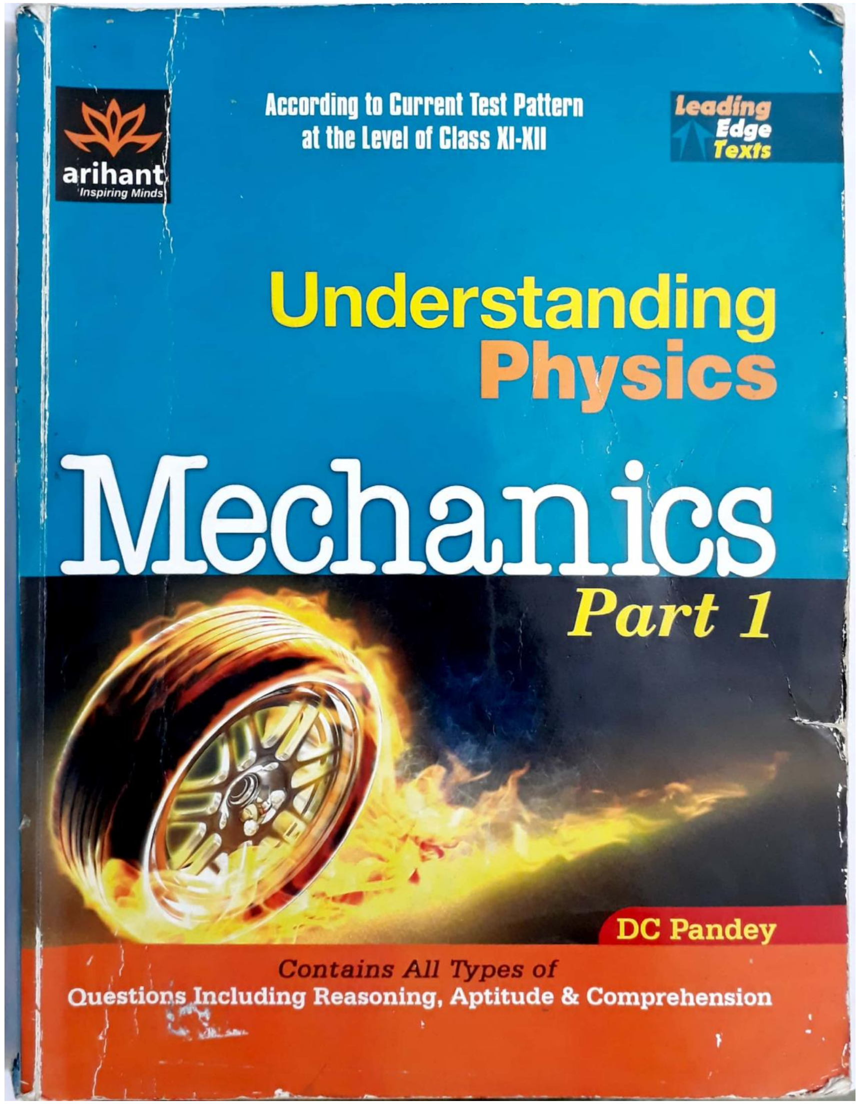
Arihant Mechanics Part 1 by 1 One of Two (Chapter 1 till 5)

Understanding Physics for IIT JEE

by

D C Pandey



Leading Edge Texts

According to Current Test Pattern
at the Level of Class XI-XII

Understanding Physics
Mechanics
Part 1

DC Pandey

arihant
Inspiring Minds.Inspiring Lives
Arihant Prakashan, Meerut

Contents

Contents

1. **Basic Mathematics & Measurements** — 1-33

- 1.1 Basic Mathematics
- 1.2 Significant Figures
- 1.3 Error Analysis
- 1.4 Length Measuring Instruments

2. **Units & Dimensions, Vectors** — 35-67

- 2.1 Units and Dimensions
- 2.2 Vector and Scalar Quantities
- 2.3 General Points Regarding Vectors
- 2.4 Addition & Subtraction of Two Vectors
- 2.5 The Components of a Vector
- 2.6 The Product of Two Vectors

3. **Motion in One Dimension** — 69-136

- 3.1 Introduction to Mechanics & Kinematics
- 3.2 Basic Definitions
- 3.3 Motion in One, Two and Three Dimensions
- 3.4 Uniformly Accelerated Motion
- 3.5 Non-Uniformly Accelerated Motion
- 3.6 Graphs
- 3.7 Relative Motion

4. **Projectile Motion** — 137-170

- 4.1 Projectile Motion
- 4.2 Projectile Motion in an Inclined Plane
- 4.3 Relative Motion Between Two Projectiles

5. **Laws of Motion** — 171-257

- 5.1 Types of Forces
- 5.2 Free Body Diagram
- 5.3 Equilibrium
- 5.4 Newton's Laws of Motion
- 5.5 Pulleys
- 5.6 Constraint Equations
- 5.7 Pseudo Forces
- 5.8 Friction

Chapter 6, 7 and 8 are in Arihant Mechanics Part 1 by 2 Two of Two

6. **Work, Energy & Power** 259-320

6.1 Introduction to Work
6.2 The Work Done
6.3 Conservative & Non-Conservative Force Field
6.4 Kinetic Energy
6.5 Work-Energy Theorem
6.6 Potential Energy
6.7 Law of Conservation of Mechanical Energy
6.8 Three Types of Equilibria
6.9 Power

7. **Circular Motion** 321-361

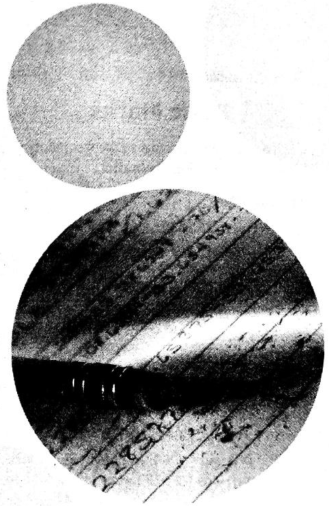
7.1 Kinematics of Circular Motion
7.2 Dynamics of Circular Motion
7.3 Motion in a Vertical Circle

8. **Centre of Mass, Conservation of Linear Momentum, Impulse and Collision** 363-437

8.1 Centre of Mass
8.2 Law of Conservation of Linear Momentum
8.3 Variable Mass
8.4 Impulse
8.5 Collision

- **Hints & Solutions** 439-483

Chapter 1 – Basic Maths and Measurement



1

Basic Maths & Measurement

Chapter Contents

1.1 Basic Maths
1.2 Significant Figures
1.3 Error Analysis
1.4 Length Measuring Instruments

1.1 Basic Maths

The following formulae are frequently used in Physics. So, the students who have just gone in class XI are advised to remember them first.

(a) Logarithms

(i) $e \approx 2.7183$

(ii) If $e^x = y$, then $x = \log_e y = \ln y$

(iii) If $10^x = y$, then $x = \log_{10} y$

(iv) $\log_{10} y = 0.4343 \log_e y = 0.4343 \ln y$

(v) $\log (ab) = \log (a) + \log (b)$

(vi) $\log \left(\frac{a}{b}\right) = \log (a) - \log (b)$

(vii) $\log a^n = n \log (a)$

(b) Trigonometry

(i) $\sin^2 \theta + \cos^2 \theta = 1$

(ii) $1 + \tan^2 \theta = \sec^2 \theta$

(iii) $1 + \cot^2 \theta = \text{cosec}^2 \theta$

(iv) $\sin 2\theta = 2 \sin \theta \cos \theta$

(v) $\cos 2\theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta = \cos^2 \theta - \sin^2 \theta$

(vi) $\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$

(vii) $\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$

(viii) $\sin C + \sin D = 2 \sin \left(\frac{C+D}{2}\right) \cos \left(\frac{C-D}{2}\right)$

(ix) $\sin C - \sin D = 2 \sin \left(\frac{C-D}{2}\right) \cos \left(\frac{C+D}{2}\right)$

(x) $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$

(xi) $\cos C - \cos D = 2 \sin \frac{D-C}{2} \sin \frac{C+D}{2}$

(xii) $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

(xiii) $\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(xiv) $\sin (90° + \theta) = \cos \theta$

(xv) $\cos (90° + \theta) = -\sin \theta$

(xvi) $\tan (90° + \theta) = -\cot \theta$

(xvii) $\sin (90° - \theta) = \cos \theta$

(xviii) $\cos (90° - \theta) = \sin \theta$

(xix) $\tan (90° - \theta) = \cot \theta$

(xx) $\sin (180° - \theta) = \sin \theta$

(xxi) $\cos (180° - \theta) = -\cos \theta$

(xxii) $\tan (180° - \theta) = -\tan \theta$

(xxiii) $\sin (180° + \theta) = -\sin \theta$

(xxiv) $\cos (180° + \theta) = -\cos \theta$

(xxv) $\tan (180° + \theta) = \tan \theta$

(xxvi) $\sin (-\theta) = -\sin \theta$

(xxvii) $\cos (-\theta) = \cos \theta$

(xxviii) $\tan (-\theta) = -\tan \theta$

(c) Differentiation

(i) $\frac{d}{dx}(\text{constant}) = 0$

(ii) $\frac{d}{dx}(x^n) = nx^{n-1}$

(iii) $\frac{d}{dx}(\log_e x)$ or $\frac{d}{dx}(\ln x) = \frac{1}{x}$

(iv) $\frac{d}{dx}(\sin x) = \cos x$

(v) $\frac{d}{dx}(\cos x) = -\sin x$

(vi) $\frac{d}{dx}(\tan x) = \sec^2 x$

(vii) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(viii) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(ix) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

(x) $\frac{d}{dx}(e^x) = e^x$

(xi) $\frac{d}{dx}\{f_1(x) . f_2(x)\} = f_1(x) \frac{d}{dx} f_2(x) + f_2(x) \frac{d}{dx} f_1(x)$

(xii) $\frac{d}{dx} \frac{f_1(x)}{f_2(x)} = \frac{f_2(x) \frac{d}{dx} f_1(x) - f_1(x) \frac{d}{dx} f_2(x)}{\{f_2(x)\}^2}$

(xiii) $\frac{d}{dx} f(ax + b) = a \frac{d}{dx} f(X)$, where $X = ax + b$

(d) Integration

(i) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$

(ii) $\int \frac{dx}{x} = \log_e x + c$ or $\ln x + c$

(iii) $\int \sin x \, dx = -\cos x + c$

(iv) $\int \cos x \, dx = \sin x + c$

(v) $\int e^x dx = e^x + c$

(vi) $\int \sec^2 x \, dx = \tan x + c$

(vii) $\int \text{cosec}^2 x \, dx = -\cot x + c$

(viii) $\int \sec x \tan x \, dx = \sec x + c$

(ix) $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x + c$

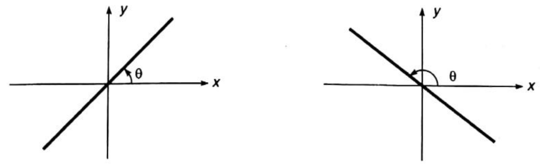
(x) $\int f(ax + b) \, dx = \frac{1}{a} \int f(X) \, dX$, where $X = ax + b$

Here, c is constant of integration.

(e) Graphs

Following graphs and their corresponding equations are frequently used in Physics.

(i) $y = mx$, represents a straight line passing through origin. Here, $m = \tan \theta$ is also called the slope of line, where θ is the angle which the line makes with positive x-axis, when drawn in anticlockwise direction from the positive x-axis towards the line.



(i) (ii)

Fig. 1.1

The two possible cases are shown in Fig. 1.1. In Fig. 1.1 (i), $\theta < 90°$. Therefore, $\tan \theta$ or slope of line is positive. In Fig. 1.1 (ii), $90° < \theta < 180°$. Therefore, $\tan \theta$ or slope of line is negative.

Note *That $y = mx$ or $y \propto x$ also means that value of y becomes 2 times if x is doubled. Or it will remain $\frac{1}{4}$ th if x becomes $\frac{1}{4}$ times.*

(ii) $y = mx + c$, represents a straight line not passing through origin. Here, m is the slope of line as discussed above and c the intercept on y-axis.

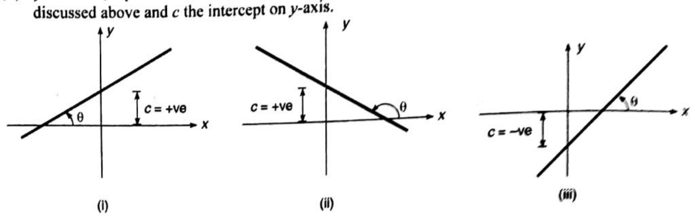


Fig. 1.2

In figure (i) : slope and intercept both are positive.
In figure (ii) : slope is negative but intercept is positive and
In figure (iii) : slope is positive but intercept is negative.

Note *That in $y = mx + c$, y does not become two times if x is doubled.*

(iii) $y \propto \frac{1}{x}$ or $y = \frac{2}{x}$ etc., represents a rectangular hyperbola in first and third quadrants. The shape of rectangular hyperbola is shown in Fig. 1.3(i).

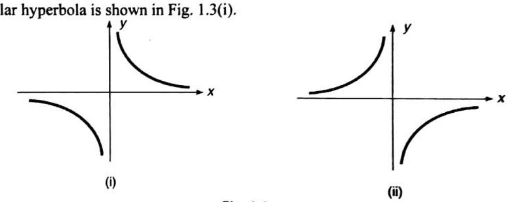


Fig. 1.3

From the figure and from the graph we can see that $y \to 0$ as $x \to \infty$ or $x \to 0$ as $y \to \infty$.
Similarly, $y = -\frac{4}{x}$ represents a rectangular hyperbola in second and fourth quadrants as shown in Fig. 1.3(ii).

Note *That in case of rectangular hyperbola if x is doubled y will become half.*

(iv) $y \propto x^2$ or $y = 2x^2$, etc., represents a parabola passing through origin as shown in Fig. 1.4(i).

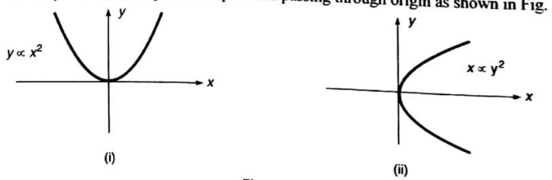


Fig. 1.4

Note *That in the parabola $y = 2x^2$ or $y \propto x^2$, if x is doubled, y will become four times.*

Graph $x \propto y^2$ or $x = 4y^2$ is again a parabola passing through origin as shown in Fig 1.4 (ii). In this case if y is doubled, x will become four times.

(v) $y = x^2 + 4$ or $x = y^2 - 6$ will represent a parabola but not passing through origin. In the first equation ($y = x^2 + 4$), if x is doubled, y will not become four times.

(vi) $y = Ae^{-Kx}$; represents exponentially decreasing graph. Value of y decreases exponentially from A to 0. The graph is shown in Fig. 1.5.

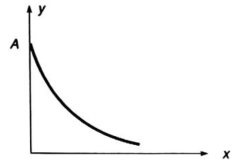


Fig. 1.5

From the graph and the equation, we can see that $y = A$ at $x = 0$ and $y \to 0$ as $x \to \infty$.

(vii) $y = A(1 - e^{-Kx})$, represents an exponentially increasing graph. Value of y increases exponentially from 0 to A. The graph is shown in Fig. 1.6.

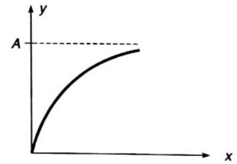


Fig. 1.6

From the graph and the equation we can see that $y = 0$ at $x = 0$ and $y \to A$ as $x \to \infty$.

(f) Maxima and Minima

Suppose y is a function of x. Or $y = f(x)$.
Then we can draw a graph between x and y. Let the graph is as shown in Fig. 1.7.

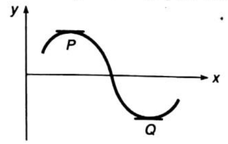


Fig. 1.7

Then from the graph we can see that at maximum or minimum value of y slope $\left(\text{or } \frac{dy}{dx}\right)$ to the graph is zero. Thus,

$$\frac{dy}{dx} = 0 \quad \text{at maximum or minimum value of } y.$$

By putting $\frac{dy}{dx} = 0$ we will get different values of x. At these values of x, value of y is maximum if $\frac{d^2y}{dx^2}$ (double differentiation of y with respect to x) is negative at this value of x. Similarly y is minimum if $\frac{d^2y}{dx^2}$ is positive. Thus,

$$\frac{d^2y}{dx^2} = -\text{ve for maximum value of } y$$

and

$$\frac{d^2y}{dx^2} = +\text{ve for minimum value of } y$$

Note *That at constant value of y also* $\frac{dy}{dx} = 0$ *but in this case* $\frac{d^2y}{dx^2}$ *is zero.*

Sample Example 1.1 *Differentiate the following functions with respect to x*

(a) $x^3 + 5x^2 - 2$ (b) $x \sin x$ (c) $(2x+3)^6$ (d) $\frac{x}{\sin x}$ (e) $e^{(5x+2)}$

Solution (a)
$$\frac{d}{dx}(x^3 + 5x^2 - 2) = \frac{d}{dx}(x^3) + 5\frac{d}{dx}(x^2) - \frac{d}{dx}(2)$$
$$= 3x^2 + 5(2x) - 0$$
$$= 3x^2 + 10x$$

(b)
$$\frac{d}{dx}(x \sin x) = x\frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(x)$$
$$= x\cos x + \sin x\,(1)$$
$$= x\cos x + \sin x$$

(c)
$$\frac{d}{dx}(2x+3)^6 = 2\frac{d}{dX}(X)^6, \quad \text{where } X = 2x+3$$
$$= 2\{6X^5\} = 12X^5$$
$$= 12(2x+3)^5$$

(d)
$$\frac{d}{dx}\left(\frac{x}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(x) - x\frac{d}{dx}(\sin x)}{(\sin x)^2}$$
$$= \frac{(\sin x)(1) - x(\cos x)}{\sin^2 x}$$
$$= \frac{\sin x - x\cos x}{\sin^2 x}$$

(e)
$$\frac{d}{dx}e^{(5x+2)} = 5\frac{d}{dX}e^X, \text{ where } X = 5x+2$$
$$= 5e^X$$
$$= 5e^{5x+2}$$

Sample Example 1.2 *Integrate the following functions with respect to x*

(a) $\int (5x^2 + 3x - 2)\,dx$ (b) $\int \left(4\sin x - \frac{2}{x}\right)dx$ (c) $\int \frac{dx}{4x+5}$ (d) $\int (6x+2)^3\,dx$

Solution (a)
$$\int (5x^2 + 3x - 2)\,dx = 5\int x^2\,dx + 3\int x\,dx - 2\int dx$$
$$= \frac{5x^3}{3} + \frac{3x^2}{2} - 2x + c$$

(b)
$$\int \left(4\sin x - \frac{2}{x}\right)dx = 4\int \sin x\,dx - 2\int \frac{dx}{x}$$
$$= -4\cos x - 2\ln x + c$$

(c)
$$\int \frac{dx}{4x+5} = \frac{1}{4}\int \frac{dX}{X}, \quad \text{where } X = 4x+5$$
$$= \frac{1}{4}\ln X + c_1 = \frac{1}{4}\ln(4x+5) + c_2$$

(d)
$$\int (6x+2)^3\,dx = \frac{1}{6}\int X^3\,dX, \quad \text{where } X = 6x+2$$
$$= \frac{1}{6}\left(\frac{X^4}{4}\right) + c_1 = \frac{(6x+2)^4}{24} + c_2$$

Sample Example 1.3 *Draw straight lines corresponding to following equations*

(a) $y = 2x$ (b) $y = -6x$ (c) $y = 4x + 2$ (d) $y = 6x - 4$

Solution (a) In $y = 2x$, slope is 2 and intercept is zero. Hence, the graph is as shown below :

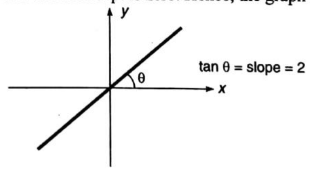


Fig. 1.8

(b) In $y = -6x$, slope is -6 and intercept is zero. Hence, the graph is as shown below :

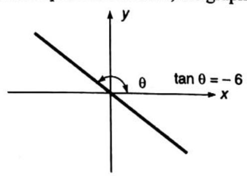


Fig. 1.9

(c) In $y = 4x + 2$, slope is + 4 and intercept is 2. The graph is as shown below :

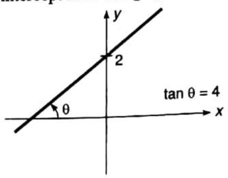


Fig. 1.10

(d) In $y = 6x - 4$, slope is + 6 and intercept is – 4. Hence, the graph is as shown in Fig. 1.11.

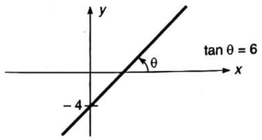


Fig. 1.11

Sample Example 1.4 *Find maximum or minimum values of the functions*

(a) $y = 25x^2 + 5 - 10x$ *(b)* $y = 9 - (x - 3)^2$

Solution (a) For maximum and minimum value, we can put $\frac{dy}{dx} = 0$.

or $$\frac{dy}{dx} = 50x - 10 = 0 \quad \therefore \; x = \frac{1}{5}$$

Further, $$\frac{d^2y}{dx^2} = 50$$

or $\frac{d^2y}{dx^2}$ has positive value at $x = \frac{1}{5}$. Therefore, y has minimum value at $x = \frac{1}{5}$. Substituting $x = \frac{1}{5}$ in given equation, we get

$$y_{\min} = 25\left(\frac{1}{5}\right)^2 + 5 - 10\left(\frac{1}{5}\right) = 4$$

(b) $$y = 9 - (x - 3)^2 = 9 - x^2 - 9 + 6x$$

or $$y = 6x - x^2$$

$\therefore$ $$\frac{dy}{dx} = 6 - 2x$$

For minimum or maximum value of y we will substitute $\frac{dy}{dx} = 0$

or $$6 - 2x = 0$$

or $$x = 3$$

To check whether value of y is maximum or minimum at $x = 3$ we will have to check whether $\frac{d^2y}{dx^2}$ is positive or negative.

$$\frac{d^2y}{dx^2} = -2$$

or $\frac{d^2y}{dx^2}$ is negative at $x = 3$. Hence, value of y is maximum. This maximum value of y is,

$$y_{\max} = 9 - (3 - 3)^2 = 9$$

1.2 Significant Figures

Significant figures in the measured value of a physical quantity tell the number of digits in which we have confidence. Larger the number of significant figures obtained in a measurement, greater is the accuracy of the measurement.

"All accurately known digits in a measurement plus the first uncertain digit together from significant figures."

For example, when we measure the length of a straight line using a metre scale and it lies between 7.4 cm and 7.5 cm, we may estimate it as $l = 7.43$ cm. This expression has three significant figures out of these 7 and 4 are precisely known but the last digit 3 is only approximately known.

Rules for counting significant figures

For counting significant figures, we use the following rules :

Rule 1. All non-zero digits are significant. For example $x = 2567$ has four significant figures.

Rule 2. The zeros appearing between two non-zero digits are counted in significant figures. For example 6.028 has 4 significant figures.

Rule 3. The zeros occurring to the left of last non-zero digit are NOT significant.
For example 0.0042 has two significant figures.

Rule 4. In a number without decimal, zeros to the right of non-zero digit are NOT significant. However when some value is recorded on the basis of actual measurement the zeros to the right of non-zero digit become significant. For example $L = 20$ m has two significant figures but $x = 200$ has only one significant figure.

Rule 5. In a number with decimal, zeros to the right of last non-zero, digit are significant. For example $x = 1.400$ has four significant figures.

Rule 6. The powers of ten are NOT counted as significant digits. For example 1.4×10^{-7} has only two significant figures 1 and 4.

Rule 7. Change in the units of measurement of a quantity does not change the number of significant figures. For example, suppose distance between two stations is 4067 m. It has four significant figures. The same distance can be expressed as 4.067 km or 4.067×10^5 cm. In all these expressions, number of significant figures continues to be four.

Example : Table 1.1

Measured value	Number of significant figures	Rule
12376	5	1
6024.7	5	2
0.071	2	3
410 m	3	4
720	2	4
2.40	3	5
1.6×10^{14}	2	6

Rounding off a digit

Following are the rules for rounding off a measurement.

Rule 1. If the number lying to the right of cut off digit is less than 5, then the cut off digit is retained as such. However if it is more than 5, then the cut off digit is increased by 1.
For example $x = 6.24$ is rounded off to 6.2 to two significant digits and $x = 5.328$ is rounded off to 5.33 to three significant digits.

Rule 2. If the digit to be dropped is 5 followed by digits other than zero then the preceding digit is increased by 1.
For example $x = 14.252$ is rounded off to $x = 14.3$ to three significant digits.

Rule 3. If the digit to be dropped is simply 5 or 5 followed by zeros, then the preceding digit it left unchanged if it is even. For example $x = 6.250$ or $x = 6.25$ becomes $x = 6.2$ after rounding off to two significant digits.

Rule 4. If the digit to be dropped is 5 or 5 followed by zeros, then the preceding digit is raised by one if it is odd.
For example $x = 6.350$ or $x = 6.35$ becomes $x = 6.4$ after rounding off to two significant digits.

Example : Table 1.2

Measured value	After rounding off to three significant digits	Rule
7.364	7.36	1
7.367	7.37	1
8.3251	8.33	2
9.445	9.44	3
9.4450	9.44	3
15.75	15.8	4
15.7500	15.8	4

Algebraic operations with significant figures

In addition, subtraction, multiplication or division inaccuracy in the measurement of any one variable affects the accuracy of the final result. Hence, in general, the final result shall have significant figures corresponding to their number in the least accurate variable involved. To understand this, let us consider a chain of which all links are strong except the one. The chain will obviously break at the weakest link. Thus, the strength of the chain cannot be more than the strength of the weakest link in the chain. Hence, we do not imply the a greater accuracy in our result than was obtained originally in our measurements.

(i) Addition and subtraction

Suppose in the measured values to be added or subtracted the least number of significant digits after the decimal is n. Then in the sum or difference also, the number of significant digits after the decimal should be n.

Example : $1.2 + 3.45 + 6.789 = 11.439 \approx 11.4$

Here, the least number of significant digits after the decimal is one. Hence, the result will be 11.4 (when rounded off to smallest number of decimal places).

Example : $12.63 - 10.2 = 2.43 \approx 2.4$

(ii) Multiplication or division

Suppose in the measured values to be multiplied or divided the least number of significant digits be n. Then in the product or quotient, the number of significant digits should also be n.

Example : $1.2 \times 36.72 = 44.064 \approx 44$

The least number of significant digits in the measured values are two. Hence, the result when rounded off to two significant digits become 44. Therefore, the answer is 44.

Example : $\dfrac{1100\ \text{ms}^{-1}}{10.2\ \text{ms}^{-1}} = 107.8431373 \approx 108$

Note *In this case answer becomes* 108. *Think why?*

1.3 Error Analysis

No measurement is perfect, as the errors involved in a measurement cannot be removed completely. Measured value is always somewhat different from the true value. The difference is called an error.

Errors can be classified in two ways. First classification is based on the cause of error. Systematic error and random errors fall in this group. Second classification is based on the magnitude of error. Absolute error, mean absolute error and relative (or fractional) error lie on this group. Now, let us discuss them separately.

(i) Systematic errors

These are the errors whose causes are known to us. Such errors can therefore be minimised. Following are few causes of these errors :

(a) Instrumental errors may be due to erroneous instruments. These errors can be reduced by using more accurate instruments and applying zero correction, when required.

(b) Sometimes errors arise on account of ignoring certain facts. For example in measuring time period of simple pendulum error may creap because no consideration is taken of air resistance. These errors can be reduced by applying proper corrections to the formula used.

(c) Change in temperature, pressure, humidity, etc., may also sometimes cause errors in the result. Relevant corrections can be made to minimise their effects.

(ii) Random errors

The causes of random errors are not known. Hence, it is not possible to remove them completely. These errors may arise due to a variety of reasons. For example the reading of a sensitive beam balance may change by the vibrations caused in the building due to persons moving in the laboratory or vehicles running nearby. The random errors can be minimised by repeating the observation a large number of times and taking the arithmetic mean of all the observations. The mean value would be very close to the most accurate reading. Thus,

$$a_{mean} = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

(iii) Absolute errors

The difference between the true value and the measured value of a quantity is called an absolute error. Usually the mean value a_m is taken as the true value. So, if

$$a_m = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

Then by definition, absolute errors in the measured values of the quantity are,

$$\Delta a_1 = a_m - a_1$$
$$\Delta a_2 = a_m - a_2$$
$$\ldots \quad \ldots \quad \ldots$$
$$\Delta a_n = a_m - a_n$$

Absolute error may be positive or negative.

(iv) Mean absolute error

Arithmetic mean of the magnitudes of absolute errors in all the measurements is called the mean absolute error. Thus,

$$\Delta a_{mean} = \frac{|\Delta a_1| + |\Delta a_2| + \ldots + |\Delta a_n|}{n}$$

The final result of measurement can be written as,

$$a = a_m \pm \Delta a_{mean}$$

This implies that value of a is likely to lie between $a_m + \Delta a_{mean}$ and $a_m - \Delta a_{mean}$.

Relative or fractional error

The ratio of mean absolute error to the mean value of the quantity measured is called relative or fractional error. Thus,

$$\text{Relative error} = \frac{\Delta a_{mean}}{a_m}$$

Relative error expressed in percentage is called as the percentage error, *i.e.*,

$$\text{Percentage error} = \frac{\Delta a_{mean}}{a_m} \times 100$$

Sample Example 1.5 *The diameter of a wire as measured by screw gauge was found to be 2.620, 2.625, 2.628 and 2.626 cm. Calculate:*

(a) mean value of diameter. *(b) absolute error in each measurement.*
(c) mean absolute error. *(d) fractional error*
(e) percentage error. *(f) Express the result in terms of percentage error.*

Solution (a) Mean value of diameter

$$a_m = \frac{2.620 + 2.625 + 2.630 + 2.628 + 2.626}{5}$$
$$= 2.6258 \text{ cm}$$
$$= 2.626 \text{ cm}$$

(rounding off to three decimal places)

(b) Taking a_m as the true value, the absolute errors in different observations are,

$$\Delta a_1 = 2.626 - 2.620 = +0.006 \text{ cm}$$
$$\Delta a_2 = 2.626 - 2.625 = +0.001 \text{ cm}$$
$$\Delta a_3 = 2.626 - 2.630 = -0.004 \text{ cm}$$
$$\Delta a_4 = 2.626 - 2.628 = -0.002 \text{ cm}$$
$$\Delta a_5 = 2.626 - 2.626 = 0.000 \text{ cm}$$

(c) Mean absolute error,

$$\Delta a_{mean} = \frac{|\Delta a_1| + |\Delta a_2| + |\Delta a_3| + |\Delta a_4| + |\Delta a_5|}{5}$$
$$= \frac{0.006 + 0.001 + 0.004 + 0.002 + 0.000}{5}$$
$$= 0.0026 = 0.003$$

(rounding off to three decimal places)

(d) Fractional error $= \pm \dfrac{\Delta a_{mean}}{a_m} = \dfrac{\pm 0.003}{2.626} = \pm 0.001$

(e) Percentage error $= \pm 0.001 \times 100 = \pm 0.1\%$

(f) Diameter of wire can be written as,

$$d = 2.626 \pm 0.1\%$$

Combination of Errors

(i) Errors in sum or difference

Let $x = a \pm b$

Further, let Δa is the absolute error in the measurement of a, Δb the absolute error in the measurement of b and Δx is the absolute error in the measurement of x.

Then, $$x + \Delta x = (a \pm \Delta a) \pm (b \pm \Delta b)$$
$$= (a \pm b) \pm (\pm \Delta a \pm \Delta b)$$
$$= x \pm (\pm \Delta a \pm \Delta b)$$

or $$\Delta x = \pm \Delta a \pm \Delta b$$

The four possible values of Δx are $(\Delta a - \Delta b)$, $(\Delta a + \Delta b)$, $(-\Delta a - \Delta b)$ and $(-\Delta a + \Delta b)$.

Therefore, the maximum absolute error in x is,

$$\Delta x = \pm(\Delta a + \Delta b)$$

i.e., the maximum absolute error in sum and difference of two quantities is equal to sum of the absolute errors in the individual quantities.

Sample Example 1.6 *The volumes of two bodies are measured to be $V_1 = (10.2 \pm 0.02)\,cm^3$ and $V_2 = (6.4 \pm 0.01)\,cm^3$. Calculate sum and difference in volumes with error limits.*

Solution $$V_1 = (10.2 \pm 0.02)\text{ cm}^3$$

and $$V_2 = (6.4 \pm 0.01)\text{ cm}^3$$

$$\Delta V = \pm(\Delta V_1 + \Delta V_2)$$
$$= \pm(0.02 + 0.01)\,\text{cm}^3$$
$$= \pm 0.03\,\text{cm}^3$$
$$V_1 + V_2 = (10.2 + 6.4)\,\text{cm}^3 = 16.6\,\text{cm}^3$$

and $\quad V_1 - V_2 = (10.2 - 6.4)\,\text{cm}^3 = 3.8\,\text{cm}^3$

Hence, sum of volumes $= (16.6 \pm 0.03)\,\text{cm}^3$

and difference of volumes $= (3.8 \pm 0.03)\,\text{cm}^3$

(ii) Errors in a product

Let $x = ab$

Then, $\quad (x \pm \Delta x) = (a \pm \Delta a)(b \pm \Delta b)$

or $\quad x\left(1 \pm \frac{\Delta x}{x}\right) = ab\left(1 \pm \frac{\Delta a}{a}\right)\left(1 \pm \frac{\Delta b}{b}\right)$

or $\quad 1 \pm \frac{\Delta x}{x} = 1 \pm \frac{\Delta b}{b} \pm \frac{\Delta a}{a} \pm \frac{\Delta a}{a} \cdot \frac{\Delta b}{b} \qquad$ (as $x = ab$)

or $\quad \pm \frac{\Delta x}{x} = \pm \frac{\Delta a}{a} \pm \frac{\Delta b}{b} \pm \frac{\Delta a}{a} \cdot \frac{\Delta b}{b}$

Here, $\frac{\Delta a}{a} \cdot \frac{\Delta b}{b}$ is a small quantity, so can be neglected. Hence,

$$\pm \frac{\Delta x}{x} = \pm \frac{\Delta a}{a} \pm \frac{\Delta b}{b}$$

Possible values of $\frac{\Delta x}{x}$ are $\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right), \left(\frac{\Delta a}{a} - \frac{\Delta b}{b}\right), \left(-\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$ and $\left(-\frac{\Delta a}{a} - \frac{\Delta b}{b}\right)$.

Hence, maximum possible value of

$$\frac{\Delta x}{x} = \pm\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$$

Therefore, maximum fractional error in product of two (or more) quantities is equal to sum of fractional errors in the individual quantities.

(iii) Errors in division

Let $\quad x = \frac{a}{b}$

Then, $\quad x \pm \Delta x = \frac{a \pm \Delta a}{b \pm \Delta b}$

or $\quad x\left(1 \pm \frac{\Delta x}{x}\right) = \frac{a\left(1 \pm \frac{\Delta a}{a}\right)}{b\left(1 \pm \frac{\Delta b}{b}\right)}$

or $\quad \left(1 \pm \frac{\Delta x}{x}\right) = \left(1 \pm \frac{\Delta a}{a}\right)\left(1 \pm \frac{\Delta b}{b}\right)^{-1} \qquad \left(\text{as } x = \frac{a}{b}\right)$

As $\frac{\Delta b}{b} << 1$, so expanding binomially, we get

$$\left(1 \pm \frac{\Delta x}{x}\right) = \left(1 \pm \frac{\Delta a}{a}\right)\left(1 \mp \frac{\Delta b}{b}\right)$$

or $\quad 1 \pm \frac{\Delta x}{x} = 1 \pm \frac{\Delta a}{a} \mp \frac{\Delta b}{b} \pm \frac{\Delta a}{a} \cdot \frac{\Delta b}{b}$

Here, $\frac{\Delta a}{a} \cdot \frac{\Delta b}{b}$ is small quantity, so can be neglected. Therefore,

$$\pm \frac{\Delta x}{x} = \pm \frac{\Delta a}{a} \mp \frac{\Delta b}{b}$$

Possible values of $\frac{\Delta x}{x}$ are $\left(\frac{\Delta a}{a} - \frac{\Delta b}{b}\right), \left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right), \left(-\frac{\Delta a}{a} - \frac{\Delta b}{b}\right)$ and $\left(-\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$. Therefore, the maximum value of

$$\frac{\Delta x}{x} = \pm\left(\frac{\Delta a}{a} + \frac{\Delta b}{b}\right)$$

or, the maximum value of fractional error in division of two quantities is equal to the sum of fractional errors in the individual quantities.

(iv) Error in quantity raised to some power

Let $\quad x = \frac{a^n}{b^m}$

Then, $\quad \ln(x) = n\ln(a) - m\ln(b)$

Differentiating both sides, we get

$$\frac{dx}{x} = n \cdot \frac{da}{a} - m\frac{db}{b}$$

In terms of fractional error we may write,

$$\pm \frac{\Delta x}{x} = \pm n\frac{\Delta a}{a} \mp m\frac{\Delta b}{b}$$

Therefore, maximum value of

$$\frac{\Delta x}{x} = \pm\left(n\frac{\Delta a}{a} + m\frac{\Delta b}{b}\right)$$

Sample Example 1.7 *The mass and density of a solid sphere are measured to be* $(12.4 \pm 0.1)\,kg$ *and* $(4.6 \pm 0.2)\,kg/m^3$. *Calculate the volume of the sphere with error limits.*

Solution Here, $m \pm \Delta m = (12.4 \pm 0.1)\,\text{kg}$

and $\quad \rho \pm \Delta\rho = (4.6 \pm 0.2)\,\text{kg/m}^3$

Volume $\quad V = \frac{m}{\rho} = \frac{12.4}{4.6} = 2.69\,\text{m}^3 = 2.7\,\text{m}^3$

(rounding off to one decimal place)

Now, $\quad \frac{\Delta V}{V} = \pm\left(\frac{\Delta m}{m} + \frac{\Delta \rho}{\rho}\right)$

or
$$\Delta V = \pm\left(\frac{\Delta m}{m} + \frac{\Delta \rho}{\rho}\right) \times V$$
$$= \pm\left(\frac{0.1}{12.4} + \frac{0.2}{4.6}\right) \times 2.7 = \pm\, 0.14$$

$\therefore$
$$V \pm \Delta V = (2.7 \pm 0.14)\,\mathrm{m}^3$$

Sample Example 1.8 *Calculate percentage error in determination of time period of a pendulum,*
$$T = 2\pi\sqrt{\frac{l}{g}}$$
where l and g are measured with ± 1% and ± 2% errors.

Solution
$$\frac{\Delta T}{T} \times 100 = \pm\left(\frac{1}{2} \times \frac{\Delta l}{l} \times 100 + \frac{1}{2} \times \frac{\Delta g}{g} \times 100\right)$$
$$= \pm\left(\frac{1}{2} \times 1 + \frac{1}{2} \times 2\right) = \pm\, 1.5\%$$

1.4 Length Measuring Instruments

Length is an elementary physical quantity. The device generally used in everyday life for measurement of the length is a metre scale. It can be used for measurement of length with an accuracy of 1 mm. So, the least count of a metre scale is 1 mm. To measure length accurately upto (1/10)th or $\left(\frac{1}{100}\right)$th of a millimetre, the following instruments are used.

(1) Vernier callipers (2) Micrometer (3) Screw gauge.

(1) Vernier Callipers

It has three parts.

(i) Main scale : It consists of a steel metallic strip M, graduated in cm and mm at one edge. It carries two fixed jaws A and C as shown in figure.

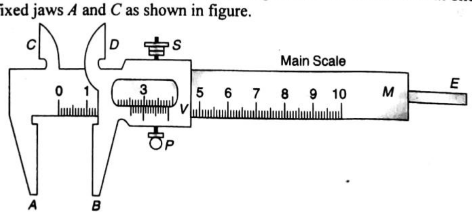


Fig. 1.12

(ii) Vernier scale : Vernier scale V slides on metallic strip M. It can be fixed in any position by screw S. The side of the vernier scale which slide over the mm sides has ten divisions over a length of 9 mm. B and D two movable jaws are fixed with it. When vernier scale is pushed towards A and C, then B touches A and

straight side of C will touch straight side of D. In this position, if the instrument is free from error, zeros of vernier scale will coincide with zeros of main scales. To measure the external diameter of an object it is held between the jaws A and B, while the straight edges of C and D are used for measuring the internal diameter of a hollow object.

(iii) Metallic strip : There is a thin metallic strip E attached to the back side of M and connected with vernier scale. When jaws A and B touch each other, the edge of E touches the edge of M. When the jaws A and B are separated E moves outwards. This strip E is used for measuring the depth of a vessel.

Principle (Theory)

In the common form, the divisions on the vernier scale V are smaller in size than the smallest division on the main scale M, but in some special cases the size of the vernier division may be larger than the main scale division.

Let n vernier scale divisions (V.S.D.) coincide with $(n-1)$ main scale divisions (M.S.D.). Then,
$$n\,\text{V.S.D.} = (n-1)\,\text{M.S.D.}$$
or
$$1\,\text{V.S.D.} = \left(\frac{n-1}{n}\right)\text{M.S.D.}$$
$$1\,\text{M.S.D.} - 1\,\text{V.S.D.} = 1\,\text{M.S.D.} - \left(\frac{n-1}{n}\right)\text{M.S.D.} = \frac{1}{n}\,\text{M.S.D.}$$

The difference between the values of one main scale division and one vernier scale division is known as **Vernier constant (V.C.)** or the **Least count (L.C.)**. This is the smallest distance that can be accurately measured with the vernier scale. Thus,
$$\text{V.C.} = \text{L.C.} = 1\,\text{M.S.D.} - 1\,\text{V.S.D.} = \left(\frac{1}{n}\right)\text{M.S.D.} = \frac{\text{Smallest division on main scale}}{\text{Number of divisions on vernier scale}}$$

In the ordinary vernier callipers one main scale division be 1 mm and 10 vernier scale divisions coincide with 9 main scale divisions.
$$1\,\text{V.S.D.} = \frac{9}{10}\,\text{M.S.D.} = 0.9\text{ mm}$$
$$\text{V.C.} = 1\,\text{M.S.D.} - 1\,\text{V.S.D.} = 1\text{ mm} - 0.9\text{ mm}$$
$$= 0.1\text{ mm} = 0.01\text{ cm}$$

Reading a vernier callipers

If we have to measure a length AB, the end A is coincided with the zero of main scale, suppose the end B lies between 1.0 cm and 1.1 cm on the main scale. Then,
$$1.0\,\text{cm} < AB < 1.1\,\text{cm}$$

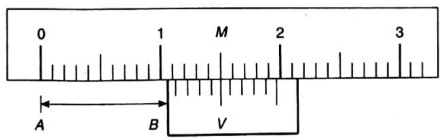


Fig. 1.13

Let 5th division of vernier scale coincides with 1.5 cm of main scale.

Then, $AB = 1.0 + 5 \times \text{V.C.} = (1.0 + 5 \times 0.01)\text{ cm} = 1.05\text{ cm}$

Thus, we can make the following formula,

$$\text{Total reading} = N + n \times \text{V.C.}$$

Here, N = main scale reading before on the left of the zero of the vernier scale.

n = number of vernier division which just coincides with any of the main scale division.

Note *That the main scale reading with which the vernier scale division coincides has no connection with reading.*

Zero error and zero correction

If the zero of the vernier scale does not coincide with the zero of main scale when jaw B touches A and the straight edge of D touches the straight edge of C, then the instrument has an error called **zero error.** Zero error is always algebraically subtracted from measured length.

Zero correction has a magnitude equal to zero error but its sign is opposite to that of the zero error. Zero correction is always algebraically added to measured length.

Zero error ⟶ algebraically subtracted

Zero correction ⟶ algebraically added

Positive and negative zero error

If zero of vernier scale lies to the right of the main scale the zero error is positive and if it lies to the left of the main scale the zero error is negative (when jaws A and B are in contact).

$$\text{Positive zero error} = (N + x \times \text{V.C.})$$

Here, N = main scale reading on the left of zero of vernier scale.

x = vernier scale division which coincides with any main scale division.

When the vernier zero lies before the main scale zero the error is said to be negative zero error. If 5th vernier scale division coincides with the main scale division, then

$$\begin{aligned}\text{Negative zero error} &= -[0.00\text{ cm} + 5 \times \text{V.C.}]\\ &= -[0.00\text{cm} + 5 \times 0.01\text{cm}]\\ &= -0.05\text{cm}\end{aligned}$$

Fig. 1.14 Positive and negative zero error

Summary

(1) $$\text{V.C.} = \text{L.C.} = \frac{1\text{M.S.D.}}{n} = \frac{\text{Smallest division on main scale}}{\text{Number of divisions on vernier scale}}$$

$$= 1\text{M.S.D.} - 1\text{V.S.D.}$$

(2) In ordinary vernier callipers, $1.\text{M.S.D.} = 1\text{ mm}$ and $n = 10$

$\therefore$ $$\text{V.C. or L.C.} = \frac{1}{10}\text{ mm} = 0.01\text{cm}$$

(3) Total reading $= (N + n \times \text{V.C})$

(4) Zero correction = – zero error

(5) Zero error is algebraically subtracted while the zero correction is algebraically added.

(6) If zero of vernier scale lies to the right of zero of main scale the error is positive. The actual length in this case is less than observed length.

(7) If zero of vernier scale lies to the left of zero of main scale the error is negative and the actual length is more than the observed length.

(8) Positive zero error $= (N + x \times \text{V.C.})$

Sample Example 1.9 *N-divisions on the main scale of a vernier callipers coincide with $N + 1$ divisions on the vernier scale. If each division on the main scale is of a units, determine the least count of the instrument.* (IIT JEE 2003)

Solution $(N + 1)$ divisions on the vernier scale $= N$ divisions on main scale

$\therefore$ $$1 \text{ division on vernier scale} = \frac{N}{N+1}\text{ divisions on main scale}$$

Each division on the main scale is of a units.

$\therefore$ $$1 \text{ division on vernier scale} = \left(\frac{N}{N+1}\right)a \text{ units} = a' \text{ (say)}$$

$$\text{Least count} = 1\text{main scale division} - 1 \text{ vernier scale division}$$

$$= a - a' = a - \left(\frac{N}{N+1}\right)a = \frac{a}{N+1}$$

Sample Example 1.10 *In the diagram shown in figure, find the magnitude and nature of zero error.*

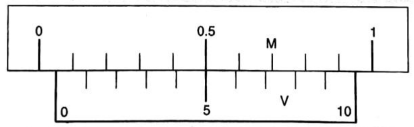


Fig. 1.15

Solution Here, zero of vernier scale lies to the right of zero of main scale, hence, it has positive zero error.

Further, $N = 0,\ x = 5,$ L.C. or V.C. $= 0.01$ cm

Hence, $$\begin{aligned}\text{Zero error} &= N + x \times \text{V.C.}\\ &= 0 + 5 \times 0.01\\ &= 0.05\text{cm}\end{aligned}$$

$$\text{Zero correction} = -0.05\text{cm}$$

$\therefore$ Actual length will be 0.05 cm less than the measured length.

(2) Principle of a Micrometer Screw

The least count of vernier callipers ordinarily available in the laboratory is 0.01cm. When lengths are to be measured with greater accuracy, say upto 0.001cm, screw gauge and spherometer are used which are based on the principle of **micrometer screw** discussed below.

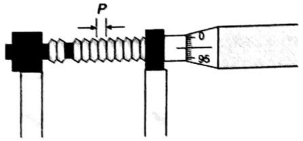


Fig. 1.16

If an accurately cut single threaded screw is rotated in a closely fitted nut, then in addition to the circular motion of the screw there is a linear motion of the screw head in the forward or backward direction, along the axis of the screw. The linear distance moved by the screw, when it is given one complete rotation is called the **pitch (p)** of the screw. This is equal to the distance between two consecutive threads as measured along the axis of the screw. In most of the cases it is either 1 mm or 0.5 mm. A circular cap is fixed on one end of the screw and the circumference of the cap is normally divided into 100 or 50 equal parts. If it is divided into 100 equal parts, then the screw moves forward or backward by $\frac{1}{100}\left(\text{or } \frac{1}{50}\right)$ of the pitch, if the circular scale (we will discuss later about circular scale) is rotated through one circular scale division. It is the minimum distance which can be accurately measured and so called the **least count (L.C.)** of the screw.

Thus, $$\text{Least count} = \frac{\text{Pitch}}{\text{Number of divisions on circular scale}}$$

If pitch is 1 mm and there are 100 divisions on circular scale then,

$$\text{L.C.} = \frac{1\,\text{mm}}{100} = 0.01\,\text{mm} = 0.001\,\text{cm} = 10\,\mu\text{m}$$

Since, L.C. is of the order of 10 μm, the screw is called micrometer screw.

(3) Screw Gauge

Screw gauge works on the principle of micrometer screw. It consists of a *U*-shaped metal frame *M*. At one end of it is fixed a small metal piece *A*. It is called stud and it has a plane face. The other end *N* of *M* carries a cylindrical hub *H*. It is graduated in millimetres and half millimetre depending upon the pitch of the screw. This scale is called **linear scale** or **pitch scale.**

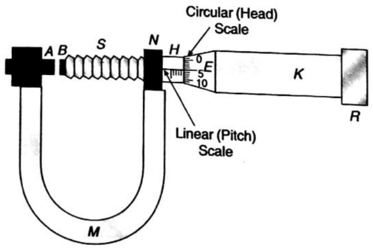


Fig. 1.17

A nut is threaded through the hub and the frame *N*. Through the nut moves a screw *S*. The front face *B* of the screw, facing the plane face *A* is also plane. A hollow cylindrical cap *K* is capable of rotating over the hub when screw is rotated. As the cap is rotated the screw either moves in or out. The surface *E* of the cap *K* is divided into 50 or 100 equal parts. It is called the **circular scale** or **head scale.** In an accurately adjusted instrument when the faces *A* and *B* are just touching each other. Zero of circular scale should coincide with zero of linear scale.

To measure diameter of a given wire using a screw gauge

If with the wire between plane faces *A* and *B*, the edge of the cap lies ahead of *N*th division of linear scale, and *n*th division of circular scale lies over reference line.

Then, $$\text{Total reading} = N + n \times \text{L.C.}$$

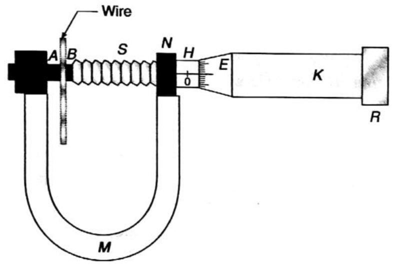


Fig. 1.18

Zero error and zero correction

If zero mark of circular scale does not coincide with the zero of the pitch scale when the faces *A* and *B* are just touching each other, the instrument is said to possess zero error. If the zero of the circular scale advances beyond the reference line the zero error is **negative** and zero correction is **positive.** If it is left behind the reference line the zero is **positive** and zero correction is **negative.** For example if zero of circular scale advances beyond the reference line by 5 divisions, zero correction $= +5 \times (\text{L.C.})$ and if the zero of circular scale is left behind the reference line by 4 divisions, zero correction $= -4 \times (\text{L.C.})$.

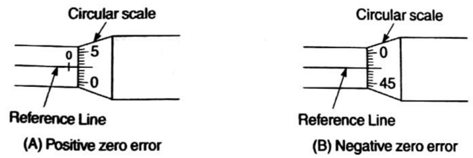


Fig. 1.19

Back lash error

When the sense of rotation of the screw is suddenly changed, the screw head may rotate, but the screw itself may not move forward or backwards. Thus, the scale reading may change even by the actual movement of the screw. This is known as back lash error. This error is due to loose fitting of the screw. This arises due to wear and tear of the threadings due to prolonged use of the screw. To reduce this error the screw must always be rotated in the same direction for a particular set of observations.

Sample Example 1.11 *The pitch of a screw gauge is 1 mm and there are 100 divisions on circular scale. When faces A and B are just touching each without putting anything between the studs 32nd division of the circular scale coincides with the reference line. When a glass plate is placed between the studs, the linear scale reads 4 divisions and the circular scale reads 16 divisions. Find the thickness of the glass plate. Zero of linear scale is not hidden from circular scale when A and B touches each other.*

Solution

$$\text{Least count L.C.} = \frac{\text{Pitch}}{\text{Number of divisions on circular scale}}$$

$$= \frac{1}{100}\text{ mm}$$

$$= 0.01\text{ mm}$$

As zero is not hidden from cricular scale when A and B touches each other. Hence, the screw gauge has positive error.

$$e = +n\,(\text{L.C.}) = 32 \times 0.01 = 0.32\text{ mm}$$

$$\text{Linear scale reading} = 4 \times (1\text{ mm}) = 4\text{ mm}$$

$$\text{Circular scale reading} = 16 \times (0.01\text{ mm}) = 0.16\text{ mm}$$

$$\therefore \quad \text{Measured reading} = (4 + 0.16)\text{ mm} = 4.16\text{ mm}$$

$$\therefore \quad \text{Absolute reading} = \text{Measured reading} - e$$

$$= (4.16 - 0.32)\text{ mm} = 3.84\text{ mm}$$

Therefore, thickness of the glass plate is 3.84 mm.

Sample Example 1.12 *The smallest division on main scale of a vernier callipers is 1 mm and 10 vernier divisions coincide with 9 scale divisions. While measuring the length of a line, the zero mark of the vernier scale lies between 10.2 cm and 10.3 cm and the third division of vernier scale coincide with a main scale division.*

(a) Determine the least count of the callipers.

(b) Find the length of the line.

Solution (a)

$$\text{Least count (L.C.)} = \frac{\text{Smallest division on main scale}}{\text{Number of divisions on vernier scale}}$$

$$= \frac{1}{10}\text{ mm} = 0.1\text{ mm}$$

$$= 0.01\text{ cm}$$

(b)

$$L = N + n\,(\text{L.C}) = (10.2 + 3 \times 0.01)\text{ cm}$$

$$= 10.23\text{ cm}$$

Sample Example 1.13 *The pitch of a screw gauge is 1 mm and there are 100 divisions on the circular scale. In measuring the diameter of a sphere there are six divisions on the linear scale and forty divisions on circular scale coincides with the reference line. Find the diameter of the sphere.*

Solution

$$\text{L.C.} = \frac{1}{100} = 0.01\text{ mm}$$

$$\text{Linear scale reading} = 6\,(\text{pitch}) = 6\text{ mm}$$

$$\text{Circular scale reading} = n\,(\text{L.C.}) = 40 \times 0.01 = 0.4\text{ mm}$$

$$\therefore \quad \text{Total reading} = (6 + 0.4) = 6.4\text{ mm}$$

Extra Points

- **Least count :** The minimum measurement that can be measured accurately by an instrument is called the least count. The least count of a meter scale graduated in milimetre mark is 1 mm. The least count of a watch having second's hand is 1 second.
- Least count of vernier callipers

  = {Value of 1 part of main scale(s)} − {Value of one part of vernier scale (V)}

  or Least count of vernier callipers = 1 MSD − 1 VSD

  where, MSD = Main scale division

  VSD = Vernier scale division
- $\text{Least count} = \dfrac{\text{Value of 1 part of main scale } (s)}{\text{Number of parts on vernier scale } (n)}$
- $\text{Least count of screw gauge} = \dfrac{\text{Pitch } (p)}{\text{Number of parts on circular scale } (n)}$

Solved Examples

Example 1 *Least count of a vernier callipers is 0.01 cm. When the two jaws of the instrument touch each other the 5th division of the vernier scale coincide with a main scale division and the zero of the vernier scale lies to the left of the zero of the main scale. Furthermore while measuring the diameter of a sphere, the zero mark of the vernier scale lies between 2.4 cm and 2.5 cm and the 6th vernier division coincides with a main scale division. Calculate the diameter of the sphere.*

Solution The instrument has a negative error,

$$e = (-5 \times 0.01)\,\text{cm}$$

or

$$e = -\,0.05\,\text{cm}$$

$$\text{Measured reading} = (2.4 + 6 \times 0.01) = 2.46\,\text{cm}$$

$$\text{True reading} = \text{Measured reading} - e$$

$$= 2.46 - (-\,0.05)$$

$$= 2.51\,\text{cm}$$

Therefore, diameter of the sphere is 2.51 cm.

Example 2 *The pitch of a screw gauge is 1 mm and there are 100 divisions on its circular scale. When nothing is put in between its jaws, the zero of the circular scale lies 6 divisions below the reference line. When a wire is placed between the jaws, 2 linear scale divisions are clearly visible while 62 divisions on circular scale coincide with the reference line. Determine the diameter of the wire.*

Solution

$$\text{L.C.} = \frac{P}{N} = \frac{1\,\text{mm}}{100} = 0.01\,\text{mm}$$

The instrument has a positive zero error,

$$e = +\,n\,(\text{L.C.}) = +\,(6 \times 0.01) = +\,0.06\,\text{mm}$$

$$\text{Linear scale reading} = 2 \times (1\,\text{mm}) = 2\,\text{mm}$$

$$\text{Circular scale reading} = 62 \times (0.01\,\text{mm}) = 0.62\,\text{mm}$$

$\therefore$

$$\text{Measured reading} = 2 + 0.62 = 2.62\,\text{mm}$$

or

$$\text{True reading} = 2.62 - 0.06$$

$$= 2.56\,\text{mm}$$

Example 3 *Write down the number of significant figures in the following*

(a) 6428 *(b)* 62.00 *m* *(c)* 0.00628 *cm* *(d)* 1200 *N*

Solution (a) 6428 has four significant figures.

(b) 62.00 m has four significant figures.

(c) 0.00628 cm has three significant figures.

(d) 1200 N has four significant figures.

Example 4 *Round off to four significant figures:*

(a) 45.689 *(b)* 2.0082

Solution (a) 45.69, (b) 2.008

Example 5 *Add* 6.75×10^3 cm *to* 4.52×10^2 cm *with regard to significant figures.*

Solution

$$a = 6.75 \times 10^3\ \text{cm}$$

$$b = 4.52 \times 10^2\ \text{cm}$$

$$= 0.452 \times 10^3\ \text{cm}$$

$$= 0.45 \times 10^3\ \text{cm} \qquad \text{(upto 2 places of decimal)}$$

$\therefore$

$$a + b = (6.75 \times 10^3 + 0.45 \times 10^3)\,\text{cm}$$

$$= 7.20 \times 10^3\ \text{cm}$$

Example 6 *A thin wire has a length of* 21.7 *cm and radius* 0.46 *mm. Calculate the volume of the wire to correct significant figures.*

Solution Given: $l = 21.7\,\text{cm}$, $r = 0.46\,\text{mm} = 0.046\,\text{cm}$.

$$\text{Volume of wire } V = \pi r^2 l$$

$$= \frac{22}{7}(0.046)^2\,(21.7)$$

$$= 0.1443\,\text{cm}^3 = 0.14\,\text{cm}^3$$

Note The result is rounded off to least number of significant figures in the given measurements i.e., 2 (in 0.46 mm).

Example 7 *The refractive index* (n) *of glass is found to have the values* 1.49, 1.50, 1.52, 1.54 *and* 1.48. *Calculate:*

(a) the mean value of refractive index.

(b) absolute error in each measurement.

(c) mean absolute error.

(d) fractional error and

(e) percentage error.

