VECTORS

Physical quantities can be divided into two classes, namely:
→ Scalars
→ Vectors

Scalars : Those physical quantities which possess only magnitude and no direction are called scalars. Like ; mass, length, time, temperature etc.

Vectors : Those physical quantities which possess both magnitude and direction are called vectors. Like: displacement, velocity, area, force, momentum etc.

Properties of vectors ;
(i) It has certain magnitude and specific direction
(ii) It must be resolved into components.
(iii) It follows vector algebra.

Representation of a vector

A vector quantity is represented by a arrowheaded straight line. The length of the line represents its magnitude and the arrowhead gives direction.

Ex: A body has a velocity $40\ kmh^{-1}$ due east.
Here 4 cm line represents $40\ kmh^{-1}$ as 1cm represents $10\ kmh^{-1}$ and the arrowhead is given along conventional east direction.

Types of vectors

(a) Fixed or localized vector :
If the initial point of a vector is fixed, then it is called fixed vector.

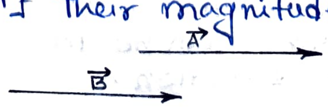
(b) Free or non-localized vector :
If the initial point of a vector is not fixed, then the vector is known as free vector or non-localized vector.

(c) Unit vector :
It is the vector whose magnitude is unity and direction is same as that of the given vector. A unit vector is represented as

$$\vec{A} = A\,\hat{n} \quad \Rightarrow \hat{n} = \frac{\vec{A}}{A}$$

Where A is the magnitude of $\vec{A}$ and $\hat{n}$ is the unit vector along $\vec{A}$.

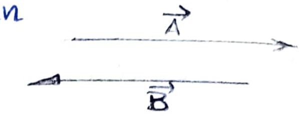
(d) Equal vectors :
Two vectors are said to be equal if their magnitude as well as direction are the same.
Here $\vec{A}$ and $\vec{B}$ are equal vectors.



$\therefore \vec{A} = \vec{B}$

(e) Negative Vector :

A vector is said to be negative of a given vector if its magnitude is the same as that of the given vector but its direction is opposite.

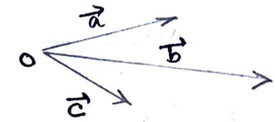


Here vector $\vec{B}$ is negative to $\vec{A}$ vector.

Therefore $\vec{A} = -\vec{B}$ or $\vec{B} = -\vec{A}$

(f) Co-initial vector

Vectors which are originated from the same point are called co-initial vectors. In the figure $\vec{a}$, $\vec{b}$ and $\vec{c}$ are co-initial vectors as they originate from the point 'O'.



(g) Co-llineare vectors:

The vectors which either act along the same or along parallel lines, are called collinear vectors.

Fig-1

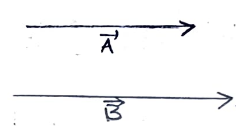


Fig-2

In the figures both $\vec{A}$ and $\vec{B}$ are collinear vector.

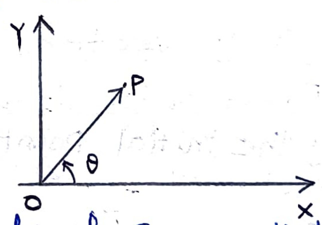
(h) Co-planar vectors :

The vectors which acts in the same plane are called co-planar vectors.

(i) Position vector :

A vector which gives position of an object with reference to some co-ordinate system is called position vector.

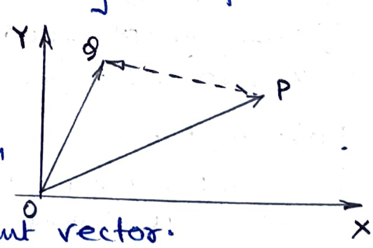
In the fig. $\vec{OP}$ is the position vector of point P in the cartesian plane.



(j) Displacement vector :

This vector helps to find the displacment of a particle in a given co-ordinate system.

In the fig. let the initial position of the particle is at P and final position is Q. Therefor $\vec{OP}$ is the initial position vector and $\vec{OQ}$ is the final position vector then $\vec{PQ}$ represents displacment vector.



(k) Zero or null vector :

It is a vector that has zero magnitude and arbitrary direction. It is represented by $\vec{0}$. Like addition of two equall and opposite vector produces a null vector.

