1 C_{V_1} + n_2 C_{V_2} \qquad (\text{as } n = n_1 + n_2)$

$\therefore \quad C_V = \frac{n_1 C_{V_1} + n_2 C_{V_2}}{n_1 + n_2}$

(iv) C_P of the mixture :

$$C_P = C_V + R$$

or
$$C_P = \frac{n_1 C_{V_1} + n_2 C_{V_2}}{n_1 + n_2} + R$$

$$= \frac{n_1 (C_{V_1} + R) + n_2 (C_{V_2} + R)}{n_1 + n_2}$$

$$= \frac{n_1 C_{P_1} + n_2 C_{P_2}}{n_1 + n_2}$$

Thus, $\quad C_P = \frac{n_1 C_{P_1} + n_2 C_{P_2}}{n_1 + n_2}$

(v) γ of the mixture : From Eq. (ii)

$$(n_1 + n_2) C_V = n_1 C_{V_1} + n_2 C_{V_2}$$

or
$$\frac{(n_1 + n_2) R}{\gamma - 1} = \frac{n_1 R}{\gamma_1 - 1} + \frac{n_2 R}{\gamma_2 - 1}$$

or
$$\frac{n_1 + n_2}{\gamma - 1} = \frac{n_1}{\gamma_1 - 1} + \frac{n_2}{\gamma_2 - 1}$$

Thus, γ of the mixture is given by above equation.

- Bulk modulus of a gas is given by

$$B = -\frac{(dP)}{(dV/V)} = V\left(-\frac{dP}{dV}\right)$$

$\left(-\frac{dP}{dV}\right) = \gamma\left(\frac{P}{V}\right)$ in an adiabatic process. Hence, adiabatic bulk modulus of an ideal gas is

$$B_s = \gamma P$$

Similarly,

$$\left(-\frac{dP}{dV}\right) = \left(\frac{P}{V}\right) \text{ in an isothermal process.}$$

Hence, isothermal bulk modulus of an ideal gas is,

$$B_T = P$$

- **Polytropic process**

When P and V bear the relation PV^x = constant, where $x \neq 1$ or γ the process is called a polytropic one. In this process the molar heat capacity is,

$$C = C_V + \frac{R}{1-x} = \frac{R}{\gamma - 1} + \frac{R}{1-x}$$

Let us now derive this relation. The molar heat capacity is defined as :

$$C = \frac{\Delta Q}{\Delta T} \text{ for 1 mole}$$

$$= \frac{\Delta U + \Delta W}{\Delta T}$$

$$= \frac{\Delta U}{\Delta T} + \frac{\Delta W}{\Delta T} \qquad \left(\frac{\Delta U}{\Delta T} = C_V\right)$$

$\therefore \quad C = C_V + \frac{\Delta W}{\Delta T} \qquad \text{...(i)}$

Here, $\quad \Delta W = \int_{V_i}^{V_f} P dV = \int_{V_i}^{V_f} kV^{-x} dV = \left[\frac{kV^{-x+1}}{-x+1}\right]_{V_i}^{V_f}$

$$= \frac{kV_f^{-x+1} - kV_i^{-x+1}}{-x+1}$$

$$= \frac{P_f V_f^x V_f^{-x+1} - P_i V_i^x V_i^{-x+1}}{1-x}$$

$$= \frac{P_f V_f - P_i V_i}{1-x} = \frac{RT_f - RT_i}{1-x}$$

$$= \frac{R\Delta T}{1-x}$$

$\therefore \quad \frac{\Delta W}{\Delta T} = \frac{R}{1-x}$

Substituting in Eq. (i), we get the result *i.e.*,

$\therefore \quad C = C_V + \frac{R}{1-x} = \frac{R}{\gamma - 1} + \frac{R}{1-x}$

Slope of *P-V* diagram

Slope of *P-V* diagram (also known as **indicator diagram**) at any point is

$$\frac{dP}{dV} = -x\frac{P}{V}$$

It x is negative, slope is positive and vice-versa.

Exercise *Derive the relation* $\frac{dP}{dV} = -x\frac{P}{V}$ *for the process* PV^x *= constant.*

- **Variation of pressure with elevation and depth**

Suppose atmospheric pressure at sea level ($y = 0$) is P_0. We are interested in finding the air pressure with elevation in the earth's atmosphere, assuming the temperature to be constant throughout.

We begin with the pressure relation that we use in the chapter of fluid mechanics,

$$\frac{dP}{dy} = -\rho g$$

The density ρ is given by, $\quad \rho = \frac{PM}{RT}$

$\therefore \quad \frac{dP}{dy} = -\left(\frac{PM}{RT}\right) g$

or $\quad \int_{P_0}^{P} \frac{dP}{P} = -\int_0^y \left(\frac{Mg}{RT}\right) dy \qquad \text{...(i)}$

or $\quad \int_{P_0}^{P} \frac{dP}{P} = -\frac{Mg}{RT} \int_0^y dy \qquad (\text{as } T = \text{constant})$

or $\quad P = P_0 e^{-\left(\frac{Mg}{RT}\right) y}$

i.e., pressure of air decreases exponentially with height (y).

With depth : We have already studied, how the pressure varies with depth (h) of water. In increases linearly with depth. The pressure at a depth h is,

$$P = P_0 + \rho_w gh$$

The P *versus* y or P *versus* h graph is shown in figure.

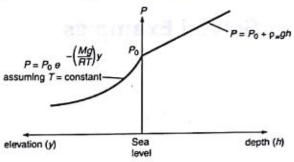


Fig. 18.29

Note *Suppose in some question* $T \neq$ constant, *but it is given a function of elevation y, then Eq. (i), will become*

$$\int_{P_0}^{P} \frac{dP}{P} = -\frac{Mg}{R} \int_0^y \frac{dy}{T}$$

Here, T is the given function of y, so it is inside the integration.

Solved Examples

Example 1 *Find the molar specific heat of the process $P = \frac{a}{T}$ for a monoatomic gas, a being constant.*

Solution We know that

$$dQ = dU + dW$$

Specific heat
$$C = \frac{dQ}{dT} = \frac{dU}{dT} + \frac{dW}{dT} \quad \ldots(i)$$

Since,
$$dU = C_V dT \quad \ldots(ii)$$

$$C = C_V + \frac{dW}{dT}$$

$$= C_V + \frac{PdV}{dT}$$

$$PV = RT$$

$\therefore$ For the given process,

$$V = \frac{RT}{P} = \frac{RT^2}{a}$$

$$\frac{dV}{dT} = \frac{2RT}{a}$$

$\therefore$
$$C = C_V + P\left(\frac{2RT}{a}\right)$$

$$= C_V + 2R$$

$$= \frac{3}{2}R + 2R = \frac{7}{2}R$$

Example 2 *At 27° C two moles of an ideal monoatomic gas occupy a volume V. The gas expands adiabatically to a volume 2V. Calculate*

(a) final temperature of the gas

(b) change in its internal energy and

(c) the work done by the gas during the process.

[R = 8.31 J/mol-K]

Solution (a) In case of adiabatic change

$$PV^{\gamma} = \text{constant with } PV = nRT$$

So that
$$T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \text{ with } \gamma = \left(\frac{5}{3}\right)$$

i.e.,
$$300 \times V^{2/3} = T(2V)^{2/3}$$

or $$T = \frac{300}{(2)^{2/3}} = 189\text{ K}$$

(b) As $$\Delta U = nC_V \Delta T = \frac{nR\Delta T}{(\gamma - 1)} \qquad \left[\text{as } C_V = \frac{R}{(\gamma-1)}\right]$$

So, $$\Delta U = 2 \times \left(\frac{3}{2}\right) \times 8.31\,(189 - 300)$$
$$= -2767.23\text{ J}$$

Negative sign means internal energy will decrease.
(c) According to first law of thermodynamics

$$Q = \Delta U + \Delta W$$

And as for adiabatic change $\Delta Q = 0$,

$$\Delta W = -\Delta U = 2767.23\text{ J}$$

Example 3 *The density versus pressure graph of one mole of an ideal monoa- tomic gas undergoing a cyclic process is shown in figure. The molecular mass of the gas is M.*

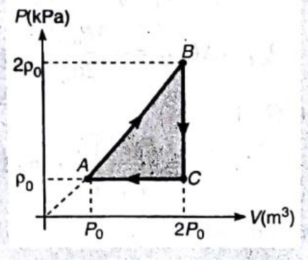


Fig. 18.30

(a) Find the work done in each process.
(b) Find heat rejected by gas in one complete cycle.
(c) Find the efficsency of the cycle.

Solution (a) As $n = 1$, $m = M$

Process AB : $\rho \propto P$, *i.e.* It is an isothermal process (T = constant), because $\rho = \dfrac{PM}{RT}$.

$\therefore$ $$W_{AB} = RT_A \ln\left(\frac{P_A}{P_B}\right) = RT_A \ln\left(\frac{1}{2}\right)$$
$$= -\frac{P_0 M}{\rho_0} \ln(2)$$
$$\Delta U_{AB} = 0$$

and $$Q_{AB} = W_{AB} = -\frac{P_0 M}{\rho_0} \ln(2)$$

Process BC is an isobaric process (P = constant)

$$W_{BC} = P_B\,(V_C - V_B) = 2P_0\left(\frac{M}{\rho_C} - \frac{M}{\rho_B}\right) = \frac{2P_0M}{2\rho_0} = \frac{P_0M}{\rho_0}$$

$$\Delta U_{BC} = C_V\Delta T$$

$$= \left(\frac{3}{2}R\right)\left[\frac{2P_0M}{\rho_0R} - \frac{2P_0M}{2\rho_0R}\right] = \frac{3P_0M}{2\rho_0}$$

$$Q_{BC} = W_{BC} + \Delta U_{BC} = \frac{5P_0M}{2\rho_0}$$

Process CA **:** As ρ = constant $\therefore$ V = constant.

So, it is an isochoric process.

$$W_{CA} = 0$$

$$\Delta U_{CA} = C_V\Delta T$$

$$= \left(\frac{3}{2}R\right)(T_A - T_C)$$

$$= \left(\frac{3}{2}R\right)\left[\frac{P_0M}{\rho_0R} - \frac{2P_0M}{\rho_0R}\right] = -\frac{3P_0M}{2\rho_0}$$

$$Q_{CA} = \Delta U_{CA} = -\frac{3P_0M}{2\rho_0}$$

(b) Heat rejected by gas $= |Q_{AB}| + |Q_{CA}|$

$$= \frac{P_0M}{\rho_0}\left[\frac{3}{2} + \ln(2)\right]$$ **Ans.**

(c) Efficiency of the cycle (in fraction)

$$\eta = \frac{\text{Total work done}}{\text{Heat supplied}} = \frac{W_{\text{Total}}}{Q_{+\text{ve}}}$$

$$= \frac{\frac{P_0M}{\rho_0}[1 - \ln(2)]}{\frac{5}{2}\left(\frac{P_0M}{\rho_0}\right)}$$

$$= \frac{2}{5}[1 - \ln(2)]$$ **Ans.**

Example 4 *Two moles of a diatomic ideal gas is taken through PT = constant. Its temperature is increased from T to $2T$. Find the work done by the system ?*

Solution

$$W = \int P\,dV$$

Here,

$$PT = P_1T_1 = P_2T_2 = c \qquad \text{(constant)}$$

$\therefore$ $$PT = c \qquad P\cdot\frac{PV}{nR} = c$$

$\therefore$ $$P^2V = ncR \;\Rightarrow\; P = \sqrt{\frac{ncR}{V}}$$

$\therefore$ $$\int P\,dV = \sqrt{ncR}\int_{V_1}^{V_2}\frac{1}{\sqrt{V}}\,dV$$

$$\sqrt{ncR}\,[2(\sqrt{V_2} - \sqrt{V_1})] = 2[\sqrt{nR\cdot P_2T_2V_2} - \sqrt{nRP_1T_1V_1}\,]$$

$$= 2nR\,(T_2 - T_1) = 8RT_0$$

Example 5 *An ideal monoatomic gas at temperature 27°C and pressure 10^6 N/m^2 occupies 10 litre volume. 10,000 cal of heat is added to the system without changing the volume. Calculate the final temperature of the gas. Given : R = 8.31 J/(mol-K) and J = 4.18 J/cal.*

Solution For n mole of gas, we have $PV = nRT$

Here, $P = 10^6\,\text{N/m}^2$, $V = 10$ litre $= 10^{-2}\,\text{m}^3$ and $T = 27°\text{C} = 300\,\text{K}$

$\therefore$ $$n = \frac{PV}{RT} = \frac{10^6\times10^{-2}}{8.31\times300} = 4.0$$

For "monoatomic" gas, $C_V = \frac{3}{2}R$

Thus, $$C_V = \frac{3}{2}\times8.31\,\text{J/mol-K}$$

$$= \frac{3}{2}\times\frac{8.31}{4.18} \approx 3\,\text{cal/(mol-K)}$$

Let ΔT be the rise in temperature when n mole of the gas is given Q cal of heat at constant volume. Then,

$$Q = nC_V\Delta T$$

or $$\Delta T = \frac{Q}{nC_V} = \frac{10{,}000\,\text{cal}}{4.0\,\text{mole}\times3\,\text{cal (mol-K)}} = 833\,\text{K}$$

Example 6 *One mole of a monoatomic ideal gas is taken through the cycle shown in figure.*

$A\to B$ Adiabatic expansion

$B\to C$ Cooling at constant volume

$C\to D$ Adiabatic compression.

$D\to A$ Heating at constant volume

The pressure and temperature at A, B etc., are denoted by $P_A, T_A; P_B, T_B$ etc. respectively.

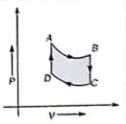


Fig. 18.31

Given $T_A = 1000K$, $P_B = \left(\frac{2}{3}\right)P_A$ and $P_C = \left(\frac{1}{3}\right)P_A$. Calculate

(a) the work done by the gas in the process $A\to B$

(b) the heat lost by the gas in the process $B\to C$ and

Given $\left(\frac{2}{3}\right)^{0.4} = 0.85$ and $R = 8.31$ J/mol-K

Solution (a) As for adiabatic change PV^γ = constant

i.e., $$P\left(\frac{nRT}{P}\right)^\gamma = \text{constant} \qquad [\text{as } PV = nRT]$$

i.e., $$\frac{T^\gamma}{P^{\gamma-1}} = \text{constant so}\left(\frac{T_B}{T_A}\right)^\gamma = \left(\frac{P_B}{P_A}\right)^{\gamma-1} \text{ where } \gamma = \frac{5}{3}$$

i.e., $$T_B = T_A\left(\frac{2}{3}\right)^{1-\frac{1}{\gamma}} = 1000\left(\frac{2}{3}\right)^{2/5} = 850\,\text{K}$$

So, $$W_{AB} = \frac{nR[T_F - T_I]}{[1-\gamma]} = \frac{1\times8.31[1000-850]}{\left[\left(\frac{5}{3}\right)-1\right]}$$

i.e., $$W_{AB} = \left(\frac{3}{2}\right)\times8.31\times150 = 1869.75\,\text{J}$$

(b) For $B\to C$, V = constant so $\Delta W = 0$

So, from first law of thermodynamics

$$\Delta Q = \Delta U + \Delta W = nC_V\Delta T + 0$$

or $$\Delta Q = 1\times\left(\frac{3}{2}R\right)(T_C - 850) \qquad \text{as } C_V = \frac{3}{2}R$$

Now, along path BC, V = constant; $P \propto T$

i.e., $$\frac{P_C}{P_B} = \frac{T_C}{T_B}$$

$$T_C = \frac{\left(\frac{1}{3}\right)P_A}{\left(\frac{2}{3}\right)P_A}\times T_B = \frac{T_B}{2} = \frac{850}{2} = 425\,\text{K} \qquad \text{...(ii)}$$

So, $$\Delta Q = 1\times\frac{3}{2}\times8.31\,(425 - 850) = -5297.625\,\text{J}$$

[Negative heat means, heat is lost by the system]

Example 7 *A gas undergoes a process such that $P \propto \frac{1}{T}$. If the molar heat capacity for this process is $C = 33.24$ J/mol-K, find the degree of freedom of the molecules of the gas.*

Solution As $$P \propto \frac{1}{T}$$

or $$PT = \text{constant} \qquad \text{...(i)}$$

We have for one mole of an ideal gas

$$PV = RT \qquad \text{...(ii)}$$

From Eqs. (i) and (ii)

$$P^2V = \text{constant}$$

or $$PV^{1/2} = K \text{ (say)} \qquad \text{...(iii)}$$

From first law of thermodynamics,

$$\Delta Q = \Delta U + \Delta W$$

or $$C\Delta T = C_V\Delta T + \Delta W$$

or $$C = C_V + \frac{\Delta W}{\Delta T} \qquad \text{...(iv)}$$

Here, $$\Delta W = \int P\,dV = K\int_{V_i}^{V_f}V^{-1/2}\,dV$$

$$= \frac{P_fV_f - P_iV_i}{1-(1/2)} = \frac{R\,(T_f - T_i)}{1/2} = \frac{R\Delta T}{1/2}$$

$\therefore$ $$\frac{\Delta W}{\Delta T} = 2R$$

Substituting in Eq. (iv), we have

$$C = C_V + 2R = \frac{R}{\gamma-1} + 2R$$

Substituting the values,

$$33.24 = R\left(\frac{1}{\gamma-1} + 2\right) = 8.31\left(\frac{1}{\gamma-1} + 2\right)$$

Solving this we get

$$\gamma = 1.5$$

Now, $$\gamma = 1 + \frac{2}{F}$$

or degree of freedom

$$F = \frac{2}{\gamma-1} = \frac{2}{1.5-1} = 4$$

Alternate Solution : In the process PV^x = constant, molar heat capacity is given by

$$C = \frac{R}{\gamma-1} + \frac{R}{1-x}$$

The given process is $PV^{1/2} = \text{constant}$

or $$x = \frac{1}{2}$$

$\therefore$ $$C = \frac{R}{\gamma - 1} + \frac{R}{1 - \frac{1}{2}} = \frac{R}{\gamma - 1} + 2R$$

Now, we may proceed in the similar manner.

Example 8 *A gaseous mixture enclosed in a vessel consists of one g mole of a gas A with $\gamma = \left(\frac{5}{3}\right)$ and some amount of gas B with $\gamma = \frac{7}{5}$ at a temperature T. The gases A and B do not react with each other and are assumed to be ideal. Find the number of g moles of the gas B if γ for the gaseous mixture is $\left(\frac{19}{13}\right)$.*

Solution As for an ideal gas, $C_P - C_V = R$ and $\gamma = \left(\frac{C_P}{C_V}\right)$

So, $$C_V = \frac{R}{(\gamma - 1)}$$

$\therefore$ $$(C_V)_1 = \frac{R}{\left(\frac{5}{3}\right) - 1} = \frac{3}{2}R; \quad (C_V)_2 = \frac{R}{\left(\frac{7}{5}\right) - 1} = \frac{5}{2}R$$

and $$(C_V)_{\text{mix}} = \frac{R}{\left(\frac{19}{13}\right) - 1} = \frac{13}{6}R$$

Now, from conservation of energy,

i.e., $$\Delta U = \Delta U_1 + \Delta U_2$$

$$(n_1 + n_2)(C_V)_{\text{mix}}\, \Delta T = [n_1 (C_V)_1 + n_2 (C_V)_2]\, \Delta T$$

i.e., $$(C_V)_{\text{mix}} = \frac{n_1 (C_V)_1 + n_2 (C_V)_2}{n_1 + n_2}$$

We have $$\frac{13}{6}R = \frac{1 \times \frac{3}{2}R + n_2 \frac{5}{2}R}{1 + n_2}$$

$$= \frac{(3 + 5n_2)R}{2(1 + n_2)}$$

or $$13 + 13n_2 = 9 + 15n_2,$$

i.e., $$n_2 = 2 \text{ g mole}$$

Example 9 *An ideal gas having initial pressure P, volume V and temperature T is allowed to expand adiabatically until its volume becomes* $5.66\,V$, while its temperature falls to $T/2$

(a) How many degrees of freedom do the gas molecules have ?

(b) Obtain the work done by the gas during the expansion as a function of the initial pressure P and volume V.

Given that $(5.66)^{0.4} = 2$

Solution (a) For adiabatic expansion

$$TV^{\gamma-1} = \text{constant}$$

i.e.,
$$TV^{\gamma-1} = T'V'^{\gamma-1} = \frac{T}{2}(5.66V)^{\gamma-1}$$

i.e.,
$$(5.66)^{\gamma-1} = 2$$

i.e.,
$$\gamma = 1.4$$

This is the value of γ of a diatomic gas. Hence the degree of freedom per molecule of a diatomic gas $= 5$.

(b) Work done during adiabatic process for one mole gas is

$$W = \frac{1}{1-\gamma}[P'V' - PV]$$

From relation,
$$\frac{PV}{T} = \frac{P'V'}{T'}$$

we get
$$P' = \frac{T'}{T}\cdot\frac{PV}{V'} = \frac{1}{2}\times\frac{1}{5.66}P = \frac{P}{11.32}$$

$\therefore$
$$W = \frac{1}{1-1.4}\left[\frac{P}{11.32}\times\frac{V}{5.66} - PV\right]$$

$$= \frac{1}{0.4}\left[1 - \frac{1}{11.32\times 5.66}\right]PV$$

$$= 2.461\,PV$$

Example 10 *Plot P-V, V-T and ρ-T graph corresponding to the P-T graph for an ideal gas shown in figure.*

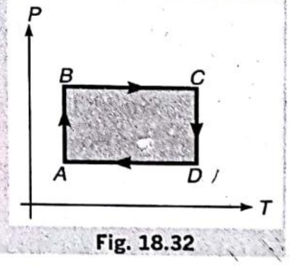


Fig. 18.32

Solution Process AB is an isothermal process with $T = \text{constant}$ and $P_B > P_A$.

P-V graph : $P \propto \frac{1}{V}$ *i.e.,* P-V graph is a hyperbola with $P_B > P_A$ and $V_B < V_A$.

V-T graph : $T = \text{constant}$. Therefore, V-T graph is a straight line parallel to V-axis with $V_B < V_A$.

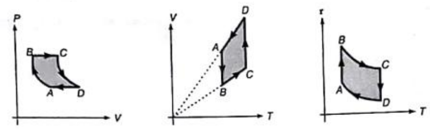


Fig. 18.33

ρ-*T* graph : $\rho = \frac{PM}{RT}$ or $\rho \propto P$.

As T is constant. Therefore, ρ-T graph is a straight line parallel to ρ-axis with $\rho_B > \rho_A$ as $P_B > P_A$.
Process BC is an isobaric process with P = constant and $T_C > T_B$.

***P-V* graph :** As P is constant. Therefore, P-V graph is a straight line parallel to V-axis with $V_C > V_B$ (because $V \propto T$ in an isobaric process)

***V-T* graph :** In isobaric process $V \propto T$, *i.e.*, V-T graph is a straight line passing through origin, with $T_C > T_B$ and $V_C > V_B$.

ρ-*T* graph : $\rho \propto \frac{1}{T}$ (when P = constant), *i.e.*, ρ-T graph is a hyperbola with $T_C > T_B$ and $\rho_C < \rho_B$.

There is no need of discussing C-D and D-A processes. As they are opposite to AB and BC respectively. The corresponding three graphs are shown above.

EXERCISES

For JEE Main

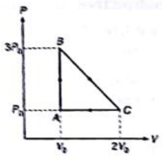
Subjective Questions

First Law of Thermodynamics

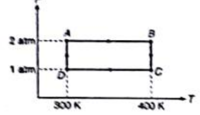
1. In a certain chemical process, a lab technician supplies 254 J of heat to a system. At the same time, 73 J of work are done on the system by its surroundings. What is the increase in the internal energy of the system?
2. One mole of an ideal monoatomic gas is initially at 300 K. Find the final temperature if 200 J of heat are added as follows :
   (a) at constant volume, (b) at constant pressure.
3. Show how internal energy U varies with T in isochoric, isobaric and adiabatic process?
4. A closed vessel 10 litres in volume contains a diatomic gas under a pressure of $10^5\ N/m^2$. What amount of heat should be imparted to the gas to increase the pressure in the vessel five times ?
5. A diatomic ideal gas is heated at constant volume until its pressure becomes three times. It is again heated at constant pressure until its volume is doubled. Find the molar heat capacity for the whole process.
6. Two moles of a certain gas at a temperature $T_0 = 300$ K were cooled isochorically so that the pressure of the gas got reduced 2 times. Then as a result of isobaric process, the gas is allowed to expand till its temperature got back to the initial value. Find the total amount of heat absorbed by gas in this process.
7. Five moles of an ideal monoatomic gas with an initial temperature of 127°C expand and in the process absorb 1200 J of heat and do 2100 J of work . What is the final temperature of the gas ?
8. An ideal gas expands while the pressure is kept constant. During this process, does heat flow into the gas or out of the gas? Justify your answer.
9. Find the change in the internal energy of 2 kg of water as it is heated from 0° C to 4° C. The specific heat capacity of water is 4200 J / kg - K and its densities at 0°C and 4°C are 999.9 kg/m^3 and 1000 kg/m^3 respectively. Atmospheric pressure $= 10^5$ Pa.
10. Calculate the increase in the internal energy of 10 g of water when it is heated from 0°C to 100°C and converted into steam at 100 kPa. The density of steam = 0.6 kg/m^3. Specific heat capacity of water = 4200 J/kg-°C and the latent heat of vaporization of water $= 2.5 \times 10^6$ J/kg.
11. One gram of water (1 cm^3) becomes 1671 cm^3 of steam when boiled at a constant pressure of 1 atm (1.013×10^5 Pa). The heat of vaporization at this pressure is $L_v = 2.256 \times 10^6$ J/kg. Compute (a) the work done by the water when it vaporizes and (b) its increase in internal energy.
12. A gas in a cylinder is held at a constant pressure of 2.30×10^5 Pa and is cooled and compressed from 1.70 m^3 to 1.20 m^3. The internal energy of the gas decreases by 1.40×10^5 J. (a) Find the work done by the gas. (b) Find the absolute value $|Q|$ of the heat flow into or out of the gas and state the direction of the heat flow. (c) Does it matter whether or not the gas is ideal ? Why or why not?

Cyclic Process and Efficiency of Cycle

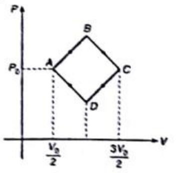
13. An ideal gas is taken through a cyclic thermodynamic process through four steps. The amounts of heat involved in these steps are $Q_1 = 5960$ J, $Q_2 = -5585$ J, $Q_3 = -2980$ J and $Q_4 = 3645$ J respectively. The corresponding quantities of work involved are $W_1 = 2200$ J, $W_2 = -825$ J, $W_3 = -1100$ J and W_4 respectively.
    (a) Find the value of W_4.
    (b) What is the efficiency of the cycle ?
14. One mole of an ideal monoatomic gas is taken round the cyclic process $ABCA$ as shown in figure. Calculate:
    (a) The work done by the gas.
    (b) The heat rejected by the gas in the path CA and heat absorbed in the path AB.
    (c) The net heat absorbed by the gas in the path BC.
    (d) The maximum temperature attained by the gas during the cycle.



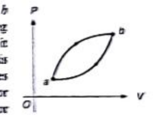
15. Two moles of helium gas undergo a cyclic process as shown in figure. Assuming the gas to be ideal, calculate the following quantities in this process.



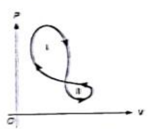
    (a) The net change in the heat energy.
    (b) The net work done.
    (c) The net change in internal energy.
16. n moles of a monoatomic gas are taken around in a cyclic process consisting of four processes along $ABCDA$ as shown. All the lines on the P-V diagram have slope of magnitude P_0/V_0. The pressure at A and C is P_0 and the volumes at A and C are $V_0/2$ and $3V_0/2$ respectively. Calculate the percentage efficiency of the cycle.



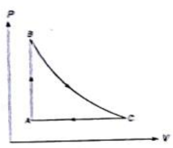
17. A system is taken around the cycle shown in figure from state a to state b and then back to state a. The absolute value of the heat transfer during one cycle is 7200 J. (a) Does the system absorb or liberate heat when it goes around the cycle in the direction shown in the figure ? (b) What is the work W done by the system in one cycle ? (c) If the system goes around the cycle in a counter-clock wise direction, does it absorb or liberate heat in one cycle? What is the magnitude of the heat absorbed or liberated in one counterclockwise cycle ?



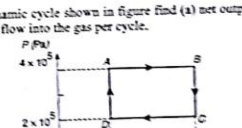
18. A thermodynamic system undergoes a cyclic process as shown in figure. The cycle consists of two closed loops, loop I and loop II. (a) Over one complete cycle, does the system do positive or negative work ? (b) In each of loops I and II, is the net work done by the system positive or negative ? (c) Over one complete cycle, does heat flow into or out of the system ? (d) In each of loops I and II, does heat flow into or out of the system ?



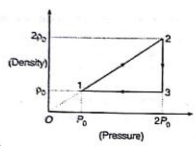
19. 1.0 k-mol of a sample of helium gas is put through the cycle of operations shown in figure. BC is an isothermal process and $P_A = 1.00$ atm, $V_A = 22.4\ m^3$, $P_B = 2.00$ atm. What are T_A, T_B and V_C?



20. For the thermodynamic cycle shown in figure find (a) net output work of the gas during the cycle, (b) net heat flow into the gas per cycle.



21. The density (ρ) versus pressure (P) graph of an ideal gas (monoatomic) undergoing a cyclic process is shown in figure. The gas taken has molecular mass M and one mole of gas is taken.



(a) Find work done in each process.
(b) Find heat rejected by gas in one complete cycle.
(c) Find the efficiency of the cycle.

22. Two moles of an ideal monoatomic gas go through the cycle *abc*. For the complete cycle 800 J of heat flows out of the gas. Process *ab* is at constant pressure and process *bc* is at constant volume. States *a* and *b* have temperatures $T_a = 200$ K and $T_b = 300$ K. (a) Sketch the *P-V* diagram for the cycle. (b) What is the work *W* for the process *ca* ?