Solution (a) Mean value of refractive index,

$$n_m = \frac{1.49 + 1.50 + 1.52 + 1.54 + 1.48}{5}$$

$$= 1.506 = 1.51 \qquad \text{(rounded off to two decimal places)}$$

(b) Taking n_m as the true value, the absolute errors in different observations are,

$$\Delta n_1 = 1.51 - 1.49 = +\,0.02$$

$$\Delta n_2 = 1.51 - 1.50 = +\,0.01$$

$$\Delta n_3 = 1.51 - 1.52 = -\,0.01$$

$$\Delta n_4 = 1.51 - 1.54 = -\,0.03$$

$$\Delta n_5 = 1.51 - 1.48 = +\,0.03$$

(c) Mean absolute error,

$$\Delta n_{\text{mean}} = \frac{|\Delta n_1| + |\Delta n_2| + |\Delta n_3| + |\Delta n_4| + |\Delta n_5|}{5}$$

$$= \frac{0.02 + 0.01 + 0.01 + 0.03 + 0.03}{5} = 0.02$$

(d) Fractional error $= \frac{\pm \Delta n_{mean}}{n_m} = \frac{\pm 0.02}{1.51} = \pm 0.0132$

(e) Percentage error $= (\pm 0.0132 \times 100) = \pm 1.32\%$

Example 8 *The radius of sphere is measured to be* $(2.1 \pm 0.5)\,cm$. *Calculate its surface area with error limits.*

Solution Surface area, $S = 4\pi r^2 = (4)\left(\frac{22}{7}\right)(2.1)^2$

$$= 55.44 = 55.4\,\text{cm}^2$$

Further, $\frac{\Delta S}{S} = 2 \cdot \frac{\Delta r}{r}$

or $\Delta S = 2\left(\frac{\Delta r}{r}\right)(S) = \frac{2 \times 0.5 \times 55.4}{2.1}$

$$= 26.38 = 26.4\,\text{cm}^2$$

$\therefore$ $S = (55.4 \pm 26.4)\,\text{cm}^2$

Example 9 *Calculate focal length of a spherical mirror from the following observations. Object distance* $u = (50.1 \pm 0.5)\,cm$ *and image distance* $v = (20.1 \pm 0.2)\,cm$.

Solution $\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$

or $f = \frac{uv}{u+v} = \frac{(50.1)(20.1)}{(50.1+20.1)} = 14.3\,\text{cm}$

Also, $\frac{\Delta f}{f} = \pm\left[\frac{\Delta u}{u} + \frac{\Delta v}{v} + \frac{\Delta u + \Delta v}{u+v}\right]$

$$= \pm\left[\frac{0.5}{50.1} + \frac{0.2}{20.1} + \frac{0.5+0.2}{50.1+20.1}\right]$$

$$= [0.00998 + 0.00995 + 0.00997]$$

$$= \pm (0.0299)$$

$\therefore$ $\Delta f = 0.0299 \times 14.3 = 0.428 = 0.4\,\text{cm}$

$\therefore$ $f = (14.3 \pm 0.4)\,\text{cm}$

EXERCISES

Section-I

Single Correct Option

1. Percentage error in the measurement of mass and speed are 2% and 3% respectively. The error in the estimate of kinetic energy obtained by measuring mass and speed will be
(a) 12% (b) 10% (c) 8% (d) 2%

2. The density of a cube is measured by measuring its mass and length of its sides. If the maximum error in the measurement of mass and length are 4% and 3% respectively, the maximum error in the measurement of density will be
(a) 7% (b) 9% (c) 12% (d) 13%

3. A force F is applied on a square plate of side L. If the percentage error in the determination of L is 2% and that in F is 4%. What is the permissible error in pressure?
(a) 8% (b) 6% (c) 4% (d) 2%

4. By what percentage should the pressure of a given mass of gas be increased, so as to decrease its volume by 10% at a constant temperature?
(a) 11.1% (b) 10.1% (c) 9.1% (d) 8.1%

5. If the error in the measurement of the momentum of a particle is (+ 100%). Then, the error in the measurement of kinetic energy is
(a) 400% (b) 300% (c) 100% (d) 200%

6. The number of significant figures in 3400 is
(a) 7 (b) 6 (c) 12 (d) 2

7. The length and breadth of a metal sheet are 3.124 m and 3.002 m respectively. The area of this sheet upto correct significant figure is
(a) $9.378\,\text{m}^2$ (b) $9.37\,\text{m}^2$ (c) $9.378248\,\text{m}^2$ (d) $9.3782\,\text{m}^2$

8. Let g be the acceleration due to gravity at earth's surface and K the rotational kinetic energy of the earth. Suppose the earth's radius decreases by 2%. Keeping all other quantities constant, then
(a) g increases by 2% and K increases by 2% (b) g increases by 4% and K increases by 4%
(c) g decreases by 4% and K decreases by 2% (d) g decreases by 2% and K decreases by 4%

9. The heat generated in a circuit is dependent upon the resistance, current and time for which the current is flown. If the error in measuring the above are 1%, 2% and 1% respectively. The maximum error in measuring the heat is
(a) 8% (b) 6% (c) 18% (d) 12%

10. The length, breadth and thickness of a block are given by $l = 12\,\text{cm}$, $b = 6\,\text{cm}$ and $t = 2.45\,\text{cm}$. The volume of the block according to the idea of significant figures should be
(a) $1 \times 10^2\,\text{cm}^3$ (b) $2 \times 10^2\,\text{cm}^3$ (c) $1.763 \times 10^2\,\text{cm}^3$ (d) None of these

11. If separation between screen and point source is increased by 2%. What would be the effect on the intensity?
(a) increases by 4% (b) increases by 2% (c) decreases by 2% (d) decreases by 4%

12. The significant figures in the number 6.0023 are
(a) 2 (b) 5 (c) 4 (d) 1

13. If error in measurement of radius of a sphere is 1%, what will be the error in measurement of volume?
(a) 1% (b) $\frac{1}{3}$% (c) 3% (d) 10%

14. The volume of a cube in m^3 is equal to the surface area of the cube in m^2. The volume of the cube is
(a) $64\ m^3$ (b) $216\ m^3$ (c) $512\ m^3$ (d) $196\ m^3$

15. The length of a simple pendulum is about 100 cm known to have an accuracy of 1 mm. Its period of oscillation is 2 s determined by measuring the time for 100 oscillations using a clock of 0.1 s resolution. What is the accuracy in the determined value of g?
(a) 0.2% (b) 0.5% (c) 0.1% (d) 2%

16. Charge on the capacitor is given by $Q = I\alpha e^{\frac{-tI}{\Delta V \varepsilon_0 \beta}}$, where α and β are constants. t = time, I = current, ΔV = potential difference. Then dimensions of $\frac{\beta}{\alpha}$ is same as the dimensions of
(a) $\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ (b) $\mu_0 \varepsilon_0$ (c) $\sqrt{\varepsilon_0 \mu_0}$ (d) $\frac{1}{\mu_0 \varepsilon_0}$

17. A physical quantity A is dependent on other four physical quantities p, q, r and s as given by $a = \frac{\sqrt{pq}}{r^2 s^3}$. The percentage error of measurement in p, q, r and s are 1%, 3%, 0.5% and 0.33% respectively, then the maximum percentrage error in A is
(a) 2% (b) 0% (c) 4% (d) 3%

18. The distance moved by the screw of a screw gauge is 2 mm in four rotations and there are 50 divisions on its cap. When nothing is put between its jaws, 30th division of circular scale coincides with reference line, with zero of circular scale lying above the reference line. When a plate is placed between the jaws, main scale reads 2 divisions and circular scale reads 20 divisions. Thickness of plate is
(a) 0.9 mm (b) 1.2 mm (c) 1.4 mm (d) 1.5 mm

More than One Correct Options

1. Given, $x = \frac{ab^2}{c^3}$. If the percentage errors in a, b and c are $\pm 1\%$, $\pm 13\%$ and $\pm 2\%$ respectively.
(a) The percentage error in x can be $\pm 13\%$ (b) The percentage error in x can be $\pm 7\%$
(c) The percentage error in x can be $\pm 18\%$ (d) The percentage error in x can be $\pm 19\%$

Assertion and Reason

Directions : *Choose the correct option.*

*(a) If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*
*(b) If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*
*(c) If **Assertion** is true, but the **Reason** is false.*
*(d) If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** A screw gauge having a smaller value of pitch has greater accuracy.
Reason : The least count of screw gauge is directly proportional to the number of divisions on circular scale.

Match the Columns

1. Match the following two columns.

Column-I	Column-II
(a) GM_eM_s	(p) $[M^2L^2T^{-3}]$
(b) $\frac{3RT}{M}$	(q) $[ML^3T^{-2}]$
(c) $\frac{F^2}{q^2B^2}$	(r) $[L^2T^{-2}]$
(d) $\frac{GM_e}{R_e}$	(s) None

Section-II

Subjective Questions

Trigonometry

1. Find the value of
(a) $\cos 120°$ (b) $\sin 240°$ (c) $\tan(-60°)$ (d) $\cot 300°$
(e) $\tan 330°$ (f) $\cos(-60°)$ (g) $\sin(-150°)$ (h) $\cos(-120°)$

2. Find the value of
(a) $\sec^2\theta - \tan^2\theta$ (b) $\text{cosec}^2\theta - \cot^2\theta - 1$
(c) $2\sin 45° \cos 15°$ (d) $2\sin 15° \cos 45°$

Calculus

3. Differentiate the following functions with respect to x
(a) $x^4 + 3x^2 - 2x$ (b) $x^2 \cos x$ (c) $(6x+7)^4$ (d) $e^x x^5$
(e) $\frac{(1+x)}{e^x}$

4. Integrate the following functions with respect to t
(a) $\int (3t^2 - 2t)\, dt$ (b) $\int (4\cos t + t^2)\, dt$ (c) $\int (2t-4)^{-4}\, dt$ (d) $\int \frac{dt}{(6t-1)}$

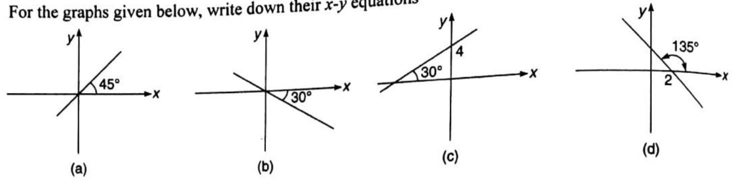
5. Integrate the following functions
(a) $\int_0^2 2t\, dt$ (b) $\int_{\pi/6}^{\pi/3} \sin x\, dx$ (c) $\int_4^{10} \frac{dx}{x}$ (d) $\int_0^{\pi} \cos x\, dx$
(e) $\int_1^2 (2t-4)\, dt$

6. Find maximum/minimum value of y in the functions given below
(a) $y = 5 - (x-1)^2$ (b) $y = 4x^2 - 4x + 7$
(c) $y = x^3 - 3x$ (d) $y = x^3 - 6x^2 + 9x + 15$
(e) $y = (\sin 2x - x)$, where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Graphs

7. Draw the graphs corresponding to the equations
   (a) $y = 4x$ (b) $y = -6x$
   (c) $y = x + 4$ (d) $y = -2x + 4$
   (e) $y = 2x - 4$ (f) $y = -4x - 6$

8. For the graphs given below, write down their x-y equations

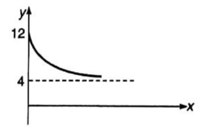


9. For the equations given below, tell the nature of graphs.
   (a) $y = 2x^2$ (b) $y = -4x^2 + 6$ (c) $y = 6e^{-4x}$ (d) $y = 4(1 - e^{-2x})$
   (e) $y = \frac{4}{x}$ (f) $y = -\frac{2}{x}$

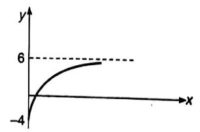
10. Value of y decreases exponentially from $y = 10$ to $y = 6$. Plot a x-y graph.

11. Value of y increases exponentially from $y = -4$ to $y = +4$. Plot a x-y graph.

12. The graph shown in figure is exponential. Write down the equation corresponding to the graph.



13. The graph shown in figure is exponential. Write down the equation corresponding to the graph.



Significant Figures

14. Write down the number of significant figures in the following.
    (a) 6428 (b) 62.00 m (c) 0.0628 cm (d) 1200 N

15. Round off to four significant figures.
    (a) 45.689 (b) 2.0082

16. Add 6.75×10^3 cm and 4.52×10^2 cm with due regards to significant figures.

17. A thin wire has length of 21.7 cm and radius 0.46 mm. Calculate the volume of the wire to correct significant figures.

18. A cube has a side of length 2.342 m. Find volume and surface area in correct significant figures.

19. Find density when a mass of 9.23 kg occupies a volume of $1.1\,m^3$. Take care of significant figures.

20. Length, breadth and thickness of a rectangular slab are 4.234 m, 1.005 m and 2.01 cm respectively. Find surface area and volume to correct significant figures.

21. Solve with due regards to significant figures

$$(4.0 \times 10^{-4} - 2.5 \times 10^{-6})$$

Error Analysis

22. The refractive index (n) of glass is found to have the values 1.49, 1.50, 1.52, 1.54 and 1.48. Calculate
    (a) the mean value of refractive index (b) absolute error is each measurement
    (c) fractional error and (d) percentage error

23. The radius of a sphere is measured to be (2.1 ± 0.5) cm. Calculate its surface area with error limits.

24. Calculate focal length of a spherical mirror from the following observations. Object distance $u = (50.1 \pm 0.5)$ cm and image distance $v = (20.1 \pm 0.2)$ cm.

25. Find the percentage error in specific resistance given by $\rho = \frac{\pi r^2 R}{l}$ where r is the radius having value (0.2 ± 0.02) cm, R is the resistance of (60 ± 2) ohm and l is the length of (150 ± 0.1) cm.

26. A physical quantity ρ is related to four variables α, β, γ and η as

$$\rho = \frac{\alpha^3 \beta^2}{\sqrt{\gamma \eta}}$$

The percentage errors of measurements in α, β, γ and η are 1%, 3%, 4% and 2% respectively. Find the percentage error in ρ.

27. The period of oscillation of a simple pendulum is $T = 2\pi\sqrt{\frac{L}{g}}$. Length L is about 10 cm and is known to 1 mm accuracy. The period of oscillation is about 0.5 s. The time of 100 oscillations is measured with wristwatch of 1 s time period. What is accuracy in the determination of g.

Vernier Callipers and Screw Gauge

28. 19 divisions on the main scale of a vernier callipers coincide with 20 divisions on the vernier scale. If each division on the main scale is of 1 cm, determine the least count of instrument.

29. The pitch of a screw guage is 1 mm and there are 100 divisions on the circular scale. While measuring the diameter of a wire, the linear scale reads 1 mm and 47th division on the circular scale coincides with the reference line. The length of the wire is 5.6 cm. Find the curved surface area (in cm^2) of the wire in appropriate number of significant figures.

30. The edge of a cube is measured using a vernier callipers. [9 divisions of the main scale is equal to 10 divisions of vernier scale and 1 main scale division is 1 mm]. The main scale division reading is 10 and 1 division of vernier scale was found to be coinciding with the main scale. The mass of the cube is 2.736 g. Calculate the density in g/cm^3 upto correct significant figures.

ANSWERS

Section-I

Single Correct Option

1.(c) 2.(d) 3.(a) 4.(a) 5.(b) 6.(d) 7.(a) 8.(b) 9.(b) 10.(b)
11.(d) 12.(b) 13.(c) 14.(b) 15.(a) 16.(a) 17.(c) 18.(d)

More than One Correct Options

1. (a,b)

Assertion and Reason

1. (c)

Match the Columns

1. (a) → q (b) → r (c) → r (d) → r

Section-II

Subjective Questions

1. (a) $-\frac{1}{2}$ (b) $-\frac{\sqrt{3}}{2}$ (c) $-\sqrt{3}$ (d) $-\frac{1}{\sqrt{3}}$ (e) $-\frac{1}{\sqrt{3}}$ (f) $\frac{1}{2}$ (g) $-\frac{1}{2}$ (h) $-\frac{1}{2}$

2. (a) 1 (b) 0 (c) $\left(\frac{\sqrt{3}+1}{2}\right)$ (d) $\left(\frac{\sqrt{3}-1}{2}\right)$

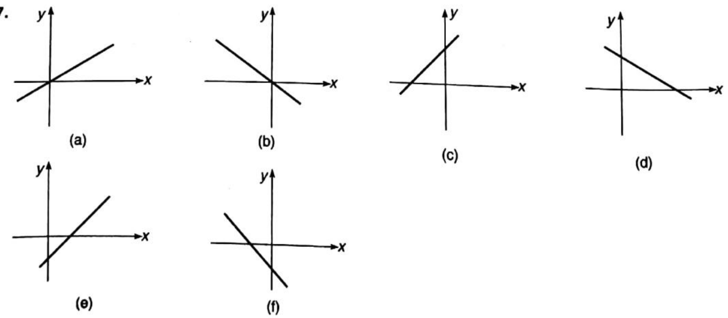
3. (a) $4x^3 + 6x - 2$ (b) $2x\cos x - x^2 \sin x$ (c) $24(6x+7)^3$ (d) $5e^x x^4 + e^x x^5$ (e) $-xe^{-x}$

4. (a) $t^3 - t^2 + C$ (b) $4\sin t + \frac{t^3}{3} + C$ (c) $-\frac{1}{6(2t-4)^3} + C$ (d) $\frac{1}{6}\ln(6t-1) + C$

5. (a) 4 (b) $\frac{(\sqrt{3}-1)}{2}$ (c) ln (5/2) (d) Zero (e) − 1

6. (a) $y_{max} = 5$ at $x = 1$ (b) $y_{min} = 6$ at $x = 1/2$ (c) $y_{min} = -2$ at $x = 1$ and $y_{max} = 2$ at $x = -1$
(d) $y_{min} = 15$ at $x = 3$ and $y_{max} = 19$ at $x = 1$ (e) $y_{min} = -\left(\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right)$ at $x = -\pi/6$ and
$y_{max} = \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$ at $x = \pi/6$

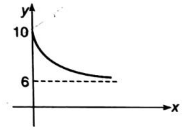
7.



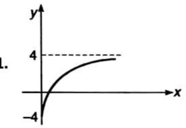
8. (a) $y = x$ (b) $y = -\frac{x}{\sqrt{3}}$ (c) $y = \frac{x}{\sqrt{3}} + 4$ (d) $y = -x + 2$

9. (a) parabola passing through origin (b) parabola not passing through origin
(c) exponentially decreasing graph (d) exponentially increasing graph
(e) Rectangular hyperbola in first and third quadrant
(f) Rectangular hyperbola in second and fourth quadrant

10.

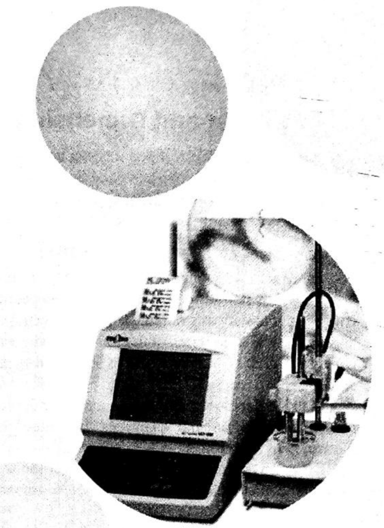


11.



12. $y = 4 + 8e^{-Kx}$ Here, K is a positive constant 13. $y = -4 + 10(1 - e^{-Kx})$ Here, K is positive constant
14. (a) Four (b) Four (c) Three (d) Four 15. (a) 45.69 (b) 2.008 16. 7.20×10^3 cm
17. 0.14 cm^3 18. Area = 5.485 m^2, volume = 12.85 m^3 19. Density = 8.4 kg/m^3
20. Area = 4.255 m^2, volume = 8.55 m^3 21. 4.0×10^{-4}
22. (a) 1.51 (b) + 0.02, + 0.01, − 0.01, − 0.03, + 0.03 (c) ± 0.0132 (d) ± 1.32% 23. (55.4 ± 26.4) cm^2
24. (14.3 ± 0.4) cm 25. 23.4% 26. 13% 27. 5% 28. 0.05 cm 29. 2.6 cm^2 30. 2.66 g/cm^3

Chapter 2 – Units & Dimensions Vectors



2

Units & Dimensions Vectors

Chapter Contents

2.1 Units and Dimensions
2.2 Vector and Scalar Quantities
2.3 General Points Regarding Vectors
2.4 Addition and Subtraction of Two Vectors
2.5 Components of a Vector
2.6 Product of Two Vectors

2.1 Units and Dimensions

To measure a physical quantity we need some standard unit of that quantity. The measurement of the quantity is mentioned in two parts, the first part gives how many times of the standard unit and the second part gives the name of the unit. Thus, suppose I say that length of this wire is 5 metres. The numeric part 5 says that it is 5 times of the unit of length and the second part metre says that unit chosen here is metre.

Fundamental and Derived Quantities

There are a large number of physical quantities and every quantity needs a unit.

However, not all the quantities are independent. For example, if a unit of length is defined, a unit of volume is automatically obtained. Thus, we can define a set of fundamental quantities and all other quantities may be expressed in terms of the fundamental quantities. Fundamental quantities are only seven in numbers. Unit of all other quantities can be expressed in terms of the units of these seven quantities by multiplication or division.

Many different choices can be made for the fundamental quantities. For example, if we take length and time as the fundamental quantities then speed is a derived quantity and if we take speed and time as fundamental quantities then length is a derived quantity.

Several system of units are in use over the world. The units defined for the fundamental quantities are called fundamental units and those obtained for derived quantities are called the derived units.

SI Units

In 1971 General Conference on Weight and Measures held its meeting and decided a system of units which is known as the International System of Units. It is abbreviated as SI from the French name Le Systeme International *d'* Unites. This system is widely used throughout the world. Table below gives the seven fundamental quantities and their SI units.

Table 2.1 Fundamental quantities and their SI units.

S. No.	Quantity	SI Unit	Symbol
1.	Length	metre	m
2.	Mass	kilogram	kg
3.	Time	second	s
4.	Electric current	ampere	A
5.	Thermodynamic temperature	kelvin	K
6.	Amount of substance	mole	mol
7.	Luminous intensity	candela	cd

Two **supplementary** units namely plane angle and solid angle are also defined. Their units are radian (rad) and steradian (st) respectively.

(i) **CGS System** : In this system, the units of length, mass and time are centimetre (cm), gram (g) and second (s) respectively. The unit of force is dyne and that of work or energy is erg.

(ii) **FPS System** : In this system, the units of length, mass and time are foot, pound and second. The unit of force in this system is poundal.

Definitions of Some Important SI Units

(i) **Metre** : 1 m = 1,650,763.73 wavelengths in vacuum, of radiation corresponding to orange-red light of krypton-86.

(ii) **Second** : 1 s = 9,192,631,770 time periods of a particular radiation from Cesium-133 atom.

(iii) **Kilogram**: 1 kg = mass of 1 litre volume of water at 4°C.

(iv) **Ampere**: It is the current which when flows through two infinitely long straight conductors of negligible cross-section placed at a distance of one metre in vacuum produces a force of 2×10^{-7} N/m between them.

(v) **Kelvin**: $1\,K = 1/273.16$ part of the thermodynamic temperature of triple point of water.

(vi) **Mole**: It is the amount of substance of a system which contains as many elementary particles (atoms, molecules, ions etc.) as there are atoms in 12 g of carbon-12.

SI Prefixes

The most commonly used prefixes are given below in tabular form.

Power of 10	Prefix	Symbol
6	mega	M
3	kilo	k
− 2	centi	c
− 3	mili	m
− 6	micro	μ
− 9	nano	n
− 12	pico	p

(vii) **Candela**: It is luminous intensity in a perpendicular direction of a surface of $\left(\frac{1}{600000}\right)$ m^2 of a black body at the temperature of freezing platinum under a pressure of 1.013×10^5 N/m^2.

(viii) **Radian**: It is the plane angle between two radii of a circle which cut-off on the circumference, an arc equal in length to the radius.

(ix) **Steradian**: The steradian is the solid angle which having its vertex at the centre of the sphere, cut-off an area of the surface of sphere equal to that of a square with sides of length equal to the radius of the sphere.

Dimensions

Dimensions of a physical quantity are the powers to which the fundamental quantities must be raised to represent the given physical quantity.

For example,
$$\text{density} = \frac{\text{mass}}{\text{volume}} = \frac{\text{mass}}{(\text{length})^3}$$

or
$$\text{density} = (\text{mass})\,(\text{length})^{-3} \qquad \ldots\text{(i)}$$

Thus, the dimensions of density are 1 in mass and − 3 in length. The dimensions of all other fundamental quantities are zero.

For convenience, the fundamental quantities are represented by one letter symbols. Generally mass is denoted by M, length by L, time by T and electric current by A.

The thermodynamic temperature, the amount of substance and the luminous intensity are denoted by the symbols of their units K, mol and cd respectively. The physical quantity that is expressed in terms of the base quantities is enclosed in square brackets.

Thus, Eq. (i) can be written as $[\text{density}] = [ML^{-3}]$

Such an expression for a physical quantity in terms of the fundamental quantities is called the dimensional formula.

Here, it is worthnoting that constants such as 5, π or trigonometrical functions such as $\sin\theta$, $\cos\theta$, etc., have no units and dimensions.

$$[\sin\theta] = [\cos\theta] = [\tan\theta] = [\log x] = [e^x] = [M^0L^0T^0]$$

Table 2.2 Dimensional formulae and SI units of some physical quantities frequently used in physics

S. No.	Physical Quantity	SI Units	Dimensional Formula
1.	Velocity = displacement/time	m/s	$[M^0LT^{-1}]$
2.	Acceleration = velocity/time	m/s^2	$[M^0LT^{-2}]$
3.	Force = mass × acceleration	kg-m/s^2 = newton or N	$[MLT^{-2}]$
4.	Work = force × displacement	kg-m^2/s^2 = N-m = joule or J	$[ML^2T^{-2}]$
5.	Energy = Work	J	$[ML^2T^{-2}]$
6.	Torque = force × perpendicular distance	N-m	$[ML^2T^{-2}]$
7.	Power = work/time	J/s or watt	$[ML^2T^{-3}]$
8.	Momentum = mass × velocity	kg-m/s	$[MLT^{-1}]$
9.	Impulse = force × time	N-s	$[MLT^{-1}]$
10.	Angle = arc/radius	radian or rad	$[M^0L^0T^0]$
11.	Strain = $\frac{\Delta L}{L}$ or $\frac{\Delta V}{V}$	no units	$[M^0L^0T^0]$
12.	Stress = force/area	N/m^2	$[ML^{-1}T^{-2}]$
13.	Pressure = force/area	N/m^2	$[ML^{-1}T^{-2}]$
14.	Modulus of elasticity = stress/strain	N/m^2	$[ML^{-1}T^{-2}]$
15.	Frequency = 1/time period	per sec or hertz (Hz)	$[M^0L^0T^{-1}]$
16.	Angular velocity = angle/time	rad/s	$[M^0L^0T^{-1}]$
17.	Moment of inertia = (mass) × (distance)^2	kg-m^2	$[ML^2T^0]$
18.	Surface tension = force/length	N/m	$[ML^0T^{-2}]$
19.	Gravitational constant = $\frac{\text{force} \times (\text{distance})^2}{(\text{mass})^2}$	N-m^2/kg^2	$[M^{-1}L^3T^{-2}]$
20.	Angular momentum	kg-m^2/s	$[ML^2T^{-1}]$
21.	Coefficient of viscosity	N-s/m^2	$[ML^{-1}T^{-1}]$
22.	Planck's constant	J-s	$[ML^2T^{-1}]$
23.	Specific heat (s)	J/kg-K	$[L^2T^{-2}\theta^{-1}]$
24.	Coefficient of thermal conductivity (K)	watt/m-K	$[MLT^{-3}\theta^{-1}]$
25.	Gas constant (R)	J/mol-K	$[ML^2T^{-2}\theta^{-1}\,mol^{-1}]$
26.	Boltzmann constant (k)	J/K	$[ML^2T^{-2}\theta^{-1}]$
27.	Wein's constant (b)	m-K	$[L\theta]$
28.	Stefan's constant (σ)	watt/m^2-K^4	$[MT^{-3}\theta^{-4}]$
29.	Electric charge	C	$[AT]$
30.	Electric intensity	N/C	$[MLT^{-3}A^{-1}]$
31.	Electric potential	volt	$[ML^2T^{-3}A^{-1}]$
32.	Capacitance	farad	$[M^{-1}L^{-2}T^4A^2]$
33.	Permittivity of free space	$C^2N^{-1}m^{-2}$	$[M^{-1}L^{-3}T^4A^2]$
34.	Electric dipole moment	C-m	$[LTA]$
35	Density	kg/m^3	$[M^1L^{-3}T^0]$

35.	Resistance	ohm	$[ML^2T^{-3}A^{-2}]$
36.	Magnetic field	tesla (T) or weber/m^2 (Wb/m^2)	$[MT^{-2}A^{-1}]$
37.	Coefficient of self induction	henry	$[ML^2T^{-2}A^{-2}]$

Check Points

- Astronomical unit
  1 AU = mean distance of earth from sun
  $\approx 1.5\times 10^{11}$ m
- Light year
  1 ly = distance travelled by light in vacuum in 1 year
  $= 9.46 \times 10^{15}$ m
- Parsec
  1 Parsec $= 3.07\times 10^{16}$ m = 3.26 light year
- X-ray unit
  1 U $= 10^{-3}$ m
- 1 shake $= 10^{-8}$ s
- 1 Bar $= 10^5$ N/m^2 $= 10^5$ Pa
- 1 torr = 1 mm of Hg = 133.3 Pa
- 1 barn $= 10^{-28}$ m^2
- 1 horse power = 746 W
- 1 pound = 453.6 g = 0.4536 kg

Sample Example 2.1 *Find the dimensional formulae of*
(a) coefficient of viscosity η *(b) charge* q
(c) potential V *(d) capacitance* C *and*
(e) resistance R
Some of the equations containing these quantities are

$$F=-\eta A\left(\frac{\Delta v}{\Delta l}\right),\quad q=It,\quad U=VIt,$$

$$q=CV \quad and \quad V=IR$$

Where A denotes the area, v the velocity, l is the length, I the electric current, t the time and U the energy.

Solution (a) $\eta=-\dfrac{F}{A}\dfrac{\Delta l}{\Delta v}$

$$\therefore \quad [\eta]=\frac{[F][l]}{[A][v]}=\frac{[MLT^{-2}][L]}{[L^2][LT^{-1}]}=[ML^{-1}T^{-1}]$$

(b) $q=It \quad \therefore \quad [q]=[I][t]=[AT]$

(c) $U=VIt \quad \therefore \quad V=\dfrac{U}{It}$ or $[V]=\dfrac{[U]}{[I][t]}=\dfrac{[ML^2T^{-2}]}{[A][T]}=[ML^2T^{-3}A^{-1}]$

(d) $q=CV \quad \therefore \quad C=\dfrac{q}{V}$ or $[C]=\dfrac{[q]}{[V]}=\dfrac{[AT]^2}{[ML^2T^{-3}A^{-1}]}=[M^{-1}L^{-2}T^4A^2]$

(e) $V=IR \quad \therefore \quad R=\dfrac{V}{I}$

or $[R]=\dfrac{[V]}{[I]}=\dfrac{[ML^2T^{-3}A^{-1}]}{[A]}=[ML^2T^{-3}A^{-2}]$

Uses of Dimensions

Theory of dimensions have following main uses:

(i) **Conversion of units** : This is based on the fact that the product of the numerical value (n) and its corresponding unit (u) is a constant, *i.e.*,

$$n[u]=\text{constant}$$

or $$n_1[u_1]=n_2[u_2]$$

Suppose the dimensions of a physical quantity are a in mass, b in length and c in time. If the fundamental units in one system are M_1, L_1 and T_1 and in the other system are M_2, L_2 and T_2 respectively. Then, we can write

$$n_1[M_1^a L_1^b T_1^c]=n_2[M_2^a L_2^b T_2^c] \qquad \text{...(i)}$$

Here, n_1 and n_2 are the numerical values in two system of units respectively. Using Eq. (i), we can convert the numerical value of a physical quantity from one system of units into the other system.

Sample Example 2.2 *The value of gravitation constant is* $G=6.67\times 10^{-11}$ *N-m^2/kg^2 in SI units. Convert it into CGS system of units.*

Solution The dimensional formula of G is $[M^{-1}L^3T^{-2}]$.

Using equation number (i), *i.e.*,

$$n_1[M_1^{-1}L_1^3T_1^{-2}]=n_2[M_2^{-1}L_2^3T_2^{-2}]$$

$$n_2=n_1\left[\frac{M_1}{M_2}\right]^{-1}\left[\frac{L_1}{L_2}\right]^3\left[\frac{T_1}{T_2}\right]^{-2}$$

Here, $n_1=6.67\times 10^{-11}$

$M_1=1\,\text{kg},\quad M_2=1\,\text{g}=10^{-3}\,\text{kg}\quad L_1=1\,\text{m},\quad L_2=1\,\text{cm}=10^{-2}\,\text{m},\quad T_1=T_2=1\text{s}$

Substituting in the above equation, we get

$$n_2=6.67\times 10^{-11}\left[\frac{1\,\text{kg}}{10^{-3}\,\text{kg}}\right]^{-1}\left[\frac{1\,\text{m}}{10^{-2}\,\text{m}}\right]^3\left[\frac{1\text{s}}{1\text{s}}\right]^{-2} \quad \text{or} \quad n_2=6.67\times 10^{-8}$$

Thus, value of G in CGS system of units is 6.67×10^{-8} dyne cm^2/g^2.

(ii) **To check the dimensional correctness of a given physical equation** : Every physical equation should be dimensionally balanced. This is called the 'Principle of Homogeneity'. The dimensions of each term on both sides of an equation must be the same. On this basis we can judge whether a given equation is correct or not. But a dimensionally correct equation may or may not be physically correct.

Sample Example 2.3 *Show that the expression of the time period T of a simple pendulum of length l given by* $T=2\pi\sqrt{\dfrac{l}{g}}$ *is dimensionally correct.*

Solution

$$T=2\pi\sqrt{\frac{l}{g}}$$

$$\text{Dimensionally } [T]=\sqrt{\frac{[L]}{[LT^{-2}]}}=[T]$$

As in the above equation, the dimensions of both sides are same. The given formula is dimensionally correct.

Principle of Homogeneity of Dimensions

This principle states that the dimensions of all the terms in a physical expression should be same. For example, in the physical expression $s = ut + \frac{1}{2}at^2$, the dimensions of s, ut and $\frac{1}{2}at^2$ all are same.

Note *The physical quantities separated by the symbols +, −, =, >, < etc., have the same dimensions.*

Sample Example 2.4 *The velocity v of a particle depends upon the time t according to the equation* $v = a + bt + \frac{c}{d+t}$. *Write the dimensions of a, b, c and d.*

Solution From principle of homogeneity

$$[a] = [v]$$

or $$[a] = [\mathrm{LT^{-1}}]$$

$$[bt] = [v]$$

or $$[b] = \frac{[v]}{[t]} = \frac{[\mathrm{LT^{-1}}]}{[\mathrm{T}]}$$

or $$[b] = [\mathrm{LT^{-2}}]$$

Similarly, $$[d] = [t] = [\mathrm{T}]$$

Further, $$\frac{[c]}{[d+t]} = [v]$$

or $$[c] = [v][d+t]$$

or $$[c] = [\mathrm{LT^{-1}}][\mathrm{T}] \quad \text{or} \quad [c] = [\mathrm{L}]$$

(iii) **To establish the relation among various physical quantities** : If we know the factors on which a given physical quantity may depend, we can find a formula relating the quantity with those factors. Let us take an example.

Sample Example 2.5 *The frequency (f) of a stretched string depends upon the tension F (dimensions of force), length l of the string and the mass per unit length μ of string. Derive the formula for frequency.*

Solution Suppose, that the frequency f depends on the tension raised to the power a, length raised to the power b and mass per unit length raised to the power c. Then,

$$f \propto [F]^a\,[l]^b\,[\mu]^c$$

or $$f = k\,[F]^a\,[l]^b\,[\mu]^c \qquad \ldots\text{(i)}$$

Here, k is a dimensionless constant. Thus,

$$[f] = [F]^a\,[l]^b\,[\mu]^c$$

or $$[\mathrm{M^0L^0T^{-1}}] = [\mathrm{MLT^{-2}}]^a\,[\mathrm{L}]^b\,[\mathrm{ML^{-1}}]^c$$

or $$[\mathrm{M^0L^0T^{-1}}] = [\mathrm{M}^{a+c}\mathrm{L}^{a+b-c}\mathrm{T}^{-2a}]$$

For dimensional balance, the dimensions on both sides should be same.

Thus, $$a + c = 0 \qquad \ldots\text{(ii)}$$

$$a + b - c = 0 \qquad \ldots\text{(iii)}$$

and $$-2a = -1 \qquad \ldots\text{(iv)}$$

Solving these three equations, we get

$$a = \frac{1}{2}, \quad c = -\frac{1}{2} \quad \text{and} \quad b = -1$$

Substituting these values in Eq. (i), we get

$$f = k(F)^{1/2}(l)^{-1}(\mu)^{-1/2} \quad \text{or} \quad f = \frac{k}{l}\sqrt{\frac{F}{\mu}}$$

Experimentally, the value of k is found to be $\frac{1}{2}$.

Hence, $$f = \frac{1}{2l}\sqrt{\frac{F}{\mu}}$$

Limitations of Dimensional Analysis

The method of dimensions has the following limitations :

(i) By this method the value of dimensionless constant cannot be calculated.

(ii) By this method the equation containing trigonometrical, exponential and logarithmic terms cannot be analysed.

(iii) If a physical quantity depends on more than three factors, then relation among them cannot be established because we can have only three equations by equalising the powers of M, L and T.

2.2 Vector and Scalar Quantities

Any physical quantity is either a scalar or a vector. A scalar quantity can be described completely by its magnitude only. Addition, subtraction, division or multiplication of scalar quantities can be done according to the ordinary rules of algebra. Mass, volume, density, etc., are few examples of scalar quantities. If a physical quantity in addition to magnitude has a specified direction as well as obeys the law of parallelogram of addition, then and then only it is said to be a vector quantity. Displacement, velocity, acceleration, etc., are few examples of vectors. Any vector quantity should have a specified direction but it is not a sufficient condition for a quantity to be a vector. For example, current flowing in a wire is shown by a direction but it is not a vector because it does not obey the law of parallelogram of vector addition. For example, in the figure shown here.

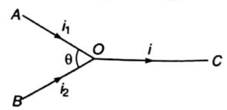


Fig. 2.1

Current flowing in wire OC = current through AO + current through BO

or $$i = i_1 + i_2$$

It would had been $$i^2 = i_1^2 + i_2^2 + 2i_1 i_2 \cos\theta$$

In case, the current would had been a vector quantity.

2.3 General Points Regarding Vectors

Vector Notation

Usually a vector is represented by a bold capital letter with an arrow over it, as $\vec{\mathbf{A}}, \vec{\mathbf{B}}, \vec{\mathbf{C}}$, etc. The magnitude of a vector $\vec{\mathbf{A}}$ is represented by A or $|\vec{\mathbf{A}}|$.

Graphical Representation of a Vector

Graphically a vector is represented by an arrow drawn to a chosen scale, parallel to the direction of the vector. The length and the direction of the arrow thus represent the magnitude and the direction of the vector respectively.

Thus, the arrow in Fig. 2.2 represents a vector $\vec{A}$ in xy-plane making an angle θ with x-axis.

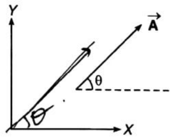


Fig. 2.2

Angle between two Vectors (θ)

To find angle between two vectors both the vectors are drawn from a point in such a manner that arrows of both the vectors are outwards from that point. Now, the smaller angle is called the angle between two vectors.

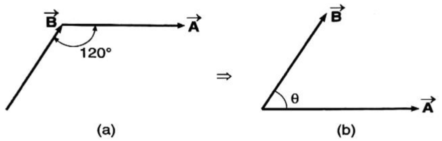


Fig. 2.3

For example in Fig. 2.3 angle between $\vec{A}$ and $\vec{B}$ is 60° not 120°. Because in figure (a) they are wrongly drawn while in figure (b) they are drawn as we desire.

Unit and Zero Vector

A vector of unit magnitude is called a unit vector and the notation for it in the direction of $\vec{A}$ is $\hat{A}$ read as 'A hat or A caret'.

Thus, $\vec{A} = A\hat{A}$

A unit vector merely indicates a direction. Unit vector along x, y and z-directions are $\hat{i}$, $\hat{j}$ and $\hat{k}$.

A vector of zero magnitude is called a zero or a null vector. Its direction is arbitrary.

Negative of a Vector

Means a vector of same magnitude but opposite in direction.

Multiplication and Division of Vectors by Scalars

The product of a vector $\vec{A}$ and a scalar m is a vector $m\vec{A}$ whose magnitude is m times the magnitude of $\vec{A}$ and which is in the direction or opposite to $\vec{A}$ according as the scalar m is positive or negative. Thus,

$$|m\vec{A}| = mA$$

Further, if m and n are two scalars, then

$$(m+n)\vec{A} = m\vec{A} + n\vec{A} \quad \text{and} \quad m(n\vec{A}) = n(m\vec{A}) = (mn)\vec{A}$$

The division of vector $\vec{A}$ by a non-zero scalar m is defined as the multiplication of $\vec{A}$ by $\frac{1}{m}$.

Equality of Vectors

All vectors with the same magnitude and direction are equal despite their different locations in space. Thus, if a vector is displaced parallel to itself, it does not change.

In Fig. 2.4, $\vec{A}$, $\vec{B}$ and $\vec{C}$ are all equal, since they have the same magnitude and direction even though they are differently located in space.

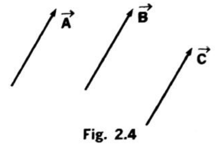


Fig. 2.4

2.4 Addition and Subtraction of Two Vectors

Addition

(i) **The Parallelogram law** : Let $\vec{R}$ be the resultant of two vectors $\vec{A}$ and $\vec{B}$. According to parallelogram law of vector addition, the resultant $\vec{R}$ is the diagonal of the parallelogram of which $\vec{A}$ and $\vec{B}$ are the adjacent sides as shown in figure. Magnitude of $\vec{R}$ is given by

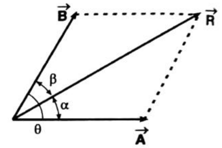


Fig. 2.5

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta} \qquad \text{...(i)}$$

Here, θ = angle between $\vec{A}$ and $\vec{B}$. The direction of $\vec{R}$ can be found by angle α or β of $\vec{R}$ with $\vec{A}$ or $\vec{B}$.

Here, $$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

and $$\tan\beta = \frac{A\sin\theta}{B + A\cos\theta} \qquad \text{...(ii)}$$

Special cases

If $\theta = 0°$, R = maximum $= A + B$

$\theta = 180°$, R = minimum $= A \sim B$

and if $\theta = 90°$, $R = \sqrt{A^2 + B^2}$

In all other cases magnitude and direction of $\vec{R}$ can be calculated by using Eqs. (i) and (ii).

(ii) **The Triangle Law** : According to this law if the tail of one vector be placed at the head of the other, their sum or resultant $\vec{R}$ is drawn from the tail end of the first to the head end of the other.

As is evident from the figure that the resultant $\vec{R}$ is the same irrespective of the order in which the vectors $\vec{A}$ and $\vec{B}$ are taken, Thus, $\vec{R} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$

Fig. 2.6

Subtraction

Negative of a vector say $-\vec{A}$ is a vector of the same magnitude as vector $\vec{A}$ but pointing in a direction opposite to that of $\vec{A}$.

Fig. 2.7

Thus, $\vec{A}-\vec{B}$ can be written as $\vec{A}+(-\vec{B})$ or $\vec{A}-\vec{B}$ is really the vector addition of $\vec{A}$ and $-\vec{B}$.

Suppose angle between two vectors $\vec{A}$ and $\vec{B}$ is θ. Then angle between $\vec{A}$ and $-\vec{B}$ will be $180-\theta$ as shown in Fig. 2.8(b).

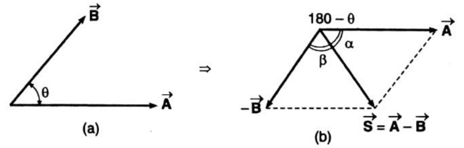


Fig. 2.8

Magnitude of $\vec{S}=\vec{A}-\vec{B}$ will be thus given by

$$S=|\vec{A}-\vec{B}|=\sqrt{A^2+B^2+2AB\cos(180-\theta)}$$

or
$$S=\sqrt{A^2+B^2-2AB\cos\theta} \qquad \text{...(i)}$$

For direction of $\vec{S}$ we will either calculate angle α or β, where,

$$\tan\alpha=\frac{B\sin(180-\theta)}{A+B\cos(180-\theta)}=\frac{B\sin\theta}{A-B\cos\theta} \qquad \text{...(ii)}$$

or
$$\tan\beta=\frac{A\sin(180-\theta)}{B+A\cos(180-\theta)}=\frac{A\sin\theta}{B-A\cos\theta} \qquad \text{...(iii)}$$

Note $\vec{A}-\vec{B}$ *or* $\vec{B}-\vec{A}$ *can also be found by making triangles as shown in Fig. 2.9 (a) and (b).*

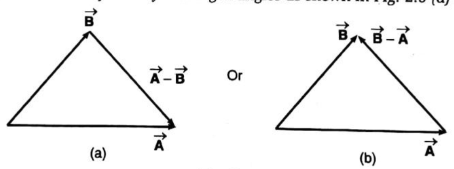


Fig. 2.9

Sample Example 2.6 *Find $\vec{A}+\vec{B}$ and $\vec{A}-\vec{B}$ in the diagram shown in figure. Given $A=4$ units and $B=3$ units.*

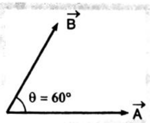


Fig. 2.10

Solution Addition :

$$R=\sqrt{A^2+B^2+2AB\cos\theta}$$
$$=\sqrt{16+9+2\times4\times3\cos60°}=\sqrt{37}\text{ units}$$

$$\tan\alpha=\frac{B\sin\theta}{A+B\cos\theta}$$
$$=\frac{3\sin60°}{4+3\cos60°}=0.472$$

$\therefore$
$$\alpha=\tan^{-1}(0.472)=25.3°$$

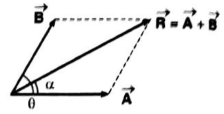


Fig. 2.11

Thus, resultant of $\vec{A}$ and $\vec{B}$ is $\sqrt{37}$ units at angle 25.3° from $\vec{A}$ in the direction shown in figure.

Subtraction :

$$S=\sqrt{A^2+B^2-2AB\cos\theta}$$
$$=\sqrt{16+9-2\times4\times3\cos60°}$$
$$=\sqrt{13}\text{ units}$$

and
$$\tan\alpha=\frac{B\sin\theta}{A-B\cos\theta}$$
$$=\frac{3\sin60°}{4-3\cos60°}=1.04$$

$\therefore$
$$\alpha=\tan^{-1}(1.04)=46.1°$$

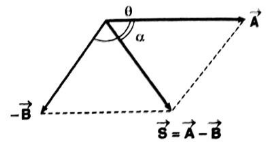


Fig. 2.12

Thus, $\vec{A}-\vec{B}$ is $\sqrt{13}$ units at 46.1° from $\vec{A}$ in the direction shown in figure.

Polygon Law of Vector Addition for more than two Vectors

This law states that if a vector polygon be drawn, placing the tail end of each succeeding vector at the head or the arrow end of the preceding one their resultant $\vec{R}$ is drawn from the tail end of the first to the head or the arrow end of the last.

Thus, in the figure $\vec{R}=\vec{A}+\vec{B}+\vec{C}$

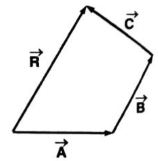


Fig. 2.13

2.5 Components of a Vector

Two or more vectors which, when compounded in accordance with the parallelogram law of vector $\vec{R}$ are said to be components of vector $\vec{R}$. The most important components with which we are concerned are mutually perpendicular or rectangular ones along the three co-ordinate axes *ox*, *oy* and *oz* respectively. Thus, a vector $\vec{R}$ can be written as $\vec{R}=R_x\,\hat{i}+R_y\,\hat{j}+R_z\,\hat{k}$.

Here, R_x, R_y and R_z are the components of $\vec{R}$ in *x*, *y* and *z*-axes respectively and $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along these directions. The magnitude of $\vec{R}$ is given by

$$R=\sqrt{R_x^2+R_y^2+R_z^2}$$

This vector $\vec{R}$ makes an angle of $\alpha=\cos^{-1}\left(\frac{R_x}{R}\right)$ with *x*-axis

$$\beta=\cos^{-1}\left(\frac{R_y}{R}\right)\text{ with } y\text{-axis}$$

and $\gamma = \cos^{-1}\left(\frac{R_z}{R}\right)$ with z-axis

Refer figure (a)

We have resolved a two dimensional vector (in xy plane) $\vec{\mathbf{R}}$ in mutually perpendicular directions x and y.

Component along x-axis $= R_x = R\cos\alpha$ or $R\sin\beta$ and component along y-axis $= R_y = R\cos\beta$ or $R\sin\alpha$.

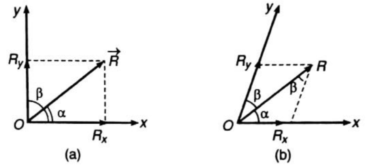


Fig. 2.14

If $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ be the unit vectors along x and y axes respectively, we can write

$$\vec{\mathbf{R}} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$$

Refer figure (b)

Vector $\vec{\mathbf{R}}$ has been resolved in two axes x and y not perpendicular to each other. Applying sine law in the triangle shown, we have

$$\frac{R}{\sin[180-(\alpha+\beta)]} = \frac{R_x}{\sin\beta} = \frac{R_y}{\sin\alpha}$$

or $$R_x = \frac{R\sin\beta}{\sin(\alpha+\beta)} \quad \text{and} \quad R_y = \frac{R\sin\alpha}{\sin(\alpha+\beta)}$$

If $\alpha+\beta = 90°$, $R_x = R\sin\beta$ and $R_y = R\sin\alpha$

Sample Example 2.7 *Resolve a weight of 10 N in two directions which are parallel and perpendicular to a slope inclined at 30° to the horizontal.*

Solution Component perpendicular to the plane

$$W_\perp = W\cos 30° = (10)\frac{\sqrt{3}}{2} = 5\sqrt{3}\text{ N}$$

and Component parallel to the plane

$$W_{||} = W\sin 30° = (10)\left(\frac{1}{2}\right) = 5\text{ N}$$

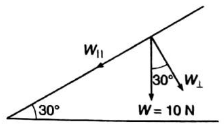


Fig. 2.15

Sample Example 2.8 *Resolve horizontally and vertically a force F = 8 N which makes an angle of 45° with the horizontal.*

Solution Horizontal component of $\vec{\mathbf{F}}$ is

$$F_H = F\cos 45° = (8)\left(\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}\text{ N}$$

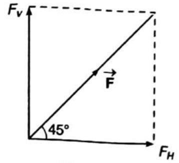


Fig. 2.16

and vertical component of $\vec{\mathbf{F}}$ is

$$F_V = F\sin 45° = (8)\left(\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}\text{ N}$$

Two vectors in the form of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ can be added, subtracted on multiplied by a scalar directly as done in the following example.

Sample Example 2.9 *Obtain the magnitude of* $2\vec{\mathbf{A}} - 3\vec{\mathbf{B}}$ *if*

$$\vec{\mathbf{A}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}} \quad \textit{and} \quad \vec{\mathbf{B}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}.$$

Solution

$$2\vec{\mathbf{A}} - 3\vec{\mathbf{B}} = 2(\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) - 3(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$$
$$= -4\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$$

$\therefore$ Magnitude of $2\vec{\mathbf{A}} - 3\vec{\mathbf{B}} = \sqrt{(-4)^2 + (5)^2 + (-7)^2} = \sqrt{16+25+49} = \sqrt{90}$

2.6 Product of Two Vectors

The product of two vectors is of two kinds.

(i) a scalar or dot product.

(ii) a vector or a cross product.

Scalar or Dot Product

The scalar or dot product of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is denoted by $\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}$ and is read as $\vec{\mathbf{A}}$ dot $\vec{\mathbf{B}}$.

It is defined as the product of the magnitudes of the two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ and the cosine of their included angle θ

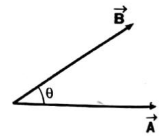


Fig. 2.17

Thus, $\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = AB\cos\theta$ (a scalar quantity)

Important Points Regarding Dot Product

The following points should be remembered regarding the dot product.

(i) $\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = \vec{\mathbf{B}}\cdot\vec{\mathbf{A}}$

(ii) $\vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}+\vec{\mathbf{C}}) = \vec{\mathbf{A}}\cdot\vec{\mathbf{B}} + \vec{\mathbf{A}}\cdot\vec{\mathbf{C}}$

(iii) $\vec{\mathbf{A}}\cdot\vec{\mathbf{A}} = A^2$

(iv) $\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = A(B\cos\theta) = A$ (Component of $\vec{\mathbf{B}}$ along $\vec{\mathbf{A}}$)

or $\vec{\mathbf{A}}\cdot\vec{\mathbf{B}} = B(A\cos\theta) = B$ (Component of $\vec{\mathbf{A}}$ along $\vec{\mathbf{B}}$)

(v) $\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = (1)(1)\cos 0° = 1$

(vi) $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = \hat{\mathbf{i}}\cdot\hat{\mathbf{k}} = (1)(1)\cos 90° = 0$

(vii) $(a_1\hat{\mathbf{i}} + b_1\hat{\mathbf{j}} + c_1\hat{\mathbf{k}})\cdot(a_2\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + c_2\hat{\mathbf{k}}) = a_1a_2 + b_1b_2 + c_1c_2$

(viii) $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{AB}$ (cosine of angle between $\vec{A}$ and $\vec{B}$)

(ix) Two vectors are perpendicular if their dot product is zero. ($\theta = 90°$)

Sample Example 2.10 *Work done by a force $\vec{F}$ on a body is $W = \vec{F}\cdot\vec{s}$, where $\vec{s}$ is the displacement of body. Given that under a force $\vec{F} = (2\hat{i} + 3\hat{j} + 4\hat{k})N$ a body is displaced from position vector $\vec{r}_1 = (2\hat{i} + 3\hat{j} + \hat{k})$ m to the position vector $\vec{r}_2 = (\hat{i} + \hat{j} + \hat{k})$ m. Find the work done by this force.*

Solution The body is displaced from $\vec{r}_1$ to $\vec{r}_2$. Therefore, displacement of the body is

$$\vec{s} = \vec{r}_2 - \vec{r}_1 = (\hat{i} + \hat{j} + \hat{k}) - (2\hat{i} + 3\hat{j} + \hat{k}) = (-\hat{i} - 2\hat{j})\text{ m}$$

Now, work done by the force is $W = \vec{F}\cdot\vec{s}$

$$= (2\hat{i} + 3\hat{j} + 4\hat{k})\cdot(-\hat{i} - 2\hat{j})$$

$$= (2)(-1) + (3)(-2) = -8\text{ J}$$

Sample Example 2.11 *Find the angle between two vectors $\vec{A} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{B} = \hat{i} - \hat{k}$.*

Solution $A = |\vec{A}| = \sqrt{(2)^2 + (1)^2\ (-1)^2} = \sqrt{6}$

$B = |\vec{B}| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$

$\vec{A}\cdot\vec{B} = (2\hat{i} + \hat{j} - \hat{k})\cdot(\hat{i} - \hat{k}) = (2)(1) + (-1)(-1) = 3$

Now, $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{AB} = \dfrac{3}{\sqrt{6}\cdot\sqrt{2}} = \dfrac{3}{\sqrt{12}} = \dfrac{\sqrt{3}}{2}$

$\therefore$ $\theta = 30°$

Sample Example 2.12 *Prove that the vectors $\vec{A} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{B} = \hat{i} + \hat{j} + \hat{k}$ are mutually perpendicular.*

Solution $\vec{A}\cdot\vec{B} = (2\hat{i} - 3\hat{j} + \hat{k})\cdot(\hat{i} + \hat{j} + \hat{k})$

$= (2)(1) + (-3)(1) + (1)(1)$

$= 0 = AB\cos\theta$

$\therefore$ $\cos\theta = 0$ (as $A \neq 0, B \neq 0$)

or $\theta = 90°$

or the vectors $\vec{A}$ and $\vec{B}$ are mutually perpendicular.

Vector or Cross Product

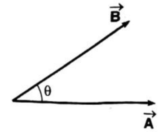
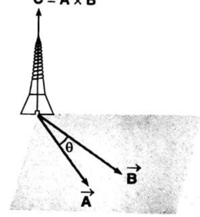


Fig. 2.18

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\times\vec{B}$ and read as $\vec{A}$ cross $\vec{B}$. It is defined as a third vector $\vec{C}$ whose magnitude is equal to the product of the magnitudes of the two vectors $\vec{A}$ and $\vec{B}$ and the sine of their included angle θ.

Thus, if $\vec{C} = \vec{A}\times\vec{B}$, then $C = AB\sin\theta$.

The vector $\vec{C}$ is normal to the plane of $\vec{A}$ and $\vec{B}$ and points in the direction in which a right handed screw would advance when rotated about an axis perpendicular to the plane of the two vectors in the direction from $\vec{A}$ to $\vec{B}$ through the smaller angle θ between them or, alternatively, we might state the rule as :



Plane of $\vec{A}$ and $\vec{B}$
Fig. 2.19

If the fingers of the right hand be curled in the direction in which vector $\vec{A}$ must be turned through the smaller included angle θ to coincide with the direction of vector $\vec{B}$, the thumb points in the direction of $\vec{C}$ as shown in Fig. 2.19.

Either of these rules is referred to as the right handed screw rule. Thus, if $\hat{n}$ be the unit vector in the direction of $\vec{C}$, we have

$$\vec{C} = \vec{A}\times\vec{B} = AB\sin\theta\ \hat{n}$$

where $0 \le \theta \le \pi$

Important Points About Vector Product

(i) $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

(ii) The cross product of two parallel vectors is zero, as $|\vec{A}\times\vec{B}| = AB\sin\theta$ and $\theta = 0°$ for two parallel vectors. Thus, $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

(iii) If two vectors are perpendicular to each other, we have $\theta = 90°$ and therefore, $\sin\theta = 1$. So that $\vec{A}\times\vec{B} = AB\ \hat{n}$. The vectors $\vec{A}$, $\vec{B}$ and $\vec{A}\times\vec{B}$ thus form a right handed system of mutually perpendicular vectors. It follows at once from the above that in case of the orthogonal triad of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ (each perpendicular to each other)

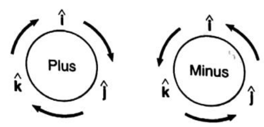


Fig. 2.20

$$\hat{i}\times\hat{j} = -\hat{j}\times\hat{i} = \hat{k}$$

$$\hat{j}\times\hat{k} = -\hat{k}\times\hat{j} = \hat{i} \quad\text{and}\quad \hat{k}\times\hat{i} = -\hat{i}\times\hat{k} = \hat{j}$$

(iv) $\vec{A}\times(\vec{B} + \vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$

(v) A vector product can be expressed in terms of rectangular components of the two vectors and put in the determinant form as may be seen from the following:

Let $\vec{A} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

and $\vec{B} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

Then, $\vec{A}\times\vec{B}=(a_1\hat{i}+b_1\hat{j}+c_1\hat{k})\times(a_2\hat{i}+b_2\hat{j}+c_2\hat{k})$

$$=a_1a_2(\hat{i}\times\hat{i})+a_1b_2(\hat{i}\times\hat{j})+a_1c_2(\hat{i}\times\hat{k})+b_1a_2(\hat{j}\times\hat{i})+b_1b_2(\hat{j}\times\hat{j})$$
$$+b_1c_2(\hat{j}\times\hat{k})+c_1a_2(\hat{k}\times\hat{i})+c_1b_2(\hat{k}\times\hat{j})+c_1c_2(\hat{k}\times\hat{k})$$

Since, $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$ and $\hat{i}\times\hat{j}=\hat{k}$, etc., we have

$$\vec{A}\times\vec{B}=(b_1c_2-c_1b_2)\hat{i}+(c_1a_2-a_1c_2)\hat{j}+(a_1b_2-b_1a_2)\hat{k}$$

or putting it in determinant form, we have

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$$

It may be noted that the scalar components of the first vector $\vec{A}$ occupy the middle row of the determinant.

