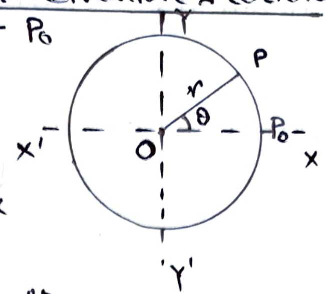
CIRCULAR MOTION

Relation between linear motion and Circular Motion.

Let a particle initially located at point P_0 on the reference axis $x'ox$, moves in a circular path of radius r. After 't' time interval the particle is located at point p after moves an angular displacement θ.



If θ is very small then we can write

$$S = r\theta \quad \text{------ (i)}$$

where s is the arc length or linear displacement between PP_0 point. [for small angle θ, s is considered to be a straight line]

Differentiating eqn (i) with respect to time (t)

$$\frac{ds}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt} \quad [\because r \text{ is Const}]$$

$$\therefore \quad v = r\omega \quad \text{------ (ii)} \quad \text{or} \quad \omega = \frac{v}{r} \quad \text{Again } \omega = \frac{2\pi}{T}$$

where $v \rightarrow$ Linear velocity and $\omega \rightarrow$ Angular velocity

$\therefore$ Linear velocity (v) = Radius (r) $\times$ Angular velocity (ω)

Again differentiating eqn (ii) with respect to time (t)

$$\frac{dv}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} \quad [r \text{ is constant}]$$

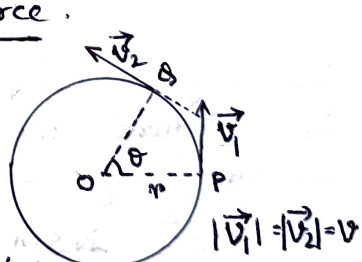
$$a = r\alpha \quad \text{------ (iii)}$$

where $a \rightarrow$ Linear acceleration and $\alpha \rightarrow$ Angular acceleration.

$\therefore$ Linear Acceleration (a) = Radius (r) $\times$ Angular acceleration.

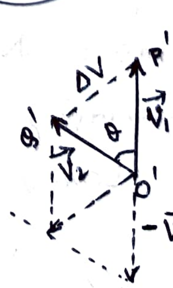
Centripetal acceleration and Centripetal force.

A particle is moving in a circular path with Constant speed. At point P and Q its velocity is $\vec{v_1}$ and $\vec{v_2}$ respectively.



$\therefore$ Change in velocity $\Delta V = \vec{v_2} - \vec{v_1}$

Now from two similar triangles OPQ and O'P'Q'



$$\frac{O'P'}{OP} = \frac{P'Q'}{PQ}$$

But from figure

$$\frac{v}{r} = \frac{\Delta v}{\Delta S} \quad \Rightarrow \quad \Delta v = \frac{\Delta S}{r} v$$

Now divide both side with Δt

$$\frac{\Delta v}{\Delta t} = \frac{1}{r}\frac{\Delta s}{\Delta t} v \quad \Rightarrow \quad \text{Taking limit on both side}$$

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{v}{r} \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \quad \Rightarrow \quad \frac{dv}{dt} = \frac{v}{r} \cdot v = \frac{v^2}{r} \quad \Rightarrow \quad a_{centri} = \frac{v^2}{r}.$$

It is very clear from the second figure, that this acceleration has direction towards the centre of the circular path. so this is known as centripetal acceleration.

$$a_{centri} = \frac{v^2}{r}$$

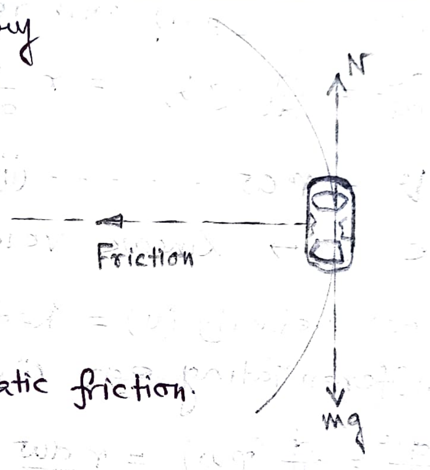
If any particle of mass 'm' moves on this circular path should experience a force

$$F_{centri} = m\, a_{centri} = \frac{mv^2}{r}$$

known as centripetal force, directed towards the centre. This is the necessary force for circular motion. If the body moves with constance speed in the circular path then the magnitude of the centripetal force also const.

Turning and Banking of a Vehicle in a curved road

For turning of the car necessary centripetal force is provided by the force of friction. The car is not to skid if the centripetal force required has the value less than the frictional force.



$$\therefore \quad \frac{mv^2}{r} < \mu_s mg$$

where $\mu_s \rightarrow$ coefficient of static friction.

$$\therefore \quad v < \sqrt{\mu_s rg}$$

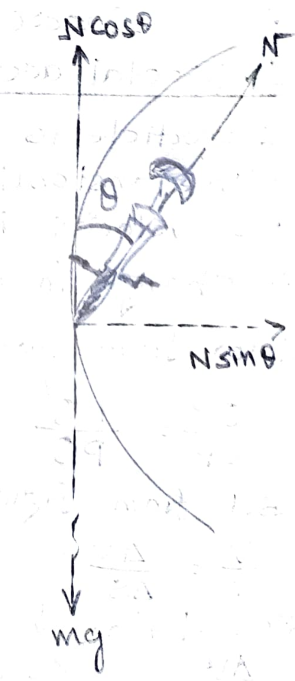
During skidding

$$\frac{mv^2}{r} = \mu_k mg \qquad \mu_k \rightarrow \text{coefficient of kinetic friction.}$$

$$\therefore \quad v = \sqrt{\mu_k rg}$$

Bending of a cyclist:

