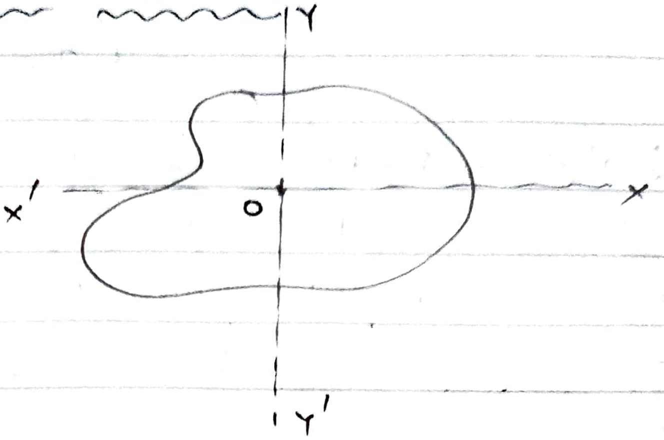
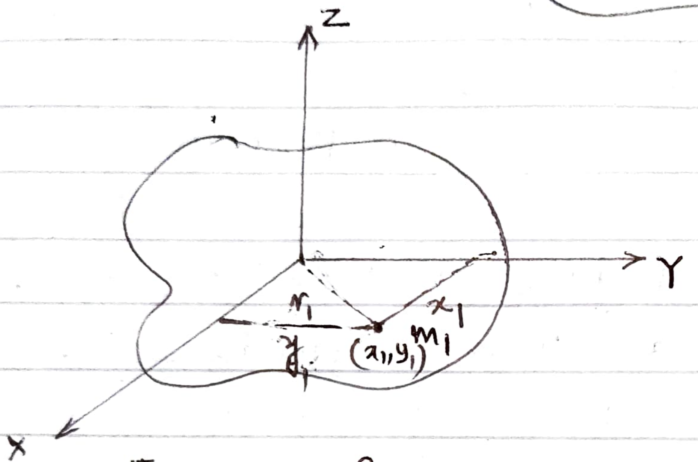
Theorems of Moment of Inertia.

Perpendiculare axis theorem

$$I_z = I_x + I_y$$



Proof



$$r_1 = \sqrt{x_1^2 + y_1^2}$$

$$I_x = m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2 + \cdots + m_n y_n^2$$
$$= \sum_{i=1}^{n} m_i y_i^2$$

Similarly $I_y = m_1 x_1^2 + m_2 x_2^2 + m_3 x_3^3 + \cdots + m_n x_n^2$
$$= \sum_{i=1}^{n} m_i x_i^2$$

$$I_z = (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots + m_n r_n^2)$$
$$= \sum_{i=1}^{n} m_i r_i^2$$

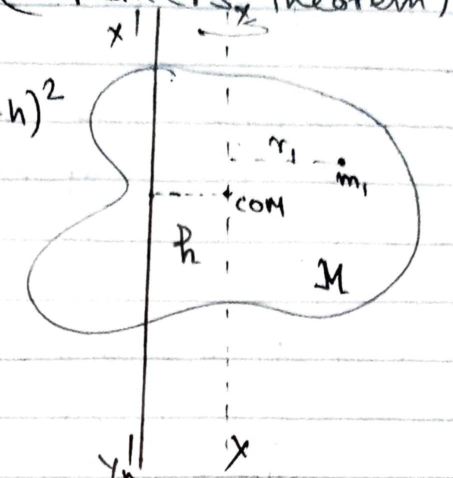
$$I_x + I_y = \sum_{i=1}^{n} m_i y_i^2 + \sum_{i=1}^{n} m_i x_i^2 = \sum_{i=1} m_i (x_i^2 + y_i^2)$$
$$= \sum_{i=1} m_i r_i^2 = I_z.$$

$$\boxed{I_x + I_y = I_z}$$

$$I_x = (I_z - I_y)$$
$$I_y = (I_z - I_x)$$

Parallel Axis theorem (Steiner's theorem)