(l) Translational vector :

Vectors can be translated without changing its magnitude and direction.

(m) Polar vectors :
Vector that reverses sign when the co-ordinate axes are reversed. Like radius vector.

(n) Axial vector or Pseudo vectors :
The vectors which are invariant under inversion of its co-ordinate axes is known as pseudo vectors. Like angular velocity (ω), angular momentum (L), torque (τ), magnetic field (H), Magnetic dipole moment (m) etc. In general these vectors represent rotational effect and acts along the axis of rotation.

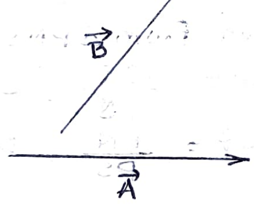
(o) Concurrent vectors :
The vectors those which pass through the same point.

Multiplication of a vector by a real number

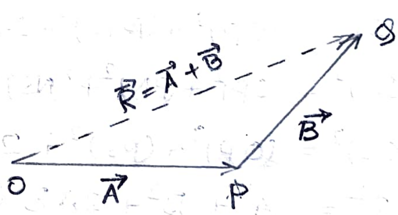
When a vector is multiplied by a real number λ, the magnitude is increased λ times. If λ is positive then $\lambda\vec{A}$ is along $\vec{A}$. If λ is negative then direction of $\lambda\vec{A}$ is opposite to $\vec{A}$.

Addition or Composition of vectors

- Triangle Laws of vector addition :
If two vectors can be represented both in magnitude and direction by the two sides of a triangle taken in the same order, then their resultant is represented completly, both in magnitude and direction, by the third side of the triangle taken in the reverse order.

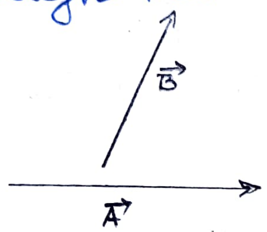


Vector $\vec{A}$ and $\vec{B}$ needs to be added

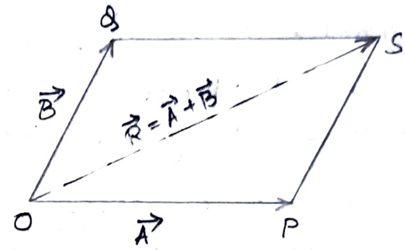


Translate the vectors, such that end of a vector must be the starting of the second vector. close the figure by drawing the thirside. Then the third side represents the resultant of two vectors.

- **Parallelogram law of vector addition:**
If two vectors can be represented both in magnitude and direction by the two adjecent sides of a parallelogram drawn from a common point, then their resultant is completly represented, both in magnitude and direction, by the diagonals of the parallelogram passing through that point.



Vector $\vec{A}$ and $\vec{B}$ needs to be added

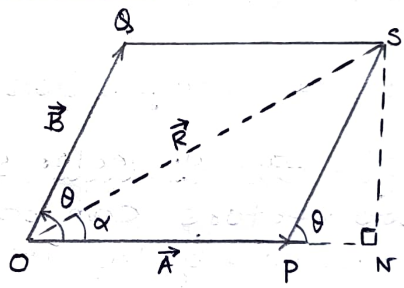


Make the two vectors co-initial, complete the parallelogram and draw the diagonal. The diagonal represents the resultant.

Analytical Method of Vector addition

In the parallelogram OPSQ, the adjecent side OP represent vector $\vec{A}$ and side OQ represent vector $\vec{B}$.

Draw SN, perpendiculare to OP produced.



Magnitude of $\vec{R}$

$\therefore$ From ΔONS

$$(OS)^2 = (ON)^2 + (NS)^2$$

$$\Rightarrow (OS)^2 = (OP+PN)^2 + (NS)^2$$

$$(OS)^2 = (OP)^2 + (PN)^2 + 2OP.PN + (NS)^2$$

$$\therefore (OS)^2 = (OP)^2 + (PN)^2 + (NS)^2 + 2\,OP.PN$$

$$\therefore (OS)^2 = (OP)^2 + (PS)^2 + 2.OP.PS\cos\theta$$

$$\therefore R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\therefore R = \sqrt{A^2 + B^2 + 2AB\cos\theta} \quad \text{-------(i)}$$

