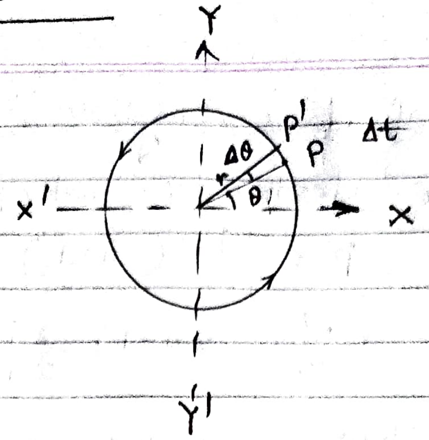
Uniform Circular Motion

UCM



$\widehat{PP'} = \Delta S = r\Delta\theta$ ---- (i)

$\Delta S = r\,\Delta\theta$

$\frac{\Delta S}{\Delta t} = r\,\frac{\Delta\theta}{\Delta t}$

$\lim_{\Delta t\to 0} \frac{\Delta S}{\Delta t} = r \lim_{\Delta t\to 0} \frac{\Delta\theta}{\Delta t}$

$\frac{ds}{dt} = r\frac{d\theta}{dt}$

$v = r\omega$ ---- (ii)

$\theta \rightarrow$ Angular position / Angular displacement

Linear velocity = $r \times$ Angular velocity

Differentiating eqn (ii) w.r.t time t.

$\frac{dv}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt}$

$a_t = r\alpha$ ~~$\frac{d\omega}{dt}$~~.

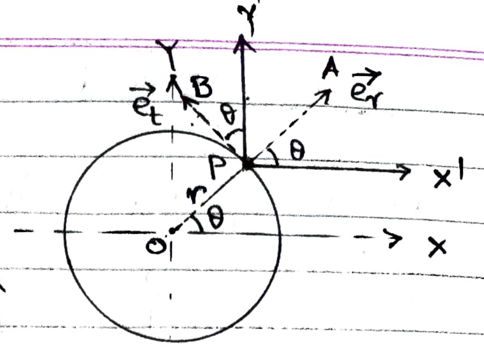
Where $a_t \rightarrow$ Linear Acceleration

$\alpha \rightarrow$ Angular Acceleration

$$\left.\begin{aligned} s &= r\theta \\ v &= r\omega \\ a_t &= r\alpha \end{aligned}\right\}$$ Relation between linear motion with angular motion.

Equation of Motion

Linear Motion	Angular motion
s, u, v, t, a	$\theta, \omega_0, \omega, t, \alpha$
$v = u \pm at$	$\omega = \omega_0 \pm \alpha t$.
$s = ut \pm \frac{1}{2}at$	$\theta = \omega_0 t \pm \frac{1}{2}\alpha t$.
$v^2 = u^2 \pm 2as$	$\omega^2 = \omega_0^2 \pm 2\alpha\theta$.



Acceleration in Circular motion

$\vec{PA} = \vec{e_r}$

$\vec{PB} = \vec{e_t}$

$$\vec{PA} = PA\cos\theta\,\hat{i} + PA\sin\theta\,\hat{j}$$

$$\vec{e_r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \quad \text{(Unit vector along radial direction)}$$

$$\vec{e_t} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\vec{r} = \vec{OP} = OP\,\vec{e_r} = r(\hat{i}\cos\theta + \hat{j}\sin\theta) \quad \text{---- (i)}$$

Differentiating both side with respect to time (t).

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[r(\hat{i}\cos\theta + \hat{j}\sin\theta)\right]$$

$$= r\left[\hat{i}\left(-\sin\theta\frac{d\theta}{dt}\right) + \hat{j}\left(\cos\theta\frac{d\theta}{dt}\right)\right]$$

$$= r\omega\left[-\hat{i}\sin\theta + \hat{j}\cos\theta\right] \quad \text{-------- (ii)}$$

$$\vec{v} = r\omega\,\vec{e_t}$$

∴ The velocity of the particle at any instance is along the tangent to the circular path.