Sample Example 2.13 *Find a unit vector perpendicular to $\vec{A}=2\hat{i}+3\hat{j}+\hat{k}$ and $\vec{B}=\hat{i}-\hat{j}+\hat{k}$ both.*

Solution As we have read, $\vec{C}=\vec{A}\times\vec{B}$ is a vector perpendicular to both $\vec{A}$ and $\vec{B}$. Hence, a unit vector $\hat{n}$ perpendicular to $\vec{A}$ and $\vec{B}$ can be written as

$$\hat{n}=\frac{\vec{C}}{C}=\frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$$

Here, $$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 1\\ 1 & -1 & 1\end{vmatrix}$$

$$=\hat{i}(3+1)+\hat{j}(1-2)+\hat{k}(-2-3)=4\hat{i}-\hat{j}-5\hat{k}$$

Further, $$|\vec{A}\times\vec{B}|=\sqrt{(4)^2+(-1)^2+(-5)^2}=\sqrt{42}$$

∴ The desired unit vector is:

$$\hat{n}=\frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}\quad\text{or}\quad\hat{n}=\frac{1}{\sqrt{42}}(4\hat{i}-\hat{j}-5\hat{k})$$

Sample Example 2.14 *Show that the vector $\vec{A}=\hat{i}-\hat{j}+2\hat{k}$ is parallel to a vector $\vec{B}=3\hat{i}-3\hat{j}+6\hat{k}$.*

Solution A vector $\vec{A}$ is parallel to an another vector $\vec{B}$ if it can be written as

$$\vec{A}=m\vec{B}$$

Here, $$\vec{A}=(\hat{i}-\hat{j}+2\hat{k})=\frac{1}{3}(3\hat{i}-3\hat{j}+6\hat{k})$$

or $$\vec{A}=\frac{1}{3}\vec{B}$$

This implies that $\vec{A}$ is parallel to $\vec{B}$ and magnitude of $\vec{A}$ is $\frac{1}{3}$ times the magnitude of $\vec{B}$.

Note *Two vectors can be shown parallel to one another if*

(i) The coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ of both the vectors bear a constant ratio. For example, a vector $\vec{A}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ is parallel to an another vector $\vec{B}=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$ if: $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

(ii) The cross product of both the vectors is zero. For instance, $\vec{A}$ and $\vec{B}$ are parallel to each other if

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}=0$$

Sample Example 2.15 *Let a force $\vec{F}$ be acting on a body free to rotate about a point O and let $\vec{r}$ the position vector of any point P on the line of action of the force. Then torque ($\vec{\tau}$) of this force about point O is defined as:*

$$\vec{\tau}=\vec{r}\times\vec{F}$$

Given, $\vec{F}=(2\hat{i}+3\hat{j}-\hat{k})N$ *and* $\vec{r}=(\hat{i}-\hat{j}+6\hat{k})m$

Find the torque of this force.

Solution $$\vec{\tau}=\vec{r}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 6\\ 2 & 3 & -1\end{vmatrix}$$

$$=\hat{i}(1-18)+\hat{j}(12+1)+\hat{k}(3+2)$$

or $$\vec{\tau}=(-17\hat{i}+13\hat{j}+5\hat{k})\text{N-m}$$

Extra Points

- Pressure, surface tension and current are not vectors.
- To qualify as a vector, a physical quantity must not only possess magnitude and direction but must also satisfy the parallelogram law of vector addition. For instance, the finite rotation of a rigid body about a given axis has magnitude (the angle of rotation) and also direction (the direction of the axis) but it is not a vector quantity. This is so for the simple reason that the two finite rotations of the body do not add up in accordance with the law of vector addition. However if the rotation be small or infinitesimal, it may be regarded as a vector quantity.
- Area can behave either as a scalar or a vector and how it behaves depends on circumstances.
- Moment of inertia is neither a vector nor a scalar as it has different values about different axes. It is tensor. Although tensor is a generalised term which is characterized by its rank. For example scalars are tensors of rank zero. Vectors are tensors of rank one.
- Area (vector), dipole moment and current density are defined as vectors with specific direction.
- Vectors associated with a linear or directional effect are called polar vectors or, usually, simply as vectors and those associated with rotation about an axis are referred to as axial vectors. Thus force, linear velocity and acceleration area polar vectors and angular velocity, angular acceleration are axial vectors.

- Examples of Dot-product and Examples of Cross-product

Examples of Dot-product	**Examples of Cross-product**
$W = \vec{F} \cdot \vec{s}$	$\vec{\tau} = \vec{r} \times \vec{F}$
$P = \vec{F} \cdot \vec{v}$	$\vec{L} = \vec{r} \times \vec{P}$
$d\phi_e = \vec{E} \cdot \overrightarrow{ds}$	$\vec{v} = \vec{\omega} \times \vec{r}$
$d\phi_B = \vec{B} \cdot \vec{ds}$	$\vec{\tau}_e = \vec{P} \times \vec{E}$
$U_e = \vec{P} \cdot \vec{E}$	$\vec{\tau}_B = \vec{M} \times \vec{B}$
$U_B = \vec{M} \cdot \vec{B}$	$\vec{F}_B = q\,(\vec{v} \times \vec{B})$
	$\vec{dB} = \dfrac{\mu_0}{4\pi} \dfrac{i\,(\vec{dl} \times \vec{r})}{r^3}$

- Students are often confused over the direction of cross product. Let us discuss a simple method. To find direction of $\vec{A} \times \vec{B}$ move from $\vec{A}$ to $\vec{B}$ through smaller angle. If it is clockwise then $\vec{A} \times \vec{B}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and away from you and if it is anti-clockwise then $\vec{A} \times \vec{B}$ is towards you perpendicular to the plane of $\vec{A}$ and $\vec{B}$.
- The area of triangle bounded by vectors $\vec{A}$ and $\vec{B}$ is $\frac{1}{2}|\vec{A} \times \vec{B}|$.

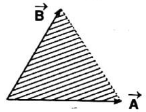


Fig. 2.21

Exercise : Prove the above result.

- **Scalar triple product :** $\vec{A} \cdot (\vec{B} \times \vec{C})$ is called scalar triple product. It is a scalar quantity. We can show that $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{B} \cdot (\vec{C} \times \vec{A})$.
- The volume of a parallelopiped bounded by vectors $\vec{A}, \vec{B}$ and $\vec{C}$ can be obtained by $(\vec{A} \times \vec{B}) \cdot \vec{C}$.
- If three vectors are coplanar then the volume of the parallelopiped bounded by these three vectors should be zero or we can say their scalar triple product should be zero.
- If $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{C} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then $\vec{A} \cdot (\vec{B} \times \vec{C})$ is also written as $[\vec{A}\,\vec{B}\,\vec{C}]$ and it has the following value,

$$[\vec{A}\,\vec{B}\,\vec{C}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

= Volume of parallelopiped whose adjacent sides are along $\vec{A}, \vec{B}$ and $\vec{C}$.

- Properties of scalar triple product :

(a) $[\vec{A}\,\vec{B}\,\vec{C}] = [\vec{B}\,\vec{C}\,\vec{A}] = [\vec{C}\,\vec{A}\,\vec{B}]$

(b) $[\vec{A}\,\vec{B}\,\vec{C}] = -[\vec{B}\,\vec{A}\,\vec{C}]$

(c) If vectors are coplanar, then $[\vec{A}\,\vec{B}\,\vec{C}] = 0$

- Area of triangle ABC if position vector of A is $\vec{a}$ position vector of B is $\vec{b}$ and position vector of C is $\vec{c}$

$$\text{Area} = \frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

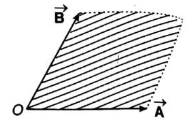


Fig. 2.22

Exercise : Prove the above relation.

- Area of parallelogram shown in figure is, Area $= |\vec{A} \times \vec{B}|$

Exercise : Prove the above relation.

- If co-ordinates of point A are (x_1, y_1, z_1) and B are (x_2, y_2, z_2). Then, position vector of A

$$= \vec{r}_A = \overrightarrow{OA} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

position vector of $B = \vec{r}_B = \overrightarrow{OB} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

and $\quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{r}_B - \vec{r}_A$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

- Suppose $\vec{A}$ and $\vec{B}$ represent the two adjacent sides of a parallelogram $OPQR$, then,

Diagonal $OQ = |\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

while diagonal $RP = |\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

We can see that $OQ = RP$ when $\theta = 90°$.

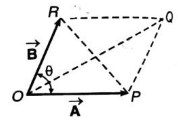


Fig. 2.23

- $\vec{a} \cdot \vec{b} = ab\cos\theta$. Here, a and b are always positive as these are the magnitudes of $\vec{a}$ and $\vec{b}$. Hence,

	$0° \le \theta < 90°$	if $\vec{a} \cdot \vec{b}$ is positive
	$90° \le \theta < 180°$	if $\vec{a} \cdot \vec{b}$ is negative.
and	$\theta = 90°$	if $\vec{a} \cdot \vec{b}$ is zero.

Solved Examples

Example 1 *Check the correctness of the relation $s = ut + \frac{1}{2}at^2$, where u is initial velocity, a the acceleration, t the time and s the displacement.*

Solution Writing the dimensions of either side of the given equation.

$$\text{LHS} = s = \text{displacement} = [M^0LT^0]$$

$$\text{RHS} = ut = \text{velocity} \times \text{time} = [M^0LT^{-1}][T] = [M^0LT^0]$$

and $$\frac{1}{2}at^2 = (\text{acceleration}) \times (\text{time})^2 = [M^0LT^{-2}][T]^2 = [M^0LT^0]$$

As LHS = RHS, formula is dimensionally correct.

Example 2 *Write the dimensions of a and b in the relation,*

$$P = \frac{b - x^2}{at}$$

where P is power, x is distance and t is time.

Solution The given equation can be written as, $Pat = b - x^2$

Now, $$[Pat] = [b] = [x^2] \quad \text{or} \quad [b] = [x^2] = [M^0L^2T^0]$$

and $$[a] = \frac{[x^2]}{[Pt]} = \frac{[L^2]}{[ML^2T^{-3}][T]} = [M^{-1}L^0T^2]$$

Example 3 *The centripetal force F acting on a particle moving uniformly in a circle may depend upon mass (m), velocity (v) and radius (r) of the circle. Derive the formula for F using the method of dimensions.*

Solution Let $F = k(m)^x (v)^y (r)^z$ ...(i)

Here, k is a dimensionless constant of proportionality. Writing the dimensions of RHS and LHS in Eq. (i), we have

$$[MLT^{-2}] = [M]^x [LT^{-1}]^y [L]^z = [M^x L^{y+z} T^{-y}]$$

Equation the powers of M, L and T of both sides, we have,

$$x = 1, \quad y = 2 \quad \text{and} \quad y + z = 1 \quad \text{or} \quad z = 1 - y = -1$$

Putting the values in Eq. (i), we get

$$F = kmv^2r^{-1} = k\frac{mv^2}{r}$$

$$F = \frac{mv^2}{r} \qquad (\text{where } k = 1)$$

Example 4 *Discuss why an infinitesimal displacement is regarded as a vector quantity.*

Solution Infinitesimal rotations may be regarded as vector quantity because the arc described by the body in a small interval of time is more or less a straight line and is thus representable as a vector.

Note *In some of the books it is written that large angular displacement does not obey the law of commutation ($\vec{\theta}_1 + \vec{\theta}_2 \neq \vec{\theta}_2 + \vec{\theta}_1$) so they cannot be treated as vectors while infinitesimal do. So, they are vectors. As per my opinion the sufficient and necessary condition to be a vector is that is should obey the law of parallelogram of vector. If this is satisfied the law of commutation is automatically satisfied. It can be explained and proved in a better way but that require three dimensional treatment. Which I feel is not necessary for any competition.*

Example 5 *Find component of vector $\vec{A} + \vec{B}$ along (i) x-axis, (ii) $\vec{C}$*

Given $\vec{A} = \hat{i} - 2\hat{j}$, $\vec{B} = 2\hat{i} + 3\hat{k}$ *and* $\vec{C} = \hat{i} + \hat{j}$.

Solution $\vec{A} + \vec{B} = (\hat{i} - 2\hat{j}) + (2\hat{i} + 3\hat{k}) = 3\hat{i} - 2\hat{j} + 3\hat{k}$

(i) Component of $\vec{A} + \vec{B}$ along x-axis is 3.

(ii) Component of $\vec{A} + \vec{B} = \vec{R}$ (say) along $\vec{C}$ is

$$R\cos\theta = \frac{\vec{R} \cdot \vec{C}}{C} = \frac{(3\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (\hat{i} + \hat{j})}{\sqrt{(1)^2 + (1)^2}} = \frac{3-2}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Example 6 *Find the angle that the vector $\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$ makes with y-axis.*

Solution

$$\cos\theta = \frac{A_y}{A} = \frac{3}{\sqrt{(2)^2 + (3)^2 + (-1)^2}} = \frac{3}{\sqrt{14}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$$

Example 7 *Explain why pressure and surface tension are not vectors?*

Solution Pressure (force per unit area normal to it) or surface tension are scalars. They have direction which is unique so need not to be specified.

Example 8 *If $\vec{a}$ and $\vec{b}$ are the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ determined by the adjacent sides of a regular hexagon. What are the vectors determined by the other sides taken in order?*

Solution Given $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$

From the method of vector addition (or subtraction) we can show that,

$$\overrightarrow{CD} = \vec{b} - \vec{a}$$

Then $$\overrightarrow{DE} = -\overrightarrow{AB} = -\vec{a}$$

$$\overrightarrow{EF} = -\overrightarrow{BC} = -\vec{b}$$

and $$\overrightarrow{FA} = -\overrightarrow{CD} = \vec{a} - \vec{b}$$

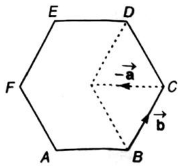


Fig. 2.24

Example 9 *If $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}\neq\vec{0}$ with $\vec{a}\neq-\vec{c}$ then show that $\vec{a}+\vec{c}=k\vec{b}$ where k is scalar.*

Solution
$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}$$
$$\vec{a}\times\vec{b}=-\vec{c}\times\vec{b}$$
$$\therefore\quad \vec{a}\times\vec{b}+\vec{c}\times\vec{b}=0$$
$$(\vec{a}+\vec{c})\times\vec{b}=0$$

$\therefore\ \vec{a}\times\vec{b}\neq 0,\ \vec{b}\times\vec{c}\neq 0,\ \vec{a},\vec{b},\vec{c}$ are non-zero vectors. $(\vec{a}+\vec{c})\neq\vec{0}$

Hence, $\vec{a}+\vec{c}$ is parallel to $\vec{b}$.
$$\therefore\quad \vec{a}+\vec{c}=k\vec{b}$$

Example 10 *If $\vec{A}=2\hat{i}-3\hat{j}+7\hat{k}$, $\vec{B}=\hat{i}+2\hat{j}$ and $\vec{C}=\hat{j}-\hat{k}$. Find $\vec{A}\cdot(\vec{B}\times\vec{C})$.*

Solution $\vec{A}\cdot(\vec{B}\times\vec{C})=[\vec{A}\,\vec{B}\,\vec{C}]$, volume of parallelopiped
$$=\begin{vmatrix}2 & -3 & 7\\ 1 & 2 & 0\\ 0 & 1 & -1\end{vmatrix}$$
$$=2(-2-0)+3(-1-0)+7(1-0)$$
$$=-4-3+7=0$$

Example 11 *Find the resultant of three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ shown in figure. Radius of circle is 'R'.*

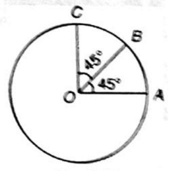


Fig. 2.25

Solution $OA=OC$

$\overrightarrow{OA}+\overrightarrow{OC}$ is along $\overrightarrow{OB}$, (bisector) and its magnitude is $2R\cos 45°=R\sqrt{2}$

$(\overrightarrow{OA}+\overrightarrow{OC})+\overrightarrow{OB}$ is along $\overrightarrow{OB}$ and its magnitude is $R\sqrt{2}+R=R(1+\sqrt{2})$

Example 12 *Prove that*
$$|\vec{a}\times\vec{b}|^2=a^2b^2-(\vec{a}\cdot\vec{b})^2$$

Solution Let $|\vec{a}|=a,\ |\vec{b}|=b$

and θ be the angle between them.

$$|\vec{a}\times\vec{b}|^2=(ab\sin\theta)^2=a^2b^2\sin^2\theta$$
$$=a^2b^2(1-\cos^2\theta)$$
$$=a^2b^2-(a\cdot b\cos\theta)^2$$
$$=a^2b^2-(\vec{a}\cdot\vec{b})^2$$

Proved.

Example 13 *Show that the vectors $\vec{a}=3\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-3\hat{j}+5\hat{k}$ and $\vec{c}=2\hat{i}+\hat{j}-4\hat{k}$ form a right angled triangle.*

Solution We have $\vec{b}+\vec{c}=(\hat{i}-3\hat{j}+5\hat{k})+(2\hat{i}+\hat{j}-4\hat{k})=3\hat{i}-2\hat{j}+\hat{k}=\vec{a}$

Hence, $\vec{a},\vec{b},\vec{c}$ are coplanar.

Also, we observe that no two of these vectors are parallel, therefore, the given vectors form a triangle.

Further, $\vec{a}\cdot\vec{c}=(3\hat{i}-2\hat{j}+\hat{k})\cdot(2\hat{i}+\hat{j}-4\hat{k})=0$

Dot product of two non-zero vectors is zero. Hence, they are perpendicular so they form a right angled triangle.

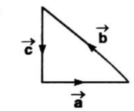


Fig. 2.26

$$|\vec{a}|=\sqrt{9+4+1}=\sqrt{14},$$
$$|\vec{b}|=\sqrt{1+9+25}=\sqrt{35}$$
and
$$|\vec{c}|=\sqrt{4+1+16}=\sqrt{21}$$

Example 14 *Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ be unit vectors. Suppose that $\vec{A}\cdot\vec{B}=\vec{A}\cdot\vec{C}=0$ and that the angle between $\vec{B}$ and $\vec{C}$ is $\frac{\pi}{6}$ then prove that $\vec{A}=\pm 2(\vec{B}\times\vec{C})$*

Solution Since, $\vec{A}\cdot\vec{B}=0,\ \vec{A}\cdot\vec{C}=0$

Hence, $(\vec{B}+\vec{C})\cdot\vec{A}=0$

So $\vec{A}$ is perpendicular to $(\vec{B}+\vec{C})\cdot\vec{A}$ is a unit vector perpendicular to the plane of vectors $\vec{B}$ and $\vec{C}$.
$$\vec{A}=\frac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|}$$
$$|\vec{B}\times\vec{C}|=|\vec{B}||\vec{C}|\sin\frac{\pi}{6}=1\times1\times\frac{1}{2}=\frac{1}{2}$$
$$\therefore\quad \vec{A}=\frac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|}=\pm 2(\vec{B}\times\vec{C})$$

Example 15 *A particle moves on a given line with a constant speed v. At a certain time it is at a point P on its straight line path. O is fixed point. Show that* $(\overrightarrow{\mathbf{OP}} \times \vec{\mathbf{v}})$ *is independent of the position P.*

Solution Let $\vec{\mathbf{v}} = v\hat{\mathbf{i}}$

$$\overrightarrow{\mathbf{OP}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

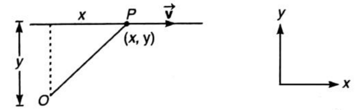


Fig. 2.27

Take $\overrightarrow{\mathbf{OP}} \times \vec{\mathbf{v}} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \times v\hat{\mathbf{i}}$

$= -yv\hat{\mathbf{k}}$ $\quad (\because y \text{ is constant})$

which is independent of position.

Example 16 *Prove that the mid-point of the hypotenuse of right angled triangle is equidistant from its vertices.*

Solution Here, $\angle CAB = 90°$, let D be the mid-point of hypotenuse, we have

$$\overrightarrow{\mathbf{BD}} = \overrightarrow{\mathbf{DC}}$$

$$\overrightarrow{\mathbf{AB}} = \overrightarrow{\mathbf{AD}} + \overrightarrow{\mathbf{DB}}$$

$$\overrightarrow{\mathbf{AC}} = \overrightarrow{\mathbf{AD}} + \overrightarrow{\mathbf{DC}} = \overrightarrow{\mathbf{AD}} + \overrightarrow{\mathbf{BD}} \quad \ldots\text{(i)}$$

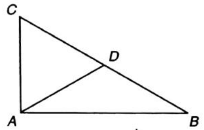


Fig. 2.28

Since, $\angle BAC = 90°$ $\quad \overrightarrow{\mathbf{AB}} \perp \overrightarrow{\mathbf{AC}}$

$$(\overrightarrow{\mathbf{AD}} + \overrightarrow{\mathbf{DB}}) \cdot (\overrightarrow{\mathbf{AD}} + \overrightarrow{\mathbf{BD}}) = 0$$

$$(\overrightarrow{\mathbf{AD}} - \overrightarrow{\mathbf{BD}}) \cdot (\overrightarrow{\mathbf{AD}} + \overrightarrow{\mathbf{BD}}) = 0$$

$$AD^2 - BD^2 = 0$$

$\therefore$ $AD = BD$ also $BD = DC$

$\because$ D is mid-point of BC.

Thus, $|AD| = |BD| = |DC|$. Hence, the result.

EXERCISES

Section-I

Single Correct Option

1. Which one of the following has the dimensions of pressure?
   (a) $[ML^{-2}T^{-2}]$ (b) $[M^{-1}L^{-1}]$ (c) $[MLT^{-2}]$ (d) $[ML^{-1}T^{-2}]$
2. Which of the following will have the dimensions of time
   (a) LC (b) $\frac{R}{L}$ (c) $\frac{L}{R}$ (d) $\frac{C}{L}$
3. The force F on a sphere of radius a moving in a medium with velocity v is given by $F = 6\pi\eta a v$. The dimensions of η are
   (a) $[ML^{-3}]$ (b) $[MLT^{-2}]$ (c) $[MT^{-1}]$ (d) $[ML^{-1}T^{-1}]$
4. The dimensional formula for magnetic flux is
   (a) $[ML^2T^{-2}A^{-1}]$ (b) $[ML^3T^{-2}A^{-2}]$ (c) $[M^0L^{-2}T^{-2}A^{-2}]$ (d) $[ML^2T^{-1}A^2]$
5. Dimensions of linear impulse are
   (a) $[ML^{-2}T^{-3}]$ (b) $[ML^{-2}]$ (c) $[MLT^{-1}]$ (d) $[MLT^{-2}]$
6. What is the dimensional formula of gravitational constant?
   (a) $[ML^2T^{-2}]$ (b) $[ML^{-1}T^{-1}]$ (c) $[M^{-1}L^3T^{-2}]$ (d) None of these
7. Using mass (M), length (L), time (T) and current (A) as fundamental quantities, the dimension of permeability is
   (a) $[M^{-1}LT^{-2}A]$ (b) $[ML^{-2}T^{-2}A^{-1}]$ (c) $[MLT^{-2}A^{-2}]$ (d) $[MLT^{-1}A^{-1}]$
8. The equation of a wave is given by
   $$y = a \sin \omega\left(\frac{x}{v} - k\right)$$
   where ω is angular velocity and v is the linear velocity. The dimensions of k will be
   (a) $[T^2]$ (b) $[T^{-1}]$ (c) $[T]$ (d) $[LT]$
9. A force is given by
   $$F = at + bt^2$$
   where t is the time. The dimensions of a and b are
   (a) $[MLT^{-4}]$ and $[MLT^1]$ (b) $[MLT^{-1}]$ and $[MLT^0]$
   (c) $[MLT^{-3}]$ and $[MLT^{-4}]$ (d) $[MLT^{-3}]$ and $[MLT^0]$

10. The dimensional formula for Planck's constant and angular momentum are
(a) $[ML^2T^{-2}]$ and $[MLT^{-1}]$ (b) $[ML^2T^{-1}]$ and $[ML^2T^{-1}]$
(c) $[ML^3T^1]$ and $[ML^2T^{-2}]$ (d) $[MLT^{-1}]$ and $[MLT^{-2}]$

11. If the energy (E), velocity (v) and force (F) be taken as fundamental quantities, then the dimensions of mass will be
(a) Fv^{-2} (b) Fv^{-1} (c) Ev^{-2} (d) Ev^2

12. The dimension of $\frac{1}{2}\varepsilon_0 E^2$ (ε_0 is the permittivity of free space and E is electric field), is
(a) $[ML^2T^{-1}]$ (b) $[ML^{-1}T^{-2}]$ (c) $[ML^2T^{-2}]$ (d) $[MLT^{-1}]$

13. The dimensions of $\frac{a}{b}$ in the equation $P = \frac{a-t^2}{bx}$, where P is pressure, x is distance and t is time, are
(a) $[M^2LT^{-3}]$ (b) $[MT^{-2}]$ (c) $[LT^{-3}]$ (d) $[ML^3\ T^{-1}]$

14. Dimension of velocity gradient is
(a) $[M^0L^0T^{-1}]$ (b) $[ML^{-1}T^{-1}]$ (c) $[M^0LT^{-1}]$ (d) $[ML^0T^{-1}]$

15. If force F, length L and time T are taken as fundamental units, the dimensional formula for mass will be
(a) $[FL^{-1}T^2]$ (b) $[FLT^{-2}]$ (c) $[FL^{-1}T^{-1}]$ (d) $[FL^5T^2]$

16. Which of the following is the dimension of the coefficient of friction?
(a) $[M^2L^2T]$ (b) $[M^0L^0T^0]$ (c) $[ML^2T^{-2}]$ (d) $[M^2L^2T^{-2}]$

17. If C and R denote capacitance and resistance, then dimensions of CR will be
(a) $[M^0L^0TA^0]$ (b) $[ML^0TA^{-2}]$ (c) $[ML^0TA^2]$ (d) $[ML^0T^2A^{-2}]$

18. The unit of permittivity of free space, ε_0 is
(a) coulomb/newton-metre (b) newton-metre^2/coulomb^2
(c) coulomb^2/newton-metre^2 (d) coulomb^2/(newton-metre)^2

19. The ratio of the dimensions of Planck's constant and that of the moment of inertia is the dimension of
(a) frequency (b) velocity (c) angular momentum (d) time

20. The velocity v of a particle at time t is given by $v = at + \frac{b}{t+c}$, where a, b and c are constants. The dimensions of a, b and c are respectively
(a) $[LT^{-2}]$, $[L]$ and $[T]$ (b) $[L^2]$, $[T]$ and $[LT^2]$
(c) $[LT^2]$, $[LT]$ and $[L]$ (d) $[L]$, $[LT]$ and $[T^2]$

21. Given that $y = A\sin\left[\left(\frac{2\pi}{\lambda}(ct-x)\right)\right]$, where y and x are measured in metres. Which of the following statements is true ?
(a) The unit of λ is same as that of x and A
(b) The unit of λ is same as that of x but not of A
(c) The unit of c is same as that of $\frac{2\pi}{\lambda}$
(d) The unit of $(ct-x)$ is same as that of $\frac{2\pi}{\lambda}$

22. The physical quantity having the dimensions $[M^{-1}L^{-3}T^3A^2]$ is
(a) resistance (b) resistivity
(c) electrical conductivity (d) electromotive force

23. The torque of force $\vec{F} = (2\hat{i} - 3\hat{j} + 4\hat{k})$ newton acting at the point $\vec{r} = (3\hat{i} + 2\hat{j} + 3\hat{k})$ metre about origin is (in N-m)
(a) $6\hat{i} - 6\hat{j} + 12\hat{k}$ (b) $17\hat{i} - 6\hat{j} - 13\hat{k}$ (c) $-6\hat{i} + 6\hat{j} - 12\hat{k}$ (d) $-17\hat{i} + 6\hat{j} + 13\hat{k}$

24. If a unit vector is represented by $0.5\hat{i} + 0.8\hat{j} + c\hat{k}$ the value of c is
(a) 1 (b) $\sqrt{0.11}$ (c) $\sqrt{0.01}$ (d) 0.39

25. If $\vec{A}$ and $\vec{B}$ are two vectors such that $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$ the angle between vectors $\vec{A}$ and $\vec{B}$ is
(a) 0° (b) 60° (c) 90° (d) 120°

26. The vector sum of two forces is perpendicular to their vector differences. In that case, the forces
(a) are not equal to each other in magnitude (b) cannot be predicted
(c) are equal to each other (d) are equal to each other in magnitude

27. Which one of the following is a scalar quantity?
(a) Displacement (b) Electric field (c) Acceleration (d) Work

28. Which one of the following is not the vector quantity?
(a) Torque (b) Displacement (c) Velocity (d) Speed

29. What is the dot product of two vectors of magnitude 3 and 5, if angle between them is 60°?
(a) 5.2 (b) 7.5 (c) 8.4 (d) 8.6

30. Two vectors $\vec{A}$ and $\vec{B}$ are such that $\vec{A} + \vec{B} = \vec{C}$ and $A^2 + B^2 = C^2$.
If θ is the angle between positive direction of $\vec{A}$ and $\vec{B}$ then the correct statement is
(a) $\theta = \pi$ (b) $\theta = \frac{2\pi}{3}$ (c) $\theta = 0$ (d) $\theta = \frac{\pi}{2}$

31. Which one is a vector quantity?
(a) Time (b) Temperature
(c) Flux density (d) Magnetic field intensity

32. Given that $P = 12$, $Q = 5$ and $R = 13$ also $\vec{P} + \vec{Q} = \vec{R}$, then the angle between $\vec{P}$ and $\vec{Q}$ will be
(a) π (b) $\frac{\pi}{2}$ (c) zero (d) $\frac{\pi}{4}$

33. The forces, which meet at one point but their lines of action do not lie in one plane, are called
(a) non-coplanar non-concurrent forces (b) non-coplanar concurrent forces
(c) coplanar concurrent forces (d) coplanar non-concurrent forces

34. Given that $\vec{P} + \vec{Q} + \vec{R} = 0$. Two out of the three vectors are equal in magnitude. The magnitude of the third vector is $\sqrt{2}$ times that of the other two. Which of the following can be the angles between these vectors?
(a) 90°, 135°, 135° (b) 45°, 45°, 90° (c) 30°, 60°, 90° (d) 45°, 90°, 135°

35. The angle between $\vec{P} + \vec{Q}$ and $\vec{P} - \vec{Q}$ will be
(a) 90° (b) between 0° and 180°
(c) 180° only (d) None of these

36. Two vectors of equal magnitude have a resultant equal to either of them, then the angle between them will be
(a) 30° (b) 120° (c) 60° (d) 45°

37. A force $(3\hat{i} + 4\hat{j})$ newton acts on a body and displaces it by $(3\hat{i} + 4\hat{j})$ metre. The work done by the force is
(a) 5 J (b) 25 J (c) 10 J (d) 30 J

38. If the vectors $\vec{P} = a\hat{i} + a\hat{j} + 3\hat{k}$ and $\vec{Q} = a\hat{i} - 2\hat{j} - \hat{k}$ are perpendicular to each other then the positive value of a is
(a) zero (b) 1 (c) 2 (d) 3

39. The angles which the vector $\vec{A} = 3\hat{i} + 6\hat{j} + 2\hat{k}$ makes with the co-ordinate axes are
(a) $\cos^{-1}\frac{3}{7}, \cos^{-1}\frac{6}{7}$ and $\cos^{-1}\frac{2}{7}$
(b) $\cos^{-1}\frac{4}{7}, \cos^{-1}\frac{5}{7}$ and $\cos^{-1}\frac{3}{7}$
(c) $\cos^{-1}\frac{3}{7}, \cos^{-1}\frac{4}{7}$ and $\cos^{-1}\frac{1}{7}$
(d) none of the above

40. Unit vector parallel to the resultant of vectors $\vec{A} = 4\hat{i} - 3\hat{j}$ and $\vec{B} = 8\hat{i} + 8\hat{j}$ will be
(a) $\frac{24\hat{i} + 5\hat{j}}{13}$ (b) $\frac{12\hat{i} + 5\hat{j}}{13}$ (c) $\frac{6\hat{i} + 5\hat{j}}{13}$ (d) None of these

41. The value of n so that vectors $2\hat{i} + 3\hat{j} - 2\hat{k}$, $5\hat{i} + n\hat{j} + \hat{k}$ and $-\hat{i} + 2\hat{j} + 3\hat{k}$ may be coplanar, will be
(a) 18 (b) 28 (c) 9 (d) 36

42. Which one of the following statement is false?
(a) A vector has only magnitude, whereas a scalar has both magnitude and direction
(b) Distance is a scalar quantity but displacement is a vector quantity
(c) Momentum, force, torque are vector quantities
(d) Mass, speed and energy are scalar quantities

43. If $\vec{a}$ and $\vec{b}$ are two vectors then the value of $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$ is
(a) $2(\vec{b} \times \vec{a})$ (b) $-2(\vec{b} \times \vec{a})$ (c) $\vec{b} \times \vec{a}$ (d) $\vec{a} \times \vec{b}$

44. The angle between the two vectors $\vec{A} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{B} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ is
(a) 60° (b) 0° (c) 90° (d) None of these

45. Maximum and minimum values of the resultant of two forces acting at a point are 7 N and 3 N respectively. The smaller force will be equal to
(a) 5 N (b) 4 N (c) 2 N (d) 1 N

46. The component of vector $\vec{A} = 2\hat{i} + 3\hat{j}$ along the vector $\hat{i} + \hat{j}$ is
(a) $\frac{5}{\sqrt{2}}$ (b) $10\sqrt{2}$ (c) $5\sqrt{2}$ (d) 5

47. The resultant of two forces $3P$ and $2P$ is R. If the first force is doubled then the resultant is also doubled. The angle between the two forces is
(a) 60° (b) 120° (c) 70° (d) 180°

48. The resultant of two forces, one double the other in magnitude, is perpendicular to the smaller of the two forces. The angle between the two forces is
(a) 120° (b) 60° (c) 90° (d) 150°

49. Three vectors satisfy the relation $\vec{A} \cdot \vec{B} = 0$ and $\vec{A} \cdot \vec{C} = 0$, then $\vec{A}$ is parallel to
(a) $\vec{C}$ (b) $\vec{B}$ (c) $\vec{B} \times \vec{C}$ (d) $\vec{B} \cdot \vec{C}$

50. The sum of two forces at a point is 16 N. If their resultant is normal to the smaller force and has a magnitude of 8 N. Then two forces are
(a) 6 N, 10 N (b) 8 N, 8 N (c) 4 N, 12 N (d) 2 N, 14 N

51. If $|\vec{A} \times \vec{B}| = \sqrt{3}\, \vec{A} \cdot \vec{B}$, then the value of $|\vec{A} + \vec{B}|$ is
(a) $(A^2 + B^2 + AB)^{1/2}$
(b) $\left(A^2 + B^2 + \frac{AB}{\sqrt{3}}\right)^{1/2}$
(c) $(A + B)$
(d) $(A^2 + B^2 + \sqrt{3}AB)^{1/2}$

52. If the angle between the vectors $\vec{A}$ and $\vec{B}$ is θ, the value of the product $(\vec{B} \times \vec{A}) \cdot \vec{A}$ is equal to
(a) $BA^2 \cos\theta$ (b) $BA^2 \sin\theta$ (c) $BA^2 \sin\theta \cos\theta$ (d) zero

53. If a vector $2\hat{i} + 3\hat{j} + 8\hat{k}$ is perpendicular to the vector $4\hat{j} - 4\hat{i} + \alpha\hat{k}$, then the value of α is
(a) -1 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1

54. Minimum number of vectors of unequal magnitudes which can give zero resultant are
(a) two (b) three (c) four (d) more than four

55. The (x, y, z) co-ordinates of two points A and B are given respectively as $(0, 3, -1)$ and $(-2, 6, 4)$. The displacement vector from A to B is given by
(a) $-2\hat{i} + 6\hat{j} + 4\hat{k}$
(b) $-2\hat{i} + 3\hat{j} + 3\hat{k}$
(c) $-2\hat{i} + 3\hat{j} + 5\hat{k}$
(d) $2\hat{i} - 3\hat{j} - 5\hat{k}$

56. The sum of two vectors $\vec{A}$ and $\vec{B}$ is at right angles to their difference. Then
(a) $A = B$
(b) $A = 2B$
(c) $B = 2A$
(d) $\vec{A}$ and $\vec{B}$ have the same direction

Match the Columns

1. Column-I shows some vector equations. Match coloumn I with the value of angle between $\vec{A}$ and $\vec{B}$ given in column II.

Column I	Column II
(a) $\lvert\vec{A} \times \vec{B}\rvert = \lvert\vec{A} \cdot \vec{B}\rvert$	(p) zero
(b) $\vec{A} \times \vec{B} = \vec{B} \times \vec{A}$	(q) $\frac{\pi}{2}$
(c) $\lvert\vec{A} + \vec{B}\rvert = \lvert\vec{A} - \vec{B}\rvert$	(r) $\frac{\pi}{4}$
(d) $\vec{A} + \vec{B} = \vec{C}$ and $A + B = C$	(s) $\frac{3\pi}{4}$

Section-II

Subjective Questions

Units and Dimensions

1. Young's modulus of steel is 2.0×10^{11} N/m^2. Express it in dyne/cm^2.
2. Surface tension of water in the CGS system is 72 dynes/cm. What is its value in SI units?
3. In the expression $y = a \sin(\omega t + \theta)$, y is the displacement and t is the time. Write the dimensions of a, ω and θ.
4. The relation between the energy E and the frequency ν of a photon is expressed by the equation $E = h\nu$, where h is Planck's constant. Write down the SI units of h and its dimensions.
5. Write the dimensions of a and b in the relation.
$$P = \frac{b - x^2}{at}$$
where P is power, x is distance and t is time.
6. Check the correctness of the relation $S_t = u + \frac{a}{2}(2t - 1)$, where u is initial velocity, a is acceleration and S_t is the displacement of by the body in t^{th} second.
7. Let x and a stand for distance. Is $\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a}\sin^{-1}\frac{a}{x}$ dimensionally correct?
8. In the equation
$$\int \frac{dx}{\sqrt{2ax - x^2}} = a^n \sin^{-1}\left(\frac{x}{a} - 1\right)$$
Find the value of n.
9. Show dimensionally that the expression, $Y = \frac{MgL}{\pi r^2 l}$ is dimensionally correct, where Y is Young's modulus of the material of wire, L is length of wire, Mg is the weight applied on the wire and l is the increase in the length of the wire.
10. The energy E of an oscillating body in simple harmonic motion depends on its mass m, frequency n and amplitude a. Using the method of dimensional analysis find the relation between E, m, n and a.
11. The centripetal force F acting on a particle moving uniformly in a circle may depend upon mass (m), velocity (v) and radius r of the circle. Derive the formula for F using the method of dimensions.
12. Taking force F, length L and time T to be the fundamental quantities, find the dimensions of (a) density, (b) pressure, (c) momentum and (d) energy.

Vectors

13. Find the cosine of the angle between the vectors $\vec{A} = (3\hat{i} + \hat{j} + 2\hat{k})$ and $\vec{B} = (2\hat{i} - 2\hat{j} + 4\hat{k})$.
14. Obtain the angle between $\vec{A} + \vec{B}$ and $\vec{A} - \vec{B}$ if $\vec{A} = 2\hat{i} + 3\hat{j}$ and $\vec{B} = \hat{i} - 2\hat{j}$.
15. Under what conditions will the vectors $\vec{A} = 3\hat{i} - 5\hat{j} + 5\hat{k}$ and $5\hat{i} - \hat{j} + b\hat{k}$ be perpendicular to each other?

16. Deduce the condition for the vectors $2\hat{i} + 3\hat{j} - 4\hat{k}$ and $3\hat{i} - a\hat{j} + b\hat{k}$ to be parallel.
17. Three vectors which are coplanar with respect to a certain rectangular co-ordinate system are given by
$\vec{a} = 4\hat{i} - \hat{j}$, $\vec{b} = -3\hat{i} + 2\hat{j}$ and $\vec{c} = -3\hat{j}$
Find
(a) $\vec{a} + \vec{b} + \vec{c}$
(b) $\vec{a} + \vec{b} - \vec{c}$
(c) Find the angle between $\vec{a} + \vec{b} + \vec{c}$ and $\vec{a} + \vec{b} - \vec{c}$
18. Find the components of a vector $\vec{A} = 2\hat{i} + 3\hat{j}$ along the directions of $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$.
19. If vectors $\vec{A}$ and $\vec{B}$ be respectively equal to $3\hat{i} - 4\hat{j} + 5\hat{k}$ and $2\hat{i} + 3\hat{j} - 4\hat{k}$. Find the unit vector parallel to $\vec{A} + \vec{B}$.
20. If two vectors are $\vec{A} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{B} = \hat{j} - 4\hat{k}$. By calculation, prove that $\vec{A} \times \vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.
21. Find the area of the parallelogram whose sides are represented by $2\hat{i} + 4\hat{j} - 6\hat{k}$ and $\hat{i} + 2\hat{k}$.
22. The resultant of two vectors $\vec{A}$ and $\vec{B}$ is at right angles to $\vec{A}$ and its magnitude is half of $\vec{B}$. Find the angle between $\vec{A}$ and $\vec{B}$.
23. The x and y-components of vector $\vec{A}$ are 4 m and 6 m respectively. The x and y-components of vector $\vec{A} + \vec{B}$ are 10 m and 9 m respectively. Calculate for the vector $\vec{B}$ the following
(a) its x and y-components
(b) its length
(c) the angle it makes with x-axis
24. Prove by the method of vectors that in a triangle
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
25. Four forces of magnitude P, $2P$, $3P$ and $4P$ act along the four sides of a square $ABCD$ in cyclic order. Use the vector method to find the resultant force.
26. If $\vec{P} + \vec{Q} = \vec{R}$ and $\vec{P} - \vec{Q} = \vec{S}$, prove that
$$R^2 + S^2 = 2(P^2 + Q^2)$$

ANSWERS

Section-I

Single Correct Option

1.(d)	**2.** (c)	**3.**(d)	**4.**(a)	**5.**(c)	**6.**(c)	**7.**(c)	**8.**(c)	**9.**(c)	**10.**(b)
11.(c)	**12.**(b)	**13.**(b)	**14.**(a)	**15.**(a)	**16.**(b)	**17.**(a)	**18.**(c)	**19.**(a)	**20.**(a)
21.(a)	**22.**(c)	**23.**(b)	**24.**(b)	**25.**(c)	**26.**(d)	**27.**(d)	**28.**(d)	**29.**(b)	**30.**(d)
31.(d)	**32.**(b)	**33.**(b)	**34.**(a)	**35.**(b)	**36.**(b)	**37.**(b)	**38.**(d)	**39.**(a)	**40.**(b)
41.(a)	**42.**(a)	**43.**(a)	**44.**(c)	**45.**(c)	**46.**(a)	**47.**(b)	**48.**(a)	**49.**(c)	**50.**(a)
51.(a)	**52.**(d)	**53.**(b)	**54.**(b)	**55.**(c)	**56.**(a)				

Match the Columns

1. (a) → r,s (b) → p (c) → q (d) → p

Section-II

Subjective Questions

1. 2.0×10^{12} dyne/cm^2 **2.** Surface tension of water = 0.072 N/m

3. $[M^0LT^0], [M^0L^0T^{-1}], [M^0L^0T^0]$, **4.** J-s, $[ML^2T^{-1}]$ **5.** $[M^0L^2T^0], [M^{-1}L^0T^2]$

7. No **8.** Zero **10.** $E = kmn^2a^2$ (k= constant) **11.** $F = \dfrac{kmv^2}{r}$

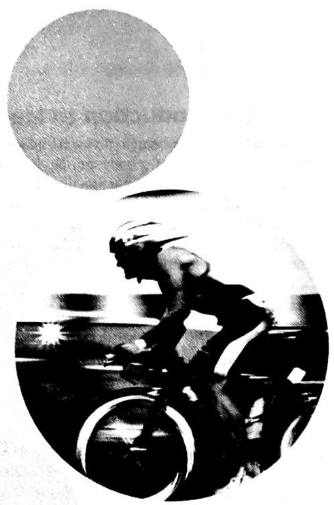
12. (a) $[FL^{-4}T^2]$ (b) $[FL^{-2}]$ (c) $[FT]$ (d) $[FL]$ **13.** $\dfrac{3}{\sqrt{21}}$ **14.** $\cos^{-1}\left(\dfrac{4}{\sqrt{65}}\right)$ **15.** $b = -4$

16. $a = -4.5, b = -6$ **17.** (a) $\hat{i} - 2\hat{j}$ (b) $\hat{i} + 4\hat{j}$ (c) $\cos^{-1}\left(\dfrac{-7}{\sqrt{85}}\right)$ **18.** $\dfrac{5}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}$

19. $\dfrac{1}{\sqrt{27}}(5\hat{i} - \hat{j} + \hat{k})$ **21.** Area = 13.4 units **22.** 150°

23. (a) 6 m, 3 m (b) $3\sqrt{5}$ m (c) $\theta = \tan^{-1}\left(\dfrac{1}{2}\right)$ **25.** $2\sqrt{2}\,P$, 225°

Chapter 3 – Motion in One Dimension



3

Motion in One Dimension

Chapter Contents

3.1 Introduction to Mechanics and Kinematics
3.2 Basic Definitions
3.3 Motion in One, Two and Three Dimensions
3.4 Uniformly Accelerated Motion
3.5 Non-Uniformly Accelerated Motion
3.6 Graphs
3.7 Relative Motion

3.1 Introduction to Mechanics and Kinematics

Mechanics is the branch of physics which deals with the motion of particles or bodies in space and time. Position and motion of a body can be determined only with respect to other bodies. Motion of the body involves position and time. For practical purposes a coordinate system, *e.g.*, the cartesian system is fixed to the reference body and position of the body is determined with respect to this reference body. For calculation of time generally clock is used.

Kinematics is the branch of mechanics which deals with the motion regardless of the causes producing it. The study of causes of motion is called **dynamics.**

3.2 Basic Definitions

(a) Position and Position Vector

In cartesian coordinate system position of any point (say A) is represented by its coordinates (x_A, y_A, z_A) with respect to an orgin O.

Position vector of point A with respect to O will now be:

$$\vec{\mathbf{r}}_A = \overrightarrow{\mathbf{OA}} = x_A \hat{\mathbf{i}} + y_A \hat{\mathbf{j}} + z_A \hat{\mathbf{k}}$$

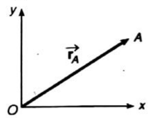


Fig. 3.1

Now, suppose coordinates of two points A and B are known to us and we want to find position vector of B with respect to A, then

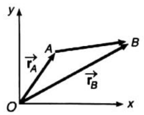


Fig. 3.2

$$\overrightarrow{\mathbf{AB}} = \vec{\mathbf{r}}_B - \vec{\mathbf{r}}_A$$

$$\overrightarrow{\mathbf{AB}} = (x_B - x_A)\hat{\mathbf{i}} + (y_B - y_A)\hat{\mathbf{j}} + (z_B - z_A)\hat{\mathbf{k}}$$

(b) Distance and Displacement

Distance is the actual path length covered by a moving particle or body in a given time interval, while displacement is the change in position vector, *i.e.*, a vector joining initial to final positions. If a particle moves from A to C (Fig. 3.3) through a path ABC. Then distance (Δs) travelled is the actual path length ABC, while the displacement is

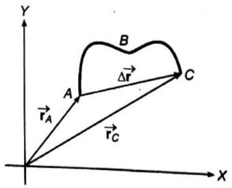


Fig. 3.3

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_C - \vec{\mathbf{r}}_A$$

If a particle moves in a straight line without change in direction, the magnitude of displacement is equal to the distance travelled, otherwise, it is always less than it. Thus,

$$|\text{displacement}| \le \text{distance}$$

(c) Average Speed and Velocity

The average speed of a particle in a given interval of time is defined as the ratio of distance travelled to the time taken while average velocity is defined as the ratio of displacement to time taken. Thus, if the distance travelled is Δs and displacement of a particle is $\Delta\vec{\mathbf{r}}$ in a given time interval Δt, then

$$v_{av} = \text{Average speed} = \frac{\Delta s}{\Delta t} \qquad \frac{\text{total distance}}{\text{total time}}.$$

and

$$\vec{\mathbf{v}}_{av} = \text{Average velocity} = \frac{\Delta\vec{\mathbf{r}}}{\Delta t} \qquad \frac{\text{total displ}}{\text{total time}}.$$

(d) Instantaneous Speed and Velocity

Instantaneous speed and velocity are defined at a particular instant and are given by

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \quad \text{and} \quad \vec{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{r}}}{\Delta t} = \frac{d\vec{\mathbf{r}}}{dt}$$

(e) Average and Instantaneous Acceleration

Average acceleration is defined as the ratio of change in velocity, *i.e.*, $\Delta\vec{\mathbf{v}}$ to the time interval Δt in which this change occures. Hence,

$$\vec{\mathbf{a}}_{av} = \frac{\Delta\vec{\mathbf{v}}}{\Delta t} \qquad \text{(x) no limits-}$$

The instantaneous acceleration is defined at a particular instant and is given by

$$\vec{\mathbf{a}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{v}}}{\Delta t} = \frac{d\vec{\mathbf{v}}}{dt}$$

Sample Example 3.1 *In one second, a particle goes from point A to point B moving in a semicircle (Fig. 3.4). Find the magnitude of the average velocity.*

Fig. 3.4

Solution

$$|\vec{v}_{av}| = \frac{AB}{\Delta t}$$
$$= \frac{2.0}{1.0} \text{ m/s}$$
$$= 2 \text{ m/s}$$

Ans.

Sample Example 3.2 *A table is given below of a particle moving along x-axis. In the table speed of particle at different time intervals is shown.*

Table 3.1

Time interval (in sec)	Speed of particle (in m/s)
0 – 2	2
2 – 5	3
5 – 10	4
10 – 15	2

Find total distance travelled by the particle and its average speed.

Solution Distance = speed × time speed x time

$\therefore$ Total distance $= (2\times 2) + (3)(3) + (5)(4) + (5)(2) = 43$ m

Total time taken is 15 s Hence,

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}} = \frac{43}{15}$$
$$= 2.87 \text{ m/s}$$

Sample Example 3.3 *(a) What does* $\left|\frac{d\vec{v}}{dt}\right|$ *and* $\frac{d|\vec{v}|}{dt}$ *represent?*

(b) Can these be equal?

(c) Can $\frac{d|\vec{v}|}{dt} = 0$ *while* $\left|\frac{d\vec{v}}{dt}\right| \neq 0$

(d) $\frac{d|\vec{v}|}{dt} \neq 0$ *while* $\left|\frac{d\vec{v}}{dt}\right| = 0$

Solution (a) $\left|\frac{d\vec{v}}{dt}\right|$ is the magnitude of total acceleration. While $\frac{d|\vec{v}|}{dt}$ represents the time rate of change of speed (called the tangential acceleration, a component of total acceleration) as $|\vec{v}| = v$.

(b) These two are equal in case of one dimensional motion (without change in direction).

(c) In case of uniform circular motion speed remains constant while velocity changes.

Hence, $\frac{d|\vec{v}|}{dt} = 0$ while $\left|\frac{d\vec{v}}{dt}\right| \neq 0$

(d) $\frac{d|\vec{v}|}{dt} \neq 0$ implies that speed of particle is not constant. Velocity cannot remain constant, if speed is changing. Hence, $\left|\frac{d\vec{v}}{dt}\right|$ cannot be zero in this case. So, it is not possible to have $\left|\frac{d\vec{v}}{dt}\right| = 0$ while $\frac{d|\vec{v}|}{dt} \neq 0$.

Sample Example 3.4 *Give examples where*

(a) the velocity is in opposite direction to the acceleration

(b) the velocity of the particle is zero but its acceleration is not zero

(c) the velocity is perpendicular to the acceleration.

Solution (a) A particle thrown upwards has its velocity in opposite direction to its acceleration (g, downwards).

(b) When the particle is released from rest from a certain height, its velocity is zero, while acceleration is g downwards. Similarly, at the extreme position of a pendulum velocity is zero, while acceleration is not zero.

(c) In uniform circular motion velocity is perpendicular to its radial or centripetal acceleration.

Introductory Exercise 3.1

1. When a particle moves with constant velocity its average velocity, its instantaneous velocity and its speed all are equal. Is this statement true or false?
2. A stone is released from an elevator going up with an acceleration of $g/2$. What is the acceleration of the stone just after release?
3. A clock has its second hand 2.0 cm long. Find the average speed and modulus of average velocity of the tip of the second hand in 15 s.
4. (a) Is it possible to be accelerating if you are travelling at constant speed?
   (b) Is it possible to move on a curved path with zero acceleration, constant acceleration, variable acceleration?
5. A particle is moving in a circle of radius 4 cm with constant speed of 1 cm/s. Find:
   (a) Time period of the particle.
   (b) Average speed, average velocity and average acceleration in a time interval from $t = 0$ to $t = T/4$. Here, T is the time period of the particle. Give only their magnitudes.
6. A particle moves in a straight line with constant speed of 4 m/s for 2 s, then with 6 m/s for 3 s. Find the average speed of the particle in the given time interval.

3.3 Motion in One, Two and Three Dimensions

Motion of a block in a straight line is one dimensional (1-D) motion. Motion of a particle in a straight line can be described by only one component of its velocity or acceleration. The motion of a particle thrown in vertical plane at some angle with horizontal ($\neq 90°$) is an example of two dimensional (2-D) motion. This is called a projectile motion. Similarly a circular motion is also an example of 2-D motion. A 2-D motion takes place in a plane and its velocity (or acceleration) can be described by two components in any two mutually perpendicular direction (v_x and v_y).

Motion of a bird (or a monkey) in space is a three dimensional (3-D) motion. In a 3-D motion velocity and acceleration of a particle can be resolved in three components (v_x, v_y, v_z, a_x, a_y and a_z). Here x, y and z are any three mutually perpendicular axes.

The position of a particle in one dimensional motion is described by one variable (say x). In a 2-D motion it involves two variables (normally x and y) and in a 3-D motion three variables are x, y and z.

3.4 Uniformly Accelerated Motion

Equations of motion for uniformly accelerated motion ($\vec{a}$ = constant) are as under,

$$\vec{v} = \vec{u} + \vec{a}\,t,\ \vec{s} = \vec{u}\,t + \frac{1}{2}\vec{a}\,t^2,\ \vec{v}\cdot\vec{v} = \vec{u}\cdot\vec{u} + 2\,\vec{a}\cdot\vec{s}$$

Note *The above formulae for one dimensional motion has been derived in article 3.5.*

Here, $\vec{u}$ = initial velocity of particle, $\vec{v}$ = velocity of particle at time t and

$\vec{s}$ = displacement of particle in time t

Note *If initial position vector of a particle is $\vec{r}_0$, then position vector at time t can be written as*

$$\vec{r} = \vec{r}_0 + \vec{s} = \vec{r}_0 + \vec{u}\,t + \frac{1}{2}\vec{a}\,t^2$$

One-dimensional Uniformly Accelerated Motion

If the motion of a particle is taking place in a straight line, there is no need of using vector addition (or subtraction) in equations of motion. We can directly use the equations.

$$v = u + at,\ s = ut + \frac{1}{2}at^2 \text{ and } v^2 = u^2 + 2as$$

Just by taking one direction as the positive (and opposite to it as negative) and then substituting u, a, etc., with sign. Normally we take vertically upward direction positive (and downward negative) and horizontally rightwards positive (or leftwards negative).

Sign convention for (a) motion in vertical direction (b) motion in horizontal direction is shown in Fig. 3.5.

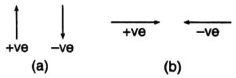


Fig. 3.5

Sample Example 3.5 *A ball is thrown upwards from the top of a tower 40 m high with a velocity of 10 m/s. Find the time when it strikes the ground. Take $g = 10\ m/s^2$.*

Solution In the problem

$$u = +10\ \text{m/s},\quad a = -10\ \text{m/s}^2$$

and $$s = -40\ \text{m}$$

(at the point where stone strikes the ground)

Substituting in $s = ut + \frac{1}{2}at^2$, we have

$$-40 = 10t - 5t^2$$

or $$5t^2 - 10t - 40 = 0$$

or $$t^2 - 2t - 8 = 0$$

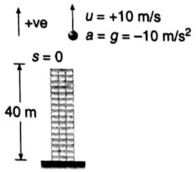


Fig. 3.6

Solving this, we have $t = 4$ s and -2 s. Taking the positive value $t = 4$ s.

Note *The significance of $t = -2$ s can be understood by following figure:*

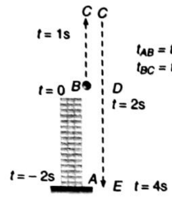


Fig. 3.7

Sample Example 3.6 *A ball is thrown upwards from the ground with an initial speed of u. The ball is at a height of 80 m at two times, the time interval being 6 s. Find u. Take $g = 10\ m/s^2$.*

Solution Here, $u = u$ m/s, $a = g = -10\ \text{m/s}^2$ and $s = 80$ m.

Substituting the values in $s = ut + \frac{1}{2}at^2$, we have

$$80 = ut - 5t^2$$

or $$5t^2 - ut + 80 = 0$$

or $$t = \frac{u + \sqrt{u^2 - 1600}}{10} \text{ and } \frac{u - \sqrt{u^2 - 1600}}{10}$$

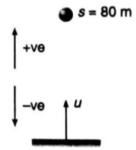


Fig. 3.8

Now, it is given that

$$\frac{u + \sqrt{u^2 - 1600}}{10} - \frac{u - \sqrt{u^2 - 1600}}{10} = 6$$

or $$\frac{\sqrt{u^2 - 1600}}{5} = 6 \text{ or } \sqrt{u^2 - 1600} = 30$$

or $$u^2 - 1600 = 900$$

$\therefore$ $$u^2 = 2500$$

or $$u = \pm 50\ \text{m/s}$$

Ignoring the negative sign, we have

$$u = 50\ \text{m/s}$$

Sample Example 3.7 *A particle of mass 1 kg has a velocity of 2 m/s. A constant force of 2 N acts on the particle for 1 s in a direction perpendicular to its initial velocity. Find the velocity and displacement of the particle at the end of 1 second.*

Solution Force acting on the particle is constant. Hence, acceleration of the particle will also remain constant.

$$a = \frac{F}{m} = \frac{2}{1} = 2\ \text{m/s}^2$$

Since, acceleration is constant. We can apply

$$\vec{v} = \vec{u} + \vec{a}\,t \quad \text{and} \quad \vec{s} = \vec{u}\,t + \frac{1}{2}\vec{a}\,t^2$$

Refer Fig. 3.9(a) $\quad \vec{v} = \vec{u} + \vec{a}\,t$

Here, $\vec{u}$ and $\vec{a}\,t$ are two mutually perpendicular vectors. So,

$$|\vec{v}| = \sqrt{(|\vec{u}|)^2 + (|\vec{a}\,t|)^2} = \sqrt{(2)^2 + (2)^2} = 2\sqrt{2}\ \text{m/s}$$

$$\alpha = \tan^{-1}\frac{|\vec{a}\,t|}{|\vec{u}|} = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = 45°$$

Thus, velocity of the particle at the end of 1s is $2\sqrt{2}$ m/s at an angle of 45° with its initial velocity.