From figure: $\vec{OP} = \vec{A} = \vec{QS}$

$\vec{OQ} = \vec{B} = \vec{PS}$

$\vec{OS} = \vec{R}$

Again from ΔPNS

$\sin\theta = \dfrac{SN}{PS} \Rightarrow SN = PS\sin\theta$

$\cos\theta = \dfrac{PN}{PS} \Rightarrow PN = PS\cos\theta$

Direction of $\vec{R}$

Again from ΔONS

$$\tan\alpha = \frac{NS}{ON} = \frac{NS}{OP+PN}$$

$$\therefore \tan\alpha = \frac{PS\sin\theta}{OP + PS\cos\theta}$$

$$\therefore \tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

$$\therefore \tan\alpha = \frac{B\sin\theta}{A+B\cos\theta} \quad \text{-------(ii)}$$

Case I: If $\theta = 0^\circ$; two vectors are parallel to each other, then

$$R = \sqrt{A^2+B^2+2AB\cos 0^\circ} = (A+B)$$

$$\tan\alpha = \frac{B\sin 0^\circ}{A+B\cos 0^\circ} = 0$$

$\therefore \alpha = 0^\circ$ Resultant is along $\vec{A}$ or $\vec{B}$

Case II: If $\theta = \pi/2$; two vectors are perpendicular to each other, then,

$$R = \sqrt{A^2+B^2+2AB\cos 90^\circ} = \sqrt{A^2+B^2}$$

$$\tan\alpha = \frac{B\sin 90^\circ}{A+B\cos 90^\circ} = \frac{B}{A}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{B}{A}\right)$$

Case III: If $\theta = \pi$; two vectors are antiparallel or opposite to eachother then,

$$R = \sqrt{A^2+B^2+2AB\cos 180^\circ} = \sqrt{A^2+B^2-2AB} = A-B$$

$$\tan\alpha = \frac{B\sin 180^\circ}{A+B\cos 180^\circ} = 0$$

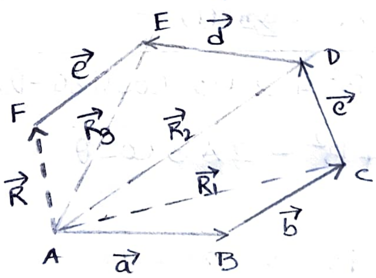
$\therefore \alpha = 0^\circ$; Resultant is either along $\vec{A}$ or $\vec{B}$ (whichever is greater).

- **Polygon Law of vectors**

If a number of vectors can be represented, both in magnitude and direction, by the sides of an open convex polygone taken in the same order, then their resultant is represented completly in magnitude and direction by the closing side of the polygone, taken in the opposite order. Polygon Law is just an extended triangle law.

From figure the sides AB, BC, CD, DE, EF represents vector $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ and $\vec{e}$ respectively. Therefore the side AF, which is closing the figure, represents the resultant vector $\vec{R}$

$$\therefore \vec{R} = \vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}$$



From fig. $\vec{AC} = \vec{AB} + \vec{BC}$

$\therefore \vec{R_1} = \vec{a} + \vec{b}$

again $\vec{AD} = \vec{AC} + \vec{CD} = \vec{AB} + \vec{BC} + \vec{CD}$

$\therefore \vec{R_2} = \vec{R_1} + \vec{c} = \vec{a} + \vec{b} + \vec{c}$

Similarly $\qquad \vec{AE} = \vec{AD} + \vec{DE} = \vec{AB} + \vec{BC} + \vec{CD} + \vec{DE}$

$$R_3 = \vec{a} + \vec{b} + \vec{c} + \vec{d}$$

Again $\qquad \vec{AF} = \vec{AE} + \vec{EF} = \vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} + \vec{EF}$

$$R = \vec{R_3} + \vec{e} = \vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e}$$

∴ Polygone rule is the extension of triangle rule.

Properties of vector Addition

(a) Vector addition is Commutative

- $\vec{A} + \vec{B} = \vec{B} + \vec{A}$.