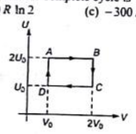
23. A monoatomic gas is expanded adiabatically from volume V_0 to $2V_0$ and then is brought back to the initial state through an isothermal and isochoric process respectively. Plot the *P-V* diagram of the complete cycle and find the efficiency of the cycle.

Objective Questions

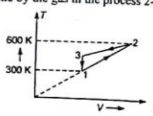
Single Correct Option

1. 1 mole of an ideal gas is taken through a cyclic process. The minimum temperature during the cycle is 300 K. Then net exchange of heat for complete cycle is

(a) $600 R \ln 2$ (b) $300 R \ln 2$ (c) $-300 R \ln 2$ (d) $900 R \ln 2$



2. Two moles of an ideal gas are undergone a cyclic process 1-2-3-1. If net heat exchange in the process is – 300 J, the work done by the gas in the process 2-3 is



(a) –500 J (b) – 5000 J (c) – 3000 J (d) None of these

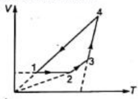
3. Two cylinders fitted with pistons contain equal amount of an ideal diatomic gas at 300 K. The piston of *A* is free to move, while that of *B* is held fixed. The same amount of heat is given to the gas in each cylinder. If the rise in temperature of the gas in *A* is 30 K, then the rise in temperature of gas in *B* is

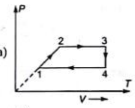
(a) 30 K (b) 18 K (c) 50 K (d) 42 K

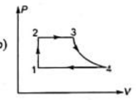
4. A gas follows a process TV^{n-1} = constant where T = absolute temperature of the gas and V = volume of the gas. The bulk modulus of the gas in the process is given by

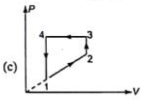
(a) $(n-1)P$ (b) $P/(n-1)$ (c) nP (d) P/n

5. A cyclic process 1-2-3-4-1 is depicted on *V-T* diagram. The *P-T* and *P-V* diagrams for this cyclic process are given below. Select the correct choices



(a)


(b)


(c)


(d) None of these

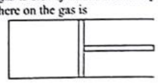
6. One mole of an ideal gas at temperature T_1 expands slowly according to the law P/V = constant. Its final temperature is T_2. The work done by the gas is

(a) $R(T_2 - T_1)$ (b) $2R(T_2 - T_1)$ (c) $\frac{R}{2}(T_2 - T_1)$ (d) $\frac{2R}{3}(T_2 - T_1)$

7. In a process the pressure of an ideal gas is proportional to square of the volume of the gas. If the temperature of the gas increases in this process, then work done by this gas :

(a) is positive (b) is negative
(c) is zero (d) may be positive or negative

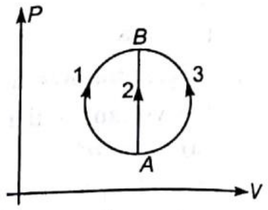
8. *n* moles of a gas are filled in a container at temperature *T*. If the gas is slowly and isothermally compressed to half its initial volume, the work done by the atmosphere on the gas is



(a) $\frac{nRT}{2}$ (b) $-\frac{nRT}{2}$
(c) $nRT \ln 2$ (d) $-nRT \ln 2$

9. 600 J of heat is added to a monoatomic gas in a process in which the gas performs a work of 150 J. The molar heat capacity for the process is
(a) $3R$ (b) $4R$ (c) $2R$ (d) $6R$

10. A gas undergoes A to B through three different processes 1, 2 and 3 as shown in the figure. The heat supplied to the gas is Q_1, Q_2 and Q_3 respectively, then

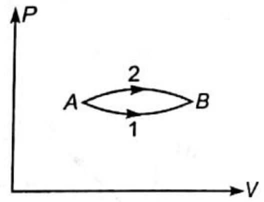


(a) $Q_1 = Q_2 = Q_3$ (b) $Q_1 < Q_2 < Q_3$
(c) $Q_1 > Q_2 > Q_3$ (d) $Q_1 = Q_3 > Q_2$

11. The internal energy of a gas is given by $U = 2PV$. It expands from V_0 to $2V_0$ against a constant pressure P_0. The heat absorbed by the gas in the process is
(a) $2P_0V_0$ (b) $4P_0V_0$ (c) $3P_0V_0$ (d) P_0V_0

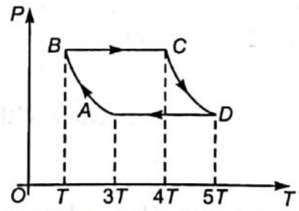
12. For an adiabatic compression the quantity PV
(a) increases (b) decreases (c) remains constant (d) depends on γ

13. The figure shows two paths for the change of state of a gas from A to B. The ratio of molar heat capacities in path 1 and path 2 is



(a) < 1 (b) > 1
(c) 1 (d) Data insufficient

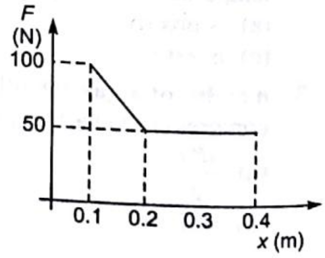
14. P-T diagram of one mole of an ideal monoatomic gas is shown. Processes AB and CD are adiabatic. Work done in the complete cycle is



(a) $2.5\ RT$ (b) $-2RT$ (c) $1.5\ RT$ (d) $-3.5\ RT$

15. An ideal monoatomic gas undergoes a process in which its internal energy U and density ρ vary as $U\rho$ = constant. The ratio of change in internal energy and the work done by the gas is
(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $\frac{3}{5}$

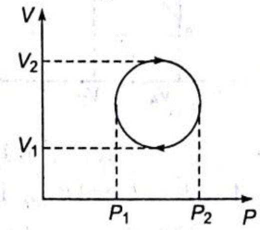
16. The given figure shows the variation of force applied by ideal gas on a piston which undergoes a process during which piston position changes from 0.1 to 0.4 m. If the internal energy of the system at the end of the process is 2.5 J higher, then the heat absorbed during the process is



(a) 15 J (b) 17.5 J
(c) 20 J (d) 22.5 J

17. A gas can expand through two processes : (i) isobaric, (ii) P/V = constant. Assuming that the initial volume is same in both processes and the final volume which is two times the initial volume is also same in both processes, which of the following is true?
(a) Work done by gas in process (i) is greater than the work done by the gas in process (ii)
(b) Work done by gas in process (i) is smaller than the work done by the gas in process (ii)
(c) Final pressure is greater in process (i)
(d) Final temperature is greater in process (i)

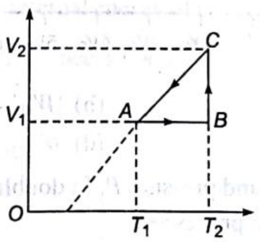
18. The cyclic process form a circle on a PV diagram as shown in figure. The work done by the gas is



(a) $\frac{\pi}{4}(P_2 - P_1)^2$ (b) $\frac{\pi}{4}(V_2 - V_1)^2$
(c) $\frac{\pi}{2}(P_2 - P_1)(V_2 - V_1)$ (d) $\frac{\pi}{4}(P_2 - P_1)(V_2 - V_1)$

19. The equation of a state of a gas is given by $P(V-b) = nRT$. If 1 mole of a gas is isothermally expanded from volume V and $2V$, the work done during the process is
(a) $RT \ln\left|\frac{2V-b}{V-b}\right|$ (b) $RT \ln\left|\frac{V-b}{V}\right|$ (c) $RT \ln\left|\frac{V-b}{2V-b}\right|$ (d) $RT \ln\left|\frac{V}{V-b}\right|$

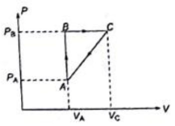
20. A cyclic process for 1 mole of an ideal gas is shown in the V-T diagram. The work done in AB, BC and CA respectively is



(a) $0, RT_2 \ln\left|\frac{V_2}{V_1}\right|, R(T_1 - T_2)$ (b) $R(T_1 - T_2), 0, RT_1 \ln\left|\frac{V_1}{V_2}\right|$
(c) $0, RT_1 \ln\left|\frac{V_1}{V_2}\right|, R(T_1 - T_2)$ (d) $0, RT_2 \ln\left|\frac{V_2}{V_1}\right|, R(T_2 - T_1)$

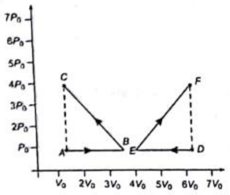
21. An ideal gas of adiabatic exponent γ is expanded so that the amount of heat transferred to the gas is equal to the decrease of its internal energy. Then the equation of the process in terms of the variables T and V is
(a) $TV^{\frac{(\gamma-1)}{2}} = C$ (b) $TV^{\frac{(\gamma-2)}{2}} = C$ (c) $TV^{\frac{(\gamma-1)}{4}} = C$ (d) $TV^{\frac{(\gamma-2)}{4}} = C$

22. A thermodynamical process is shown in the figure with $P_A = 3 \times P_{atm}$, $V_A = 2 \times 10^{-4}\ m^3$, $P_B = 8 \times P_{atm}$, $V_C = 5 \times 10^{-4}\ m^3$. In the process AB and BC, 600 J and 200 J heat are added to the system. Find the change in internal energy of the system in the process CA. [1 $P_{atm} = 10^5\ N/m^2$]



(a) 560 J (b) −560 J (c) −240 J (d) +240 J

23. If W_{ABC} is the work done in process $A \to B \to C$ and W_{DEF} is work done in process $D \to E \to F$ as shown in the figure, then

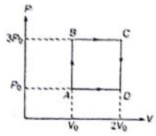


(a) $|W_{DEF}| > |W_{ABC}|$ (b) $|W_{DEF}| < |W_{ABC}|$
(c) $W_{DEF} = W_{ABC}$ (d) $W_{DEF} = -W_{ABC}$

24. An ideal gas has initial volume V and pressure P. In doubling its volume the minimum work done will be in the process (of the given processes)
(a) isobaric process (b) isothermal process
(c) adiabatic process (d) same in all given processes

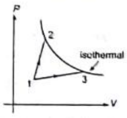
25. Ten moles of a diatomic perfect gas are allowed to expand at constant pressure. The initial volume and temperature are V_0 and T_0 respectively. If $\frac{7}{2}RT_0$ heat is transferred to the gas then the final volume and temperature are
(a) $1.1V_0, 1.1T_0$ (b) $0.9V_0, 0.9T_0$ (c) $1.1V_0, \frac{10}{11}T_0$ (d) $0.9V_0, \frac{10}{9}T_0$

26. An ideal monoatomic gas is carried around the cycle $ABCDA$ as shown in the figure. The efficiency of the gas cycle is



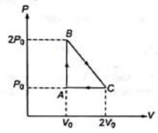
(a) $\frac{4}{21}$ (b) $\frac{2}{21}$ (c) $\frac{4}{31}$ (d) $\frac{2}{31}$

27. A gas takes part in two processes in which it is heated from the same initial state 1 to the same final temperature. The processes are shown on the P-V diagram by the straight lines 1-3 and 1-2. 2 and 3 are the points on the same isothermal curve. Q_1 and Q_2 are the heat transfer along the two processes. Then



(a) $Q_1 = Q_2$ (b) $Q_1 < Q_2$ (c) $Q_1 > Q_2$ (d) Insufficient data

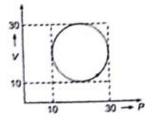
28. In the process shown in figure, the internal energy of an ideal gas decreases by $\frac{3P_0V_0}{2}$ in going from point C to A. Heat transfer along the process CA is



(a) $(-3\,P_0V_0)$ (b) $(-5\,P_0V_0/2)$ (c) $(-3\,P_0V_0/2)$ (d) zero

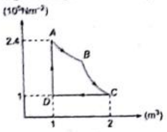
29. A closed system receives 200 kJ of heat at constant volume. It then rejects 100 kJ of heat while it has 50 kJ of work done on it at constant pressure. If an adiabatic process can be found which will restore the system to its initial state, the work done by the system during this process is
(a) 100 kJ (b) 50 kJ (c) 150 kJ (d) 200 kJ

30. Heat energy absorbed by a system in going through a cyclic process is shown in the figure [V in litres and P in k Pa] is



(a) $10^7\,\pi$ J (b) $10^4\,\pi$ J (c) $10^2\,\pi$ J (d) $10^3\,\pi$ J

31. 100 moles of an ideal monatomic gas undergoes the thermodynamic process as shown in the figure



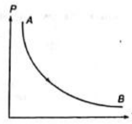
$A \to B$: isothermal expansion $B \to C$: adiabatic expansion
$C \to D$: isobaric compression $D \to A$: isochoric process

The heat transfer along the process AB is 9×10^4 J. The net work done by the gas during the cycle is [Take $R = 8\ JK^{-1}\ mol^{-1}$]
(a) -0.5×10^4 J (b) $+0.5 \times 10^4$ J (c) -5×10^4 J (d) $+5 \times 10^4$ J

32. One mole of an ideal monoatomic gas at temperature T_0 expands slowly according to the law $\frac{P}{V}$ = constant. If the final temperature is $2T_0$, heat supplied to the gas is
(a) $2RT_0$ (b) $\frac{3}{2}RT_0$ (c) RT_0 (d) $\frac{1}{2}RT_0$

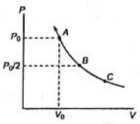
33. A mass of gas is first expanded isothermally and then compressed adiabatically to its original volume. What further simplest operation must be performed on the gas to restore it to its original state ?
(a) An isobaric cooling to bring its temperature to initial value
(b) An isochoric cooling to bring its pressure to its initial value
(c) An isothermal process to take its pressure to its initial value
(d) An isochoric heating to bring its temperature to initial value

34. Two moles of an ideal monoatomic gas are expanded according to the equation PT = constant from its initial state (P_0, V_0) to the final state due to which its pressure becomes half of the initial pressure. The change in internal energy is



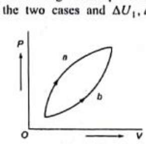
(a) $\frac{3P_0V_0}{4}$ (b) $\frac{3P_CV_0}{2}$ (c) $\frac{9P_0V_0}{2}$ (d) $\frac{5P_0V_0}{2}$

35. The state of an ideal gas is changed through an isothermal process at temperature T_0 as shown in figure. The work done by gas in going from state B to C is double the work done by gas in going from state A to B. If the pressure in the state B is $\frac{P_0}{2}$, then the pressure of the gas in state C is



(a) $\frac{P_0}{3}$ (b) $\frac{P_0}{4}$ (c) $\frac{P_0}{6}$ (d) $\frac{P_0}{8}$

36. Figure shows two processes a and b for a given sample of a gas. If $\Delta Q_1, \Delta Q_2$ are the amounts of heat absorbed by the system in the two cases and $\Delta U_1, \Delta U_2$ are changes in internal energies respectively, then



(a) $\Delta Q_1 = \Delta Q_2; \Delta U_1 = \Delta U_2$ (b) $\Delta Q_1 > \Delta Q_2; \Delta U_1 > \Delta U_2$
(c) $\Delta Q_1 < \Delta Q_2; \Delta U_1 < \Delta U_2$ (d) $\Delta Q_1 > \Delta Q_2; \Delta U_1 = \Delta U_2$

37. A Carnot engine works between 600 K and 300 K. The efficiency of the engine is
(a) 50% (b) 70% (c) 20% (d) 80%

38. Air in a cylinder is suddenly compressed by a piston which is then maintained at the same position. As the time passes pressure of the gas
(a) increases
(b) decreases
(c) remains the same
(d) may increase or decrease depending on the nature of the gas

39. A cycle pump becomes hot near the nozzle after a few quick strokes even if they are smooth because
(a) the volume of air decreases
(b) the number of air molecules increases
(c) the compression is adiabatic
(d) collision between air particles increases

40. A monoatomic ideal gas, initially at temperature T_1, is enclosed in a cylinder fitted with a frictionless piston. The gas is allowed to expand adiabatically to a temperature T_2 by releasing the piston suddenly. If L_1 and L_2 are the lengths of gas column before and after expansion respectively, then $\frac{T_1}{T_2}$ is given by
(a) $\left(\frac{L_1}{L_2}\right)^{2/3}$
(b) $\frac{L_1}{L_2}$
(c) $\frac{L_2}{L_1}$
(d) $\left(\frac{L_2}{L_1}\right)^{2/3}$

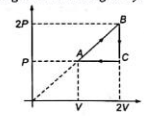
41. In an adiabatic change, the pressure P and temperature T of a diatomic gas are related by the relation $P \propto T^{\alpha}$ where α equals
(a) 1.67
(b) 0.4
(c) 0.6
(d) 3.5

42. A diatomic gas obeys the law PV^x = constant. For what value of x, it has negative molar specific heat ?
(a) $x > 1.4$
(b) $x < 1.4$
(c) $1 < x < 1.4$
(d) $0 < x < 1$

43. The molar specific heat at constant volume of gas mixture is $\frac{13R}{6}$. The gas mixture consists of
(a) 2 moles of O_2 and 4 moles of H_2
(b) 2 moles of O_2 and 4 moles of argon
(c) 2 moles of argon and 4 moles of O_2
(d) 2 moles of CO_2 and 4 moles of argon

Passage : (Q. 44 and 45)

One mole of a monoatomic ideal gas is taken along the cycle ABCA as shown in the diagram.

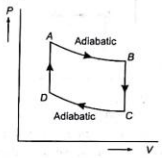


44. The net heat absorbed by the gas in the given cycle is
(a) PV
(b) $\frac{PV}{2}$
(c) $2PV$
(d) $4PV$

45. The ratio of specific heat in the process CA to the specific heat in the process BC is
(a) 2
(b) $\frac{5}{3}$
(c) 4
(d) None of these

Passage : (Q. 46 - Q. 48)

One mole of a monoatomic ideal gas is taken through the cycle ABCDA as shown in the figure. $T_A = 1000\,K$ and $2P_A = 3P_B = 6P_C$.



$$\left[Assume \left(\frac{2}{3}\right)^{0.4} = 0.85 \text{ and } R = \frac{25}{3}\,JK^{-1}\,mol^{-1}\right]$$

46. The temperature at B is
(a) 350 K
(b) 1175 K
(c) 850 K
(d) 577 K

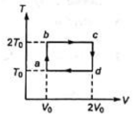
47. Work done by the gas in the process $A \rightarrow B$ is
(a) 5312.5 J
(b) 1875 J
(c) zero
(d) 8854 J

48. Heat lost by the gas in the process $B \rightarrow C$ is
(a) 5312.5 J
(b) 1875 J
(c) zero
(d) 8854 J

49. An ideal gas is taken from the state A (pressure P, volume V) to the state B (pressure $\frac{P}{2}$, volume $2V$) along a straight line path in the P-V diagram. Select the correct statement(s) from the following.
(a) The work done by the gas in the process A to B is negative
(b) In the T-V diagram, the path AB becomes a part of a parabola
(c) In the P-T diagram, the path AB becomes a part of a hyperbola
(d) In going from A to B, the temperature T of the gas first increases to a maximum value and then decreases

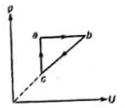
50. In the process PV^2 = constant, if temperature of gas is increased
(a) change in internal energy of gas is positive
(b) work done by gas is positive
(c) heat is given to the gas
(d) heat is taken out from the gas

51. T-V diagram of two moles of a monoatomic gas is as shown in figure :



For the process $abcda$ choose the correct options given below :
(a) $\Delta U = 0$
(b) work done by gas > 0
(c) heat given to the gas is $4RT_0$
(d) heat given to the gas is $2RT_0$

52. Density (ρ) versus internal energy (U) graph of a gas is as shown in figure. Choose the correct options



(a) $Q_{bc} = 0$
(b) $W_{bc} = 0$
(c) $W_{ca} < 0$
(d) $Q_{ab} > 0$

Here W is work done by gas and Q is heat given to the gas.

53. Temperature of a monoatomic gas is increased from T_0 to $2T_0$ in three different processes : isochoric, isobaric and adiabatic. Heat given to the gas in these three processes are Q_1, Q_2 and Q_3 respectively. Then choose the correct option.
(a) $Q_1 > Q_3 > Q_2$
(b) $Q_2 > Q_1 > Q_3$
(c) $Q_1 > Q_2 > Q_3$
(d) $Q_2 > Q_3 > Q_1$

For **JEE Advanced**

Assertion and Reason

Directions : *Choose the correct option.*
*(a) If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*
*(b) If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*
*(c) If **Assertion** is true, but the **Reason** is false.*
*(d) If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** In adiabatic expansion, temperature of gas always decreases.
Reason : In adiabatic process exchange of heat is zero.

2. **Assertion :** In a thermodynamic process, initial volume of gas is equal to final volume of gas. Work done by gas in this process should be zero.
Reason : Work done by gas in isochoric process is zero.

3. **Assertion :** First law of thermodynamics can be applied for ideal gases only.
Reason : First law is simply, law of conservation of energy.

4. **Assertion :** When ice melts, work is done by atmosphere on (ice + water) system.
Reason : On melting of ice volume decreases.

5. **Assertion :** Between two thermodynamic states, the value of $(Q - W)$ is constant for any process.
Reason : Q and W are path functions.

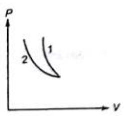
6. **Assertion :** Efficiency of a heat engine can't be greater than efficiency of Carnot's engine.
Reason : Efficiency of any engine is never 100%.

7. **Assertion :** In the process PT = constant, if temperature of gas is increased work done by the gas is positive.
Reason : For the given process $V \propto T$

8. **Assertion :** In free expansion of a gas inside an adiabatic chamber Q, W and ΔU all are zero.
Reason : In such an expansion $P \propto \frac{1}{V}$.

9. **Assertion :** For an ideal gas in a cyclic process and in an isothermal process change in internal energy is zero.
Reason : In both processes there is no change in temperature.

10. **Assertion :** Isothermal and adiabatic, two processes are shown on P-V diagram. Process-1 is adiabatic and process-2 is isothermal.
Reason : At a given point, slope of adiabatic process $= \gamma \times$ slope of isothermal process.

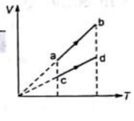


Match the Columns

1. Temperature of 2 moles of a monoatomic gas is increased from T to $2T$. Match the following two columns.

Column I	Column II
(a) Work done by gas in isobaric process	(p) $3RT$
(b) Change in internal energy in isobaric process	(q) $4RT$
(c) Work done by gas in adiabatic process	(r) $2RT$
(d) Change in internal energy in an adiabatic process	(s) None of these

2. For V-T diagrams of two processes $a - b$ and $c - d$ shown in figure for same gas match the following two columns.



Column I	Column II
(a) Work-done	(p) is more in process ab
(b) Change in internal energy	(q) is more in process ca
(c) Heat exchange	(r) is same in both processes
(d) Molar heat capacity	(s) Can't say anything

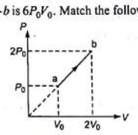
3. Temperature of a monoatomic gas is increased by ΔT in process P^2V = constant. Match the following two columns.

Column I	Column II
(a) Work done by gas	(p) $2nR\Delta T$
(b) Change in internal energy of gas	(q) $5nR\Delta T$
(c) Heat taken by the gas	(r) $3nR\Delta T$
(d) Work done on the gas	(s) None of these

4. Match the following two columns.

Column I	Column II
(a) Isobaric expansion	(p) $W > \Delta U$
(b) Isochoric cooling	(q) $W < \Delta U$
(c) Adiabatic expansion	(r) $Q = \Delta U$
(d) Isothermal expansion	(s) $Q < \Delta U$

5. Heat taken by a gas in process a-b is $6P_0V_0$. Match the following columns.



Column I	Column II
(a) W_{ab}	(p) $2P_0V_0$
(b) ΔU_{ab}	(q) $4P_0V_0$
(c) Molar heat capacity in given process	(r) $2R$
(d) C_V of gas	(s) None of these

Subjective Questions

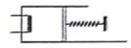
1. A cylinder of ideal gas is closed by an 8 kg movable piston of area $60\,\text{cm}^2$. The atmospheric pressure is 100 kPa. When the gas is heated from $30°\text{C}$ to $100°\text{C}$ the piston rises 20 cm. The piston is then fastened in the place and the gas is cooled back to $30°\text{C}$. If ΔQ_1 is the heat added to the gas during heating and ΔQ_2 is the heat lost during cooling, find the difference.

2. Three moles of an ideal gas $\left(C_P = \frac{7}{2}R\right)$ at pressure P_0 and temperature T_0 is isothermally expanded to twice its initial volume. It is then compressed at a constant pressure to its original volume.
(a) Sketch P-V and P-T diagram for complete process.
(b) Calculate net work done by the gas.
(c) Calculate net heat supplied to the gas during complete process.
(Write your answer in terms of gas constant $= R$)

3. Two moles of a gas $(\gamma = 5/3)$ are initially at temperature $27°\text{C}$ and occupy a volume of 20 litres. The gas is first expanded at constant pressure until the volume is doubled. Then it is subjected to an adiabatic change until the temperature returns to its initial value.

(a) Sketch the process on a P-V diagram.
(b) What are final volume and pressure of the gas?
(c) What is the work done by the gas?

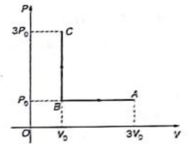
4. A ideal monoatomic gas is confined by a spring loaded massless piston of cross-section $8.0 \times 10^{-3}\,\text{m}^2$. Initially the gas is at 300 K and occupies a volume of $2.4 \times 10^{-3}\,\text{m}^3$ and the spring is in its relaxed state. The gas is heated by an electric heater until the piston moves out slowly without friction by 0.1m. Calculate :



(a) the final temperature of the gas and
(b) the heat supplied by the heater.
The force constant of the spring is 8000 N/m, atomspheric pressure is $1.0 \times 10^5\,\text{N/m}^2$. The cylinder and the piston are thermally insulated.

5. An ideal diatomic gas $\left(\gamma = \frac{7}{5}\right)$ undergoes a process in which its internal energy relates to the volume as $U = \alpha\sqrt{V}$, where α is a constant.
(a) Find the work performed by the gas to increase its internal energy by 100 J.
(b) Find the molar specific heat of the gas.

6. P-V diagram of an ideal gas for a process ABC is as shown in the figure.
(a) Find total heat absorbed or released by the gas during the process ABC.
(b) Change in internal energy of the gas during the process ABC.
(c) Plot pressure *versus* density graph of the gas for the process ABC.



7. For an ideal gas the molar heat capacity varies as $C = C_V + 3aT^2$. Find the equation of the process in the variables (T, V) where a is a constant.

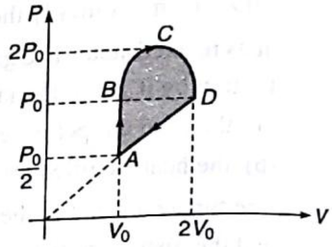
8. One mole of an ideal monoatomic gas undergoes the process $P = \alpha T^{1/2}$, where α is a constant.
(a) Find the work done by the gas if its temperature increases by 50 K.
(b) Also, find the molar specific heat of the gas.

9. One mole of a gas is put under a weightless piston of a vertical cylinder at temperature T. The space over the piston opens into atmosphere. Initially piston was in equilibrium. How much work should be performed by some external force to increase isothermally the volume under the piston to twice the volume? (Neglect friction of piston).

10. An ideal monoatomic gas undergoes a process where its pressure is inversely proportional to its temperature.
(a) Calculate the molar specific heat for the process.
(b) Find the work done by two moles of gas if the temperature changes from T_1 to T_2.

11. The volume of one mole of an ideal gas with the adiabatic exponent γ is changed according to the relation $V = \frac{a}{T}$, where a is a constant. Find the amount of heat absorbed by the gas in the process, if the temperature is increased by ΔT.

12. Two moles of a monoatomic ideal gas undergo a cyclic process $ABCDA$ as shown in figure. BCD is a semicircle. Find the efficiency of the cycle.



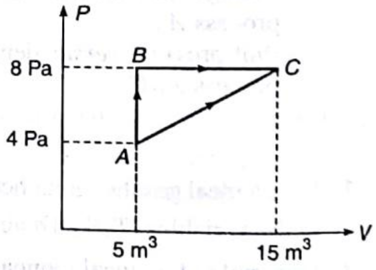
13. Pressure P, volume V and temperature T for a certain gas are related by

$$P = \frac{\alpha T - \beta T^2}{V}$$

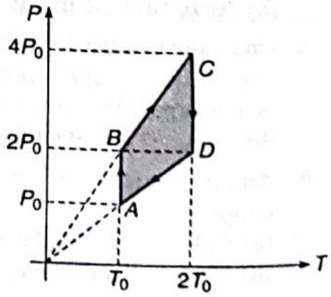
where α and β are constants. Find the work done by the gas if the temperature changes from T_1 to T_2 while the pressure remains the constant.

14. An ideal gas has a specific heat at constant pressure $C_P = \frac{5R}{2}$. The gas is kept in a closed vessel of volume V_0 at temperature T_0 and pressure P_0. An amount of $10\,P_0V_0$ of heat is supplied to the gas.
(a) Calculate the final pressure and temperature of the gas.
(b) Show the process on P-V diagram.

15. In the given graph an ideal gas changes its state from state A to state C by two paths ABC and AC.
(a) Find the path along which work done is less.
(b) The internal energy of gas at A is 10 J and the amount of heat supplied in path AC is 200 J. Calculate the internal energy of gas at C.
(c) The internal energy of gas at state B is 20 J. Find the amount of heat supplied to the gas to go from A to B.



16. Pressure versus temperature (P-T) graph of n moles of an ideal gas is shown in figure. Plot the corresponding:
(a) density versus volume (ρ-V) graph,
(b) pressure versus volume (P-V) graph and
(c) density versus pressure (ρ-P) graph.



ANSWERS

Introductory Exercise 18.1

1. (a) -6.8×10^4 J (b) 1.78×10^5 J, out of gas (c) No

2. (a) Positive (b) Into the system (c) In loop 1, into the system, In loop 2, out of the system

3. (a) No (b) No (c) No **4.** 2.67×10^{-2} mol **5.** 11609.99762 J

Introductory Exercise 18.2

1. (a) 316 K (b) 310 K **3.** (a) -150 J (b) -400 J

4. (a) 241 K (b) 383 K (c) 1770 J (d) 1770 J **5.** (a) 200 J (b) 0.80°C

7.