Again differentiating eqn (ii) w.r.t time

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[r\omega(-\hat{i}\sin\theta + \hat{j}\cos\theta)\right]$$

$$= r\left[\omega\frac{d}{dt}(-\hat{i}\sin\theta + \hat{j}\cos\theta) + (-\hat{i}\sin\theta + \hat{j}\cos\theta)\frac{d\omega}{dt}\right]$$

$$= \omega r\left[-\hat{i}\cos\theta\cdot\frac{d\theta}{dt} - \hat{j}\sin\theta\cdot\frac{d\theta}{dt}\right] + r\frac{d\omega}{dt}(-\hat{i}\sin\theta + \hat{j}\cos\theta)$$

$$= -r\omega^2\,\vec{e_r} + r\frac{d\omega}{dt}\,\vec{e_t}$$

$$= -\omega^2 r\,\vec{e_r} + \frac{d}{dt}(r\omega)\,\vec{e_t} \qquad [\text{As } r\omega = v]$$

$$\vec{a} = -\omega^2 r\,\vec{e_r} + \frac{dv}{dt}\,\vec{e_t} \qquad \therefore \vec{a} = -a_r\,\vec{e_r} + a_t\,\vec{e_t}$$

$$\therefore a_r = \omega^2 r \qquad a_t = \frac{dv}{dt}$$

In case of Uniform Circular motion.

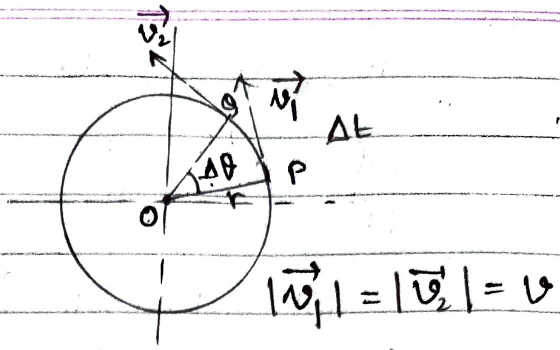
$$\frac{dv}{dt} = 0 \qquad \text{As } v \text{ is const (speed const)}$$

$$\vec{a} = -\omega^2 r\,\vec{e_r} = -\frac{v^2}{r}\,\vec{e_r}.$$

$$|\vec{a}| = \frac{v^2}{r}$$

Centripetal Acceleration.
Directed towards the centre of the circular path.

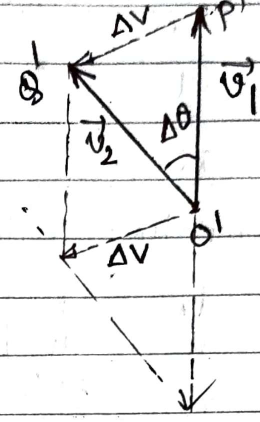
Alternative Method



$|\vec{v_1}| = |\vec{v_2}| = v$

From figure change in velocity

$\Delta v = \vec{v_2} - \vec{v_1}$



From ΔOPQ and $\Delta O'P'Q'$ (similar triangle) $\frac{PQ}{OP} = \frac{P'Q'}{O'P}$

$$\frac{PQ}{r} = \frac{\Delta v}{v}$$

$$\frac{\Delta s}{r} = \frac{\Delta v}{v}$$

$$\Rightarrow \Delta v = \frac{v}{r}\Delta s$$

Divide both side with Δt

$$\frac{\Delta v}{\Delta t} = \frac{v}{r} \cdot \frac{\Delta s}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{v}{r} \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

$$\frac{dv}{dt} = \frac{v}{r}\frac{ds}{dt} = \frac{v^2}{r}$$

$$\boxed{a = \frac{v^2}{r}} = \boxed{\omega^2 r}$$

From figure the direction of Δv is along the centre of the circular path

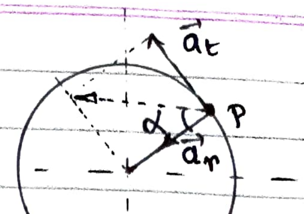
∴ The acceleration (a) acts along the centre of the circular path.