M.I about $x'y'$ axis



$$I = m_1(r_1+h)^2 + m_2(r_2+h)^2 + m_3(r_3+h)^2 + \cdots + m_n(r_n+h)^2$$

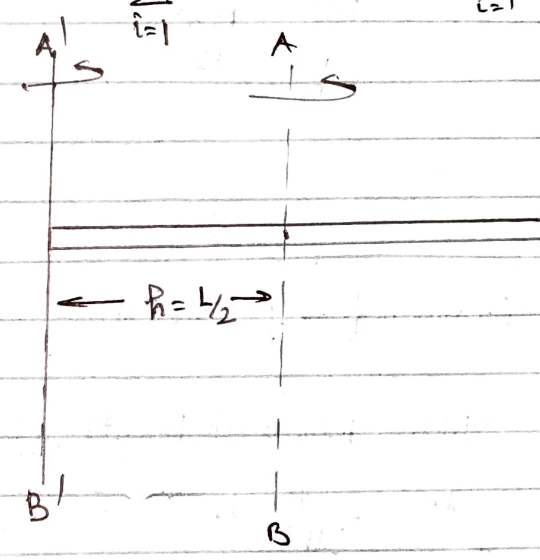
$$I' = \sum m_i(r_i+h)^2$$

$$= \sum_{i=1}^{n} m_i\left(r_i^2 + 2r_i h + h^2\right)$$

$$= \sum_{i=1}^{n} m_i r_i^2 + 2h\sum_{i=1}^{n} m_i r_i + h^2\sum_{i=1}^{n} m_i$$

Now, $\sum_{i=1}^{n} m_i r_i^2 = I_{cm}$; $\sum_{i=1}^{n} m_i = M$ $\quad \sum_{i=1}^{n} m_i r_i = 0$

$$\boxed{I' = I_{CM} + Mh^2}$$



$$I_{CM} = \frac{ML^2}{12}$$

$$I' = I_{CM} + Mh^2$$

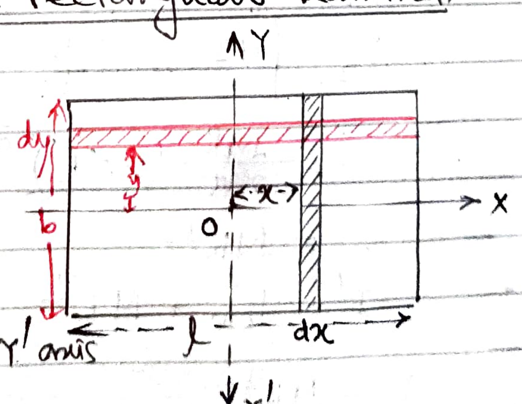
$$= \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$$

$$= \frac{ML^2}{12} + \frac{ML^2}{4}$$

$$= \frac{ML^2(1+3)}{12} = \frac{ML^2}{3}$$

Moment of Inertia of a rectangulare Lamina.

$$\int dm\, r^2$$



Mass per unit length $= \frac{M}{l}$

$\therefore$ Mass of the element strip

$$dm = \frac{M}{l}dx$$

M.I of the strip about YOY' axis

$$dI = dm\, x^2 = \frac{M}{l}dx\, x^2$$

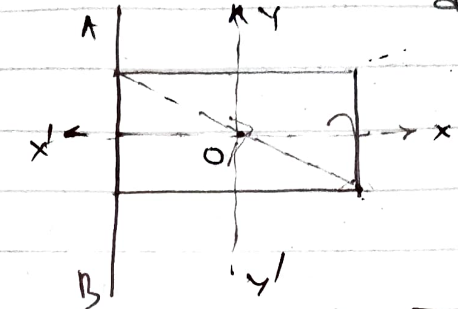
$$I_y = \int_{-l/2}^{l/2} \frac{M}{l}x^2 dx = \frac{M}{l}\int_{-l/2}^{l/2} x^2 dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{l/2}$$

$$= \frac{M}{3l}\left[\frac{l^3}{8} + \frac{l^3}{8}\right] = \frac{M}{3l}\frac{l^3}{4} = \frac{Ml^2}{12}.$$