Process	Q	W	ΔU
BC	–	0	–
CA	–	–	–
AB	+	+	+

8. 166 J **9.** 1.18 MJ

Introductory Exercise 18.3

1. 1.4 **2.** 0.6 kJ, 1.0 kJ, 1.6 **3.** 1.68×10^3 W

For JEE Main

Subjective Questions

1. 327 J **2.** (a) 316 K (b) 310 K **3.** $\Delta U=\dfrac{nR\Delta T}{\gamma-1}$ for all processes **4.** 10^4 J

5. $C=3.1R$ **6.** 2.49 kJ **7.** 113°C **8.** Into gas **9.** (33599.98) J

10. 2.75×10^4 J **11.** (a) 169 J (b) 2087 J

12. (a) -1.15×10^5 J (b) 2.55×10^5 J out of gas (c) No **13.** (a) 765 J (b) 10.82%

14. (a) P_0V_0 (b) $\dfrac{5}{2}P_0V_0$, $3P_0V_0$ (c) $\dfrac{P_0V_0}{2}$ (d) $\dfrac{25P_0V_0}{8R}$ **15.** (a) 1153 J (b) 1153 J (c) zero

16. 18.18% **17.** (a) absorbs (b) 7200 J (c) liberates 7200 J

18. (a) positive (b) I→ positive, II→ negative (c) into the system
(d) I→ into the system II→ out of the system.

19. $T_A=273$ K, $T_B=546$ K, $V_C=44.8$ m^3 **20.** (a) 0.50 J (b) 0.50 J

21. (a) $W_{12}=\dfrac{-P_0M}{\rho_0}\ln(2)$, $W_{23}=\dfrac{P_0M}{\rho_0}$, $W_{31}=0$ (b) $\dfrac{P_0M}{\rho_0}\left(\dfrac{3}{2}+\ln 2\right)$ (c) $\dfrac{2}{5}(1-\ln 2)$

22. (b) -2460 J **23.** 10.82%

Objective Questions

1.(b)	2.(d)	3.(d)	4.(c)	5.(a,b)	6.(c)	7.(a)	8.(c)	9.(c)	10.(c)
11(c)	12.(a)	13.(a)	14.(a)	15.(a)	16.(c)	17.(b)	18.(d)	19.(a)	20.(a)
21.(a)	22.(b)	23.(d)	24.(c)	25.(a)	26.(a)	27.(c)	28.(b)	29.(c)	30.(c)
31.(d)	32.(a)	33.(b)	34.(b)	35.(d)	36.(d)	37.(a)	38.(b)	39.(c)	40.(d)
41.(d)	42.(c)	43.(c)	44.(b)	45.(b)	46.(c)	47.(b)	48.(a)	49.(b,d)	50.(a,c)
51.(a,b)	52.(c,d)	53.(b)							

For JEE Advanced

Assertion and Reason

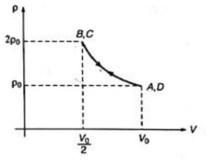
1. (b) 2. (d) 3. (d) 4. (a) 5. (b) 6. (d) 7. (c) 8. (b) 9. (a) 10. (a or b)

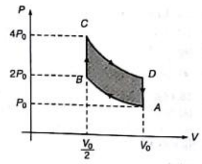
Match the Columns

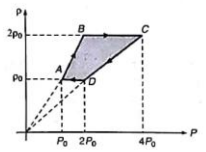
1. (a) → r (b) → p (c) → s (d) → p
2. (a) → s (b) → s (c) → s (d) → r
3. (a) → p (b) → s (c) → s (d) → s
4. (a) → q (b) → p,r (c) → p (d) → p
5. (a) → s (b) → s (c) → r (d) → s

Subjective Questions

1. 136 J 2. (b) $3RT_0 \ln(2) - \frac{3}{2}RT_0$ (c) $3RT_0 \ln(2) - \frac{21}{4}RT_0$
3. (b) 113.1 L, 0.44×10^5 N/m^2 (c) 12479 J 4. (a) 800 K (b) 720 J
5. (a) 80 J (b) $\frac{9R}{2}$ 6. (a) $Q_{ABC} = -2P_0V_0$ (b) $\Delta U_{ABC} = 0$ 7. $Ve^{-\left(\frac{3a}{2R}\right)V^2}$ = constant
8. (a) 207.75 J (b) $2R$ 9. $RT(1 - \ln 2)$ 10. $\frac{7R}{2}$, $4R(T_2 - T_1)$
11. $\frac{(2-\gamma)R\Delta T}{(\gamma - 1)}$ 12. 25.8% 13. $\alpha(T_2 - T_1) - \beta(T_2^2 - T_1^2)$ 14. (a) $\frac{23}{3}P_0, \frac{23}{3}T_0$
15. (a) AC (b) 150 J (c) 10 J
16.







19

CALORIMETRY & HEAT TRANSFER

Chapter Contents

19.1 Specific Heat

When heat energy flows into a substance, the temperature of the substance usually rises. An exception occurs during a change in phase, as when water freezes or evaporates. The amount of heat required to produce the same temperature increase for a given amount of substance varies from one substance to another. The relationship between heat exchanged and the corresponding temperature change is characterized by the **specific heat** c of a substance. If the temperature of a substance of mass m changes from T to $T + dT$ when it exchanges an amount of heat dQ with its surroundings then its specific heat is

$$c = \frac{1}{m} \cdot \frac{dQ}{dT} \qquad \text{...(i)}$$

The SI unit of specific heat is J/kg-K. Because heat is so frequently measured in calories, the unit cal/g-°C is also used quite often. The specific heat capacity of water is approximately 1 cal/g-°C.

From Eq. (i), we can define the specific heat of a substance as "the amount of energy needed to raise the temperature of unit mass of that substance by 1°C (or 1 K)". A closely related quantity is the **Molar heat capacity** C. It is defined as,

$$C = \frac{1}{n} \cdot \frac{dQ}{dT} \qquad \text{...(ii)}$$

where n is the number of moles of the substance. If M is the molecular mass of the substance, then $n = \dfrac{m}{M}$

where m is the mass of the substance and,

$$C = \frac{M}{m} \cdot \frac{dQ}{dT} \qquad \text{...(iii)}$$

The SI unit of molar heat capacity C is, J/mol-K and it can be defined as "the amount of energy needed to raise the temperature of one mole of a substance by 1°C (or 1 K) sometimes the product of mc is also written as C, simply the **heat capacity**, which is defined as the energy needed to raise the temperature of the whole substance by 1°C (or K)".

Thus,
$$C = mc = \frac{dQ}{dT} \qquad \text{...(iv)}$$

The SI units of C are J/K.

Note **1.** *In general, if c varies with temperature over the interval, then the corresponding expression for Q is,*

$$Q = m\int_{T_1}^{T_2} c \cdot dT$$

2. *The specific heat of water is much larger than that of most other substances. Consequently, for the same amount of added heat, the temperature change of a given mass of water is generally less than that for the same mass of another substance. For this reason a large body of water moderates the climate of nearby land. In the winter the water cools off more slowly than the surrounding land and tends to warm the land. In the summer, the opposite effect occurs, as the water heats up more slowly than the land.*

Sample Example 19.1 *When 400 J of heat are added to a 0.1 kg sample of metal, its temperature increases by 20°C. What is the specific heat of the metal?*

Solution Using
$$c = \frac{1}{m} \cdot \frac{\Delta Q}{\Delta T}$$

we have
$$c = \left(\frac{1}{0.1}\right)\left(\frac{400}{20}\right)$$
$$= 200 \text{ J/kg-}^\circ\text{C}$$

19.2 Phase Changes, Latent Heat

Suppose that we slowly heat a cube of ice whose temperature is below 0°C at atmospheric pressure, what changes do we observe in the ice? Initially we find that its temperature increases according to equation

$$Q = mc\,(T_2 - T_1)$$

Once 0°C is reached, the additional heat does not increase the temperature of the ice. Instead, the ice melts and temperature remains at 0°C. The temperature of the water then starts to rise and eventually reaches 100°C, whereupon the water vaporizes into steam at this same temperature.

During phase transitions (solid to liquid or liquid to gas) the added heat causes a change in the positions of the molecules relative to one another, without affecting the temperature.

The heat necessary to change a unit mass of a substance from one phase to another is called the **latent heat (L)**. Thus, the amount of heat required for melting and vaporizing a substance of mass m are given by,

$$Q = mL \qquad \text{...(i)}$$

For a solid-liquid transition, the latent heat is known as the **latent heat of fusion (L_f)** and for the liquid-gas transition, it is known as the **latent heat of vaporization (L_v)**.

For water at 1 atmosphere latent heat of fusion is 80.0 cal/g. This simply means 80.0 cal of heat are required to melt 1.0 g of water or 80.0 cal heat is liberated when 1.0 g of water freezes at 0°C. Similarly latent heat of vaporization for water at 1 atmosphere is 539 cal/g.

Figure shows how the temperature varies when we add heat continuously to a specimen of ice with an initial temperature below 0°C. Suppose we have taken 1 g of ice at −20°C specific heat of ice is 0.53 cal/g-°C.

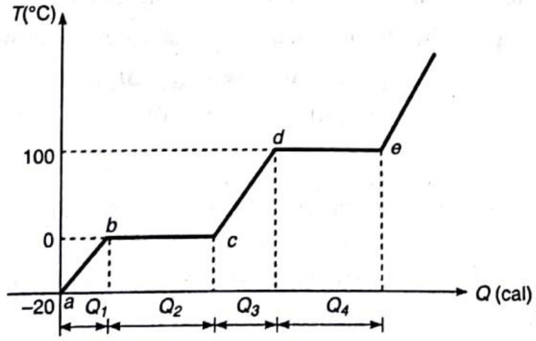


Fig. 19.1

In the figure :

***a* to *b* :** Temperature of ice increases until it reaches its melting point 0°C.

$$Q_1 = mc_{\text{ice}}\,[0 - (-20)] = (1)\,(0.53)\,(20) = 10.6 \text{ cal}$$

b to c : Temperature remains constant until all the ice has melted.

$$Q_2 = mL_f = (1)(80) = 80 \text{ cal}$$

c to d : Temperature of water again rises until it reaches its boiling point 100°C.

$$Q_3 = mc_{\text{water}}[100 - 0] = (1)(1.0)(100) = 100 \text{ cal}$$

d to e : Temperature is again constant until all the water is transformed into the vapour phase.

$$Q_4 = mL_v = (1)(539) = 539 \text{ cal}$$

Thus, the net heat required to convert 1 g of ice at $-20°$C into steam at 100°C is

$$Q = Q_1 + Q_2 + Q_3 + Q_4 = 729.6 \text{ cal}$$

Sample Example 19.2 *How much heat is required to convert 8.0 g of ice at – 15°C to steam at 100°C? (Given $c_{ice} = 0.53$ cal/g-°C, $L_f = 80$ cal/g and $L_v = 539$ cal/g, and $c_{water} = 1$ cal/g-°C)*

Solution

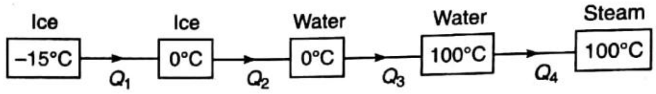


Fig. 19.2

$$Q_1 = mc_{\text{ice}}(T_f - T_i) = (8.0)(0.53)[0 - (-15)] = 63.6 \text{ cal}$$
$$Q_2 = mL_f = (8)(80) = 640 \text{ cal}$$
$$Q_3 = mc_{\text{water}}(T_f - T_i) = (8.0)(1.0)[100 - 0] = 800 \text{ cal}$$
$$Q_4 = mL_v = (8.0)(539) = 4312 \text{ cal}$$

$\therefore$ Net heat required

$$Q = Q_1 + Q_2 + Q_3 + Q_4 = 5815.6 \text{ cal}$$

Sample Example 19.3 *10 g of water at 70°C is mixed with 5 g of water at 30°C. Find the temperature of the mixture in equilibrium.*

Solution Let $t°$C be the temperature of the mixture. From energy conservation,

Heat given by 10 g of water = Heat taken by 5 g of water

or
$$m_1 c_{\text{water}} |\Delta t_1| = m_2 c_{\text{water}} |\Delta t_2|$$

$\therefore$
$$(10)(70 - t) = 5(t - 30)$$

$\therefore$
$$t = 56.67°\text{C}$$

Sample Example 19.4 *In a container of negligible mass 30 g of steam at 100°C is added to 200 g of water that has a temperature of 40°C. If no heat is lost to the surroundings, what is the final temperature of the system? Also find masses of water and steam in equilibrium. Take $L_v = 539$ cal/g and $c_{water} = 1$ cal/g-°C.*

Solution Let Q be the heat required to convert 200 g of water at 40°C into 100°C, then

$$Q = mc\Delta T$$
$$= (200)(1.0)(100 - 40)$$
$$= 12{,}000 \text{ cal}$$

Now, suppose m_0 mass of steam converts into water to liberate this much amount of heat, then

$$m_0 = \frac{Q}{L} = \frac{12000}{539} = 22.26\text{ g}$$

Since it is less than 30 g, the temperature of the mixture is 100°C. **Ans.**

Mass of steam in the mixture $= 30 - 22.26 = 7.74$ g **Ans.**

and mass of water in the mixture $= 200 + 22.26 = 222.26$ g **Ans.**

Introductory Exercise 19.1

Take $c_{ice} = 0.53$ cal/g-°C, $c_{water} = 1.0$ cal/g-°C, $(L_f)_{water} = 80$ cal/g and $(L_v)_{water} = 529$ cal/g unless given in the question.

1. In a container of negligible mass 140 g of ice initially at –15° C is added to 200 g of water that has a temperature of 40°C. If no heat is lost to the surroundings, what is the final temperature of the system and masses of water and ice in mixture?

2. The temperatures of equal masses of three different liquids A, B and C are 12°C, 19°C and 28°C respectively. The temperature when A and B are mixed is 16°C and when B and C are mixed is 23°C. What would be the temperature when A and C are mixed?

3. Equal masses of ice (at 0°C) and water are in contact. Find the temperature of water needed to just melt the complete ice.

4. A closely thermally insulated vessel contains 100 g of water at 0°C. If the air from this vessel is rapidly pumped out, intensive evaporation will produce cooling and as a result of this water freeze. How much ice will be formed by this method? If latent heat of fusion is 80 cal/g and of evaporation 560 cal/g.
[Hint : If m g ice is formed, $mL_f = (100 - m) L_v$]

5. A nuclear power plant generates 500 MW of waste heat that must be carried away by water pumped from a lake. If the water temperature is to rise by 10°C, what is the required flow rate in kg/s?

19.3 Heat Transfer

Heat can be transferred from one place to the other by any of three possible ways : **conduction, convection** and **radiation.** In the first two processes, a medium is necessary for the heat transfer. Radiation, however, does no have this restriction. This is also the fastest mode of heat transfer, in which heat is transferred from one place to the other in the form of electromagnetic radiation. In competition examinations problems are asked only in first and last. So, we will discuss conduction and radiation in detail.

Conduction

Figure shows a rod whose ends are in thermal contact with a hot reservoir at temperature T_1 and a cold reservoir at temperature T_2. The sides of the rod are covered with insulating medium, so the transport of heat is along the rod, not through the sides. The molecules at the hot reservoir have greater vibrational energy. This energy is transferred by collisions to the atoms at the end face of the rod. These atoms in turn transfer energy to their neighbours further along the rod. Such transfer of heat through a substance in which heat is transported without direct mass transport is called conduction.

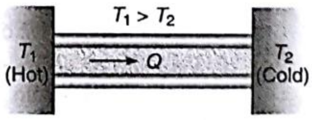


Fig. 19.3

Most metals use another, more effective mechanism to conduct heat. The free electrons, which move throughout the metal can rapidly carry energy from the hotter to cooler regions, so metals are generally good conductors of heat. The presence of 'free' electrons also causes most metals to be good electrical conductors. A metal rod at 5°C feels colder than a piece of wood at 5°C because heat can flow more easily from your hand into the metal.

Heat transfer occurs only between regions that are at different temperatures, and the rate of heat flow is $\frac{dQ}{dt}$. This rate is also called the **heat current**, denoted by H. Experiments show that the heat current is proportional to the cross-section area A of the rod and to the temperature gradient $\frac{dT}{dx}$, which is the rate of change of temperature with distance along the bar. In general

$$H = \frac{dQ}{dt} = -kA\frac{dT}{dx} \qquad \ldots\text{(i)}$$

The negative sign is used to make $\frac{dQ}{dt}$ a positive quantity since $\frac{dT}{dx}$ is negative. The constant k, called the **thermal conductivity** is a measure of the ability of a material to conduct heat.

A substance with a large thermal conductivity k is a good heat conductor. The value of k depends on the temperature, increasing slightly with increasing temperature, but k can be taken to be practically constant throughout a substance if the temperature difference between its ends is not too great.

Let us apply Eq. (i) to a rod of length L and constant cross sectional area A in which a steady state has been reached. In a steady state the temperature at each point is constant in time. Hence,

$$-\frac{dT}{dx} = T_1 - T_2$$

Therefore, the heat ΔQ transferred in time Δt is

$$\Delta Q = kA\left(\frac{T_1 - T_2}{L}\right)\Delta t \qquad \ldots\text{(ii)}$$

Thermal Resistance (R)

Eq. (ii) in differential form can be written as

$$\frac{dQ}{dt} = H = \frac{\Delta T}{l/kA} = \frac{\Delta T}{R} \qquad \ldots\text{(iii)}$$

Here, ΔT = temperature difference (TD) and

$$R = \frac{l}{kA} = \text{thermal resistance of the rod.}$$

• *Important Points in* CONDUCTION

1. Consider a section ab of a rod as shown in figure. Suppose Q_1 heat enters into the section at 'a' and Q_2 leaves at 'b', then $Q_2 < Q_1$. Part of the energy $Q_1 - Q_2$ is utilized in raising the temperature of section ab and the remaining is lost to atmosphere through ab. If heat is continuously supplied from the left end of the rod, a stage comes when temperature of the section becomes constant. In that case, $Q_1 = Q_2$ if rod is insulated from the surroundings (or loss through ab is zero). This is called the **steady state** condition. Thus, in steady state temperature of different sections of the rod becomes constant (but not same).

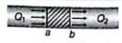
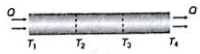


Fig. 19.4

Hence, in the figure :



Insulated rod in steady state

Fig. 19.5

$$T_1 = \text{constant}, \quad T_2 = \text{constant etc.}$$

and

$$T_1 > T_2 > T_3 > T_4$$

Now, a natural question arises, why the temperature of whole rod not becomes equal when heat is being continuously supplied? The answer is : there must be a temperature difference in the rod for the heat flow, same as we require a potential difference across a resistance for the current flow through it.

In steady state, the temperature varies linearly with distance along the rod if it is insulated.

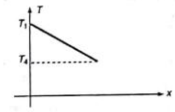


Fig. 19.6

2. Comparing equation number (iii), *i.e.*, heat current

$$H = \frac{dQ}{dt} = \frac{\Delta T}{R} \qquad \left(\text{where } R = \frac{l}{kA}\right)$$

with the equation, of current flow through a resistance,

$$i = \frac{dq}{dt} = \frac{\Delta V}{R} \qquad \left(\text{where } R = \frac{l}{\sigma A}\right)$$

We find the following similarities in heat flow through a rod and current flow through a resistance.

Table 19.1

Heat flow through a conducting rod	Current flow through a resistance
Heat current $H = \frac{dQ}{dt}$ = rate of heat flow	Electric current $i = \frac{dq}{dt}$ = rate of charge flow
$H = \frac{\Delta T}{R} = \frac{\text{TD}}{R}$	$i = \frac{\Delta V}{R} = \frac{\text{PD}}{R}$
$R = \frac{l}{kA}$	$R = \frac{l}{\sigma A}$
k = thermal conductivity	σ = electrical conductivity

From the above table it is evident that flow of heat through rods in series and parallel is analogous to the flow of current through resistances in series and parallel. This analogy is of great importance in solving complicated problems of heat conduction.

Convection

Although conduction does occur in liquids and gases also, heat is transported in these media mostly by convection. In this process, the actual motion of the material is responsible for the heat transfer. Familiar examples include hot-air and hot-water home heating systems, the cooling system of an automobile engine and the flow of blood in the body.

You probably have warmed your hands by holding them over an open flame. In this situation, the air directly above the flame is heated and expands. As a result, the density of this air decreases and then air rises. When the movement results from differences in density, as with air around fire, it is referred to as **natural convection.** Air flow at a beach is an example of natural convection. When the heated substance is forced to move by a fan or pump, the process in called **forced convection.** If it were not for convection currents, it would be very difficult to boil water. As water is heated in a kettle, the heated water expands and rises to the top because its density is lowered. At the same time, the denser, cool water at the surface sinks to the bottom of the kettle and is heated. Heating a room by a radiator is an example of forced convection.

It is possible to write an equation for the thermal energy transported by convection and define a coefficient of convection, but the analysis of practical problems is very difficult and will not be treated here. To some approximation, the heat transferred from a body to its surroundings is proportional to the area of the body and to the difference in temperature between the body and the surrounding fluid.

Radiation

The third means of energy transfer is radiation which does not require a medium. The best known example of this process is the radiation from sun. All objects radiate energy continuously in the form of electromagnetic waves. The rate at which an object radiates energy is proportional to the fourth power of its absolute temperature. This is known as the **Stefan's law** and is expressed in equation form as

$$P = \sigma A e T^4$$

Here P is the power in watts (J/s) radiated by the object, A is the surface area in m^2, e is a fraction between 0 and 1 called the **emissivity** of the object and σ is a universal constant called **Stefan's constant,** which has the value

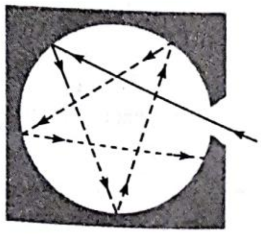
$$\sigma = 5.67 \times 10^{-8}\ \mathrm{W/m^2\text{-}K^4}$$

Now, let us define few terms before studying the other topics.

(a) Perfectly black body

A body that absorbs all the radiation incident upon it and has an emissivity equal to 1 is called a perfectly black body. A black body is also an ideal radiator. It implies that if a black body and an identical another body are kept at the same temperature, then the black body will radiate maximum power as is obvious from equation $P = eA\sigma T^4$ also. Because $e = 1$ for a perfectly black body while for any other body $e < 1$.

Materials like black velvet or lamp black come close to being ideal black bodies, but the best practical realization of an ideal black body is a small hole leading into a cavity, as this absorbs 98% of the radiation incident on them.



Cavity approximating an ideal black body. Radiation entering the cavity has little chance of leaving before it is completely absorbed.

Fig. 19.7

(b) Absorptive power 'a'

"It is defined as the ratio of the radiant energy absorbed by it in a given time to the total radiant energy incident on it in the same interval of time."

$$a = \frac{\text{energy absorbed}}{\text{energy incident}}$$

As a perfectly black body absorbs all radiations incident on it, the absorptive power of a perfectly black body is maximum and unity.

(c) Spectral absorptive power 'a_λ'

The absorptive power 'a' refers to radiations of all wavelengths (or the total energy) while the spectral absorptive power is the ratio of radiant energy absorbed by a surface to the radiant energy incident on it for a particular wavelength λ. It may have different values for different wavelengths for a given surface. Let us take an example, suppose $a = 0.6$, $a_\lambda = 0.4$ for 1000 Å and $a_\lambda = 0.7$ for 2000 Å for a given surface. Then it means that this surface will absorbs only 60% of the total radiant energy incident on it. Similarly it absorbs 40% of the energy incident on it corresponding to 1000 Å and 70% corresponding to 2000 Å. The spectral absorptive power a_λ is related to absorptive power a through the relation

$$a = \int_0^\infty a_\lambda \, d\lambda$$

(d) Emissive power 'e'

(Don't confuse it with the emissivity e which is different from it, although both have the same symbols e).

"For a given surface it is defined as the radiant energy emitted per second per unit area of the surface." It has the units of W/m^2 or $J/s\text{-}m^2$. For a black body $e = \sigma T^4$.

(e) Spectral emissive power 'e_λ'

"It is emissive power for a particular wavelength λ." Thus,

$$e = \int_0^\infty e_\lambda \, d\lambda$$

Kirchhoff's law : "According to this law the ratio of emissive power to absorptive power is same for all surfaces at the same temperature."

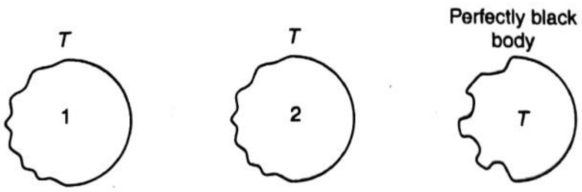


Fig. 19.8

Hence, $$\frac{e_1}{a_1} = \frac{e_2}{a_2} = \left(\frac{e}{a}\right)_{\text{Perfectly black body}}$$

but $$(a)_{\text{black body}} = 1$$

and $$(e)_{\text{black body}} = E \quad \text{(say)}$$

Then, $$\left(\frac{e}{a}\right)_{\text{for any surface}} = \text{constant} = E$$

Similarly, for a particular wavelength λ,

$$\left(\frac{e_\lambda}{a_\lambda}\right)_{\text{for any body}} = E_\lambda$$

Here, E = emissive power of black body at temperature T

$= \sigma T^4$

From the above expression, we can see that

$$e_\lambda \propto a_\lambda$$

i.e., **good absorbers for a particular wavelength are also good emitters of the same wavelength.**

Cooling by radiation

Consider a hot body at temperature T placed in an environment at a lower temperature T_0. The body emits more radiation than it absorbs and cools down while the surroundings absorb radiation from the body and warm up. The body is losing energy by emitting radiations at a rate.

$$P_1 = eA\sigma T^4$$

and is receiving energy by absorbing radiations at a rate

$$P_2 = aA\sigma T_0^4$$

Here, 'a' is a pure number between 0 and 1 indicating the relative ability of the surface to absorb radiation from its surroundings. Note that this 'a' is different from the absorptive power 'a'. In thermal equilibrium, both the body and the surrounding have the same temperature (say T_c) and,

$$P_1 = P_2$$

or $$eA\sigma T_c^4 = aA\sigma T_c^4$$

or $$e = a$$

Thus, when $T > T_0$, the net rate of heat transfer from the body to the surroundings is,

$$\frac{dQ}{dt} = eA\sigma (T^4 - T_0^4)$$

or $$mc\left(-\frac{dT}{dt}\right) = eA\sigma (T^4 - T_0^4)$$

$\Rightarrow$ Rate of cooling

$$\left(-\frac{dT}{dt}\right) = \frac{eA\sigma}{mc}(T^4 - T_0^4)$$

or $$-\frac{dT}{dt} \propto (T^4 - T_0^4)$$

Newton's law of cooling

According to this law, if the temperature T of the body is not very different from that of the surroundings T_0, then rate of cooling $-\frac{dT}{dt}$ is proportional to the temperature difference between them.

To prove it let us assume that

$$T = T_0 + \Delta T$$

So that $$T^4 = (T_0 + \Delta T)^4 = T_0^4\left(1 + \frac{\Delta T}{T_0}\right)^4$$

$$\approx T_0^4\left(1 + \frac{4\Delta T}{T_0}\right) \quad \text{(from binomial expansion)}$$

$\therefore$ $$(T^4 - T_0^4) = 4T_0^3(\Delta T)$$

or $$(T^4 - T_0^4) \propto \Delta T \quad \text{(as } T_0 = \text{constant)}$$

Now, we have already shown that rate of cooling

$$\left(-\frac{dT}{dt}\right) \propto (T^4 - T_0^4)$$

and here we have shown that

$$(T^4 - T_0^4) \propto \Delta T,$$

if the temperature difference is small.

Thus, rate of cooling

$$-\frac{dT}{dt} \propto \Delta T \quad \text{or} \quad -\frac{d\theta}{dt} \propto \Delta\theta$$

as $$dT = d\theta \quad \text{or} \quad \Delta T = \Delta\theta$$

Variation of temperature of a body according to Newton's law

Suppose a body has a temperature θ_i at time $t = 0$. It is placed in an atmosphere whose temperature is θ_0. We are interested in finding the temperature of the body at time t, assuming Newton's law of cooling to hold good or by assuming that the temperature difference is small. As per this law,

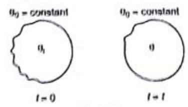


Fig. 19.9

rate of cooling $\propto$ temperature difference

or $$\left(-\frac{d\theta}{dt}\right)=\left(\frac{eA\sigma}{mc}\right)(4\theta_0^3)(\theta-\theta_0)$$

or $$\left(-\frac{d\theta}{dt}\right)=\alpha\,(\theta-\theta_0)$$

Here $$\alpha=\left(\frac{4eA\sigma\theta_0^3}{mc}\right)\text{ is a constant}$$

$\therefore$ $$\int_{\theta_i}^{\theta}\frac{d\theta}{\theta-\theta_0}=-\alpha\int_0^t dt$$

$\therefore$ $$\theta=\theta_0+(\theta_i-\theta_0)\,e^{-\alpha t}$$

From this expression we see that $\theta=\theta_i$ at $t=0$ and $\theta=\theta_0$ at $t=\infty$, *i.e.*, temperature of the body varies exponentially with time from θ_i to θ_0 $(<\theta_i)$. The temperature versus time graph is as shown in figure.