∴ If a body of mass (m) moves in a circular path with constant speed then acceleration experiences by it is $\frac{v^2}{r}$ and force experiences by it is $F = \frac{mv^2}{r}$ this accelration and the force is known as centripetal acceleration and centripetal force respectively.

∴ Centripetal Acceleration $(a_c) = \frac{v^2}{r} = \omega^2 r$

Centripetal force $(F_c) = \frac{mv^2}{r} = m\omega^2 r$.

Non-Uniform Circular Motion

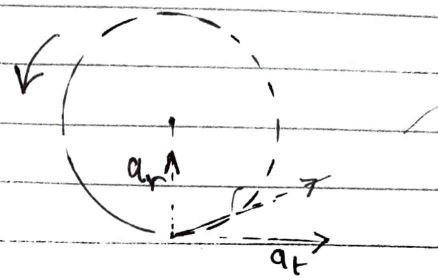


$$a_r = -\omega^2 r = -\frac{v^2}{r}$$

$$a_t = \frac{dv}{dt}$$

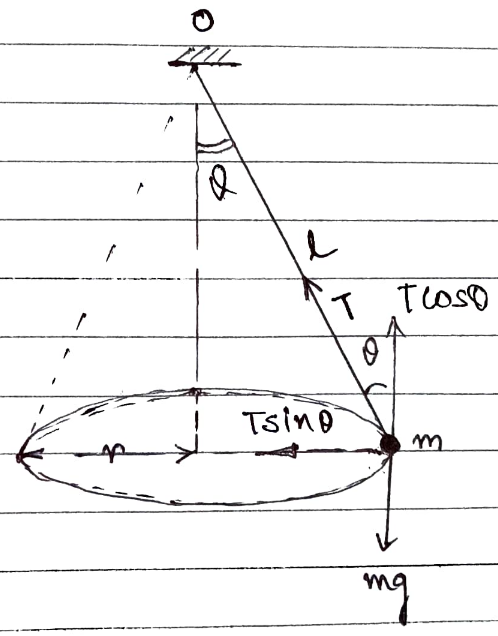
$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$$

$$\tan\alpha = \frac{a_t}{a_r} = \frac{(dv/dt)}{v^2/r}$$



$$\therefore \alpha = \tan^{-1}\frac{(dv/dt)}{(v^2/r)}$$

Conical Pendulum:



Necessary Centripetal force is provided by $T\sin\theta$

$$\frac{mv^2}{r} = T\sin\theta \quad \text{---- (i)}$$

Again the wight of the body is balanced by $T\cos\theta$

$$mg = T\cos\theta \quad \text{---- (ii)}$$

$$\therefore \quad \tan\theta = \frac{v^2}{rg}$$

$$v = \sqrt{rg\tan\theta}.$$

$$\text{Time period } (T) = \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{rg\tan\theta}} = 2\pi\sqrt{\frac{r}{g\tan\theta}}$$

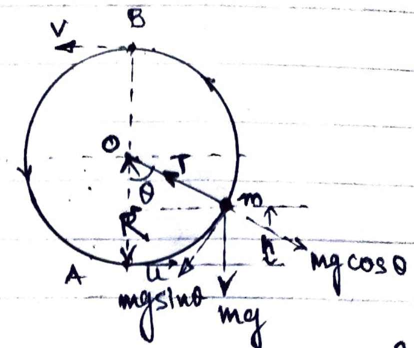
From fig $r = l\sin\theta$

$$T = 2\pi\sqrt{\frac{l\sin\theta}{g\tan\theta}} = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

For very small angular amplitude $(\theta \simeq 0)$ $\cos\theta \simeq 1$

$$T = 2\pi\sqrt{\frac{l}{g}}$$ Time period for simple pendulum

Motion in a Verticle plane



Eqn. of motion of the particle at any instance

$$T - mg\cos\theta = \frac{mv^2}{R} \quad \text{---(i)}$$

At point A $(\theta = 0°)$

$$T_A - mg\cos 0° = \frac{mv_A^2}{R} \quad \text{---(ii)} \Rightarrow T_A - mg = \frac{mu^2}{R}$$

At Point B $(\theta = 180°)$

$$T_B - mg\cos 180° = \frac{mv_B^2}{R} \Rightarrow T_B + mg = \frac{mv^2}{R} \quad \text{---(iii)}$$

The condition for loop the loop $T_B = 0$

$$mg = \frac{mv^2}{R}$$

$$\therefore \boxed{v = \sqrt{Rg}}$$

Velocity at point B to complete the loop.