Refer Fig. 3.9(b) $\quad \vec{s} = \vec{u}\,t + \frac{1}{2}\vec{a}\,t^2$

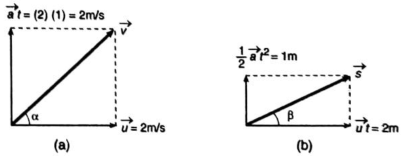


Fig. 3.9

Here, $\vec{u}\,t$ and $\frac{1}{2}\vec{a}\,t^2$ are also two mutually perpendicular vectors. So,

$$|\vec{s}| = \sqrt{(|\vec{u}\,t|)^2 + (|\tfrac{1}{2}\vec{a}\,t^2|)^2}$$

$$= \sqrt{(2)^2 + (1)^2} = \sqrt{5}\ \text{m}$$

and

$$\beta = \tan^{-1}\frac{|\frac{1}{2}\vec{a}\,t^2|}{|\vec{u}\,t|} = \tan^{-1}\left(\frac{1}{2}\right)$$

Thus, displacement of the particle at the end of 1 s is $\sqrt{5}$ m at an angle of $\tan^{-1}\left(\frac{1}{2}\right)$ from its initial velocity.

Sample Example 3.8 *Velocity and acceleration of a particle at time $t = 0$ are $\vec{u} = (2\hat{i} + 3\hat{j})$ m/s and $\vec{a} = (4\hat{i} + 2\hat{j})\ m/s^2$ respectively. Find the velocity and displacement of particle at $t = 2$ s.*

Solution Here, acceleration $\vec{a} = (4\hat{i} + 2\hat{j})\ \text{m/s}^2$ is constant. So, we can apply

$$\vec{v} = \vec{u} + \vec{a}\,t \quad \text{and} \quad \vec{s} = \vec{u}\,t + \frac{1}{2}\vec{a}\,t^2$$

Substituting the proper values, we get

$$\vec{v} = (2\hat{i} + 3\hat{j}) + (2)(4\hat{i} + 2\hat{j})$$

$$= (10\hat{i} + 7\hat{j})\ \text{m/s}$$

and

$$\vec{s} = (2)(2\hat{i} + 3\hat{j}) + \frac{1}{2}(2)^2(4\hat{i} + 2\hat{j})$$

$$= (12\hat{i} + 10\hat{j})\ \text{m}$$

Therefore, velocity and displacement of particle at $t = 2$s are $(10\hat{i} + 7\hat{j})$ m/s and $(12\hat{i} + 10\hat{j})$ m respectively.

Following points are worthnoting in case of one dimensional motion with constant acceleration.

(i) This type of motion can be observed when either $u = 0$, $\vec{u}\uparrow\uparrow\vec{a}$ or $\vec{u}\uparrow\downarrow\vec{a}$

(ii) In the first two cases when either $u = 0$ or $\vec{u}\uparrow\uparrow\vec{a}$ motion is only accelerated.

(iii) When $\vec{u}\uparrow\downarrow\vec{a}$ motion is first retarded (till the velocity becomes zero) and then accelerated in opposite direction.

(iv) As per our convention (vertically upward positive) acceleration due to gravity g is always negative whether the particle is moving upwards or downwards. We are now left with the sign of u and s. Displacement s is measured from the point of projection.

(v) For fast calculation in objective problems, remember the following results.

(a) Maximum height attained by a particle, thrown upwards from ground

$$h = \frac{u^2}{2g}$$

(b) Velocity of particle at the time of striking the ground when released ($u = 0$) from a height h is,

$$v = \sqrt{2gh}$$

(c) In (b) time of collision with ground

$$t = \sqrt{\frac{2h}{g}}$$

(d) Displacement of particle in t^{th} second of its motion,

$$s_t = u + at - \frac{1}{2}a$$

EXERCISE : *Derive the above four relations for h, v, t and s_t.*

Difference between distance (*d*) and displacement (*s*)

The s in equations of motion

$$\left(s = ut + \frac{1}{2}at^2 \quad \text{and} \quad v^2 = u^2 + 2as\right)$$

is really the displacement not the distance. They have different values only when u and a are of opposite sign or $u\uparrow\downarrow a$.

Let us take the following two cases.

Case 1. When u is either zero or parallel to a, then motion is simply accelerated and in this case distance is equal to displacement. So, we can write

$$d = s = ut + \frac{1}{2}at^2$$

Case 2. When u is antiparallel to a, the motion is first retarded then accelerated in opposite direction. So distance is either greater than or equal to displacement ($d \geq |s|$). In this case first find the time when velocity becomes zero. Say it is t_0.

$$0 = u - at_0$$

$$\therefore \qquad t_0 = \left|\frac{u}{a}\right|$$

Now, if the given time $t \leq t_0$: distance and displacement are equal. So

$$d = s = ut + \frac{1}{2}at^2$$

For $t \leq t_0$ (with u positive and a negative)

For $t > t_0$: distance is greater than displacement.

$$d = d_1 + d_2$$

Here, $\quad d_1$ = distance travelled before coming to rest $= \left|\dfrac{u^2}{2a}\right|$

d_2 = distance travelled in remaining time $t - t_0$

$$= \frac{1}{2}|a(t - t_0)^2|$$

$$\therefore \qquad d = \left|\frac{u^2}{2a}\right| + \frac{1}{2}|a(t - t_0)^2|$$

Note *The displacement is still* $\quad s = ut + \frac{1}{2}at^2 \quad$ *with u positive and a negative.*

Sample Example 3.9 *A particle is projected vertically upwards with velocity 40 m/s. Find the displacement and distance travelled by the particle in*

(a) 2 s (b) 4 s (c) 6 s

Take $g = 10\,m/s^2$.

Solution Here, u is positive (upwards) and a is negative (downwards). So, first we will find t_0, the time when velocity becomes zero.

$$t_0 = \left|\frac{u}{a}\right| = \frac{40}{10} = 4\text{ s}$$

+ve

−ve

(a) $t < t_0$. Therefore, distance and displacement are equal.

$$d = s = ut + \frac{1}{2}at^2$$

$$= 40 \times 2 - \frac{1}{2} \times 10 \times 4 = 60\text{ m}$$

(b) $t = t_0$. So, again distance and displacement are equal.

$$d = s = 40 \times 4 - \frac{1}{2} \times 10 \times 16 = 80\text{ m}$$

(c) $t > t_0$. Hence, $d > s$

$$s = 40 \times 6 - \frac{1}{2} \times 10 \times 36 = 60\text{ m}$$

While $\qquad d = \left|\frac{u^2}{2a}\right| + \frac{1}{2}|a(t - t_0)^2|$

$$= \frac{(40)^2}{2 \times 10} + \frac{1}{2} \times 10 \times (6 - 4)^2 = 100\text{ m}$$

3.5 Non-Uniformly Accelerated Motion

When acceleration of particle is not constant, we go for basic equations of velocity and acceleration, *i.e.,*

(i) $\vec{v} = \dfrac{d\vec{s}}{dt}$ or sometimes $\vec{v} = \dfrac{d\vec{r}}{dt}$ (ii) $\vec{a} = \dfrac{d\vec{v}}{dt}$ (iii) $d\vec{s} = \vec{v}\,dt$ (iv) $d\vec{v} = \vec{a}\,dt$

For one dimensional motion, above relations can be written as under.

(i) $v = \dfrac{ds}{dt}$ (ii) $a = \dfrac{dv}{dt} = v\dfrac{dv}{ds}$ (iii) $ds = v\,dt$ and (iv) $dv = adt$ or $v\,dv = a\,ds$

Such, problems can be solved either by differentiation or integration (with some boundary conditions).

$s - t \longrightarrow v - t \longrightarrow a - t$ (Differentiation)

$a - t \longrightarrow v - t \longrightarrow s - t$ (Integration with boundary conditions)

Note (i) *By boundary condition we mean that velocity or displacement at some time (usually at t = 0) should be known to us. Otherwise we cannot find constant of integration.*

(ii) *Equation* $a = v\dfrac{dv}{ds}$ *or* $v\,dv = a\,ds$ *is useful when acceleration displacement equation is known and velocity displacement equation is required.*

Derivation of Equation of Motion ($v = u + at$ etc.)

For one dimensional motion with a = constant.

We can write,

$$dv = a\,dt \qquad \left(\text{as } a = \frac{dv}{dt}\right)$$

Integrating both sides, we have

$$\int dv = a\int dt \qquad (\text{as } a = \text{constant})$$

At $t = 0$, velocity is u and at $t = t$ velocity is v. Hence,

$$\int_u^v dv = a\int_0^t dt$$

$$\therefore \qquad [v]_u^v = a[t]_0^t$$

or $\qquad v - u = at$

$$\therefore \qquad v = u + at$$

Hence proved.

Further, we can write

$$ds = v\,dt \qquad \left(\text{as } v = \frac{ds}{dt}\right)$$

$$= (u + at)\,dt \qquad (\text{as } v = u + at)$$

At time $t = 0$ suppose $s = 0$ and at $t = t$, displacement is s, then

$$\int_0^s ds = \int_0^t (u + at)\,dt$$

$\therefore$ $$[s]_0^s = \left[ut + \frac{1}{2}at^2\right]_0^t$$

or $$s = ut + \frac{1}{2}at^2$$ **Hence proved.**

We can also write, $$v \cdot dv = a \cdot ds \qquad \left(\text{as } a = v \cdot \frac{dv}{ds}\right)$$

When $s = 0$, v is u and at $s = s$, velocity is v. Therefore,

$$\int_u^v v \cdot dv = a \int_0^s ds \qquad (\text{as } a = \text{constant})$$

or $$\left[\frac{v^2}{2}\right]_u^v = a[s]_0^s$$

$\therefore$ $$\frac{v^2}{2} - \frac{u^2}{2} = as$$

or $$v^2 = u^2 + 2as$$ **Hence proved.**

Sample Example 3.10 *Displacement time equation of a particle moving along x-axis is*

$$x = 20 + t^3 - 12t \qquad (SI\ Units)$$

(a) Find, position and velocity of particle at time $t = 0$.
(b) State whether the motion is uniformly accelerated or not.
(c) Find position of particle when velocity of particle is zero.

Solution (a) $$x = 20 + t^3 - 12t \qquad \ldots(i)$$

At $t = 0$, $$x = 20 + 0 - 0 = 20\text{ m}$$

Velocity of particle at time t can be obtained by differentiating Eq. (i) w.r.t. time *i.e.*,

$$v = \frac{dx}{dt} = 3t^2 - 12 \qquad \ldots(ii)$$

At $t = 0$, $$v = 0 - 12 = -12\text{ m/s}$$

(b) Differentiating Eq. (ii) w.r.t. time t, we get the acceleration

$$a = \frac{dv}{dt} = 6t$$

As acceleration is a function of time, the motion is non-uniformly accelerated.

(c) Substituting $v = 0$ in Eq. (ii), we have

$$0 = 3t^2 - 12$$

Positive value of t comes out to be 2 second from this equation. Substituting $t = 2$ second in Eq. (i), we have

$$x = 20 + (2)^3 - 12(2) \quad \text{or} \quad x = 4\text{ m}$$

Sample Example 3.11 *Velocity-time equation of a particle moving in a straight line is,*

$$v = (10 + 2t + 3t^2) \qquad (SI\ Units)$$

Find :
(a) displacement of particle from the mean position at time $t = 1$ second, if it is given that displacement is 20 m at time $t = 0$.
(b) acceleration-time equation.

Solution (a) The given equation can be written as,

$$v = \frac{ds}{dt} = (10 + 2t + 3t^2)$$

or $$ds = (10 + 2t + 3t^2)\,dt$$

or $$\int_{20}^s ds = \int_0^1 (10 + 2t + 3t^2)\,dt$$

or $$s - 20 = [10t + t^2 + t^3]_0^1$$

or $$s = 20 + 12 = 32\text{ m}$$

(b) Acceleration-time equation can be obtained by differentiating the given equation w.r.t. time. Thus,

$$a = \frac{dv}{dt} = \frac{d}{dt}(10 + 2t + 3t^2)$$

or $$a = 2 + 6t$$

Introductory Exercise 3.2

1. A ball is thrown vertically upwards. Which quantity remains constant among, speed, kinetic energy, velocity and acceleration?
2. Equation $s_t = u + at - \frac{1}{2}a$ does not seem dimensionally correct, why?
3. Can the speed of a particle increase as its acceleration decreases? If yes give an example.
4. The velocity of a particle moving in a straight line is directly proportional to 3/4th power of time elapsed. How does its displacement and acceleration depend on time?
5. A particle is projected vertically upwards with an initial velocity of 40 m/s. Find the displacement and distance covered by the particle in 6 seconds. Take $g = 10\text{ m/s}^2$.
6. Velocity of a particle moving along positive x-direction is $v = (40 - 10t)$ m/s. Here, t is in seconds. At time $t = 0$, the x coordinate of particle is zero. Find the time when the particle is at a distance of 60 m from origin.
7. A particle moves rectilinearly with initial velocity u and a constant acceleration a. Find the average velocity of the particle in a time interval from $t = 0$ to $t = t$ second of its motion.
8. A particle moves in a straight line with uniform acceleration. Its velocity at time $t = 0$ is v_1 and at time $t = t$ is v_2. The average velocity of the particle in this time interval is $\frac{v_1 + v_2}{2}$. Is this statement true or false?
9. Find the average velocity of a particle released from rest from a height of 125 m over a time interval till it strikes the ground. $g = 10\text{ m/s}^2$.

10. Velocity of a particle moving along x-axis varies with time as, $v = (10 + 5t - t^2)$
At time $t = 0$, $x = 0$. Find
(a) acceleration of particle at $t = 2\text{ s}$ (b) x-coordinate of particle at $t = 3\text{ s}$

11. Velocity of a particle at time $t = 0$ is 2 m/s. A constant acceleration of 2 m/s^2 acts on the particle for 2 seconds at an angle of 60° with its initial velocity. Find the magnitude of velocity and displacement of particle at the end of $t = 2$ s.

12. Velocity of a particle at any time t is $\vec{v} = (2\hat{i} + 2t\,\hat{j})$ m/s. Find acceleration and displacement of particle at $t = 1$ s. Can we apply $\vec{v} = \vec{u} + \vec{a}t$ or not?

13. The coordinates of a particle moving in x-y plane at any time t are $(2t, t^2)$.. Find: (a) the trajectory of the particle, (b) velocity of particle at time t and (c) acceleration of particle at any time t.

3.6 Graphs

The theory of graphs can be generalised and summarised in following six points :

(i) A linear equation between x and y represents a straight line, *e.g.*, $y = 4x - 2$, $y = 5x + 3$, $3x = y - 2$ equations represent straight line on x-y graph.

(ii) $x \propto y$ or $y = kx$ represents a straight line passing through origin.

(iii) $x \propto \frac{1}{y}$ represents a rectangular hyperbola in x-y graph. Shape of rectangular hyperbola is as shown in Fig. 3.10.

e.g., $P \propto \frac{1}{V}$ in an isothermal process (T = constant). Hence, P-V graph in isothermal process is a rectangular hyperbola as shown in Fig 3.11.

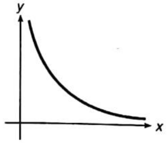


Fig. 3.10

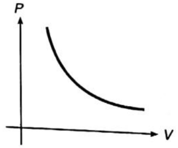


Fig. 3.11

(iv) A quadratic equation in x and y represents a parabola in x-y graph, *e.g.*, $y = 3x^2 + 2$, $y^2 = 4x$, $x^2 = y - 2$ equations represent parabola in x-y graph.

(v) If $z = \frac{dy}{dx}$ or $\frac{y}{x}$, then the value of z at any point on x-y graph can be obtained by the slope of the graph at that point.

(vi) If $z = yx$ or $y(dx)$ or $x(dy)$, then value of z between x_1 and x_2 or between y_1 and y_2 can be obtained by the area of graph between x_1 and x_2 or y_1 and y_2.

From the above six points we may conclude that in case of a one dimensional motion :

(a) slope of displacement-time graph gives velocity $\left(\text{as } v = \frac{ds}{dt}\right)$.

(b) slope of velocity-time graph gives acceleration $\left(\text{as } a = \frac{dv}{dt}\right)$.

(c) area under velocity-time graph gives displacement (as $ds = v\,dt$).

(d) area under acceleration-time graph gives change in velocity (as $dv = a\,dt$).

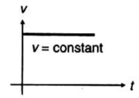
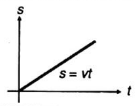
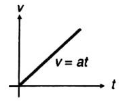
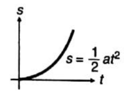
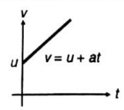
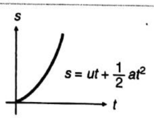
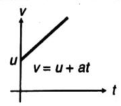
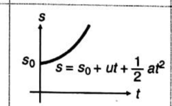
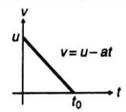
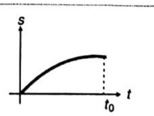
(e) displacement-time graph in uniform motion is a straight line passing through origin, if displacement is zero at time $t = 0$ (as $s = vt$).

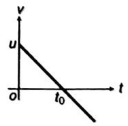
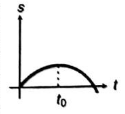
(f) velocity-time graph is a straight line passing through origin in a uniformly accelerated motion if initial velocity $u = 0$ and a straight line not passing through origin if initial velocity $u \neq 0$ (as $v = u + at$).

(g) displacement-time graph in uniformly accelerated or retarded motion is a parabola $\left(\text{as } s = ut \pm \frac{1}{2}at^2\right)$.

Now, we can plot v-t and s-t graphs of some standard results in tabular form as under. But note that all the following graphs are drawn for one-dimensional motion with uniform velocity or with constant acceleration.

Table 3.2

S. No.	Different Cases	v-t Graph	s-t Graph	Important Points
1.	Uniform motion			(i) Slope of v-t graph $= v =$ constant (ii) In s-t graph $s = 0$ at $t = 0$
2.	Uniformly accelerated motion with $u = 0$ and $s = 0$ at $t = 0$			(i) $u = 0$, *i.e.*, $v = 0$ at $t = 0$ (ii) a or slope of v-t graph is constant (iii) $u = 0$, *i.e.*, slope of s-t graph at $t = 0$, should be zero
3.	Uniformly accelerated motion with $u \neq 0$ but $s = 0$ at $t = 0$			(i) $u \neq 0$, *i.e.*, v or slope of v-t graph at $t = 0$ is not zero (ii) s or slope of s-t graph gradually goes on increasing
4.	Uniformly accelerated motion with $u \neq 0$ and $s = s_0$ at $t = 0$			(i) $v = u$ at $t = 0$ (ii) $s = s_0$ at $t = 0$
5.	Uniformly retarded motion till velocity becomes zero			(i) Slope of s-t graph at $t = 0$ gives u (ii) Slope of s-t graph at $t = t_0$ becomes zero (iii) In this case u can't be zero

6.	Uniformly retarded then accelerated in opposite direction			(i) At time $t = t_0$, $v = 0$ or slope of s-t graph is zero (ii) In s-t graph slope or velocity first decreases then increases with opposite sign.

Important Points in Graphs

- Slopes of v-t or s-t graphs can never be infinite at any point, because infinite slope of v-t graph means infinite acceleration. Similarly, infinite slope of s-t graph means infinite velocity. Hence, the following graphs are not possible :

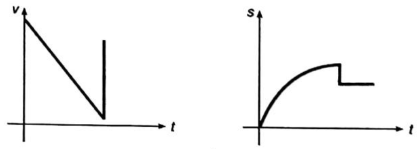


Fig. 3.12

- At one time, two values of velocity or displacement are not possible. Hence, the following graphs are not acceptable :

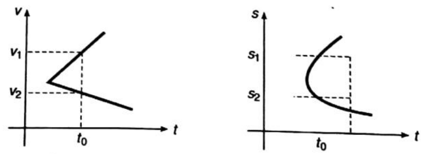


Fig. 3.13

- Different values of displacements in s-t graph corresponding to given v-t graph can be calculated just by calculating areas under v-t graph. There is no need of using equations like $v = u + at$, etc.

Sample Example 3.12 *Displacement-time graph of a particle moving in a straight line is as shown in figure. State whether the motion is accelerated or not. Describe the motion in detail. Given $s_0 = 20$ m and $t_0 = 4$ second.*

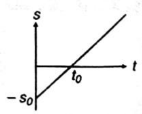


Fig. 3.14

Solution Slope of s-t graph is constant. Hence, velocity of particle is constant. Further at time $t = 0$, displacement of the particle from the mean position is $-s_0$ or -20 m. Velocity of particle,

$$v = \text{slope} = \frac{s_0}{t_0} = \frac{20}{4} = 5 \text{ m/s}$$

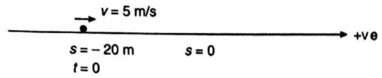


Fig. 3.15

Motion of the particle is as shown in figure. At $t = 0$ particle is at -20 m and has a constant velocity of 5 m/s. At $t_0 = 4$ second particle will pass through its mean position.

Sample Example 3.13 *A car accelerates from rest at a constant rate α for some time, after which it decelerates at a constant rate β, to come to rest. If the total time elapsed is t second evaluate (a) the maximum velocity reached and (b) the total distance travelled.*

Solution (a) Let the car accelerates for time t_1 and decelerates for time t_2. Then,

$$t = t_1 + t_2 \qquad \text{...(i)}$$

and corresponding velocity-time graph will be as shown in Fig. 3.16.

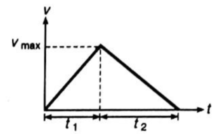


Fig. 3.16

From the graph,

$$\alpha = \text{slope of line } OA = \frac{v_{max}}{t_1}$$

or $$t_1 = \frac{v_{max}}{\alpha} \qquad \text{...(ii)}$$

and $$\beta = -\text{ slope of line } AB = \frac{v_{max}}{t_2}$$

or $$t_2 = \frac{v_{max}}{\beta} \qquad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$\frac{v_{max}}{\alpha} + \frac{v_{max}}{\beta} = t \quad \text{or} \quad v_{max}\left(\frac{\alpha + \beta}{\alpha\beta}\right) = t$$

or $$v_{max} = \frac{\alpha\beta t}{\alpha + \beta}$$

(b) Total distance = displacement = area under v-t graph

$$= \frac{1}{2} \times t \times v_{max}$$

$$= \frac{1}{2} \times t \times \frac{\alpha\beta t}{\alpha + \beta}$$

or $$\text{Distance} = \frac{1}{2}\left(\frac{\alpha\beta t^2}{\alpha + \beta}\right)$$

Note *This problem can also be solved by using equations of motion ($v = u + at$ etc.). Try it yourself.*

Sample Example 3.14 *A rocket is fired vertically upwards with a net acceleration of 4 m/s^2 and initial velocity zero. After 5 s its fuel is finished and it decelerates with g. At the highest point its velocity becomes zero. Then it accelerates downwards with acceleration g and return back to ground. Plot velocity-time and displacement-time graphs for the complete journey. Take g = 10 m/s^2.*

Solution In the graphs,

$$v_A = at_{OA} = (4)(5) = 20\ m/s$$
$$v_B = 0 = v_A - gt_{AB}$$

$\therefore$
$$t_{AB} = \frac{v_A}{g} = \frac{20}{10} = 2\text{s}$$

$\therefore$
$$t_{OAB} = (5+2)\text{s} = 7\text{s}$$

Now,
$$s_{OAB} = \text{area under } v\text{-}t \text{ graph between 0 to 7 s}$$
$$= \frac{1}{2}(7)(20) = 70\text{ m}$$

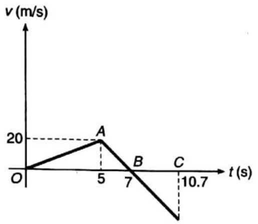


Fig. 3.17

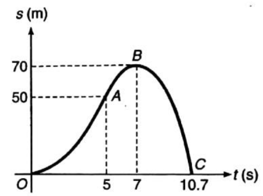


Fig. 3.18

Now,
$$|s_{OAB}| = |s_{BC}| = \frac{1}{2} g t_{BC}^2$$

$\therefore$
$$70 = \frac{1}{2}(10)\, t_{BC}^2$$

$\therefore$
$$t_{BC} = \sqrt{14} = 3.7\text{ s}$$

$\therefore$
$$t_{OABC} = 7 + 3.7 = 10.7\text{ s}$$

Also,
$$s_{OA} = \text{area under } v\text{-}t \text{ graph between } OA$$
$$= \frac{1}{2}(5)(20) = 50\text{ m}$$

Sample Example 3.15 *Acceleration-time graph of a particle moving in a straight line is shown in Fig. 3.19. Velocity of particle at time $t = 0$ is 2 m/s. Find velocity at the end of fourth second.*

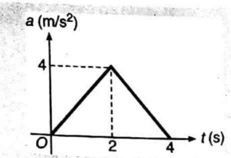


Fig. 3.19

Solution
$$dv = a\,dt$$

or
$$\text{change in velocity} = \text{area under } a\text{-}t \text{ graph}$$

Hence,
$$v_f - v_i = \frac{1}{2}(4)(4) = 8\text{ m/s}$$

$\therefore$
$$v_f = v_i + 8 = (2+8)\text{ m/s} = 10\text{ m/s}$$

Sample Example 3.16 *Velocity-time graph of a particle moving in a straight line is shown in Fig. 3.20.*

Plot the corresponding displacement-time graph of the particle if at time $t = 0$, displacement $s = 0$.

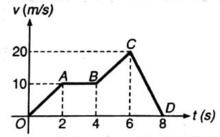


Fig. 3.20

Solution Displacement = Area under velocity-time graph

Hence,
$$s_{OA} = \frac{1}{2} \times 2 \times 10 = 10\text{ m}$$
$$s_{AB} = 2 \times 10 = 20\text{ m}$$

or
$$s_{OAB} = 10 + 20 = 30\text{ m}$$
$$s_{BC} = \frac{1}{2} \times 2(10+20) = 30\text{ m}$$

or
$$s_{OABC} = 30 + 30 = 60\text{ m}$$

and
$$s_{CD} = \frac{1}{2} \times 2 \times 20 = 20\text{ m}$$

or
$$s_{OABCD} = 60 + 20 = 80\text{ m}$$

Between 0 to 2 s and 4 to 6 s motion is accelerated, hence displacement-time graph is a parabola. Between 2 to 4 s motion is uniform, so displacement-time graph will be a straight line. Between 6 to 8 s motion is decelerated hence displacement-time graph is again a parabola but inverted in shape. At the end of 8 s velocity is zero, therefore, slope of displacement-time graph should be zero. The corresponding graph is shown in Fig. 3.21.

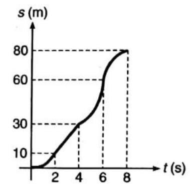


Fig. 3.21

Introductory Exercise 3.3

1. Figure shows the displacement-time graph of a particle moving in a straight line. Find the signs of velocity and acceleration of particle at time $t = t_1$ and $t = t_2$.

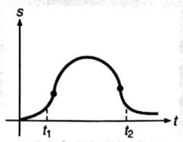


Fig. 3.22

2. A particle of mass m is released from a certain height h with zero initial velocity. It strikes the ground elastically (direction of its velocity is reversed but magnitude remains the same). Plot the graph between its kinetic energy and time till it returns to its initial position.
3. A ball is dropped from a height of 80 m on a floor. At each collision, the ball loses half of its speed. Plot the speed-time graph and velocity-time graph of its motion till two collisions with the floor. [Take $g = 10\text{ m/s}^2$]

4. Figure shows the acceleration-time graph of a particle moving along a straight line. After what time the particle acquires its initial velocity?

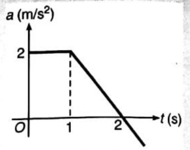


Fig. 3.23

3.7 Relative Motion

The word 'relative' is a very general term, which can be applied to physical, nonphysical, scalar or vector quantities. For example, my height is five feet and six inches while my wife's height is five feet and four inches. If I ask you how high I am relative to my wife, your answer will be two inches. What you did? You simply subtracted my wife's height from my height. The same concept is applied everywhere, whether it is a relative velocity, relative acceleration or anything else. So, from the above discussion we may now conclude that relative velocity of A with respect to B (written as $\vec{v}_{AB}$) is

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

Similarly, relative acceleration of A with respect to B is

$$\vec{a}_{AB} = \vec{a}_A - \vec{a}_B$$

If it is a one dimensional motion we can treat the vectors as scalars just by assigning the positive sign to one direction and negative to the other. So, in case of a one dimensional motion the above equations can be written as

$$v_{AB} = v_A - v_B$$

and $$a_{AB} = a_A - a_B$$

Further, we can see that

$$\vec{v}_{AB} = -\vec{v}_{BA} \quad \text{or} \quad \vec{a}_{BA} = -\vec{a}_{AB}$$

Sample Example 3.17 *Seeta is moving due east with a velocity of 1 m/s and Geeta is moving due west with a velocity of 2 m/s. What is the velocity of Seeta with respect to Geeta?*

Solution It is a one dimensional motion. So, let us choose the east direction as positive and the west as negative. Now, given that

$$v_S = \text{velocity of Seeta} = 1 \text{ m/s}$$

and v_G = velocity of Geeta = −2 m/s

Thus, $$v_{SG} = \text{velocity of Seeta with respect to Geeta}$$
$$= v_S - v_G = 1 - (-2) = 3 \text{ m/s}$$

Hence, velocity of Seeta with respect to Geeta is 3 m/s due east.

Sample Example 3.18 *Car A has an acceleration of 2 m/s² due east and car B, 4 m/s² due north. What is the acceleration of car B with respect to car A?*

Solution It is a two dimensional motion. Therefore,

$\vec{a}_{BA}$ = acceleration of car B with respect to car A

$$= \vec{a}_B - \vec{a}_A$$

Here, $\vec{a}_B$ = acceleration of car $B = 4 \text{ m/s}^2$ (due north)

and $\vec{a}_A$ = acceleration of car $A = 2 \text{ m/s}^2$ (due east)

$$|\vec{a}_{BA}| = \sqrt{(4)^2 + (2)^2} = 2\sqrt{5} \text{ m/s}^2$$

and $$\alpha = \tan^{-1}\left(\frac{4}{2}\right) = \tan^{-1}(2)$$

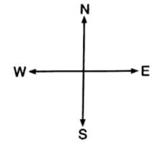


Fig. 3.24

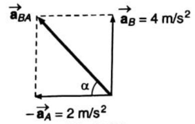


Fig. 3.25

Thus, $\vec{a}_{BA}$ is $2\sqrt{5}$ m/s² at an angle of $\alpha = \tan^{-1}(2)$ from west towards north.

The topic 'relative motion' is very useful in two and three dimensional motion. Questions based on relative motion are usually of following four types :

(a) Minimum distance between two bodies in motion
(b) River-boat problems
(c) Aircraft-wind problems
(d) Rain problems

(a) Minimum distance between two bodies in motion

When two bodies are in motion, the questions like, the minimum distance between them or the time when one body overtakes the other can be solved easily by the principle of relative motion. In these type of problems one body is assumed to be at rest and the relative motion of the other body is considered. By assuming so two body problem is converted into one body problem and the solution becomes easy. Following example will illustrate the statement.

Sample Example 3.19 *Car A and car B start moving simultaneously in the same direction along the line joining them. Car A with a constant acceleration $a = 4$ m/s², while car B moves with a constant velocity $v = 1$ m/s. At time $t = 0$, car A is 10 m behind car B. Find the time when car A overtakes car B.*

Solution Given : $u_A = 0$, $u_B = 1$ m/s, $a_A = 4$ m/s² and $a_B = 0$

Assuming car B to be at rest, we have

$$u_{AB} = u_A - u_B = 0 - 1 = -1 \text{ m/s}$$
$$a_{AB} = a_A - a_B = 4 - 0 = 4 \text{ m/s}^2$$

Now, the problem can be assumed in simplified form as follows :

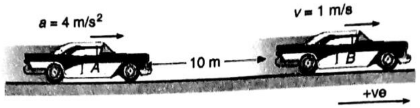


Fig. 3.26

Substituting the proper values in equation

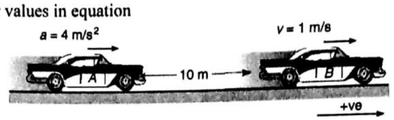


Fig. 3.27

$$s = ut + \frac{1}{2}at^2$$

we get $$10 = -t + \frac{1}{2}(4)(t^2)$$

or $$2t^2 - t - 10 = 0$$

or $$t = \frac{1 \pm \sqrt{1+80}}{4} = \frac{1 \pm \sqrt{81}}{4}$$

$$= \frac{1 \pm 9}{4} \quad \text{or} \quad t = 2.5\,\text{s} \quad \text{and} \quad -2\,\text{s}$$

Ignoring the negative value, the desired time is 2.5 s.

Note *The above problem can also be solved without using the concept of relative motion as under.*
At the time when A overtakes B,

$$s_A = s_B + 10$$

$\therefore$ $$\frac{1}{2} \times 4 \times t^2 = 1 \times t + 10$$

or $$2t^2 - t - 10 = 0$$

Which on solving gives t = 2.5 s and –2 s, the same as we found above.
As per my opinion, this approach (by taking absolute values) is more suitable in case of two body problem in one dimensional motion. Let us see one more example in support of it.

Sample Example 3.20 *An open lift is moving upwards with velocity 10 m/s. It has an upward acceleration of 2 m/s². A ball is projected upwards with velocity 20 m/s relative to ground. Find :*
(a) time when ball again meets the lift.
(b) displacement of lift and ball at that instant.
(c) distance travelled by the ball upto that instant. Take $g = 10\,m/s^2$

Solution (a) At the time when ball again meets the lift,

$$s_L = s_B$$

$\therefore$ $$10t + \frac{1}{2} \times 2 \times t^2 = 20t - \frac{1}{2} \times 10t^2$$

Solving this equation, we get

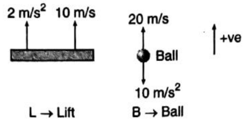


Fig. 3.28

$$t = 0 \quad \text{and} \quad t = \frac{5}{3}\,\text{s}$$

$\therefore$ Ball will again meet the lift after $\frac{5}{3}$ s.

(b) At this instant

$$s_L = s_B = 10 \times \frac{5}{3} + \frac{1}{2} \times 2 \times \left(\frac{5}{3}\right)^2 = \frac{175}{9}\,\text{m} = 19.4\,\text{m}$$

(c) For the ball $u \uparrow \downarrow a$. Therefore, we will first find t_0, the time when its velocity becomes zero.

$$t_0 = \left|\frac{u}{a}\right| = \frac{20}{10} = 2\,\text{s}$$

As $t\left(= \frac{5}{3}\,\text{s}\right) < t_0$, distance and displacement are equal

or $$d = 19.4\,\text{m}$$

Concept of relative motion is more useful in two body problem in two (or three) dimensional motion. This can be understood by the following example.

Sample Example 3.21 *Two ships A and B are 10 km apart on a line running south to north. Ship A farther north is streaming west at 20 km/h and ship B is streaming north at 20 km/h. What is their distance of closest approach and how long do they take to reach it?*

Solution Ships A and B are moving with same speed 20 km/h in the directions shown in figure. It is a two dimensional, two body problem with zero acceleration. Let us find $\vec{v}_{BA}$

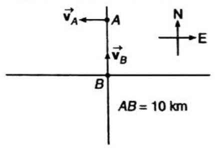


Fig. 3.29

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$

Here, $$|\vec{v}_{BA}| = \sqrt{(20)^2 + (20)^2}$$

$$= 20\sqrt{2}\,\text{km/h}$$

i.e., $\vec{v}_{BA}$ is $20\sqrt{2}$ km/h at an angle of 45° from east towards north. Thus, the given problem can be simplified as :

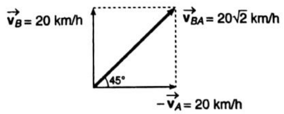


Fig. 3.30

A is at rest and B is moving with $\vec{v}_{BA}$ in the direction shown in Fig. 3.31. Therefore, the minimum distance between the two is

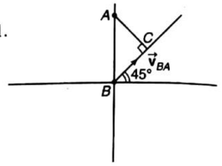


Fig. 3.31

$$s_{\min} = AC = AB \sin 45°$$

$$= 10\left(\frac{1}{\sqrt{2}}\right) \text{km} = 5\sqrt{2} \text{ km}$$

and the desired time is

$$t = \frac{BC}{|\vec{v}_{BA}|} = \frac{5\sqrt{2}}{20\sqrt{2}} \qquad (BC = AC = 5\sqrt{2} \text{ km})$$

$$= \frac{1}{4} \text{h} = 15 \text{ min}$$

(b) River-Boat Problems

In river-boat problems we come across the following three terms:

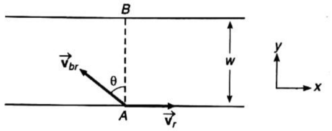


Fig. 3.32

$\vec{v}_r$ = absolute velocity of river

$\vec{v}_{br}$ = velocity of boatman with respect to river or velocity of boatman in still water

and $\vec{v}_b$ = absolute velocity of boatman.

Here, it is important to note that $\vec{v}_{br}$ is the velocity of boatman with which he steers and $\vec{v}_b$ is the actual velocity of boatman relative to ground.

Further, $\vec{v}_b = \vec{v}_{br} + \vec{v}_r$

Now, let us derive some standard results and their special cases.

A boatman starts from point A on one bank of a river with velocity $\vec{v}_{br}$ in the direction shown in Fig. 3.32. River is flowing along positive x-direction with velocity $\vec{v}_r$. Width of the river is w, then

$$\vec{v}_b = \vec{v}_r + \vec{v}_{br}$$

Therefore, $v_{bx} = v_{rx} + v_{brx} = v_r - v_{br} \sin\theta$

and $v_{by} = v_{ry} + v_{bry}$

$$= 0 + v_{br} \cos\theta = v_{br} \cos\theta$$

Now, time taken by the boatman to cross the river is :

$$t = \frac{w}{v_{by}} = \frac{w}{v_{br} \cos\theta}$$

or $$t = \frac{w}{v_{br}\cos\theta} \qquad \text{...(i)}$$

Further, displacement along x-axis when he reaches on the other bank (also called drift) is:

$$x = v_{bx}\, t = (v_r - v_{br}\sin\theta)\frac{w}{v_{br}\cos\theta}$$

or $$x = (v_r - v_{br}\sin\theta)\frac{w}{v_{br}\cos\theta} \qquad \text{...(ii)}$$

Three special cases are:

(i) Condition when the boatman crosses the river in shortest interval of time

From Eq. (i) we can see that time (t) will be minimum when $\theta = 0°$, *i.e.*, the boatman should steer his boat perpendicular to the river current.

Also, $t_{\min} = \dfrac{w}{v_{br}}$ as $\cos\theta = 1$

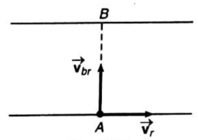


Fig. 3.33

(ii) Condition when the boatman wants to reach point B, *i.e.*, at a point just opposite from where he started

In this case, the drift (x) should be zero.

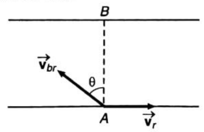


Fig. 3.34

∴ $x = 0$

or $(v_r - v_{br}\sin\theta)\dfrac{w}{v_{br}\cos\theta} = 0$

or $v_r = v_{br}\sin\theta$

or $\sin\theta = \dfrac{v_r}{v_{br}}$ or $\theta = \sin^{-1}\left(\dfrac{v_r}{v_{br}}\right)$

Hence, to reach point B the boatman should row at an angle $\theta = \sin^{-1}\left(\dfrac{v_r}{v_{br}}\right)$ upstream from AB.

Further, since $\sin\theta \not> 1$.

So, if $v_r \geq v_{br}$, the boatman can never reach at point B. Because if $v_r = v_{br}$, $\sin\theta = 1$ or $\theta = 90°$ and it is just impossible to reach at B if $\theta = 90°$. Moreover it can be seen that $v_b = 0$ if $v_r = v_{br}$ and $\theta = 90°$. Similarly, if $v_r > v_{br}$, $\sin\theta > 1$, *i.e.*, no such angle exists. Practically it can be realized in this manner that it is not possible to reach at B if river velocity (v_r) is too high.

(iii) Shortest path

Path length travelled by the boatman when he reaches the opposite shore is

$$s=\sqrt{w^2+x^2}$$

Here, w = width of river is constant. So for s to be minimum modulus of x (drift) should be minimum. Now two cases are possible.

When $v_r < v_{br}$: In this case $x=0$, when $\theta=\sin^{-1}\left(\frac{v_r}{v_{br}}\right)$

or $$s_{min}=w \quad \text{at} \quad \theta=\sin^{-1}\left(\frac{v_r}{v_{br}}\right)$$

When $v_r > v_{br}$: In this case x is minimum, where $\frac{dx}{d\theta}=0$

or $$\frac{d}{d\theta}\left\{\frac{w}{v_{br}\cos\theta}(v_r - v_{br}\sin\theta)\right\}=0$$

or $$-v_{br}\cos^2\theta-(v_r-v_{br}\sin\theta)(-\sin\theta)=0$$

or $$-v_{br}+v_r\sin\theta=0$$

or $$\theta=\sin^{-1}\left(\frac{v_{br}}{v_r}\right)$$

Now, at this angle we can find x_{min} and then s_{min} which comes out to be

$$s_{min}=w\left(\frac{v_r}{v_{br}}\right) \quad \text{at} \quad \theta=\sin^{-1}\left(\frac{v_{br}}{v_r}\right)$$

Sample Example 3.22 *A man can row a boat with 4 km/h in still water. If he is crossing a river where the current is 2 km/h.*

(a) In what direction will his boat be headed, if he wants to reach a point on the other bank, directly opposite to starting point?

(b) If width of the river is 4 km, how long will the man take to cross the river, with the condition in part (a)?

(c) In what direction should he head the boat if he wants to cross the river in shortest time and what is this minimum time?

(d) How long will it take him to row 2 km up the stream and then back to his starting point?

Solution (a) Given, that $v_{br}=4$ km/h and $v_r=2$ km/h

$$\therefore \quad \theta=\sin^{-1}\left(\frac{v_r}{v_{br}}\right)=\sin^{-1}\left(\frac{2}{4}\right)=\sin^{-1}\left(\frac{1}{2}\right)=30^\circ$$

Hence, to reach the point directly opposite to starting point he should head the boat at an angle of 30° with AB or 90° + 30° = 120° with the river flow.

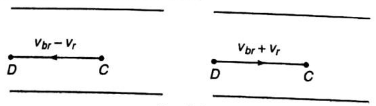


Fig. 3.35

(b) Time taken by the boatman to cross the river

$$w = \text{width of river} = 4 \text{ km}$$

$$v_{br}=4 \text{ km/h} \quad \text{and} \quad \theta=30^\circ$$

$$\therefore \quad t=\frac{4}{4\cos 30^\circ}=\frac{2}{\sqrt{3}}\text{ h}$$

(c) For shortest time $\theta=0^\circ$

and $$t_{min}=\frac{w}{v_{br}\cos 0^\circ}=\frac{4}{4}=1\text{ h}$$

Hence, he should head his boat perpendicular to the river current for crossing the river in shortest time and this shortest time is 1 h.

(d) $t=t_{CD}+t_{DC}$

or $$t=\frac{CD}{v_{br}-v_r}+\frac{DC}{v_{br}+v_r}$$

$$=\frac{2}{4-2}+\frac{2}{4+2}=1+\frac{1}{3}=\frac{4}{3}\text{ h}$$

(c) Aircraft Wind Problems

This is similar to river boat problem. The only difference is that $\vec{v}_{br}$ is replaced by $\vec{v}_{aw}$ (velocity of aircraft with respect to wind or velocity of aircraft in still air), $\vec{v}_r$ is replaced by $\vec{v}_w$ (velocity of wind) and $\vec{v}_b$ is replaced by $\vec{v}_a$ (absolute velocity of aircraft). Further, $\vec{v}_a=\vec{v}_{aw}+\vec{v}_w$. The following example will illustrate the theory.

Sample Example 3.23 *An aircraft flies at 400 km/h in still air. A wind of $200\sqrt{2}$ km/h is blowing from the south. The pilot wishes to travel from A to a point B north east of A. Find the direction he must steer and time of his journey if AB = 1000 km.*

Solution Given that $v_w=200\sqrt{2}$ km/h

$v_{aw}=400$ km/h and $\vec{v}_a$ should be along AB or in north-east direction. Thus, the direction of $\vec{v}_{aw}$ should be such as the resultant of $\vec{v}_w$ and $\vec{v}_{aw}$ is along AB or in north-east direction.

Let $\vec{v}_{aw}$ makes an angle α with AB as shown in Fig. 3.36. Applying sine law in triangle ABC, we get

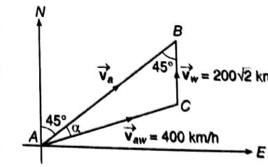


Fig. 3.36

$$\frac{AC}{\sin 45^\circ}=\frac{BC}{\sin\alpha}$$

or $$\sin\alpha=\left(\frac{BC}{AC}\right)\sin 45^\circ=\left(\frac{200\sqrt{2}}{400}\right)\frac{1}{\sqrt{2}}=\frac{1}{2}$$

$$\therefore \quad \alpha=30^\circ$$

Therefore, the pilot should steer in a direction at an angle of $(45^\circ+\alpha)$ or 75° from north towards east.

Further,
$$\frac{|\vec{v}_a|}{\sin(180^\circ - 45^\circ - 30^\circ)} = \frac{400}{\sin 45^\circ}$$

or
$$|\vec{v}_a| = \frac{\sin 105^\circ}{\sin 45^\circ} \times (400)\frac{\text{km}}{\text{h}}$$

$$= \left(\frac{\cos 15^\circ}{\sin 45^\circ}\right)(400)\frac{\text{km}}{\text{h}} = \left(\frac{0.9659}{0.707}\right)(400)\frac{\text{km}}{\text{h}}$$

$$= 546.47 \text{ km/h}$$

$\therefore$ The time of journey from A to B is

$$t = \frac{AB}{|\vec{v}_a|} = \frac{1000}{546.47}\text{ h}$$

$$t = 1.83 \text{ h}$$

(d) Rain Problems

In these type of problems we again come across three terms $\vec{v}_r$, $\vec{v}_m$ and $\vec{v}_{rm}$. Here,

$\vec{v}_r$ = velocity of rain

$\vec{v}_m$ = velocity of man (it may be velocity of cyclist or velocity of motorist also)

and $\vec{v}_{rm}$ = velocity of rain with respect to man.

Here, $\vec{v}_{rm}$ is the velocity of rain which appears to the man. Now, let us take one example of this.

Sample Example 3.24 *To a man walking at the rate of 3 km/h the rain appears to fall vertically. When he increases his speed to 6 km/h it appears to meet him at an angle of 45° with vertical. Find the speed of rain.*

Solution Let $\hat{i}$ and $\hat{j}$ be the unit vectors in horizontal and vertical directions respectively.

Let velocity of rain

$$\vec{v}_r = a\hat{i} + b\hat{j} \quad \text{...(i)}$$

Then speed of rain will be

$$|\vec{v}_r| = \sqrt{a^2 + b^2} \quad \text{...(ii)}$$

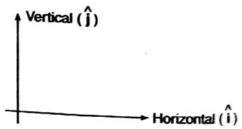


Fig. 3.37

In the first case

$$\vec{v}_m = \text{velocity of man} = 3\hat{i}$$

$\therefore$
$$\vec{v}_{rm} = \vec{v}_r - \vec{v}_m = (a-3)\hat{i} + b\hat{j}$$

It seems to be in vertical direction. Hence,

$$a - 3 = 0 \quad \text{or} \quad a = 3$$

In the second case
$$\vec{v}_m = 6\hat{i}$$

$\therefore$
$$\vec{v}_{rm} = (a-6)\hat{i} + b\hat{j} = -3\hat{i} + b\hat{j}$$

This seems to be at 45° with vertical.

Hence, $|b| = 3$

Therefore, from Eq. (ii) speed of rain is

$$|\vec{v}_r| = \sqrt{(3)^2 + (3)^2} = 3\sqrt{2} \text{ km/h}$$

Introductory Exercise 3.4

1. Two balls A and B are projected vertically upwards with different velocities. What is the relative acceleration between them?
2. In the above problem what is the shape of the graph between distance between the balls and time before either of the two collide with ground?
3. A river 400 m wide is flowing at a rate of 2.0 m/s. A boat is sailing at a velocity of 10.0 m/s with respect to the water in a direction perpendicular to the river.
   (a) Find the time taken by the boat to reach the opposite bank.
   (b) How far from the point directly opposite to the starting point does the boat reach the opposite bank?
4. An aeroplane has to go from a point A to another point B, 500 km away due 30° east of north. Wind is blowing due north at a speed of 20 m/s. The air-speed of the plane is 150 m/s. (a) Find the direction in which the pilot should head the plane to reach the point B. (b) Find the time taken by the plane to go from A to B.
5. Two particles A and B start moving simultaneously along the line joining them in the same direction with acceleration of 1 m/s^2 and 2 m/s^2 and speeds 3 m/s and 1 m/s respectively. Initially A is 10 m behind B. What is the minimum distance between them?

Extra Points

- If y (may be velocity, acceleration etc.) is a function of time or $y = f(t)$ and we want to find the average value of y between a time interval of t_1 and t_2. Then,

$$\langle y \rangle_{t_1 \text{ to } t_2} = \text{average value of } y \text{ between } t_1 \text{ and } t_2$$
$$= \frac{\int_{t_1}^{t_2} f(t)\,dt}{t_2 - t_1}$$

or
$$\langle y \rangle_{t_1 \text{ to } t_2} = \frac{\int_{t_1}^{t_2} f(t)\,dt}{t_2 - t_1}$$

But if $f(t)$ is a linear function of t then

$$y_{av} = \frac{y_f + y_i}{2}$$

Here, y_f = final value of y and y_i = initial value of y

At the same time we should not forget that

$$v_{av} = \frac{\text{total displacement}}{\text{total time}}$$

and
$$a_{av} = \frac{\text{change in velocity}}{\text{total time}}$$

Example : In one dimensional uniformly accelerated motion. Find average velocity from $t = 0$ to $t = t$.

Solution : We can solve this problem by three methods.

Method 1. $v = u + at$

$$\therefore \quad \langle v \rangle_{0-t} = \frac{\int_0^t (u + at)\,dt}{t - 0}$$

$$= u + \frac{1}{2}at$$

Method 2. Since v is a linear function of time, we can write

$$v_{av} = \frac{v_f + v_i}{2} = \frac{(u + at) + u}{2}$$

$$= u + \frac{1}{2}at$$

Method 3.

$$v_{av} = \frac{\text{Total displacement}}{\text{Total time}}$$

$$= \frac{ut + \frac{1}{2}at^2}{t} = u + \frac{1}{2}at$$

- A particle is thrown upwards with velocity u. Suppose it takes time t to reach its highest point, then distance travelled in last second is independent of u.
This is because this distance is equal to the distance travelled in first second of a freely falling object. Thus,

$$s = \frac{1}{2}g \times (1)^2$$

$$= \frac{1}{2} \times 10 \times 1 = 5 \text{ m}$$

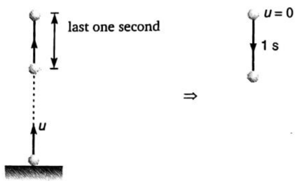


Fig. 3.38

Exercise : A particle is thrown upwards with velocity u (>20 m/s). Prove that distance travelled in last 2 second is 20 m.

- Suppose we have given velocity-time v-t graph. We want to plot corresponding displacement-time s-t graph then values of displacements at different times can be found just by adding the corresponding areas under v-t graph.
- The modulus of velocity is really the speed or

$$|\vec{v}| = v$$

- Rate of change of velocity is acceleration, while rate of change of speed is the tangential acceleration (component of acceleration along velocity). Thus,

$$\frac{d\vec{v}}{dt} = \vec{a}$$

while

$$\frac{dv}{dt} = \frac{d\,|\vec{v}|}{dt} = a_t$$

- Angle between velocity vector $\vec{v}$ and acceleration vector $\vec{a}$ decides whether the speed of particle is increasing, decreasing or constant.

Speed increases, if	$0° \le \theta < 90°$
Speed decreases, if	$90° < \theta \le 180°$
Speed is constant, if	$\theta = 90°$

The angle θ between $\vec{v}$ and $\vec{a}$ can be obtained by the relation,

$$\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{a}}{va}\right)$$

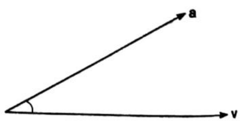


Fig. 3.39

Exercise : Prove that speed of a particle increases if dot product of $\vec{v}$ and $\vec{a}$ is positive, speed decreases, if the dot product is negative and speed remains constant if dot product is zero.

- It is found that problems of two (or three) dimensional motion become easy by writing the different vectors in terms of $\hat{i}$, $\hat{j}$ and $\hat{k}$ rather by using graphical method of vector addition or subtraction. For details you can refer Sample Example 3.24 in article 3.7.
- The magnitude of instantaneous velocity is called the instantaneous speed, *i.e.*,

$$v = |\vec{v}| = \left|\frac{d\vec{r}}{dt}\right|$$

Speed is not equal to $\frac{dr}{dt}$, *i.e.*, $\quad v \ne \frac{dr}{dt}$

where r is the modulus of radius vector $\vec{r}$ because in general case $|d\vec{r}| \ne dr$. For example when $\vec{r}$ changes only in direction, *i.e.*, if a point moves in a circle, then r = constant, $dr = 0$ but $|d\vec{r}| \ne 0$.

- I have personally felt that the concept which I am going to present here becomes very useful while dealing with addition or subtraction of two vectors. Suppose a vector $\vec{C}$ is a vector sum of two vectors $\vec{A}$ and $\vec{B}$ and the direction of $\vec{C}$ is given to us. Say the vector $\vec{C}$ has to be along a line PQ. Then $\vec{A} + \vec{B}$ should be along PQ or sum of components of $\vec{A}$ and $\vec{B}$ perpendicular to line PQ should be zero. Similarly, if $\vec{C} = \vec{A} - \vec{B}$ and $\vec{C}$ has to be along the line PQ, then the sum of components of $\vec{A}$ and $-\vec{B}$ perpendicular to line PQ should be zero.
For instance, in example 2.29, $\vec{v}_a$ has to be along AB and we know that $\vec{v}_a = \vec{v}_{aw} + \vec{v}_w$. Therefore, sum of components of $\vec{v}_{aw}$ and $\vec{v}_w$ perpendicular to line AB (shown as dotted) should be zero.

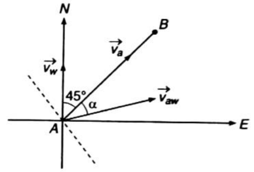


Fig. 3.40

or $\quad |\vec{v}_{aw}| \sin\alpha = |\vec{v}_w| \sin 45°$

or $\quad \sin\alpha = \frac{|\vec{v}_w|}{|\vec{v}_{aw}|} \sin 45°$

$$= \left(\frac{200\sqrt{2}}{400}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$

$\therefore \quad \alpha = 30°$

Now, $\quad |\vec{v}_a| = |\vec{v}_{aw}| \cos\alpha + |\vec{v}_w| \cos 45°$

$$= (400)\cos 30° + (200\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) = (400)\frac{\sqrt{3}}{2} + 200$$

$$= 346.47 + 200 = 546.47 \text{ km/h}$$

$\therefore$ Time of journey from A to B will be

$$t = \frac{AB}{|\vec{v}_a|} = \frac{1000}{546.47} = 1.83 \text{ h}$$

- I have found students often confused over the sign of 'g'. As per our sign convention (positive upwards and negative downwards) it is always negative, whether the particle is moving upwards or downwards. Now if u is upwards (*i.e.*, $u \uparrow \downarrow g$) motion is retarded and if u is either zero or downwards ($u \uparrow\uparrow g$) motion is accelerated.

Solved Examples

Level 1

Example 1 *A farmer has to go 500 m due north, 400 m due east and 200 m due south to reach his field. If he takes 20 minutes to reach the field.*
(a) What distance he has to walk to reach the field ?
(b) What is the displacement from his house to the field ?
(c) What is the average speed of farmer during the walk ?
(d) What is the average velocity of farmer during the walk ?

Solution

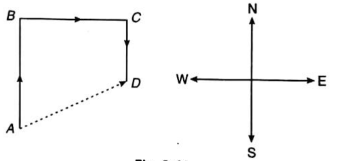


Fig. 3.41

(a) Distance $= AB + BC + CD = (500 + 400 + 200) = 1100\ \text{m}$

(b) Displacement $= AD = \sqrt{(AB - CD)^2 + BC^2}$

$$= \sqrt{(500 - 200)^2 + (400)^2}$$

$$= 500\ \text{m}$$

(c) Average speed $= \dfrac{\text{Total distance}}{\text{Total time}} = \dfrac{1100}{20} = 55\ \text{m/min}$

(d) Average velocity $= \dfrac{AD}{t} = \dfrac{500}{20} = 25\ \text{m/min (along } AD)$

Example 2 *A particle starts with an initial velocity and passes successively over the two halves of a given distance with accelerations a_1 and a_2 respectively. Show that the final velocity is the same as if the whole distance is covered with a uniform acceleration $\dfrac{(a_1 + a_2)}{2}$.*

Solution In the first case,

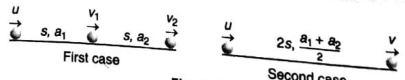


Fig. 3.42

$$v_1^2 = u^2 + 2a_1 s \qquad \ldots\text{(i)}$$

$$v_2^2 = v_1^2 + 2a_2 s \qquad \ldots\text{(ii)}$$

Adding Eqs. (i) and (ii), we have

$$v_2^2 = u^2 + 2\left(\frac{a_1 + a_2}{2}\right)(2s) \qquad \ldots\text{(iii)}$$

In the second case

$$v^2 = u^2 + 2\left(\frac{a_1 + a_2}{2}\right)(2s) \qquad \ldots\text{(iv)}$$

From Eqs. (iii) and (iv), we can see that

$$v_2 = v$$

Hence proved.

Example 3 *In a car race, car A takes a time t less than car B at the finish and passes the finishing point with speed v more than that of the car B. Assuming that both the cars starts from rest and travel with constant acceleration a_1 and a_2 respectively. Show that $v = \sqrt{a_1 a_2}\, t$.*

Solution Let A takes t_1 second, then according to the given problem B will take $(t_1 + t)$ seconds. Further, let v_1 be the velocity of B at finishing point, then velocity of A will be $(v_1 + v)$. Writing equations of motion for A and B.

$$v_1 + v = a_1 t_1 \qquad \ldots\text{(i)}$$

$$v_1 = a_2 (t_1 + t) \qquad \ldots\text{(ii)}$$

From these two equations, we get

$$v = (a_1 - a_2) t_1 - a_2 t \qquad \ldots\text{(iii)}$$

Total distance travelled by both the cars is equal.

or $\qquad s_A = s_B$

or $\qquad \dfrac{1}{2} a_1 t_1^2 = \dfrac{1}{2} a_2 (t_1 + t)^2$

or $\qquad t_1 = \dfrac{\sqrt{a_2}\, t}{\sqrt{a_1} - \sqrt{a_2}}$

Substituting this value of t_1 in Eq. (iii), we get the desired result

or $\qquad v = (\sqrt{a_1 a_2})\, t$

Example 4 *A particle is moving with a velocity of $v = (3 + 6t + 9t^2)$ cm/s. Find out :*
(a) the acceleration of the particle at $t = 3$ s.
(b) the displacement of the particle in the interval $t = 5$ s to $t = 8$ s.