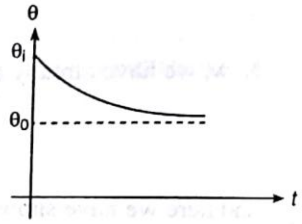


Fig. 19.10

Note *If the body cools by radiation from θ_1 to θ_2 in time t, then taking the approximation*

$$\left(-\frac{d\theta}{dt}\right)=\frac{\theta_1-\theta_2}{t}\quad\text{and}\quad\theta=\theta_{av}=\left(\frac{\theta_1+\theta_2}{2}\right)$$

The equation $\left(-\frac{d\theta}{dt}\right)=\alpha\,(\theta-\theta_0)$ *becomes*

$$\frac{\theta_1-\theta_2}{t}=\alpha\left(\frac{\theta_1+\theta_2}{2}-\theta_0\right)$$

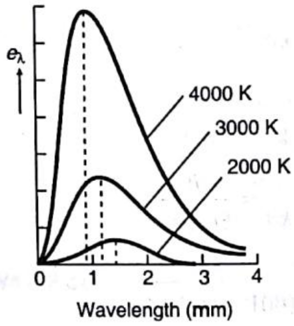
This form of the law helps in solving numerical problems related to **Newton's law of cooling.**

Wien's displacement law

At ordinary temperatures (below about 600°C) the thermal radiation emitted by a body is not visible, most of it is concentrated in wavelengths much longer than those of visible light.

Figure shows how the energy of a black body radiation varies with temperature and wavelength. As the temperature of the black body increases, two distinct behaviours are observed. The first effect is that the peak of the distribution shifts to shorter wavelengths. This shift is found to obey the following relationship called **Wien's displacement law.**

$$\lambda_{max}T=b$$



Power of black body radiation *versus* wavelength at three temperatures. Note that the amount of radiation emitted (the area under a curve) increase with increasing temperature.

Fig. 19.11

Here, b is a constant called Wien's constant. The value of this constant in SI unit is 2.898×10^{-3} m-K. Thus,

$$\lambda_{max} \propto \frac{1}{T}$$

Here, λ_{max} is the wavelength corresponding to the maximum spectral emissive power e_λ.

The second effect is that the total amount of energy the black body emits per unit area per unit time $(=\sigma T^4)$ increases with fourth power of absolute temperature T. This is also known as the emissive power. We know

$$e = \int_0^\infty e_\lambda \, d\lambda = \text{Area under } e_\lambda\text{-}\lambda \text{ graph} = \sigma T^4$$

or

$$\text{Area} \propto T^4$$

$$A_2 = (2)^4 A_1 = 16 A_1$$

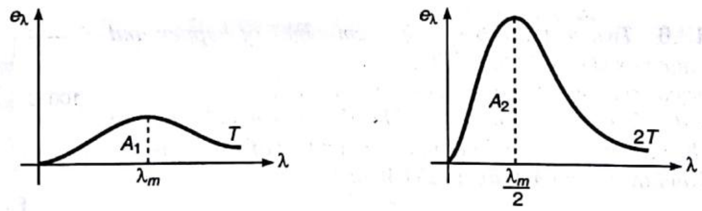


Fig. 19.12

Thus, if the temperature of the black body is made two fold, λ_{max} remains half while the area becomes 16 times.

Sample Example 19.5 *A copper rod 2 m long has a circular cross section of radius 1 cm. One end is kept at 100°C and the other at 0°C, and the surface is insulated so that negligible heat is lost through the surface. Find :*

(a) the thermal resistance of the bar

(b) the thermal current H

(c) the temperature gradient $\frac{dT}{dx}$ *and*

(d) the temperature 25 cm from the hot end.

Thermal conductivity of copper is 401 W/m-K.

Solution (a) Thermal resistance $R = \frac{l}{kA} = \frac{l}{k(\pi r^2)}$

or $$R = \frac{(2)}{(401)(\pi)(10^{-2})^2} = 15.9 \text{ K/W}$$ **Ans.**

(b) $$\text{Thermal current, } H = \frac{\Delta T}{R} = \frac{\Delta\theta}{R} = \frac{100}{15.9}$$

or $$H = 6.3 \text{ W}$$ **Ans.**

(c) Temperature gradient

$$= \frac{0-100}{2} = -50 \text{ K/m} = -50°\text{C/m}$$ **Ans.**

(d) Let θ_c° be the temperature at 25 cm from the hot end, then

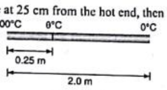


Fig. 19.13

$$(\theta - 100) = (\text{temperature gradient}) \times (\text{distance})$$

or $$\theta - 100 = (-50)(0.25)$$

or $$\theta = 87.5°\text{C}$$ **Ans.**

Sample Example 19.6 *Two metal cubes with 3 cm-edges of copper and aluminium are arranged as shown in figure. Find :*

(a) the total thermal current from one reservoir to the other

(b) the ratio of the thermal current carried by the copper cube to that carried by the aluminium cube. Thermal conductivity of copper is 401 W/m-K and that of aluminium is 237 W/m-K.

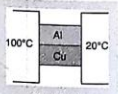


Fig. 19.14

Solution (a) Thermal resistance of aluminium cube $R_1 = \frac{l}{kA}$

or $$R_1 = \frac{(3.0 \times 10^{-2})}{(237)(3.0 \times 10^{-2})^2} = 0.14 \text{ K/W}$$

and thermal resistance of copper cube $R_2 = \frac{l}{kA}$

or $$R_2 = \frac{(3.0 \times 10^{-2})}{(401)(3.0 \times 10^{-2})^2} = 0.08 \text{ K/W}$$

As these two resistances are in parallel, their equivalent resistance will be

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

$$= \frac{(0.14)(0.08)}{(0.14) + (0.08)}$$

$$= 0.05 \text{ K/W}$$

$\therefore$ $$\text{Thermal current } H = \frac{\text{Temperature difference}}{\text{Thermal resistance}}$$

$$= \frac{(100-20)}{0.05} = 1.6 \times 10^3 \text{ W}$$

(b) In parallel thermal current distributes in the inverse ratio of resistance. Hence,

$$\frac{H_{Cu}}{H_{Al}} = \frac{R_{Al}}{R_{Cu}} = \frac{R_1}{R_2} = \frac{0.14}{0.08} = 1.75$$

Sample Example 19.7 *One end of a copper rod of length 1 m and area of cross section* 4.0×10^{-4} m^2 *is maintained at 100°C. At the other end of the rod ice is kept at 0°C. Neglecting the loss of heat from the surroundings find the mass of ice melted in 1 h. Given* $k_{Cu} = 401$ *W/m-K and* $L_f = 3.35 \times 10^5$ *J/kg.*

Solution Thermal resistance of the rod,

Fig. 19.15

$$R = \frac{l}{kA} = \frac{1.0}{(401)(4 \times 10^{-4})} = 6.23 \text{ K/W}$$

$\therefore$ $$\text{Heat current } H = \frac{\text{Temperature difference}}{\text{Thermal resistance}}$$

$$= \frac{(100-0)}{6.23} = 16 \text{ W}$$

Heat transferred in 1 h,

$$Q = Ht \qquad \left(H = \frac{Q}{t}\right)$$

$$= (16)(3600) = 57600 \text{ J}$$

Now, let m mass of ice melts in 1 h, then

$$m = \frac{Q}{L} \qquad (Q = mL)$$

$$= \frac{57600}{3.35 \times 10^5} = 0.172 \text{ kg} \quad \text{or} \quad 172 \text{ g}$$

Sample Example 19.8 *A body cools in 10 minutes from 60°C to 40°C. What will be its temperature after next 10 minutes? The temperature of the surroundings is 10°C.*

Solution According to Newton's law of cooling

$$\left(\frac{\theta_1 - \theta_2}{t}\right) = \alpha\left[\left(\frac{\theta_1 + \theta_2}{2}\right) - \theta_0\right]$$

For the given conditions,

$$\frac{60-40}{10} = \alpha\left[\frac{60+40}{2} - 10\right] \quad \text{...(i)}$$

Let θ be the temperature after next 10 minutes. Then,

$$\frac{40-\theta}{10} = \alpha\left[\frac{40+\theta}{2} - 10\right] \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$\theta = 28°\text{C}$$

Sample Example 19.9 *Two bodies A and B have thermal emissivities of 0.01 and 0.81 respectively. The outer surface areas of the two bodies are same. The two bodies emit total radiant power at the same rate. The wavelength* λ_B *corresponding to maximum spectral radiancy from B is shifted from the wavelength corresponding to maximum spectral radiancy in the radiation from A by 1.0 μm. If the temperature of A is 5802 K, calculate*

(a) the temperature of B, *(b) wavelength* λ_B.

Solution (a) $$P_A = P_B$$

$\therefore$ $$e_A \sigma A_A T_A^4 = e_B \sigma A_B T_B^4$$

$\therefore$ $$T_B = \left(\frac{e_A}{e_B}\right)^{1/4} T_A \qquad (\text{as } A_A = A_B)$$

Substituting the values

$$T_B = \left(\frac{0.01}{0.81}\right)^{1/4} (5802) = 1934 \text{ K}$$

(b) According to Wein's displacement law,

$$\lambda_A T_A = \lambda_B T_B$$

$\therefore$ $$\lambda_B = \left(\frac{5802}{1934}\right)\lambda_A$$

or $$\lambda_B = 3\lambda_A$$

Also, $$\lambda_B - \lambda_A = 1 \mu\text{m}$$

or $$\lambda_B - \left(\frac{1}{3}\right)\lambda_B = 1 \mu\text{m}$$

or $$\lambda_B = 1.5 \mu\text{m}$$

Introductory Exercise 19.2

1. Suppose a liquid in a container is heated at the top rather than at the bottom. What is the main process by which the rest of the liquid becomes hot?
2. The inner and outer surfaces of a hollow spherical shell of inner radius 'a' and outer radius 'b' are maintained at temperatures T_1 and T_2 ($< T_1$). The thermal conductivity of material of the shell is k. Find the rate of heat flow from inner to outer surface.
3. Show that the SI units of thermal conductivity are W/m-K.
4. A carpenter builds an outer house wall with a layer of wood 2.0 cm thick on the outside and a layer of an insulation 3.5 cm thick as the inside wall surface. The wood has $k = 0.08$ W/m-K and the insulation has $k = 0.01$ W/m-K. The interior surface temperature is 19°C and the exterior surface temperature is −10°C.
   (a) What is the temperature at the plane where the wood meets the insulation?
   (b) What is the rate of heat flow per square metre through this wall?
5. A pot with a steel bottom 1.2 cm thick rests on a hot stove. The area of the bottom of the pot is 0.150 m^2. The water inside the pot is at 100°C and 0.440 kg are evaporated every 5.0 minute. Find the temperature of the lower surface of the pot, which is in contact with the stove. Take $L_v = 2.256 \times 10^6$ J/kg and $k_{steel} = 50.2$ W/m-K.
6. A layer of ice of thickness y is on the surface of a lake. The air is at a constant temperature $-\theta$°C and the ice water interface is at 0°C. Show that the rate at which the thickness increases is given by,
$$\frac{dy}{dt} = \frac{k\theta}{L\rho y}$$
   where k is the thermal conductivity of the ice, L the latent heat of fusion and ρ is the density of the ice.
7. The emissivity of tungsten is 0.4. A tungsten sphere with a radius of 4.0 cm is suspended within a large evacuated enclosure whose walls are at 300 K. What power input is required to maintain the sphere at a temperature of 3000 K if heat conduction along supports is neglected? Take $\sigma = 5.67 \times 10^{-8}$ $W/m^2\text{-}K^4$.
8. Find SI units of thermal resistance.

Extra Points

- The general expression for the heat involved in phase change is
$$Q = \pm mL$$
The plus sign (heat entering) is used when the material melts, the minus sign (heat leaving) is used when it freezes.
- For JEE remember the heat equations in differential forms as under :
$$\frac{dQ}{dt} = mc\frac{dT}{dt}$$
and $$\frac{dQ}{dt} = \pm L\frac{dm}{dt}$$
Here, $\frac{dQ}{dt}$ is the time rate of heat transfer,
$\frac{dT}{dt}$ is the rate of change of temperature

and $\frac{dm}{dt}$ is the rate of mass transfer from one phase of matter to the other.

- While solving the problems of heat flow, remember the following equation :

$$\frac{dQ}{dt}=\frac{\text{T.D.}}{R}=mc\cdot\frac{dT}{dt}=L\cdot\frac{dm}{dt}$$

of these we will have to choose the appropriate relation according to the problem.

For instance, one end of a rod of length l, area A and thermal conductivity k is maintained at 100°C and at the other end ice is melting at 0°C. We are interested in finding the mass of ice which transforms into water in unit time. For this, we will take

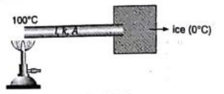


Fig. 19.16

$$\frac{\text{T.D.}}{R}=L_f\,\frac{dm}{dt}$$

$\therefore$ $$\left(\frac{dm}{dt}\right)=\frac{\text{T.D}}{(L_f)(R)}=\text{rate of ice which transforms into water in unit time.}$$

Here, T.D. = temperature difference

$$R=\frac{l}{kA}\quad\text{and}\quad L_f=\text{latent heat of fusion}$$

- **Growth of ice on ponds :** When temperature of the atmosphere falls below 0°C, the water in the pond starts freezing. Let at time t thickness of ice in the pond is y and atmospheric temperature is $-T$°C. The temperature of water in contact with the lower surface of ice will be 0°C. Using

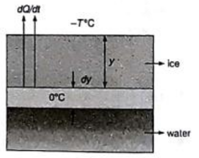


Fig. 19.17

$$\frac{dQ}{dt}=L_f\left(\frac{dm}{dt}\right)$$

or $$\frac{\text{T.D.}}{R}=L_f\,\frac{d}{dt}(A\rho y)\quad(A=\text{area of pond})$$

$\therefore$ $$\frac{[0-(-T)]}{(y/kA)}=L_fA\rho\cdot\frac{dy}{dt}\qquad(k=\text{thermal conductivity of ice})$$

$\therefore$ $$\frac{dy}{dt}=\frac{kT}{\rho L_f}\cdot\frac{1}{y}$$

and hence time taken by ice to grow a thickness y,

$$t=\frac{\rho L_f}{kT}\int_0^y y\,dy$$

or $$t=\frac{1}{2}\frac{\rho L_f}{kT}y^2$$

Time does not depend on the area of pond.

Solved Examples

For JEE Main

Example 1 *5 g of water at 30°C and 5 g of ice at −20°C are mixed together in a calorimeter. Find the final temperature of mixture. Water equivalent of calorimeter is negligible, specific heat of ice = 0.5 cal/g°C and latent heat of ice = 80 cal/g.*

Solution In this case heat is given by water and taken by ice

Heat available with water to cool from 30°C to 0°C

$$=ms\Delta\theta=5\times1\times30=150\text{ cal}$$

Heat required by 5 g ice to increase its temperature up to 0°C

$$ms\Delta\theta=5\times0.5\times20=50\text{ cal}$$

Out of 150 cal heat available, 50 cal is used for increasing temperature of ice from −20°C to 0°C. The remaining heat 100 cal is used for melting the ice.

If mass of ice melted is m g then

$$m\times80=100\;\Rightarrow\;m=1.25\text{ g}$$

Thus, 1.25 g ice out of 5 g melts and mixture of ice and water is at 0°C.

Example 2 *A bullet of mass 10 g moving with a speed of 20 m/s hits an ice block of mass 990 g kept on a frictionless floor and gets stuck in it. How much ice will melt if 50% of the lost kinetic energy goes to ice? (Temperature of ice block = 0°C).*

Solution Velocity of bullet + ice block,

$$V=\frac{(10\text{ g})\times(20\text{ m/s})}{1000\text{ g}}=0.2\text{ m/s}$$

$$\text{Loss of KE}=\frac{1}{2}mv^2-\frac{1}{2}(m+M)V^2$$

$$=\frac{1}{2}[0.01\times(20)^2-1\times(0.2)^2]$$

$$=\frac{1}{2}[4-0.04]=1.98\text{ J}$$

$\therefore$ $$\text{Heat received by ice block}=\frac{1.98}{4.2\times2}\text{ cal}$$

$$=0.24\text{ cal}$$

$\therefore$ $$\text{Mass of ice melted}=\frac{(0.24\text{ cal})}{(80\text{ cal/g})}$$

$$=0.003\text{ g}$$

Example 3 *The temperature of equal masses of three different liquids A, B and C are 12°C, 19°C and 28°C respectively. The temperature when A and B are mixed is 16°C and when B and C are mixed it is 23°C. What should be the temperature when A and C are mixed ?*

Solution Let m be the mass of each liquid and S_A, S_B, S_C specific heats of liquids A, B and C respectively. When A and B are mixed. The final temperature is 16°C.

$\therefore$ $$\text{Heat gained by } A=\text{heat lost by } B$$

i.e., $$mS_A(16-12)=mS_B(19-16)$$

i.e., $$S_B=\frac{4}{3}S_A \qquad \text{...(i)}$$

When B and C are mixed. Heat gained by B = heat lost by C

i.e., $$mS_B(23-19)=mS_C(28-23)$$

i.e., $$S_C=\frac{4}{5}S_B \qquad \text{...(ii)}$$

From Eqs. (i) and (ii)

$$S_C=\frac{4}{5}\times\frac{4}{3}S_A=\frac{16}{15}S_A$$

When A and C are mixed, let the final temperature be θ

$\therefore$ $$mS_A(\theta-12)=mS_C(28-\theta)$$

i.e., $$\theta-12=\frac{16}{15}(28-\theta)$$

By solving, we get $$\theta=\frac{628}{31}=20.26\text{°C}$$

Example 4 *At 1 atmospheric pressure, 1.000 g of water having a volume of 1.000 cm^3 becomes 1671 cm^3 of steam when boiled. The heat of vaporization of water at 1 atmosphere is 539 cal/g. What is the change in internal energy during the process?*

Solution Heat spent during vaporisation

$$Q=mL=1.000\times539=539\text{ cal}$$

Work done $$W=P(V_v-V_l)$$

$$=1.013\times10^5\times(1671-1.000)\times10^{-6}$$

$$=169.2\text{ J}=\frac{169.2}{4.18}\text{ cal}=40.5\text{ cal}$$

$\therefore$ Change in internal energy

$\therefore$ $$U=539\text{ cal}-40.5\text{ cal}=498.5\text{ cal}$$

Example 5 *At 1 atmospheric pressure, 1.000 g of water having a volume of 1.000 cm^3 becomes 1.091 cm^3 of ice on freezing. The heat of fusion of water at 1 atmosphere is 80.0 cal/g. What is the change in internal energy during the process?*

Solution Heat given out during freezing

$$Q=-mL=-1\times80=-80\text{ cal}$$

External work done $$W=P(V_{\text{ice}}-V_{\text{water}})$$

$$=1.013\times10^5\times(1.091-1.000)\times10^{-6}$$

$$=9.22\times10^{-3}\text{ J}$$

$$=\frac{9.22\times10^{-3}}{4.18}\text{ cal}=0.0022\text{ cal}$$

$\therefore$ Change in internal energy

$$\Delta U=Q-W=-80-0.0022=-80.0022\text{ cal}$$ **Ans.**

Example 6 *Two bodies A and B have thermal emissivities of 0.01 and 0.81 respectively. The outer surface area of the two bodies are the same. The two bodies emit total radiant power at the same rate. The wavelengths λ_A and λ_B corresponding to maximum spectral radiancy in the radiation from A and B respectively differ by 1.00 μm. If the temperature of A is 5802 K, Find (a) the temperature of B (b) and λ_B.*

Solution Given $e_A=0.01$, $e_B=0.81$ and $T_A=5802$K

From Wien's displacement law

$$\lambda_mT=\text{constant}\quad\therefore\quad\lambda_AT_A=\lambda_BT_B$$

Power radiated $P=e\sigma T^4A$ as $P_1=P_2$ and $A_1=A_2$

we have $$e_AT_A^4=e_BT_B^4$$

$\therefore$ $$T_B=\left(\frac{e_A}{e_B}\right)^{1/4}T_A=\left(\frac{0.01}{0.81}\right)^{1/4}\times5802$$

$$=1934\text{ K}$$

as $$T_B<T_A,\ \lambda_B>\lambda_A$$

$\therefore$ $$\lambda_B-\lambda_A=1\,\mu\text{m} \qquad \text{(given)}$$

$$\lambda_B-\lambda_A=1\times10^{-6}\text{ m} \qquad \text{...(i)}$$

and $$\frac{\lambda_B}{\lambda_A}=\frac{T_A}{T_B}=\frac{5802}{1934}=3$$

$\therefore$ $$\lambda_B=3\lambda_A \qquad \text{...(ii)}$$

From Eqs. (i) and (ii) $$\lambda_B=1.5\times10^{-6}\text{ m}$$ **Ans.**

Example 7 *Two plates each of area A, thickness L_1 and L_2 thermal conductivities K_1 and K_2 respectively are joined to form a single plate of thickness (L_1+L_2). If the temperatures of the free surfaces are T_1 and T_2. Calculate*

(a) Rate of flow of heat
(b) Temperature of interface and
(c) Euivalent thermal conductivity.

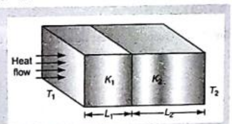


Fig. 19.18

Solution (a) If the thermal resistance of the two plates are R_1 and R_2 respectively then as plates are in series.

$$R_S = R_1 + R_2 = \frac{L_1}{AK_1} + \frac{L_2}{AK_2} \quad \text{as } R = \frac{L}{KA}$$

and so
$$H = \frac{dQ}{dt} = \frac{\Delta\theta}{R} = \frac{(T_1 - T_2)}{(R_1 + R_2)} = \frac{A(T_1 - T_2)}{\left[\frac{L_1}{K_1} + \frac{L_2}{K_2}\right]}$$

(b) If T is the common temperature of interface then as in series rate of flow of heat remains same, *i.e.*, $H = H_1 \ (= H_2)$

$$\frac{T_1 - T_2}{R_1 + R_2} = \frac{T_1 - T}{R_1},$$

i.e.,
$$T = \frac{T_1 R_2 + T_2 R_1}{(R_1 + R_2)}$$

or
$$T = \frac{\left[T_1 \frac{L_2}{K_2} + T_2 \frac{L_1}{K_1}\right]}{\left[\frac{L_1}{K_1} + \frac{L_2}{K_2}\right]} \quad \text{as } R = \frac{L}{KA}$$

(c) If K is the equivalent conductivity of composite slab, *i.e.*, slab of thickness $L_1 + L_2$ and cross-sectional area A, then as in series

$$R_S = R_1 + R_2 \quad \text{or} \quad \frac{(L_1 + L_2)}{AK_{eq}} = R_1 + R_2$$

i.e.,
$$K_{eq} = \frac{L_1 + L_2}{A(R_1 + R_2)} = \frac{L_1 + L_2}{\left[\frac{L_1}{K_1} + \frac{L_2}{K_2}\right]} \quad \text{as } R = \frac{L}{KA}$$

Example 8 *One end of a rod of length 20 cm is inserted in a furnace at 800 K. The sides of the rod are covered with an insulating material and the other end emits radiation like a black body. The temperature of this end is 750 K in the steady state. The temperature of the surrounding air is 300 K. Assuming radiation to be the only important mode of energy transfer between the surrounding and the open end of the rod, find the thermal conductivity of the rod. Stefan constant* $\sigma = 6.0 \times 10^{-8}\ W/m^2\text{-}K^4$.

Solution Quantity of heat flowing through the rod per second in steady state:

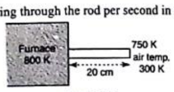


Fig. 19.19

$$\frac{dQ}{dt} = \frac{K.A.d\theta}{x} \qquad ...(i)$$

Quantity of heat radiated from the end of the rod per second in steady state:

$$\frac{dQ}{dt} = A\sigma\,(T^4 - T_0^4) \qquad ...(ii)$$

From Eqs. (i) and (ii),

$$\frac{K.d\theta}{x} = \sigma\,(T^4 - T_0^4)$$

$$\frac{K \times 50}{0.2} = 6.0 \times 10^{-8}\,[(7.5)^4 - (3)^4] \times 10^8$$

or
$$K = 74 \text{ W/mK}$$ **Ans.**

For JEE Advanced

Example 1 *Three rods of material x and three rods of material y are connected as shown in figure. All the rods are of identical length and cross-sectional area. If the end A is maintained at 60°C and the junction E at 10°C, calculate temperature of junctions B, C and D. The thermal conductivity of x is 0.92 cal/cm-s°C and that of y is 0.46 cal/cm-s°C.*

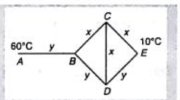


Fig. 19.20

Solution Thermal resistance $R = \frac{l}{kA}$

$\therefore$
$$\frac{R_x}{R_y} = \frac{k_y}{k_x} \qquad (\text{as } l_x = l_y \text{ and } A_x = A_y)$$
$$= \frac{0.46}{0.92} = \frac{1}{2}$$

So, if $R_x = R$ then $R_y = 2R$

CEDB forms a balanced Wheatstone bridge, *i.e.*, $T_C = T_D$ and no heat flows through *CD*

$\therefore$
$$\frac{1}{R_{BE}} = \frac{1}{R + R} + \frac{1}{2R + 2R}$$

or
$$R_{BE} = \frac{4}{3}R$$

The total resistance between A and E will be,

$$R_{AE} = R_{AB} + R_{BE} = 2R + \frac{4}{3}R = \frac{10}{3}R$$

$\therefore$ Heat current between A and E is

$$H = \frac{(\Delta T)_{AE}}{R_{AE}} = \frac{(60 - 10)}{(10/3)\,R} = \frac{15}{R}$$

Now, if T_B is the temperature at B,

$$H_{AB} = \frac{(\Delta T)_{AB}}{R_{AB}}$$

or
$$\frac{15}{R} = \frac{60 - T_B}{2R}$$

or
$$T_B = 30°\text{C}$$ **Ans.**

Further,

$$H_{AB} = H_{BC} + H_{BD}$$

or
$$\frac{15}{R} = \frac{30 - T_C}{R} + \frac{30 - T_D}{2R} \qquad [T_C = T_D = T\ (\text{say})]$$

or
$$15 = (30 - T) + \frac{(30 - T)}{2}$$

Solving this, we get
$$T = 20°\text{C}$$

or
$$T_C = T_D = 20°\text{C}$$ **Ans.**

Example 2 *A cylinder of radius R made of a material of thermal conductivity K_1 is surrounded by cylindrical shell of inner radius R and outer radius 2R made of a material of thermal conductivity K_2. The two ends of the combined system are maintained at two different temperatures. There is no loss of heat across the cylindrical surface and system is in steady state. What is the effective thermal conductivity of the system?*

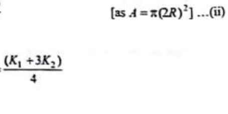


Fig. 19.21

Solution In this situation a rod of length L and area of cross-section πR^2 and another of same length L and area of cross-section $\pi\,[(2R)^2 - R^2] = 3\pi R^2$ will conduct heat simultaneously so total heat flowing per sec will be

$$\frac{dQ}{dt} = \frac{dQ_1}{dt} + \frac{dQ_2}{dt}$$
$$= \frac{K_1 \pi R^2\,(\theta_1 - \theta_2)}{L} + \frac{K_2 3\pi R^2\,(\theta_1 - \theta_2)}{L} \qquad ...(i)$$

Now, if the equivalent conductivity is K. Then,

$$\frac{dQ}{dt} = K\frac{4\pi R^2\,(\theta_1 - \theta_2)}{L} \qquad [\text{as } A = \pi(2R)^2] \ ...(ii)$$

So, from Eqs. (i) and (ii), we have

$$4K = K_1 + 3K_2 \quad \text{i.e., } K = \frac{(K_1 + 3K_2)}{4}$$

Example 3 *A hollow sphere of glass whose external and internal radii are 11 cm and 9 cm respectively is completely filled with ice at 0°C and placed in a bath of boiling water. How long will it take for the ice to melt completely? Given that density of ice* $= 0.9\ g/cm^3$, *latent heat of fusion of ice* $= 80\ cal/g$ *and thermal conductivity of glass* $= 0.002\ cal/cm\text{-}s°C$.

Solution

In steady state, rate of heat flow
$$H = \frac{4\pi k r_1 r_2 \Delta T}{r_2 - r_1}$$

Substituting the values,
$$H = \frac{(4)(\pi)(0.002)(11)(9)(100 - 0)}{(11 - 9)}$$

or
$$\frac{dQ}{dt} = 124.4 \text{ cal/s}$$

This rate should be equal to, $L\frac{dm}{dt}$

$\therefore$
$$\left(\frac{dm}{dt}\right) = \frac{dQ/dt}{L} = \frac{124.4}{80} = 1.555 \text{ g/s}$$

Total mass of ice,
$$m = \rho_{ice}\,(4\pi r_1^2)$$
$$= (0.9)(4)(\pi)(9)^2$$
$$= 916 \text{ g}$$

$\therefore$ Time taken for the ice to melt completely

$$t = \frac{m}{(dm/dt)} = \frac{916}{1.555} = 589 \text{ s}$$ **Ans.**

Example 4 *A point source of heat of power P is placed at the centre of a spherical shell of mean radius R. The material of the shell has thermal conductivity k. Calculate the thickness of the shell if temperature difference between the outer and inner surfaces of the shell in steady state is T.*

Solution Consider a concentric spherical shell of radius r and thickness dr as shown in figure. In steady state, the rate of heat flow (heat current) through this shell will be,

$$H = \frac{\Delta T}{R} = \frac{(-d\theta)}{\frac{dr}{(k)(4\pi r^2)}} \qquad \left(R = \frac{l}{kA}\right)$$

or
$$H = -(4\pi k r^2)\frac{d\theta}{dr}$$

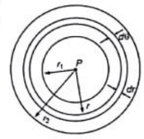


Fig. 19.22

Here, negative sign is used because with increase in r, θ decreases.

