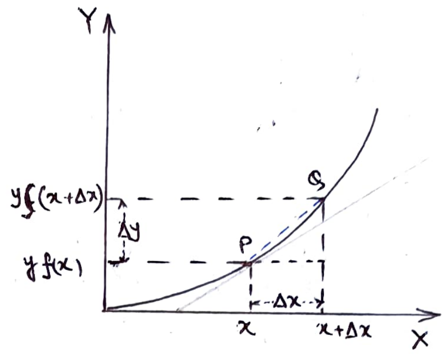
# BASIC MATHEMATICS

## Differential Calculus

Consider two quantities $y$ and $x$ interrelated in such a way that for each value of $x$ there is one and only one value of $y$. Figure represents the graphical relation between $x$ and $y$.




To find the slope at point P, let consider another point Q located close to P, such that the quadrinate of Q point is $\{(x+\Delta x), y f(x+\Delta x)\}$ or $\{(x+\Delta x), (y+\Delta y)\}$.

$$\therefore \quad \text{Slope } (\tan\theta) = \frac{\text{Change in } y}{\text{Change in } x} = \frac{y f(x+\Delta x) - y f(x)}{(x+\Delta x) - x} = \frac{(y+\Delta y) - y}{(x+\Delta x) - x}$$

$$\tan\theta = \frac{\Delta y}{\Delta x}$$

However, this can not be the precise definition of slope. Because the slope also varies between the point P and Q. Thus to get the exact value of the slope, we have to take $\Delta x$ very small. However small we take $\Delta x$, as long as it is not zero the slope may vary with in the small part of the curve. However if we go on drawing the point Q closer to A and everytime calculate $\frac{\Delta y}{\Delta x} = \tan\theta$, we shall see that as $\Delta x$ is made smaller and smaller the slope of the line PQ approaches the slope of the tangent at P. This slope of the tangent at point P thus gives the rate of change of $y$ with respect to $x$ at point P.

Thus
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

If the function $y$ increases with an increase in $x$ at a point, $\frac{dy}{dx}$ is positive. If the function $y$ decreases with an increase in $x$ at a point, $\frac{dy}{dx}$ is negative.

## Meaning of $\frac{dy}{dx}$ and its property

- Differentiation of function $y$ with respect to $x$.
- Rate of change in $y$ with respect to $x$.
- Slope of $y$ vs $x$ graph.
- $\frac{d}{dx}$ is called diffrential operator, which when operate on a function, give the rate of change of that function with respect to $x$.

  Example: $\frac{d}{dt}(s) = v$ $\quad \because \frac{d}{dt}$ diffrential operator when operates on displacement, gives rate of change in displacement with time.
- Any function $y$ is diffrentiable with respect to $x$ if and only if $y$ has functional dependency on $x$, means $y$ must depends on the value of $x$.

# Formulas for Differentiation

- $\frac{d}{dx}(\text{const}) = 0$
- $\frac{d}{dx}(x^n) = n x^{n-1}$ • $\frac{d}{dx}(a^x) = a^x \log_e a$ . $[a > 0]$
- $\frac{d}{dx}(kx^n) = k \cdot n x^{n-1}$ $[\because k \text{ is a const}]$
- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(\log_e x) = \frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ • $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
- $\frac{d}{dx}\left(\frac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- Chain rule:
  If $y = f(z)$ and $z = g(x)$ then
  $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

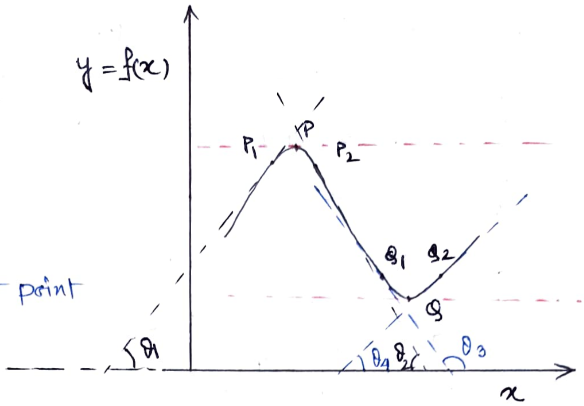
Find the differentiation of the following function with respect to $x$.