Solution (a) Acceleration of particle,

$$a = \frac{dv}{dt} = (6 + 18t)\ \text{cm/s}^2$$

At $t = 3$ s,

$$a = (6 + 18 \times 3)\ \text{cm/s}^2$$

$$= 60\ \text{cm/s}^2$$

(b) Given, $\qquad v = (3 + 6t + 9t^2)$ cm/s

or $\qquad \dfrac{ds}{dt} = (3 + 6t + 9t^2)$

or $\qquad ds = (3 + 6t + 9t^2)\, dt$

$$\therefore \quad \int_0^s ds = \int_5^8 (3 + 6t + 9t^2)\, dt$$

$$\therefore \quad s = [3t + 3t^2 + 3t^3]_5^8$$

or $$s = 1287 \text{ cm}$$

Example 5 *The motion of a particle along a straight line is described by the function $x = (2t - 3)^2$ where x is in metres and t is in seconds.*
(a) Find the position, velocity and acceleration at $t = 2$ s.
(b) Find velocity of the particle at origin.

Solution (a) Position, $x = (2t - 3)^2$

Velocity, $$v = \frac{dx}{dt} = 4(2t - 3) \text{ m/s}$$

and acceleration, $$a = \frac{dv}{dt} = 8 \text{ m/s}^2$$

At $t = 2$ s, $$x = (2 \times 2 - 3)^2 = 1.0 \text{ m}$$

$$v = 4(2 \times 2 - 3) = 4 \text{ m/s}$$

and $$a = 8 \text{ m/s}^2$$

At origin, $x = 0$

or $$(2t - 3) = 0$$

$$\therefore \quad v = 4 \times 0 = 0$$

Example 6 *An open elevator is ascending with zero acceleration. The speed $v = 10$ m/s. A ball is thrown vertically up by a boy when he is at a height $h = 10$ m from the ground. The velocity of projection is $v = 30$ m/s with respect to elevator. Find:*
(a) the maximum height attained by the ball.
(b) the time taken by the ball to meet the elevator again.
(c) time taken by the ball to reach the ground after crossing the elevator.

Solution (a) Absolute velocity of ball = 40 m/s (upwards)

$$\therefore \quad h_{max} = h_i + h_f$$

Here, h_i = initial height = 10 m

and h_f = further height attained by ball

$$= \frac{u^2}{2g} = \frac{(40)^2}{2 \times 10} = 80 \text{ m}$$

$$\therefore \quad h_{max} = (10 + 80) \text{ m} = 90 \text{ m}$$

(b) The ball will meet the elevator again when displacement of lift = displacement of ball

or $$10 \times t = 40 \times t - \frac{1}{2} \times 10 \times t^2$$

or $$t = 6 \text{ s}$$

(c) Let t_0 be the total time taken by the ball to reach the ground. Then,

$$-10 = 40 \times t_0 - \frac{1}{2} \times 10 \times t_0^2$$

Therefore, time taken by the ball to reach the ground after crossing the elevator,

$$= (t_0 - t) = 2.24 \text{ s}$$

Example 7 *From an elevated point A, a stone is projected vertically upwards. When the stone reaches a distance h below A, its velocity is double of what it was at a height h above A. Show that the greatest height attained by the stone is $\frac{5}{3}h$.*

Solution Let u be the velocity with which the stone is projected vertically upwards.

Given that, $$v_{-h} = 2v_h$$

or $$(v_{-h})^2 = 4v_h^2$$

$$\therefore \quad u^2 - 2g(-h) = 4(u^2 - 2gh)$$

$$\therefore \quad u^2 = \frac{10gh}{3}$$

Now, $$h_{max} = \frac{u^2}{2g} = \frac{5h}{3}$$ **Hence proved.**

Example 8 *A man crosses a river in a boat. If he cross the river in minimum time he takes 10 min with a drift 120 m. If he crosses the river taking shortest path, he takes 12.5 min, find :*
(a) width of the river.
(b) velocity of the boat with respect to water.
(c) speed of the current.

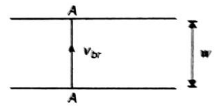
Solution Let v_r = velocity of river

v_{br} = velocity of river in still water and

w = width of river

Given, $$t_{min} = 10 \text{ min}$$

or $$\frac{w}{v_{br}} = 10 \qquad \text{...(i)}$$



For minimum time

Fig. 3.43

Drift in this case will be,

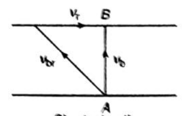
$$x = v_r t$$

$$\therefore \quad 120 = 10 v_r \qquad \text{...(ii)}$$

Shortest path is taken when v_b is along AB. In this case,

$$v_b = \sqrt{v_{br}^2 - v_r^2}$$

Now, $$12.5 = \frac{w}{v_b} = \frac{w}{\sqrt{v_{br}^2 - v_r^2}} \qquad \text{...(iii)}$$



Shortest path

Fig. 3.44

Solving these three equations, we get

$$v_{br} = 20 \text{ m/min},$$

$$v_r = 12 \text{ m/min}$$

and $$w = 200 \text{ m}$$

Example 9 *The acceleration versus time graph of a particle moving along a straight line is shown in the figure. Draw the respective velocity-time graph. Given $v = 0$ at $t = 0$.*

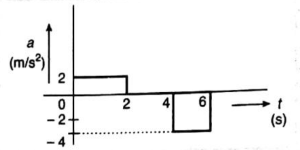


Fig. 3.45

Solution From $t = 0$ to $t = 2$ s, $a = +2\,\text{m/s}^2$

$\therefore$
$$v = at = 2t$$

or v-t graph is a straight line passing through origin with slope $2\,\text{m/s}^2$.

At the end of 2 s,

$$v = 2 \times 2 = 4\,\text{m/s}$$

From $t = 2$ to 4 s, $a = 0$. Hence, $v = 4$ m/s will remain constant.

From $t = 4$ to 6 s, $a = -4\,\text{m/s}^2$. Hence,

$$v = u - at = 4 - 4t \qquad \text{(with } t = 0 \text{ at 4 s)}$$

$v = 0$ at $t = 1$ s or at 5 s from origin.

At the end of 6 s (or $t = 2$ s) $v = -4$ m/s. Corresponding v-t graph is as shown below :

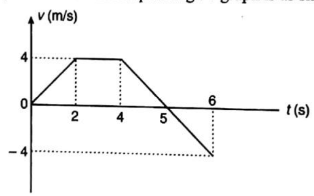


Fig. 3.46

Example 10 *A ball is thrown upward with an initial velocity of $100\,ms^{-1}$. After how much time will it return? Draw velocity-time graph for the ball and find from the graph (i) the maximum height attained by the ball and (ii) height of the ball after 15 s. Take $g = 10\,ms^{-2}$.*

Solution Here, $\quad u = 100\,\text{ms}^{-1}, \quad g = -10\,\text{ms}^{-2}$

At highest point, $\quad v = 0$

As $\quad v = u + gt \quad \therefore 0 = 100 - 10 \times t$

$\therefore$ Time taken to reach highest point,

$$t = \frac{100}{10} = 10\,\text{s}$$

The ball will return to the ground at $t = 20$ s.

Corresponding velocity-time graph of the ball is shown in Fig. 3.47.

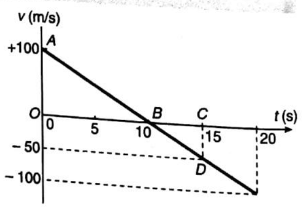


Fig. 3.47

(i) Maximum height attained by the ball = Area of ΔAOB

$$= \frac{1}{2} \times 10 \times 100 = 500\,\text{m}$$

(ii) Height attained after 15 s = Area of ΔAOB + Area of ΔBCD

$$= 500 + \frac{1}{2}(15 - 10) \times (-50) = 500 - 125 = 375\,\text{m}$$

Example 11 *A rocket is fired vertically up from the ground with a resultant vertical acceleration of $10\,m/s^2$. The fuel is finished in 1 min and it continues to move up.*

(a) What is the maximum height reached? (b) After how much time from then will the maximum height be reached? (Take $g = 10\,m/s^2$)

Solution (a) The distance travelled by the rocket in 1 min (= 60 s) in which resultant acceleration is vertically upwards and $10\,\text{m/s}^2$ will be

$h = ut - \frac{1}{2} g t^2$

$$h_1 = (1/2) \times 10 \times 60^2 = 18000\,\text{m} = 18\,\text{km} \qquad \text{...(i)}$$

and velocity acquired by it will be $\quad v = 10 \times 60 = 600\,\text{m/s} \qquad$ ...(ii)

Now, after 1 min the rocket moves vertically up with velocity of 600 m/s and acceleration due to gravity opposes its motion. So, it will go to a height h_2 till its velocity becomes zero such that

$$0 = (600)^2 - 2gh_2 \qquad v^2 = u^2 - 2gh.$$

or
$$h_2 = 18000\,\text{m} \quad [\text{as } g = 10\,\text{m/s}^2] \qquad \text{...(iii)}$$
$$= 18\,\text{km}$$

So, from Eqs. (i) and (iii) the maximum height reached by the rocket from the ground

$$h = h_1 + h_2 = 18 + 18 = 36\,\text{km}$$ **Ans.**

(b) As after burning of fuel the initial velocity from Eq. (ii) is 600 m/s and gravity opposes the motion of rocket, so the time taken by it to reach the maximum height (for which $v = 0$),

$$0 = 600 - gt \quad \text{or} \quad t = 60\,\text{s}$$

i.e., after finishing fuel the rocket further goes up for 60 s, or 1 min. **Ans.**

Level 2

Note 1. *In one dimensional motion, if a = constant, we use $v = u + at$, etc., with proper signs and if $a \neq$ constant, we go for integration and differentiation.*

2. *Sometimes the standard results are written in different manners and the students unnecessarily go on integrating or differentiating. The standard results which are usually altered are :*

(i) $v = u + at$
(ii) $s = ut + \frac{1}{2}at^2$
(iii) $v^2 = u^2 + 2as$

} *These are the equations of motion in one dimension with constant acceleration*

(iv) $v = \omega\sqrt{A^2 - x^2}$
(v) $a = -\omega^2 x$

} *These are the equations of simple harmonic motion.*

The above point will be more clear after going through following two examples 1 and 2.

Example 1 *Velocity of a particle moving in a straight line varies with its displacement as* $v=(\sqrt{4+4s})$ *m/s. Displacement of particle at time* $t=0$ *is* $s=0$. *Find displacement of particle at time* $t=2$ *s.*

Solution Squaring the given equation, we get

$$v^2=4+4s$$

Now, comparing it with

$$v^2=u^2+2as$$

we get,

$$u=2\text{ m/s} \quad \text{and} \quad a=2\text{ m/s}^2$$

$\therefore$ Displacement at $t=2$s is

$$s=ut+\frac{1}{2}at^2 \quad \text{or} \quad s=(2)(2)+\frac{1}{2}(2)(2)^2$$

or

$$s=8\text{ m}$$

Example 2 *Velocity of a particle varies with its displacement as*

$$v=(\sqrt{9-x^2})\text{ m/s}$$

Find the magnitude of maximum acceleration of the particle.

Solution Comparing the given equation with standard velocity-displacement equation of simple harmonic motion, *i.e.*, $v=\omega\sqrt{A^2-x^2}$, we get

$$\omega=1\text{ rad/s} \quad \text{and} \quad A=3\text{ m}$$

The magnitude of maximum acceleration of the particle in SHM is $=\omega^2 A$

$$=(1)^2(3)\text{ m/s}^2=3\text{ m/s}^2$$

Example 3 *Fig. 3.48 shows a rod of length l resting on a wall and the floor. Its lower end A is pulled towards left with a constant velocity v. Find the velocity of the other end B downward when the rod makes an angle θ with the horizontal.*

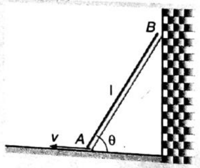


Fig. 3.48

Solution In such type of problems, when velocity of one part of a body is given and that of other is required, we first find the relation between the two displacements, then differentiate them with respect to time. Here, if the distance from the corner to the point A is x and that up to B is y. Then,

$$v=\frac{dx}{dt}$$

and

$$v_B=-\frac{dy}{dt} \quad (-\text{ sign denotes that } y \text{ is decreasing})$$

Further,

$$x^2+y^2=l^2$$

Differentiating with respect to time t

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$xv=yv_B$$

$$v_B=\frac{x}{y}v=v\cot\theta$$

Example 4 *A particle is moving in a straight line with constant acceleration. If x, y and z be the distances described by a particle during the pth, qth and rth second respectively, prove that*

$$(q-r)x+(r-p)y+(p-q)z=0$$

Solution As

$$s_{nth}=u+an-\frac{1}{2}a=u+\frac{a}{2}(2n-1)$$

$\therefore$

$$x=u+\frac{a}{2}(2p-1) \quad \text{...(i)}$$

$$y=u+\frac{a}{2}(2q-1) \quad \text{...(ii)}$$

$$z=u+\frac{a}{2}(2r-1) \quad \text{...(iii)}$$

Subtracting Eq. (iii) from Eq. (ii),

$$y-z=\frac{a}{2}(2q-2r) \quad \text{or} \quad q-r=\frac{y-z}{a}$$

or

$$(q-r)x=\frac{1}{a}(yx-zx) \quad \text{...(iv)}$$

Similarly, we can show that

$$(r-p)y=\frac{1}{a}(zy-xy) \quad \text{...(v)}$$

and

$$(p-q)z=\frac{1}{a}(xz-yz) \quad \text{...(vi)}$$

Adding Eqs. (iv), (v) and (vi), we get

$$(q-r)x+(r-p)y+(p-q)z=0$$

Example 5 *A balloon starts rising from the earth's surface. The ascension rate is constant and equal to* v_0. *Due to the wind, the balloon gathers the horizontal velocity component* $v_x=ky$, *where k is a constant and y is the height of ascent. Find how the following quantities depend on the height of ascent:*
(a) the horizontal drift of the balloon $x(y)$.
(b) the total, tangential and normal accelerations of the balloon.

Solution (a)

$$\frac{dy}{dt}=v_0 \quad \text{...(i)}$$

$$\frac{dx}{dt}=ky \quad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{dy}{dx}=\frac{v_0}{ky} \quad \text{or} \quad dx=\frac{ky}{v_0}dy$$

Intergrating, we get $$x = \frac{k}{v_0}\left(\frac{y^2}{2}\right)$$

This is the desired trajectory of the particle, which is an equation of a parabola.

(b) For finding the tangential and normal accelerations, we require an expression for the speed as a function of height y

$$v_y = v_0 \quad \text{and} \quad v_x = ky$$

$\therefore$ $$v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_0^2 + k^2 y^2}$$

Therefore, tangential acceleration,

$$a_t = \frac{dv}{dt} = \frac{k^2 y}{\sqrt{v_0^2 + k^2 y^2}} \frac{dy}{dt}$$

$$= \frac{k^2 y v_0}{\sqrt{v_0^2 + k^2 y^2}}$$

or $$a_t = \frac{k^2 y}{\sqrt{1 + k^2 y^2 / v_0^2}}$$

Now, the total acceleration is,

$$a = \sqrt{\left(\frac{dv_y}{dt}\right)^2 + \left(\frac{dv_x}{dt}\right)^2}$$

$$= \frac{dv_x}{dt} = k\frac{dy}{dt} = kv_0$$

$\therefore$ Normal acceleration, $$a_n = \sqrt{a^2 - a_t^2} = \frac{kv_0}{\sqrt{1 + \left(\frac{ky}{v_0}\right)^2}}$$

Example 6 *Three particles A, B and C are situated at the vertices of an equilateral triangle ABC of side d at time $t = 0$. Each of the particles moves with constant speed v. A always has its velocity along AB, B along BC and C along CA. At what time will the particles meet each other?*

Solution Velocity of A is v along AB. The velocity of B is along BC. Its component along BA is $v \cos 60° = v/2$. Thus, the separation AB decreases at the rate

$$v + \frac{v}{2} = \frac{3v}{2}$$

Since, this rate is constant, the time taken in reducing the separation AB from d to zero is

$$t = \frac{d}{(3v/2)} = \frac{2d}{3v}$$

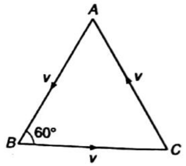


Fig. 3.49

Example 7 *An elevator car whose floor to ceiling distance is equal to 2.7 m starts ascending with constant acceleration 1.2 m/s^2. 2 s after the start a bolt begins falling from the ceiling of the car. Find:*

(a) the time after which bolt hits the floor of the elevator.

(b) the net displacement and distance travelled by the bolt, with respect to earth.

(Take $g = 9.8\ m/s^2$)

Solution (a) If we consider elevator at rest, then relative acceleration of the bolt is $a_r = 9.8 + 1.2$

$$= 11\ \text{m/s}^2 \text{ (downwards)}$$

After 2 s velocity of lift is $v = at = (1.2)(2) = 2.4$ m/s. Therefore initial velocity of the bolt is also 2.4 m/s and it gets accelerated with relative acceleration 11 m/s^2. With respect to elevator initial velocity of bolt is zero and it has to travel 2.7 m with 11 m/s^2. Thus, time taken can be directly given as

$$\sqrt{\frac{2s}{a}} = \sqrt{\frac{2 \times 2.7}{11}} = 0.7\ \text{s.}$$

(b) Displacement of bolt relative to ground in 0.7 s.

$$s = ut + \frac{1}{2}at^2$$

or $$s = (2.4)(0.7) + \frac{1}{2}(-9.8)(0.7)^2$$

$$s = -0.72\ \text{m}$$

Velocity of bolt will become zero after a time

$$t_0 = \frac{u}{g} \qquad (v = u - gt)$$

$$= \frac{2.4}{9.8} = 0.245\ \text{s}$$

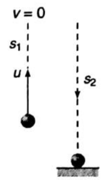


Fig. 3.50

Therefore, distance travelled by the bolt $= s_1 + s_2$

$$= \frac{u^2}{2g} + \frac{1}{2}g(t - t_0)^2$$

$$= \frac{(2.4)^2}{2 \times 9.8} + \frac{1}{2} \times 9.8(0.7 - 0.245)^2 = 1.3\ \text{m}$$

Example 8 *A man wants to reach point B on the opposite bank of a river flowing at a speed as shown in figure. What minimum speed relative to water should the man have so that he can reach point B? In which direction should he swim?*

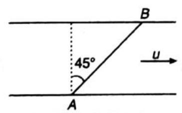


Fig. 3.51

Solution Let v be the speed of boatman in still water.

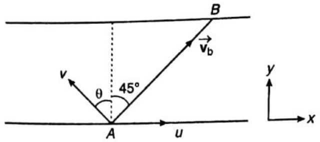


Fig. 3.52

Resultant of v and u should be along AB. Components of $\vec{v}_b$ (absolute velocity of boatman) along x and y-direction are,

$$v_x = u - v\sin\theta \quad \text{and} \quad v_y = v\cos\theta$$

Further,

$$\tan 45° = \frac{v_y}{v_x}$$

or

$$1 = \frac{v\cos\theta}{u - v\sin\theta}$$

$\therefore$

$$v = \frac{u}{\sin\theta + \cos\theta} = \frac{u}{\sqrt{2}\sin(\theta + 45°)}$$

v is minimum at,

$$\theta + 45° = 90° \quad \text{or} \quad \theta = 45°$$

and

$$v_{min} = \frac{u}{\sqrt{2}}$$

EXERCISES

AIEEE Corner

Subjective Questions (Level 1)

Basic Definitions

1. A car moves with 60 km/h in first one hour and with 80 km/h in next half an hour. Find :
(a) total distance travelled by the car, (b) average speed of car in total 1.5 hours.

2. A particle moves in a straight line with initial velocity 4 m/s and a constant acceleration of $6\ m/s^2$. Find the average velocity of the particle in a time interval from
(a) $t = 0$ to $t = 2$ s (b) $t = 2$ s to $t = 4$ s.

3. A particle is projected upwards from the roof of a tower 60 m high with velocity 20 m/s. Find :
(a) the average speed and
(b) average velocity of the particle upto an instant when it strikes the ground. Take $g = 10\ m/s^2$.

4. A block moves in a straight line with velocity v for time t_0. Then, its velocity becomes $2v$ for next t_0 time. Finally its velocity becomes $3v$ for time T. If average velocity during the complete journey was $2.5\ v$, then find T in terms of t_0.

5. A particle starting from rest has a constant acceleration of $4\ m/s^2$ for 4 s. It then retards uniformly for next 8 s and comes to rest. Find during the motion of particle :
(a) average acceleration, (b) average speed,
(c) average velocity.

6. A particle moves in a circle of radius $R = \frac{21}{22}$ m with constant speed 1m/s. Find :
(a) magnitude of average velocity and (b) magnitude of average acceleration in 2 s.

7. A particle is moving in x-y plane. At time $t = 0$, particle is at (1m, 2m) and has velocity $(4\hat{i} + 6\hat{j})$ m/s. At $t = 4$ s, particle reaches at (6m, 4m) and has velocity $(2\hat{i} + 10\hat{j})$ m/s. In the given time interval, find :
(a) average velocity,
(b) average acceleration and
(c) from the given data, can you find average speed?

Uniform Acceleration

(a) One dimensional motion

8. Two diamonds begin a free fall from rest from the same height, 1.0 s apart. How long after the first diamond begins to fall will the two diamonds be 10 m apart? Take $g = 10\ m/s^2$.

9. Two bodies are projected vertically upwards from one point with the same initial velocity v_0. The second body is projected t_0 s after the first. How long after will the bodies meet?

10. A stone is dropped from the top of a tower. When it crosses a point 5 m below the top, another stone is let fall from a point 25 m below the top. Both stones reach the bottom of the tower simultaneously. Find the height of the tower. Take $g = 10\ \text{m/s}^2$.

11. A point mass starts moving in a straight line with constant acceleration. After time t_0 the acceleration changes its sign, remaining the same in magnitude. Determine the time t from the beginning of motion in which the point mass returns to the initial position.

12. A football is kicked vertically upward from the ground and a student gazing out of the window sees it moving upwards past her at 5.00 m/s. The window is 15.0 m above the ground. Air resistance may be ignored. Take $g = 10\ \text{m/s}^2$.
(a) How high does the football go above ground?
(b) How much time does it take to go from the ground to its highest point?

13. A car moving with constant acceleration covered the distance between two points 60.0 m apart in 6.00 s. Its speed as it passes the second point was 15.0 m/s.
(a) What is the speed at the first point?
(b) What is the acceleration?
(c) At what prior distance from the first was the car at rest?
(d) Graph s *versus* t and v *versus* t for the car, from rest ($t = 0$).

14. A train stopping at two stations 4 km apart takes 4 min on the journey from one of the station to the other. Assuming that it first accelerates with a uniform acceleration x and then that of uniform retardation y, prove that $\frac{1}{x} + \frac{1}{y} = 2$.

15. A particle moves along the x-direction with constant acceleration. The displacement, measured from a convenient position, is 2m at time $t = 0$ and is zero when $t = 10$ s. If the velocity of the particle is momentary zero when $t = 6$ s, determine the acceleration a and the velocity v when $t = 10$ s.

(b) Two or three dimensional motion

16. Net force acting on a particle of mass 2 kg is 10 N in north direction. At $t = 0$, particle was moving eastwards with 10 m/s. Find displacement and velocity of particle after 2 s.

17. At time $t = 0$, a particle is at (2m, 4m). It starts moving towards positive x-axis with constant acceleration $2\ \text{m/s}^2$ (initial velocity = 0). After 2 s an acceleration of $4\ \text{m/s}^2$ starts acting on the particle in negative y-direction also. Find after next 2 s :
(a) velocity and (b) coordinates of particle.

18. A particle moving in x-y plane is at origin at time $t = 0$. Velocity of the particle is $(2\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$ m/s and acceleration is $(4\hat{\mathbf{i}} + \hat{\mathbf{j}})\ \text{m/s}^2$. Find at $t = 2$ s :
(a) velocity of particle and (b) coordinates of particle.

19. A particle starts from the origin at $t = 0$ with a velocity of $8.0\hat{\mathbf{j}}$ m/s and moves in the x-y plane with a constant acceleration of $(4.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})\ \text{m/s}^2$. At the instant the particle's x-coordinate is 29 m, what are :
(a) its y-coordinate and (b) its speed ?

20. At time $t = 0$, the position vector of a particle moving in the x-y plane is $5\hat{\mathbf{i}}$ m. By time $t = 0.02$ s, its position vector has become $(5.1\hat{\mathbf{i}} + 0.4\hat{\mathbf{j}})$ m determine the magnitude v_{av} of the average velocity during this interval and the angle θ made by the average velocity with the positive x-axis.

Non-uniform Acceleration

21. x-coordinate of a particle moving along this axis is : $x = (2 + t^2 + 2t^3)$. Here, x is in metre and t in seconds. Find :
(a) position of particle from where it started its journey,
(b) initial velocity of particle and
(c) acceleration of particle at $t = 2$ s.

22. The velocity of a particle moving in a straight line is decreasing at the rate of 3 m/s per metre of displacement at an instant when the velocity is 10 m/s. Determine the acceleration of the particle at this instant.

23. The position of a particle along a straight line is given by $s = (t^3 - 9t^2 - 15t)$ m, here t is in seconds. Determine its maximum acceleration during the time interval $0 \le t \le 10$ s.

24. The acceleration of a particle is given by $a(t) = (3.00\ \text{m/s}^2) - (2.00\ \text{m/s}^3)t$.
(a) Find the initial speed v_0 such that the particle will have the same x-coordinate at $t = 5.00$ s as it had at $t = 0$.
(b) What will be the velocity at $t = 5.00$ s?

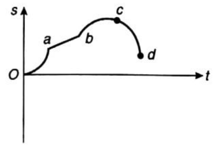
25. A particle moves along a horizontal path, such that its velocity is given by $v = (3t^2 - 6t)$ m/s, where t is the time in seconds. If it is initially located at the origin O, determine the distance travelled by the particle in time interval from $t = 0$ to $t = 3.5$ s and the particle's average velocity and average speed druing the same time interval.

26. A particle travels in a straight line, such that for a short time $2\text{s} \le t \le 6\text{s}$, its motion is described by $v = (4/a)$ m/s, where a is in m/s^2. If $v = 6$ m/s when $t = 2$ s, determine the particle's acceleration when $t = 3$ s.

27. If the velocity v of a particle moving along a straight line decreases linearly with its displacement from 20 m/s to a value approaching zero at $s = 30$ m, determine the acceleration of the particle when $s = 15$ m.

Graphs

28. Displacement-time graph of a particle moving in a straight line is as shown in figure.



(a) Find the sign of velocity in regions oa, ab, bc and cd.
(b) Find the sign of acceleration in the above region.

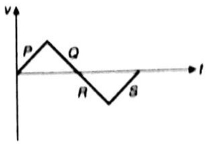
29. Let us call a motion as :
$M_1 \rightarrow$ if velocity and acceleration both are positive.

$M_2 \rightarrow$ if velocity is positive but acceleration is negative.

$M_3 \rightarrow$ if velocity and acceleration both are negative.

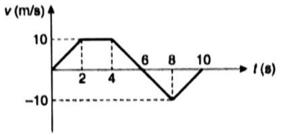
$M_4 \rightarrow$ if velocity is negative but acceleration is positive.

(a) State, in which of the above four motions, magnitude of velocity is increasing and in which it is decreasing.

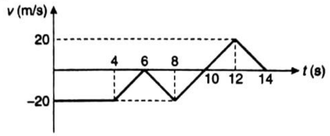


(b) v-t graph of a particle moving in a straight line is as shown in figure. The whole graph is made up of four straight lines P, Q, R and S. These four straight lines indicate four type of motions ($M_1 ... M_4$) discussed above. State, which straight line corresponds to which type of motion.

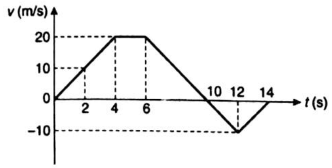
30. Velocity-time graph of a particle moving in a straight line is shown in figure. At time $t = 0$, $s = -10$ m. Plot corresponding a-t and s-t graphs.



31. Velocity-time graph of a particle moving in a straight line is shown in figure. At time $t = 0$, $s = 20$ m. Plot a-t and s-t graphs of the particle.

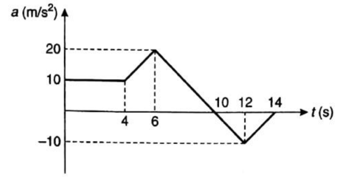


32. Velocity-time graph of a particle moving in a straight line is shown in figure. In the time interval from $t = 0$ to $t = 14$ s, find :



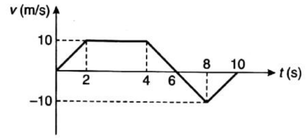
(a) average velocity and
(b) average speed of the particle

33. Acceleration-time graph of a particle moving in a straight line is as shown in figure. At time $t = 0$, velocity of the particle is zero. Find :



(a) average acceleration in a time interval from $t = 6$ s to $t = 12$ s,
(b) velocity of the particle at $t = 14$ s.

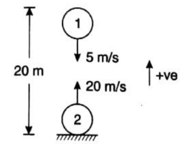
34. Velocity-time graph of a particle moving in a straight line is shown in figure. At time $t = 0$, displacement of the particle from mean position is 10 m. Find :



(a) acceleration of particle at $t = 1$ s, 3 s and 9 s.
(b) position of particle from mean position at $t = 10$ s.
(c) write down s-t equation for time interval :
(i) $0 \le t \le 2$ s, (ii) $4\text{ s} \le t \le 8$ s

Relative Motion

35. Two particles 1 and 2 are thrown in the directions shown in figure simultaneously with velocities 5 m/s and 20 m/s. Initially particle 1 is at height 20 m from the ground. Taking upwards as the positive direction, find :



(a) acceleration of 1 with respect to 2
(b) initial velocity of 2 with respect to 1
(c) velocity of 1 with respect to 2 after time $t = 1/2$ s
(d) time when the particles will collide.

36. A person walks up a stalled 15 m long escalator in 90 s. When standing on the same escalator, now moving, the person is carried up in 60 s. How much time would it take that person to walk up the moving escalator? Does the answer depend on the length of the escalator?

37. A ball is thrown vertically upward from the 12 m level with an initial velocity of 18 m/s. At the same instant an open platform elevator passes the 5 m level, moving upward with a constant velocity of 2 m/s. Determine :

(a) when and where the ball will meet the elevator,

(b) the relative velocity of the ball with respect to the elevator when the ball hits the elevator.

38. An automobile and a truck start from rest at the same instant, with the automobile initially at some distance behind the truck. The truck has a constant acceleration of 2.2 m/s^2 and the automobile has an acceleration of 3.5 m/s^2. The automobile overtakes the truck when it (truck) has moved 60 m.

(a) How much time does it take the automobile to overtake the truck ?

(b) How far was the automobile behind the truck initially ?

(c) What is the speed of each during overtaking ?

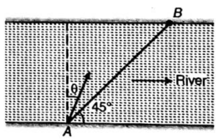
39. A body is thrown up in a lift with a velocity u relative to the lift and the time of flight is t. Show that the lift is moving up with an acceleration $\frac{2u - tg}{t}$.

40. A river is 20 m wide. River speed is 3 m/s. A boat starts with velocity $2\sqrt{2}$ m/s at angle 45° from the river current (relative to river)

(a) Find the time taken by the boat to reach the opposite bank.

(b) How far from the point directly opposite to the starting point does the boat reach the opposite bank?

41. Given $|\vec{v}_{br}| = 4$ m/s = magnitude of velocity of boatman with respect to river, $\vec{v}_r = 2$ m/s in the direction shown. Boatman wants to reach from point A to point B. At what angle θ should he row his boat.



42. An aeroplane has to go from a point P to another point Q, 1000 km away due north. Wind is blowing due east at a speed of 200 km/h. The air speed of plane is 500 km/h.

(a) Find the direction in which the pilot should head the plane to reach the point Q.

(b) Find the time taken by the plane to go from P to Q.

Objective Questions (Level 1)

Single Correct Option

1. A packet is released from a rising balloon accelerating upward with acceleration a. The acceleration of the stone just after the release is

(a) a upward (b) g downward (c) $(g - a)$ downward (d) $(g + a)$ downward

2. A ball is thrown vertically upwards from the ground. If T_1 and T_2 are the respective time taken in going up and coming down, and the air resistance is not ignored, then

(a) $T_1 > T_2$ (b) $T_1 = T_2$ (c) $T_1 < T_2$ (d) nothing can be said

3. The length of a seconds hand in watch is 1 cm. The change in velocity of its tip in 15 s is

(a) zero (b) $\frac{\pi}{30\sqrt{2}}$ cms^{-1} (c) $\frac{\pi}{30}$ cms^{-1} (d) $\frac{\pi\sqrt{2}}{30}$ cms^{-1}

4. A particle moving along a straight line travels half of the distance with uniform speed 30 ms^{-1} and the remaining half of the distance with speed 60 ms^{-1}. The average speed of the particle is

(a) 45 ms^{-1} (b) 42 ms^{-1} (c) 40 ms^{-1} (d) 50 ms^{-1}

5. A boat is moving with a velocity $(3\hat{i} + 4\hat{j})$ with respect to ground. The water in the river is moving with a velocity $-3\hat{i} - 4\hat{j}$ with respect to ground. The relative velocity of the boat with respect to water is

(a) $8\hat{j}$ (b) $-6\hat{i} - 8\hat{j}$ (c) $6\hat{i} + 9\hat{j}$ (d) zero

6. During the first 18 min of a 60 min trip, a car has an average speed of 11 ms^{-1}. What should be the average speed for remaining 42 min so that car is having an average speed of 21 ms^{-1} for the entire trip ?

(a) 25.3 ms^{-1} (b) 29.2 ms^{-1} (c) 31 ms^{-1} (d) 35.6 ms^{-1}

7. A particle moves along a straight line. Its position at any instant is given by $x = 32t - \frac{8t^3}{3}$ where x is in metre and t in second. Find the acceleration of the particle at the instant when particle is at rest.

(a) $-16\, ms^{-2}$ (b) $-32\, ms^{-2}$ (c) 32 ms^{-2} (d) 16 ms^{-2}

8. A car starts from rest and accelerates at constant rate in a straight line. In the first second the car covers a distance of 2 m. The velocity of the car at the end of second (sec) will be

(a) 4.0 ms^{-1} (b) 8.0 ms^{-1} (c) 16 ms^{-1} (d) None of these

9. A particle is moving along x-axis whose position is varying with time according to the relation $x = -3t + t^3$ where x is in metre and t is in second. The displacement of particle for $t = 1$ s to $t = 3$ s is

(a) +16 m (b) −16 m (c) + 20 m (d) −20 m

10. The acceleration of a particle is increasing linearly with time t as bt. The particle starts from the origin with an initial velocity v_0. The distance travelled by the particle in time t will be

(a) $v_0t + \frac{1}{6}bt^3$ (b) $v_0t + \frac{1}{3}bt^3$ (c) $v_0t + \frac{1}{3}bt^2$ (d) $v_0t + \frac{1}{2}bt^2$

11. Water drops fall at regular intervals from a tap 5 m above the ground. The third drop is leaving the tap, the instant the first drop touches the ground. How far above the ground is the second drop at that instant. $(g = 10\, ms^{-2})$

(a) 1.25 m (b) 2.50 m (c) 3.75 m (d) 4.00 m

12. A stone is dropped from the top of a tower and one second later, a second stone is thrown vertically downward with a velocity 20 ms^{-1}. The second stone will overtake the first after travelling a distance of $(g = 10\, ms^{-2})$

(a) 13 m (b) 15 m (c) 11.25 m (d) 19.5 m

13. When a ball is thrown up vertically with velocity v_0, it reaches a maximum height of h. If one wishes to triple the maximum height then the ball should be thrown with velocity

(a) $\sqrt{3}\, v_0$ (b) $3\, v_0$ (c) $9\, v_0$ (d) $3/2\, v_0$

14. A particle moves in the x-y plane with velocity $v_x = 8t - 2$ and $v_y = 2$. If it passes through the point $x = 14$ and $y = 4$ at $t = 2$ s, the equation of the path is

(a) $x = y^2 - y + 2$ (b) $x = y^2 - 2$ (c) $x = y^2 + y - 6$ (d) None of these

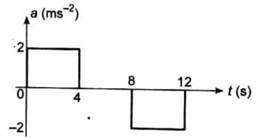
15. The horizontal and vertical displacements of a particle moving along a curved line are given by $x = 5t$ and $y = 2t^2 + t$. Time after which its velocity vector makes an angle of 45° with the horizontal is
(a) 0.5 s (b) 1 s (c) 2 s (d) 1.5 s

16. The height y and the distance x along the horizontal plane of a projectile on a certain planet (with no surrounding atmosphere) are given by $y = (8t - 5t^2)$ metre and $x = 6t$ metre where t is in second. The velocity of projection is
(a) 8 ms^{-1} (b) 6 ms^{-1} (c) 10 ms^{-1} (d) data insufficient

17. A ball is released from the top of a tower of height h metre. It takes T second to reach the ground. What is the position of the ball in $T/3$ second ?
(a) $\frac{h}{9}$ metre from the ground (b) $(7h/9)$ metre from the ground
(c) $(8h/9)$ metre from the ground (d) $(17h/18)$ metre form the ground

18. An ant is at a corner of a cubical room of side a. The ant can move with a constant speed u. The minimum time taken to reach the farthest corner of the cube is
(a) $\frac{3a}{u}$ (b) $\frac{\sqrt{3}a}{u}$ (c) $\frac{\sqrt{5}a}{u}$ (d) $\frac{(\sqrt{2}+1)a}{u}$

19. A lift starts from rest. Its acceleration is plotted against time. When it comes to rest its height above its starting point is



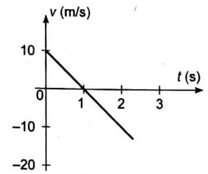
(a) 20 m (b) 64 m (c) 32 m (d) 36 m

20. A lift performs the first part of its ascent with uniform acceleration a and the remaining with uniform retardation $2a$. If t is the time of ascent, find the depth of the shaft.
(a) $\frac{at^2}{4}$ (b) $\frac{at^2}{3}$ (c) $\frac{at^2}{2}$ (d) $\frac{at^2}{8}$

21. Two objects are moving along the same straight line. They cross a point A with an acceleration a, $2a$ and velocity $2u$, u at time $t = 0$. The distance moved by the object when one overtakes the other is
(a) $\frac{6u^2}{a}$ (b) $\frac{2u^2}{a}$ (c) $\frac{4u^2}{a}$ (d) $\frac{8u^2}{a}$

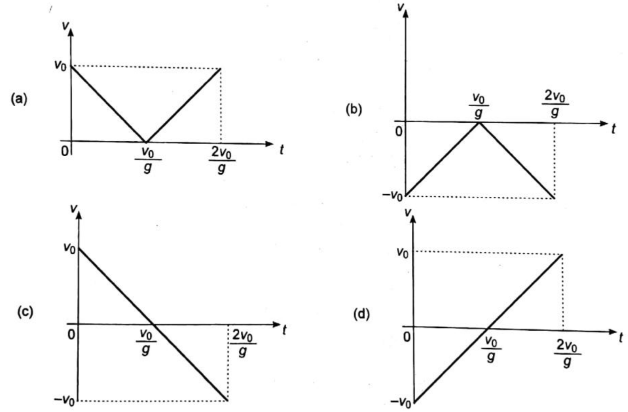
22. A cart is moving horizontally along a straight line with constant speed 30 ms^{-1}. A particle is to be fired vertically upwards from the moving cart in such a way that it returns to the cart at the same point from where it was projected after the cart has moved 80 m. At what speed (relative to the cart) must the projectile be fired? (Take $g = 10\ ms^{-2}$)
(a) 10 ms^{-1} (b) $10\sqrt{8}$ ms^{-1} (c) $\frac{40}{3}$ ms^{-1} (d) None of these

23. The figure shows velocity–time graph of a particle moving along a straight line. Identify the correct statement.



(a) The particle starts from the origin
(b) The particle crosses it initial position at $t = 2$s
(c) The average speed of the particle in the time interval, $0 \le t \le 2$s is zero
(d) All of the above

24. A ball is thrown vertically upwards with a velocity v_0. If the vertical downward direction is considered to be positive, then the correct variation of velocity with time t is represented by



25. A boy standing in an elevator accelerating upward throws a ball upward with a velocity v_0. The ball returns in his hands after a time t. The acceleration of the lift is
(a) $\frac{2v_0 - gt}{t}$ (b) $\frac{2v_0 + gt}{t}$ (c) $\frac{v_0 - gt}{t}$ (d) $\frac{v_0 + gt}{t}$

26. A body starts moving with a velocity $v_0 = 10\ ms^{-1}$. It experiences a retardation equal to $0.2v^2$. Its velocity after 2s is given by
(a) $+2\ ms^{-1}$ (b) $+4\ ms^{-1}$ (c) $-2\ ms^{-1}$ (d) $+6\ ms^{-1}$

27. Two trains are moving with velocities $v_1 = 10\ ms^{-1}$ and $v_2 = 20\ ms^{-1}$ on the same track in opposite directions. After the application of brakes if their retarding rates are $a_1 = 2\ ms^{-2}$ and $a_2 = 1\ ms^{-2}$ respectively, then the minimum distance of separation between the trains to avoid collision is
(a) 150 m (b) 225 m (c) 450 m (d) 300 m

28. Two identical balls are shot upward one after another at an interval of 2s along the same vertical line with same initial velocity of 40 ms^{-1}. The height at which the balls collide is

(a) 50 m (b) 75 m (c) 100 m (d) 125 m

29. A particle is projected vertically upwards and reaches the maximum height H in time T. The height of the particle at any time t will be

(a) $g(t-T)^2$ (b) $H-g(t-T)^2$ (c) $\frac{1}{2}g(t-T)^2$ (d) $H-\frac{1}{2}g(t-T)^2$

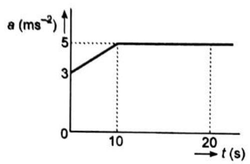
30. A particle moves along the curve $y=\frac{x^2}{2}$. Here x varies with time as $x=\frac{t^2}{2}$. Where x and y are measured in metre and t in second. At $t=2$s, the velocity of the particle (in ms^{-1}) is

(a) $2\hat{i}-4\hat{j}$ (b) $2\hat{i}+4\hat{j}$ (c) $4\hat{i}+2\hat{j}$ (d) $4\hat{i}-2\hat{j}$

31. If the displacement of a particle varies with time as $\sqrt{x}=t+3$

(a) velocity of the particle is inversely proportional to t
(b) velocity of particle varies linearly with t
(c) velocity of particle is proportional to $\sqrt{t}$
(d) initial velocity of the particle is zero

32. The graph describes an airplane's acceleration during its take-off run. The airplane's velocity when it lifts off at $t=20$s is



(a) 40 ms^{-1} (b) 50 ms^{-1} (c) 90 ms^{-1} (d) 180 ms^{-1}

33. A particle moving in a straight line has velocity-displacement equation as $v=5\sqrt{1+s}$. Here v is in ms^{-1} and s in metre. Select the correct alternative.

(a) Particle is initially at rest
(b) Initially velocity of the particle is 5 m/s and the particle has a constant acceleration of 12.5 ms^{-2}
(c) Particle moves with a uniform velocity
(d) None of these

34. A particle is thrown upwards from ground. It experiences a constant resistance force which can produce a retardation of 2 ms^{-2}. The ratio of time of ascent to time of descent is ($g=10\,ms^{-2}$)

(a) 1 : 1 (b) $\sqrt{\frac{2}{3}}$ (c) $\frac{2}{3}$ (d) $\sqrt{\frac{3}{2}}$

35. A river is flowing from west to east with a speed of 5 m min^{-1}. A man can swim in still water with a velocity 10 m min^{-1}. In which direction should the man swim so as to take the shortest possible path to go to the south?

(a) 30° east of south (b) 30° west of south (c) South (d) 30° west of north

36. A body of mass 10 kg is being acted upon by a force $3t^2$ and an opposing constant force of 32 N. The initial speed is 10 ms^{-1}. The velocity of body after 5 s is

(a) 14.5 ms^{-1} (b) 6.5 ms^{-1} (c) 3.5 ms^{-1} (d) 4.5 ms^{-1}

37. A stone is thrown vertically upwards. When stone is at a height half of its maximum height, its speed is 10 ms^{-1}; then the maximum height attained by the stone is ($g=10\,ms^{-2}$)

(a) 25 m (b) 10 m (c) 15 m (d) 20 m

38. A ball is thrown vertically upwards from the ground and a student gazing out of the window sees it moving upward past him at 10 ms^{-1}. The window is at 15 m above the ground level. The velocity of ball 3 s after it was projected from the ground is [Take $g=10\,ms^{-2}$]

(a) 10 m/s, up (b) 20 ms^{-1}, up (c) 20 ms^{-1}, down (d) 10 ms^{-1}, down

JEE Corner

Assertion and Reason

Directions : *Choose the correct option.*

(a) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*
(b) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*
(c) *If **Assertion** is true, but the **Reason** is false.*
(d) *If **Assertion** is false but the **Reason** is true.*

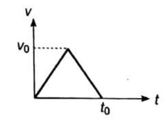
1. Assertion : Velocity and acceleration of a particle are given as,

$$\vec{v}=\hat{i}-\hat{j} \text{ and } \vec{a}=-2\hat{i}+2\hat{j}$$

This is a two dimensional motion with constant acceleration.
Reason : Velocity and acceleration are two constant vectors.

2. Assertion : Displacement-time graph is a parabola corresponding to straight line velocity-time graph.
Reason : If $v=u+at$, then $s=ut+\frac{1}{2}at^2$

3. Assertion : In v-t graph shown is figure, average velocity in time interval from 0 t_0 to depends only on v_0. It is independent of t_0.
Reason : In the given time interval average velocity is $\frac{v_0}{2}$.



4. Assertion : We know the relation $a=v.\frac{dv}{ds}$. Therefore, if velocity of a particle is zero, then acceleration is also zero.
Reason : In the above equation, a is the instantaneous acceleration.

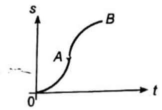
5. Assertion : Speed of a particle may decrease, even if acceleration is increasing.
Reason : This will happen if acceleration is positive.

6. Assertion : Starting from rest with zero acceleration if acceleration of particle increases at a constant rate of 2 ms^{-3} then velocity should increase at constant rate of 1 ms^{-2}.
Reason : For the given condition.

$$\frac{da}{dt}=2\,ms^{-3} \qquad \therefore \quad a=2t.$$

7. Assertion : Average velocity can't be zero in case of uniform acceleration.
Reason : For average velocity to be zero, velocity should not remain constant.

8. **Assertion :** In displacement-time graph of a particle as shown in figure. Velocity of particle changes its direction at point A.



Reason : Sign of slope of s-t graph decides the direction of velocity.

9. **Assertion :** Displacement-time equation of two particles moving in a straight line are, $s_1 = 2t - 4t^2$ and $s_2 = -2t + 4t^2$. Relative velocity between the two will go on increasing.

Reason : If velocity and acceleration are of same sign then speed will increase.

10. **Assertion :** Acceleration of a moving particle can change its direction without any change in direction of velocity.

Reason : If the direction of change in velocity vector changes, the direction of acceleration vector also changes.

11. **Assertion :** A body is dropped from height h and another body is thrown vertically upwards with a speed $\sqrt{gh}$. They meet at height $h/2$.

Reason : The time taken by both the blocks in reaching the height $h/2$ is same.

12. **Assertion :** Two bodies of unequal masses m_1 and m_2 are dropped from the same height. If the resistance offered by air to the motion of both bodies is the same, the bodies will reach the earth at the same time.

Reason : For equal air resistance, acceleration of fall of masses m_1 and m_2 will be different.

Objective Questions (Level 2)

Single Correct Option

1. When a man moves down the inclined plane with a constant speed 5 ms^{-1} which makes an angle of 37° with the horizontal, he finds that the rain is falling vertically downward. When he moves up the same inclined plane with the same speed, he finds that the rain makes an angle $\theta = \tan^{-1}\left(\frac{7}{8}\right)$ with the horizontal. The speed of the rain is

(a) $\sqrt{116}$ ms^{-1} (b) $\sqrt{32}$ ms^{-1} (c) 5 ms^{-1} (d) $\sqrt{73}$ ms^{-1}

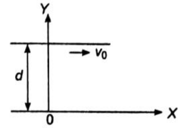
2. Equation of motion of a body is $\frac{dv}{dt} = -4v + 8$, where v is the velocity in ms^{-1} and t is the time in second. Initial velocity of the particle was zero. Then

(a) the initial rate of change of acceleration of the particle is 8 ms^{-2}
(b) the terminal speed is 2 ms^{-1}
(c) Both (a) and (b) are correct
(d) Both (a) and (b) are wrong

3. Two particles A and B are placed in gravity free space at (0, 0, 0)m and (30, 0, 0)m respectively. Particle A is projected with a velocity $(5\hat{i} + 10\hat{j} + 5\hat{k})$ ms^{-1}, while particle B is projected with a velocity $(10\hat{i} + 5\hat{j} + 5\hat{k})$ ms^{-1} simultaneously. Then

(a) they will collide at (10, 20, 10) m
(b) they will collide at (10, 10, 10) m
(c) they will never collide
(d) None of the above

4. Velocity of the river with respect to ground is given by v_0. Width of the river is d. A swimmer swims (with respect to water) perpendicular to the current with acceleration $a = 2t$ (where t is time) starting from rest from the origin O at $t = 0$. The equation of trajectory of the path followed by the swimmer is



(a) $y = \frac{x^3}{3v_0^3}$ (b) $y = \frac{x^2}{2v_0^2}$

(c) $y = \frac{x}{v_0}$ (d) $y = \sqrt{\frac{x}{v_0}}$

5. The relation between time t and displacement x is $t = \alpha x^2 + \beta x$ where α and β are constants. The retardation is

(a) $2\alpha v^3$ (b) $2\beta v^3$ (c) $2\alpha\beta v^3$ (d) $2\beta^2 v^3$

6. A street car moves rectilinearly from station A to the next station B with an acceleration varying according to the law $f = a - bx$, where a and b are constants and x is the distance from station A. The distance between the two stations and the maximum velocity are

(a) $x = \frac{2a}{b}, v_{max} = \frac{a}{\sqrt{b}}$ (b) $x = \frac{b}{2a}, v_{max} = \frac{a}{b}$ (c) $x = \frac{a}{2b}, v_{max} = \frac{b}{\sqrt{a}}$ (d) $x = \frac{a}{b}, v_{max} = \frac{\sqrt{a}}{b}$

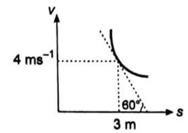
7. A particle of mass m moves on positive x-axis under the influence of force acting towards the origin given by $-kx^2\hat{i}$. If the particle starts from rest at $x = a$, the speed it will attain when it crosses the origin is

(a) $\sqrt{\frac{k}{ma}}$ (b) $\sqrt{\frac{2k}{ma}}$ (c) $\sqrt{\frac{ma}{2k}}$ (d) $\sqrt{\frac{2ka^3}{m}}$

8. A particle is moving in X-Y plane such that $v_x = 4 + 4t$ and $v_y = 4t$. If the initial position of the particle is (1, 2). Then the equation of trajectory will be

(a) $y^2 = 4x$ (b) $y = 2x$ (c) $x^2 = \frac{y}{2}$ (d) None of these

9. A particle is moving along a straight line whose velocity-displacement graph is as shown in the figure. What is the magnitude of acceleration when displacement is 3 m ?



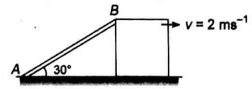
(a) $4\sqrt{3}$ ms^{-2} (b) $3\sqrt{3}$ ms^{-2}

(c) $\sqrt{3}$ ms^{-2} (d) $\frac{4}{\sqrt{3}}$ ms^{-2}

10. A particle is falling freely under gravity. In first t second it covers distance x_1 and in the next t second it covers distance x_2, then t is given by

(a) $\sqrt{\frac{x_2 - x_1}{g}}$ (b) $\sqrt{\frac{x_2 + x_1}{g}}$ (c) $\sqrt{\frac{2(x_2 - x_1)}{g}}$ (d) $\sqrt{\frac{2(x_2 + x_1)}{g}}$

11. A rod AB is shown in figure. End A of the rod is fixed on the ground. Block is moving with velocity 2 ms^{-1} towards right. The vertical component of velocity of end B of rod at the instant shown in figure is



(a) $\sqrt{3}$ ms^{-1} (b) 2 ms^{-1} (c) $2\sqrt{3}$ ms^{-1} (d) 4 ms^{-1}

12. A thief in a stolen car passes through a police check post at his top speed of 90 kmh^{-1}. A motorcycle cop, reacting after 2 s, accelerates from rest at 5 ms^{-2}. His top speed being 108 kmh^{-1}. Find the maximum separation between policemen and thief.
(a) 112.5 m (b) 115 m (c) 116.5 m (d) None of these

13. Sachin (*S*) hits a ball along the ground with a speed *u* in a direction which makes an angle 30° with the line joining him and the fielder Prem (*P*). Prem runs to intercept the ball with a speed $\frac{2u}{3}$. At what angle θ should he run to intercept the ball ?

(a) $\sin^{-1}\left[\frac{\sqrt{3}}{2}\right]$ (b) $\sin^{-1}\left[\frac{2}{3}\right]$ (c) $\sin^{-1}\left[\frac{3}{4}\right]$ (d) $\sin^{-1}\left[\frac{4}{5}\right]$

14. A car is travelling on a straight road. The maximum velocity the car can attain is 24 ms^{-1}. The maximum acceleration and deceleration it can attain are 1 ms^{-2} and 4 ms^{-2} respectively. The shortest time the car takes from rest to rest in a distance of 200 m is,
(a) 22.4 s (b) 30 s (c) 11.2 s (d) 5.6 s

15. A car is travelling on a road. The maximum velocity the car can attain is 24 ms^{-1} and the maximum deceleration is 4 ms^{-2}. If car starts from rest and comes to rest after travelling 1032 m in the shortest time of 56 s, the maximum acceleration that the car can attain is
(a) 6 ms^{-2} (b) 1.2 ms^{-2} (c) 12 ms^{-2} (d) 3.6 ms^{-2}

16. Two particles are moving along two long straight lines, in the same plane with same speed equal to 20 cms^{-1}. The angle between the two lines is 60° and their intersection point is *O*. At a certain moment, the two particles are located at distances 3m and 4m from *O* and are moving towards *O*. Subsequently, the shortest distance between them will be
(a) 50 cm (b) $40\sqrt{2}$ cm (c) $50\sqrt{2}$ cm (d) $50\sqrt{3}$ cm

Passage : Q 17 to 20

An elevator without a ceiling is ascending up with an acceleration of 5 ms^{-2}. A boy on the elevator shoots a ball in vertical upward direction from a height of 2 m above the floor of elevator. At this instant the elevator is moving up with a velocity of 10 ms^{-1}, and floor of the elevator is at a height of 50 m from the ground. The initial speed of the ball is 15 ms^{-1} with respect to the elevator. Consider the duration for which the ball strikes the floor of elevator in answering following questions. ($g = 10$ ms^{-2})

17. The time in which the ball strikes the floor of elevator is given by
(a) 2.13 s (b) 2.0 s (c) 1.0 s (d) 3.12 s

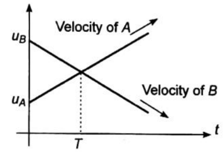
18. The maximum height reached by ball, as measured from the ground would be
(a) 73.65 m (b) 116.25 m (c) 82.56 m (d) 63.25 m

19. Displacement of ball with respect to ground during its flight would be
(a) 16.25 m (b) 8.76 m (c) 20.24 m (d) 30.56 m

20. The maximum separation between the floor of elevator and the ball during its flight would be
(a) 12 m (b) 15 m (c) 9.5 m (d) 7.5 m

Passage : Q 21 to Q 23

A situation is shown in which two objects *A* and *B* start their motion from same point in same direction. The graph of their velocities against time is drawn. u_A and u_B are the initial velocities of *A* and *B* respectively. *T* is the time at which their velocities become equal after start of motion. You cannot use the data of one question while solving another question of the same set. So all the questions are independent of each other.



21. If the value of *T* is 4 s, then the times after which *A* will meet *B* is
(a) 12 s (b) 6 s (c) 8 s (d) data insufficient

22. Let v_A and v_B be the velocities of the particles *A* and *B* respectively at the moment *A* and *B* meet after start of the motion. If $u_A = 5\,\text{ms}^{-1}$ and $u_B = 15\,\text{ms}^{-1}$, then the magnitude of the difference of velocities v_A and v_B is
(a) 5 ms^{-1} (b) 10 ms^{-1} (c) 15 ms^{-1} (d) data insufficient

23. After 10 s of the start of motion of both objects *A* and *B*, find the value of velocity of *A* if $u_A = 6\,\text{ms}^{-1}$, $u_B = 12\,\text{ms}^{-1}$ and at *T* velocity of *A* is 8 ms^{-1} and $T = 4$ s
(a) 12 ms^{-1} (b) 10 ms^{-1} (c) 15 ms^{-1} (d) None of these

More than One Correct Options

1. A particle having a velocity $v = v_0$ at $t = 0$ is decelerated at the rate $|a| = \alpha\sqrt{v}$, where α is a positive constant.
(a) The particle comes to rest at $t = \frac{2\sqrt{v_0}}{\alpha}$
(b) The particle will come to rest at infinity.
(c) The distance travelled by the particle before coming to rest is $\frac{2v_0^{3/2}}{\alpha}$
(d) The distance travelled by the particle before coming to rest is $\frac{2v_0^{3/2}}{3\alpha}$

2. At time $t = 0$, a car moving along a straight line has a velocity of 16 ms^{-1}. It slows down with an acceleration of $-0.5t$ ms^{-2}, where *t* is in second. Mark the correct statement (s).
(a) The direction of velocity changes at $t = 8$ s
(b) The distance travelled in 4 s is approximately 58.67 m
(c) The distance travelled by the particle in 10 s is 94 m
(d) The speed of particle at $t = 10$ s is 9 ms^{-1}

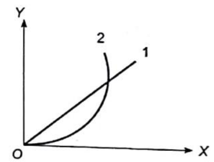
3. An object moves with constant acceleration $\vec{a}$. Which of the following expressions are also constant ?
(a) $\frac{d|\vec{v}|}{dt}$ (b) $\left|\frac{d\vec{v}}{dt}\right|$ (c) $\frac{d(v^2)}{dt}$ (d) $\frac{d\left(\frac{\vec{v}}{|\vec{v}|}\right)}{dt}$

4. Ship A is located 4 km north and 3 km east of ship B. Ship A has a velocity of 20 kmh^{-1} towards the south and ship B is moving at 40 kmh^{-1} in a direction 37° north of east. X and Y-axes are along east and north directions, respectively
(a) Velocity of A relative to B is $-32\hat{\mathbf{i}}-44\hat{\mathbf{j}}$
(b) Position of A relative to B as a function of time is given by
$$\vec{\mathbf{r}}_{AB}=(3-32t)\hat{\mathbf{i}}+(4-44t)\hat{\mathbf{j}}$$
(c) Velocity of A relative to B is $32\hat{\mathbf{i}}-44\hat{\mathbf{j}}$
(d) Position of A relative to B as a function of time is given by $(32t\hat{\mathbf{i}}-44t\hat{\mathbf{j}})$

5. Starting from rest a particle is first accelerated for time t_1 with constant acceleration a_1 and then stops in time t_2 with constant retardation a_2. Let v_1 be the average velocity in this case and s_1 the total displacement. In the second case it is accelerating for the same time t_1 with constant acceleration $2a_1$ and come to rest with constant retardation a_2 in time t_3. If v_2 is the average velocity in this case and s_2 the total displacement, then
(a) $v_2=2v_1$ (b) $2v_1<v_2<4v_1$ (c) $s_2=2s_1$ (d) $2s_1<s_2<4s_1$

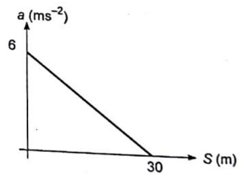
6. A particle is moving along a straight line. The displacement of the particle becomes zero in a certain time $(t>0)$. The particle does not undergo any collision.
(a) The acceleration of the particle may be zero always
(b) The acceleration of the particle may be uniform
(c) The velocity of the particle must be zero at some instant
(d) The acceleration of the particle must change its direction

7. A particle is resting over a smooth horizontal floor. At $t=0$, a horizontal force starts acting on it. Magnitude of the force increases with time according to law $F=\alpha t$, where α is a positive constant. From figure, which of the following statements are correct ?