From figure the necessary centripetal force is provided by $N\sin\theta$.



$$\therefore \quad \frac{mv^2}{r} = N\sin\theta \qquad \text{and}$$

$$mg = N\cos\theta$$

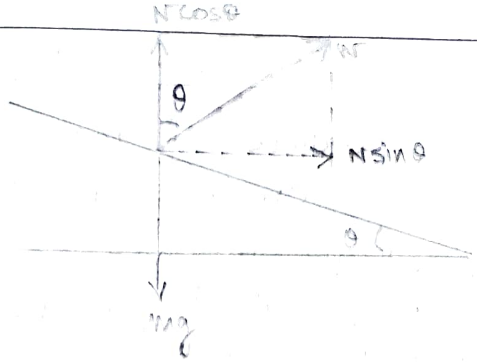
By dividing the two equation

$$\tan\theta = \frac{v^2}{rg}$$

or

$$v = \sqrt{rg\tan\theta}.$$

Banking of a road



From figure necessary centripetal force is provided by N sin θ

$\therefore \quad \frac{mv^2}{r} = N \sin\theta$

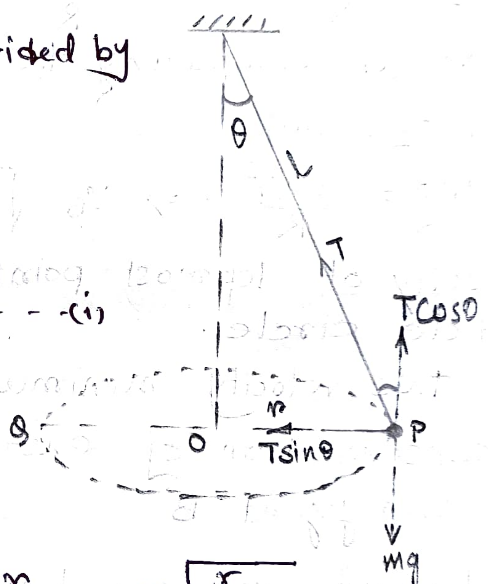
$mg = N\cos\theta$

$\therefore$ On dividing the two equations

$$\tan\theta = \frac{v^2}{rg}$$

or $$v = \sqrt{rg\tan\theta}$$

Conical Pendulum



Necessary centripetal force is provided by

$T\sin\theta = \frac{mv^2}{r}$

$T\cos\theta = mg$

$\therefore \tan\theta = \frac{v^2}{rg} \Rightarrow v = \sqrt{rg\tan\theta}$ ---- (i)

But from fig. $r = l\sin\theta$ --- (ii)

$\therefore$ Time period of oscillation

$$T = \frac{\text{Total Circumference}}{\text{velocity}} = \frac{2\pi r}{\sqrt{rg\tan\theta}} = 2\pi\sqrt{\frac{r}{g\tan\theta}} \quad \text{---- (iii)}$$

Now substitute eqn (ii) in (iii)

$$T = 2\pi\sqrt{\frac{l\sin\theta}{g\tan\theta}} = 2\pi\sqrt{\frac{l\cos\theta}{g}}$$

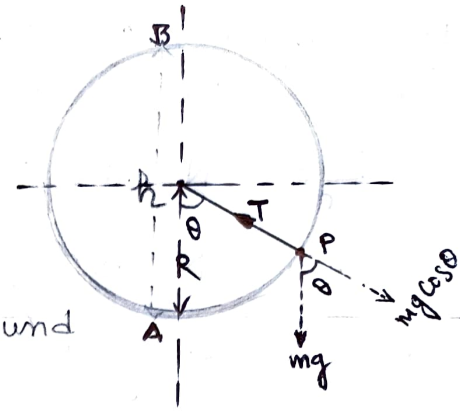
For very small angular amplitude [θ is very small] $\cos\theta \simeq 1$.

$$\therefore \quad T = 2\pi\sqrt{\frac{l}{g}}$$

$\therefore$ For very small angular amplitude a conical pendulum converts to a simple pendulum.

Motion in a verticle plane (Non-Uniform Circular Motion)

Let a particle of mass m moves on a verticle circle of radius R.



At point P the necessary Centripetal force is provided by the resultant force of T and mg cos θ

$$\therefore \quad T - mg\cos\theta = \frac{mv^2}{r} \quad \text{------ (i)}$$

Eqn (i) for point A.

$$T_A - mg\cos 0^\circ = \frac{mv_A^2}{R} \quad \text{--- (ii)} \Rightarrow T_A - mg = \frac{mv_A^2}{R}$$

Eqn (ii) for point B

$$T_B - mg\cos 180^\circ = \frac{mv_B^2}{R} \quad \text{--- (iii)} \Rightarrow T_B + mg = \frac{mv_B^2}{R}$$

Condition for completing the loop or verticle circle is $T_B = 0$

∴ From eqn (iii)

$$mg = \frac{mv_B^2}{R} \Rightarrow v_B = \sqrt{Rg}$$

∴ Velocity at topmost point $\sqrt{Rg}$ is essential to complete the verticle circle.

∴ For this velocity minimum velocity at lower most point A is

By conservation of energy, energy at A must be same as total energy at B

$$\therefore \quad \frac{1}{2}mv_A^2 + mg \times 0 = \frac{1}{2}mv_B^2 + mg(2R) \Rightarrow v_A^2 = v_B^2 + 4gR$$

∴ Now substitute $v_B = \sqrt{Rg}$

$$v_A^2 = 5gR$$

$$\Rightarrow \quad v_A = \sqrt{5Rg}$$

∴ Minimum velocity requires to Complete the vertical circle at the lower most point is $\sqrt{5gR}$.