1. $y = 2^x \cdot x^5$
2. $y = x^3 + 5x^2 + 2$
3. $y = x \sin x$
4. $y = \frac{x}{\sin x}$
5. $y = x^2 \sin x$
6. $y = \frac{\sin x}{x}$
7. $y = \sin x^2$
8. $y = \sqrt{x} \sec x \tan x$
9. $y = e^{5x+2}$
10. $y = \frac{\cot x + \operatorname{cosec} x}{\cot x - \operatorname{cosec} x}$
11. $y = \frac{x \sin x + \cos x}{x \cos x - \sin x}$
12. $y = \frac{x^2 - 3x + 4}{x + 3}$
13. $y = \sqrt{3x} - \sqrt{\frac{3}{x}}$
14. $y = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$
15. $y = \sin(\log_e x)$
16. $y = e^{\sqrt{x}}$
17. If $y = \operatorname{cosec}\theta + \cot\theta$ then prove that
    $2\frac{dy}{d\theta} + y^2 + 1 = 0$
18. If $y = \frac{x}{x+4}$ then prove that
    $x\frac{dy}{dx} + y(y-1) = 0$
19. $y = 2^{\log(ax^2+bx+c)}$
20. $y = \sqrt[3]{\tan e^{x^2}}$

# Maxima and Minima

At point maxima & minima slope becomes zero. Therefore first differentiation gives zero result.

$$\left(\frac{dy}{dx}\right)_{\text{at maxima or Minima}} = 0$$




But at maxima for its adjecent point slope changes its sign from Positive to negative and at minima slope changes sign from negative to positive. Therefore the rate of change of slope or the Second differentiation of $y=f(x)$ with respect to $x$ gives negative for maxima point and gives positive for minima Point. Therefore

Condition of Maxima

$$\frac{dy}{dx} = 0$$

and $$\frac{d^2y}{dx^2} = -ve$$

Condition for minima

$$\frac{dy}{dx} = 0$$

and $$\frac{d^2y}{dx^2} = +ve$$

Note: If the second derivative becomes zero then the point may be a inflection point. It is a point on the graph at which the function changes from cancave up to concave down (or vice versa). The second derivative must be zero at that point. But it is not true that if second derivative is zero then the function has point of inflextion at that point. Then it need to be tested for its adjecent neighbourhood.

- A function $f(x)$ is said to have local maxima at $x = c$; if $f(c) > f(x)$ for all values of $x$ in the interval $c-h < x < c+h$, where $h$ is a positive quantity, however small.
- A function $f(x)$ is said to have a local minima at $x = c$ if $f(c) < f(x)$ for all values of $x$ in the interval $c-h < x < c+h$.
- Let, $f(x)$ be continuous in $a \leq x \leq b$ and differentiable in $a < x < b$. Suppose, $x = c$ is a point in $a < x < b$ such that $f'(c) = 0$ and $f''(c) \neq 0$. Then the function $f(x)$ has local maxima at $x = c$, when $f''(c) < 0$ and a local minima at $x = c$, when $f''(c) > 0$.

## Problems on Maxima & Minima

- Find the maximum or Minimum value of $y$ in the function given below.

a) $y = 5 - (x-1)^2$

b) $y = 4x^2 - 4x + 7$

c) $y = x^3 - 3x$

d) $y = 2x^3 + 3x - 36 + 10$

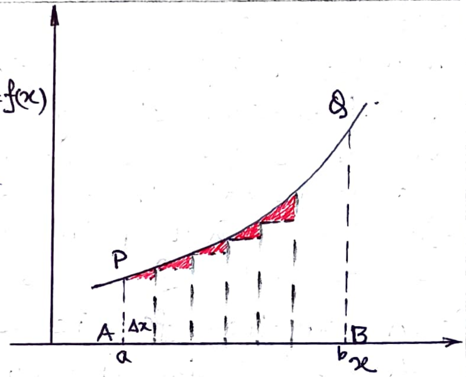
e) If $y = \frac{\sin(x+\alpha)}{\sin(x+\beta)}$ has neither a maxima nor a minima for all values of $x$ then show the relation between $\alpha$ and $\beta$. [use: $\sin(A+B) = \sin A\cos B - \cos A \sin B$]

f) Show that, the maximum value of $2x + \frac{1}{2x}$ is less than its minimum value.