(b) Vector addition is Associative

- $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

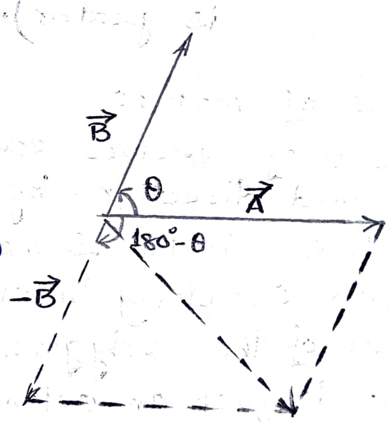
(c) Vector addition is Distributive

- $m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$

Subtraction of Vectors :

The subtraction of a vector $\vec{B}$ from a vector $\vec{A}$ is equivalent to the addition of vector $\vec{A}$ and the negative vector $\vec{B}$ $(-\vec{B})$.

∴ $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$



Now the vector addition of $\vec{A}$ and $\vec{B}$ is -

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta} \quad \text{---(i)}$$

Now from fig. angle between $\vec{A}$ and $-\vec{B}$ is $(180° - \theta)$

∴ Put it in the eqn (i)

$$R = \sqrt{A^2 + B^2 + 2AB\cos(180° - \theta)}$$

$$R = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

Again,

$$\tan\alpha = \frac{B\sin(180° - \theta)}{A + B\cos(180° - \theta)}$$

∴ $$\tan\alpha = \frac{B}{A - B\cos\theta}.$$

Properties of vector Subtraction:

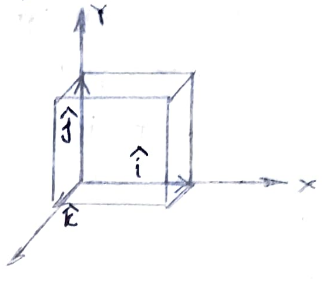
(a) Vector subtraction is non-Commutative

$$\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$$

(b) Vector subtraction is Non-Associative

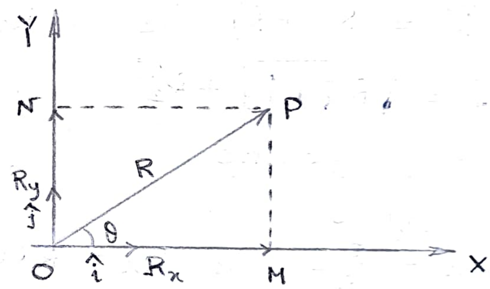
$$\vec{A} - (\vec{B} - \vec{C}) \neq (\vec{A} - \vec{B}) - \vec{C}$$

Orthogonal unit vector



In a right-handed Cartesian co-ordinate system, three unit vectors $\hat{i}, \hat{j}, \hat{k}$ are used to represent the positive direction of X-axis, Y-axis and Z-axis respectively. These three mutually perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$ are collectively known as orthogonal triad of unit vectors. Thus $|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$

Resolution of vector in a plane (In two dimension)



Suppose we wish to resolve vector $\vec{OP}$ ($\vec{R}$) along X and Y axis. Drop projection on X and Y axis from point P.

Now from figure in ΔOMP.

$OM = OP \cos\theta$

$\therefore R_x = R\cos\theta$

$\therefore \vec{R_x} = R\cos\theta\,\hat{i}$ ----(i)

Similarly from ΔOMP

$MP = ON = OP \sin\theta$

$\therefore R_y = R\sin\theta$

$\therefore \vec{R_y} = R\sin\theta\,\hat{j}$ ----(ii)

$\therefore$ From Parallelogram Law $\quad \vec{OP} = \vec{OM} + \vec{ON}$

$$\vec{R} = \vec{R_x} + \vec{R_y}$$

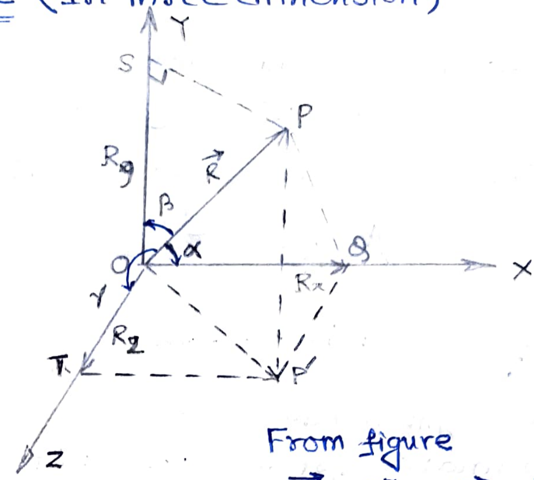
$$\therefore \quad \vec{R} = R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j}$$

