Dimensional Analysis

Dimension : Dimension of a derived physical quantity are the power to which the fundamental quantities are raised to represent the given quantity.

Dimensional Formula : Dimensional formula of a derived Physical quantity is an expression showing which of the fundamental quantities, along with their powers, are required to represent that quantity.

Classification of Physical Quantity:

(i) **Dimensional variable**

These are the physical quantities which possess dimensions. and do not have a constant value, like area, volume, velocity, force, Coefficient of viscosity etc.

(ii) **Non-dimensional Variable**

These are the physical quantities which have neither dimensions nor a constant value, like angle, Sp. gravity, Strain, refractive index etc.

(iii) **Dimensional Constant**

These physical quantities have a constant value and possess dimension, like Speed of light in vacuum (c), Universal gravitational Constant (G), Plank Constant (h) permittivity of free space (ϵ_0), Boltzmann Constant (K_B). etc.

(iv) **Non-dimensional Constant**

These include constant quantities having no dimensions like pure numbers, e, π etc.

Dimensional Equation : It is the equation obtained by equating that quantity to its dimensional formula.

Like, $S = ut + \frac{1}{2}at^2$ Equation of Motion

$$[L] = [LT^{-1}][T] + [LT^{-2}][T^2] = [L] + [L]$$ Dimensional Equation.

Principle of homogeneity : When a physical equation consists of a number of terms, each of these terms must be of the same dimensions in each of the fundamental units. simply physical equation must be dimensionally homogeneous. This is Know as principle of homogenity. Like

$A = B + C$. or $a = bc + d$

In this equation A, B and C must have same dimensions or in the other equation a, bc and d must have same dimensions.

Dimensional Analysis and its uses

(a) Checking the dimensional correctness of a physical equation:

Ex: Chek dimensionally, the correctness of the equation

$$T = 2\pi\sqrt{l/g}$$

$\therefore$ LHS: $[T]$

RHS: $\dfrac{[L]^{1/2}}{[L/T^2]^{1/2}} = [T]$

$\therefore$ As L.H.S = R.H.S (dimensionally), therefore the given eqn is dimensionally correct.

(b) Deriving relationship between different physical quantities:

Ex: Obtain an expression for kinetic energy (E) of a body of mass (m) and moving with velocity (v)

$$\therefore E \propto m^a v^b$$

$$\therefore E = K m^a v^b$$

$$\therefore [ML^2T^{-2}] = [M^a][LT^{-1}]^b$$

$$\therefore [ML^2T^{-2}] = [M^a L^b T^{-b}]$$

By applying principle of homogenity

$$a = 1 \qquad -b = 2 \text{ or } b = 2$$

$$\therefore E = K m v^2$$

The value of K generally obtain from experiment.

(c) Conversion of one system of units into another.

Let M_1, L_1, T_1 represents the fundamental unit in one system of units and M_2, L_2, T_2 to be corresponding units in another system:

$$u_1 = [M_1^a L_1^b T_1^c] \quad \text{and} \quad u_2 = [M_2^a L_2^b T_2^c]$$

If n_1 is the numerical value of the quantity in one system and n_2 is for the other system then

$$n_1 u_1 = n_2 u_2$$

$$n_1 [M_1^a L_1^b T_1^c] = n_2 [M_2^a L_2^b T_2^c]$$

$$\therefore n_2 = n_1 \left(\frac{M_1}{M_2}\right)^a \left(\frac{L_1}{L_2}\right)^b \left(\frac{T_1}{T_2}\right)^c$$

Ex: Convert 1 joule to erg.

$$\therefore n_2 = 1\left(\frac{1\,kg}{1\,g}\right)\left(\frac{1\,m}{1\,cm}\right)^2\left(\frac{1\,s}{1\,s}\right)^{-2} = \left(\frac{1000\,g}{1\,g}\right)\left(\frac{100\,cm}{1\,cm}\right)^2\left(\frac{1\,s}{1\,s}\right)^{-2} = 10^3 \times 10^4 = 10^7$$

Hence 1 joule $= 10^7$ erg.

Limitations of Dimensional Analysis:

1. The method does not give any information about the dimensionless constant K.
2. It fails when a physical quantity depends on more than 3 physical quantities.
3. It fails when a physical quantity (eg $s = ut + \frac{1}{2}at^2$) is the sum or difference of two or more quantities.
4. It fails to derive relationships which involve trigonometric, logarithmic or exponential functions.
5. Sometimes, it is difficult to identify the factors on which the physical quantity depends. The method becomes more complicated when dimensional constant like G, h, etc are involved.
6. The method enables us to check only the dimensional correctness of a formula and not its overall correctness.

Problems on Dimensional Analysis:

1. Test the dimensional consistency of the following equations
   (i) $v = u + at$ (ii) $v^2 = u^2 + 2as$
2. The viscous force 'F' acting on a small sphere of radius r moving with velocity v through a liquid is given by $F = 6\pi\eta r v$. Calculate the dimension of η, the coefficient of viscosity.
   Ans: $ML^{-1}T^{-1}$
3. The rate of flow (V) of a liquid flowing through a pipe of radius r and a pressure gradient (P/l) is given by Poiseuille's equation: $V = \frac{\pi}{8}\frac{Pr^4}{\eta l}$. check the dimensional consistency of this equation.
4. Find the dimensions of the quantity q from the expression
   $$T = 2\pi\sqrt{\frac{ml^3}{3Yq}}$$
   where T is the time period of a bar of length l, mass m and Young's modulus Y.
5. A body of mass m hung at one end of the spring executes SHM. Prove that the relation $T = 2\pi m/k$ is incorrect, where k is the force constant of the spring. Also derive the correct relation.
   Ans: $T = K\sqrt{m/k}$
6. The critical angular velocity ω_c of a cylinder inside another cylinder containing a liquid at which its turbulence occurs depends on viscosity η, density ρ and the distance 'd' between the walls of the cylinder. Find the expression for ω_c.
   Ans: $\omega_c = \frac{K\eta}{\rho d^2}$

7. The equation of a wave is given by $y = a \sin\omega\left(\frac{x}{v} - k\right)$ where ω is angular velocity and v is the linear velocity. Find the dimension of k.
Ans $[T^{-1}]$

8. If the energy (E), velocity (v) and force (F) be taken as fundamental quantities, then find the dimension of mass.
Ans: $[Ev^{-2}]$

9. Write the dimension of a and b in the relation.

$$P = \frac{b - x^2}{at}$$

Where P is power, x is distance and t is time.
Ans:

10 The equation of state of some gas can be expressed as $(P + a/V^2)(V - b) = RT$ where P is the pressure, V, volume, T absolute temperature and a, b, R are constant. Find the dimension of a and b.

11. During discharging of capacitor, the potential drop across its plate is given by

$$V_c = V_0 e^{-t/RC}$$

where R is resistance, t is time and V_c and V_0 are potential drop. Find the dimension of C.

12. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a} \sin^{-1}\frac{a}{x}$ Prove wheather this is dimensionally correct or not, where x and a stands for distance.