$\therefore$
$$\int_{r_1}^{r_2} \frac{dr}{r^2} = -\frac{4\pi k}{H}\int_{\theta_1}^{\theta_2} d\theta$$

This equation gives, $$H = \frac{4\pi k r_1 r_2 (\theta_1 - \theta_2)}{(r_2 - r_1)}$$

In steady state, $H = P$, $r_1 r_2 \approx R^2$ and $\theta_1 - \theta_2 = T$

$\therefore$ Thickness of shell, $$r_2 - r_1 = \frac{4\pi k R^2 T}{P}$$ **Ans.**

Example 5 *A steam pipe of radius 5 cm carries steam at 100°C. The pipe is covered by a jacket of insulating material 2 cm thick having a thermal conductivity 0.07 W/m-K. If the temperature at the outer wall of the pipe jacket is 20°C, how much heat is lost through the jacket per metre length in an hour?*

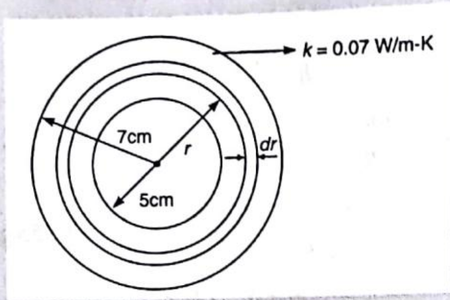


Fig. 19.23

Solution Thermal resistance per metre length of an element at distance r of thickness dr is

$$dR = \frac{dr}{k\,(2\pi r)} \qquad \left(R = \frac{l}{kA}\right)$$

$$\therefore \quad \text{Total resistance } R = \int_{r_1 = 5\text{ cm}}^{r_2 = 7\text{ cm}} dR$$

$$= \frac{1}{2\pi k}\int_{5.0\times10^{-2}\text{ m}}^{7.0\times10^{-2}\text{ m}} \frac{dr}{r}$$

$$= \frac{1}{2\pi k}\ln\left(\frac{7}{5}\right)$$

$$= \frac{1}{(2\pi)(0.07)}\ln(1.4)$$

$$= 0.765 \text{ K/W}$$

$$\text{Heat current } H = \frac{\text{Temperature difference}}{\text{Thermal resistance}}$$

$$= \frac{(100 - 20)}{0.765} = 104.6 \text{ W}$$

$$\therefore \quad \text{Heat lost in one hour} = \text{Heat current} \times \text{time}$$

$$= (104.6)(3600) \text{ J}$$

$$= 3.76 \times 10^5 \text{ J}$$ **Ans.**

EXERCISES

For JEE Main

Subjective Questions

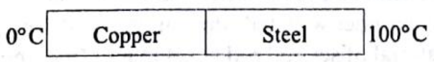
Calorimetry

1. 10 g ice at 0°C is converted into steam at 100°C. Find total heat required. ($L_f = 80$ cal/g, $S_w = 1$ cal/g °C, $L_v = 540$ cal/g)
2. 15 g ice at 0°C is mixed with 10 g water at 40°C. Find the temperature of mixture. Also, find mass of water and ice in the mixture.
3. Three liquids P, Q and R are given. 4 kg of P at 60°C and 1 kg of R at 50°C when mixed produce a resultant temperature 55°C. A mixture of 1 kg of P at 60°C and 1 kg of Q at 50°C shows a temperature of 55°C. What will be the resulting temperature when 1 kg of Q at 60°C is mixed with 1 kg of R at 50°C?
4. A certain amount of ice is supplied heat at a constant rate for 7 minutes. For the first one minute the temperature rises uniformly with time. Then it remains constant for the next 4 minutes and again the temperature rises at uniform rate for the last two minutes. Calculate the final temperature at the end of seven minutes. (Given: L of ice $= 336 \times 10^3$ J/kg and specific heat of water = 4200 J/kg-K)
5. A lead bullet penetrates into a solid object and melts. Assuming that 50% of its kinetic energy was used to heat it, calculate the initial speed of the bullet. The initial temperature of the bullet is 27°C and its melting point is 327°C. Latent heat of fusion of lead $= 2.5 \times 10^4$ J/kg and specific heat capacity of lead = 125 J / kg-K.
6. A ball is dropped on a floor from a height of 2.0 m. After the collision it rises upto a height of 1.5 m. Assume that 40% of the mechanical energy lost goes as thermal energy into the ball. Calculate the rise in the temperature of the ball in the collision. Specific heat of the ball is 800 J/K.

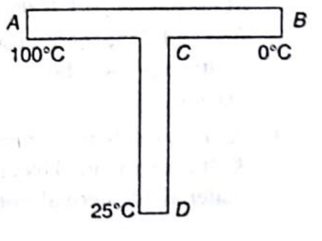
Heat Transfer

(a) Conduction

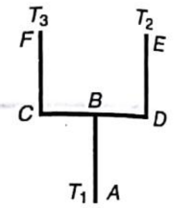
7. Figure shows a copper rod joined to a steel rod. The rods have equal length and equal cross-sectional area. The free end of the copper rod is kept at 0°C and that of the steel rod is kept at 100°C. Find the temperature at the junction of the rods. Conductivity of copper = 390 W/m-°C and that of steel = 46 W/m-°C.



8. A rod CD of thermal resistance 5.0 K/W is joined at the middle of an identical rod AB as shown in figure. The ends A, B and D are maintained at 100°C and 25°C respectively. Find the heat current in CD.



9. Four identical rods AB, CD, CF and DE are joined as shown in figure. The length, cross sectional area and thermal conductivity of each rod are l, A and K respectively. The ends A, E and F are maintained at temperatures T_1, T_2 and T_3 respectively. Assuming no loss of heat to the atmosphere. Find the temperature at B, the mid point of CD.



10. Three rods each of same length and cross section are joined in series. The thermal conductivity of the materials are $k, 2k$ and $3k$ respectively. If one end is kept at $200°C$ and the other at $100°C$. What would be the temperature of the junctions in the steady state? Assume that no heat is lost due to radiation from the sides of the rods.

11. The ends of a copper rod of length 1 m and area of cross-section $1\,cm^2$ are maintained at $0°C$ and $100°C$. At the centre of the rod there is a source of heat of power 25 W. Calculate the temperature gradient in the two halves of the rod in steady state. Thermal conductivity of copper is $400\ W\ m^{-1}K^{-1}$.

(b) Radiation

12. A thin square steel plate 10 cm on a side is heated in a black smith's forge to temperature of $800°C$. If the emissivity is 0.60, what is the total rate of radiation of energy. ?

13. A copper sphere is suspended in an evacuated chamber maintained at 300 K. The sphere is maintained at a constant temperature of 500 K by heating it electrically . A total of 210 W of electric power is needed to do it. When the surface of the copper sphere is completely blackened, 700 W is needed to maintain the same temperature of the sphere. Calculate the emissivity of copper.

14. A liquid takes 5 minutes to cool from $80°C$ to $50°C$. How much time will it take to cool from $60°C$ to $30°C$? The temperature of surrounding is $20°C$.

Objective Questions

Single Correct Option

1. A wall has two layers A and B each made of different materials. The layer A is 10 cm thick and B is 20 cm thick. The thermal conductivity of A is thrice that of B. Under thermal equilibrium temperature difference across the wall is 35°C. The difference of temperature across the layer A is
(a) 30°C (b) 14°C (c) 8.75°C (d) 5°C

2. The intensity of radiation emitted by the sun has its maximum value at a wavelength of 510 nm and that emitted by the North star has the maximum value at 350 nm. If these stars behave like black bodies, then the ratio of the surface temperatures of the sun and the north star is
(a) 1.46 (b) 0.69 (c) 1.21 (d) 0.83

3. A cylindrical rod with one end in a steam chamber and the other end in ice results in melting of 0.1 g of ice per second. If the rod is replaced by another with half the length and double the radius of the first and if the thermal conductivity of material of second rod is 1/4 that of first, the rate at which ice melts in g/s will be
(a) 0.4 (b) 0.05 (c) 0.2 (d) 0.1

4. A hollow sphere of inner radius a and outer radius $2a$ is made of a material of thermal conductivity K. It is surrounded by another hollow sphere of inner radius $2a$ and outside radius $3a$ made of same material of thermal conductivity K. The inside of smaller sphere is maintained at 0°C and the

outside of bigger sphere at 100°C. The system is in steady state. The temperature of the interface will be

(a) 50°C (b) 60°C (c) 75°C (d) 25°C

5. Two sheets of thickness d and $3d$, are touching each other. The temperature just outside the thinner sheet is T_1 and on the side of the thicker sheet is T_3. The interface temperature is T_2. T_1, T_2 and T_3 are in arithmetic progression. The ratio of thermal conductivity of thinner sheet to thicker sheet is

(a) 1 : 3 (b) 3 : 1 (c) 2 : 3 (d) 3 : 9

6. The end of two rods of different material with their thermal conductivities, area of cross-section and lengths all in the ratio 1 : 2 are maintained at the same temperature difference. If the rate of flow of heat in the first rod is 4 cal/s. Then in the second rod rate of heat flow in cal/s will be

(a) 1 (b) 2 (c) 8 (d) 16

7. A long rod has one end at 0°C and other end at a high temperature. The coefficient of thermal conductivity varies with distance from the low temperature end as $k = k_0(1 + ax)$, where $k_0 = 10^2$ SI unit and $a = 1\ \text{m}^{-1}$. At what distance from the first end the temperature will be 100°C ? The area of cross-section is $1\ \text{cm}^2$ and rate of heat conduction is 1 W.

(a) 2.7 m (b) 1.7 m (c) 3 m (d) 1.5 m

8. For an enclosure maintained at 2000 K, the maximum radiation occurs at wavelength λ_m. If the temperature is raised to 3000 K, the peak will shift to

(a) $0.5\lambda_m$ (b) λ_m (c) $\frac{2}{3}\lambda_m$ (d) $\frac{3}{2}\lambda_m$

9. 1 g of ice is mixed with 1 g of steam. After thermal equilibrium is achieved, the temperature of the mixture is

(a) 100°C (b) 55°C (c) 75°C (d) 0°C

10. A substance cools from 75°C to 70°C in T_1 minute, from 70°C to 65°C in T_2 minute and from 65°C to 60°C in T_3 minute, then

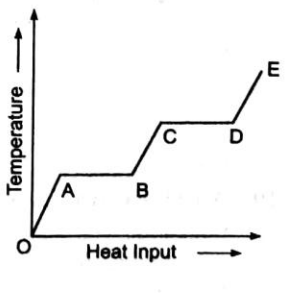
(a) $T_1 = T_2 = T_3$ (b) $T_1 < T_2 < T_3$ (c) $T_1 > T_2 > T_3$ (d) $T_1 < T_2 > T_3$

11. Two ends of rods of length L and radius R of the same material are kept at the same temperature. Which of the following rods conducts the maximum heat ?

(a) $L = 50\ \text{cm}, R = 1\ \text{cm}$ (b) $L = 100\ \text{cm}, R = 2\ \text{cm}$

(c) $L = 25\ \text{cm}, R = 0.5\ \text{cm}$ (d) $L = 75\ \text{cm}, R = 1.5\ \text{cm}$

12. A solid material is supplied heat at a constant rate. The temperature of material is changing with heat input as shown in the figure. What does the slope of DE represent ?



(a) Latent heat of liquid

(b) Latent heat of vaporization

(c) Heat capacity of vapour

(d) Inverse of heat capacity of vapour

13. The specific heat of a metal at low temperatures varies according to $S = aT^3$, where a is a constant and T is absolute temperature. The heat energy needed to raise unit mass of the metal from temperature $T = 1$ K to $T = 2$ K is

(a) $3a$ (b) $\frac{15a}{4}$ (c) $\frac{2a}{3}$ (d) $\frac{13a}{4}$

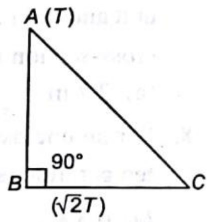
14. Two rods are of same material and having same length and area. If heat ΔQ flows through them for 12 min when they are joined side by side. If now both the rods are joined in parallel, then the same amount of heat ΔQ will flow in

(a) 24 min (b) 3 min (c) 12 min (d) 6 min

15. Two liquids are at temperatures 20°C and 40°C. When same mass of both of them is mixed, the temperature of the mixture is 32°C. What is the ratio of their specific heats?

(a) 1/3 (b) 2/5 (c) 3/2 (d) 2/3

16. Three rods of identical cross-sectional area and made from the same metal form the sides of an isosceles triangle ABC right angled at B as shown in the figure. The points A and B are maintained at temperatures T and $\sqrt{2}\,T$ respectively in the steady state. Assuming that only heat conduction takes place, temperature of point C will be



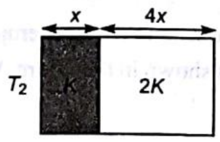
(a) $\frac{T}{\sqrt{2}-1}$ (b) $\frac{T}{\sqrt{2}+1}$

(c) $\frac{3T}{\sqrt{2}+1}$ (d) $\frac{T}{\sqrt{3}(\sqrt{2}-1)}$

17. A kettle with 2 litre water at 27°C is heated by operating coil heater of power 1 kW. The heat is lost to the atmosphere at constant rate 160 J/s, when its lid is open. In how much time will water heated to 77°C with the lid open? (specific heat of water = 4.2 kJ/°C-kg)

(a) 8 min 20 s (b) 6 min 2 s (c) 14 min (d) 7 min

18. The temperature of the two outer surfaces of a composite slab consisting of two materials having coefficients of thermal conductivity K and $2K$ and thickness x and $4x$ respectively are T_2 and T_1 ($T_2 > T_1$). The rate of heat transfer through the slab in steady state is $\frac{AK(T_2 - T_1)}{x} f$. where f is equal to



(a) 1 (b) 1/2 (c) 2/3 (d) 1/3

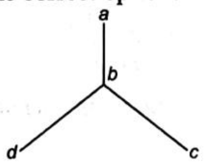
19. A wall has two layers A and B each made of different materials. Both the layers have the same thickness. The thermal conductivity of material A is twice of B. Under thermal equilibrium the temperature difference across the layer B is 36°C. The temperature difference across layer A is

(a) 6°C (b) 12°C (c) 18°C (d) 24°C

More than One Correct Options

20. A solid sphere and a hollow sphere of the same material and of equal radii are heated to the same temperature
    (a) both will emit equal amount of radiation per unit time in the beginning
    (b) both will absorb equal amount of radiation per second from the surrounding in the beginning
    (c) the initial rate of cooling will be the same for both the spheres
    (d) the two spheres will have equal temperatures at any instant

21. Three identical conducting rods are connected as shown in figure. Given that $\theta_a = 40°C$, $\theta_c = 30°C$ and $\theta_d = 20°C$. Choose the correct options



    (a) temperature of junction b is 15°C
    (b) temperature of junction b is 30°C
    (c) heat will flow from c to b
    (d) heat will flow from b to d

22. Two liquids of specific heat ratio 1 : 2 are at temperatures 2θ and θ
    (a) if equal amounts of them are mixed, then temperature of mixture is $1.5\,\theta$
    (b) if equal amounts of them are mixed, then temperature of mixture is $\frac{4}{3}\theta$
    (c) for their equal amounts, the ratio of heat capacities is 1 : 1
    (d) for their equal amounts, the ratio of their heat capacities is 1 : 2

23. Two conducting rods when connected between two points at constant but different temperatures separately the rate of heat flow through them is q_1 and q_2
    (a) When they are connected in series the net rate of heat flow will be $q_1 + q_2$
    (b) When they are connected in series, the net rate of heat flow is $\frac{q_1 q_2}{q_1 + q_2}$
    (c) When they are connected in parallel, the net rate of heat flow is $q_1 + q_2$
    (d) When they are connected in parallel, the net rate of heat flow is $\frac{q_1 q_2}{q_1 + q_2}$

24. Choose the correct options.
    (a) Good absorbers of a particular wavelength are good emitters of same wavelength. This statement was given by Kirchhoff
    (b) At low temperature of a body the rate of cooling is directly proportional to temperature of the body. This statement was given by the Newton
    (c) Emissive power of a perfectly black body is 1
    (d) Absorptive power of a perfectly black body is 1

For JEE Advanced

Assertion and Reason

Directions : *Choose the correct option.*

(a) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*
(b) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*
(c) *If **Assertion** is true, but the **Reason** is false.*
(d) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** Specific heat of any substance remains constant at all temperatures.
   Reason : It is given by $S = \frac{1}{m} \cdot \frac{dQ}{dT}$

2. **Assertion :** When temperature of a body is increased, in radiant energy number of low wavelength photons get increased.
   Reason : According to Wien's displacement law $\lambda_m \propto \frac{1}{T}$.

3. **Assertion :** Warming a room by a heat blower is an example of forced convection.
   Reason : Natural convection takes place due to gravity.

4. **Assertion :** A conducting rod is placed between boiling water and ice. If rod is broken into two equal parts and two parts are connected side by side, then rate of melting of ice will increase to four times.
   Reason : Thermal resistance will become four times.

5. **Assertion :** A normal body can radiate energy more than a perfectly black body.
   Reason : A perfectly black body is always black in colour.

6. **Assertion :** According to Newton's law, good conductors of electricity are also good conductors of heat.
   Reason : At a given temperature, $e_\lambda \propto a_\lambda$ for any body.

7. **Assertion :** Good conductors of electricity are also good conductors of heat due to large number of free electrons.
   Reason : It is easy to conduct heat from free electrons.

8. **Assertion :** Emissivity of any body (e) is equal to its absorptive power (a).
   Reason : Both the quantities are dimensionless.

9. **Assertion :** Heat is supplied at constant rate from one end of a conducting rod. In steady state, temperature of all points of the rod become uniform.
   Reason : In steady state temperature of rod does not increase.

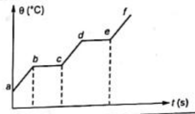
10. **Assertion :** A solid sphere and a hollow sphere of same material and same radius are kept at same temperatures in atmosphere. Rate of cooling of hollow sphere will be more.
    Reason : If all other conditions are same, then rate of cooling is inversely proportional to mass of body.

Match the Columns

1. Match the following two columns.

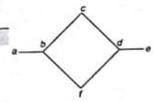
Column I	Column II
(a) Stefan's constant	(p) $[L\theta]$
(b) Wien's constant	(q) $[ML^2T^{-3}\theta^{-2}]$
(c) Emissive power	(r) $[MT^{-3}]$
(d) Thermal resistance	(s) None of these

2. Heat is supplied to a substance in solid state. Its temperature varies with time as shown in figure. Match the following two columns.



Column I	Column II
(a) Slope of line ab	(p) de
(b) Length of line bc	(q) cd
(c) Solid + liquid state	(r) directly proportional to mass
(d) Only liquid state	(s) None of these

3. Six identical conducting rods are connected as shown in figure. In steady state temperature of point a is fixed at 100°C and temperature of e at −80°C. Match the following two columns.



Column I	Column II
(a) Temperature of b	(p) 10°C
(b) Temperature of c	(q) 40°C
(c) Temperature of f	(r) −20°
(d) Temperature of d	(s) None of these

4. Three liquids A, B and C having same specific heats have masses m, $2m$ and $3m$. Their temperatures are, θ, 2θ and 3θ respectively. For temperature of mixture, match the following two columns.

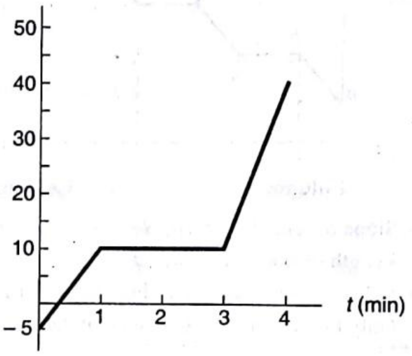
Column I	Column II
(a) When A and B are mixed	(p) $\frac{5}{2}\theta$
(b) When A and C are mixed	(q) $\frac{5}{3}\theta$
(c) When B and C are mixed	(r) $\frac{7}{3}\theta$
(d) When A, B and C all are mixed	(s) $\frac{13}{5}\theta$

5. Match the following two columns.

Column I	Column II
(a) Specific heat	(p) watt
(b) Heat capacity	(q) J/kg-°C
(c) Heat current	(r) J/sec
(d) Latent heat	(s) none

Subjective Questions

1. As a physicist, you put heat into a 500 g solid sample at the rate of 10.0 kJ/min, while recording its temperature as a function of time. You plot your data and obtain the graph shown in figure.
   (a) What is the latent heat of fusion for this solid?



   (b) What is the specific heat of solid state of the material?

2. A hot body placed in air is cooled according to Newton's law of cooling, the rate of decrease of temperature being k times the temperature difference from the surroundings. Starting from $t = 0$, find the time in which the body will lose half the maximum temperature it can lose.

3. Three rods of copper, brass and steel are welded together to form a Y-shaped structure. The cross-sectional area of each rod is $4\ \text{cm}^2$. The end of copper rod is maintained at $100°$ C and the ends of the brass and steel rods at $80°$ C and $60°$ C respectively. Assume that there is no loss of heat from the surfaces of the rods. The lengths of rods are : copper 46 cm, brass 13 cm and steel 12 cm.
   (a) What is the temperature of the junction point?
   (b) What is the heat current in the copper rod ?

$$k(\text{copper}) = 0.92,\ k(\text{steel}) = 0.12 \quad \text{and} \quad k(\text{brass}) = 0.26\ \text{cal/cm-s-°C}$$

4. Ice at $0°$ C is added to 200 g of water initially at $70°$ C in a vacuum flask. When 50 g of ice has been added and has all melted the temperature of the flask and contents is $40°$ C. When a further 80 g of ice has been added and has all melted the temperature of the whole becomes $10°$ C. Find the latent heat of fusion of ice.

5. A copper cube of mass 200 g slides down a rough inclined plane of inclination $37°$ at a constant speed. Assuming that the loss in mechanical energy goes into the copper block as thermal energy. Find the increase in temperature of the block as it slides down through 60 cm. Specific heat capacity of copper is equal to 420 J / kg-K. (Take $g = 10\ \text{m/s}^2$)

6. A cylindrical block of length 0.4 m and area of cross-section $0.04\ m^2$ is placed coaxially on a thin metal disc of mass 0.4 kg and of the same cross section. The upper face of the cylinder is maintained at a constant temperature of 400 K and the initial temperature of the disc is 300 K. If the thermal conductivity of the material of the cylinder is 10 watt/m-K and the specific heat of the material of the disc is 600 J/kg-K, how long will it take for the temperature of the disc to increase to 350 K? Assume for purpose of calculation the thermal conductivity of the disc to be very high and the system to be thermally insulated except for the upper face of the cylinder.

7. A metallic cylindrical vessel whose inner and outer radii are r_1 and r_2 is filled with ice at 0° C. The mass of the ice in the cylinder is m. Circular portions of the cylinder is sealed with completely adiabatic walls. The vessel is kept in air. Temperature of the air is 50° C. How long will it take for the ice to melt completely. Thermal conductivity of the cylinder is k and its length is l. Latent heat of fusion is L.

8. An electric heater is placed inside a room of total wall area $137\ m^2$ to maintain the temperature inside at 20° C . The outside temperature is $-10°$ C. The walls are made of three composite materials. The inner most layer is made of wood of thickness 2.5 cm the middle layer is of cement of thickness 1 cm and the exterior layer is of brick of thickness 2.5 cm. Find the power of electric heater assuming that there is no heat losses through the floor and ceiling . The thermal conductivities of wood, cement and brick are 0.125 W/m°C, 1.5 W/m°C and 1.0 W/m°C respectively.

9. A 2 m long wire of resistance 4 Ω and diameter 0.64 mm is coated with plastic insulation of thickness 0.06 mm. A current of 5 A flows through the wire. Find the temperature difference across the insulation in the steady state. Thermal conductivity of plastic $= 0.16 \times 10^{-2}$ cal/s cm°C.

10. Two chunks of metal with heat capacities C_1 and C_2 are interconnected by a rod of length l and cross-sectional area A and fairly low conductivity k. The whole system is thermally insulated from the environment. At a moment $t = 0$, the temperature difference between two chunks of metal equals $(\Delta T)_0$. Assuming the heat capacity of the rod to be negligible, find the temperature difference between the chunks as a function of time .

11. A rod of length l with thermally insulated lateral surface consists of material whose heat conductivity coefficient varies with temperature as $k = a / T$, where a is a constant. The ends of the rod are kept at temperatures T_1 and T_2. Find the function $T(x)$ where x is the distance from the end whose temperature is T_1.

12. One end of a uniform brass rod 20 cm long and $10\ cm^2$ cross-sectional area is kept at 100°C. The other end is in perfect thermal contact with another rod of identical cross-section and length 10 cm. The free end of this rod is kept in melting ice and when the steady state has been reached, it is found that 360 g of ice melts per hour. Calculate the thermal conductivity of the rod, given that the thermal conductivity of brass is 0.25 cal/s cm °C and $L = 80$ cal/g.

13. Heat flows radially outward through a spherical shell of outside radius R_2 and inner radius R_1. The temperature of inner surface of shell is θ_1 and that of outer is θ_2. At what radial distance from centre of shell the temperature is just half way between θ_1 and θ_2?

ANSWERS

Introductory Exercise 19.1

1. 0°C, mass of ice is 54 g and that of water is 286 g 2. 20.25°C 3. 80°C 4. 87.5 g
5. 1.2×10^4 kg/s

Introductory Exercise 19.2

1. Conduction 2. $\frac{dQ}{dt} = 4\pi kab\left(\frac{T_1 - T_2}{b - a}\right)$ 4. (a) −8.1°C (b) 7.7 W/m^2 5. 105°C
7. 3.7×10^4 W 8. KW^{-1}

For JEE Main

Subjective Questions

1. 7200 cal 2. 0°C, $m_w = 15$ g, $m_i = 10$ g 3. 52°C 4. 40°C 5. 500 m/s
6. 2.5×10^{-3}°C 7. 10.6°C 8. 4 W 9. $\frac{3T_1 + 2(T_2 + T_3)}{7}$ 10. 145.5°C, 118.2°C
11. 412°C/m, 212°C/m 12. 900 W 13. 0.3 14. 9 min.

Objective Questions

1.(d) 2.(b) 3.(c) 4.(c) 5.(a) 6.(c) 7.(b) 8.(c) 9.(a) 10.(b)
11.(b) 12.(d) 13.(b) 14.(b) 15.(d) 16.(c) 17.(a) 18.(d) 19.(c)

More than One Correct Options

20.(a,b) 21.(b,d) 22.(b,d) 23.(b,c) 24.(a,d)

For JEE Advanced

Assertion and Reason

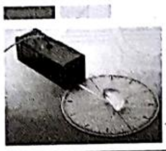
1. (d) 2. (a) 3. (b) 4. (c) 5. (c) 6. (d) 7. (a) 8. (b) 9. (d) 10. (a or b)

Match The Columns

1. (a) → s (b) → p (c) → r (d) → s
2. (a) → s (b) → r (c) → s (d) → q
3. (a) → q (b) → p (c) → p (d) → r
4. (a) → q (b) → p (c) → s (d) → r
5. (a) → q (b) → s (c) → p,r (d) → s

Subjective Questions

1. (a) 40 kJ/kg (b) 1.33 kJ/kg-°C 2. ln(2)/k 3. (a) 84°C (b) 1.28 cal/s 4. 90 cal/g
5. 8.6×10^{-3}°C 6. 166 s 7. $t = mL\ln(r_2/r_1)/100\pi kl$ 8. 17647 W 9. 2.23°C
10. $\Delta T = (\Delta T)_0 e^{-\alpha t}$, where $\alpha = \frac{kA(C_1 + C_2)}{lC_1C_2}$ 11. $T = T_1\left(\frac{T_2}{T_1}\right)^{t/t}$ 12. 0.222 cal/cm-s-°C
13. $\frac{2R_1R_2}{R_1 + R_2}$



EXPERIMENTAL SKILLS

Chapter Contents

- Speed of Sound
- Specific Heat Capacity
- Law of Cooling

1. Speed of Sound Using Resonance Tube

Apparatus

Figure shows a resonance tube. It consists of a long vertical glass tube T. A metre scale S (graduated in mm) is fixed adjacent to this tube. The zero of the scale coincides with the upper end of the tube. The lower end of the tube T is connected to a reservoir R of water tube through a pipe P. The water level in the tube can be adjusted by the adjustabel screws attached with the reservoir. The vertical adjustment of the tube can be made with the help of levelling screws. For fine adjustments of the water level in the tube, the pinchcock is used.

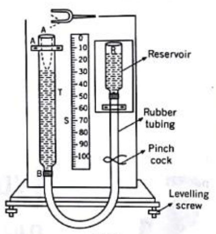


Fig. 1

Principle

If a fibrating tuning fork (of known frequency) is held over the open end of the resonance tube T, then resonance is obtained at some position as the level of water is lowered. If e is the end correction of the tube and l_1 is the length from the water level to the top of the tube, then

$$l_1 + e = \frac{\lambda}{4}$$

$$= \frac{1}{4}\left(\frac{v}{f}\right) \quad \text{...(i)}$$

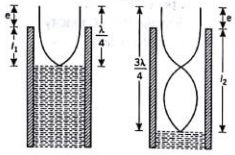


Fig. 2

Here v is the speed of sound in air and f is the frequency of tuning fork (or air column). Now, the water level is further lowered until a resonance is again obtained. If l_2 is the new length of air column.

Then

$$l_2 + e = \frac{3\lambda}{4}$$

$$= \frac{3}{4}\left(\frac{v}{f}\right) \quad \text{...(ii)}$$

Subtracting Eq. (i) from Eq. (ii), we get

$$l_2 - l_1 = \frac{1}{2}\left(\frac{v}{f}\right)$$

or $$v = 2f\,(l_2 - l_1) \quad \text{...(iii)}$$

So, from Eq. (iii) we can find speed of sound v.

Note *that we have nothing to do with the end correction e, as far as v is concerned.*

2. Specific Heat Capacity

(i) Determination of specific heat capacity of a given solid

Specific Heat

"Amount of heat energy required to raise the temperature of unit mass of a solid by one Kelvin or 1°C is known as specific heat of the solid."

Mathematical Expression

If ΔQ is the amount of heat required to raise the temperature of m kg of the solid through ΔT Kelvin. Then

$$\Delta Q \propto m \quad \text{...(i)}$$

$$\Delta Q \propto \Delta T \quad \text{...(ii)}$$

or $$\Delta Q = mc\Delta T$$

where c is the constant called specific heat of solid.

or $$c = \Delta Q / m\Delta T$$

Unit of Specific Heat

Unit of specific heat is J/kg-K in S.I. system. Experiments show that specific heat of a particular material varies with temperature.