$$\therefore \quad R = \sqrt{R^2\cos^2\theta + R^2\sin\theta} = \sqrt{R_x^2 + R_y^2}$$

$$\therefore \quad \tan\theta = \frac{MP}{OM} = \frac{R_y}{R_x}$$

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

Resolution of vector in space (In three dimension)



Suppose we wish to resolve vector $\vec{OP}$ ($\vec{R}$). along x, y and z axis.
Drop the projection on X, Y and Z axis at Q, S and T point respectively.
Again drop the projection of point P on X, Z plane at point P'. From P' point drop projection on OX and OZ line at points Q and T respectively.

From figure

$\vec{OQ} = \vec{R_x}$, $\vec{OS} = \vec{R_y}$

$\vec{OT} = \vec{R_z}$.

α, β and γ are the angles made by $\vec{OP}$ with X, Y and Z axis respectively.

$\therefore$ From fig, $\vec{OP} = \vec{OP'} + \vec{P'P}$ ------ (i)

Again $\vec{OP'} = \vec{OQ} + \vec{OT}$ ------ (ii)

$\therefore$ $\vec{OP} = \vec{OQ} + \vec{OT} + \vec{P'P}$ ------ (iii)

$\therefore$ $\vec{R} = \vec{R_x} + \vec{R_y} + \vec{R_z} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$

$\therefore$ $R = \sqrt{R_x^2 + R_y^2 + R_z^2}$

Directional Cosine

From figure

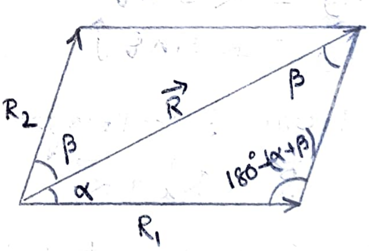
$\cos\alpha = \frac{R_x}{R}$ or $R_x = R\cos\alpha$ or $\vec{R_x} = R\cos\alpha\,\hat{i}$

$\cos\beta = \frac{R_y}{R}$ or $R_y = R\cos\beta$ or $\vec{R_y} = R\cos\beta\,\hat{j}$

$\cos\gamma = \frac{R_z}{R}$ or $R_z = R\cos\gamma$ or $\vec{R_z} = R\cos\gamma\,\hat{k}$

$\therefore$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

A vector can be resolved into infinite number of components



$\therefore$ $\vec{R} = \vec{R_1} + \vec{R_2} + \vec{R_3} + \cdots + \vec{R_n}$

$\therefore$ $\vec{R_x} = \vec{R_{1x}} + \vec{R_{2x}} + \vec{R_{3x}} + \cdots + \vec{R_{nx}}$ Along x axis

$\vec{R_y} = \vec{R_{1y}} + \vec{R_{2y}} + \vec{R_{3y}} + \cdots + \vec{R_{ny}}$ Along Y axis

$\vec{R_z} = \vec{R_{1z}} + \vec{R_{2z}} + \vec{R_{3z}} + \cdots + \vec{R_{nz}}$ Along z-axis

By applying sine law of triangle

$$\frac{R_1}{\sin\beta} = \frac{R_2}{\sin\alpha} = \frac{R}{\sin\{180° - (\alpha+\beta)\}}$$