Similarly $I_x = \frac{Mb^2}{12}$ $\quad I_y = \frac{Ml^2}{12}$

$$I_z = I_x + I_y$$

$$I_z = \frac{M}{12}\left[l^2 + b^2\right] = \frac{M}{12} D^2 \quad \left(D = \sqrt{l^2+b^2} \text{ diagonal}\right)$$



$$I = I_z + M\left(\frac{L}{2}\right)^2$$

$$= \frac{M}{12}(l^2+b^2) + \frac{ML^2}{4}$$

$$= \frac{ML^2}{12} + \frac{ML^2}{4} + \frac{Mb^2}{12}$$

$$= Ml^2\left(\frac{1+3}{12}\right) + \frac{Mb^2}{12}$$

$$= \frac{Ml^2}{3} + \frac{Mb^2}{12}$$

$$h = \frac{\sqrt{l^2+b^2}}{2}$$

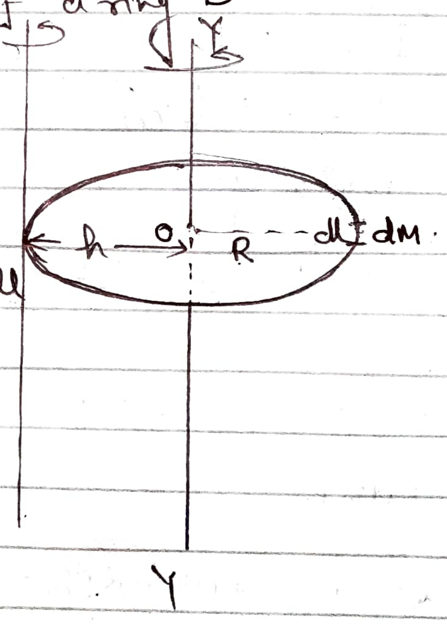
$$I' = I_z + Mh^2$$

$$= \frac{M}{12}(l^2+b^2) + M\frac{l^2+b^2}{4}$$

$$= M(l^2+b^2)\left(\frac{1}{12}+\frac{1}{4}\right)$$

$$= \frac{M(l^2+b^2)}{3}$$

Moment of inertia of a ring



Mass per unit Length $\frac{M}{2\pi R}$

$$dm = \frac{M}{2\pi R} \cdot dl$$

$$dI = dm\, R^2 = \frac{M}{2\pi R} \cdot R^2 dl$$

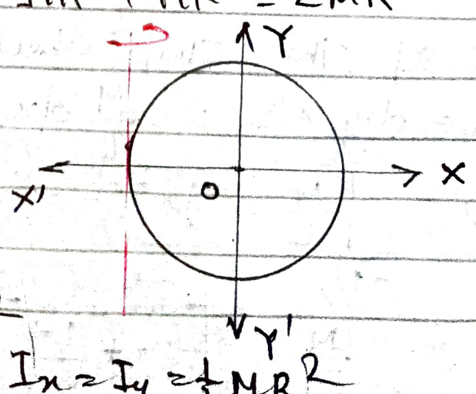
$$= \frac{MR}{2\pi} dl$$

$$I_z = \int dI = \frac{MR}{2\pi} \int_0^{2\pi R} dl$$

$$= \frac{MR}{2\pi}(2\pi R)$$

$$= MR^2$$

$$I' = I_{cm} + MR^2 = MR^2 + MR^2 = 2MR^2$$



$$\frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

$I_z = MR^2$

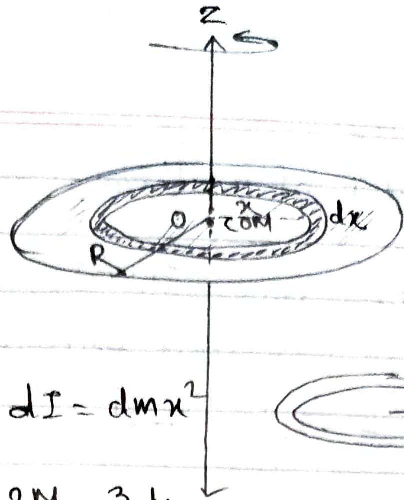
$I_x = ?$ $\quad I_y = ?$

$I_z = I_x + I_y$

As $I_x = I_y = I$

$\therefore I = \frac{I_z}{2} = \frac{1}{2}MR^2$

$I_x = I_y = \frac{1}{2}MR^2$



Mass per unit area

$$\frac{M}{\pi R^2}$$