Determination of Specific Heat

Specific heat of a solid can be determined by the **"Method of Mixture"** using the concept of the **"law of Heat Exchange"** *i.e.*,

Heat lost by hot body = Heat gained by cold body

The method of mixture is based on the fact that when a hot solid is mixed with a cold body, the hot body loses heat and the cold body absorbs heat until thermal equilibrium is attained. At equilibrium, final temperature of mixture is measured. The specific heat of the solid is calculated with the help of the law of heat exchange.

Let

Mass of solid = m_s kg
Mass of liquid = m_l kg
Mass of calorie meter = m_c kg

Initial temperature of solid = T_s K
Initial temperature of liquid = $T_1 K$
Initial temperature of the calorie meter = $T_c K$
Specific heat of solid = c_s
Specific heat of liquid = c_l
Specific heat of the material of the calorie meter = c_c
Final temperature of the mixture = TK

According to the law of heat exchange

$$Q_{\text{Lost by solid}} = Q_{\text{Gained by Liquid}} + Q_{\text{Gained by calorie meter}}$$

$$M_s c_s (T_s - T) = m_1 c_1 (T - T_1) + m_c c_c (T - T_c)$$

$$c_s = \frac{m_1 c_1 (T - T_1) + m_c c_c (T - T_c)}{M_s (T_s - T)}$$

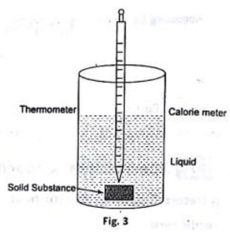


Fig. 3

Which is the required value of specific heat of solid in J/kg K.

(ii) Determination of specific heat capacity of the given liquid by the method of mixtures.

To determine the specific heat capacity of a liquid by the method of mixtures a solid of known specific heat capacity is taken and the given liquid is taken in the calorimeter in place of water. Suppose a solid of mass m_s and specific heat capacity c_s is heated to T_2 °C and then mixed with m_1 mass of liquid of specific heat capacity c_1 at temperature T_1. The temperature of the mixture is T. Then

Heat lost by the solid = $m_s c_s (T_2 - T)$
Heat gained by the liquid plus calorimeter = $(m_1 c_1 + m_c c_c)(T - T_1)$
By Law of heat exchange,

Heat Lost = Heat Gained

$$\therefore \quad m_s c_s (T_2 - T) = (m_1 c_1 + m_c c_c)(T - T_1)$$

From this equation we calculate the value of c_1. However the procedure remains exactly the same as done previously.

3. Law of Cooling

Aim

To study the relationship between temperature of a hot body (allowed to cool) and time, by plotting a cooling curve.

Theory

According to Newton's law of cooling, the rate at which a hot body loses heat is directly proportional to the excess of temperature of hot body over that of its surroundings (provided the difference of temperature is not too large). This law can be expressed mathematically as follows :

Consider a body of mass m and specific heat s losing heat (at temperature θ), at the rate of $\frac{dQ}{dt}$. Let θ_0 be the temperature of its surroundings. Then according to Newton's law of cooling,

$$\frac{dQ}{dt} = -ms\frac{d\theta}{dt} \propto (\theta - \theta_0) \text{ or } \frac{dQ}{dt} = -ms\frac{d\theta}{dt} = k(\theta - \theta_0) \quad \text{...(i)}$$

$$\therefore \quad \frac{d\theta}{dt} = -\frac{k}{ms}(\theta - \theta_0) \quad \text{...(ii)}$$

where k is the constant of proportionality. The equation (ii) can be expressed in the form

$$\frac{d\theta}{dt} = -K(\theta - \theta_0) \quad \text{...(iii)}$$

where K is another constant given as

$$K = \frac{k}{ms}$$

Further modifying the relation (iii) as

$$\frac{d\theta}{\theta - \theta_0} = -Kdt$$

or $$\int \frac{d\theta}{\theta - \theta_0} = -K\int dt \quad \text{...(iv)}$$

or $$\log_e (\theta - \theta_0) = -Kt + C \quad \text{...(v)}$$

where C is constant of integration. Equation (iv) shows that the graph between $\log_e (\theta - \theta_0)$ and t will be a straight line. This has to be borne in mind that this law holds for excess of temperatures of the body over that of its surroundings, maximum up to about 30°C.

If initial temperature of body is θ_i and temperature at time t is θ, then equation (iv) can be written as :

$$\int_{\theta_i}^{\theta} \frac{d\theta}{\theta - \theta_0} = -K\int_0^t dt$$

Solving these equation we get,

$\theta = \theta_0 + (\theta_i - \theta_0)e^{-Kt}$ i.e., θ *versus* t graph is exponentially decreasing.

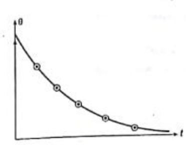


Fig. 4

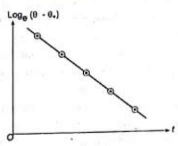


Fig. 5

EXERCISES

Experimental Skills & General Physics

Sound

1. Do the prongs and stem of a tuning fork execute same type of vibrations ?
2. Is the frequency of these vibrations different ?
3. What about the amplitude of two kinds of vibrations ?
4. How does the frequency of tuning fork vary with increase of length of the prong ?
5. Suppose you have been given two tuning forks of the same metal on which, marks of the frequency have disappeared. How will you detect the one with higher frequency ?
6. If the prongs of the fork are rubbed with a file slightly, will it affect the frequency ?
7. If a prong of the tuning fork is loaded with wax, will its frequency change ?
8. Why are the forks made of some standard frequencies like 256, 288, 320, 341.5, 384, 426.6, 484, 512, Hz etc. ?
9. What is the significance of the letters of English alphabet engraved on the tuning forks ?
10. What do you mean by a note ? How does it differ from a tone ?
11. The vibrations of a fork stop when its prongs are touched but they do not stop if the stem is touched. Why?
12. Why a tuning fork has two prongs ?
13. Why should a tuning fork not be struck with a great force ?
14. Why do both the prongs vibrate when we strike only one ?
15. Can sound also travel in vacuum ?
16. In which medium is the velocity of sound higher, in oxygen or hydrogen ?
17. Which particular column in your apparatus, do you call resonance column ?
18. What is the role of water in this apparatus ?
19. Why do you use water; can't you use mercury instead of water ?
20. What types of waves are there in air above water in this resonance column ?
21. How are these waves produced ?
22. How do you get the position of first resonance ?
23. Where are the nodes and antinodes situated ?
24. Is antinode situated exactly at the open end ?
25. How can you get wavelength of sound by this first resonating length ?
26. Can we use a resonance tube of square cross-section for the experiment ?

27. Why do we use a long tube ?
28. How do you keep the vibrating tuning fork near the open end of the tube ?
29. What do you mean by second resonance ?
30. Why is the second resonance found feebler than the first ?
31. Where are the nodes and antinodes in this case ?
32. What will be the wavelength of sound in this case ?
33. Can't you eliminate this end correction ?
34. Can you also determine the velocity of sound by this method ?
35. If n is the frequency of the tuning fork used to excite the air column in resonating air column apparatus, l_1 first resonating length and l_2 second resonating length, then the velocity of sound in air is given by the formula
    (a) $v = n(l_2 - l_1)$
    (b) $v = n(l_1 - l_2)$
    (c) $v = 2n(l_2 - l_1)$
    (d) $v = 2n(l_1 - l_2)$
36. In resonating air column apparatus while comparing the frequencies of the two tuning forks, if

    l_1 is the 1st resonating length of air column with 1st fork of frequency n_1

    l_2 is the 2nd resonating length of air column with 1st fork of frequency n_1

    l_1' is the 1st resonating length of air column with 2nd fork of frequency n_2

    l_2' is the 2nd resonating length of air column with 2nd fork of frequency n_2

    Then :
    (a) $\frac{n_1}{n_2} = \frac{l_2 - l_1}{l_2' - l_1'}$
    (b) $\frac{n_1}{n_2} = \frac{l_2' - l_2}{l_1' - l_1}$
    (c) $\frac{n_1}{n_2} = \frac{l_1' - l_1}{l_2' - l_2}$
    (d) $\frac{n_1}{n_2} = \frac{l_2' - l_1'}{l_2 - l_1}$
37. The end correction (e) is (l_1 = length of air column at first resonance and l_2 is length of air column at second resonance)
    (a) $e = \frac{l_2 - 3l_1}{2}$
    (b) $e = \frac{l_1 - 3l_2}{2}$
    (c) $e = \frac{l_2 - 2l_1}{2}$
    (d) $e = \frac{l_1 - 3l_2}{2}$
38. In the resonating air column experiment, l_1 represents 1st resonating length and l_2 represents 1 2nd resonating length, the relation between 1st and 2nd resonating lengths is
    (a) $l_2 = l_1$
    (b) $l_2 = 2l_1$
    (c) $l_2 = 3l_1$
    (d) $l_2 = 4l_1$
39. In resonating air column apparatus the approximate relation between end correction and diameter of the tube is
    (a) $0.3d$
    (b) $0.15d$
    (c) $0.6d$
    (d) $0.4d$
40. In resonance column apparatus the reason for hearing booming sound is because
    (a) the air column in the tube and the tuning fork vibrate with the same frequency
    (b) the air column in the tube vibrates with frequency which is greater than the frequency of the tuning fork
    (c) the air column in the tube vibrates with frequency which is less than the frequency of the tuning fork.
    (d) velocity of sound in air column is greater than the velocity of sound in atmospheric air

41. The resonating lengths of an air column in the first and second modes of vibration are 37 cm and 97 cm. If velocity of sound is 300 ms^{-1} the frequency of the tuning fork is
(a) 1000 Hz (b) 500 Hz (c) 250 Hz (d) 125 Hz

42. The end correction of a resonance tube is 1 cm. If shortest resonating length is 15 cm, the next resonating length will be
(a) 47 cm (b) 45 cm (c) 50 cm (d) 33 cm

43. A tuning fork of frequency 340 Hz is excited and held above a cylindrical tube of length 120 cm. It is slowly filled with water. The minimum height of water column required for resonance to be first heard (Velocity of sound = 340 ms^{-1}) is
(a) 25 cm (b) 75 cm (c) 45 cm (d) 105 cm

44. Two unknown frequency tuning forks are used in resonance column apparatus. When only first tuning fork is excited the 1st and 2nd resonating lengths noted are 10 cm and 30 cm respectively. When only second tuning fork is excited the 1st and 2nd resonating lengths noted are 30 cm and 90 cm respectively. The ratio of the frequency of the 1st to 2nd tuning fork is
(a) 1 : 3 (b) 1 : 2 (c) 3 : 1 (d) 2 : 1

45. In a resonance tube, the first resonance is obtained when the level of water in the tube is at 16 cm from the open end. Neglecting end correction, the next resonance will be obtained when the level of water from the open end is
(a) 24 cm (b) 32 cm (c) 48 cm (d) 64 cm

Heat

46. Is specific heat of a substance is a constant quantity ?

47. What is meant by thermal capacity of a body ?

48. What are the units of water equivalent and thermal capacity of a body ?

49. Why do we use generally a calorimeter made of copper ?

50. In an experiment to determine the specific heat of aluminium, piece of aluminium weighing 500g is heated to 100 °C. It is then quickly transferred into a copper calorimeter of mass 500g containing 300g of water at 30 °C. The final temperature of the mixture is found to be 46.8 °C. If specific heat of copper is 0.093 cal g^{-1} °C^{-1}, then the specific heat of aluminium is
(a) 0.11 cal g^{-1} °C^{-1} (b) 0.22 cal g^{-1} °C^{-1} (c) 0.33 cal g^{-1} °C^{-1} (d) 0.44 cal g^{-1} °C^{-1}

51. The mass of a copper calorimeter is 40 g and its specific heat in SI units is 4.2×10^{2} J kg^{-1} °C^{-1}. The thermal capacity is
(a) 4 J °C^{-1} (b) 18.6 J (c) 16.8 J/kg (d) 16.8J °C^{-1}

52. When 0.2 kg of brass at 100 °C is dropped into 0.5 kg of water at 20 °C, the resulting temperature is 23 °C. The specific heat of brass is
(a) 0.41×10^{3} Jkg^{-1} °C^{-1} (b) 0.41×10^{2} Jkg^{-1} °C^{-1}
(c) 0.41×10^{4} Jkg^{-1} °C^{-1} (d) 0.41 Jkg^{-1} °C^{-1}

53. In an experiment to determine the specific heat of a metal, a 0.20 kg block of the metal at 150 °C is dropped in a copper calorimeter (of water equivalent 0.025 kg) containing 150 cm^3 of water at 27 °C. The final temperature is 40 °C. The specific heat of the metal is
(a) 0.1 Jg^{-1} °C^{-1} (b) 0.2 Jg^{-1} °C^{-1}
(c) 0.3 cal g^{-1} °C^{-1} (d) 0.1 cal g^{-1} °C^{-1}

54. Is Newton's law of cooling true for all differences of temperature between the body losing heat and that of its surroundings ?

55. How do you express this law mathematically ?

56. What is the shape of the graph of $\log(\theta-\theta_0)$ *versus* t ?

ANSWERS

Experimental Skills & General Physics

35.(c)	36.(d)	37.(a)	38.(c)	39.(a)	40.(a)	41.(c)	42.(a)	43.(c)	44.(c)
45.(c)	50.(b)	51.(c)	52.(a)	53.(d)					

HINTS & SOLUTIONS

Chapter 14 — Wave Motion

JEE Advanced (Subjective Questions)

1. (a) $$v_P = -v\left(\frac{dy}{dx}\right)$$

As v_P and $(\text{slope})_P$ are both positive, v must be negative. Hence, the wave is moving in negative x-axis.

(b) $$y = A\sin(\omega t - kx + \phi) \quad \text{...(i)}$$
$$k = \frac{2\pi}{\lambda} = \frac{\pi}{2}\ \text{cm}^{-1}$$
$$A = 4 \times 10^{-3}\ \text{m} = 0.4\ \text{cm}$$

At $t = 0, x = 0$, slope $\frac{dy}{dx} = +\text{ ve}$

$\therefore$ $$v_P = -v(\text{slope}) = +\text{ ve}$$

Further at $$t = 0, x = 0, y = +\text{ ve}$$

$\therefore$ $$\phi = \frac{\pi}{4}$$

Further, $$20\sqrt{3} = -v\tan 60^\circ$$

$\therefore$ $$v = -20\ \text{cm/s}$$
$$f = \frac{v}{\lambda} = 5\ \text{Hz}$$

$\therefore$ $$\omega = 2\pi f = 10\pi$$

$\therefore$ $$y = (0.4\ \text{cm})\sin\left(10\pi t + \frac{\pi}{2}x + \frac{\pi}{4}\right)$$ **Ans.**

(c) $$P = 2\pi^2 A^2 f^2 \mu v$$

$\therefore$ Energy carried per cycle $$E = PT = \frac{P}{f} = 2\pi^2 A^2 f \mu v$$

Substituting the values, we have $$E = 1.6 \times 10^{-5}\ \text{J}$$ **Ans.**

2. (a) $$v = \sqrt{\frac{T}{\rho S}} = \sqrt{\frac{64}{12.5 \times 10^3 \times 0.8 \times 10^{-6}}} = 80\ \text{m/s}$$ **Ans.**

(b) $$\omega = 2\pi f = 2\pi(20) = 40\pi\ \text{rad/s}$$
$$k = \frac{\omega}{v} = \frac{40\pi}{80} = \frac{\pi}{2}\ \text{m}^{-1}$$

$\therefore$ $$y = (1.0\ \text{cm})\cos\left[(40\pi s^{-1})t - \left(\frac{\pi}{2}m^{-1}\right)x\right]$$ **Ans.**

(c) Substituting $x = 0.5$ m and $t = 0.05$ s, we get
$$y = \frac{1}{\sqrt{2}}\ \text{cm}$$ **Ans.**

(d) Particle velocity at time t,
$$v_P = \frac{\partial y}{\partial t} = -(40\pi\ \text{cm/s})\sin\left[(40\pi s^{-1})t - \left(\frac{\pi}{2}\text{m}^{-1}\right)x\right]$$

Substituting $x = 0.5$ m and $t = 0.05$ s, we get
$$v_P = 89\ \text{cm/s}$$ **Ans.**

3. $$k_{\text{eff}} = 2k = 1.0\ \text{N/m}$$
$$f = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{m}} = \frac{10}{2\pi}\ \text{s}^{-1}$$
$$v = 0.1\ \text{m/s}, \quad A = 0.02\ \text{m}$$
$$\lambda = \frac{v}{f} = \frac{0.1}{\left(\frac{10}{2\pi}\right)} = \frac{2\pi}{100}\ \text{m}$$
$$y = A\cos\left(\frac{2\pi}{\lambda}\right)(vt - x)$$
$$= 0.02\cos 100(0.1t - x)$$
$$= 0.02\cos(10t - 100x)\ \text{m}$$ **Ans.**

The distance between two successive maximas
$$= \lambda = \frac{2\pi}{100} = 0.0628\ \text{m}$$ **Ans.**

4. (a) Dimensions of A and Y are same. Similary dimensions of a and x are same.

(b) As the wave is travelling towards positive x-axis, there should be netative sign between term of x and term of t.

Further, $$\text{speed of wave } v = \frac{\text{coefficient of } t}{\text{coefficient of } x}$$

$\therefore$ coefficient of $t = (v) \times$ coefficient of x.

5. From the given figure we can see that :

(a) Amplitude $A = 1.0$ mm

(b) Wavelength $\lambda = 4$ cm

(c) Wave number $k = \frac{2\pi}{\lambda} = 1.57\ \text{cm}^{-1} \approx 1.6\ \text{cm}^{-1}$

(d) Frequency $f = \frac{v}{\lambda} = \frac{20}{4} = 5$ Hz

6. $v = \sqrt{\frac{T}{\rho S}}$ or $v \propto \frac{1}{\sqrt{\rho}}$

$\therefore$ $$\frac{v_1}{v_2} = \sqrt{\frac{\rho_2}{\rho_1}}$$

$\therefore$ $$\frac{\rho_1}{\rho_2} = \left(\frac{v_2}{v_1}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

7. $v_1 = \sqrt{\frac{T_1}{\mu_1}} = \sqrt{\frac{4.8}{1.2 \times 10^{-2}}} = 20$ m/s
$$v_2 = \sqrt{\frac{T_2}{\mu_2}} = \sqrt{\frac{7.5}{1.2 \times 10^{-2}}} = 25\ \text{m/s}$$

Pulses will meet when $x_A = x_B$

or $$20t = 25(t - 0.02)$$

$\therefore$ $$t = 0.1\ \text{s}$$

and $$x_A \text{ or } x_B = 20 \times 0.1 = 2\ \text{m}$$

8. (a) $P = \frac{1}{2}\rho\omega^2 A^2 sv$ or $A = \frac{1}{\omega}\sqrt{\frac{2P}{\rho sv}}$ ...(i)

Here $$\rho s = \mu = \text{mass per unit length} = \frac{6 \times 10^{-3}}{8}\ \text{kg/m}$$
$$\omega = 2\pi f = \frac{2\pi v}{\lambda} = \frac{2\pi \times 30}{0.2}$$

Substituting these values in Eq. (i) we have,
$$A = \frac{0.2}{2\pi \times 30}\sqrt{\frac{2 \times 50 \times 8}{6 \times 10^{-3} \times 30}} = 0.0707\ \text{m}$$
$$= 7.07\ \text{cm}$$

(b) $P \propto v\omega^2$ or $P \propto v(v^2)$

or $$P \propto v^3$$

When wave speed is double, power will become eight times.

9. $-dT = (dm)x\omega^2 = \left(\frac{m}{L}.dx\right)x\omega^2$

or $$-\int_0^T dT = \frac{m}{L}\omega^2 \int_{x=L}^{x} x.\,dx$$

$\therefore$ $$-T = \frac{m}{L}\omega^2\left(\frac{x^2}{2} - \frac{L^2}{2}\right)$$

or $$T = \frac{m\omega^2}{2L}(L^2 - x^2)$$
$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{\frac{m\omega^2}{2L}(L^2 - x^2)}{m/L}} = \omega\sqrt{\frac{L^2 - x^2}{2}}$$

or $$\frac{dx}{dt} = \frac{\omega}{\sqrt{2}}\sqrt{L^2 - x^2}$$

$\therefore$ $$\int_0^t dt = \frac{\sqrt{2}}{\omega}\int_0^L \frac{dx}{\sqrt{L^2 - x^2}}$$

$\therefore$ $$t = \frac{\sqrt{2}}{\omega}\left[\sin^{-1}\left(\frac{x}{L}\right)\right]_0^L = \frac{\sqrt{2}}{\omega}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{2}\omega}$$ **Ans.**

Chapter 15 — Superposition of Waves

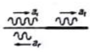
JEE Advanced (Subjective Questions)

1. (a) $v_1 = \sqrt{T/\mu_1}$, $v_2 = \sqrt{F/4\mu_1} = \frac{1}{2}\sqrt{F/\mu_1}$, $v_3 = \sqrt{F/(\mu_1/4)} = 2\sqrt{F/\mu_1}$

$\therefore$ $$t = t_1 + t_2 + t_3 = \frac{L}{v_1} + \frac{L}{v_2} + \frac{L}{v_3} = \frac{7L}{2}\sqrt{\frac{\mu_1}{F}}$$ **Ans.**

2. Let a_i and a_r be the amplitudes of incident and reflected waves.

Then $$\frac{a_i + a_r}{a_i - a_r} = 6$$ (given)



Hence $$\frac{a_r}{a_i} = \frac{5}{7}$$

Now $$\frac{E_r}{E_i} = \left(\frac{a_r}{a_i}\right)^2 = \left(\frac{5}{7}\right)^2 = 0.51$$

or percentage of energy reflected is
$$100 \times \frac{E_r}{E_i} = 51\%.$$

So, percentage of energy transmitted will be $(100 - 51)\%$ or 49%. **Ans.**

3. Amplitude at a distance x is
$$A = a\sin kx$$

First node can be obtained at $x = 0$,

and the second at $x = \pi/k$

At position x, mass of the element PQ is

$$dm = (\rho S)dx$$

its amplitude is $A = a\sin kx$

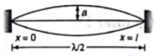
Hence mechanical energy stored in this element is

(energy of particle in SHM) $$dE = \frac{1}{2}(dm)A^2\omega^2$$

or $$dE = \frac{1}{2}(\rho S A^2\omega^2).\,dx$$
$$= \frac{1}{2}(\rho S a^2\omega^2\sin^2 kx)\,dx$$

Therefore, total energy stored between two adjacent nodes will be
$$E = \int_{x=0}^{x=\pi/k} dE$$

Solving this, we get
$$E = \frac{\pi S\rho\omega^2 a^2}{4k}$$ **Ans.**



4. $l = \frac{\lambda}{2}$ or $\lambda = 2l, k = \frac{2\pi}{\lambda} = \frac{\pi}{l}$

The amplitude at a distance x from $x = 0$ is given by

$$A = a \sin kx$$

Total mechanical energy at x of length dx is

$$dE = \frac{1}{2}(dm)A^2\omega^2$$

$$= \frac{1}{2}(\mu dx)(a \sin kx)^2(2\pi f)^2$$

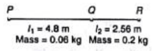
or $$dE = 2\pi^2 \mu f^2 a^2 \sin^2 kx\, dx \qquad ...(i)$$

Here $$f = \frac{v^2}{\lambda^2} = \frac{\left(\frac{T}{\mu}\right)}{(4l^2)} \text{ and } k = \frac{\pi}{l}$$

Substituting these values in Eq. (i) and integrating it from $x = 0$ to $x = l$, we get total energy of string.

$$E = \frac{\pi^2 a^2 T}{4l} \qquad \textbf{Ans.}$$

5. Tension $T = 80$ N



Amplitude of incident wave, $A_i = 3.5$ cm

Mass per unit length of wire PQ is

$$m_1 = \frac{0.06}{4.8} = \frac{1}{80} \text{ kg/m}$$

and mass per unit length of wire QR is

$$m_2 = \frac{0.2}{2.56} = \frac{1}{12.8} \text{ kg/m}$$

(a) Speed of wave in wire PQ is

$$v_1 = \sqrt{T/m_1} = \sqrt{\frac{80}{1/80}} = 80 \text{ m/s}$$

and speed of wave in wire QR is

$$v_2 = \sqrt{T/m_2} = \sqrt{\frac{80}{1/12.8}} = 32 \text{ m/s}$$

$\therefore$ Time teken by the wave pulse to reach from P to R is

$$t = \frac{4.8}{v_1} + \frac{2.56}{v_2} = \left(\frac{4.8}{80} + \frac{2.56}{32}\right) \text{s} = 0.14 \text{ s} \qquad \textbf{Ans.}$$

(b) The expressions for reflected and transmitted amplitudes (A_r and A_t) in terms of v_1, v_2 and A_i are as follows

$$A_r = \frac{v_2 - v_1}{v_2 + v_1} A_i \text{ and } A_t = \frac{2v_2}{v_1 + v_2} A_i$$

Substituting the values, we get

$$A_r = \left(\frac{32 - 80}{32 + 80}\right)(3.5) = -1.5 \text{ cm}$$

i.e., the amplitude of reflected wave will be 1.5 cm. Negative sign of A_r indicates that there will be a phase change of π in reflected wave.

Similarly

$$A_t = \left(\frac{2 \times 32}{32 + 80}\right)(3.5) = 2.0 \text{ cm}$$

i.e., the amplitude of transmitted wave will be 2.0 cm.

The expressions of A_r and A_t are derived as below.

Derivation

Suppose the incident wave of amplitude A_i and angular frequency ω is travelling in positive x-direction with velocity v_1 then we can write

$$y_i = A_i \sin \omega[t - x/v_1] \qquad ...(i)$$

In reflected as well as transmitted wave, ω will not change, therefore, we can write

$$y_r = A_r \sin \omega[t + x/v_1] \qquad (ii)$$

and $$y_t = A_t \sin \omega[t - x/v_2] \qquad ...(iii)$$

Now as wave is continuous, so at the boundary ($x = 0$).

Continuity of displacement requires

$$y_i + y_r = y_t \text{ for } x = 0$$

Substituting from (i), (i) and (iii) in the above, we get

$$A_i + A_r = A_t \qquad ...(iv)$$

Also at the boundary, slope of wave will be continuous, *i.e.*,

$$\frac{\partial y_i}{\delta x} + \frac{\partial y_r}{\delta x} = \frac{\partial y_t}{\partial x} \qquad \text{for } x = 0$$

which gives

$$A_i - A_r = \left(\frac{v_1}{v_2}\right) A_t \qquad ...(v)$$

Solving Eq. (iv) and (v) for A_r and A_t we get the required equations, *i.e.*,

$$A_r = \frac{v_2 - v_1}{v_2 + v_1} A_i$$

and $$A_t = \frac{2v_2}{v_2 + v_1} A_i$$

6. **When A is a node :** Suppose n_1 and n_2 are the complete loops formed on left and right side of point A. Then

$$f_1 = f_2$$

or $$n_1\left(\frac{v_1}{2L}\right) = n_2\left(\frac{v_2}{2L}\right)$$

or $$\frac{n_1}{n_2} = \left(\frac{v_2}{v_1}\right) = \sqrt{\frac{\mu_1}{\mu_2}} = \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, ..., \text{etc.} \qquad \left(\text{as } v \propto \frac{1}{\sqrt{\mu}}\right)$$

$\therefore$ Possible frequencies are,

$$\frac{v_1}{2L}, 2\left(\frac{v_1}{2L}\right), \frac{3v_1}{2L}, ..., \text{etc.} \qquad \left(v_1 = \sqrt{\frac{T}{\mu}}\right)$$

or $$\frac{1}{2L}\sqrt{\frac{T}{\mu}}, \frac{1}{L}\sqrt{\frac{T}{\mu}}, \frac{3}{2L}\sqrt{\frac{T}{2\mu}}, ..., \text{etc.}$$

When A is an antinode : Suppose n_1 and n_2 are complete loops on left and right side of point A.

$$n_1\frac{\lambda_1}{2} + \frac{\lambda_1}{4} = L \quad \text{or} \quad f_1 = \frac{v_1}{L}\left(\frac{n_1}{2} + \frac{1}{4}\right)$$

$$n_2\frac{\lambda_2}{2} + \frac{\lambda_2}{4} = L \quad \text{or} \quad f_2 = \frac{v_2}{L}\left(\frac{n_2}{2} + \frac{1}{4}\right)$$

Substituting, $f_1 = f_2$, we get

$$\frac{2n_1 + 1}{2n_2 + 1} = \frac{1}{3}$$

$\therefore$ For
$$n_1 = 1, \; n_2 = 4$$
$$n_1 = 2, \; n_2 = 7$$
$$n_1 = 3, \; n_2 = 10 \quad \text{etc.}$$
$$\dots \quad \dots \quad \dots$$

Therefore, the possible frequencies are

$$\frac{v_1}{L}\left(\frac{1}{2} + \frac{1}{4}\right), \frac{v_1}{L}\left(\frac{2}{2} + \frac{1}{4}\right), \frac{v_1}{L}\left(\frac{3}{2} + \frac{1}{4}\right), ..., \text{etc.}$$

or $$\frac{3}{4L}\sqrt{\frac{T}{\mu}}, \frac{5}{4L}\sqrt{\frac{T}{\mu}}, \frac{7}{4L}\sqrt{\frac{T}{\mu}}, ..., \text{etc.}$$

7. $$\frac{L}{4} = \frac{\lambda}{4}$$

$\therefore$ $$L = \lambda$$

In the next higher mode there will be total 6 loops and the desired frequency is

$$\left(\frac{6}{2}\right)(100) = 300 \text{ Hz} \qquad \textbf{Ans.}$$

8. $$k = \frac{\omega}{v} = \frac{\pi}{3}$$

$$y_1 = 0.06\,(\pi t - kx) = 0.06 \sin\left(\pi t - \frac{\pi}{3} \times 12\right)$$

$$= 0.06 \sin(\pi t - 4\pi)$$

Similarly, $$y_2 = 0.02 \sin(\pi t - kx') = 0.02 \sin\left(\pi t - \frac{\pi}{3} \times 8\right)$$

$$= 0.02 \sin\left(\pi t - \frac{8\pi}{3}\right)$$

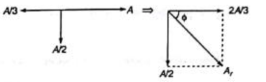
$$y = y_1 + y_2$$

$$= 0.06 \sin \pi t \cos 4\pi - 0.06 \cos \pi t \sin 4\pi + 0.02 \sin \pi t \cos \frac{8\pi}{3} - 0.02 \cos \pi t \sin \frac{8\pi}{3}$$

$$= 0.05 \sin \pi t - 0.0173 \cos \pi t \qquad \textbf{Ans.}$$

9. Resultant amplitude $A_r = \sqrt{\left(\frac{2A}{3}\right)^2 + \left(\frac{A}{2}\right)^2}$

$$= \frac{5}{6}A \qquad \textbf{Ans.}$$



$$\tan\phi = -\frac{A/2}{2A/3} = -\frac{3}{4}$$

or $$\phi = -\tan^{-1}(3/4) \qquad \textbf{Ans.}$$

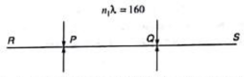
10. Speed of longitudinal waves in the rod

$$v = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{1.6 \times 10^{11}}{2500}} = 8000 \text{ m/s}$$

At the clamped position nodes will be formed. Between the clamps integer number of loops will be formed. Hence,

$$n_1\frac{\lambda}{2} = 80$$

or $$n_1\lambda = 160 \qquad ...(i)$$



Between P and R, P is a fixed end and R is the free end. It means the number of loops between P and R will be odd multiple of $\frac{\lambda}{4}$. Then

$$\frac{(2n_2 - 1)}{2}\frac{\lambda}{2} = 5$$

or $$(2n_2 - 1)\lambda = 20 \qquad ...(ii)$$

Also between Q and S

$$(2n_3 - 1)\lambda = 60 \qquad ...(iii)$$

From Eqs. (i) and (ii)

$$\frac{n_1}{2n_2 - 1} = \frac{160}{20} = 8 \qquad ...(iv)$$

and from Eqs. (i) and (iii)

$$\frac{n_1}{2n_3 - 1} = \frac{160}{60} = \frac{8}{3} \qquad ...(v)$$

For minimum frequency n_1, n_2 and n_3 should be least from Eqs. (iv) and (v)

We get, $n_1 = 8$, $n_2 = 1$, $n_3 = 2$

$$\lambda = \frac{20}{2n_2 - 1} = 20 \text{ cm} \qquad \text{[from Eq. (ii)]}$$

$$= 0.2 \text{ m}$$

$\therefore$ $$f_{min} = \frac{v}{\lambda} = \frac{8000}{0.2} = 40 \text{ kHz} \qquad \textbf{Ans.}$$

Next higher frequency corresponds to

$$n_1 = 24, \; n_2 = 2 \quad \text{and} \quad n_3 = 5$$

and $$f = 120 \text{ kHz} \qquad \textbf{Ans.}$$

11. (a) Distance between two nodes is $\lambda/2$ or π/k. The volume of string between two nodes is therefore,

$$V = \frac{\pi}{k} s \quad \text{...(ii)}$$

Energy density (energy per unit volume) of each wave will be,

$$u_1 = \frac{1}{2}\rho\omega^2(8)^2 = 32\rho\omega^2$$

and $$u_2 = \frac{1}{2}\rho\omega^2(6)^2 = 18\rho\omega^2$$

$\therefore$ Total mechanical energy between two consecutive nodes will be,

$$E = (u_1 + u_2)V$$
$$= 50\frac{\pi}{k}\rho\omega^2 s$$

(b) $$y = y_1 + y_2$$
$$= 8\sin(\omega t - kx) + 6\sin(\omega t + kx)$$
$$= 2\sin(\omega t - kx) + \{6\sin(\omega t + kx) + 6\sin(\omega t + kx)\}$$
$$= 2\sin(\omega t - kx) + 12\cos kx \sin \omega t$$

Thus, the resultant wave will be a sum of standing wave and a travelling wave.