(a) Curve 1 can be the plot of acceleration against time
(b) Curve 2 can be the plot of velocity against time
(c) Curve 2 can be the plot of velocity against acceleration
(d) Curve 1 can be the plot of displacement against time

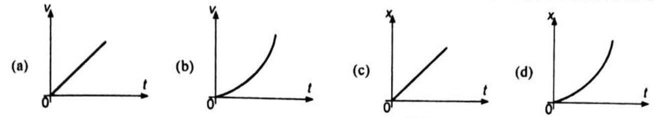
8. A train starts from rest at $S=0$ and is subjected to acceleration as shown



(a) velocity at the end of 10 m displacement is 20 ms^{-1}
(b) velocity of the train at $S=10$ m is 10 ms^{-1}
(c) The maximum velocity attained by train is $\sqrt{180}$ ms^{-1}
(d) The maximum velocity attained by the train is 15 ms^{-1}

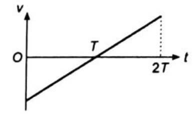
9. For a moving particle which of the following options may be correct?
(a) $|\vec{\mathbf{v}}_{av}|<v_{av}$ (b) $|\vec{\mathbf{v}}_{av}|>v_{av}$ (c) $\vec{\mathbf{v}}_{av}=0$ but $v_{av}\neq 0$ (d) $\vec{\mathbf{v}}_{av}\neq 0$ but $\vec{\mathbf{v}}_{av}=0$

10. Identify the correct graph representing the motion of a particle along a straight line with constant acceleration.



11. A man who can swim at a velocity v relative to water wants to cross a river of width b, flowing with a speed u.
(a) The minimum time in which he can cross the river is $\frac{b}{v}$
(b) He can reach a point exactly opposite on the bank in time $t=\frac{d}{\sqrt{v^2-u^2}}$ if $v>u$
(c) He cannot reach the point exactly opposite on the bank if $u>v$
(d) He cannot reach the point exactly opposite on the bank if $v>u$

12. The figure shows the velocity (v) of a particle plotted against time (t).



(a) The particle changes its direction of motion at some points
(b) The acceleration of the particle remains constant
(c) The displacement of the particle is zero
(d) The initial and final speeds of the particle are the same

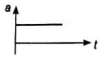
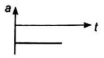
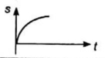
13. The speed of a train increases at a constant rate α from zero to v and then remains constant for an interval and finally decreases to zero at a constant rate β. The total distance travelled by the train is l. The time taken to complete the journey is t. Then
(a) $t=\frac{l(\alpha+\beta)}{\alpha\beta}$ (b) $t=\frac{l}{v}+\frac{v}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$
(c) t is minimum when $v=\sqrt{\frac{2l\alpha\beta}{(\alpha-\beta)}}$ (d) t is minimum when $v=\sqrt{\frac{2l\alpha\beta}{(\alpha+\beta)}}$

14. A particle moves in x-y plane and at time t is at the point (t^2, t^3-2t), then which of the following is/are correct?
(a) At $t=0$, particle is moving parallel to y-axis
(b) At $t=0$, direction of velocity and acceleration are perpendicular.
(c) At $t=\sqrt{\frac{2}{3}}$, particle is moving parallel to x-axis.
(d) At $t=0$, particle is at rest.

15. A car is moving with uniform acceleration along a straight line between two stops X and Y. Its speed at X and Y are 2 ms^{-1} and 14 ms^{-1}, Then
(a) its speed at mid-point of XY is 10 ms^{-1}
(b) its speed at a point A such that $XA:AY=1:3$ is 5 ms^{-1}
(c) the time to go from X to the mid-point of XY is double of that to go from mid-point to Y
(d) the distance travelled in first half of the total time is half of the distance travelled in the second half of the time

Match the Columns

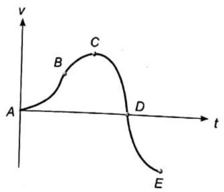
1. Match the following two columns.

	Column I	Column II
(a)		(p) speed must be increasing
(b)		(q) speed must be decreasing
(c)		(r) speed may be increasing
(d)		(s) speed may be decreasing

2. Match the following two columns.

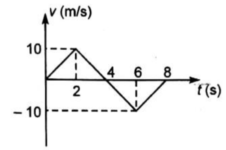
Column I	Column II
(a) $\vec{v} = -2\hat{i}, \vec{a} = -4\hat{j}$	(p) speed increasing
(b) $\vec{v} = 2\hat{i}, \vec{a} = 2\hat{i} + 2\hat{j}$	(q) speed decreasing
(c) $\vec{v} = -2\hat{i}, \vec{a} = +2\hat{i}$	(r) speed constant
(d) $\vec{v} = 2\hat{i}, \vec{a} = -2\hat{i} + 2\hat{j}$	(s) Nothing can be said

3. The velocity-time graph of a particle moving along X-axis is shown in figure. Match the entries of Column I with entries of Column II.



Column I	Column II
(a) For AB, particle is	(p) Moving in +ve X-direction with increasing speed
(b) For BC, particle is	(q) Moving in +ve X-direction with decreasing speed
(c) For CD, particle is	(r) Moving in –ve X-direction with increasing speed
(d) For DE, particle is	(s) Moving in +ve X-direction with decreasing speed

4. Corresponding to velocity-time graph in one dimensional motion of a particle is shown in figure, match the following two columns.



Column I	Column II
(a) Average velocity between zero sec and 4 s	(p) 10 SI units
(b) Average acceleration between 1 s and 4 s	(q) 2.5 SI units
(c) Average speed between zero sec and 6 s	(r) 5 SI units
(d) Rate of change of speed at 4 s	(s) None

5. A particle is moving along x-axis. Its x-coordinate varies with time as :
$x = -20 + 5t^2$

For the given equation match the following two columns

Column I	Column II
(a) Particle will cross the origin at	(p) zero sec
(b) At what time velocity and acceleration are equal	(q) 1 s
(c) At what time particle changes its direction of motion	(r) 2 s
(d) At what time velocity is zero	(s) None

6. x and y-coordinates of particle moving in x-y plane are,

$$x = 1 - 2t + t^2 \text{ and } y = 4 - 4t + t^2$$

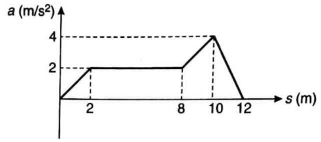
For the given situation match the following two columns.

Column I	Column II
(a) y-component of velocity when it crosses the y-axis	(p) + 2 SI unit
(b) x-component of velocity when it crosses the x-axis	(q) – 2 SI units
(c) Initial velocity of particle	(r) + 4 SI units
(d) Initial acceleration of particle	(s) None

Subjective Questions (Level 2)

1. To test the quality of a tennis ball, you drop it onto the floor from a height of 4.00 m. It rebounds to a height of 2.00 m. If the ball is in contact with the floor for 12.0 ms, what is its average acceleration during that contact? Take $g = 9.8\ \text{m/s}^2$.

2. The acceleration-displacement graph of a particle moving in a straight line is as shown in figure, initial velocity of particle is zero. Find the velocity of the particle when displacement of the particle is $s = 12$ m.



3. At the initial moment three points A, B and C are on a horizontal straight line at equal distances from one another. Point A begins to move vertically upward with a constant velocity v and point C vertically downward without any initial velocity but with a constant acceleration a. How should point B move vertically for all the three points to be constantly on one straight line. The points begin to move simultaneously.

4. A particle moves in a straight line with constant acceleration a. The displacements of particle from origin in times t_1, t_2 and t_3 are s_1, s_2 and s_3 respectively. If times are in AP with common difference d and displacements are in GP, then prove that $a = \dfrac{(\sqrt{s_1} - \sqrt{s_3})^2}{d^2}$.

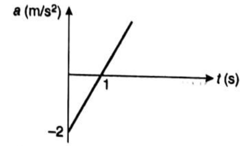
5. A car is to be hoisted by elevator to the fourth floor of a parking garage, which is 14 m above the ground. If the elevator can have maximum acceleration of $0.2\ \text{m/s}^2$ and maximum deceleration of $0.1\ \text{m/s}^2$ and can reach a maximum speed of 2.5 m/s, determine the shortest time to make the lift, starting from rest and ending at rest.

6. To stop a car, first you require a certain reaction time to begin braking; then the car slows under the constant braking deceleration. Suppose that the total distance moved by your car during these two phases is 56.7 m when its initial speed is 80.5 km/h and 24.4 m when its initial speed is 48.3 km/h. What are :
   (a) your reaction time and
   (b) the magnitude of the deceleration?

7. An elevator without a ceiling is ascending with a constant speed of 10 m/s. A boy on the elevator shoots a ball directly upward, from a height of 2.0 m above the elevator floor. At this time the elevator floor is 28 m above the ground. The initial speed of the ball with respect to the elevator is 20 m/s. (Take $g = 9.8\ \text{m/s}^2$)
   (a) What maximum height above the ground does the ball reach?
   (b) How long does the ball take to return to the elevator floor?

8. A particle moves along a straight line and its velocity depends on time as $v = 3t - t^2$. Here, v is in m/s and t in second. Find :
   (a) average velocity and
   (b) average speed for first five seconds.

9. The acceleration of particle varies with time as shown.

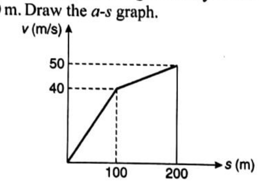


   (a) Find an expression for velocity in terms of t.
   (b) Calculate the displacement of the particle in the interval from $t = 2$ s to $t = 4$ s. Assume that $v = 0$ at $t = 0$.

10. A man wishes to cross a river of width 120 m by a motorboat. His rowing speed in still water is 3 m/s and his maximum walking speed is 1 m/s. The river flows with velocity of 4 m/s.
    (a) Find the path which he should take to get to the point directly opposite to his starting point in the shortest time.
    (b) Also, find the time which he takes to reach his destination.

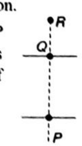
11. The current velocity of river grows in proportion to the distance from its bank and reaches the maximum value v_0 in the middle. Near the banks the velocity is zero. A boat is moving along the river in such a manner that the boatman rows his boat always perpendicular to the current. The speed of the boat in still water is u. Find the distance through which the boat crossing the river will be carried away by the current, if the width of the river is c. Also determine the trajectory of the boat.

12. The v-s graph for an airplane travelling on a straight runway is shown. Determine the acceleration of the plane at $s = 50$ m and $s = 150$ m. Draw the a-s graph.



13. A river of width a with straight parallel banks flows due north with speed u. The points O and A are on opposite banks and A is due east of O. Coordinate axes Ox and Oy are taken in the east and north directions respectively. A boat, whose speed is v relative to water, starts from O and crosses the river. If the boat is steered due east and u varies with x as : $u = x(a - x)\dfrac{v}{a^2}$. Find :
    (a) equation of trajectory of the boat,
    (b) time taken to cross the river,
    (c) absolute velocity of boatman when he reaches the opposite bank,
    (d) the displacement of boatman when he reaches the opposite bank from the initial position.

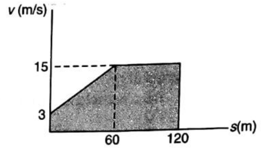
14. A river of width ω is flowing with a uniform velocity v. A boat starts moving from point P also with velocity v relative to the river. The direction of resultant velocity is always perpendicular to the line joining boat and the fixed point R. Point Q is on the opposite side of the river. P, Q and R are in a straight line. If $PQ = QR = \omega$, find :
    (a) the trajectory of the boat,



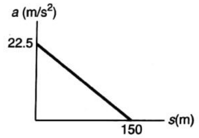
(b) the drifting of the boat and

(c) the time taken by the boat to cross the river.

15. The v-s graph describing the motion of a motorcycle is shown in figure. Construct the a-s graph of the motion and determine the time needed for the motorcycle to reach the position $s = 120$ m. Given $\ln 5 = 1.6$



16. The jet plane starts from rest at $s = 0$ and is subjected to the acceleration shown. Determine the speed of the plane when it has travelled 60 m.



17. A particle leaves the origin with an initial velocity $\vec{\mathbf{v}} = (3.00\hat{\mathbf{i}})$ m/s and a constant acceleration $\vec{\mathbf{a}} = (-1.00\hat{\mathbf{i}} - 0.500\hat{\mathbf{j}})$ m/s^2. When the particle reaches its maximum x coordinate, what are

(a) its velocity and

(b) its position vector?

18. The speed of a particle moving in a plane is equal to the magnitude of its instantaneous velocity, $v = |\vec{\mathbf{v}}| = \sqrt{v_x^2 + v_y^2}$.

(a) Show that the rate of change of the speed is $dv/dt = (v_x a_x + v_y a_y)/\sqrt{v_x^2 + v_y^2}$.

(b) Show that the rate of change of speed can be expressed as $dv/dt = \vec{\mathbf{v}} \cdot \vec{\mathbf{a}}/v$, and use this result to explain why dv/dt is equal to a_t the component of $\vec{\mathbf{a}}$ that is parallel to $\vec{\mathbf{v}}$.

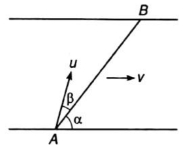
19. A man with some passengers in his boat, starts perpendicular to flow of river 200 m wide and flowing with 2 m/s. Speed of boat in still water is 4 m/s. When he reaches half the width of river the passengers asked him that they want to reach the just opposite end from where they have started.

(a) Find the direction due which he must row to reach the required end.

(b) How many times more time, it would take to that if he would have denied the passengers?

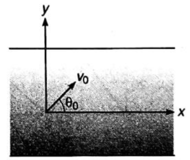
20. A child in danger of drowning in a river is being carried downstream by a current that flows uniformly at a speed of 2.5 km/h. The child is 0.6 km from shore and 0.8 km upstream of a boat landing when a rescue boat sets out. If the boat proceeds at its maximum speed of 20 km/h with respect to the water, what angle does the boat velocity v make with the shore? How long will it take boat to reach the child?

21. A launch plies between two points A and B on the opposite banks of a river always following the line AB. The distance S between points A and B is 1200 m. The velocity of the river current $v = 1.9$ m/s is constant over the entire width of the river. The line AB makes an angle $\alpha = 60°$ with the direction of the current. With what velocity u and at what angle β to the line AB should the launch move to cover the distance AB and back in a time $t = 5$ min? The angle β remains the same during the passage from A to B and from B to A.

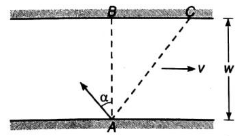


22. The slopes of wind screen of two cars are $\alpha_1 = 30°$ and $\alpha_2 = 15°$ respectively. At what ratio $\frac{v_1}{v_2}$ of the velocities of the cars will their drivers see the hail stones bounced back by the wind screen on their cars in vertical direction? Assume hail stones fall vertically downwards and collisions to be elastic.

23. A projectile of mass m is fired into a liquid at an angle θ_0 with an initial velocity v_0 as shown. If the liquid develops a frictional or drag resistance on the projectile which is proportional to its velocity, *i.e.*, $F = -kv$ where k is a positive constant, determine the x and y components of its velocity at any instant. Also find is the maximum distance x_{max} that it travels?



24. A man in a boat crosses a river from point A. If he rows perpendicular to the banks he reaches point C ($BC = 120$ m) in 10 min. If the man heads at a certain angle α to the straight line AB (AB is perpendicular to the banks) against the current he reaches point B in 12.5 min. Find the width of the river w, the rowing velocity u, the speed of the river current v and the angle α. Assume the velocity of the boat relative to water to be constant and the same magnitude in both cases.



ANSWERS

Introductory Exercise 3.1

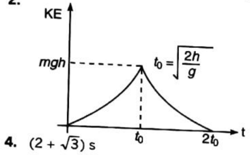
1. True 2. g (downwards) 3. $\frac{\pi}{15}$ cm/s, $\frac{2\sqrt{2}}{15}$ cm/s
4. (a) Yes, in uniform circular motion (b) No, yes (projectile motion), yes
5. (a) 25.13 s (b) 1 cm/s, 0.9 cm/s, 0.23 cm/s^2 6. 5.2 m/s

Introductory Exercise 3.2

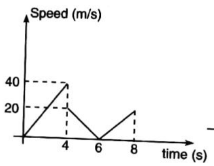
1. acceleration 2. $s_t = \left\{ut + \frac{1}{2}at^2\right\} - \left\{u(t-1) + \frac{1}{2}a(t-1)^2\right\} = (u \cdot 1) + (at)(1) - \frac{1}{2}a(1)^2$
3. Yes, in simple harmonic motion 4. $t^{7/4}, t^{-1/4}$ 5. 60 m, 100 m 6. 2 s, 6 s, $2(2+\sqrt{7})$ s
7. $u + \frac{1}{2}at$ 8. True 9. 25 m/s (downwards) 10. (a) 1 m/s^2 (b) 43.5 m 11. $2\sqrt{7}$ m/s, $4\sqrt{3}$ m
12. $(2\hat{j})$ m/s^2, $(2\hat{i} + \hat{j})$ m, yes 13. (a) $x^2 = 4y$ (b) $(2\hat{i} + 2t\hat{j})$ units (c) $(2\hat{j})$ units

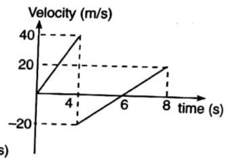
Introductory Exercise 3.3

1. v_{t_1}, a_{t_1} and a_{t_2} are positive while v_{t_2} is negative
2.



3.





4. $(2+\sqrt{3})$ s

Introductory Exercise 3.4

1. zero 2. straight line passing through origin 3. (a) 40 s (b) 80 m
4. (a) $\sin^{-1}\left(\frac{1}{15}\right)$ east of the line AB (b) 50 min 5. 8 m

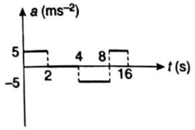
AIEEE Corner

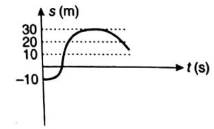
Subjective Questions (Level 1)

1. (a) 100 km (b) 66.67 kmh^{-1} 2. (a) 10 ms^{-1} (b) 22 ms^{-1} 3. (a) 16.67 ms^{-1} (b) 10 ms^{-1} (downwards)
4. $T = 4t_0$ 5. (a) zero (b) 8 ms^{-1} (c) 8 ms^{-1} 6. (a) $\frac{21\sqrt{3}}{44}$ ms^{-1} (b) $(\sqrt{3}/2)$ ms^{-2}
7. (a) $(1.25\hat{i} + 0.5\hat{j})$ ms^{-1} (b) $(-0.5\hat{i} + \hat{j})$ ms^{-2} (c) No 8. 1.5 s 9. $\frac{v_0}{g} + \frac{t_0}{2}$ 10. 45 m
11. $(3.414)\,t_0$ 12. (a) 16.25 m (b) 1.8 s 13. (a) 5 ms^{-1} (b) 1.67 ms^{-2} (c) 7.5 m
15. 0.2 ms^{-2}, 0.8 ms^{-1}
16. $10\sqrt{5}$ m at $\cos^{-1}(2)$ from east towards north, $10\sqrt{2}$ ms^{-1} at 45° from east towards north.
17. (a) $(8\hat{i} - 8\hat{j})$ ms^{-1} (b) (18 m, – 4m) 18. (a) $(10\hat{i} - 2\hat{j})$ ms^{-1} (b) (12 m, – 6 m)
19. (a) 45 m (b) 22 ms^{-1} 20. 20.6 ms^{-1}, $\tan^{-1}(4)$ 21. (a) $x = 2.0$ m (b) zero (c) 26 ms^{-2}
22. – 30 ms^{-2} 23. 42 ms^{-2} 24. (a) 0.833 ms^{-1} (b) – 9.17 ms^{-1} 25. 14.125 m, 1.75 ms^{-1}, 4.03 ms^{-1}

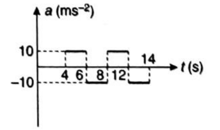
26. 0.603 ms^{-2} 27. (– 20/3) ms^{-2}
28. (a) positive, positive, positive, negative (b) positive, zero, negative, negative
29. (a) In M_1 and M_3 magnitude is increasing, in M_2 and M_4 magnitude is decreasing
(b) $P \to M_1$; $Q \to M_2$; $R \to M_3$; $S \to M_4$

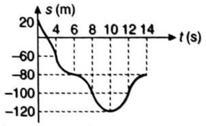
30.





31.





32. (a) (50/7) ms^{-1} (b) 10 ms^{-1} 33. (a) 5 ms^{-2} (b) 90 ms^{-1}
34. (a) 5 ms^{-2}, zero, 5 ms^{-2} (b) $s = 30$ m (c) (i) $s = 10 + 2.5t^2$ (ii) $s = 40 + 10(t-4) - 2.5(t-4)^2$
35. (a) zero (b) 25 ms^{-1} (c) – 25 ms^{-1} (d) 0.8 s 36. 36 s, No
37. (a) 3.65 s, at 12.30 m level (b) 19.8 ms^{-1} (downwards)
38. (a) 7.39 s (b) 35.5 m (c) automobile 25.9 ms^{-1}, truck 16.2 ms^{-1} 40. (a) 10 s (b) 50 m
41. $45° - \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right) \approx 24.3°$ 42. (a) at an angle $\theta = \sin^{-1}(0.4)$ west of north (b) $\frac{10}{\sqrt{21}}h$

Objective Questions (Level 1)

1.(b)	2.(c)	3.(d)	4.(c)	5.(c)	6.(a)	7.(b)	8.(b)	9.(c)	10.(a)
11.(c)	12.(c)	13.(a)	14.(a)	15.(b)	16.(c)	17.(c)	18.(c)	19.(b)	20.(b)
21.(a)	22.(c)	23.(b)	24.(d)	25.(a)	26.(a)	27.(b)	28.(b)	29.(d)	30.(b)
31.(b)	32.(c)	33.(b)	34.(b)	35.(b)	36.(b)	37.(b)	38.(d)		

JEE Corner

Assertion and Reason

1.(d)	2.(d)	3.(a)	4.(d)	5.(c)	6.(d)	7.(d)	8.(d)	9.(d)	10.(a,b)
11.(a)	12.(d)								

Objective Questions (Level 2)

1.(b)	2.(b)	3.(c)	4.(a)	5.(a)	6.(a)	7.(d)	8.(d)	9.(a)	10.(a)
11.(c)	12.(a)	13.(c)	14.(a)	15.(b)	16.(d)	17.(a)	18.(c)	19.(d)	20.(c)
21.(c)	22.(b)	23.(d)							

More than One Correct Options

1.(a,d)	2.(all)	3.(b)	4.(a,b)	5.(a,d)	6.(b,c)	7.(a,b)	8.(b,c)	9.(a,c)	10.(a,d)
11.(a,b,c)	12.(all)	13.(b,d)	14.(a,b,c)	15.(a,c)					

Match the Columns

1.	(a) → r,s	(b) → r,s	(c) → p	(d) → q
2.	(a) → p	(b) → p	(c) → q	(d) → q
3.	(a) → p	(b) → p	(c) → q	(d) → r
4.	(a) → (r)	(b) → (s)	(c) → (r)	(d) → (r)
5.	(a) → (r)	(b) → (q)	(c) → (s)	(d) → (p)
6.	(a) → (q)	(b) → (p)	(c) → (s)	(d) → (s)

Subjective Questions (Level 2)

1. $1.26 \times 10^3\ ms^{-2}$ (upward) **2.** $4\sqrt{3}\ ms^{-1}$

3. B moves up with initial velocity $\frac{v}{2}$ and downward acceleration $-\frac{a}{2}$ **5.** 20.5 s

6. (a) 0.74 s (b) $6.2\ ms^{-2}$ **7.** (a) 76 m (b) 4.2 s **8.** (a) $-0.833\ ms^{-1}$ (b) $2.63\ ms^{-1}$

9. (a) $v = t^2 - 2t$ (b) 6.67 m **10.** (a) $90° + \sin^{-1}(3/5)$ from river current (b) 2 min 40 s

11. $\frac{cv_0}{2u}, y^2 = \frac{ucx}{v_0}$ **12.** $8\ ms^{-2}$, $4.5\ ms^{-2}$ **13.** (a) $y = \frac{x^2}{2a} - \frac{x^3}{3a^2}$ (b) $\frac{a}{v}$ (c) v (due east) (d) $a\hat{\mathbf{i}} + \frac{a\hat{\mathbf{j}}}{6}$

14. (a) circle (b) $\sqrt{3}\ \omega$ (c) $\frac{1.317\ \omega}{v}$ **15.** 12.0 s **16.** $46.47\ ms^{-1}$

17. (a) $(-1.5\hat{\mathbf{j}})\ ms^{-1}$ (b) $(4.5\hat{\mathbf{i}} - 2.25\hat{\mathbf{j}})$ m

19. (a) At an angle $(90° + 2\theta)$ from river current (upstream). Here : $\theta = \tan^{-1}\left(\frac{1}{2}\right)$ (b) $\frac{4}{3}$

20. 37°, 3 min **21.** $u = 8\ ms^{-1}, \beta = 12°$ **22.** $\frac{v_1}{v_2} = 3$

23. $v_x = v_0 \cos\theta_0\, e^{-kt/m}, v_y = \frac{m}{k}\left[\left(\frac{k}{m}v_0 \sin\theta_0 + g\right)e^{-\frac{kt}{m}} - g\right] X_m = \frac{mv \cos\theta}{k}$

24. 200 m, $20\ m\ min^{-1}$, $12\ m\ min^{-1}$, 36°50′.

Chapter – 4 Projectile Motion

4

Projectile Motion

Chapter Contents

4.1 Projectile Motion
4.2 Projectile Motion in Inclined Plane
4.3 Relative Motion between Two Projectiles

4.1 Projectile Motion

If a constant force (and hence constant acceleration) acts on a particle at an angle θ ($\neq 0°$ or $180°$) with the direction of its initial velocity ($\neq$ zero), the path followed by the particle is a parabola and the motion of the particle is called projectile motion. Projectile motion is a two dimensional motion, *i.e.*, motion of the particle is constrained in a plane.

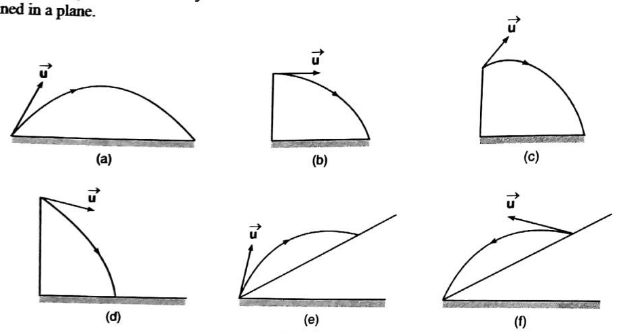


Fig. 4.1

When a particle is thrown obliquely near the earth's surface it moves in a parabolic path, provided the particle remains close to the surface of earth and the air resistance is negligible. This is an example of projectile motion. The different types of projectile motion we come across are shown in Fig. 4.1.

In all the above cases acceleration of the particle is g downwards.

Let us first make ourselves familiar with certain terms used in projectile motion.

Fig. 4.2 shows a particle projected from the point O with an initial velocity u at an angle α with the horizontal. It goes through the highest point A and falls at B on the horizontal surface through O. The point O is called the **point of projection**, the angle α is called the **angle of projection**, the distance OB is called the **horizontal range (R)** or simply range and the vertical height AC is called the **maximum height (H)**. The total time taken by the particle in describing the path OAB is called the **time of flight (T)**.

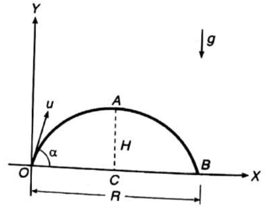


Fig. 4.2

As we have already discussed, projectile motion is a two dimensional motion with constant acceleration (normally g). Problems related to projectile motion of any type can be solved by selecting two appropriate mutually perpendicular directions (x and y) and substituting the proper values in equations

$$v_x = u_x + a_x t, \qquad s_x = u_x t + \frac{1}{2} a_x t^2,$$

$$v_x^2 = u_x^2 + 2a_x s_x, \qquad v_y = u_y + a_y t,$$

$$s_y = u_y t + \frac{1}{2} a_y t^2 \quad \text{and} \quad v_y^2 = u_y^2 + 2a_y s_y$$

In any problem of projectile motion we usually follow the three steps given below :

Step 1. Select two mutually perpendicular directions x and y.

Step 2. Write down the proper values of u_x, a_x, u_y and a_y with sign.

Step 3. Apply those equations from the six listed above which are required in the problem.

What should be the directions x and y or which equations are to be used, this you will learn after solving some problems of projectile motion. Using the above methodology let us first prove the three standard results of time of flight (T), horizontal range (R) and the maximum height (H).

Time of Flight (T)

Refer Fig. 4.2. Here, x and y-axes are in the directions shown in figure. Axis x is along horizontal direction and axis y is vertically upwards. Thus,

$$u_x = u\cos\alpha, \quad u_y = u\sin\alpha, a_x = 0 \quad \text{and} \quad a_y = -g$$

At point B, $s_y = 0$. So, applying

$$s_y = u_y t + \frac{1}{2} a_y t^2, \text{ we have}$$

$$0 = (u\sin\alpha)t - \frac{1}{2} g t^2$$

$$\therefore \quad t = 0, \ \frac{2u\sin\alpha}{g}$$

Both $t = 0$ and $t = \frac{2u\sin\alpha}{g}$ correspond to the situation where $s_y = 0$. The time $t = 0$ corresponds to point O and time $t = \frac{2u\sin\alpha}{g}$ corresponds to point B. Thus, time of flight of the projectile is :

$$T = t_{OAB} \quad \text{or} \quad T = \frac{2u\sin\alpha}{g}$$

Horizontal Range (R)

Distance OB is the range R. This is also equal to the displacement of particle along x-axis in time $t = T$. Thus, applying $s_x = u_x t + \frac{1}{2} a_x t^2$, we get

$$R = (u\cos\alpha)\left(\frac{2u\sin\alpha}{g}\right) + 0$$

as $\quad a_x = 0 \quad$ and $\quad t = T = \frac{2u\sin\alpha}{g}$

$$\therefore \quad R = \frac{2u^2\sin\alpha\cos\alpha}{g} = \frac{u^2\sin 2\alpha}{g}$$

or

$$R = \frac{u^2\sin 2\alpha}{g}$$

Here, two points are important regarding the range of a projectile.

(i) Range is maximum where $\sin 2\alpha = 1$ or $\alpha = 45°$ and this maximum range is:

$$R_{max} = \frac{u^2}{g} \qquad (\text{at } \alpha = 45°)$$

(ii) For given value of u range at α and range at $90° - \alpha$ are equal although times of flight and maximum heights may be different. Because

$$R_{90°-\alpha} = \frac{u^2 \sin 2(90° - \alpha)}{g} = \frac{u^2 \sin (180° - 2\alpha)}{g}$$

$$= \frac{u^2 \sin 2\alpha}{g} = R_\alpha$$

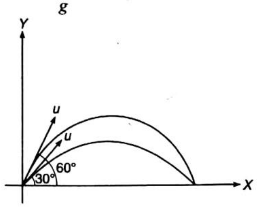


Fig. 4.3

So, $R_{30°} = R_{60°}$ or $R_{20°} = R_{70°}$

This is shown in Fig. 4.3.

Maximum Height (*H*)

At point A vertical component of velocity becomes zero, *i.e.*, $v_y = 0$. Substituting the proper values in

$$v_y^2 = u_y^2 + 2a_y s_y$$

we have,

$$0 = (u \sin \alpha)^2 + 2(-g)(H)$$

$$\therefore \quad H = \frac{u^2 \sin^2 \alpha}{2g}$$

Check Points

- As we have seen in the above derivations that $a_x = 0$, *i.e.*, motion of the projectile in horizontal direction is uniform. Hence, horizontal component of velocity $u \cos \alpha$ does not change during its motion.
- Motion in vertical direction is first retarded then accelerated in opposite direction. Because u_y is upwards and a_y is downwards. Hence, vertical component of its velocity first decreases from O to A and then increases from A to B. This can be shown as in Fig. 4.4.

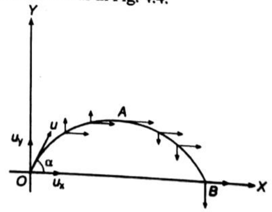


Fig. 4.4

$R = \frac{u^2 \sin 2\theta}{g}$; $\frac{2u^2 \sin\theta \cos\theta}{g}$, $\frac{2u^2 \cos^2\theta}{g}$

- The coordinates and velocity components of the projectile at time t are

$$x = s_x = u_x t = (u \cos \alpha) t$$

$$y = s_y = u_y t + \frac{1}{2} a_y t^2 = (u \sin \alpha) t - \frac{1}{2} g t^2$$

$$v_x = u_x = u \cos \alpha$$

and

$$v_y = u_y + a_y t = u \sin \alpha - gt$$

Therefore, speed of projectile at time t is $\boxed{v = \sqrt{v_x^2 + v_y^2}}$ and the angle made by its velocity vector with positive x-axis is

$$\boxed{\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)}$$

- Equation of trajectory of projectile

$$x = (u \cos \alpha) t \qquad \therefore \quad t = \frac{x}{u \cos \alpha}$$

Substituting this value of t in, $y = (u \sin \alpha) t - \frac{1}{2} g t^2$, we get

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}$$

$$= x \tan \alpha - \frac{gx^2}{2u^2} \sec^2 \alpha$$

$$= x \tan \alpha - \frac{gx^2}{2u^2} (1 + \tan^2 \alpha)$$

$y = x \tan\theta - \frac{g x^2}{2u^2 \cos^2\theta}$

These are the standard equations of trajectory of a projectile. The equation is quadratic in x. This is why the path of a projectile is a parabola. The above equation can also be written in terms of range (R) of projectile as

$$y = x\left(1 - \frac{x}{R}\right) \tan \alpha$$

Now, let us take few examples based on the above theory.

- Projectile motion is a two dimensional motion with constant acceleration (g). So, we can use $\vec{v} = \vec{u} + \vec{a}\, t$, $\vec{s} = \vec{u}\, t + \frac{1}{2} \vec{a}\, t^2$, etc., in projectile motion as well. Here,

$$\vec{u} = u \cos \alpha \hat{i} + u \sin \alpha \hat{j} \qquad \text{and} \qquad \vec{a} = -g\hat{j}$$

Now, suppose we want to find velocity at time t.

$$\vec{v} = \vec{u} + \vec{a}\, t$$

$$= (u \cos \alpha \hat{i} + u \sin \alpha \hat{j}) - gt\hat{j}$$

or

$$\vec{v} = u \cos \alpha \hat{i} + (u \sin \alpha - gt)\hat{j}$$

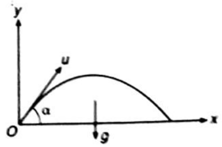


Fig. 4.5

Similarly, displacement at time t will be

$$\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$$

$$= (u\cos\alpha\hat{i} + u\sin\alpha\hat{j})t - \frac{1}{2}gt^2\hat{j}$$

$$= ut\cos\alpha\hat{i} + \left(ut\sin\alpha - \frac{1}{2}gt^2\right)\hat{j}$$

Sample Example 4.1 *Find the angle of projection of a projectile for which the horizontal range and maximum height are equal.*

Solution Given, $R = H$

$$\therefore \quad \frac{u^2\sin 2\alpha}{g} = \frac{u^2\sin^2\alpha}{2g}$$

or
$$2\sin\alpha\cos\alpha = \frac{\sin^2\alpha}{2}$$

or
$$\frac{\sin\alpha}{\cos\alpha} = 4 \quad \text{or} \quad \tan\alpha = 4$$

$$\therefore \quad \alpha = \tan^{-1}(4)$$

Sample Example 4.2 *Prove that the maximum horizontal range is four times the maximum height attained by the projectile; when fired at an inclination so as to have maximum horizontal range.*

Solution For $\theta = 45°$, the horizontal range is maximum and is given by

$$R_{max} = \frac{u^2}{g}$$

$\sin 45 = \frac{1}{\sqrt{2}}$

Maximum height attained

$$H_{max} = \frac{u^2\sin^2 45°}{2g} = \frac{u^2}{4g} = \frac{R_{max}}{4}$$

$\sin^2 45 = \frac{1}{2}$

or
$$R_{max} = 4H_{max}$$

Proved.

Sample Example 4.3 *There are two angles of projection for which the horizontal range is the same. Show that the sum of the maximum heights for these two angles is independent of the angle of projection.*

Solution There are two angles of projection α and $90° - \alpha$ for which the horizontal range R is same.

Now,
$$H_1 = \frac{u^2\sin^2\alpha}{2g}$$

and
$$H_2 = \frac{u^2\sin^2(90° - \alpha)}{2g} = \frac{u^2\cos^2\alpha}{2g}$$

Therefore,
$$H_1 + H_2 = \frac{u^2}{2g}(\sin^2\alpha + \cos^2\alpha) = \frac{u^2}{2g}$$

Clearly the sum of the heights for the two angles of projection is independent of the angles of projection.

Sample Example 4.4 *Show that there are two values of time for which a projectile is at the same height. Also show mathematically that the sum of these two times is equal to the time of flight.*

Solution For vertically upward motion of a projectile,

$$y = (u\sin\alpha)t - \frac{1}{2}gt^2$$

or
$$\frac{1}{2}gt^2 - (u\sin\alpha)t + y = 0$$

This is a quadratic equation in t. Its roots are

$$t_1 = \frac{u\sin\alpha - \sqrt{u^2\sin^2\alpha - 2gy}}{g}$$

and
$$t_2 = \frac{u\sin\alpha + \sqrt{u^2\sin^2\alpha - 2gy}}{g}$$

$$\therefore \quad t_1 + t_2 = \frac{2u\sin\alpha}{g} = T \quad \text{(time of flight of the projectile)}$$

Sample Example 4.5 *A projectile is fired horizontally with a velocity of 98 m/s from the top of a hill 490 m high. Find:*
(a) the time taken by the projectile to reach the ground,
(b) the distance of the point where the particle hits the ground from foot of the hill and
(c) the velocity with which the projectile hits the ground.
$(g = 9.8\ m/s^2)$

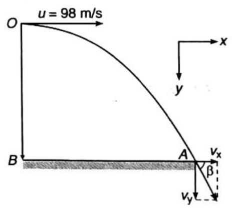


Fig. 4.6

Solution In this problem we cannot apply the formulae of R, H and T directly. We will have to follow the three steps discussed in the theory. Here, it will be more convenient to choose x and y directions as shown in figure.

Here, $u_x = 98$ m/s, $a_x = 0$, $u_y = 0$ and $a_y = g$

(a) At A, $s_y = 490$ m. So, applying

$$s_y = \cancel{u_y t} + \frac{1}{2}a_y t^2$$

$$\therefore \quad 490 = 0 + \frac{1}{2}(9.8)t^2$$

$$\therefore \quad t = 10\text{s}$$

(b)
$$BA = s_x = u_x t + \frac{1}{2}a_x t^2$$

or
$$BA = (98)(10) + 0$$

or
$$BA = 980\text{ m}$$

(c)
$$v_x = u_x = 98 \text{ m/s}$$
$$v_y = u_y + a_y t = 0 + (9.8)(10) = 98 \text{ m/s}$$
$\therefore$
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(98)^2 + (98)^2} = 98\sqrt{2} \text{ m/s}$$
and
$$\tan\beta = \frac{v_y}{v_x} = \frac{98}{98} = 1$$
$\therefore$
$$\beta = 45°$$

Thus, the projectile hits the ground with a velocity $98\sqrt{2}$ m/s at an angle of $\beta = 45°$ with horizontal as shown in Fig. 4.6.

Sample Example 4.6 *A body is thrown horizontally from the top of a tower and strikes the ground after three seconds at an angle of 45° with the horizontal. Find the height of the tower and the speed with which the body was projected. Take $g = 9.8 \, m/s^2$.*

Solution As shown in the figure of Sample Example 4.5.

$$u_y = 0 \quad \text{and} \quad a_y = g = 9.8 \text{ m/s}^2 \qquad s_y = u_y t + \frac{1}{2} a_y t^2$$
$$= 0 \times 3 + \frac{1}{2} \times 9.8 \times (3)^2$$
$$= 44.1 \text{ m}$$

Further,
$$v_y = u_y + a_y t = 0 + (9.8)(3)$$
$$= 29.4 \text{ m/s}$$

As the resultant velocity v makes an angle of 45° with the horizontal, so

$$\tan 45° = \frac{v_y}{v_x} \quad \text{or} \quad 1 = \frac{29.4}{v_x}$$
$$v_x = 29.4 \text{ m/s}$$

Therefore, the speed with which the body was projected (horizontally) is 29.4 m/s.

Introductory Exercise 4.1

1. Projectile motion is a 3-dimensional motion. Is this statement true or false.
2. Projectile motion (at low speeds) is uniformly accelerated motion. Is this statement true or false.
3. A particle is projected with speed u at angle θ with vertical. Find :
   (a) time of flight (b) maximum height
   (c) range (d) maximum range and corresponding value of θ.
4. A particle is projected from ground with velocity $40\sqrt{2}$ m/s at 45°. Find :
   (a) velocity and
   (b) displacement of the particle after 2 s. ($g = 10 \text{ m/s}^2$)
5. A particle is projected from ground with velocity $20\sqrt{2}$ m/s at 45°. At what time particle is at height 15 m from ground? ($g = 10 \text{ m/s}^2$)

6. A particle is projected from ground with velocity 40 m/s at 60° with horizontal. Find speed of particle when its velocity is making 45° with horizontal. Also find the times (s) when it happens. ($g = 10 \text{ m/s}^2$)
7. What is the average velocity of a particle projected from the ground with speed u at an angle α with the horizontal over a time interval from beginning till it strikes the ground again?
8. What is the change in velocity in the above question?
9. Under what conditions the formulae of range, time of flight and maximum height can be applied directly in case of a projectile motion?
10. A body is projected up such that its position vector varies with time as $\vec{r} = \{3t\hat{i} + (4t - 5t^2)\hat{j}\}$ m. Here, t is in seconds.
    Find the time and x-coordinate of particle when its y-coordinate is zero.
11. A particle is projected at an angle 60° with horizontal with a speed $v = 20$ m/s. Taking $g = 10 \text{ m/s}^2$. Find the time after which the speed of the particle remains half of its initial speed.

4.2 Projectile Motion in Inclined Plane

Here, two cases arise. One is up the plane and the other is down the plane. Let us discuss both the cases separately.

(i) Up the Plane : In this case direction x is chosen up the plane and direction y is chosen perpendicular to the plane. Hence,

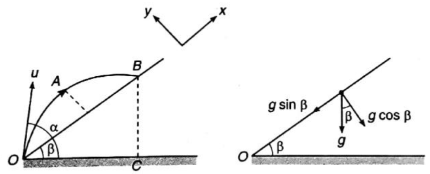


Fig. 4.7

$$u_x = u\cos(\alpha - \beta), \qquad a_x = -g\sin\beta$$
$$u_y = u\sin(\alpha - \beta) \quad \text{and} \quad a_y = -g\cos\beta$$

Now, let us derive the expressions for time of flight (T) and range (R) along the plane.

Time of Flight

At point B displacement along y-direction is zero. So, substituting the proper values in $s_y = u_y t + \frac{1}{2} a_y t^2$, we get

$$0 = ut\sin(\alpha - \beta) + \frac{1}{2}(-g\cos\beta)t^2$$
$\therefore$
$$t = 0 \quad \text{and} \quad \frac{2u\sin(\alpha - \beta)}{g\cos\beta}$$

$t = 0$, corresponds to point O and $t = \dfrac{2u\sin(\alpha - \beta)}{g\cos\beta}$ corresponds to point B. Thus,

$$T = \frac{2u\sin(\alpha - \beta)}{g\cos\beta}$$

Note *Substituting $\beta = 0$, in the above expression, we get $T = \frac{2u \sin \alpha}{g}$ which is quite obvious because $\beta = 0$ is the situation shown in Fig. 4.8.*

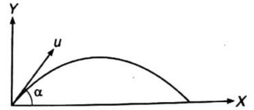


Fig. 4.8

Range

Range (R) or the distance OB can be found by following two methods:

Method 1. Horizontal component of initial velocity is:

$$u_H = u \cos \alpha \qquad (\text{as } a_H = 0)$$

$$\therefore \quad OC = u_H T = \frac{(u \cos \alpha) 2u \sin (\alpha - \beta)}{g \cos \beta} = \frac{2u^2 \sin (\alpha - \beta) \cos \alpha}{g \cos \beta}$$

$$\therefore \quad R = OB = \frac{OC}{\cos \beta} = \frac{2u^2 \sin (\alpha - \beta) \cos \alpha}{g \cos^2 \beta}$$

Using, $\sin C - \sin D = 2 \sin \left(\frac{C-D}{2}\right) \cos \left(\frac{C+D}{2}\right)$,

Range can also be written as,

$$R = \frac{u^2}{g \cos^2 \beta} [\sin (2\alpha - \beta) - \sin \beta]$$

This range will be maximum when

$$2\alpha - \beta = \frac{\pi}{2} \quad \text{or} \quad \alpha = \frac{\pi}{4} + \frac{\beta}{2}$$

and

$$R_{max} = \frac{u^2}{g \cos^2 \beta} [1 - \sin \beta]$$

Here, also we can see that for $\beta = 0$, range is maximum at $\alpha = \frac{\pi}{4}$ or $\alpha = 45°$

and

$$R_{max} = \frac{u^2}{g \cos^2 0°} (1 - \sin 0°) = \frac{u^2}{g}$$

Method 2. Range (R) or the distance OB is also equal to the displacement of projectile along x-direction in time $t = T$. Therefore,

$$R = s_x = u_x T + \frac{1}{2} a_x T^2$$

Substituting the values of u_x, a_x and T, we get the same result.

(ii) Down the Plane : Here, x and y-directions are down the plane and perpendicular to plane respectively as shown in Fig. 4.9. Hence,

$$u_x = u \cos (\alpha + \beta), \quad a_x = g \sin \beta$$
$$u_y = u \sin (\alpha + \beta), \quad a_y = -g \cos \beta$$

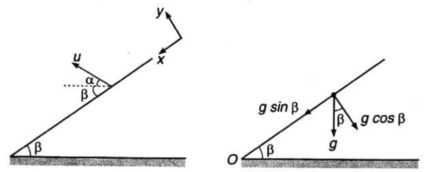


Fig. 4.9

Proceeding in the similar manner, we get the following results:

$$T = \frac{2u \sin (\alpha + \beta)}{g \cos \beta}, \quad R = \frac{u^2}{g \cos^2 \beta} [\sin (2\alpha + \beta) + \sin \beta]$$

From the above expressions, we can see that if we replace β by $-\beta$, the equations of T and R for up the plane and down the plane are interchanged provided α (angle of projection) in both the cases is measured from the horizontal not from the plane.

Sample Example 4.7 *A man standing on a hill top projects a stone horizontally with speed v_0 as shown in figure. Taking the co-ordinate system as given in the figure. Find the co-ordinates of the point where the stone will hit the hill surface.*

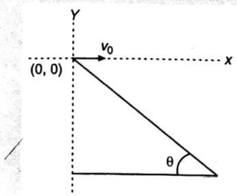


Fig. 4.10

Solution Range of the projectile on an inclined plane (down the plane) is,

$$R = \frac{u^2}{g \cos^2 \beta} [\sin (2\alpha + \beta) + \sin \beta]$$

Here, $u = v_0$, $\alpha = 0$ and $\beta = \theta$

$$\therefore \qquad R = \frac{2v_0^2 \sin\theta}{g\cos^2\theta}$$

Now,
$$x = R\cos\theta = \frac{2v_0^2 \tan\theta}{g}$$

and
$$y = -R\sin\theta = -\frac{2v_0^2 \tan^2\theta}{g}$$

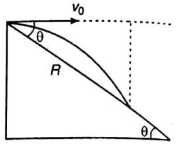


Fig. 4.11

4.3 Relative Motion between Two Projectiles

Let us now discuss the relative motion between two projectiles or the path observed by one projectile of the other. Suppose that two particles are projected from the ground with speeds u_1 and u_2 at angles α_1 and α_2 as shown in Fig. 4.13 and 4.14. Acceleration of both the particles is g downwards. So, relative acceleration between them is zero because

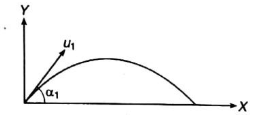


Fig. 4.12

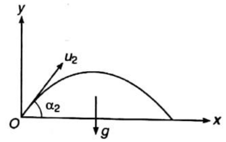


Fig. 4.13

$$a_{12} = a_1 - a_2 = g - g = 0$$

i.e., the relative motion between the two particles is uniform. Now,

$$u_{1x} = u_1\cos\alpha_1, \qquad u_{2x} = u_2\cos\alpha_2$$

$$u_{1y} = u_1\sin\alpha_1 \quad \text{and} \quad u_{2y} = u_2\sin\alpha_2$$

Therefore,
$$u_{12x} = u_{1x} - u_{2x} = u_1\cos\alpha_1 - u_2\cos\alpha_2$$

and
$$u_{12y} = u_{1y} - u_{2y} = u_1\sin\alpha_1 - u_2\sin\alpha_2$$

u_{12x} and u_{12y} are the x and y components of relative velocity of 1 with respect to 2.

Hence, relative motion of 1 with respect to 2 is a straight line at an angle $\theta = \tan^{-1}\left(\frac{u_{12y}}{u_{12x}}\right)$ with positive x-axis.

Now, if $u_{12x} = 0$ or $u_1\cos\alpha_1 = u_2\cos\alpha_2$, the relative motion is along y-axis or in vertical direction (as $\theta = 90°$). Similarly, if $u_{12y} = 0$ or $u_1\sin\alpha_1 = u_2\sin\alpha_2$, the relative motion is along x-axis or in horizontal direction (as $\theta = 0°$).

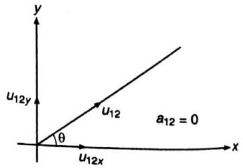


Fig. 4.14

Note *Relative acceleration between two projectiles is zero. Relative motion between them is uniform. Therefore, condition of collision of two particles in air is that relative velocity of one with respect to the other should be along line joining them, i.e., if two projecticles A and B collide in mid air, then $\vec{v}_{AB}$ should be along AB or $\vec{v}_{BA}$ along BA.*

Sample Example 4.8 *A particle A is projected with an initial velocity of 60 m/s at an angle 30° to the horizontal. At the same time a second particle B is projected in opposite direction with initial speed of 50 m/s from a point at a distance of 100 m from A. If the particles collide in air, find (a) the angle of projection α of particle B, (b) time when the collision takes place and (c) the distance of P from A, where collision occurs.* ($g = 10\ m/s^2$)

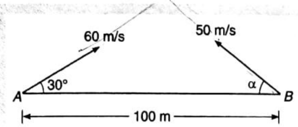


Fig. 4.15

Solution (a) Taking x and y-directions as shown in figure.

Here,
$$\vec{a}_A = -g\hat{j}$$
$$\vec{a}_B = -g\hat{j}$$
$$u_{Ax} = 60\cos 30° = 30\sqrt{3}\ \text{m/s}$$
$$u_{Ay} = 60\sin 30° = 30\ \text{m/s}$$
$$u_{Bx} = -50\cos\alpha$$

and
$$u_{By} = 50\sin\alpha$$

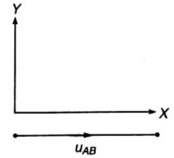


Fig. 4.16

Relative acceleration between the two is zero as $\vec{a}_A = \vec{a}_B$. Hence, the relative motion between the two is uniform. It can be assumed that B is at rest and A is moving with $\vec{u}_{AB}$. Hence, the two particles will collide, if $\vec{u}_{AB}$ is along AB. This is possible only when

$$u_{Ay} = u_{By}$$

i.e., component of relative velocity along y-axis should be zero.

or
$$30 = 50\sin\alpha$$

$$\therefore \qquad \alpha = \sin^{-1}(3/5)$$

(b) Now,
$$|\vec{u}_{AB}| = u_{Ax} - u_{Bx}$$
$$= (30\sqrt{3} + 50\cos\alpha)\ \text{m/s}$$
$$= \left(30\sqrt{3} + 50 \times \frac{4}{5}\right)\ \text{m/s}$$
$$= (30\sqrt{3} + 40)\ \text{m/s}$$

Therefore, time of collision is

$$t = \frac{AB}{|\vec{u}_{AB}|} = \frac{100}{30\sqrt{3} + 40}$$

or
$$t = 1.09\ \text{s}$$

(c) Distance of point P from A where collision takes place is

$$s=\sqrt{(u_{Ax}t)^2+\left(u_{Ay}t-\frac{1}{2}gt^2\right)^2}$$

$$=\sqrt{(30\sqrt{3}\times 1.09)^2+\left(30\times 1.09-\frac{1}{2}\times 10\times 1.09\times 1.09\right)^2}$$

or $$s = 62.64 \text{ m}$$

Introductory Exercise 4.2

1. A particle is projected along an inclined plane as shown in figure. What is the speed of the particle when it collides at point A? $(g = 10 \text{ m/s}^2)$

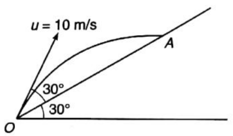


Fig. 4.17

2. In the above problem what is the component of its velocity perpendicular to the plane when it strikes at A?

3. Two particles A and B are projected simultaneously from the two towers of height 10 m and 20 m respectively. Particle A is projected with an initial speed of $10\sqrt{2}$ m/s at an angle of 45° with horizontal, while particle B is projected horizontally with speed 10 m/s. If they collide in air, what is the distance d between the towers?

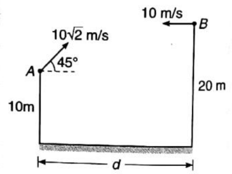


Fig. 4.18

4. Two particles A and B are projected from ground towards each other with speeds 10 m/s and $5\sqrt{2}$ m/s at angles 30° and 45° with horizontal from two points separated by a distance of 15 m. Will they collide or not?

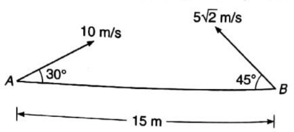


Fig. 4.19

5. A particle is projected from the bottom of an inclined plane of inclination 30°. At what angle α (from the horizontal) should the particle be projected to get the maximum range on the inclined plane.

6. A particle is projected from the bottom of an inclined plane of inclination 30° with velocity of 40 m/s at an angle of 60° with horizontal. Find the speed of the particle when its velocity vector is parallel to the plane. Take $g = 10 \text{ m/s}^2$.

7. Two particles A and B are projected simultaneously in the directions shown in figure with velocities $v_A = 20$ m/s and $v_B = 10$ m/s respectively. They collide in air after $\frac{1}{2}$ s. Find :

(a) the angle θ (b) the distance x.

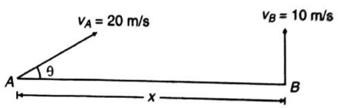


Fig. 4.20

Extra Points

- In projectile motion speed (and hence kinetic energy) is minimum at highest point.

  Speed = $(\cos\theta)$ times the speed of projection

  and kinetic energy = $(\cos^2\theta)$ times the initial kinetic energy

  Here, θ = angle of projection

- Path of a particle depends on nature of acceleration and the angle between initial velocity $\vec{u}$ and acceleration $\vec{a}$. Following are few paths which are observed frequently.

  (a) If $\vec{a}$ = constant and θ is either 0° or 180°, then path of the particle is straight line.

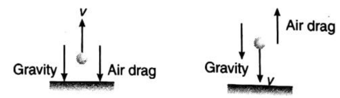
  (b) If $\vec{a}$ = constant but θ is other then 0° or 180°, then path of the particle is parabola (as in projectile motion).

  (c) If $|\vec{a}|$ = constant and $\vec{a}$ is always perpendicular to velocity vector $\vec{v}$, then path of the particle is a circle.

- In projectile motion it is sometimes better to write the equations of H, R and T in terms of u_x and u_y as under.

$$T=\frac{2u_y}{g},\quad H=\frac{u_y^2}{2g}\quad \text{and}\quad R=\frac{2u_xu_y}{g}$$

- In projectile motion $H = R$, when $u_y = 4u_x$ or $\tan\theta = 4$.

- 

  If a particle is projected vertically upwards, then during upward journey gravity forces (weight) and air drag both are acting downwards. Hence, |retardation| > $|g|$. During its downward journey air drag is upwards while gravity is downwards. Hence, acceleration < g. Therefore we may conclude that,

  time of ascent < time of descent

  Exercise : In projectile motion, if air drag is taken into consideration than state whether the H, R and T will increase, decrease or remain same.

- In JEE problems from the present chapter kinematics are often based on collision of two particles. Following are two tips to solve such problems.

  (a) At the time of collision coordinates of particles should be same, *i.e.*,

  $$x_1 = x_2,\quad \text{and}\quad y_1 = y_2 \quad \text{(for a 2- D motion)}$$

  Similarly $$x_1 = x_2,\quad y_1 = y_2 \quad \text{and}\quad z_1 = z_2 \quad \text{(for a 3-D motion)}$$

  (b) Two particles collide at the same moment. Of course their time of journies may be different, *i.e.*, they may start at different times (t_1 and t_2 may be different). If they start together then $t_1 = t_2$.