$$\therefore \; dM = \frac{M}{\pi R^2} 2\pi x \, dx$$

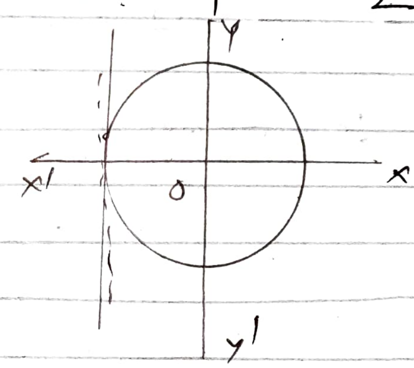
$$= \frac{2M}{R^2} x \, dx \qquad dI = dm x^2$$

$$dI = \left(\frac{2M}{R^2} x \, dx\right) x^2 = \frac{2M}{R^2} x^3 dx$$

$$I = \int dI = \frac{2M}{R^2} \int_0^R x^3 \, dx$$

$$= \frac{2M}{R^2} \left[\frac{x^4}{4}\right]_0^R = \frac{2M}{R^2} \cdot \frac{R^4}{4} = \frac{MR^2}{2}$$

$$I_x = I_y = \frac{I_z}{2} = \frac{MR^2}{4}$$



Prob $\quad m = 200 \, kg$

$K = 0.6 \, m$. (Radius of gyration)

150 rpm in 90 revolutions. Starting from rest. Find τ

$$\tau = I\alpha \qquad I = mk^2$$

$$f = 150 \text{ rpm} \qquad f = \frac{150}{60} \text{ rps (Hz)}$$

$$\omega = 2\pi f = 2\pi \times \frac{150}{60} \text{ rad/s}$$

$$\tau = I\alpha \qquad \cancel{\omega = \omega_0 + \alpha t} \qquad \omega^2 = \omega_0^2 + 2\alpha\theta$$

$$= m \times k^2 \times \frac{5\pi}{72} \qquad \cancel{\alpha = \frac{\omega}{t}} = \qquad \alpha = \frac{\omega^2}{2\theta} = \frac{\omega^2}{2 \times 180\pi}$$

$$= 200 \times (0.6)^2 \times \frac{5\pi}{72} \qquad \theta = 90 \times 2\pi = 180\pi$$

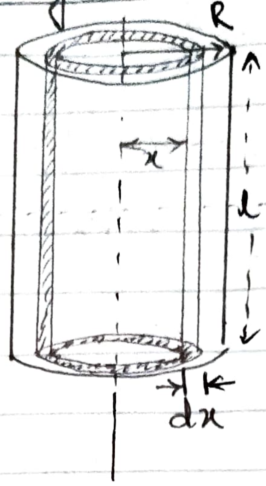
$$= 200 \times \frac{36}{100} \times \frac{5\pi}{72}$$

$$= 5\pi$$

$$\alpha = \frac{(5\pi)^2}{2 \times 180 \times \pi} = \frac{25\pi^2}{360 \times \pi}$$

$$= \frac{5\pi}{72} \text{ rad/s}^2$$

M.I of uniform solid cylinder.