$\therefore$ $R_1 = R\frac{\sin\beta}{\sin(\alpha+\beta)}$

$R_2 = R\frac{\sin\alpha}{\sin(\alpha+\beta)}$

For orthogonal component $(\alpha+\beta) = 90°$

$\therefore$ $R_1 = R_x = R\sin\beta = R\cos\alpha$

$R_2 = R_y = R\sin\alpha$

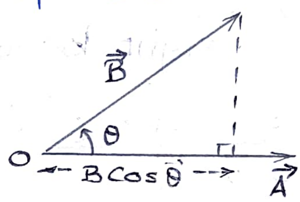
Multiplication of vectors

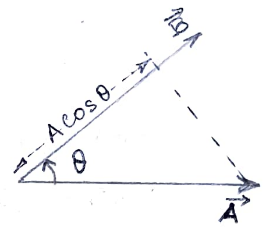
Scalar product of two vectors

The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and cosine of the angle θ between them. Thus

$$\vec{A}\cdot\vec{B} = AB\cos\theta = |\vec{A}||\vec{B}|\cos\theta$$

Geometrical Interpretation :





$$\vec{A}\cdot\vec{B} = AB\cos\theta = A(B\cos\theta)$$

= A × Magnitude of Component of $\vec{B}$ along the direction $\vec{A}$

$$\vec{A}\cdot\vec{B} = AB\cos\theta = (A\cos\theta)B$$

= B × Magnitude of component of $\vec{A}$ along $\vec{B}$

Thus the scalar product of two vectors is equal to the product of magnitude of one vector and the magnitude of component of other vector in the direction of first vector.

Scalar product in terms of rectangular components

Let $\vec{A}$ and $\vec{B}$ are two vectors, expressed in their rectangular components.

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\therefore \quad \vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= (A_xB_x + A_yB_y + A_zB_z) \quad \left[\because \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \text{ and } \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0\right]$$

Angle θ between two vectors

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\therefore \cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2+A_y^2+A_z^2}\cdot\sqrt{B_x^2+B_y^2+B_z^2}}$$

Properties of scalar product

- Scalar product is commutative : $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
- scalar product is distributive: $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$
- If $\vec{A}\cdot\vec{B} = 0$, then angle θ between them is 90°, provided $\vec{A}$ and $\vec{B}$ are not null vectors.

Vector product of two vectors

The vector or cross product of two vectors is defined as the vector whose magnitude is equal to the product of the magnitudes of two vectors and sine of the angle between them and whose direction is perpendicular to the plane of the two vectors. Thus

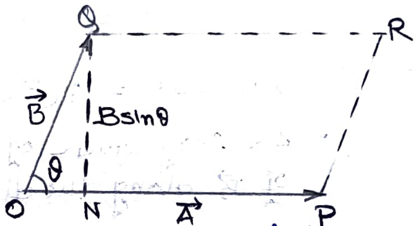
$$\vec{A} \times \vec{B} = AB\sin\theta \ \hat{n}$$

$\hat{n}$ is the unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$

The direction of the vector $\vec{A} \times \vec{B}$ can be determined by right handed screw rule or right handed thumb rule.

Geometrical Interpretation of vector product.

$$\vec{A} \times \vec{B} = AB\sin\theta = (OP)(OQ)\sin\theta$$
$$= (OP)(QN)$$
= Area of the parallelogram OPRQ.



Thus the magnitude of the vector product of two vectors is equal to area of the parallelogram formed by the two vectors as its adjacent side.

Vector product in terms of rectangular component.

We can express $\vec{A}$ and $\vec{B}$ in terms of their rectangular components as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \qquad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\therefore \quad \vec{A} \times \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

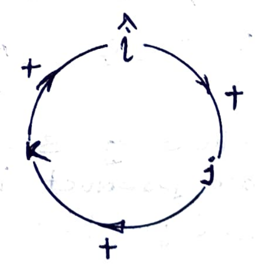
$$= A_xB_x(\hat{i}\times\hat{i}) + A_xB_y(\hat{i}\times\hat{j}) + A_xB_z(\hat{i}\times\hat{k}) +$$
$$A_yB_x(\hat{j}\times\hat{i}) + A_yB_y(\hat{j}\times\hat{j}) + A_yB_z(\hat{j}\times\hat{k}) +$$
$$A_zB_x(\hat{k}\times\hat{i}) + A_zB_y(\hat{k}\times\hat{j}) + A_zB_z(\hat{k}\times\hat{k})$$

$$= A_xB_y\ \hat{k} + A_xB_z\ \hat{j} - A_yB_x\ \hat{k} + A_yB_z\ \hat{i} + A_zB_x\hat{j} - A_zB_y\hat{i}$$

$$= (A_yB_z - A_zB_y)\hat{i} + (A_xB_z - A_zB_x)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

Rule used:

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$$
$$\hat{i}\times\hat{j} = \hat{k}, \quad \hat{j}\times\hat{k} = \hat{i}, \quad \hat{k}\times\hat{i} = \hat{j}$$
$$\hat{j}\times\hat{i} = -\hat{k}, \quad \hat{k}\times\hat{j} = -\hat{i}, \quad \hat{i}\times\hat{k} = -\hat{j}$$



Determinant Method:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}(A_yB_z - A_zB_y) - \hat{j}(A_xB_z - A_zB_x) + \hat{k}(A_xB_z - A_zB_x)$$

$$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_z - A_zB_x)\hat{k}$$

Properties of Vector product

- Vector product is anti-commutative

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

- Vector product is distributive over addition.

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

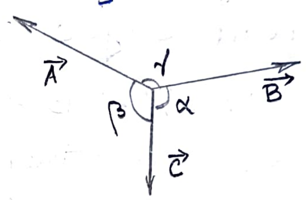
- Vector product of two parallel or antiparallel vector is a null vector. Thus

$$\vec{A}\times\vec{B} = AB\sin(0^\circ \text{ or } 180^\circ)\,\hat{n} = \vec{0}$$

Lami's Theorem

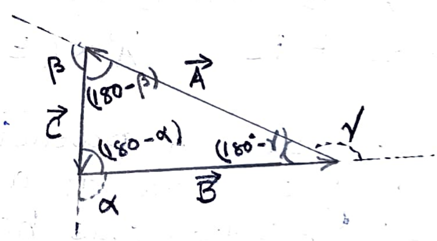
When three forces (concurrent) acting at a point are in equilibrium then each force is proportional to the sine of the angle between the other two forces. In mathematical form, it is expressed as

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\gamma}$$



Proof:

If three concurrent forces $\vec{A}$, $\vec{B}$ and $\vec{C}$ produces zero net force, then these three vector can be represented by three consecutive sides of a triangle taken in the same order. Therefore



$$\vec{A}+\vec{B}+\vec{C} = 0 \quad \text{------ (i)}$$

cross product on both side with $\vec{A}$.

$$\vec{A}\times(\vec{A}+\vec{B}+\vec{C}) = 0 \quad \text{--- (ii)}$$

$$\vec{A}\times\vec{A}+\vec{A}\times\vec{B}+\vec{A}\times\vec{C} = 0$$

$$\therefore \vec{A}\times\vec{B} = \vec{C}\times\vec{A}$$

$$\therefore AB\sin(180-\gamma) = CA\sin(180^\circ-\beta)$$

$$B\sin\gamma = C\sin\beta$$

$$\therefore \frac{B}{\sin\beta} = \frac{C}{\sin\gamma} \quad \text{------ (iii)}$$

Again cross product on both side of eq (i) with $\vec{B}$

$$\vec{B}\times(\vec{A}+\vec{B}+\vec{C}) = 0$$

$$\vec{B}\times\vec{A}+\vec{B}\times\vec{B}+\vec{B}\times\vec{C} = 0$$

$$\therefore \vec{B}\times\vec{C} = -\vec{B}\times\vec{A}$$

$$\vec{B}\times\vec{C} = \vec{A}\times\vec{B}$$

$$BC\sin\alpha = AB\sin(180^\circ-\gamma)$$

$$\therefore C\sin\alpha = A\sin\gamma$$

$$\therefore \frac{A}{\sin\alpha} = \frac{C}{\sin\gamma} \quad \text{-------- (iv)}$$

$\therefore$ From eq^n (iii) & (iv) $\dfrac{A}{\sin\alpha} = \dfrac{B}{\sin\beta} = \dfrac{C}{\sin\gamma}$

Lagrange's Identity

$$|\vec{A}\times\vec{B}|^2 + (\vec{A}\cdot\vec{B})^2 = |\vec{A}|^2|\vec{B}|^2$$

Proof

we know $|\vec{A}\times\vec{B}| = AB\sin\theta$ and $\vec{A}\cdot\vec{B} = AB\cos\theta$

$$\therefore \quad |\vec{A}\times\vec{B}|^2 + (\vec{A}\cdot\vec{B})^2 = A^2B^2\sin^2\theta + A^2B^2\cos^2\theta$$
$$= A^2B^2(\sin^2\theta + \cos^2\theta)$$
$$= A^2B^2 = |\vec{A}|^2|\vec{B}|^2$$