Solved Examples

Level 1

Example 1 *A particle is projected from horizontal making an angle 60° with initial velocity 40 m/s. Find the time taken to the particle to make angle 45° from horizontal.*

Solution At 45°, $v_x = v_y$

or $$u_x = u_y - gt$$

$\therefore$ $$t = \frac{u_y - u_x}{g} = \frac{40\,(\sin 60° - \sin 30°)}{9.8}$$

$$= 1.5\,\text{s}$$

Example 2 *A ball rolls off the edge of a horizontal table top 4 m high. If it strikes the floor at a point 5 m horizontally away from the edge of the table, what was its speed at the instant it left the table.*

Solution Using $h = \frac{1}{2}gt^2$, we have

$$h_{AB} = \frac{1}{2}gt^2_{AC}$$

or $$t_{AC} = \sqrt{\frac{2h_{AB}}{g}} = \sqrt{\frac{2\times 4}{9.8}} = 0.9\,\text{s}$$

Further, $$BC = vt_{AC}$$

or $$v = \frac{BC}{t_{AC}} = \frac{5.0}{0.9} = 5.55\,\text{m/s}$$

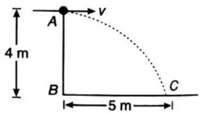


Fig. 4.21

Example 3 *An aeroplane is flying in a horizontal direction with a velocity 600 km/h at a height of 1960 m. When it is vertically above the point A on the ground, a body is dropped from it. The body strikes the ground at point B. Calculate the distance AB.*

Solution From $h = \frac{1}{2}gt^2$

we have, $$t_{OB} = \sqrt{\frac{2h_{OA}}{g}} = \sqrt{\frac{2\times 1960}{9.8}} = 20\,\text{s}$$

Horizontal distance $$AB = vt_{OB}$$

$$= \left(600 \times \frac{5}{18}\,\text{m/s}\right)(20\,\text{s})$$

$$= 3333.33\,\text{m} = 3.33\,\text{km}$$

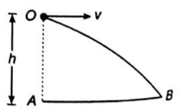


Fig. 4.22

Example 4 *A particle moves in the plane xy with constant acceleration a directed along the negative y-axis. The equation of motion of the particle has the form $y = px - qx^2$ where p and q are positive constants. Find the velocity of the particle at the origin of co-ordinates.*

Solution Comparing the given equation with the equation of a projectile motion,

$$y = x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta)$$

we find that $$g = a,\quad \tan\theta = p \quad\text{and}\quad \frac{a}{2u^2}(1+\tan^2\theta) = q$$

$\therefore$ $$u = \text{velocity of particle at origin}$$

$$= \sqrt{\frac{a(1+\tan^2\theta)}{2q}} = \sqrt{\frac{a(1+p^2)}{2q}}$$

Level 2

Example 1 *A car accelerating at the rate of $2\,m/s^2$ from rest from origin is carrying a man at the rear end who has a gun in his hand. The car is always moving along positive x-axis. At $t = 4$ s, the man fires from the gun and the bullet hits a bird at $t = 8$s. The bird has a position vector $40\hat{\mathbf{i}} + 80\hat{\mathbf{j}} + 40\hat{\mathbf{k}}$. Find velocity of projection of the bullet. Take the y-axis in the horizontal plane. ($g = 10\,m/s^2$)*

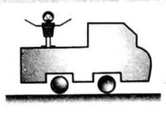


Fig. 4.23

Solution Let velocity of bullet be,

$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

At $t = 4$ s, x-coordinate of car is

$$x_c = \frac{1}{2}at^2 = \frac{1}{2}\times 2\times 16 = 16\,\text{m}$$

x-coordinate of bird is $x_b = 40$ m

$\therefore$ $$x_b = x_c + v_x\,(8-4)$$

or $$40 = 16 + 4v_x$$

$\therefore$ $$v_x = 6\,\text{m/s}$$

Similarly, $$y_b = y_c + v_y\,(8-4)$$

or $$80 = 0 + 4v_y$$

or $$v_y = 20\,\text{m/s}$$

and $$z_b = z_c + v_z\,(8-4) - \frac{1}{2}g\,(8-4)^2$$

or $$40 = 0 + 4v_z - \frac{1}{2}\times 5\times 16$$

or

$$v_z = 20\,\text{m/s}^2$$

$\therefore$ Velocity of projection of bullet

$$\vec{v} = (6\hat{i} + 20\hat{j} + 20\hat{k})\,\text{m/s}$$

Example 2 *The velocity of a projectile when it is at the greatest height is $\sqrt{2/5}$ times its velocity when it is at half of its greatest height. Determine its angle of projection.*

Solution Suppose the particle is projected with velocity u at an angle θ with the horizontal. Horizontal component of its velocity at all height will be $u\cos\theta$.

At the greatest height, the vertical component of velocity is zero, so the resultant velocity is

$$v_1 = u\cos\theta$$

At half the greatest height during upward motion,

$$y = h/2, \quad a_y = -g, \quad u_y = u\sin\theta$$

Using $$v_y^2 - u_y^2 = 2a_y y$$

we get, $$v_y^2 - u^2\sin^2\theta = 2(-g)\frac{h}{2}$$

or $$v_y^2 = u^2\sin^2\theta - g\times\frac{u^2\sin^2\theta}{2g} = \frac{u^2\sin^2\theta}{2} \qquad \left[\because h = \frac{u^2\sin^2\theta}{2g}\right]$$

or $$v_y = \frac{u\sin\theta}{\sqrt{2}}$$

Hence, resultant velocity at half of the greatest height is

$$v_2 = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2\cos^2\theta + \frac{u^2\sin^2\theta}{2}}$$

Given, $$\frac{v_1}{v_2} = \sqrt{\frac{2}{5}}$$

$\therefore$ $$\frac{v_1^2}{v_2^2} = \frac{u^2\cos^2\theta}{u^2\cos^2\theta + \dfrac{u^2\sin^2\theta}{2}} = \frac{2}{5} \quad \text{or} \quad \frac{1}{1 + \frac{1}{2}\tan^2\theta} = \frac{2}{5}$$

or $$2 + \tan^2\theta = 5 \quad \text{or} \quad \tan^2\theta = 3$$

or $$\tan\theta = \sqrt{3}$$

$\therefore$ $$\theta = 60°$$

Example 3 *A particle is thrown over a triangle from one end of a horizontal base and after grazing the vertex falls on the other end of the base. If α and β be the base angles and θ the angle of projection, prove that $\tan\theta = \tan\alpha + \tan\beta$.*

Solution The situation is shown in Fig. 4.24.

From figure, we have

$$\tan\alpha + \tan\beta = \frac{y}{x} + \frac{y}{R-x}$$

$$\tan\alpha + \tan\beta = \frac{yR}{x(R-x)} \qquad \ldots\text{(i)}$$

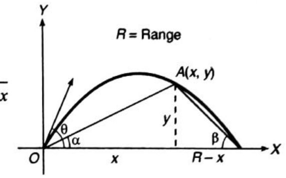


Fig. 4.24

Equation of trajectory is $$y = x\tan\theta\left[1 - \frac{x}{R}\right]$$

or, $$\tan\theta = \frac{yR}{x(R-x)} \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we have

$$\tan\theta = \tan\alpha + \tan\beta$$

Example 4 *Two inclined planes OA and OB having inclinations 30° and 60° with the horizontal respectively intersect each other at O, as shown in figure. A particle is projected from point P with velocity $u = 10\sqrt{3}$ m/s along a direction perpendicular to plane OA. If the particle strikes plane OB perpendicular at Q. Calculate*

(a) time of flight,

(b) velocity with which the particle strikes the plane OB,

(c) height h of point P from point O,

(d) distance PQ. (Take $g = 10\,m/s^2$)

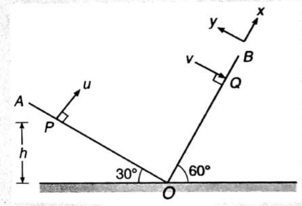


Fig. 4.25

Solution Let us choose the x and y directions along OB and OA respectively. Then,

$$u_x = u = 10\sqrt{3}\,\text{m/s}, \; u_y = 0$$

$$a_x = -g\sin 60° = -5\sqrt{3}\,\text{m/s}^2 \quad \text{and} \quad a_y = -g\cos 60° = -5\,\text{m/s}^2$$

(a) At point Q, x-component of velocity is zero. Hence, substituting in

$$v_x = u_x + a_x t$$

$$0 = 10\sqrt{3} - 5\sqrt{3}t$$

or $$t = \frac{10\sqrt{3}}{5\sqrt{3}} = 2\,\text{s}$$

(b) At point Q, $$v = v_y = u_y + a_y t$$

$\therefore$ $$v = 0 - (5)(2) = -10\,\text{m/s}$$

Here, negative sign implies that velocity of particle at Q is along negative y direction.

(c) Distance $PO = |\text{displacement of particle along } y\text{-direction}| = |s_y|$

Here, $$s_y = u_y t + \frac{1}{2}a_y t^2 = 0 - \frac{1}{2}(5)(2)^2 = -10\,\text{m}$$

$\therefore$ $$PO = 10\,\text{m}$$

Therefore, $h = PO \sin 30° = (10)\left(\frac{1}{2}\right)$

or $h = 5$ m

(d) Distance OQ = displacement of particle along x-direction $= s_x$

Here, $s_x = u_x t + \frac{1}{2} a_x t^2$

$$= (10\sqrt{3})(2) - \frac{1}{2}(5\sqrt{3})(2)^2 = 10\sqrt{3} \text{ m}$$

or $OQ = 10\sqrt{3}$ m

$\therefore$ $PQ = \sqrt{(PO)^2 + (OQ)^2}$

$$= \sqrt{(10)^2 + (10\sqrt{3})^2}$$

$$= \sqrt{100 + 300} = \sqrt{400}$$

$\therefore$ $PQ = 20$ m

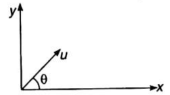
EXERCISES

AIEEE Corner

Subjective Questions (Level 1)

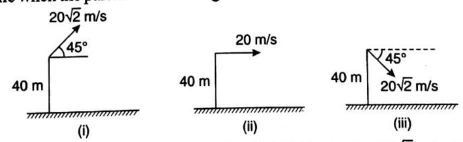
Projectile Motion from Ground to Ground

1. A particle is projected from ground with initial velocity $u = 20\sqrt{2}$ m/s at $\theta = 45°$. Find :



(a) R, H and T,
(b) velocity of particle after 1 s
(c) velocity of particle at the time of collision with the ground (x-axis).

2. In the figures shown, three particles are thrown from a tower of height 40 m as shown in figure. In each case find the time when the particles strike the ground and the distance of this point from foot of tower.



3. A particle is projected from ground at angle 45° with initial velocity $20\sqrt{2}$ m/s. Find :
(a) change in velocity,
(b) magnitude of average velocity in a time interval from $t = 0$ to $t = 3$ s.

4. The coach throws a baseball to a player with an initial speed of 20 m/s at an angle of 45° with the horizontal. At the moment the ball is thrown, the player is 50 m from the coach. At what speed and in what direction must the player run to catch the ball at the same height at which it was released? $(g = 10 \text{ m/s}^2)$

5. At time $t = 0$ a smail ball is projected from point A with a velocity of 60 m/s at 60° angle with horizontal. Neglect atmospheric resistance and determine the two times t_1 and t_2 when the velocity of the ball makes an angle of 45° with the horizontal x-axis.

6. A particle moves in the xy-plane with constant acceleration a directed along the negative y-axis. The equation of path of the particle has the form $y = bx - cx^2$, where b and c are positive constants. Find the velocity of the particle at the origin of coordinates.

7. A ball is shot from the ground into the air. At a height of 9.1 m, its velocity is observed to be $\vec{v} = 7.6\hat{i} + 6.1\hat{j}$ in metre per second ($\hat{i}$ is horizontal, $\hat{j}$ is upward). Give the approximate answers.
   (a) To what maximum height does the ball rise?
   (b) What total horizontal distance does the ball travel?
   What are :
   (c) the magnitude and
   (d) the direction of the ball's velocity just before it hits the ground?

8. Two particles move in a uniform gravitational field with an acceleration g. At the initial moment the particles were located over a tower at one point and moved with velocities $v_1 = 3$ m/s and $v_2 = 4$ m/s horizontally in opposite directions. Find the distance between the particles at the moment when their velocity vectors become mutually perpendicular.

9. A ball is thrown from the ground to clear a wall 3 m high at a distance of 6 m and falls 18 m away from the wall. Find the angle of projection of ball.

10. A particle is projected with velocity $2\sqrt{gh}$, so that it just clears two walls of equal height h which are at a distance of $2h$ from each other. Show that the time of passing between the walls is $2\sqrt{\frac{h}{g}}$.

   [**Hint :** First find velocity at height h. Treat it as initial velocity and $2h$ as the range.]

11. A particle is projected at an angle of elevation α and after t second it appears to have an elevation of β as seen from the point of projection. Find the initial velocity of projection.

12. A projectile aimed at a mark, which is in the horizontal plane through the point of projection, falls a cm short of it when the elevation is α and goes b cm far when the elevation is β. Show that, if the speed of projection is same in all the cases the proper elevation is :

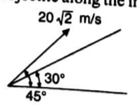
$$\frac{1}{2}\sin^{-1}\left[\frac{b\sin 2\alpha + a\sin 2\beta}{a+b}\right]$$

13. Two particles are simultaneously thrown in horizontal direction from two points on a riverbank, which are at certain height above the water surface. The initial velocities of the particles are $v_1 = 5$ m/s and $v_2 = 7.5$ m/s respectively. Both particles fall into the water at the same time. First particle enters the water at a point $s = 10$ m from the bank. Determine :
   (a) the time of flight of the two particles,
   (b) the height from which they are thrown,
   (c) the point where the second particle falls in water.

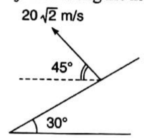
14. A balloon is ascending at the rate $v = 12$ km/h and is being carried horizontally by the wind at $v_w = 20$ km/h. If a ballast bag is dropped from the balloon at the instant $h = 50$ m, determine the time needed for it to strike the ground. Assume that the bag was released from the balloon with the same velocity as the balloon. Also, find the speed with which bag the strikes the ground?

Projectile Motion in Inclined Plane

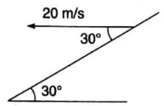
15. Find time of flight and range of the projectile along the inclined plane as shown in figure. ($g = 10$ m/s^2)



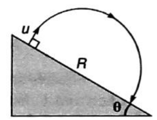
16. Find time of flight and range of the projectile along the inclined plane as shown in figure. ($g = 10$ m/s^2)



17. Find time of flight and range of the projectile along the inclined plane as shown in figure. ($g = 10$ m/s^2)



18. A projectile is fired with a velocity u at right angles to the slope, which is inclined at an angle θ with the horizontal. Derive an expression for the distance R to the point of impact.



Relative Motion in Projectiles

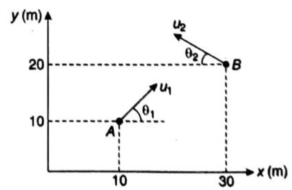
Note *The problems can also be solved without using the concept of relative motion.*

19. A particle is projected upwards with velocity 20 m/s. Simultaneously another particle is projected with velocity $20\sqrt{2}$ m/s at 45°. ($g = 10$ m/s^2)
   (a) What is acceleration of first particle relative to the second?
   (b) What is initial velocity of first particle relative to the other?
   (c) What is distance between two particles after 2 s?

20. Passenger of a train just drops a stone from it. The train was moving with constant velocity. What is path of the stone as observed by
   (a) the passenger itself,
   (b) a man standing on ground?

21. An elevator is going up with an upward acceleration of 1 m/s^2. At the instant when its velocity is 2 m/s, a stone is projected upward from its floor with a speed of 2 m/s relative to the elevator, at an elevation of 30°.
   (a) Calculate the time taken by the stone to return to the floor.
   (b) Sketch the path of the projectile as observed by an observer outdside the elevator.
   (c) If the elevator was moving with a downward acceleration equal to g, how would the motion be altered?

22. Two particles A and B are projected simultaneously in a vertical plane as shown in figure. They collide at time t in air. Write down two necessary equations for collision to take place.



Objective Questions (Level 1)

Single Correct Option

1. A particle has initial velocity, $\vec{v} = 3\hat{i} + 4\hat{j}$ and a constant force $\vec{F} = 4\hat{i} - 3\hat{j}$ acts on it. The path of the particle is
(a) straight line (b) parabolic (c) circular (d) elliptical

2. Identify the correct statement related to the projectile motion.
(a) It is uniformly accelerated everywhere
(b) It is uniformly accelerated everywhere except at the highest position where it is moving with constant velocity
(c) Acceleration is never perpendicular to velocity
(d) None of the above

3. A ball is projected with a velocity 20 ms^{-1} at an angle to the horizontal. In order to have the maximum range. Its velocity at the highest position must be
(a) 10 ms^{-1} (b) 14 ms^{-1} (c) 18 ms^{-1} (d) 16 ms^{-1}

4. Two bodies are thrown with the same initial velocity at angles θ and $(90° - \theta)$ respectively with the horizontal, then their maximum heights are in the ratio
(a) 1 : 1 (b) $\sin\theta : \cos\theta$ (c) $\sin^2\theta : \cos^2\theta$ (d) $\cos\theta : \sin\theta$

5. A gun is firing bullets with velocity v_0 by rotating through 360° in the horizontal plane. The maximum area covered by the bullets is
(a) $\frac{\pi v_0^2}{g}$ (b) $\frac{\pi^2 v_0^2}{g}$ (c) $\frac{\pi v_0^4}{g^2}$ (d) $\frac{\pi^2 v_0^4}{g}$

6. A body is projected at an angle 60° with the horizontal with kinetic energy K. When the velocity makes an angle 30° with the horizontal, the kinetic energy of the body will be
(a) $K/2$ (b) $K/3$ (c) $2K/3$ (d) $3K/4$

7. The range of a projectile at an angle θ is equal to half of the maximum range if thrown at the same speed. The angle of projection θ is given by
(a) 15° (b) 30° (c) 60° (d) data insufficient

8. If T_1 and T_2 are the times of flight for two complementary angles, then the range of projectile R is given by
(a) $R = 4gT_1T_2$ (b) $R = 2gT_1T_2$ (c) $R = \frac{1}{4}gT_1T_2$ (d) $R = \frac{1}{2}gT_1T_2$

9. A grass hopper can jump maximum distance 1.6 m. It spends negligible time on ground. How far can it go in $10\sqrt{2}$ s.
(a) 45 m (b) 30 m (c) 20 m (d) 40 m

10. Average velocity of a particle in projectile motion between its starting point and the highest point of its trajectory is (projection speed = u, angle of projection from horizontal = θ)
(a) $u\cos\theta$ (b) $\frac{u}{2}\sqrt{1+3\cos^2\theta}$ (c) $\frac{u}{2}\sqrt{2+\cos^2\theta}$ (d) $\frac{u}{2}\sqrt{1+\cos^2\theta}$

11. A train is moving on a track at 30 ms^{-1}. A ball is thrown from it perpendicular to the direction of motion at 30 ms^{-1} at 45° from horizontal. Find the distance of ball from the point of projection on train to the point where it strikes the ground.
(a) 90 m (b) $90\sqrt{3}$ m (c) 60 m (d) $60\sqrt{3}$ m

12. A body is projected at time $t = 0$ from a certain point on a planet's surface with a certain velocity at a certain angle with the planet's surface (assumed horizontal). The horizontal and vertical displacements x and y (in metre) respectively vary with time t in second as, $x = (10\sqrt{3})t$ and $y = 10t - t^2$. The maximum height attained by the body is
(a) 200 m (b) 100 m (c) 50 m (d) 25 m

13. A particle is fired horizontally from an inclined plane of inclination 30° with horizontal with speed 50 ms^{-1}. If $g = 10\ ms^{-2}$, the range measured along the incline is
(a) 500 m (b) $\frac{1000}{3}$ m (c) $200\sqrt{2}$ m (d) $100\sqrt{3}$ m

14. Two stones are projected with the same speed but making different angles with the horizontal. Their horizontal ranges are equal. The angle of projection of one is $\frac{\pi}{3}$ and the maximum height reached by it is 102 m. Then the maximum height reached by the other in metres is
(a) 336 (b) 224 (c) 56 (d) 34

15. A ball is projected upwards from the top of a tower with a velocity 50 ms^{-1} making an angle 30° with the horizontal. The height of tower is 70 m. After how many seconds from the instant of throwing, will the ball reach the ground. ($g = 10\ ms^{-2}$)
(a) 2 s (b) 5 s (c) 7 s (d) 9 s

16. A fixed mortar fires a bomb at an angle of 53° above the horizontal with a muzzle velocity of 80 ms^{-1}. A tank is advancing directly towards the mortar on level ground at a constant speed of 5m/s. The initial separation (*at the instant mortar is fired*) between the mortar and tank, so that the tank would be hit is [Take $g = 10\ ms^{-2}$]
(a) 662.4 m (b) 526.3 m (c) 486.6 m (d) None of these

JEE Corner

Assertion and Reason

Directions : *Choose the correct option.*

(a) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*
(b) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*
(c) *If **Assertion** is true, but the **Reason** is false.*
(d) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** A particle can follow a parabolic path only if acceration is constant.
Reason : In projectile motion path is parabolic, as acceleration is assumed to be constant at low heights.

2. **Assertion :** Projectile motion is called a two dimensional motion, although it takes place in space.
Reason : In space it takes place in a plane.

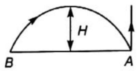
3. **Assertion :** If time of flight in a projectile motion is made two times, its maximum height will become four times.
Reason : In projectile motion $H \propto T^2$, where H is maximum height and T the time of flight.

4. **Assertion :** A particle is projected with velocity $\vec{u}$ at angle 45° with ground. Let $\vec{v}$ be the velocity of particle at time t ($\neq 0$), then value of $\vec{u}\cdot\vec{v}$ can be zero.
Reason : Value of dot product is zero when angle between two vectors is 90°.

5. **Assertion :** A particle has constant acceleration is x-y plane. But neither of its acceleration components (a_x and a_y) is zero. Under this condition particle can not have parabolic path.
Reason : In projectile motion, horizontal component of acceleration is zero.

6. **Assertion :** In projectile motion at any two positions $\dfrac{\vec{v}_2-\vec{v}_1}{t_2-t_1}$ always remains constant.
Reason : The given quantity is average acceleration, which should remain constant as acceleration is constant.

7. **Assertion :** Particle A is projected upwards. Simultaneouly particle B is projected as projectile as shown. Particle A returns to ground is 4 s. At the same time particle B collides with A. Maximum height H attained by B would be 20 m. ($g = 10\,\text{ms}^{-2}$)



Reason : Speed of projection of both the particles should be same under the given condition.

8. **Assertion :** Two projectiles have maximum heights $4H$ and H respectively. The ratio of their horizontal components of velocities should be 1 : 2 for their horizontal ranges to be same.
Reason : Horizontal range = horizontal component of velocity X time of flight.

9. **Assertion :** If $g = 10\,\text{m/s}^2$ then in projectile motion speed of particle in every second will change by $10\,\text{ms}^{-1}$.
Reason : Acceleration is nothing but rate of change of velocity.

10. **Assertion:** In projectile motion if particle is projected with speed u, then speed of particle at height h would be $\sqrt{u^2-2gh}$.
Reason : If particle is projected with vertical component of velocity u_y. Then vertical component at the height h would be $\pm\sqrt{u_y^2-2gh}$.

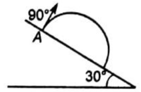
Objective Questions (Level 2)

Single Correct Option

1. Two bodies were thrown simultaneously from the same point, one straight up, and the other, at an angle of $\theta = 30°$ to the horizontal. The initial velocity of each body is $20\,\text{ms}^{-1}$. Neglecting air resistance, the distance between the bodies at $t = 1.2$ later is
(a) 20 m (b) 30 m (c) 24 m (d) 50 m

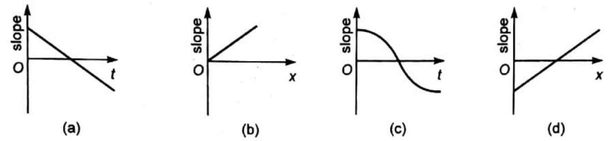
2. A particle is dropped from a height h. Another particle which is initially at a horizontal distance d from the first is simultaneously projected with a horizontal velocity u and the two particles just collide on the ground. Then :
(a) $d^2 = \dfrac{u^2h}{2h}$ (b) $d^2 = \dfrac{2u^2h}{g}$
(c) $d = h$ (d) $gd^2 = u^2h$

3. A ball is projected from point A with velocity $10\,\text{ms}^{-1}$ perpendicular to the inclined plane as shown in figure. Range of the ball on the inclined plane is



(a) $\dfrac{40}{3}$ m (b) $\dfrac{20}{3}$ m
(c) $\dfrac{12}{3}$ m (d) $\dfrac{60}{3}$ m

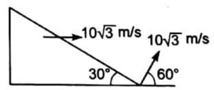
4. A heavy particle is projected with a velocity at an angle with the horizontal into the uniform gravitational field. The slope of the trajectory of the particle varies as



5. A particle starts from the origin of coordinates at time $t = 0$ and moves in the xy plane with a constant acceleration α in the y-direction. Its equation of motion is $y = \beta x^2$. Its velocity component in the x-direction is
(a) variable (b) $\sqrt{\dfrac{2\alpha}{\beta}}$ (c) $\dfrac{\alpha}{2\beta}$ (d) $\sqrt{\dfrac{\alpha}{2\beta}}$

6. A projectile is projected with speed u at an angle of 60° with horizontal from the foot of an inclined plane. If the projectile hits the inclined plane horizontally, the range on inclined plane will be
(a) $\dfrac{u^2\sqrt{21}}{2g}$ (b) $\dfrac{3u^2}{4g}$ (c) $\dfrac{u^2}{2\beta}$ (d) $\dfrac{\sqrt{21}\,u^2}{8g}$

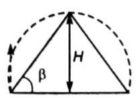
7. A particle is projected at an angle 60° with speed $10\sqrt{3}$ m/s, from the point A, as shown in the figure. At the same time the wedge is made to move with speed $10\sqrt{3}$ m/s towards right as shown in the figure. Then the time after which particle will strike with wedge is



(a) 2 s (b) $2\sqrt{3}$ s (c) $\dfrac{4}{\sqrt{3}}$ s (d) None of these

8. A particle moves along the parabolic path $x = y^2 + 2y + 2$ in such a way that Y-component of velocity vector remains $5\,\text{ms}^{-1}$ during the motion. The magnitude of the acceleration of the particle is
(a) $50\,\text{ms}^{-2}$ (b) $100\,\text{ms}^{-2}$
(c) $10\sqrt{2}\,\text{ms}^{-2}$ (d) $0.1\,\text{ms}^{-2}$

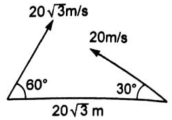
9. A shell fired from the base of a mountain just clears it. If α is the angle of projection, then the angular elevation of the summit β is



(a) $\dfrac{\alpha}{2}$ (b) $\tan^{-1}\left(\dfrac{1}{2}\right)$
(c) $\tan^{-1}\left(\dfrac{\tan\alpha}{2}\right)$ (d) $\tan^{-1}(2\tan\alpha)$

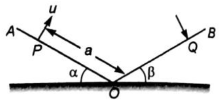
10. In the figure shown, the two projectiles are fired simultaneously. The minimum distance between them during their flight is

(a) 20 m
(b) $10\sqrt{3}$ m
(c) 10 m
(d) None of these



Passage (Q. No. 11-12)

Two inclined planes OA and OB intersect in a horizontal plane having their inclinations α and β with the horizontal as shown in figure. A particle is projected from point P with velocity u along a direction perpendicular to plane OA. The particle strikes plane OB perpendicularly at Q.



11. If $\alpha = 30°, \beta = 30°$, the time of flight from P to Q is

(a) $\frac{u}{g}$
(b) $\frac{\sqrt{3}u}{g}$
(c) $\frac{\sqrt{2}u}{g}$
(d) $\frac{2u}{g}$

12. If $\alpha = 30°, \beta = 30°$ and $a = 4.9$ m, the initial velocity of projection is

(a) 9.8 ms^{-1}
(b) 4.9 ms^{-1}
(c) $4.9\sqrt{2}$ ms^{-1}
(d) 19.6 ms^{-1}

More than One Correct Options

1. Two particles projected from the same point with same speed u at angles of projection α and β strike the horizontal ground at the same point. If h_1 and h_2 are the maximum heights attained by the projectile, R is the range for both and t_1 and t_2 are their times of flights, respectively, then

(a) $\alpha + \beta = \frac{\pi}{2}$
(b) $R = 4\sqrt{h_1 h_2}$
(c) $\frac{t_1}{t_2} = \tan\alpha$
(d) $\tan\alpha = \sqrt{\frac{h_1}{h_2}}$

2. A ball is dropped from a height of 49 m. The wind is blowing horizontally. Due to wind a constant horizontal acceleration is provided to the ball. Choose the correct statement (s). [Take $g = 9.8\ ms^{-2}$]

(a) Path of the ball is a straight line
(b) Path of the ball is a curved one
(c) The time taken by the ball to reach the ground is 3.16 s
(d) Actual distance travelled by the ball is more then 49 m

3. A particle is projected from a point P with a velocity v at an angle θ with horizontal. At a certain point Q it moves at right angles to its initial direction. Then

(a) velocity of particle at Q is $v\sin\theta$
(b) velocity of particle at Q is $v\cot\theta$
(c) time of flight from P to Q is $(v/g)\operatorname{cosec}\theta$
(d) time of flight from P to Q is $(v/g)\sec\theta$

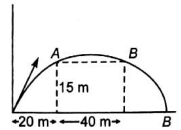
4. At a height of 15 m from ground velocity of a projectile is $\vec{v} = (10\hat{i} + 10\hat{j})$. Here, $\hat{j}$ is vertically upwards and $\hat{i}$ is along horizontal direction then $(g = 10\ ms^{-2})$

(a) particle was projected at an angle of 45° with horizontal
(b) time of flight of projectile is 4 s
(c) horizontal range of projectile is 100 m
(d) maximum height of projectile from ground is 20 m

5. Which of the following quantities remain constant during projectile motion?

(a) Average velocity between two points
(b) Average speed between two points
(c) $\frac{d\vec{v}}{dt}$
(d) $\frac{d^2\vec{v}}{dt^2}$

6. In the projectile motion shown is figure, given $t_{AB} = 2$ s then $(g = 10\ ms^{-2})$



(a) particle is at point B at 3 s
(b) maximum height of projectile is 20 m
(c) initial vertical component of velocity is 20 ms^{-1}
(d) horizontal component of velocity is 20 ms^{-1}

Match the Columns

1. Particle-1 is just dropped from a tower. 1 s later particle-2 is thrown from the same tower horizontally with velocity 10 ms^{-1}. Taking $g = 10\ ms^{-2}$, match the following two columns at $t = 2$ s.

Column I	Column II
(a) Horizontal displacement between two	(p) 10 SI units
(b) Vertical displacement between two	(q) 20 SI units
(c) Magnitude of relative horizontal component of velocity	(r) $10\sqrt{2}$ SI units
(d) Magnitude of relative vertical component of velocity	(s) None

2. In a projectile motion, given $H = \frac{R}{2} = 20$ m. Here, H is maximum height and R the horizontal range. For the given condition match the following two columns.

Column I	Column II
(a) Time of flight	(p) 1
(b) Ratio of vertical component of velocity and horizontal component of velocity	(q) 2
(c) Horizontal component of velocity	(r) 10
(d) Vertical component of velocity	(s) None

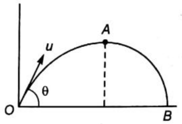
3. A particle can be thrown at a constant speed at different angles. When it is thrown at 15° with horizontal, it falls at a distance of 10 m from point of projection. For this speed of particle match following two columns.

Column I	Column II
(a) Maximum horizontal range which can be taken with this speed	(p) 10 m
(b) Maximum height	(q) 20 m
(c) Range at 75°	(r) 15 m
(d) Height at 30°	(s) None

4. In projectile motion if vertical component of velocity is increased to two times, keeping horizontal component unchanged, then

Column I	Column II
(a) Time of flight	(p) will remain same
(b) Maximum height	(q) will become two times
(c) Horizontal range	(r) will become four times
(d) Angle of projection with horizontal	(s) None

5. In projectile motion shown in figure.

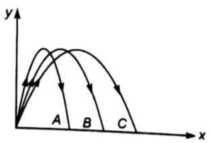


Column I	Column II
(a) Change in velocity between O and A	(p) $u\cos\theta$
(b) Average velocity between O and A	(q) $u\sin\theta$
(c) Change in velocity between O and B	(r) $2u\cos\theta$
(d) Average velocity between O and B	(s) None

6. Particle-1 is projected from ground (take it origin) at time $t = 0$, with velocity $(30\hat{\mathbf{i}} + 30\hat{\mathbf{j}})\,\text{ms}^{-1}$. Particle-2 is projected from (130 m, 75 m) at time $t = 1$ second with velocity $(-20\hat{\mathbf{i}} + 20\hat{\mathbf{j}})\,\text{ms}^{-1}$. Assuming $\hat{\mathbf{j}}$ to be vertically upward and $\hat{\mathbf{i}}$ to be in horizontal direction, match the following two columns at $t = 2\,\text{s}$.

Column I	Column II
(a) horizontal distance between two	(p) 30 SI units
(b) vertical distance between two	(q) 40 SI units
(c) relative horizontal component of velocity between two	(r) 50 SI units
(d) relative vertical component of velocity between two	(s) None

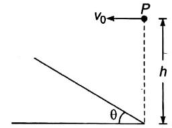
7. The trajectories of the motion of 3 particles are shown in the figure. Match the entries of column I with the entries of column II.



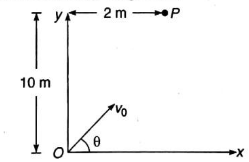
Column I	Column II
(a) Time of flight is least for	(p) A
(b) Vertical component of velocity is greatest for	(q) B
(c) Horizontal component of velocity is greatest for	(r) C
(d) Launch speed is least for	(s) can't say

Subjective Questions (Level 2)

1. Determine the horizontal velocity v_0 with which a stone must be projected horizontally from a point P, so that it may hit the inclined plane perpendicularly. The inclination of the plane with the horizontal is θ and point P is at a height h above the foot of the incline, as shown in the figure.



2. A particle is dropped from point P at time $t = 0$. At the same time another particle is thrown from point O as shown in the figure and it collides with the particle P. Acceleration due to gravity is along the negative y-axis. If the two particles collide 2 s after they start, find the initial velocity v_0 of the particle which was projected from O. Point O is not necessarily on ground.



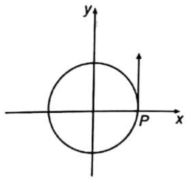
3. Two particles are simultaneously projected in the same vertical plane from the same point with velocities u and v at angles α and β with horizontal. Find the time that elapses when their velocities are parallel.

4. A projectile takes off with an initial velocity of 10 m/s at an angle of elevation of 45°. It is just able to clear two hurdles of height 2 m each, separated from each other by a distance d. Calculate d. At what distance from the point of projection is the first hurdle placed? Take $g = 10\,\text{m/s}^2$.

5. A stone is projected from the ground in such a direction so as to hit a bird on the top of a telegraph post of height h and attains the maximum height of $2h$ above the ground. If at the instant of projection, the bird were to fly away horizontally with a uniform speed, find the ratio between the horizontal velocity of bird and the horizontal component of velocity of stone, if the stone hits the bird while descending.

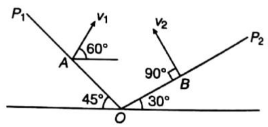
6. A particle is released from a certain height $H = 400$m. Due to the wind the particle gathers the horizontal velocity component $v_x = ay$ where $a = \sqrt{5}\ s^{-1}$ and y is the vertical displacement of the partricle from the point of release, then find :
(a) the horizontal drift of the particle when it stikes the ground,
(b) the speed with which particle strikes the ground.
(Take $g = 10\ m/s^2$)

7. A train is moving with a constant speed of 10 m/s in a circle of radius $\frac{16}{\pi}$ m. The plane of the circle lies in horizontal x-y plane. At time $t = 0$ train is at point P and moving in counter-clockwise direction. At this instant a stone is thrown from the train with speed 10 m/s relative to train towards negative x-axis at an angle of 37° with vertical z-axis. Find

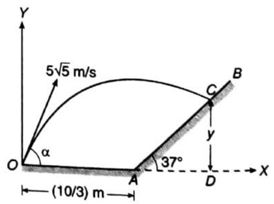


(a) the velocity of particle relative to train at the highest point of its trajectory.
(b) the co-ordinates of points on the ground where it finally falls and that of the highest point of its trajectory.
Take $g = 10\ m/s^2$, $\sin 37° = \frac{3}{5}$

8. A particle is projected from an inclined plane OP_1 from A with velocity $v_1 = 8\ ms^{-1}$ at an angle 60° with horizontal. An another particle is projected at the same instant from B with velocity $v_2 = 16\ ms^{-1}$ and perpendicular to the plane OP_2 as shown in figure. After time $10\sqrt{3}$ sec there separation was minimum and found to be 70 m. Then find distance AB.



9. A particle is projected from point O on the ground with velocity $u = 5\sqrt{5}$ m/s at angle $\alpha = \tan^{-1}(0.5)$. It strikes at a point C on a fixed smooth plane AB having inclination of 37° with horizontal as shown in figure. If the particle does not rebound, calculate
(a) coordinates of point C in reference to coordinate system as shown in the figure.
(b) maximum height from the ground to which the particle rises. ($g = 10\ m/s^2$).



10. A plank fitted with a gun is moving on a horizontal surface with speed of 4 m/s along the positive x-axis. The z-axis is in vertically upward direction. The mass of the plank including the mass of the gun is 50 kg. When the plank reaches the origin, a shell of mass 10 kg is fired at an angle of 60° with the positive x-axis with a speed of $v = 20$ m/s with respect to the gun in x-z plane. Find the position vector of the shell at $t = 2$ s after firing it. Take $g = 9.8\ m/s^2$.

ANSWERS

Introductory Exercise 4.1

1. False 2. True 3. (a) $T = \frac{2u\cos\theta}{g}$ (b) $H = \frac{u^2\cos^2\theta}{2g}$ (c) $R = \frac{u^2\sin 2\theta}{g}$ (d) $R_{max} = \frac{u^2}{g}$ at $\theta = 45°$
4. $20\sqrt{5}$ m/s at angle $\tan^{-1}\left(\frac{1}{2}\right)$ with horizontal (b) 100 m 5. 1 s and 3 s
6. $20\sqrt{2}$ m/s, $2(\sqrt{3} \pm 1)$ s 7. $u\cos\alpha$ 8. $2u\sin\alpha$ (downwards)
9. Between two points lying on the same horizontal line. 10. time = 0, 0.8 s, x-coordinate = 0, 2.4 m
11. $\sqrt{3}$ s

Introductory Exercise 4.2

1. $\frac{10}{\sqrt{3}}$ m/s 2. 5 m/s 3. 20 m 4. No 5. 60° 6. $\frac{40}{\sqrt{3}}$ m/s 7. (a) 30° (b) $5\sqrt{3}$ m

AIEEE Corner

Subjective Questions (Level 1)

1. (a) 80 m, 20 m, 4 s (b) $(20\hat{i} + 10\hat{j})\ ms^{-1}$ (c) $(20\hat{i} - 20\hat{j})\ ms^{-1}$
2. 5.46 s, 2.83 s, 1.46 s, 109.2 m, 56.6 m, 29.2 m
3. (a) 30 ms^{-1} (vertically downwards) (b) 20.62 ms^{-1} 4. $\frac{5}{\sqrt{2}}\ ms^{-1}$ 5. $t_1 = 2.19$ s, $t_2 = 8.20$ s
6. $v = \sqrt{\frac{a}{2c}(1 + b^2)}$ 7. (a) 11 m (b) 23 m (c) 16.6 ms^{-1} (d) $\tan^{-1}(2)$, below horizontal
8. 2.5 m 9. $\tan^{-1}\left(\frac{2}{3}\right)$ 11. $u = \frac{gt\cos\beta}{2\sin(\alpha - \beta)}$ 13. (a) 2 s (b) 19.6 m (c) 15 m
14. 3.55 s, 32.0 m/s 15. $T = \frac{4(\sqrt{3} + 1)}{\sqrt{3}}$ s = 1.69 s, $R = \frac{160}{3}(\sqrt{3} - 1)$ m = 39 m
16. $T = \frac{4(\sqrt{3} + 1)}{\sqrt{3}}$ s = 6.31 s, $\frac{160}{3}(\sqrt{3} + 1)$ m = 145.71 m 17. $T = \frac{4}{\sqrt{3}}$ s = 2.31 s, $R = \frac{160}{3}$ m = 53.33 m
18. $R = \frac{2u^2}{g}\tan\theta\sec\theta$ 19. (a) zero (b) 20 ms^{-1} in horizontal direction (c) 40 m
20. (a) A vertical straight line (b) A parabola 21. (a) 0.18 s
22. $(u_1\cos\theta_1 + u_2\cos\theta_2)\ t = 20$ ...(i) $(u_1\sin\theta_1 - u_2\sin\theta_2)\ t = 10$ ...(ii)

Objective Questions (Level 1)

1.(b)	2.(a)	3.(b)	4.(c)	5.(c)	6.(b)	7.(a)	8.(d)	9.(d)	10.(b)
11.(a)	12.(d)	13.(b)	14.(d)	15.(c)	16.(d)				

JEE Corner

Assertion and Reason

1.(d)	2.(a)	3.(a)	4.(b)	5.(d)	6.(a)	7.(c)	8.(a or b)	9.(d)	10.(b)

Objective Questions (Level 2)

1.(c) 2.(b) 3.(a) 4.(a) 5.(d) 6.(d) 7.(a) 8.(a) 9.(c) 10.(b)
11.(b) 12.(a)

More than One Correct Options

1. (all) 2. (a,c,d) 3. (b,c) 4. (a,b,d) 5. (c,d) 6. (all)

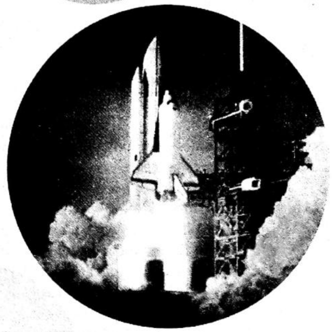
Match the Columns

1. (a) → (p) (b) → (s) (c) → (p) (d) → (p)
2. (a) → (s) (b) → (q) (c) → (r) (d) → (s)
3. (a) → (q) (b) → (p) (c) → (p) (d) → (s)
4. (a) → (q) (b) → (r) (c) → (q) (d) → (s)
5. (a) → (q) (b) → (s) (c) → (s) (d) → (p)
6. (a) → (r) (b) → (r) (c) → (r) (d) → (s)
7. (a) → (s) (b) → (s) (c) → (r) (d) → (p)

Subjective Questions (Level 2)

1. $V_0 = \sqrt{\dfrac{2gh}{2 + \cot^2 \theta}}$ 2. $\sqrt{26}$ ms^{-1} at angle $\theta = \tan^{-1}(5)$ with x-axis 3. $t = \dfrac{uv \sin(\alpha - \beta)}{g(v \cos \beta - u \cos \alpha)}$
4. 4.47 m, 2.75 m 5. $\dfrac{2}{\sqrt{2} + 1}$ 6. (a) 2.67 km (b) 0.9 kms^{-1}
7. (a) $(-6\hat{i} + 10\hat{j})$ ms^{-1} (b) (–4.5 m, 16 m, 0), (0.3 m, 8.0 m, 3.2 m) 8. (a) 250 m
9. (a) (5 m, 1.25 m) (b) 4.45 m 10. $[24\hat{i} + 15\hat{k}]$ m.

Chapter – 5 Laws of Motion



5

Laws of Motion

Chapter Contents

5.1 Types of Forces
5.2 Free Body Diagram
5.3 Equilibrium
5.4 Newton's Laws of Motion
5.5 Pulleys
5.6 Constraint Equations
5.7 Pseudo Force
5.8 Friction

5.1 Types of Forces

There are basically three forces which are commonly encountered in mechanics.

(a) Field Forces

These are the forces in which contact between two objects is not necessary. Gravitational force between two bodies and electrostatic force between two charges are two examples of field forces. Weight ($W = mg$) of a body comes in this category.

(b) Contact Forces

Two bodies in contact exert equal and opposite forces on each other. If the contact is frictionless the contact force is perpendicular to the common surface and known as **normal reaction.**

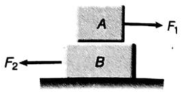


Fig. 5.1

If, however, the objects are in rough contact and move (or have a tendency to move) relative to each other without losing contact then **frictional force** arise which oppose such motion. Again each object exerts a frictional force on the other and the two forces are equal and opposite. This force is perpendicular to normal reaction. Thus, the contact force (F) between two objects is made up of two forces.

(i) Normal reaction (N) (ii) Force of friction (f)

and since these two forces are mutually perpendicular.

$$F = \sqrt{N^2 + f^2}$$

Consider two wooden blocks A and B being rubbed against each other.

In Fig. 5.1, A is being moved to the right while B is being moved leftward. In order to see more clearly which forces act on A and which on B, a second diagram is drawn showing a space between the blocks but they are still supposed to be in contact.

In Fig. 5.2 the two normal reactions each of magnitude N are perpendicular to the surface of contact between the blocks and the two frictional forces each of magnitude f act along that surface, each in a direction opposing the motion of the block upon which it acts.

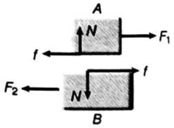


Fig. 5.2

Note *Forces on block B from the ground are not shown in the figure.*

(c) Attachment to Another Body

Tension (T) in a string and **spring force** ($F = kx$) come in this group. Regarding the tension and string, the following three points are important to remember.

1. If a string is inextensible the magnitude of acceleration of any number of masses connected through string is always same.

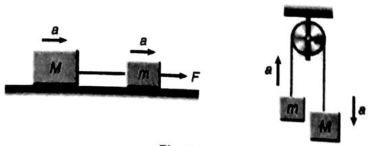


Fig. 5.3

2. If a string is massless, the tension in it is same everywhere. However, if a string has a mass, tension at different points will be different.
3. If there is friction between string and pulley, tension is different on two sides of the pulley, but if there is no friction between pulley and string, tension will be same on both sides of the pulley.

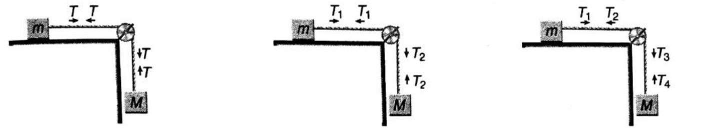


Fig. 5.4

Last two points can be understood in diagram as follows:

Spring force ($F = kx$) has been discussed in detail in the chapter of work, energy and power.

5.2 Free Body Diagram

No system, natural or man made, consists of a single body alone or is complete in itself. A single body or a part of the system can, however be isolated from the rest by appropriately accounting for its effect on the remaining system.

A free body diagram (FBD) consists of a diagrammatic representation of a single body or a sub-system of bodies isolated from its surroundings showing all the forces acting on it.

Consider, for example, a book lying on a horizontal surface.

A free body diagram of the book alone would consist of its weight ($W = mg$), acting through the centre of gravity and the reaction (N) exerted on the book by the surface.

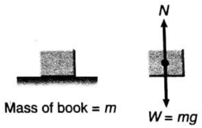


Fig. 5.5

Sample Example 5.1 *A cylinder of weight w is resting on a V-groove as shown in figure. Draw its free body diagram.*

N N

mg


Fig. 5.6(a)

Solution The free body diagram of the cylinder is as shown in Fig. 5.6(b)

Here, w = weight of cylinder and N_1 and N_2 are the normal reactions between the cylinder and the two inclined walls.

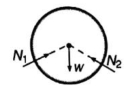


Fig. 5.6(b)

Sample Example 5.2 *Three blocks A, B and C are placed one over the other as shown in figure. Draw free body diagrams of all the three blocks.*

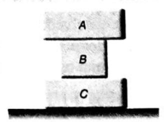


Fig. 5.7

Solution Free body diagrams of A, B and C are shown below:

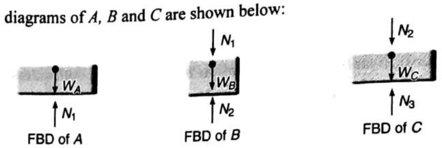


Fig. 5.8

Here, N_1 = normal reaction between A and B.

N_2 = normal reaction between B and C.

N_3 = normal reaction between C and ground.

Sample Example 5.3 *A block of mass m is attached with two strings as shown in figure. Draw the free body diagram of the block.*

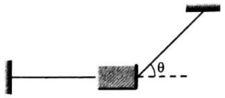


Fig. 5.9

Solution The free body diagram of the block is as shown in Fig. 5.10.

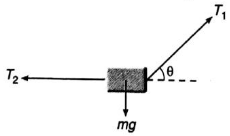


Fig. 5.10

5.3 ~~Equilibrium~~

Forces which have zero linear resultant and zero turning effect will not cause any change in the motion of the object to which they are applied. Such forces (and the object) are said to be in equilibrium. For understanding the equilibrium of an object under two or more concurrent or coplanar forces let us first discuss the resolution of force and moment of a force about some point.

(a) Resolution of a Force

When a force is replaced by an equivalent set of components, it is said to be resolved. One of the most useful ways in which to resolve a force is to choose only two components (although a force may be resolved in three or more components also) which are at right angles also. The magnitude of these components can be very easily found using trigonometry.

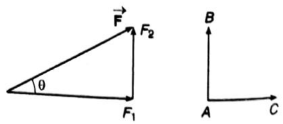


Fig. 5.11

In Fig. 5.11,

$$F_1 = F\cos\theta = \text{component of } \vec{F} \text{ along } AC$$

$$F_2 = F\sin\theta = \text{component of } \vec{F} \text{ perpendicular to } AC \text{ or along } AB$$

Finding such components is referred to as resolving a force in a pair of perpendicular directions. Note that the component of a force in a direction perpendicular to itself is zero. For example, if a force of 10 N is applied on an object in horizontal direction then its component along vertical is zero. Similarly, the component of a force in a direction parallel to the force is equal to the magnitude of the force. For example component of the above force in the direction of force (horizontal) will be 10 N.

Sample Example 5.4 *Resolve a weight of 10 N in two directions which are parallel and perpendicular to a slope inclined at 30° to the horizontal.*

Solution Component perpendicular to the plane

$$W_\perp = W\cos 30° = (10)\frac{\sqrt{3}}{2} = 5\sqrt{3}\text{ N}$$

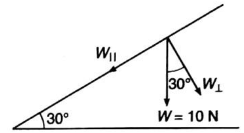


Fig. 5.12

and component parallel to the plane

$$W_{||} = W\sin 30° = (10)\left(\frac{1}{2}\right) = 5\text{ N}$$

Sample Example 5.5 *Resolve horizontally and vertically a force $F = 8$ N which makes an angle of 45° with the horizontal.*

Solution Horizontal component of $\vec{F}$ is

$$F_H = F\cos 45° = (8)\left(\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}\text{ N}$$

and vertical component of $\vec{F}$ is $F_V = F\sin 45°$

$$= (8)\left(\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}\text{ N}$$

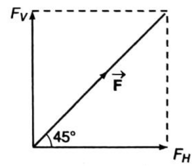


Fig. 5.13

Sample Example 5.6 *A body is supported on a rough plane inclined at 30° to the horizontal by a string attached to the body and held at an angle of 30° to the plane. Draw a diagram showing the forces acting on the body and resolve each of these forces:*

(a) horizontally and vertically,

(b) parallel and perpendicular to the plane.

Solution The forces are :

The tension in the string T.

The normal reaction with the plane N.

The weight of the body W

and the friction f.

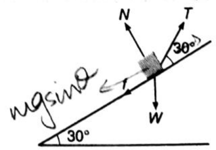


Fig. 5.14

(a) Resolving horizontally and vertically:

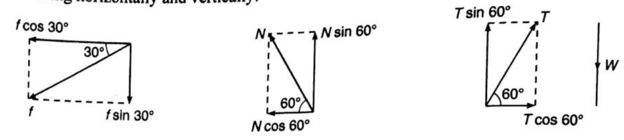


Fig. 5.15

(b) Resolving parallel and perpendicular to the plane:

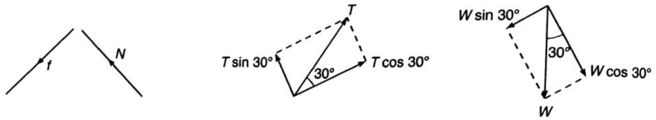


Fig. 5.16

Resolving horizontally and vertically in the senses OX and OY as shown, the components are:

Force	Components	
	Parallel to OX (horizontal)	Parallel to OY (vertical)
f	$-f\cos 30°$	$-f\sin 30°$
N	$-N\cos 60°$	$N\sin 60°$
T	$T\cos 60°$	$T\sin 60°$
W	0	$-W$

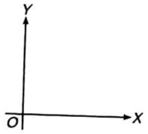


Fig. 5.17

Resolving parallel and perpendicular to the plane in the senses OX' and OY' as shown, the components are :

Force	Components	
	Parallel to OX' (parallel to plane)	Parallel to OY' (perpendicular to plane)
f	$-f$	0
N	0	N
T	$T\cos 30°$	$T\sin 30°$
W	$-W\sin 30°$	$-W\cos 30°$

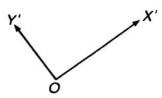


Fig. 5.18

(b) Moment of a Force

The general name given to any turning effect is **torque.** The magnitude of torque, also known as the moment of a force F is calculated by multiplying together the magnitude of the force and its perpendicular distance $r_\perp$ from the axis of rotation. This is denoted by C or τ (tau).

i.e., $$C = Fr_\perp \quad \text{or} \quad \tau = Fr_\perp$$

Direction of Torque

The angular direction of a torque is the sense of the rotation it would cause.

Consider a lamina that is free to rotate in its own plane about an axis perpendicular to the lamina and passing through a point A on the lamina. In the diagram the moment about the axis of rotation of the force F_1 is F_1r_1 anticlock-wise and the moment of the force F_2 is F_2r_2 clockwise. A convenient way to differentiate between clockwise and anticlock-wise torques is to allocate a positive sign to one sense (usually, but not invariably, this is anticlockwise) and negative sign to the other.

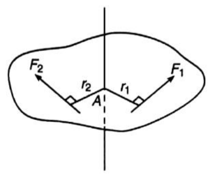


Fig. 5.19

With this convention, the moments of F_1 and F_2 are $+F_1r_1$ and $-F_2r_2$ (when using a sign convention in any problem it is advisable to specify the chosen positive sense).

Zero moment → doubt

If the line of action of a force passes through the axis of rotation, its perpendicular distance from the axis is zero. Therefore, its moment about that axis is also zero.

Note *Later in the chapter of rotation we will see that torque is a vector quantity.*

Sample Examle 5.7 *ABCD is a square of side* 2 m *and O is its centre. Forces act along the sides as shown in the diagram. Calculate the moment of each force about:*

(a) an axis through A and perpendicular to the plane of square.
(b) an axis through O and perpendicular to the plane of square.

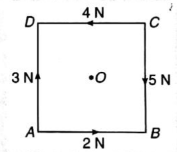


Fig. 5.20

Solution Taking anticlockwise moments as positive we have:

(a)	**Magnitude of force**	**2 N**	**5 N**	**4 N**	**3 N**
	Perpendicular distance from A	0	2 m	2 m	0
	Moment about A	0	−10 N-m	+8 N-m	0
(b)	**Magnitude of force**	**2 N**	**5 N**	**4 N**	**3 N**
	Perpendicular distance from O	1 m	1 m	1 m	1 m
	Moment about O	+2 N-m	−5 N-m	+4 N-m	−3 N-m

Sample Example 5.8 *Forces act as indicated on a rod AB which is pivoted at A. Find the anticlockwise moment of each force about the pivot.*

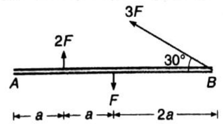


Fig. 5.21

Solution

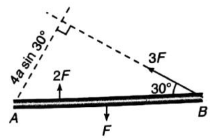


Fig. 5.22

Magnitude of force	2F	F	3F
Perpendicular distance from A	a	$2a$	$4a \sin 30° = 2a$
Anticlockwise moment about A	$+2\,Fa$	$-2\,Fa$	$+6\,Fa$

Coplanar Forces in Equilibrium

When an object is in equilibrium under the action of a set of two or more coplanar forces, each of three factors which comprise the possible movement of the object must be zero, *i.e.*, the object has

(i) no linear movement along any two mutually perpendicular directions ox and oy.
(ii) no rotation about any axis.

The set of forces must, therefore, be such that

(a) the algebraic sum of the components parallel to ox is zero or $\Sigma F_x = 0$
(b) the algebraic sum of the components parallel to oy is zero or $\Sigma F_y = 0$
(c) the resultant moment about any specified axis is zero or $\Sigma \tau_{\text{any axis}} = 0$

Thus, for the equilibrium of a set of two or more coplanar forces:

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0 \quad \text{and} \quad \Sigma \tau_{\text{any axis}} = 0$$

Using the above three conditions, we get only three set of equations. So, in a problem number of unknowns should not be more than three.

Sample Example 5.9 *A rod AB rests with the end A on rough horizontal ground and the end B against a smooth vertical wall. The rod is uniform and of weight W. If the rod is in equilibrium in the position shown in figure. Find :*

(a) frictional force at A
(b) normal reaction at A
(c) normal reaction at B.

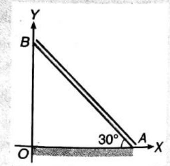


Fig. 5.23

Solution Let length of the rod be $2l$. Using the three conditions of equilibrium. Anticlockwise moment is taken as positive.

(i) $\Sigma F_x = 0$

$\therefore \quad N_B - f_A = 0$

or $\quad N_B = f_A \quad$ ...(i)

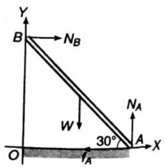


Fig. 5.24

(ii) $\Sigma F_y = 0$

$\therefore \quad N_A - W = 0$

or $\quad N_A = W \quad$ ...(ii)

(iii) $\Sigma \tau_o = 0$

$\therefore \quad N_A\,(2l \cos 30°) - N_B\,(2l \sin 30°) - W(l \cos 30°) = 0$

or $\quad \sqrt{3} N_A - N_B - \dfrac{\sqrt{3}}{2} W = 0 \quad$ ...(iii)

Solving these three equations, we get

(a) $f_A = \dfrac{\sqrt{3}}{2} W$ (b) $N_A = W$ (c) $N_B = \dfrac{\sqrt{3}}{2} W$

Equilibrium of Concurrent Coplanar Forces

If an object is in equilibrium under two or more concurrent coplanar forces the algebraic sum of the components of forces in any two mutually perpendicular directions ox and oy should be zero, *i.e.*, the set of forces must be such that:

(a) the algebraic sum of the components parallel to ox is zero, *i.e.*, $\Sigma F_x = 0$.
(b) the algebraic sum of the components parallel to oy is zero, *i.e.*, $\Sigma F_y = 0$.

Thus, for the equilibrium of two or more concurrent coplanar forces:

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$

The third condition of zero moment about any specified axis is automatically satisfied if the moment is taken about the point of intersection of the forces. So, here we get only two equations. Thus, number of unknown in any problem should not be more than two.

Sample Example 5.10 *An object is in equilibrium under four concurrent forces in the directions shown in figure. Find the magnitude of* $\vec{F}_1$ *and* $\vec{F}_2$.

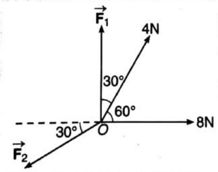


Fig. 5.25

Solution The object is in equilibrium. Hence,

(i) $\Sigma F_x = 0$

$\therefore \quad 8 + 4\cos 60° - F_2 \cos 30° = 0$

or $\quad 8 + 2 - F_2 \dfrac{\sqrt{3}}{2} = 0$

or $\quad F_2 = \dfrac{20}{\sqrt{3}}\,\text{N}$

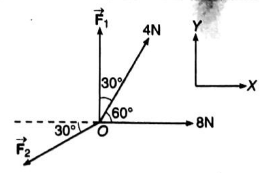


Fig. 5.26

(ii) $\Sigma F_y = 0$

$\therefore$ $$F_1 + 4\sin 60° - F_2 \sin 30° = 0$$

or $$F_1 + \frac{4\sqrt{3}}{2} - \frac{F_2}{2} = 0$$

or $$F_1 = \frac{F_2}{2} - 2\sqrt{3} = \frac{10}{\sqrt{3}} - 2\sqrt{3}$$

or $$F_1 = \frac{4}{\sqrt{3}}\ \text{N}$$

Lami's Theorem : If an object O is in equilibrium under three concurrent forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ as shown in figure. Then,

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}$$

This property of three concurrent forces in equilibrium is known as Lami's theorem and is very useful method of solving problems related to three concurrent forces in equilibrium.