Volume(V) = $\pi R^2 l$

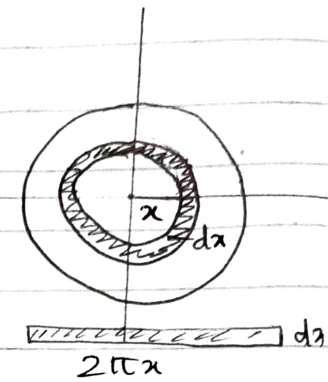
Mass per unit volume

$(\rho) = \frac{M}{V} = \frac{M}{\pi R^2 l}$

Mass of the element

$dm = (2\pi x \, dx \, l) \cdot \frac{M}{\pi R^2 l}$

$= \frac{2M}{R^2} x \, dx$



M.I of the mass element

$dI = \left(\frac{2M}{R^2} x \, dx\right) x^2$

M.I of the solid cylinder

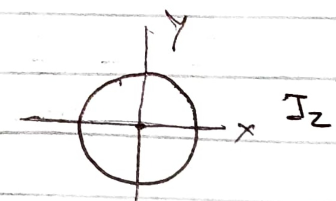
$I = \int dI = \int_0^R \frac{2M}{R^2} x^3 \, dx$

$= \frac{2M}{R^2} \left[\frac{x^4}{4}\right]_0^R = \frac{2M}{R^2} \cdot \frac{R^4}{4} = \frac{MR^2}{2}$

$I_x = I_y \quad \therefore I_z = 2I_x = 2I_y$

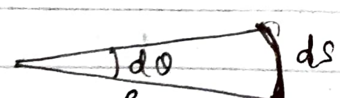
$I_x = I_y = \frac{I_z}{2}$

$I_x = I_y = \frac{MR^2}{4}$.



$I_x + I_y = I_z$

$F_B = V \rho g$



$ds = R \, d\theta$.

$\int ds = R \int d\theta$

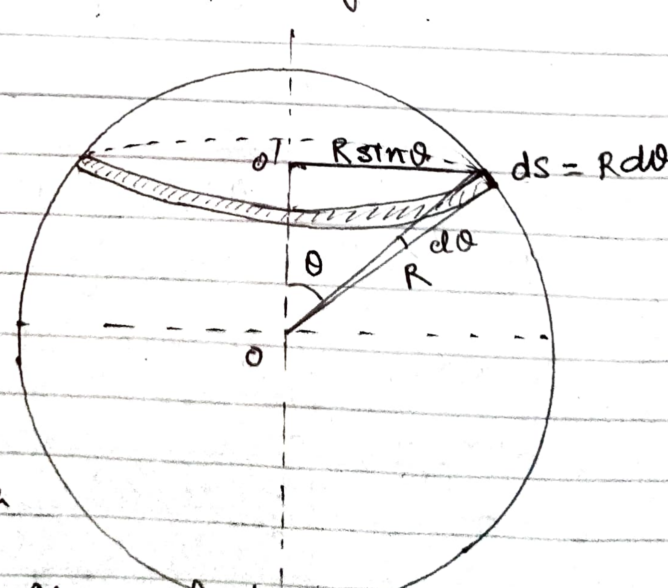
= Circumference

$= R \, 2\pi$

$= 2\pi R$

Mass per unit area

$\frac{M}{4\pi R^2}$



Mass of the element

$dm = 2\pi(R\sin\theta) \cdot R\,d\theta \cdot \frac{M}{4\pi R^2}$

$$dM = \frac{M}{2}\sin\theta\, d\theta$$

$$dI = dM\, r^2 = \frac{M}{2}\sin\theta \cdot d\theta \cdot (R\sin\theta)^2$$

$$= \frac{M}{2}R^2 \sin^3\theta \cdot d\theta$$

$$I = \int_0^\pi \frac{M}{2} R^2 \sin^3\theta\, d\theta$$

$$= \frac{MR^2}{2}\int_0^\pi \sin^3\theta\, d\theta$$

$$= \frac{MR^2}{2}\left[\int_0^\pi (1-\cos^2\theta)\sin\theta\, d\theta\right]$$

Let $\cos\theta = z$ $\quad$ $\cancel{\cos\theta = z}$

$\Rightarrow -\sin\theta \cdot d\theta = dz$ $\quad$ If $\theta = 0$, $z = 1$; $\theta = \pi$, $z = -1$