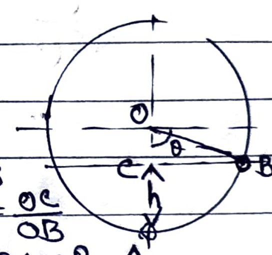
By conservation of energy

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mg(2R)$$

$$u^2 = v^2 + 4gR$$

$$\Rightarrow u^2 = Rg + 4Rg = 5gR$$

$$\boxed{u = \sqrt{5gR}}$$



ΔOCB

$\cos\theta = \frac{OC}{OB}$

$OC = OB\cos\theta$

$OC = R\cos\theta$

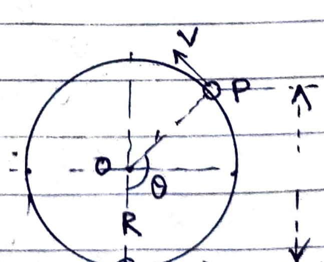
$h = (OA - OC) = R - R\cos\theta$

$h = R(1 - \cos\theta)$

$\therefore$ To loop the loop the minimum velocity at lowest point is $\sqrt{5}$ times the velocity at the topmost point.

Condition of Leaving the Circle

If $u < \sqrt{5gR}$, then the tension in the string will become zero before reaching the highest point.



As we know $T - mg\cos\theta = \frac{mv^2}{R}$

As $T = 0$

$$\cos\theta = -\frac{v^2}{Rg}$$

[$v \rightarrow$ velocity at the instance when $T = 0$]

$$v^2 = u^2 - 2gh$$

$$\boxed{\cos\theta = \frac{2gh - u^2}{Rg}}$$

Again $h = R(1 - \cos\theta)$

$$\frac{h}{R} = 1 - \frac{2gh - u^2}{Rg}$$

$$\Rightarrow \frac{h}{R} = \frac{Rg - 2gh + u^2}{Rg}$$

$$\Rightarrow gh = Rg - 2gh + u^2$$

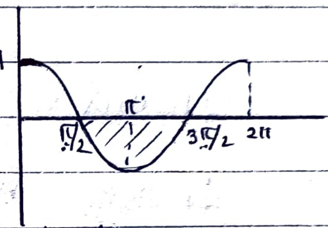
$$\Rightarrow 3gh = Rg + u^2$$

$$h = \frac{u^2 + Rg}{3g}$$

Let say $h = h_1$

$$\boxed{h_1 = \frac{u^2 + Rg}{3g}}$$ The height at which $T = 0$

The point at which $v = 0$



$$0 = u^2 - 2gh$$

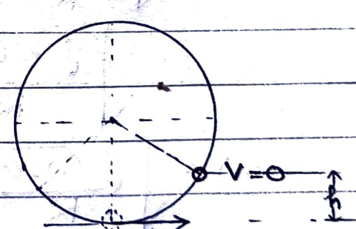
$$h = \frac{u^2}{2g} = h_2 \text{ (say)}.$$

$$\boxed{h_2 = \frac{u^2}{2g}}$$ The hight at which $v = 0$

If $u^2 > 2gR$ then $h > R$ $\therefore 0^\circ < \theta < 180^\circ$

Condition of Oscillation

The particle will oscillate, if velocity of the particle becomes zero but tension in the string is not zero.



$$h_2 < h_1$$

$$\frac{u^2}{2g} < \frac{u^2 + Rg}{3g}$$

$$3u^2 < 2u^2 + 2Rg$$

$$u^2 < 2Rg$$

$$u < \sqrt{2Rg}$$

If $h_1 = h_2$, $u = \sqrt{2gR}$, Tension and velocity both becomes zero simultaneously.

$\therefore 0 < u^2 \leq 2gR$, then $0^\circ < \theta \leq 90^\circ$