Energy crossing through a node per second = power or travelling wave

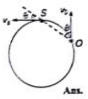
$\therefore$ $$P = \frac{1}{2}\rho\omega^2(2)^2 sv$$
$$= \frac{1}{2}\rho\omega^2(4)(s)\left(\frac{\omega}{k}\right)$$
$$= \frac{2\rho\omega^3 s}{k}$$

Chapter 16 Sound Waves

JEE Advanced (Subjective Questions)

1. $v_s = v_0 = v$

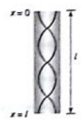
Let u be the speed of sound. Then



$$f' = f\left(\frac{u + v_0\cos\theta}{u + v_s\cos\theta}\right)$$
$$= f\left(\frac{u + v\cos\theta}{u + v\cos\theta}\right)$$

or $$f' = f$$ **Ans.**

2. Given length of pipe $l = 3$ m



Third harmonic implies that $3\left(\frac{\lambda}{2}\right) = l$

or $$\lambda = \frac{2l}{3} = \frac{2\times 3}{3} = 2\text{ m}$$

The angular frequency is $\omega = 2\pi f$

$$= \frac{2\pi v}{\lambda} = \frac{(2\pi)(332)}{2}$$

or $$\omega = 332\pi \text{ rad/s}$$

The particle displacement $y(x, t)$ can be written as

$$y(x,t) = A\cos kx \sin \omega t$$
$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{(2l/3)} = \frac{3\pi}{l}$$

and $$\omega = kv = \frac{3\pi v}{l} \qquad \left(v = \frac{\omega}{k}\right)$$

$\therefore$ $$y(x,t) = A\cos\left(\frac{3\pi x}{l}\right)\cdot\sin\left(\frac{3\pi v}{l}\right)t$$

The longitudinal oscillations of an air column can be viewed as oscillations of particle displacement or pressure wave or density wave. Pressure variation is related to particle displacement as

$$P(x,t) = -B\frac{\partial y}{\partial x} \qquad (B = \text{Bulk modulus})$$
$$= \left(\frac{3BA\pi}{l}\right)\sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3\pi v}{l}\right)t$$

The amplitude of pressure variation is

$$P_{max} = \frac{3BA\pi}{l}$$
$$v = \sqrt{\frac{B}{\rho}} \text{ or } B = \rho v^2$$

$\therefore$ $$P_{max} = \frac{3\rho v^2 A\pi}{l}$$

or $$A = \frac{P_{max} l}{3\rho v^2 \pi}$$

Here $P_{max} = 1\%$ or $P_0 = 10^3$ N/m^2

Substituting the values

$$A = \frac{(10^3)(3)}{(3)(1.03)(332)^2\pi}$$
$$= 0.0028\text{ m}$$

or $$A = 0.28\text{ cm}$$ **Ans.**

According to definition of Bulk modulus (B)

$$B = \frac{-dP}{(dV/V)} \quad \text{...(i)}$$

$$\text{Volume} = \frac{\text{mass}}{\text{density}} \text{ or } V = \frac{m}{\rho}$$

or $$dV = -\frac{m}{\rho^2}.d\rho = -\frac{V.d\rho}{\rho}$$

or $$\frac{dV}{V} = -\frac{d\rho}{\rho}$$

Substituting in Eq. (i), we get

$$d\rho = \frac{\rho(dP)}{B}$$

or amplitude of density oscillation is

$$d\rho_{max} = \frac{\rho}{B}.P_{max}$$
$$= \frac{P_{max}}{v^2} \qquad \left(\frac{\rho}{B} = \frac{1}{v^2}\right)$$
$$= \frac{10^3}{(332)^2}$$
$$= 9\times 10^{-3}\text{ kg/m}^3$$

3. Sound level (in dB)

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

where $$I_0 = 10^{-12}\text{ W/m}^2$$
$$L = 60\text{ dB}$$

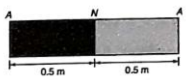
Hence $$I = (10^6)I_0 = 10^{-6}\text{ W/m}^2$$

Intensity, $$I = \frac{\text{Power}}{\text{Area}} = \frac{P}{4\pi r^2}$$

$\Rightarrow$ $$P = I(4\pi r^2)$$
$$P = (10^{-6})(4\pi)(500)^2 = 3.14\text{ W}$$

$\therefore$ Time, $$t = \frac{(1.0\times 10^3)\left(\frac{30}{100}\right)}{3.14}$$
$$t = 95.5\text{ s}$$

4. Let n_1 harmonic of the chamber containing H_2 is equal to n_2 harmonic of the chamber containing O_2. Then



$$n_1\left(\frac{v_{H_2}}{4l}\right) = n_2\left(\frac{v_{O_2}}{4l}\right)$$

$\therefore$ $$\frac{n_1}{n_2} = \frac{v_{O_2}}{v_{H_2}} = \frac{300}{1100} = \frac{3}{11}$$

$\therefore$ $$f_{min} = 3\left(\frac{v_{H_2}}{4l}\right) = 3\left(\frac{1100}{4\times 0.5}\right) = 1650\text{ Hz}$$ **Ans.**

5. This problem is a Doppler-effect analogy.

(a) Here, $$f = 20\text{ min}^{-1}$$
$$v = 300\text{ m/min}$$
$$v_s = 0 \text{ and } v_0 = 0$$

Spacing between the pies $= \frac{300}{20} = 15$ m **Ans.**

and $$f' = f = 20\text{ min}^{-1}$$

(b) $v_s = 30$ m/min

Spacing between the pies will be

$$\frac{300-30}{20} \quad \text{or} \quad 13.5\text{ m}$$

and $$f' = f\left(\frac{v}{v - v_s}\right) = (20)\left(\frac{300}{300-30}\right)$$
$$= 22.22\text{ min}^{-1}$$ **Ans.**

6. (a) Comparing with the equation of a travelling wave

$$y = a\sin(kx - \omega t)$$
$$k = 15\pi \quad \text{and} \quad \omega = 6000\pi$$

$\therefore$ velocity of the sound $v = \frac{\omega}{k}$

$$= \frac{6000\pi}{15\pi} = 400\text{ ms}^{-1}$$

as $$v = \sqrt{\frac{B}{\rho}}$$

Hence, $$\rho = \frac{B}{v^2} = \frac{1.6\times 10^5}{(400)^2} = 1\text{ kg/m}^3$$ **Ans.**

(b) Pressure amplitude $P_0 = BAk$

Hence
$$A = \frac{P_0}{Bk}$$
$$= \frac{24\pi}{1.6\times10^5\times15\pi}$$
$$= 10^{-5}\text{ m} = 10\,\mu\text{m}$$ **Ans.**

(c) Intensity received by the person
$$I = \frac{W}{4\pi R^2} = \frac{W}{4\pi(10)^2}$$

Also,
$$I = \frac{P_0^2}{2\rho v}$$
$$\frac{W}{4\pi(10)^2} = \frac{(24\pi)^2}{2\times1\times400}$$
$$W = 288\pi^3\text{ W}$$ **Ans.**

7. (a) Path difference and hence phase difference at P from both the sources is $0°$, whether $\theta = 45°$, or $\theta = 60°$. So, both the wave will interfere constructively. Or maximum intensity will be obtained. From
$$I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2 \text{ we have}$$
$$I_0 = (\sqrt{I} + \sqrt{I})^2 = 4I \quad (I_1 = I_2 = I\text{, say})$$
$\therefore$
$$I = I_0/4$$
When one source is switched off, no interference will be obtained. Intensity will be due to a single source, or $I_0/4$.

(b) At $\theta = 60°$, also maximum intensity or I_0 will be observed.

8. (a) Frequency of second harmonic in pipe A = frequency of third harmonic in pipe B

$\therefore$
$$2\left(\frac{v_A}{2l_A}\right) = 3\left(\frac{v_B}{4l_B}\right)$$
or
$$\frac{v_A}{v_B} = \frac{3}{4}$$
$$\frac{\sqrt{\frac{\gamma_A RT_A}{M_A}}}{\sqrt{\frac{\gamma_B RT_B}{M_B}}} = \frac{3}{4}$$
or
$$\sqrt{\frac{\gamma_A}{\gamma_B}}\sqrt{\frac{M_B}{M_A}} = \frac{3}{4} \qquad (\text{as } T_A = T_B)$$
$\therefore$
$$\frac{M_A}{M_B} = \frac{\gamma_A}{\gamma_B}\left(\frac{16}{9}\right)$$
$$= \left(\frac{5/3}{7/5}\right)\left(\frac{16}{9}\right)$$

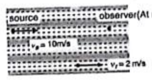
or
$$\frac{M_A}{M_B} = \left(\frac{25}{21}\right)\left(\frac{16}{9}\right) = \frac{400}{189}$$

(b) Ratio of fundamental frequency in pipe A and in pipe B is :
$$\frac{f_A}{f_B} = \frac{v_A/2l_A}{v_B/2l_B}$$
$$= \frac{v_A}{v_B} \qquad (\text{as } l_A = l_B)$$
$$= \sqrt{\frac{\frac{\gamma_A RT_A}{M_A}}{\frac{\gamma_B RT_B}{M_B}}}$$
$$= \sqrt{\frac{\gamma_A}{\gamma_B}\cdot\frac{M_B}{M_A}} \qquad (\text{as } T_A = T_B)$$

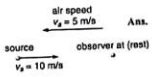
Substituting $\frac{M_B}{M_A} = \frac{189}{400}$ from part (a), we get
$$\frac{f_A}{f_B} = \sqrt{\frac{25}{21}\times\frac{189}{400}} = \frac{3}{4}$$ **Ans.**

9. Velocity of sound in water is
$$v_w = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{2.088\times10^9}{10^3}} = 1445\text{ m/s}$$

Frequency of sound in water will be
$$f_0 = \frac{v_w}{\lambda_w} = \frac{1445}{14.45\times10^{-3}}\text{ Hz}$$
$$f_0 = 10^5\text{ Hz}$$



(a) Frequency of sound detected by receiver (observer) at rest would be
$$f_1 = f_0\left(\frac{v_w + v_r}{v_w + v_r - v_s}\right)$$
$$= 10^5\left(\frac{1445 + 2}{1445 + 2 - 10}\right)\text{ Hz}$$
$$f_1 = 1.0069\times10^5\text{ Hz}$$ **Ans.**



(b) Velocity of sound in air is
$$v_a = \sqrt{\frac{\gamma RT}{M}}$$
$$= \sqrt{\frac{(1.4)(8.31)(20+273)}{28.8\times10^{-3}}}$$
$$= 344\text{ m/s}$$

Frequency does not depend on the medium. Therefore, frequency in air is also $f_0 = 10^5$ Hz.

$\therefore$ Frequency of sound detected by receiver (observer) in air would be
$$f_2 = f_0\left(\frac{v_a - v_w}{v_a - v_w - v_s}\right)$$
$$= 10^5\left[\frac{344-5}{344-5-10}\right]\text{ Hz}$$
$$f_2 = 1.0304\times10^5\text{ Hz}$$ **Ans.**

10. Frequency of fundamental mode of closed pipe
$$f_1 = \frac{v}{4l} = 200\text{ Hz}$$
Decreasing the tension in the string decrease the beat frequency.
Hence the first overtone frequency of the string should be 208 Hz (not 192 Hz)

$\therefore$
$$208 = \frac{1}{l}\sqrt{\frac{T}{\mu}}$$
$\therefore$
$$T = \mu\cdot l^2(208)$$
$$= \left(\frac{2.5\times10^{-3}}{0.25}\right)(0.25)^2(208)^2$$
$$= 27.04\text{ N}$$ **Ans.**

11. (a) Wavelength of sound ahead of the source is :
$$\lambda' = \frac{v - v_s}{f} = \frac{332-32}{1000} = 0.3\text{ m}$$ **Ans.**

(b)
$$f' = f\left(\frac{v + v_0}{v - v_s}\right) = 1000\left(\frac{332+64}{332-32}\right)$$
$$= 1320\text{ Hz}$$ **Ans.**

(c) Speed of reflected wave will remain 332 m/s. **Ans.**

(d) Wavelength of reflected wave.
$$\lambda'' = \frac{v - v_0}{f'} = \frac{332-64}{1320} = 0.2\text{ m}$$ **Ans.**

Chapter 17 Thermometry, Thermal Expansion and Kinetic Theory of Gases

JEE Advanced (Subjective Questions)

1. (a) $v_{rms} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3\times8.31\times300}{28\times10^{-3}}} = 517$ m/s

Pressure = force per unit area = change in momentum per unit area.

$\therefore$
$$P = 2mnv_{rms}$$
Here n = number of molecules striking per unit area per second.

$\therefore$
$$n = \frac{P}{2mv_{rms}} = \frac{P}{2\left(\frac{M}{N}\right)v_{rms}} = \frac{PN}{2Mv_{rms}} = \frac{(2\times1.01\times10^5)(6.02\times10^{26})}{2\times28\times517}$$
$$= 4.2\times10^{27}$$ **Ans.**

(b) $\frac{1}{2}mv^2 = nC_V\Delta T$
$$v = \sqrt{\frac{2nC_V\Delta T}{m}} \qquad (m = \text{mass of gas})$$
$$= \sqrt{\frac{2C_V\Delta T}{M}} \qquad \left(\frac{n}{m} = \frac{1}{M}\right)$$
$$= \sqrt{\frac{2\times\frac{5}{2}\times8.31\times2}{28\times10^{-3}}} = 54.5\text{ m/s}$$ **Ans.**

2. In the process PV^x = constant
$$C = \frac{R}{\gamma-1} + \frac{R}{1-x}\text{, Given } C = R \text{ and } \gamma = 7/5$$
substituting we get,
$$x = \frac{5}{3}$$
Now,
$$PV^{5/3} = \text{constant}$$
or
$$P \propto \frac{1}{(V)^{5/3}}$$
By increasing volume to two times pressure will decrease $(2)^{5/3}$ times.
$$v_{rms} \propto \sqrt{T} \quad \text{or} \quad v_{rms} \propto \sqrt{PV}$$
or v_{rms} will become $\sqrt{\frac{(2)}{(2)^{5/3}}}$ times

or v_{rms} will become $(2)^{-1/3}$ times or $\frac{1}{(2)^{1/3}}$ times.

Now, $P \propto$ (no. of collisions) (v_{rms})

$$\frac{1}{(2)^{3/2}} \propto \text{(no. of collisions)} \frac{1}{(2)^{1/2}}$$

or number of collisions will decrease $(2)^{1}$ times. **Ans.**

3. $$P_1V = n_1RT$$

$$P_2V = n_2RT \qquad (T = 273\text{ K})$$

$\therefore$ $$(P_1 - P_2)V = (n_1 - n_2)RT = \left(\frac{m_1 - m_2}{M}\right)RT$$

$$(\Delta P)V = \frac{\Delta m}{M}RT \qquad \text{...(i)}$$

In the initial condition (at STP)

$$\frac{RT}{M} = \frac{P_0}{\rho} \qquad \text{...(ii)}$$

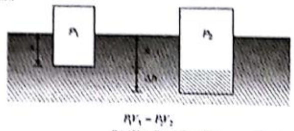
$$(T = 273\text{ K})$$

From Eqs. (i) and (ii), we get

$$\Delta m = \frac{\rho V\,\Delta P}{P_0} = \frac{1.2 \times 0.4 \times 0.24}{1.0} = 0.1152\text{ kg}$$

$$= 115.2\text{ g}$$ **Ans.**

4. P_2 = pressure at depth x



$$P_1V_1 = P_2V_2$$

or $$P_1(Ah) = P_2(A)(h - \Delta h)$$

$\therefore$ $$P_1 = \frac{P_2(h - \Delta h)}{h} \qquad \text{...(i)}$$

Initially $$mg = \rho gxA$$

Now, $$P_2 = P_0 + \rho gx = P_0 + \frac{mg}{A}$$

Substituting in Eq. (i), we have

$$P_1 = \left(P_0 + \frac{mg}{A}\right)\left(1 - \frac{\Delta h}{h}\right)$$ **Ans.**

5. (a) In 1 sec, molecules make 500 hit in a cubical vessel of side 1m. Therefore

$$v_{rms} = 1000\text{ m/s}$$

Because between two successive collisions a molecule will travel 2m.

Using $$v_{rms} = \sqrt{\frac{3RT}{M}}$$

$\therefore$ $$T = \frac{Mv_{rms}^2}{3R} = \frac{(4 \times 10^{-3})(10^3)^2}{3 \times 8.31} = 160\text{ K}$$ **Ans.**

(b) Average kinetic energy per atom $= \frac{3}{2}kT$ (monoatomic)

$$= \frac{3}{2} \times 1.38 \times 10^{-23} \times 160\text{ J}$$

$$= 3.31 \times 10^{-21}\text{ J}$$ **Ans.**

(c) $PV = nRT = \frac{m}{M}RT$

$\therefore$ $$m = \frac{PMV}{RT} = \frac{(100)(4 \times 10^{-3})(1)^3}{8.31 \times 160} = 3.0 \times 10^{-4}\text{ kg}$$ **Ans.**

6. Under isothermal conditions :

$$P_1V_1 = P_2V_2$$

$$V_2 = \left(\frac{P_1}{P_2}\right)V_1$$

P_1 = pressure at depth 11 m

$$= P_0 + \rho gh = (1.01 \times 10^5) + (10^3 \times 10 \times 11) = 2.11 \times 10^5\text{ N/m}^2$$

$$P_2 = P_0 = 1.01 \times 10^5\text{ N/m}^2$$

$\therefore$ $$V_2 = \left(\frac{2.11 \times 10^5}{1.01 \times 10^5}\right)V = 2.089\,V$$

7. $$P = \frac{RT}{V} \qquad (n = 1)$$

or $$P = \frac{R}{V}(T_0 + \alpha V^2) \qquad \text{...(i)}$$

For minimum attainable pressure

$$\frac{dP}{dV} = 0 \text{ or } \frac{-RT_0}{V^2} + \alpha R = 0$$

or $$V = \sqrt{\frac{T_0}{\alpha}}$$

At this volume we can see that $\frac{d^2P}{dV^2}$ is positive or P is minimum.

From Eq. (i)

$$P_{min} = \frac{RT_0}{\sqrt{T_0/\alpha}} + \alpha R\sqrt{T_0/\alpha} = 2R\sqrt{\alpha T_0}$$ **Ans.**

8. At 50°C, density of solid = density of liquid. ...(i)

At 0°C, fraction submerged $\left(\frac{\rho_s}{\rho_l}\right)_{0°C} \times 100$

$$(\rho)_{50} = \frac{\rho_0}{1 + 50\gamma}$$

$\therefore$ $$\rho_0 = \rho_{50}(1 + 50\gamma)$$

Substituting in Eq. (i) we have

$$\%\text{ of fraction submerged} = \left(\frac{1 + 50\gamma_s}{1 + 50\gamma_l}\right) \times 100 = \left(\frac{1 + 0.3 \times 10^{-5}}{1 + 8 \times 10^{-5}}\right) \times 100$$

$$= 99.99\%$$ **Ans.**

9. $$\rho_{100}gh_{100} = \rho_0 gh_0 \quad \text{or} \quad \frac{\rho_{100}}{\rho_0} = \frac{h_0}{h_{100}}$$

$$\left(\frac{1}{1 + 100\gamma}\right) = \frac{39.2}{40}$$

Solving this equation we get :

$$\gamma = 2.0 \times 10^{-4}\text{ per °C}$$ **Ans.**

10. Let Δl be the change in length. (Let $\Delta l_s > \Delta l > \Delta l_a$)

$$\text{Strain in steel} = \frac{\Delta l_s - \Delta l}{l_0}$$

and $$\text{strain in aluminum} = \frac{\Delta l - \Delta l_a}{l_0}$$

In equilibrium : $$2F_s = F_a$$

$$2\left[\frac{\Delta l_s - \Delta l}{l_0}\right]Y_sA = \left[\frac{\Delta l - \Delta l_a}{l_0}\right]Y_aA$$

or $$2(l_0\alpha_s\theta - \Delta l)Y_s = (\Delta l - l_0\alpha_a\theta)Ya$$

Solving this equation we get,

$$\Delta l = \left(\frac{2\alpha_sY_s + \alpha_aY_a}{2Ys + Y_a}\right)l_0\theta$$

$\therefore$ Total length $= l_0 + \Delta l = l_0\left[1 + \left(\frac{\alpha_aY_a + 2\alpha_sY_s}{2Y_s + Y_a}\right)\theta\right]$ **Ans.**

11. From $\Delta l = l\alpha\Delta\theta$ we have,

$$0.05 = 25\alpha_A(100)$$

$\therefore$ $$\alpha_A = 0.00002\text{ per°C}$$

$$0.04 = 40\alpha_B(100)$$

$$\alpha_B = 0.00001\text{ per°C}$$

In third case let l is the length of rod A. Then length of rod B will be $(50 - l)$ cm.

$$\Delta l = \Delta l_1 + \Delta l_2$$

or $$0.03 = l(0.00002)(50) + (50 - l)(0.00001)(50)$$

Solving we get $l = 10$ cm and $50 - l = 40$ cm **Ans.**

Chapter 18 First Law of Thermodynamics

JEE Advanced (Subjective Questions)

1. First process is isobaric

$\therefore$ $$\Delta Q_1 = nC_P\Delta T + P\Delta V$$

Second process is isochoric

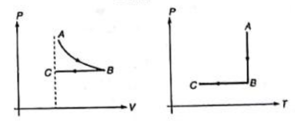
$\therefore$ $$\Delta Q_2 = nC_V\Delta T$$

$$\Delta Q_1 - \Delta Q_2 = P\Delta V = \left[P_0 + \frac{mg}{A}\right][Ax] = (P_0A + mg)x$$

$$= [10^5 \times 60 \times 10^{-4} + 8 \times 10](0.2)$$

$$= 136\text{ J}$$ **Ans.**

2. (a)



(b) For the process AB $$\frac{P_0V_0}{T_0} = \frac{P_B(2V_0)}{T_0}$$

$\therefore$ $$P_B = \frac{P_0}{2}$$

$$\Delta U = 0$$

$$Q = W + \Delta U = nRT\ln\frac{V_B}{V_A} = 3RT_0\ln 2$$

For the process BC $$\frac{2V_0}{T_0} = \frac{V_0}{T_C}$$

$\therefore$ $$T_C = \frac{T_0}{2}$$

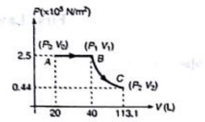
$$W = nR\Delta T = 3R\left(\frac{T_0}{2} - T_0\right) = -\frac{3}{2}RT_0$$

$$Q = nC_P\Delta T = -\frac{21}{4}RT_0$$

$\therefore$ $$W_{Total} = 3RT_0\ln(2) - \frac{3}{2}RT_0$$

(c) $$Q_{Total} = 3RT_0\ln(2) - \frac{21}{4}RT_0$$

3. (a)



(b) $$P_0 = \frac{nRT_0}{V_0} = \frac{2 \times 8.31 \times 300}{20 \times 10^{-3}} = 2.5 \times 10^5 \text{ N/m}^2$$

$$P_1 = P_0 = 2.5 \times 10^5 \text{ N/m}^2$$

$$V_1 = 2V = 40 \times 10^{-3} \text{ m}^3$$

Process AB : $$V \propto T$$ $\therefore\ T_1 = 2T_0 = 600$ K

Process BC : Using $T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$, we get

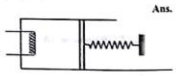
$$V_2 = 2\sqrt{2}V_1 = 113.1 \times 10^{-3} \text{ m}^3$$ **Ans.**

and $$P_2 = \frac{nRT_2}{V_2} = \frac{(2)(8.31)(300)}{113.1 \times 10^{-3}} = 0.44 \times 10^5 \text{ N/m}^2$$ **Ans.**

(c) $$W_{\text{Total}} = W_1 + W_2 = P_0(V_1 - V_0) + \frac{nR}{\gamma - 1}(T_1 - T_2)$$

Substituting the values, we get $$W_{\text{Total}} = 12479 \text{ J}$$ **Ans.**

4. (a) $$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$$



$$T_2 = \left(\frac{P_2V_2T_1}{P_1V_1}\right) \qquad ...(i)$$

Here, $$P_1 = 1.0 \times 10^5 \text{ N/m}^2,\ \ V_1 = 2.4 \times 10^{-3} \text{ m}^3,\ \ T_1 = 300 \text{ K}$$

$$P_2 = P_1 + \frac{kx}{A} = 1.0 \times 10^5 + \frac{8000 \times 0.1}{8 \times 10^{-3}} = 2.0 \times 10^5 \text{ N/m}^2$$

$$V_2 = V_1 + Ax = 2.4 \times 10^{-3} + 8 \times 10^{-3} \times 0.1 = 3.2 \times 10^{-3} \text{ m}^3$$

Substituting in Eq. (i), we get

$$T_2 = 800 \text{ K}$$ **Ans.**

(b) Heat supplied by the heater.

$$Q = W + \Delta U$$

Here, $$\Delta U = nC_V\,\Delta T = \left(\frac{P_1V_1}{RT_1}\right)\left(\frac{3}{2}R\right)(800 - 300)$$

$$= \frac{(1.0 \times 10^5)(2.4 \times 10^{-3})(1.5)(500)}{(300)}$$

$$= 600 \text{ J}$$

$$W = \frac{1}{2}kx^2 + P_1\Delta V$$

$$= \frac{1}{2} \times (8000)(0.1)^2 + (1.0 \times 10^5)(0.1)(8 \times 10^{-3})$$

$$= (40 + 80) \text{ J}$$

$$= 120 \text{ J}$$

$\therefore$ $$Q = 600 + 120 = 720 \text{ J}$$ **Ans.**

5. $$U \propto \sqrt{V}$$

As $$U \propto T$$

$\therefore$ $$T \propto V^{1/2}$$

or $$TV^{-1/2} = \text{constant}$$

or $$PV^{1/2} = \text{constant}$$

Comparing with PV^x = constant, we have

$$x = \frac{1}{2}$$

$\therefore$ Molar specific heat $$C = \frac{R}{\gamma - 1} + \frac{R}{1 - x}$$

$$= \frac{R}{7/5 - 1} + \frac{R}{1 - 1/2} \qquad (\gamma = 7/5)$$

$$= \frac{5}{2}R + 2R = \frac{9R}{2}$$ **Ans.**

$$Q = nC\Delta T$$

$$\Delta U = nC_V\Delta T$$

$\therefore$ $$W = Q - \Delta U = n(C - C_V)\Delta T$$

$$\frac{W}{\Delta U} = \frac{C - C_V}{C_V}$$

$\therefore$ $$W = \left(\frac{C - C_V}{C_V}\right)\Delta U = \left(\frac{9/2 - 5/2}{5/2}\right)(100) = 80 \text{ J}$$ **Ans.**

6. $$(PV)_A = (PV)_C = 3P_0V_0$$

$\therefore$ $$T_A = T_C$$

or $$\Delta T = 0$$

$\therefore$ $$\Delta U_{ABC} = 0$$

$$Q_{ABC} = W_{ABC} = \text{Area under the graph}$$

$$= -2P_0V_0$$ **Ans.**

i.e., Heat is released during the process.