$\sin\theta \cdot d\theta = -dz$

$$-\int_0^\pi (1-z^2)\, dz$$

$$\int_0^\pi (z^2-1)\, dz$$

$$\Rightarrow \int_0^\pi z^2 dz - \int_{+1}^{-1} dz$$

$$= \left[\frac{z^3}{3} - z\right]_{1}^{-1}$$

$$= \left(-\frac{1}{3}+1\right) - \left(\frac{1}{3}-1\right)$$

$$= -\frac{1}{3}+1-\frac{1}{3}+1$$

$$= 2-\frac{2}{3} = 2\left(1-\frac{1}{3}\right) = \frac{4}{3}$$

$$I = \frac{MR^2}{2}\times\frac{4}{3}$$

$$\boxed{I = \frac{2}{3}MR^2}$$

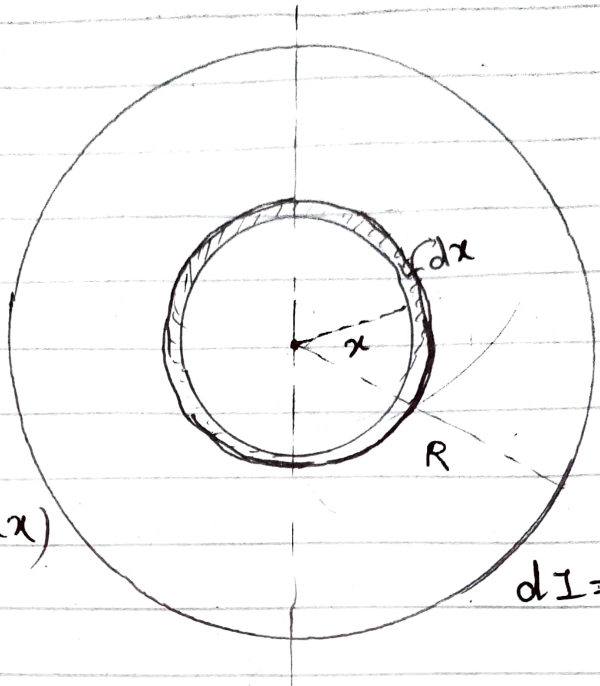
Moment of Inertia of a solid Sphere

M per unit volume

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

volume of the element

$$= 4\pi x^2 dx$$



Mass of the element

$$dM = \left(\frac{3M}{4\pi R^3}\right)(4\pi x^2 dx)$$

$$= \frac{3M}{R^3} x^2 dx$$

$$dI = \frac{2}{3} dM x^2$$

M.I of the volume element about the axis

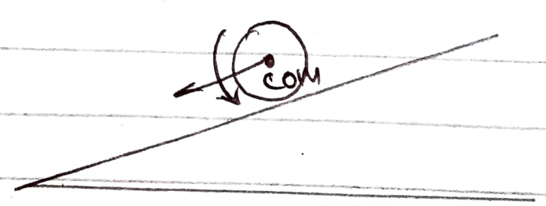
$$dI = dm\, x^2 = \frac{2}{3}\left(\frac{3M}{R^3} x^2 dx\right) x^2$$

$$dI = \frac{2M}{R^3} x^4 dx$$

$$I = \int dI = \int_0^R \frac{2M}{R^3} x^4 dx = \frac{2M}{R^3}\left[\frac{x^5}{5}\right]_0^R$$

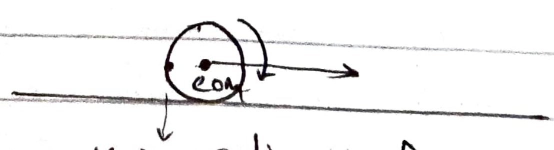
$$= \frac{2M}{R^3} \cdot \frac{R^5}{5} = \frac{2}{5} MR^2$$