g) $y = (\sin 2x - x)$; $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

- If the hypotenuse of the right angle triangle is given, then show that the area is maximum when the triangle is isosceles.
- Find the dimension of a rectangle with perimeter 1000 m, so that the area of the rectangle is maximum.
- Find the maximum area of a rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Assume that the sides of the rectangle are parallel to the axes.

## Integral Calculus




Let PQ represent a curve which represent the relation between $y$ and $x$. To find area under the curve within the value $x = a$ to $x = b$, let divide the whole are in $N$ equal elements each of length $\Delta x = \frac{b-a}{N}$.

$\therefore$ Area under the curve

$$I = f(a)\,\Delta x + f(a+\Delta x)\cdot\Delta x + f(a+2\Delta x)\cdot\Delta x + \dots + f[a+(N-1)\Delta x]\,\Delta x.$$

This may be written as

$$I = \sum_{i=1}^{N} f(x_i)\,\Delta x$$ where $x_i$ takes the values $a, (a+\Delta x), (a+2\Delta x), \dots (b-\Delta x)$

As $N$ tends to infinity the total area of the small triangles dereases and tens to zero. In such limit the sum becomes area I of PABQ. Thus we may write.

$$I = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\,\Delta x \quad \text{or} \quad I = \int_a^b f(x)\,dx.$$

a and b is lower limit and upper limit of integration.

## Formula for Integration

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$
- $\int \frac{1}{x}\,dx = \ln x + c$
- $\int a\,dx = ax + c$
- $\int e^x\,dx = e^x + c$
- $\int a^x\,dx = \frac{a^x}{\ln(a)} + c$
- $\int \ln x\,dx = x\ln(x) - x + c$
- $\int \sin x\,dx = -\cos x + c$
- $\int \cos x\,dx = \sin x + c$
- $\int (f+g)\,dx = \int f\,dx + \int g\,dx$
- $\int (f-g)\,dx = \int f\,dx - \int g\,dx$
- $\int \sec^2 x\,dx = \tan x + c$
- $\int \sec x \cdot \tan x\,dx = \sec x + c$
- $\int \operatorname{cosec} x \cdot \cot x\,dx = -\operatorname{cosec} x + c$

## Problems on Integration

- Integrate the following functions with respect to $t$

a) $\int (3t^2 - 2t)\, dt$

b) $\int (4\cos t + t^2)\, dt$

c) $\int (2t-4)^{-4}\, dt$

d) $\int \frac{dt}{(6t-1)}$

- Find the value of the following definit integral

a) $\int_0^2 2t\, dt$

b) $\int_{\pi/6}^{\pi/3} \sin x\, dx$

c) $\int_4^{10} \frac{dx}{x}$

d) $\int_1^2 (2t-4)\, dt$.

- Find the area under the curve $y = x^2$ with in the limit $x=0$ to $x=6$
- Evaluate $\int_0^t A \sin \omega t\, dt$ where $A$ and $\omega$ are constant.
- The velocity $v$ and displacement $x$ of a particle executing S.H.M are related as $v \frac{dv}{dx} = -\omega^2 x$

At $x=0$, $v = v_0$. Find the velocity $v$ when the displacement becomes $x$.
- The charge flown through a circuit in the time interval between $t$ and $t+dt$ is given by $dq = e^{-t/\tau}\, dt$, where $\tau$ is a constant. Find the total charge flown through the circuit between $t=0$ to $t=\tau$.