7. $$dQ = dU + dW$$

$$C\,dT = C_V\,dT + P\,dV$$

$$(C_V + 3aT^2)\,dT = C_V\,dT + P\,dV$$

$\therefore$ $$3aT^2\,dT = P\,dV = \left(\frac{RT}{V}\right).\,dV$$

$\therefore$ $$\left(\frac{3a}{R}\right)T\,dT = \frac{dV}{V}$$

Integrating, we get

$$\left(\frac{3aT^2}{2R}\right) = \ln V - \ln C$$

$$V = Ce^{\frac{3aT^2}{2R}} \quad \text{or} \quad Ve^{-\frac{3aT^2}{2R}} = \text{constant}$$

8. (a) $$P = \alpha T^{1/2}$$

or $$PT^{-1/2} = \text{constant}$$

$$P^{1/2}V^{-1/2} = \text{constant}$$

$$PV^{-1} = \text{constant}$$

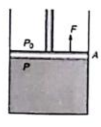
$\therefore$ $$x = -1$$

$$\Delta W = \left(\frac{R}{1 - x}\right)(\Delta T) = \left(\frac{8.31}{2}\right)(50)$$

$$= 207.75 \text{ J}$$ **Ans.**

(b) $$C = C_V + \frac{R}{1 - x} = \frac{3}{2}R + \frac{R}{2} = 2R$$ **Ans.**

9. $$F + PA = P_0A$$



$$F = (P_0 - P)A$$

$$W = \int_V^{2V}(P_0 - P)A\,dx$$

$$= \int_V^{2V}P_0\,dV - \int_V^{2V}P\,dV$$

$$= P_0V - \int_V^{2V}RT\left(\frac{dV}{V}\right)$$

$$= P_0V - RT\ln(2)$$

$$= RT - RT\ln(2)$$

$$= RT(1 - \ln 2)$$ **Ans.**

10. (a) $$P \propto \frac{1}{T} \text{ or } PT = \text{constant}$$

$\therefore$ $$P(PV)) = \text{constant}$$

or $$PV^{1/2} = \text{constant}$$

In the process PV^x = constant, molar heat capacity is,

$$C = C_V + \frac{R}{1 - x}$$

or $$C = \frac{3}{2}R + \frac{R}{1 - \frac{1}{2}}$$

$$= \frac{3}{2}R + 2R = \frac{7}{2}R$$ **Ans.**

(b) $W = Q - \Delta U = nC\Delta T - nC_V\Delta T = n(C - C_V)\Delta T$

$$= 2\left[\frac{7}{2}R - \frac{3}{2}R\right](T_2 - T_1) = 4R(T_2 - T_1)$$ **Ans.**

11. $V = \frac{a}{T}$ or VT = constant or $V(PV)$ = constant

$\therefore$ $$PV^2 = \text{constant}$$

In the process PV^x = constant, molar heat capacity is

$$C = \frac{R}{\gamma - 1} + \frac{R}{1 - x}$$

Here $$x = 2$$

$\therefore$ $$C = \frac{R}{\gamma - 1} + \frac{R}{1 - 2} = \left(\frac{2 - \gamma}{\gamma - 1}\right)R$$

Now $$Q = nC\Delta T = (1)\left(\frac{2 - \gamma}{\gamma - 1}\right)R\Delta T = \left(\frac{2 - \gamma}{\gamma - 1}\right)R\Delta T$$ **Ans.**

12. Process AB is isochoric (V = constant). Hence

$\Delta W_{AB} = 0$

$$\Delta W_{BCD} = P_0V_0 + \frac{\pi}{2}(P_0)\left(\frac{V_0}{2}\right)$$

$$= \left(\frac{\pi}{4} + 1\right)P_0V_0$$

$$\Delta W_{DA} = -\frac{1}{2}\left(\frac{P_0}{2} + P_0\right)(2V_0 - V_0)$$

$$= -\frac{3}{4}P_0V_0$$

$$\Delta U_{AB} = nC_V\Delta T = (2)\left(\frac{3}{2}R\right)(T_B - T_A) \qquad \left(n = 2, C_V = \frac{3}{2}R\right)$$

$$= 3R\left(\frac{P_0V_0}{2R} - \frac{P_0V_0}{4R}\right)$$

$$= \frac{3}{4}P_0V_0 = \Delta Q_{AB} \qquad \left(T = \frac{PV}{nR}\right)$$

$$\Delta U_{BCD} = nC_V\Delta T = (2)\left(\frac{3}{2}R\right)(T_D - T_B)$$

$$= (3R)\left(\frac{2P_0V_0}{2R} - \frac{P_0V_0}{2R}\right) = \frac{3}{2}P_0V_0$$

Hence $$\Delta Q_{BCD} = \Delta U_{BCD} + \Delta W_{BCD}$$

$$= \left(\frac{\pi}{4} + \frac{5}{2}\right)P_0V_0$$

$$\Delta U_{DA} = nC_V\Delta T$$

$$= (2)\left(\frac{3}{2}R\right)(T_A - T_D)$$

$$= (3R)\left(\frac{P_0V_0}{4R} - \frac{2P_0V_0}{2R}\right)$$

$$= -\frac{9}{4}P_0V_0$$

$\therefore$
$$\Delta Q_{DA} = \Delta U_{DA} + \Delta W_{DA}$$
$$= -\frac{9}{4}P_0V_0 - \frac{3}{4}P_0V_0$$
$$= -3P_0V_0$$

Net work done is

$$W_{net} = \left(\frac{\pi}{4} + 1 - \frac{3}{4}\right)P_0V_0$$
$$= 1.04P_0V_0$$

and heat absorbed is

$$Q_{ab} = \Delta Q_{+ve}$$
$$= \left(\frac{3}{4} + \frac{\pi}{4} + \frac{5}{2}\right)P_0V_0 = 4.03P_0V_0$$

Hence efficiency of the cycle is

$$\eta = \frac{W_{net}}{Q_{ab}} \times 100$$
$$= \frac{1.04\,P_0V_0}{4.03P_0V_0} \times 100 = 25.8\%$$ **Ans.**

13. $$P = \frac{\alpha T - \beta T^2}{V}$$ (P = constant)

Hence $$V = \frac{\alpha T - \beta T^2}{P}$$

or $$dV = \left(\frac{\alpha - 2\beta T}{P}\right)dT$$

$$W = \int P\,dV = \int_{T_1}^{T_2} P\left(\frac{\alpha - 2\beta T}{P}\right)dT$$

or $$W = [\alpha T - \beta T^2]_{T_1}^{T_2}$$
$$= \alpha(T_2 - T_1) - \beta(T_2^2 - T_1^2)$$ **Ans.**

14. (a) First law of thermodynamics for the given process from state 1 to state 2

$$Q_{12} - W_{12} = U_2 - U_1$$

Here, $$Q_{12} = +\,10P_0V_0 \text{ joule}$$

$W_{12} = 0$ (Volume remains constant)

$$U_2 - U_1 = nC_V\,(T_2 - T_1)$$
$$nC_V\,(T_2 - T_1) = 10P_0V_0$$

For an ideal gas

$$P_0V_0 = nRT_0$$

and $$C_P - C_V = R$$

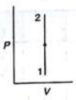
$\therefore$ $$C_V = C_P - R = \frac{5R}{2} - R = \frac{3R}{2}$$

$\therefore$ $$n\left(\frac{3R}{2}\right)(T_2 - T_0) = 10nRT_0$$
$$T_2 = \frac{23}{3}T_0$$

As $P \propto T$ for constant volume

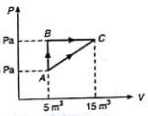
$$P_2 = \frac{23}{3}P_0$$

(b)



15. (a) W_{AC} is less than W_{ABC} as area under graph is less.

(b) For process A to C



$$Q = 200\text{ J}$$

Work done

$$W_{AC} = \text{area under } AC$$
$$= \frac{1}{2}(8 + 4) \times 10 = 60\text{ J}$$

From first law of thermodynamics.

$$\Delta U = Q - W_{AC}$$
$$U_C - U_A = 200 - 60$$

$\therefore$
$$U_C = U_A + 140$$
$$= 10 + 140 = 150\text{ J}$$ **Ans.**

(c) From A to B

$$U_B = 20\text{ J}$$

$\therefore$
$$\Delta U = Q - W_{AB}$$
$$U_B - U_A = Q - 0$$
$$20 - 10 = Q$$

$\therefore$ $$Q = 10\text{ J}$$ **Ans.**

16. **Process A-B** is an isothermal process *i.e.* T = constant.

Hence $P \propto \frac{1}{V}$ or P - V graph will a rectangular hyperbola with increasing P and decreasing V.

$\rho \propto \frac{1}{V}$. Hence ρ - V graph is also a rectangular hyperbola with decreasing V and hence increasing ρ.

$$\rho \propto P$$ $\left[\rho = \frac{PM}{RT}\right]$

Hence ρ - P graph will be a straight line passing through origin, with increasing ρ and P.

Process B - C is an isochoric process, because P - T graph is a straight line passing through origin

i.e. $$V = \text{constant}$$

Hence P - V graph will be a straight line parallel to P-axis with increasing P.

Since V = constant hence ρ will also be constant

Hence ρ-V graph will be a dot.

ρ-P graph will be a straight line parallel to P-axis with increasing P, because

$$\rho = \text{constant}$$

Process C-D is inverse of A-B and D-A is inverse of B-C.

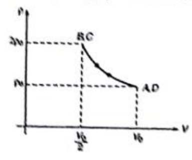
Different values of P, V, T and ρ in tabular form are shown below

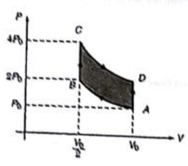
	P	V	T	ρ
A	P_0	V_0	T_0	ρ_0
B	$2P_0$	$\frac{V_0}{2}$	T_0	$2\rho_0$
C	$4P_0$	$\frac{V_0}{2}$	$2T_0$	$2\rho_0$
D	$2P_0$	V_0	$2T_0$	ρ_0

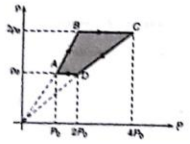
Here $$V_0 = nR\left(\frac{T_0}{P_0}\right)$$

and $$\rho_0 = \frac{P_0M}{RT_0}$$

The corresponding graphs are as follows :







Chapter 19 Calorimetry and Heat Transfer

JEE Advanced (Subjective Questions)

(a) Between $t = 1$ min to $t = 3$ min, there is no rise in the temperature of substance. Therefore solid melts in this time.

$$L = \frac{Q}{m} = \frac{Ht}{m} = \frac{10 \times 2}{0.5} = 40\text{ kJ/kg}$$

(b) From $Q = ms\Delta T$ or $s = \frac{Q}{m\Delta T}$

Specific heat in solid state $s = \frac{10 \times 1}{0.5 \times 15} = 1.33\text{ kJ/kg-°C}$

2. Let T_0 be the temperature of surrounding and T the temperature of hot body at some instant. Then :

$$-\frac{dT}{dt} = k(T - T_0)$$

or $$\int_{T_m}^{T} \frac{dT}{T - T_0} = -k\int_0^t dT$$ (T_m = temperature at $t = 0$)

Solving this equation, we get

$$T = T_0 + (T_m - T_0)e^{-kt}$$ (ii)

Maximum temperature it can loose is $(T_m - T_0)$

From Eq. (i)

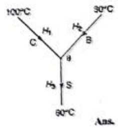
$$T - T_0 = (T_m - T_0)e^{-kt}$$

Given that $$T - T_0 = \frac{T_m - T_0}{2} = (T_m - T_0)e^{-kt}$$

Solving this equation we get

$$t = \frac{\ln(2)}{k}$$ **Ans.**

3. Let θ be the temperature of junction. Then :



$$H_1 + H_2 = H_3$$

$\therefore$ $$\frac{100 - \theta}{(46/0.92 \cdot 4)} + \frac{80 - \theta}{(13/0.26 \cdot 4)} = \frac{\theta - 60}{(12/0.12 \cdot 4)}$$

Solving this equation we have, $\theta = 84°C$

(b) Heat current in copper rod $H_1 = \frac{100 - \theta}{(46/0.92 \times 4)} = 1.28$ cal/s **Ans.**

4. Let heat capacity of flask be C and latent heat of fusion of ice L.

Then, $$C(70 - 40) + 200 \times 1 \times (70 - 40) = 50L + 50 \times 1 \times (40 - 0)$$ ...(i)

or $$3C - 5L = -400$$

Further, $$C(40 - 10) + 250 \times 1 \times (40 - 10) = 80L + 80 \times 1 \times (10 - 0)$$ ...(ii)

or $$3C - 8L = -670$$

Solving Eq. (i) and (ii), we have,

$$L = 90\text{ cal/g}$$ **Ans.**

5.
$$ms\Delta\theta = \text{work done against friction} = (\mu mg\cos\theta).d \qquad (\text{but } \mu mg\cos\theta = mg\sin\theta)$$

$$\Delta\theta = \frac{(\mu g\cos\theta)d}{s} = \frac{(g\sin\theta)d}{s}$$

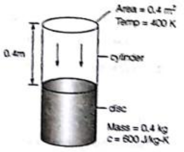
$$= \frac{(10)\left(\frac{3}{5}\right)(0.6)}{420}$$

$$= 8.57\times10^{-3}\ ^\circ C$$

$$\approx 8.6\times10^{-3}\ ^\circ C$$

Ans.

6. Using $\frac{dQ}{dt} = mc\frac{d\theta}{dt} = \frac{KA(\theta_1-\theta_2)}{l}$ we have



$$mc\left(\frac{d\theta}{dt}\right) = \frac{KA(400-\theta)}{0.4}$$

$$(0.4)(600)\frac{d\theta}{dt} = \frac{(10)(0.04)(400-\theta)}{0.4}$$

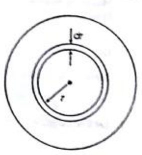
∴ $$\left(\frac{d\theta}{400-\theta}\right) = \frac{1}{240}dt$$

∴ $$\int_0^t dt = 240\int_{300}^{350}\frac{d\theta}{400-\theta}$$

Solving this, we get $t \approx 166$ s

Ans.

7. Consider a differential cylinder



$$H = \frac{dQ}{dt} = kA\frac{d\theta}{dr} = (2\pi krl)\frac{d\theta}{dr}$$

∴ $$\frac{H}{2\pi kl}\int_{r_1}^{r_2}\frac{dr}{r} = \int_0^{50} d\theta$$

or $$\frac{H}{2\pi kl}\ln\left(\frac{r_2}{r_1}\right) = 50$$

∴ $$H = \frac{100\pi kl}{\ln(r_2/r_1)}$$

Now $$Ht = mL$$

∴ $$t = \frac{mL}{H} = \frac{mL\ln(r_2/r_1)}{100\pi kl}$$

Ans.

8. Three thermal resistances are in series. $\left(R = \frac{l}{KA}\right)$

∴ $$R = R_1 + R_2 + R_3 = \frac{2.5\times10^{-2}}{0.125\times137} + \frac{1.0\times10^{-2}}{1.5\times137} + \frac{2.5\times10^{-2}}{1.0\times137}$$

$$= 0.0017\ \frac{^\circ C - s}{J}$$

Now heat current $$H = \frac{\text{Temperature difference}}{\text{Net thermal resistance}}$$

$$= \frac{30}{0.0017} = 17647 \text{ W}.$$

9. Thermal resistance of plastic coating : $\left(R_t = \frac{l}{KA}\right)$

$$R_t = \frac{l}{K(\pi d)l} \qquad (d = \text{diameter})$$

$$= \frac{0.06\times10^{-3}}{(0.26\times10^{-2}\times4.18\times10^{2})(\pi)(0.64\times10^{-3})(2)} = 0.0223\ ^\circ C\text{-s/J}$$

Now $$i^2R_t = \frac{TD}{R_t}$$

∴ $$TD = i^2R_tR_t$$

$$= (5)^2(4)(0.0223)$$

$$= 2.23^\circ C$$

Ans.

10. Let T_1 be the temperature of C_1 and T_2 the temperature of C_2 at some instant of time. Further let T be the temperature difference at that instant.

Then, $$C_1\left(-\frac{dT_1}{dt}\right) = H = \frac{T}{l/KA} = \frac{KA}{l}(T) \qquad T = T_1 - T_2$$

and $$C_2\left(+\frac{dT_2}{dt}\right) = H = \frac{T}{l/KA} = \frac{KA}{l}(T) \qquad \therefore -\frac{dT}{dt} = -\frac{dT_1}{dt} + \frac{dT_2}{dt}$$

∴ $$-\frac{dT_1}{dt} = \frac{KA}{lC_1}(T) \quad \text{and} \quad +\frac{dT_2}{dt} = \frac{KA}{lC_2}$$

Further, $$-\frac{dT}{dt} = -\frac{dT_1}{dt} + \frac{dT_2}{dt} = \frac{KA(C_1+C_2)}{lC_1C_2}T$$

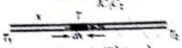
or $$\int_{\Delta T_0}^{T}\frac{dT}{T} = -\frac{KA(C_1+C_2)}{lC_1C_2}\int_0^t dt$$

Solving we get :

$$T = \Delta T_0 e^{-\alpha t}$$

where $$\alpha = \frac{KA(C_1+C_2)}{lC_1C_2}$$

11.



$$H = \frac{TD}{R} = \frac{(-dT)}{(dx)/KA} = -\left(\frac{dT}{dx}\right)\left(\frac{\alpha}{T}A\right) = \text{constant} \qquad ...(i)$$

∴ $$\int_{T_1}^{T_2} -\frac{dT}{T} = \frac{H}{\alpha A}\int_0^l dx$$

$$\ln\left(\frac{T_1}{T_2}\right) = \frac{Hl}{\alpha A} \quad \text{or} \quad H = \frac{\alpha A}{l}\ln(T_1/T_2)$$

Substituting in Eq. (i), we have

$$\frac{\alpha A}{l}\ln\left(\frac{T_1}{T_2}\right) = -\left(\frac{dT}{dx}\right)\frac{\alpha A}{T}$$

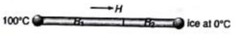
or $$\int_{T_1}^{T}\frac{dT}{T} = -\frac{\ln\left(\frac{T_1}{T_2}\right)}{l}\int_0^x dx$$

∴ $$\ln\left(\frac{T}{T_1}\right) = -\frac{x}{l}\ln(T_1/T_2) = \ln\left(\frac{T_1}{T_2}\right)^{-\frac{x}{l}}$$

or $$\frac{T}{T_1} = \left(\frac{T_1}{T_2}\right)^{-\frac{x}{l}} \quad \text{or} \quad T = T_1\left(\frac{T_1}{T_2}\right)^{-\frac{x}{l}} = T_1\left(\frac{T_2}{T_1}\right)^{\frac{x}{l}}$$

Ans.

12. $$L\left(\frac{dm}{dt}\right) = \frac{TD}{R_1 + R_2}$$



or $$\frac{80\times360}{3600} = \frac{100}{\frac{20}{0.25\times10} + \frac{10}{k\times10}}$$

Solving this equation we get :

$$k = 0.222 \text{ cal/cm-s-}^\circ C$$

Ans.

13. $$H = \frac{-d\theta}{dr/K(4\pi r^2)} = \left(-\frac{d\theta}{dr}\right)(4\pi kr^2) \qquad ...(i)$$

or $$\int_{R_1}^{R_2}\frac{dr}{r^2} = -\frac{4\pi k}{H}\cdot\int_{\theta_1}^{\theta_2}d\theta$$

or $$\frac{R_2-R_1}{R_1R_2} = \frac{4\pi k}{H}(\theta_1-\theta_2)$$

∴ $$H = \frac{4\pi k(R_1R_2)(\theta_1-\theta_2)}{R_2-R_1}$$

Substituting this value of H in Eq. (i) we have,

$$\frac{R_1R_2}{R_2-R_1}(\theta_1-\theta_2) = -\frac{d\theta}{dr}(r^2)$$

∴ $$\int_{\theta_1}^{\left(\frac{\theta_1+\theta_2}{2}\right)}d\theta = -\frac{R_1R_2}{R_2-R_1}(\theta_1-\theta_2)\int_{R_1}^{r}\frac{dr}{r^2}$$

or $$\frac{\theta_1+\theta_2}{2} - \theta_1 = \frac{R_1R_2}{R_2-R_1}(\theta_1-\theta_2)\left[\frac{1}{r}-\frac{1}{R_1}\right]$$

or $$\frac{\theta_1-\theta_2}{2} = \frac{R_1R_2}{R_2-R_1}(\theta_1-\theta_2)\left(\frac{1}{R_1}-\frac{1}{r}\right)$$

$$\frac{1}{R_1} - \frac{1}{r} = \frac{R_2-R_1}{2R_1R_2}$$

$$\frac{1}{r} = \frac{1}{R_1} - \frac{R_2-R_1}{2R_1R_2} = \frac{R_1+R_2}{2R_1R_2}$$

or $$r = \frac{2R_1R_2}{R_1+R_2}$$

Ans.

Experimental Skills

1. No, prongs execute transverse vibrations as the vibrations and the stem executes longitudinal vibrations.
2. No, frequency is exactly the same for both kinds of vibrations.
3. Amplitudes in the two cases are different ; prongs have larger amplitudes than that of the stem.
4. The frequency of a tuning fork decreases with the increase in length of its prongs.
5. The one which has smaller and thicker prongs will be of higher frequency.
6. The frequency will increase.
7. Yes, its frequency will decrease.
8. These are the frequencies identical with those of the major diatonic scale which is a musical scale (C, D, E, F, G, A, B, C_1). Frequency ratio of the first and eight note is 1 : 2.
9. It signifies the note of the major diatonic scale (C, D, E, F, G, A, B, C_1). It means that the letter C stands for first note of the scale and corresponds to a frequency of 256 Hz Similarly D corresponds to a frequency of 288 Hz and so on.
10. A tone is a simple sound resulting from a pure harmonic motion. On the other hand a note is a complex sound made up of a complex periodic motion as obtained by the superposition of a number of pure simple harmonic motions.
11. This is because the prongs are vibrating perpendicular to their length whereas the stem is vibrating along its length.
12. (i) If there is only one prong, then the vibrations will die out quickly as and when the stem is touched and (ii) we want to have a node in between the two antinodes which exist at the free ends of the two prongs.
13. If we strike a tuning fork with a great force, a pure note may not be produced, but overtones may be produced.
14. Energy from one prong is transmitted to the other prong through the body of the tuning fork and as such the other prong also starts vibrating.
15. No, sound can't travel in vacuum because sound needs a material medium for its propagation.
16. In hydrogen its velocity is higher because velocity of sound is inversely proportional to square root of the density of the medium. Since hydrogen is a lighter medium than oxygen, sound will have a higher velocity in hydrogen.
17. The air column above water over which the vibrating tuning fork is held, is the resonance column.
18. It acts as a rigid wall for the reflection of sound waves. We can also adjust the length of air column by adjusting the level of water.
19. Yes, mercury can be used; rather mercury will be better in the sense that air above it will not become wet. But mercury is costly and dangerous for health. That's why water is used.
20. Longitudinal stationary waves.
21. A vibrating tuning fork is placed near the end of the resonance column which gives rise to the longitudinal waves in air column. These waves are reflected from the water surface. Thus due to superposition of the direct and reflected waves, stationary waves are formed.
22. The vibrating tuning fork is placed near the open end of the resonance column. The length of air column between the open end and the water surface is increased starting from a minimum, such that loudness of sound increases to a quite large extent. The length of the air column above the water surface up to the end of the tube is the first resonating length.
23. Node is at the water surface and antinode is at the open end of the tube.
24. No, it is slightly above the open end.

25. The distance between a node and the nearest antinode is given as $\lambda/4$, where λ is the wavelength of the sound. Here in this case, this distance is $l_1 + x$, where l_1 is the length of the air column above the water surface and $x = 0.3\,d$, is known as end correction.
Therefore,
$$\lambda/4 = l_1 + x$$
or
$$\lambda = 4\,(l_1 + x)$$
26. Yes, a long tube having cross-section of any shape will serve the purpose.
27. In order to obtain two resonance positions which enable us to eliminate end correction.
28. It is kept horizontally just above the open end of the resonating tube in such a manner that the direction of vibrations of the prongs of the tuning fork is along the length of the air column.
29. When after first resonance is reached, if we increase the length of air column, a situation of resonance is again reached when the resonating length is approximately thrice the first resonant length. This is called second resonance.
30. In the case of second resonance, energy gets distributed over a larger region and as such second resonance becomes feebler.
31. In between the node at the water surface and antinode at the open end of the tube, there is one more node and one more antinode. These, node and antinode, are at distances $l_2/3$ and $2l_2/3$ from the open end respectively (neglecting the end correction), where l_2 is the length of the resonating column in the second case.
32. In this case
$$\frac{3\lambda}{4} = l_2 + x \text{ or } \lambda = \frac{4}{3}(l_2 + x)$$
where x is the end correction.
33. Yes, we can. If we take both the positions of resonance then we get
$$\frac{3\lambda}{4} - \frac{\lambda}{4} = (l_2 + x) - (l_1 - x)$$
or
$$\frac{\lambda}{2} = l_2 + l_1 \text{ or } \lambda = 2(l_2 - l_1)$$
and hence λ so obtained is free from the end correction.
34. Yes. Once λ of waves in air column is known and frequency f of the tuning fork is the same as that of resonating air column then by using the relation $v = f\lambda$, v can be determined.
36. $n_1 = \dfrac{v}{2(l_2 - l_1)}$
$$n_2 = \frac{v}{2\,(l_2' - l_1')} \quad \therefore \quad \frac{n_1}{n_2} = \frac{l_2' - l_1'}{l_2 - l_1}$$
41. $v = 2n\,(l_2 - l_1)$
$$300 = 2n\left(\frac{97-37}{100}\right) \quad \therefore \quad 300 = 2n\left(\frac{60}{100}\right)$$
$\therefore$ $$n = 250 \text{ Hz}$$ **Ans.**
42. $e = \dfrac{l_2 - 3l_1}{2}$
$\therefore$ $$e = 1 \text{ cm} = \frac{1}{100}\text{ m}$$
$\therefore$ $$\frac{1}{100} = \frac{l_2 - 3\,(0.15)}{2}$$
$\therefore$ $$l_2 = 0.47 \text{ m}$$
$\therefore$ $$l = 47 \text{ cm}$$ **Ans.**

43. Maximum length of air column $l = \dfrac{3\lambda}{4}$
$\therefore$ $$l = \frac{3v}{4f}$$
where $v = 340 \text{ ms}^{-1}$ and $f = 340$ Hz
$\therefore$ $$l = \frac{3}{4}\text{ m} = 0.75 \text{ m} = 75 \text{ cm}$$
So, length of water column is $l_{water} = 120 \text{ cm} - 75 \text{ cm} = 45 \text{ cm}$ **Ans.**
44. $\dfrac{n_1}{n_2} = \dfrac{l_2' - l_1'}{l_2 - l_1} = \dfrac{90-30}{30-10} = \dfrac{3}{1}$
45. $l_2 = 3l_1 = 48$ cm
46. No, the specific heat of a substance changes with temperature.
47. Thermal capacity of a body is the amount of heat required to raise the temperature of the whole body through 1 °C.
Thermal capacity = Mass × Specific heat = $m \times s$
48. Thermal capacity has the unit calorie/gm and water equivalent gm.
49. Since specific heat of copper is small, so for a certain amount of heat transferred, rise in temperature is large and the change in temperature can be measured accurately.
50. Heat lost by aluminium = $500 \times c \times (100 - 46.8)$ cal
$\therefore$ Heat lost = 26600 c
Heat gained by water and calorimeter = $300 \times 1 \times (46.8 - 30) + 500 \times 0.093 \times (46.8 - 30)$
$\therefore$ Heat gained = $5040 + 781.2 = 5821.2$
Now, heat lost = Heat gained
$\therefore$ $$26600\,c = 5821.2$$
$\therefore$ $$c = 0.22 \text{ cal g}^{-1}\,(°C)^{-1}$$
51. Thermal capacity = $(0.04 \text{ kg})\,(4.2 \times 10^2 \text{ J kg}^{-1}\,°C^{-1})$
$= 16.8$ J/kg **Ans.**
52. Heat lost = Heat gained
$\therefore$ $$m_1c_1\Delta T_1 = m_2c_2\Delta T_2$$
$\therefore$ $$c_1 = \frac{m_2c_2\Delta T_2}{m_1\Delta T_1} = \frac{0.5 \times 4.2 \times 10^3 \times 3}{0.2 \times 77}\text{ J kg}^{-1}\,°C^{-1}$$
$\therefore$ $$c_1 = 0.41 \times 10^3 \text{ J kg}^{-1}\,°C^{-1}$$ **Ans.**
53. Heat lost = Heat gained
$$0.20 \times 10^3 \times c(150 - 40)$$
$$= 150 \times 1 \times (40 - 27) + 0.025 \times 10^3 \times (40 - 27) \qquad [\because c_{water} = 1 \text{ cal g}^{-1}\,(°C)^{-1}]$$
$\therefore$ $$22000\,c = 1950 + 325$$
$\therefore$ $$22000\,c = 2275$$
$\therefore$ $$c = 0.10 \text{ cal g}^{-1}\,(°C)^{-1}$$ **Ans.**
54. No, it holds good only when the difference in temperatures is small, in the range (20°C – 30°C).
55. $\dfrac{dQ}{dt} \propto (\theta - \theta_0)$
where the symbols have their usual meanings.
56. It is a straight line.