Kinetic Energy of a rolling Motion



E_K = K.E of translation + K.E of rotation.

$$E_K = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I\omega^2$$

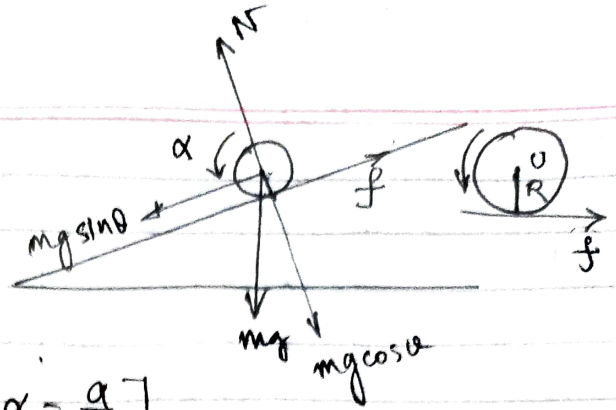


$$v_{CM} = R\omega \qquad I = mK^2$$

K → radius of gyration

$$K = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} mK^2 \left(\frac{v_{cm}}{R}\right)^2$$

$$= \frac{1}{2} m v_{cm}^2 \left(1 + \frac{K^2}{R^2}\right)$$



$$N = mg\cos\theta$$
$$F = Ma = Mg\sin\theta - f$$
$$\tau = f\cdot R = I\alpha$$
$$f = \frac{I\alpha}{R} = \frac{Ia}{R^2} \quad \left[\alpha = \frac{a}{R}\right]$$

$F = ma$

$$Ma = Mg\sin\theta - \frac{Ia}{R^2}$$
$$\cancel{\alpha\left(M + \frac{I}{R^2}\right)} \quad a = g\sin\theta - \frac{Ia}{MR^2}$$
$$a\left[1 + \frac{I}{MR^2}\right] = g\sin\theta$$
$$a = \frac{g\sin\theta}{\left[1 + \frac{I}{MR^2}\right]}$$

For solid cylinder

$$a = \frac{g\sin\theta}{\left[1 + \frac{MR^2}{2MR^2}\right]} = \frac{2}{3} g\sin\theta$$

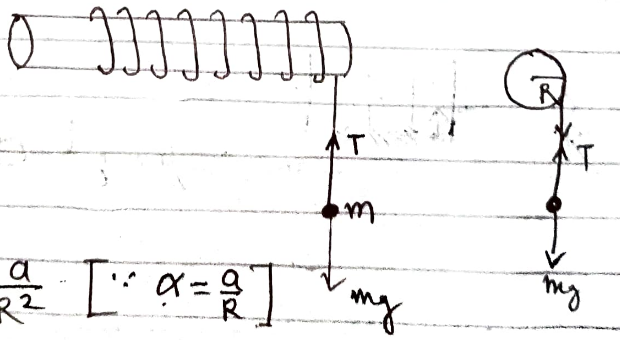
For hollow sphere

$$a = \frac{g\sin\theta}{\left[1 + \frac{2}{3}\frac{MR^2}{MR^2}\right]} = \frac{3}{5} g\sin\theta$$

For solid sphere

$$a = \frac{g\sin\theta}{\left[1 + \frac{2}{5}\frac{MR^2}{MR^2}\right]} = \frac{5}{7} g\sin\theta$$

Mass attached on string wound on a cylinder.



$$mg - T = ma$$
$$\tau = T\cdot R = I\alpha$$
$$T = \frac{I\alpha}{R} = \frac{Ia}{R^2} \quad \left[\because \alpha = \frac{a}{R}\right]$$
$$ma = mg - \frac{Ia}{R^2}$$
$$a = g - \frac{Ia}{mR^2} \Rightarrow a\left(1 + \frac{I}{mR^2}\right) = g$$
$$a = \frac{g}{\left[1 + I/mR^2\right]}$$