## Binomial Theorem

We know that
$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$\therefore$ In General we can write
$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_{n-1} a b^{n-1} + {}^nC_n b^n$$

where ${}^nC_r = \frac{\lfloor n}{\lfloor n-r \cdot \lfloor r}$, $0 \le r \le n$, $n$ a non-negative integer

Also, ${}^nC_n = 1 = {}^nC_0$ here $n \in N$ (natural number).

## Problems on Binomial theorem

- Find the number of terms in the following and expand the functions.

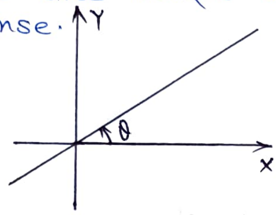
a) $(1-x)^4$

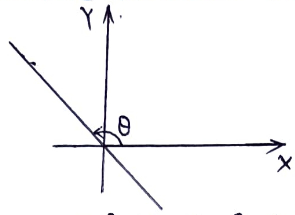
b) $(2x^2 + 3y)^5$.

# Basic Concept of Graph

## 1. Linear Graph

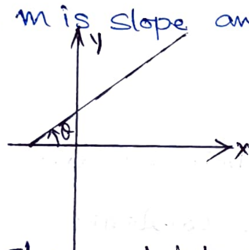
(i) $y = mx$ represents a straight line. Here $m = \tan\theta$ is also called gradient or the slope of the graph. $\theta$ is the angle which the line makes with positive x-axis, taken in anticlockwise sense.



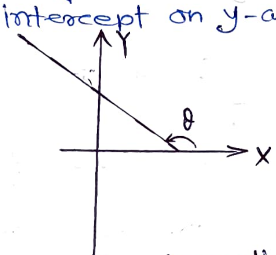




$\theta < 90°$ $\therefore \tan\theta$ is +ve $\qquad$ $90° < \theta < 180°$ $\therefore \tan\theta$ is −ve.

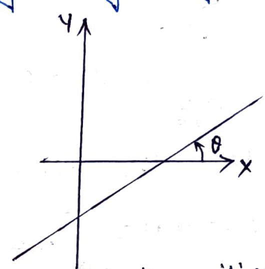
(ii) $y = mx + c$, represents a straight line not passing through origin. Here $m$ is slope and $c$ is intercept on y-axis




Slope and intercept both are positive

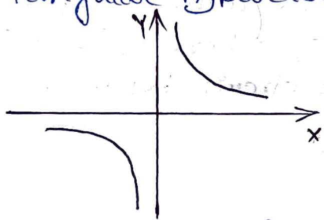


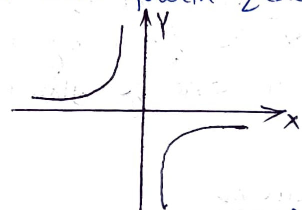

Slope is negative
Intercept is positive



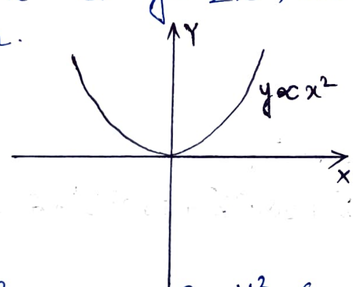

Slope is positive
Intercept is negative.

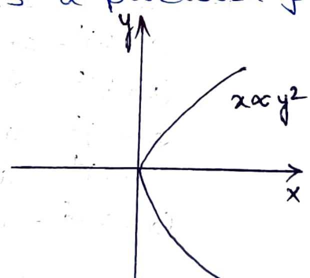
(iii) $y \propto \frac{1}{x}$ or $y = \frac{2}{x}$ etc. represents a retangular hyperabola in the first and the third quadrent but $y = -\frac{4}{x}$ or etc. represents a retangular hyparabola in second and fourth quadrants.




(iv) $y \propto x^2$ or $y = 2x^2$, etc represents a parabola passing through origin.




(v) $y = x^2 + 4$ or $x = y^2 - 6$ will represent a parabola but not passing through origin. For first case it passes through (0,4) and in the second case it passes through (−6,0)

(vi) $y = Ae^{-kx}$ represents exponential decreasing graph. Similarly $y = A(1 - e^{-kx})$, represents an exponential increasing graph.