Scalar Triple Product

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three vectors. then scalar triple product is $\vec{A}\cdot(\vec{B}\times\vec{C})$. It is equal to the volume (V) of a parallelopiped formed by the vector $\vec{A}$, $\vec{B}$ and $\vec{C}$ as its concurrent edges. It gives a scalar quantity.

$$V = \vec{A}\cdot(\vec{B}\times\vec{C})$$

If $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$, then the three vectors are coplaner. Therefore the condition for coplanarity of three vectors is that their scalar triple product is zero.

Vector Triple product

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three vectors, then vector triple product is $\quad \vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

For example: A particle in circular motion at angular velocity ω and at a position r measured from a point on the rotation axis has centripetal acceleration

$$\vec{a}_c = \vec{\omega}\times(\vec{\omega}\times\vec{r}) \qquad [\because \vec{\omega}\times\vec{v} = \vec{a}_c]$$

VECTORS (DOSE-I)

1. Calculate the magnitude of vector $\vec{r} = (3\hat{i} + 4\hat{j} + 7\hat{k})$

2. Find the unit vector parallel to the resultant of the vectors $\vec{A} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{B} = \hat{i} + 2\hat{j} + 3\hat{k}$

3. Determine the value of m so that $\vec{A} = 2\hat{i} + m\hat{j} + \hat{k}$ and $\vec{B} = 4\hat{i} - 2\hat{j} - 2\hat{k}$ are perpendicular.
   Ans: $m = 3$

4. Find the area of a triangle $P(1,3,2)$, $Q(2,-1,1)$ and $R(-1,2,3)$
   Ans: $\frac{1}{2}\sqrt{107}$

5. The edges of a parallelopiped are given by the vector $(\hat{i} + 2\hat{j} + 3\hat{k})$, $5\hat{j}$ and $(4\hat{j} + m\hat{k})$. What should be the value of m in order that the volume of the parallelopiped be 20 units?
   Ans: $m = 4$

6. Two vectors, both equal in magnitude, have their resultant equal in magnitude of the either. Find the angle between the two vectors.
   Ans: $\theta = 120^\circ$

7. Find the angle between the following pairs of vector
   (i) $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{B} = -2\hat{i} - 2\hat{j} - 2\hat{k}$
   (ii) $A = -2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{B} = 3\hat{i} + 6\hat{j} + 2\hat{k}$
   Ans: (i) 180°
   (ii) 79°

8. The sum and difference of two vectors $\vec{A}$ and $\vec{B}$ are $2\hat{i} + 6\hat{j} + \hat{k}$ and $4\hat{i} + 2\hat{j} - 11\hat{k}$. Find the magnitude of each vector and their scalar product.
   Ans: $\sqrt{50}, \sqrt{41}, -25$

9. Find $|\vec{A} \times \vec{B}|$ if $|\vec{A}| = 10$, $|\vec{B}| = 2$ and $\vec{A} \cdot \vec{B} = 12$.

10. The maximum resultant of two vectors $\vec{A}$ and $\vec{B}$ $(A > B)$ is n times their least resultant. If θ be the angle between the vectors and their resultant be half the sum of the vectors, then show that $\cos\theta = -\dfrac{n^2+2}{2(n^2-1)}$

11. The vector sum of two vectors $\vec{P}$ and $\vec{Q}$ is $\vec{R}$. If the vector Q is reversed, the resultant becomes $\vec{S}$. Show that $R^2 + S^2 = 2(P^2 + Q^2)$

12. Show that the condition that the three points determined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear is
$$(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) = 0$$

13. The diagonals of the parallelogram are represented by vectors $\vec{P} = (5\hat{i} - 4\hat{j} + 3\hat{k})$ and $\vec{q} = (3\hat{i} + 2\hat{j} - \hat{k})$. Find the area of the parallelogram.
   Ans: $\sqrt{171}$

14. Prove that $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$

15. The weight is suspended from the middle of a rope whose ends are at the same level. The rope is no longer horizontal. Find the minimum tension required to completly straighten the rope.
   Ans: Infinity.