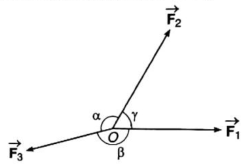


Fig. 5.27

Sample Example 5.11 *One end of a string 0.5 m long is fixed to a point A and the other end is fastened to a small object of weight 8 N. The object is pulled aside by a horizontal force F, until it is 0.3 m from the vertical through A. Find the magnitudes of the tension T in the string and the force F.*

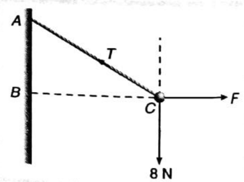


Fig. 5.28

Solution $AC = 0.5\,\text{m}$, $BC = 0.3\,\text{m}$

$\therefore$ $$AB = 0.4\ \text{m}$$

and if $$\angle BAC = \theta.$$

Then $$\cos\theta = \frac{AB}{AC} = \frac{0.4}{0.5} = \frac{4}{5}$$

and $$\sin\theta = \frac{BC}{AC} = \frac{0.3}{0.5} = \frac{3}{5}$$

Here, the object is in equilibrium under three concurrent forces. So, we can apply Lami's theorem.

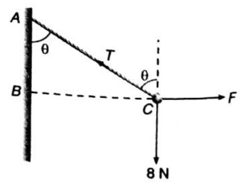


Fig. 5.29

or $$\frac{F}{\sin(180° - \theta)} = \frac{8}{\sin(90° + \theta)} = \frac{T}{\sin 90°}$$

or $$\frac{F}{\sin\theta} = \frac{8}{\cos\theta} = T$$

$\therefore$ $$T = \frac{8}{\cos\theta} = \frac{8}{4/5} = 10\,\text{N}$$

and $$F = \frac{8\sin\theta}{\cos\theta} = \frac{(8)(3/5)}{(4/5)} = 6\,\text{N}$$ **Ans.**

Introductory Exercise 5.1

1. The diagram shows a rough plank resting on a cylinder with one end of the plank on rough ground. Neglect friction between plank and cylinder. Draw diagrams to show:
   (a) the forces acting on the plank,
   (b) the forces acting on the cylinder.

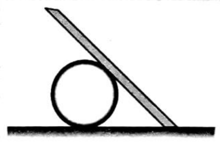


Fig. 5.30

2. Two spheres A and B are placed between two vertical walls as shown in figure. Friction is absent everywhere. Draw the free body diagrams of both the spheres.

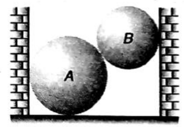


Fig. 5.31

3. A point A on a sphere of weight W rests in contact with a smooth vertical wall and is supported by a string joining a point B on the sphere to a point C on the wall. Draw free body diagram of the sphere.

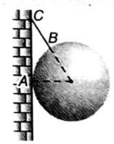


Fig. 5.32

4. Write down the components of four forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$ and $\vec{F}_4$ along ox and oy directions as shown in Fig. 5.33.

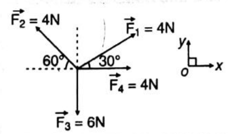


Fig. 5.33

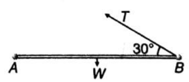


Fig. 5.34

5. A uniform rod AB of weight W is hinged to a fixed point at A. It is held in the horizontal position by a string, one end of which is attached to B as shown in Fig. 5.34. Find in terms of W, the tension in the string.
6. In Question 3 of the same exercise the radius of the sphere is a. The length of the string is also a. Find tension in the string.
7. Find the values of the unknown forces if the given set of forces shown in Fig. 5.35 are in equilibrium.

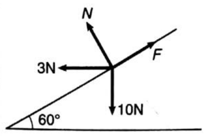


Fig. 5.35

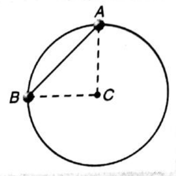


Fig. 5.36

8. Two beads of equal masses m are attached by a string of length $\sqrt{2}a$ and are free to move in a smooth circular ring lying in a vertical plane as shown in Fig. 5.36. Here, a is the radius of the ring. Find the tension and acceleration of B just after the beads are released to move.

5.4 Newton's Laws of Motion

It is interesting to read Newton's original version of the laws of motion.

Law I. Every body continues in its state of rest or in uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

Law II. The change of motion is proportional to the magnitude of force impressed; and is made in the direction of the straight line in which that force is impressed.

Law III. To every action there is always an equal and opposite reaction; or, the mutual actions of two bodies upon each other are always directed to contrary parts.

The modern version of these laws is:

1. A body continues in its initial state of rest or motion with uniform velocity unless acted on by an unbalanced external force.
2. The acceleration of a body is inversely proportional of its mass and directly proportional to the resultant external force acting on it, *i.e.*,

$$\Sigma \vec{F} = \vec{F}_{net} = m\vec{a} \quad \text{or} \quad \vec{a} = \frac{\vec{F}_{net}}{m}$$

3. Forces always occur in pairs. If body A exerts a force on body B, an equal but opposite force is exerted by body B on body A.

Working with Newton's First and Second Laws

Normally any problem relating to Newton's laws is solved in following four steps:

1. First of all we decide the system on which the laws of motion are to be applied. The system may be a single particle, a block or a combination of two or more blocks, two blocks connected by a string, etc. The only restriction is that all parts of the system should have the same acceleration.
2. Once the system is decided, we make the list of all the forces acting on the system. Any force applied by the system on other bodies is not included in the list of the forces.
3. Then we make a free body diagram of the system and indicate the magnitude and directions of all the forces listed in step 2 in this diagram.

4. In the last step we choose any two mutually perpendicular axes say x and y in the plane of the forces in case of coplanar forces. Choose the x-axis along the direction in which the system is known to have or is likely to have the acceleration. A direction perpendicular to it may be chosen as the y-axis. If the system is in equilibrium, any mutually perpendicular directions may be chosen. Write the components of all the forces along the x-axis and equate their sum to the product of the mass of the system and its acceleration, *i.e.*,

$$\Sigma F_x = ma \qquad \ldots\text{(i)}$$

This gives us one equation. Now, we write the components of the forces along the y-axis and equate the sum to zero. This gives us another equation, *i.e.*,

$$\Sigma F_y = 0 \qquad \ldots\text{(ii)}$$

Note (i) *If the system is in equilibrium we will write the two equations as:*

$$\Sigma F_x = 0 \quad \text{and} \quad \Sigma F_y = 0$$

(ii) *If the forces are collinear, the second equation, i.e.,* $\Sigma F_y = 0$ *is not needed.*

Sample Example 5.12 *Two blocks of mass 4 kg and 2 kg are placed side by side on a smooth horizontal surface as shown in the figure. A horizontal force of 20 N is applied on 4 kg block. Find :*

(a) the acceleration of each block.

(b) the normal reaction between two blocks.

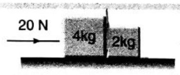


Fig. 5.37

Solution (a) Since, both the blocks will move with same acceleration (say a) in horizontal direction.

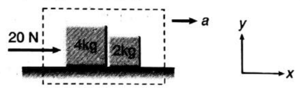


Fig. 5.38

$20 = 6a$

$a = \frac{10}{3}$

Let us take both the blocks as a system. Net external force on the system is 20 N in horizontal direction.

Using
$$\Sigma F_x = ma_x$$
$$20 = (4+2)a = 6a$$
or
$$a = \frac{10}{3}\ \text{m/s}^2$$

(b) The free body diagram of both the blocks are as shown in Fig. 5.39.

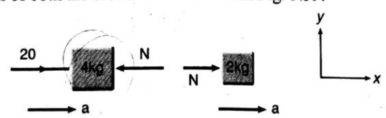


Fig. 5.39

$20 - N = 4a$

Using $\Sigma F_x = ma_x$

For 4 kg block
$$20 - N = 4a = 4 \times \frac{10}{3}$$
$$N = 20 - \frac{40}{3} = \frac{20}{3}\ \text{N}$$

This can also be solved as under

For 2 kg block $\qquad N = 2a = 2 \times \frac{10}{3} = \frac{20}{3}$ N

Here, N is the normal reaction between the two blocks.

Note *In free body diagram of the blocks we have not shown the forces acting on the blocks in vertical direction, because normal reaction between the blocks and acceleration of the system can be obtained without using $\Sigma F_y = 0$.*

Sample Example 5.13 *Three blocks of mass 3 kg, 2 kg and 1 kg are placed side by side on a smooth surface as shown in figure. A horizontal force of 12 N is applied on 3 kg block. Find the net force on 2 kg block.*

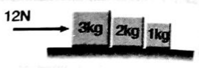


Fig. 5.40

Solution Since, all the blocks will move with same acceleration (say a) in horizontal direction. Let us take all the blocks as a system.

Net external force on the system is 12 N in horizontal direction.

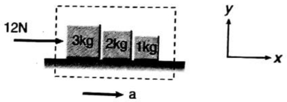


Fig. 5.41

Using $\qquad \Sigma F_x = ma_x$, we get

$$12 = (3 + 2 + 1)a = 6a$$

or $\qquad a = \frac{12}{6} = 2 \text{ m/s}^2$

Now, let F be the net force on 2 kg block in x-direction, then using $\Sigma F_x = ma_x$ for 2 kg block, we get

$$F = (2)(2) = 4 \text{ N}$$

Note *Here, net force F on 2 kg block is the resultant of N_1 and N_2 $(N_1 > N_2)$*
where $\;N_1$ = normal reaction between 3 kg and 2 kg block,
and $\;N_2$ = normal reaction between 2 kg and 1 kg block.
Thus, $\;F = N_1 - N_2$

Sample Example 5.14 *In the arrangement shown in figure. The strings are light and inextensible. The surface over which blocks are placed is smooth. Find :*

(a) the acceleration of each block,
(b) the tension in each string.

Fig. 5.42

Solution (a) Let 'a' be the acceleration of each block and T_1 and T_2 be the tensions, in the two strings as shown in figure.

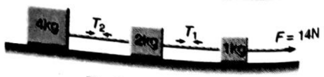


Fig. 5.43

Taking the three blocks and the two strings as the system.

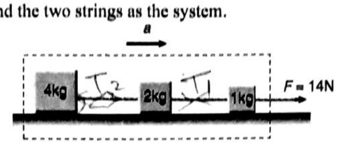


Fig. 5.44

F − T = ma

Using $\qquad \Sigma F_x = ma_x$

or $\qquad 14 = (4 + 2 + 1)a$

or $\qquad a = \frac{14}{7} = 2 \text{ m/s}^2$

(b) Free body diagram (showing the forces in x-direction only) of 4 kg block and 1 kg block are shown in figure.

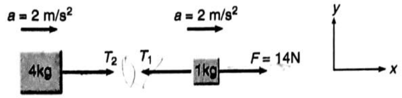


Fig. 5.45

Using $\qquad \Sigma F_x = ma_x$

For 1 kg block, $\qquad F - T_1 = (1)(a)$

or $\qquad 14 - T_1 = (1)(2) = 2$

$\therefore \qquad T_1 = 14 - 2 = 12$ N

For 4 kg block, $\qquad T_2 = (4)(a)$

$\therefore \qquad T_2 = (4)(2) = 8$ N

T₂ · ma

Sample Example 5.15 *Two blocks of mass 4 kg and 2 kg are attached by an inextensible light string as shown in figure. Both the blocks are pulled vertically upwards by a force $F = 120$ N. Find :*

(a) the acceleration of the blocks,
(b) tension in the string. (Take $g = 10$ m/s^2).

Fig. 5.46

Solution (a) Let a be the acceleration of the blocks and T the tension in the string as shown in figure.

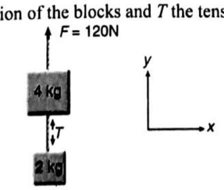


Fig. 5.47

F

120 $\frac{120}{6}$ = 20

Taking the two blocks and the string as the system.
Using $\Sigma F_y = ma_y$, we get

$$F - 4g - 2g = (4 + 2)a$$

or $$120 - 40 - 20 = 6a$$

or $$60 = 6a$$

$\therefore$ $$a = 10 \text{ m/s}^2$$

(b) Free body diagram of 2 kg block is as shown in Fig. 5.49.

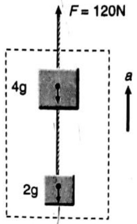


Fig. 5.48

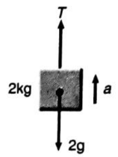


Fig. 5.49

Using $\Sigma F_y = ma_y$ we get

$$T - 2g = 2a$$

or $$T - 20 = (2)(10)$$

$\therefore$ $$T = 40 \text{ N}$$

Note *If the string is having some mass, tension in it is different at different points. Under such condition tension on the string at some point is calculated as under :*

In the adjoining figure the length of the string connecting the two blocks is 2 m and mass is 2 kg. Tension at A, B and C (centre point) can be calculated by considering the motion of system below A, B and C. For example :

$$a = \frac{F - \text{weight of 2 kg} - \text{weight of 4 kg} - \text{weight of string}}{\text{mass of 2 kg} + \text{mass of 4 kg} + \text{mass of string}}$$

$$= \frac{100 - 20 - 40 - 20}{2 + 4 + 2}$$

$$= \frac{20}{8} = 2.5 \text{ m/s}^2$$

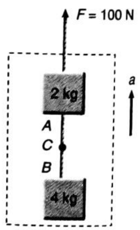


Fig. 5.50

$(g = 10 \text{ m/s}^2)$

Refer Fig. (a)

$$T_A - m_{AB}g - 40 = (m_{AB} + 4)a$$

or $$T_A - 20 - 40 = (2 + 4)(2.5)$$

or $$T_A = 75 \text{ N}$$

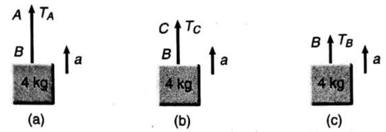


Fig. 5.51

Refer Fig. (b)

$$T_C - m_{BC}g - 40 = (m_{BC} + 4)a$$

or $$T_C - 10 - 40 = (1 + 4)(2.5)$$

or $$T_C = 62.5 \text{ N}$$

Refer Fig. (c) $$T_B - 40 = 4a$$

or $$T_B = 40 + 4 \times 2.5$$

or $$T_B = 50 \text{ N}$$

Introductory Exercise 5.2

1. Three blocks of mass 1 kg, 4 kg and 2 kg are placed on a smooth horizontal plane as shown in figure. Find:
   (a) the acceleration of the system,
   (b) the normal force between 1 kg block and 4 kg block,
   (c) the net force on 2 kg block.

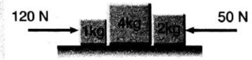


Fig. 5.52

2. Two blocks of mass 2 kg and 4 kg are released from rest over a smooth inclined plane of inclination 30° as shown in figure. What is the normal force between the two blocks?

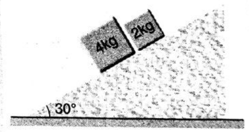


Fig. 5.53

3. What should be the acceleration 'a' of the box shown in figure so that the block of mass m exerts a force $\frac{mg}{4}$ on the floor of the box?

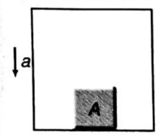


Fig. 5.54

4. A plumb bob of mass 1 kg is hung from the ceiling of a train compartment. The train moves on an inclined plane with constant velocity. If the angle of incline is 30°. Find the angle made by the string with the normal to the ceiling. Also, find the tension in the string. $(g = 10 \text{ m/s}^2)$
5. Repeat both parts of the above question, if the train moves with an acceleration $a = g/2$ up the plane.

6. Two blocks of mass 1 kg and 2 kg are connected by a string AB of mass 1 kg. The blocks are placed on a smooth horizontal surface. Block of mass 1 kg is pulled by a horizontal force F of magnitude 8 N. Find the tension in the string at points A and B.

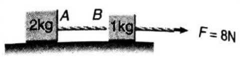


Fig. 5.55

5.5 Pulleys

As an author I personally feel that problems based on pulleys become very simple using **pulling force** method. Now, let us see what is this pulling force method with the help of an example.

Suppose two unequal masses m and $2m$ are attached to the ends of a light inextensible string which passes over a smooth massless pulley. We have to find the acceleration of the system. We can assume that the mass $2m$ is pulled downwards by a force equal to its weight, *i.e.*, $2mg$. Similarly, the mass m is being pulled by a force of mg downwards. Therefore, net pulling force on the system is $2mg - mg = mg$ and total mass being pulled is $2m + m = 3m$.

$\therefore$ Acceleration of the system is

$$a = \frac{\text{Net pulling force}}{\text{Total mass to be pulled}}$$

$$= \frac{mg}{3m} = \frac{g}{3}$$

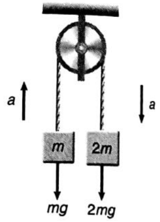


Fig. 5.56

Let us now take few examples based on pulling force method.

Note *While considering net pulling force, take the forces (or their components) which are in the direction of motion (or opposite to it) and are single (i.e., they are not forming pairs of equal and oppsite forces). For example weight (mg) or some applied force F. Tension makes an equal and opposite pair. So, they will not be included, unless the system in broken at some place and only one tension is considered on the remaining system.*

Sample Example 5.16 *In the system shown in figure pulley is smooth. String is massless and inextensible. Find acceleration of the system a, tensions T_1 and T_2. $(g = 10\ m/s^2)$*

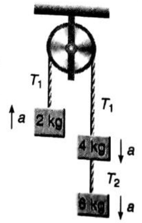


Fig. 5.57

Solution Here, net pulling force will be:

Weight of 4 kg and 6 kg blocks on one side – weight of 2 kg block on the other side. Therefore,

$$a = \frac{\text{Net pulling force}}{\text{Total mass}}$$

$$= \frac{(6 \times 10) + (4 \times 10) - (2)(10)}{6 + 4 + 2}$$

$$= \frac{20}{3}\ \text{m/s}^2$$

For T_1, let us consider FBD of 2 kg block. Writing equation of motion, we get

$$T_1 - 20 = 2a$$

or $$T_1 = 20 + 2 \times \frac{20}{3} = \frac{100}{3}\ \text{N}$$

For T_2, we may consider FBD of 6 kg block. Writing equation of motion, we get

$$60 - T_2 = 6a$$

$\therefore$ $$T_2 = 60 - 6a$$

$$= 60 - 6\left(\frac{20}{3}\right)$$

$$= \frac{60}{3}\ \text{N}$$

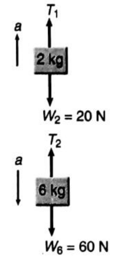


Fig. 5.58

Exercise : Draw FBD of 4 kg block. Write down the equation of motion for it and check whether the values calculated above are correct or not.

Sample Example 5.17 *In the system shown in figure all surfaces are smooth. String is massless and inextensible. Find acceleration a of the system and tension T in the string. $(g = 10\ m/s^2)$*

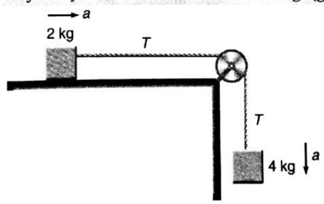


Fig. 5.59

Solution Here, weight of 2 kg is perpendicular to motion (or a). Hence, it will not contribute in net pulling force. Only weight of 4 kg block will be included.

$\therefore$ $$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{(4)(10)}{(4 + 2)}$$

$$= \frac{20}{3}\ \text{m/s}^2$$

For T, consider FBD of 4 kg block. Writing equation of motion.

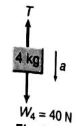


Fig. 5.60

$$40 - T = 4a$$

$$\therefore \quad T = 40 - 4a$$

$$= 40 - 4\left(\frac{20}{3}\right)$$

$$= \frac{40}{3}\text{ N}$$

Exercise : Draw FBD of 2 kg block and write down equation of motion for it. Check whether the values calculated above are correct or not.

Sample Example 5.18 *In the adjacent figure, mass of A, B and C are* 1 *kg,* 3 *kg and* 2 *kg respectively. Find :*

(a) the acceleration of the system and

(b) tension in the string.

Neglect friction. $(g = 10\ m/s^2)$

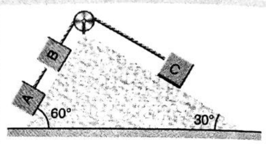


Fig. 5.61

Solution (a) In this case net pulling force

$$= m_A g \sin 60° + m_B g \sin 60° - m_C g \sin 30°$$

$$= (1)(10)\frac{\sqrt{3}}{2} + (3)(10)\left(\frac{\sqrt{3}}{2}\right) - (2)(10)\left(\frac{1}{2}\right)$$

$$= 24.64 \text{ N}$$

Total mass being pulled $= 1 + 3 + 2 = 6$ kg

$\therefore$ Acceleration of the system $a = \dfrac{21.17}{6} = 4.1 \text{ m/s}^2$

(b) For the tension in the string between A and B.

FBD of A

$$m_A\ g \sin 60° - T_1 = (m_A)(a)$$

$$\therefore \quad T_1 = m_A\ g \sin 60° - m_A\ a$$

$$= m_A\ (g \sin 60° - a)$$

$$\therefore \quad T_1 = (1)\left(10 \times \frac{\sqrt{3}}{2} - 4.1\right)$$

$$= 4.56 \text{ N}$$

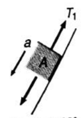


Fig. 5.62

For the tension in the string between B and C.

FBD of C

$$T_2 - m_C g \sin 30° = m_C\, a$$

$$\therefore \quad T_2 = m_C\,(a + g \sin 30°)$$

$$\therefore \quad T_2 = 2\left[3.53 + 10\left(\frac{1}{2}\right)\right]$$

$$= 18.2 \text{ N}$$

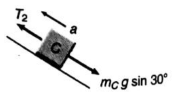


Fig. 5.63

Introductory Exercise 5.3

1. In the arrangement shown in figure what should be the mass of block A, so that the system remains at rest? Neglect friction and mass of strings.

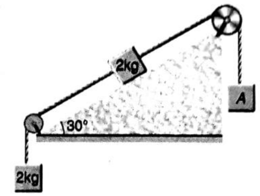


Fig. 5.64

2. In the arrangement shown in figure, find the ratio of tensions in the strings attached with 4 kg block and that with 1 kg block.

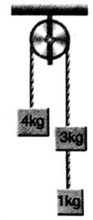


Fig. 5.65

3. Two unequal masses of 1 kg and 2 kg are connected by a string going over a clamped light smooth pulley as shown in figure. The system is released from rest. The larger mass is stopped for a moment 1.0 s after the system is set in motion. Find the time elapsed before the string is tight again.

Fig. 5.66

4. Two unequal masses of 1 kg and 2 kg are connected by an inextensible light string passing over a smooth pulley as shown in figure. A force $F = 20$ N is applied on 1 kg block. Find the acceleration of either block. $(g = 10\ \text{m/s}^2)$.

Fig. 5.67

5.6 Constraint Equations

These equations basically establish the relation between accelerations (or velocities) of different masses attached by string(s). Usually it is observed that the number of constraint equations are as many as the number of strings in the system under consideration. From the following few examples we can better understand the method.

Sample Example 5.19 *Using constraint method find the relation between accelerations of* 1 *and* 2.

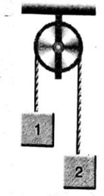


Fig. 5.68

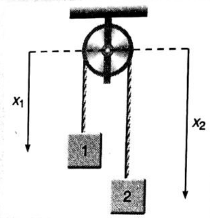


Fig. 5.69

Solution At any instant of time let x_1 and x_2 be the displacements of 1 and 2 from a fixed line (shown dotted).

Then, $$x_1 + x_2 = \text{constant}$$

or $$x_1 + x_2 = l \qquad \text{(length of string)}$$

Differentiating with respect to time, we have

$$v_1 + v_2 = 0 \quad \text{or} \quad v_1 = -v_2$$

Again differentiating with respect to time, we get

$$a_1 + a_2 = 0 \quad \text{or} \quad a_1 = -a_2$$

This is the required relation between a_1 and a_2, *i.e.*, accelerations of 1 and 2 are equal but in opposite directions.

Sample Example 5.20 *Find the constraint relation between a_1, a_2 and a_3.*

Fig. 5.70

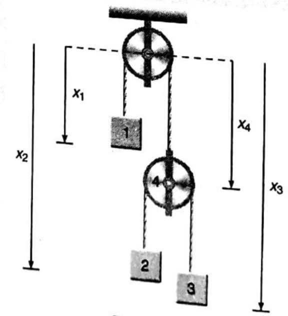


Fig. 5.71

Solution Points 1, 2, 3 and 4 are movable. Let their displacements from a fixed line be x_1, x_2, x_3 and x_4.

We have,

$$x_1 + x_4 = l_1 \qquad \text{(length of first string)} \quad \ldots\text{(i)}$$

and $$(x_2 - x_4) + (x_3 - x_4) = l_2 \qquad \text{(length of second string)}$$

or $$x_2 + x_3 - 2x_4 = l_2 \qquad \ldots\text{(ii)}$$

On double, differentiating with respect to time, we get

$$a_1 + a_4 = 0 \qquad \ldots\text{(iii)}$$

and $$a_2 + a_3 - 2a_4 = 0 \qquad \ldots\text{(iv)}$$

But since $$a_4 = -a_1 \qquad \text{[From Eq. (iii)]}$$

We have, $$a_2 + a_3 + 2a_1 = 0$$

This is the required constraint relation between a_1, a_2 and a_3.

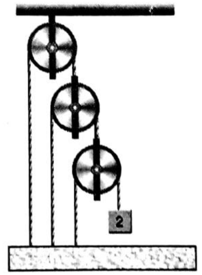


Fig. 5.72

Sample Example 5.21 *Using constraint equations find the relation between a_1 and a_2.*

Solution Points 1, 2, 3 and 4 are movable. Let their displacements from a fixed line be x_1, x_2, x_3 and x_4

$$x_1 + x_3 = l_1$$

$$(x_1 - x_3) + (x_4 - x_3) = l_2$$

$$(x_1 - x_4) + (x_2 - x_4) = l_3$$

On double differentiating with respect to time, we will get following three constraint relations

$$a_1 + a_3 = 0 \qquad \ldots\text{(i)}$$

$$a_1 + a_4 - 2a_3 = 0 \qquad \ldots\text{(ii)}$$

$$a_1 + a_2 - 2a_4 = 0 \qquad \ldots\text{(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$a_2 = -7a_1$$

Which is the desired relation between a_1 and a_2.

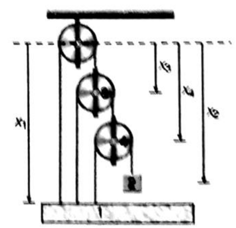


Fig. 5.73

Sample Example 5.22 *At certain moment of time, velocities of* 1 *and* 2 *both are* 1 m/s *upwards. Find the velocity of* 3 *at that moment.*

Solution In Instance 20, we have found

$$a_2 + a_3 + 2a_1 = 0$$

Similarly, we can find

$$v_2 + v_3 + 2v_1 = 0$$

Taking, upward direction as positive we are given:

$$v_1 = v_2 = 1 \text{ m/s}$$

$$\therefore \quad v_3 = -3 \text{ m/s}$$

i.e., velocity of block 3 is 3 m/s (downwards).

Fig. 5.74

Introductory Exercise 5.4

1. Consider the situation shown in figure. Both the pulleys and the string are light and all the surfaces are smooth.
   (a) Find the acceleration of 1 kg block.
   (b) Find the tension in the string.
   $(g = 10 \text{ m/s}^2)$.

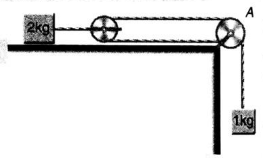


Fig. 5.75

2. Calculate the acceleration of either blocks and tension in the string shown in figure. The pulley and the string are light and all surfaces are smooth.

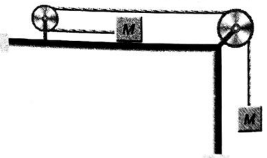


Fig. 5.76

3. Find the mass M so that it remains at rest in the adjoining figure. Both the pulley and string are light and friction is absent everywhere. $(g = 10 \text{ m/s}^2)$.

Fig. 5.77

4. In figure assume that there is negligible friction between the blocks and table. Compute the tension in the cord connecting m_2 and the pulley and acceleration of m_2 if $m_1 = 300g$, $m_2 = 200g$ and $F = 0.40N$.

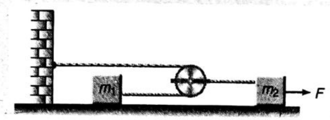


Fig. 5.78

5.7 Pseudo Force

Before studying pseudo force let us first discuss frame of reference. A system of coordinate axes which defines the position of a particle or an event in two or three dimensional space is called a frame of reference. The simplest frame of reference is, of course, the familiar cartesian system of coordinates, in which the position of the particle is specified by its three coordinates x, y and z. Frame of references are of two types :

(a) Inertial frame of reference

A non-accelerating frame of reference is called an inertial frame of reference. A frame of reference moving with a constant velocity is an inertial frame of reference.

(b) Non-inertial frame of reference

An accelerating frame of reference is called a non-inertial frame of reference.

Note *A rotating frame of reference is a non-inertial frame of reference, because it is also an accelerating one.*

Now, let us come to the pseudo force. Newton's first two laws hold good in an intertial frame only. However, we people spend most of our time on the earth which is an (approximate) inertial frame. We are so familiar with the Newton's laws that we will still like to use 'total force equals mass times acceleration' even when we use a non-inertial frame. This can be done if we agree to call $(-m\vec{a}_0)$ a force acting on the particle. Then while preparing the list of the forces acting on a particle P, we include all the (real) forces acting on P by all other objects and also include an imaginary force $-m\vec{a}_0$. Here, $\vec{a}_0$ is the acceleration of the non-inertial frame under consideration. After applying this additional imaginary force (called pseudo force) $-m\vec{a}_0$ we can now use 'total force equals mass time acceleration' even in non-inertial frames also. Now, with the help of a simple example let us see what problem arises if we don't apply the pseudo force $-m\vec{a}_0$ while using $\vec{F} = m\vec{a}$ (second law) in non-inertial frame. Suppose a block A of mass m is placed on a lift ascending with an acceleration a_0. Let N be the normal reaction between the block and the floor of the lift. Free body diagram of A in ground frame of reference (inertial) is shown in Fig. 5.80.

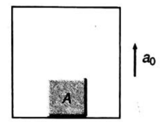


Fig. 5.79

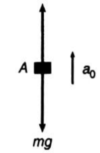


Fig. 5.80

$$\therefore \quad N - mg = ma_0$$

$$\text{or} \quad N = m(g + a_0) \qquad \ldots\text{(i)}$$

But if we draw the free body diagram of A with respect to the elevator (a non-inertial frame of reference) without applying the pseudo force, as shown in Fig. 5.81, we get

$$N' - mg = 0 \quad \text{or} \quad N' = mg \qquad \ldots\text{(ii)}$$

Fig. 5.81

Since, $N' \neq N$, either of the equations is wrong. But if we apply a pseudo force in non-inertial frame of reference, N' becomes equal to N as shown in Fig. 5.82. Acceleration of block with respect to elevator is zero.

$\therefore \quad N' - mg - ma_0 = 0$

or $\quad N' = m(g + a_0) \qquad \ldots\text{(iii)}$

$\therefore \quad N' = N$

Pseudo force is given by $\vec{F}_P = -m\vec{a}_0$. Here, $\vec{a}_0$ is the acceleration of the non-inertial frame of reference and m the mass of the body under consideration. In the whole chapter, we will show the pseudo force by $\vec{F}_P$.

Thus, we may conclude that pseudo force is not a real force. When we draw the free body diagram of a mass, with respect to an inertial frame of reference we apply only the real forces (forces which are actually acting on the mass), but when the free body diagram is drawn from a non-inertial frame of reference a pseudo force (in addition to all real forces) has to be applied to make the equation $\vec{F} = m\vec{a}$, valid in this frame also.

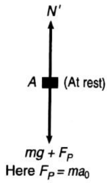


Fig. 5.82

Note *In case of rotating frame of reference this pseudo force is called the centrifugal force when applied for centripetal acceleration. Let us take few examples of pseudo forces.*

Sample Example 5.23 *In the adjoining figure, the coefficient of friction between wedge (of mass M) and block (of mass m) is μ. Find the minimum horizontal force F required to keep the block stationary with respect to wedge.*

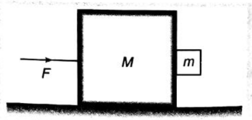


Fig. 5.83

Solution Such problems can be solved with or without using the concept of pseudo force. Let us, solve the problem by both the methods.

$$a = \text{acceleration of (wedge + block) in horizontal direction}$$
$$= \frac{F}{M + m}.$$

Inertial frame of reference (Ground)

FBD of block with respect to ground (only real forces have to applied) with respect to ground block is moving with an acceleration 'a'. Therefore,

$$\Sigma F_y = 0 \quad \text{and} \quad \Sigma F_x = ma$$
$$mg = \mu N \quad \text{and} \quad N = ma$$

$\therefore \quad a = \dfrac{g}{\mu}$

$\therefore \quad F = (M + m)a = (M + m)\dfrac{g}{\mu}$

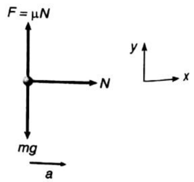


Fig. 5.84

Non-inertial frame of reference (Wedge)

FBD of 'm' with respect to wedge (real + one pseudo force) with respect to wedge block is stationary.

$\therefore \quad \Sigma F_x = 0 = \Sigma F_y$

$\therefore \quad mg = \mu N \quad \text{and} \quad N = ma$

$\therefore \quad a = \dfrac{g}{\mu}$

and $\quad F = (M + m)a$

$\quad = (M + m)\dfrac{g}{\mu}$

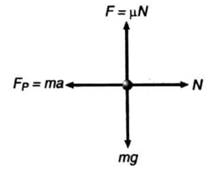


Fig. 5.85

From the above discussion, we can see that from both the methods results are same.

Sample Example 5.24 *All surfaces are smooth in following figure. Find F, such that block remains stationary with respect to wedge.*

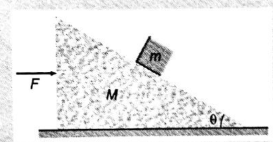


Fig. 5.86

Solution Acceleration of (block + wedge) $a = \dfrac{F}{(M + m)}$

Let us solve the problem by both the methods.

From inertial frame of reference (Ground)

FBD of block w.r.t. ground (Apply real forces):

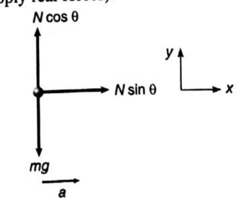


Fig. 5.87

With respect to ground block is moving with an acceleration a.

$\therefore \quad \Sigma F_y = 0 \Rightarrow N\cos\theta = mg \qquad \ldots\text{(i)}$

and $\quad \Sigma F_x = ma \Rightarrow N\sin\theta = ma \qquad \ldots\text{(ii)}$

From Eqs. (i) and (ii)

$$a = g \tan\theta$$

$$\therefore \quad F = (M+m)a$$

$$= (M+m)g \tan\theta$$

From non-inertial frame of reference (Wedge)

FBD of block w.r.t. wedge (real forces + pseudo force)

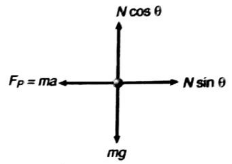


Fig. 5.88

w.r.t. wedge, block is stationary

$$\therefore \quad \Sigma F_y = 0 \Rightarrow N\cos\theta = mg \quad \text{...(iii)}$$

$$\Sigma F_x = 0 \Rightarrow N\sin\theta = ma \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we will get the same result

i.e., $$F = (M+m)g\tan\theta.$$

Sample Example 5.25 *A bob of mass m is suspended from the ceiling of a train moving with an acceleration a as shown in figure. Find the angle θ in equilibrium position.*

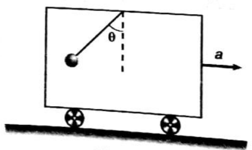


Fig. 5.89

Solution This problem can also be solved by both the methods.

Inertial frame of reference (Ground)

FBD of bob w.r.t. ground (only real forces) :

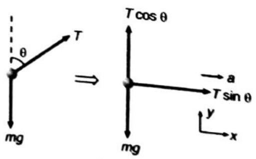


Fig. 5.90

With respect to ground, bob is also moving with an acceleration 'a'.

$$\therefore \quad \Sigma F_x = 0 \Rightarrow T\sin\theta = ma \quad \text{...(i)}$$

and $$\Sigma F_y = 0 \Rightarrow T\cos\theta = mg \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\tan\theta = \frac{a}{g} \text{ or } \theta = \tan^{-1}\left(\frac{a}{g}\right)$$

Non-inertial frame of reference (Train)

FBD of bob w.r.t train. (real forces + pseudo force):

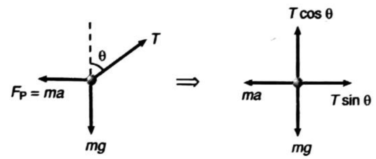


Fig. 5.91

with respect to train, bob is in equilibrium

$$\therefore \quad \Sigma F_x = 0 \Rightarrow T\sin\theta = ma \quad \text{...(iii)}$$

$$\therefore \quad \Sigma F_y = 0 \Rightarrow T\cos\theta = mg \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get the same result, *i.e.*,

$$\theta = \tan^{-1}\left(\frac{a}{g}\right)$$

Sample Example 5.26 *In the adjoining figure, a wedge is fixed to an elevator moving upwards with an acceleration a. A block of mass m is placed over the wedge. Find the acceleration of the block with respect to wedge. Neglect friction.*

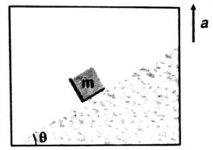


Fig. 5.92

Solution Since, acceleration of block w.r.t. wedge (an accelerating or non-inertial frame of reference) is to be find out.

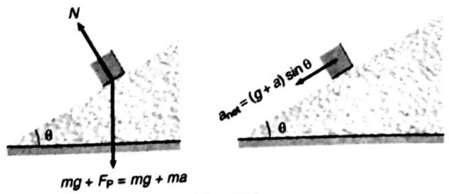


Fig. 5.93

FBD of 'block' w.r.t. 'wedge' is shown in Fig. 5.93.

The acceleration would had been $g \sin \theta$ (down the plane) if the lift were stationary or when only weight (*i.e.*, mg) acts downwards.

Here, downward force is $m(g + a)$.

$\therefore$ Acceleration of the block (of course w.r.t. wedge) will be $(g + a) \sin \theta$ down the plane.

5.8 Friction

As we have discussed in Article 5.1 friction is the parallel component of contact force between two bodies in contact. These forces are basically electromagnetic in nature. Friction can operate between a given pair of solids between a solid and a fluid or between a pair of fluids. Frictional force exerted by fluids is called viscous force. When two bodies slip over each other the force of friction is called kinetic friction, but when they do not slip but have a tendency to do so the force of friction is called static friction.

Regarding friction it is worth noting that:

1. If a body is at rest and no pulling force is acting on it, force of friction on it is zero.
2. If a force is applied to pull the body and it does not move, the friction acts which is equal in magnitude and opposite in direction to the applied force, *i.e.*, friction is self adjusting force. Further, as the body is at rest the friction is called static friction.
3. If the applied force is increased the force of static friction also increases. If the applied force exceeds a certain (maximum) value, the body starts moving. This maximum force of static friction upto which body does not move is called limiting friction. Thus, static friction is a self adjusting force with an upper limit called limiting friction.
4. This limiting force of friction (f_L) is found experimentally to depend on normal reaction (N). Hence,

$$f_L \propto N$$

or

$$f_L = \mu_s N$$

Here, μ_s is a dimensionless constant and called coefficient of static friction, which depends on nature of surfaces in contact.

5. If the applied force is further increased, the friction opposing the motion is called kinetic or sliding friction. Experimentally, it is well established that kinetic friction is lesser than limiting friction and is given by

$$f_k = \mu_k N$$

where μ_k is coefficient of kinetic friction and less than μ_s.

Note (i) *In problems if μ_s and μ_k are separately not given but only μ is given. Then use*

$$f_L = f_k = \mu N$$

(ii) *If more than two blocks are placed one over the other on a horizontal ground then normal reaction between two blocks will be equal to the weight of the blocks over the common surface.*

For example N_1 = *normal reaction between A and B*

$= m_A g$

N_2 = *normal reaction between B and C*

$= (m_A + m_B) g$ *and so on.*

The theory of static and kinetic friction can be better understood by the following simple example.

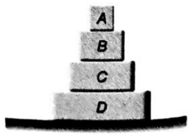


Fig. 5.94

Sample Example 5.27 *Suppose a block of mass 1 kg is placed over a rough surface and a horizontal force F is applied on the block as shown in figure. Now, let us see what are the values of force of friction f and acceleration of the block a if the force F is gradually increased. Given that $\mu_s = 0.5$, $\mu_k = 0.4$ and $g = 10 m/s^2$.*

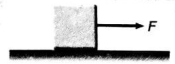


Fig. 5.95

Solution Free body diagram of block is

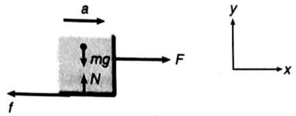


Fig. 5.96

$$\Sigma F_y = 0$$

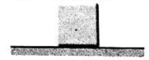
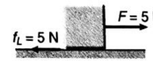
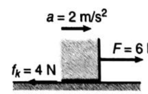
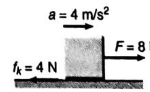
$\therefore$ $$N - mg = 0$$

or $$N = mg = (1)(10) = 10\,\text{N}$$

$$f_L = \mu_s N = (0.5)(10) = 5\,\text{N}$$

and $$f_k = \mu_k N = (0.4)(10) = 4\,\text{N}$$

Below is explained in tabular form, how the force of friction f depends on the applied force F.

F	f	$F_{net} = F - f$	Acceleration of block $a = \frac{F_{net}}{m}$	Diagram
0	0	0	0	
2 N	2 N	0	0	
4 N	4 N	0	0	
5 N	5 N	0	0	
6 N	4 N	2 N	2 m/s^2	
8 N	4 N	4 N	4 m/s^2	

Graphically this can be understood as under :

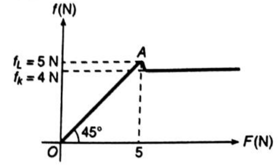


Fig. 5.97

Note that $f = F$ till $F \leq f_L$. Therefore, slope of line OA will be 1 ($y = mx$) or angle of line OA with F-axis is 45°.

Here, $a = 0$ for $F \leq 5$ N

and $$a = \frac{F - f_K}{m} = \frac{F - 4}{1} = F - 4 \text{ for } F > 5 \text{ N}$$

a-F graph is shown in figure. When F is slightly increased from 5 N, acceleration of block increases from 0 to 1 m/s^2. Think why?

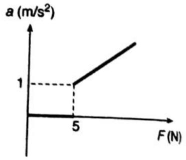


Fig. 5.98

Note *Henceforth, we will take coefficient of friction as μ unless and until specially mentioned in the question μ_s and μ_k separately.*

Angle of friction (λ)

At a point of rough contact, where slipping is about to occur the two forces acting on each object are the normal reaction N and frictional force μN.

The resultant of these two forces is F and it makes an angle λ with the normal where

$$\tan \lambda = \frac{\mu N}{N} = \mu$$

or $$\lambda = \tan^{-1}(\mu) \qquad \ldots\text{(i)}$$

This angle λ is called the angle of friction.

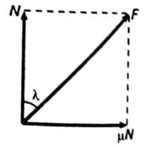


Fig. 5.99

Angle of repose (α)

Suppose a block of mass m is placed on an inclined plane whose inclination θ can be increased or decreased. Let, μ be, the coefficient of friction between the block and the plane. At a general angle θ,

Normal reaction $N = mg \cos \theta$

Limiting friction $f_L = \mu N = \mu mg \cos \theta$

and the driving force (or pulling force)

$$F = mg \sin \theta \qquad \text{(Down the plane)}$$

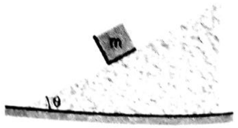


Fig. 5.100

From these three equations we see that when θ is increased from 0° to 90°, normal reaction N and hence, the limiting friction f_L is decreased while the driving force F is increased. There is a critical angle called angle of repose (α) at which these two forces are equal. Now, if θ is further increased, then the driving force F becomes more than the limiting friction f_L and the block starts sliding.

Thus, $f_L = F$ at $\theta = \alpha$

or $\mu\, mg \cos \alpha = mg \sin \alpha$

or $\tan \alpha = \mu$

or $$\alpha = \tan^{-1}(\mu) \qquad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we see that angle of friction (λ) is numerically equal to the angle of repose.

or $$\lambda = \alpha$$

From the above discussion we can conclude that

If $\theta < \alpha$, $F < f_L$ the block is stationary

If $\theta = \alpha$, $F = f_L$ the block is on the verge of sliding

and if $\theta > \alpha, F > f_L$ the block slides down with acceleration

$$a = \frac{F - f_L}{m} = g(\sin \theta - \mu \cos \theta)$$

How, N, f_L and F varies with θ, this can be shown graphically as shown in Fig. 5.101.

$N = mg \cos \theta$

or $N \propto \cos \theta$

$f_L = \mu mg \cos \theta$

or $f_L \propto \cos \theta$

$F = mg \sin \theta$

or $F \propto \sin \theta$

Normally $\mu < 1$,

So, $f_L < N$.

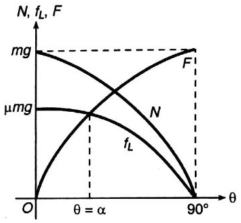


Fig. 5.101

Sample Example 5.28 *A particle of mass 1 kg rests on rough contact with a plane inclined at 30° to the horizontal and is just about to slip. Find the coefficient of friction between the plane and the particle.*

Solution The given angle 30° is really the angle of repose α. Hence,

$$\mu = \tan 30° = \frac{1}{\sqrt{3}}$$

Sample Example 5.29 *A block of weight W rests on a horizontal plane with which the angle of friction is λ. A force P inclined at an angle θ to the plane is applied to the plane until it is on the point of moving. Find the value of θ for which the value of P will be least.*

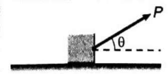


Fig. 5.102

Solution In the limiting case contact force F is inclined at λ to the normal. Only three forces act on the block. Applying Lami's theorem, we get

$$\frac{P}{\sin(180° - \lambda)} = \frac{W}{\sin(90° - \theta + \lambda)}$$

$$= \frac{F}{\sin(90° + \theta)}$$

or $$\frac{P}{\sin\lambda} = \frac{W}{\cos(\theta - \lambda)}$$

or $$P = \frac{W \sin\lambda}{\cos(\theta - \lambda)}$$

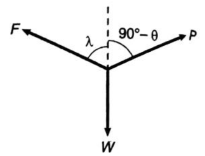


Fig. 5.103

P will be least when $\cos(\theta - \lambda)$ is greatest because W and λ are constant.

i.e., when $\cos(\theta - \lambda) = 1$ and $\theta - \lambda = 0°$

or $\theta = \lambda$ **Ans.**

Sample Example 5.30 *Figure shows two blocks in contact sliding down an inclined surface of inclination $30°$. The friction coefficient between the block of mass 2.0 kg and the incline is $\mu_1 = 0.20$ and that between the block of mass 4.0 kg and the incline is $\mu_2 = 0.30$. Find the acceleration of 2.0 kg block. ($g = 10\ \text{m/s}^2$).*

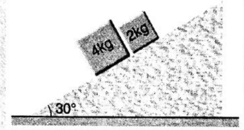


Fig. 5.104

Solution Since, $\mu_1 < \mu_2$, acceleration of 2 kg block down the plane will be more than the acceleration of 4 kg block, if allowed to move separately. But as the 2.0 kg block is behind the 4.0 kg block both of them will move with same acceleration say 'a'. Taking both the blocks as a system.

Force down the plane on the system $= (4 + 2) g \sin 30°$

$$= (6)(10)\left(\frac{1}{2}\right) = 30\ \text{N}$$

Force up the plane on the system

$$= \mu_1 (2)(g) \cos 30° + \mu_2 (4)(g) \cos 30°$$
$$= (2\mu_1 + 4\mu_2) g \cos 30°$$
$$= (2 \times 0.2 + 4 \times 0.3)(10)(0.86)$$
$$\approx 13.76\ \text{N}$$

$\therefore$ Net force down the plane is $F = 30 - 13.76 = 16.24\ \text{N}$

$\therefore$ Acceleration of both the blocks down the plane will be a.

$$a = \frac{F}{4 + 2} = \frac{16.24}{6} = 2.7\ \text{m/s}^2$$

Introductory Exercise 5.5

1. In figure $m_1 = 1$ kg and $m_2 = 4$ kg. Find the mass M of the hanging block which will prevent the smaller block from slipping over the triangular block. All the surfaces are frictionless and the strings and the pulleys are light

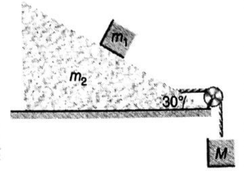


Fig. 5.105

Note *In exercises 2 to 4 the situations described take place in a box car which has initial velocity $v = 0$ but acceleration $\vec{a} = (5\ \text{m/s}^2)\hat{\mathbf{i}}$. (Take $g = 10\ \text{m/s}^2$)*

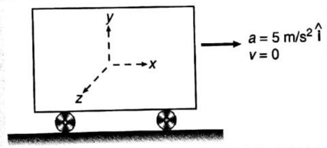


Fig. 5.106

2. A 2 kg object is slid along the frictionless floor with initial velocity $(10\ \text{m/s})\hat{\mathbf{i}}$ (a) Describe the motion of the object relative to car (b) when does the object reach its original position relative to the box car.

3. A 2 kg object is slid along the frictionless floor with initial transverse velocity $(10\ \text{m/s})\hat{\mathbf{k}}$. Describe the motion (a) in car's frame (b) in ground frame.

4. A 2 kg object is slid along a rough floor (coefficient of sliding friction = 0.3) with initial velocity $(10\ \text{m/s})\hat{\mathbf{i}}$. Describe the motion of the object relative to car assuming that the coefficient of static friction is greater than 0.5.

5. A block is placed on an inclined plane as shown in figure. What must be the frictional force between block and incline if the block is not to slide along the incline when the incline is accelerating to the right at $3\ \text{m/s}^2 \left(\sin 37° = \frac{3}{5}\right)$? (Take $g = 10\ \text{m/s}^2$)

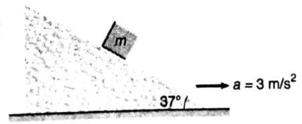


Fig. 5.107

Extra Points

- The common problem which I feel, students often face is the resolution of forces. Following two rules can be made in this regard.
  Rule 1 : If the body is in equilibrium, you can resolve the forces in any direction. Net force should be zero in all directions. A body moving with constant velocity is also in equilibrium.
  Rule 2 : If the body is accelerated, resolve the forces along acceleration and perpendicular to it. Net force along acceleration $= m\vec{a}$ and net force perpendicular to acceleration is zero.
- To find net force on a body find the acceleration of the body. Net force = mass × acceleration.

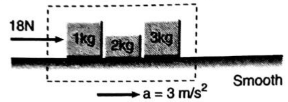


Fig. 5.108

For example, in the figure shown,

$$a = \frac{18}{1+2+3} = 3\ \text{m/s}^2$$

Therefore,net force on 1 kg block $= 1 \times 3 = 3$ N.
This 3 N is the resultant of the applied force 18 N and normal reaction between 1 kg block and 2 kg block.
Similarly, net force on 2 kg block $= 2 \times 3 = 6$ N.
This is the resultant of normal reactions (1 kg and 2 kg) and (2 kg and 3 kg) blocks.

- Force of friction does not oppose the motion of a body but it opposes the relative motion between two bodies in contact.
  As far as motion of individual body is concerned it is sometimes friction which is responsible for its motion.
  For example, in the figure shown the 1 kg block moves with 2 kg block only due to friction.

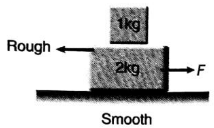


Fig. 5.109

- Mathematically a body is said to be in equilibrium, if
  (a) Net force acting on it is zero, *i.e.*, $\vec{F}_{net} = 0$.
  (b) Net moments of all the forces acting on it about any axis is zero. Physically the body at rest is said to be in equilibrium, if it is permanently at rest (unless some other force is applied on it, which may disturb its equilibrium). If a body is at rest just for a moment, it does not mean it is in equilibrium.

  For example, when a ball is thrown upwards, at highest point of its journey it momentarily comes at rest, but there it is not in equilibrium. A net force (equal to its weight) is acting downward. Due to that force it moves downwards.
  If a problem is asked on equilibrium, check whether the body is in equilibrium (permanent rest) or it is at rest just for a moment.
  Now, if the body is in equilibrium, you may resolve the forces in any direction (x, y, z whatsoever). Net force on the body should be zero in all directions.
  But if the body is **momentarily at rest but not in equilibrium** (I call it Mr. Ne. momentary rest not equilibrium); you will have to be very careful. First see in what direction will the body move just after few seconds. Obviously the net force on the body should point in that particular direction. Therefore components of all the forces in a direction perpendicular to the net force or perpendicular to the direction in which motion is likely to occur after few seconds should be zero.

- If a pulley is massless, net force on it is zero even, if it is accelerated. For example in the adjoining figure: $T_1 = 2T_2$ whether the pulley is accelerated on not provided the pulley is massless. This is because $\vec{F}_{net}$ = mass x acceleration and $\vec{F}_{net}$ will be zero if pulley is massless.

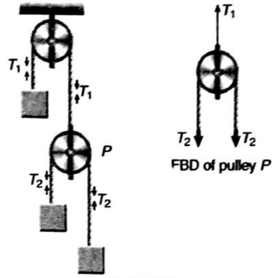


Fig. 5.110

- The direction of friction force on each of them is such as it either stops the relative motion or attempts to do so. For example, if a force F is applied on block A of a two block system, the direction of frictional forces at different contacts on different bodies will be as shown:
  Here, f_1 = force of friction between A and B
  and f_2 = force of friction between B and ground

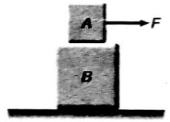


Fig. 5.111

- Force of friction $f = 0$, if no driving force is applied.
  $$f \le f_L \quad (= \mu_s N),$$
  If driving force is applied but no relative motion is there.
  and $$f = \mu_k N$$
  If relative motion is there.
  So, apply kinetic friction whenever you see the relative motion between two bodies in contact. But don't apply $f = f_L$ in case there is no relative motion. Because being a self adjusting force only that much amount will act which is required for stopping the relative motion. So, it may be less than f_L also.

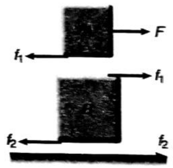


Fig. 5.112

- A common mistake which the students err in hurry is that they always write $f_L = \mu mg$ (in case of horizontal ground) or $f_L = \mu mg \cos\theta$ (in inclined surface). The actual formula is $f_L = \mu N$. Here, N is equal to mg or $mg \cos\theta$ upto when no force is acting at some angle ($\neq 0°$) with the plane.

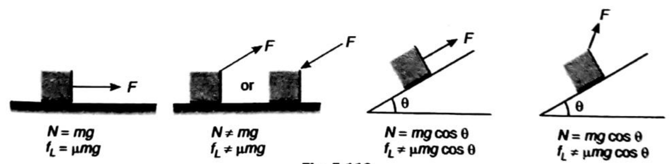


Fig. 5.113

- A car (or any vehicle) accelerates and decelerates by friction. So, maximum acceleration or deceleration of a car on horizontal ground can be μg, unless some external force is applied.
  Exercise : Think about maximum acceleration or retardation on an inclined road.

Solved Examples

Level 1

Example 1 *Determine the tensions T_1 and T_2 in the strings as shown in figure.*

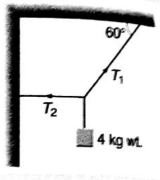


Fig. 5.114

Solution Resolving the tension T_1 along horizontal and vertical directions. As the body is in equilibrium,

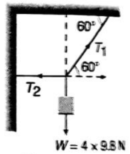


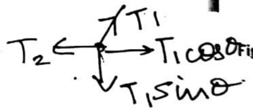
Fig. 5.115

$$T_1 \sin 60° = 4 \times 9.8 \text{ N} \quad \ldots(\text{i})$$

$$T_1 \cos 60° = T_2 \quad \ldots(\text{ii})$$

$$T_1 = \frac{4 \times 9.8}{\sin 60°}$$

$$= \frac{4 \times 9.8 \times 2}{\sqrt{3}} = 45.26 \text{ N}$$



$$T_2 = T_1 \cos 60° = 45.26 \times 0.5 = 22.63 \text{ N}$$

Example 2 *A ball of mass 1 kg hangs in equilibrium from two strings OA and OB as shown in figure. What are the tensions in strings OA and OB? (Take $g = 10\ m/s^2$).*

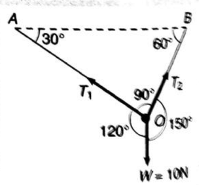


Fig. 5.116

Solution Various forces acting on the ball are as shown in Fig 5.116. The three concurrent forces are in equilibrium. Using Lami's theorem,

$$\frac{T_1}{\sin 150°} = \frac{T_2}{\sin 120°} = \frac{10}{\sin 90°}$$

or

$$\frac{T_1}{\sin 30°} = \frac{T_2}{\sin 60°} = \frac{10}{1}$$

$\therefore$

$$T_1 = 10 \sin 30° = 10 \times 0.5 = 5 \text{ N}$$

and

$$T_2 = 10 \sin 60° = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3} \text{ N}$$

Example 3 *A 4 m long ladder weighing 25 kg rests with its upper end against a smooth wall and lower end on rough ground. What should be the minimum coefficient of friction between the ground and the ladder for it to be inclined at 60° with the horizontal without slipping? (Take $g = 10\ m/s^2$).*

Solution In Fig 5.117 AB is a ladder of weight W which acts at its centre of gravity G.

$$\angle ABC = 60°$$

$\therefore$

$$\angle BAC = 30°$$

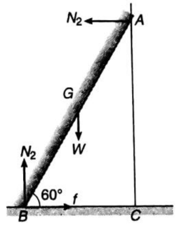


Fig. 5.117

Let N_1 be the reaction of the wall, and N_2 the reaction of the ground.
Force of friction f between the ladder and the ground acts along BC.
For horizontal equilibrium,

$$f = N_1 \quad \ldots(\text{i})$$

For vertical equilibrium,

$$N_2 = W \quad \ldots(\text{ii})$$

Taking moments about B, we get for equilibrium,

$$N_1 (4 \cos 30°) - W (2 \cos 60°) = 0 \quad \ldots(\text{iii})$$

Here, $W = 250$N

Solving these three equations, we get

$$f = 72.17\text{N} \quad \text{and} \quad N_2 = 250\text{N}$$

$\therefore$

$$\mu = \frac{f}{N_2} = \frac{72.17}{250} = 0.288$$

Example 4 *A block of mass 25 kg is raised by a 50 kg man in two different ways as shown in figure. What is the action on the floor by the man in the two cases? If the floor yields to a normal force of 700 N, which mode should the man adopt to lift the block without the floor yielding? (Take $g = 9.8\ m/s^2$).*

Solution In mode (a), the man applies a force equal to 25 kg weight in upward direction. According to Newton's third law of motion, there will be a downward force of reaction on the floor.

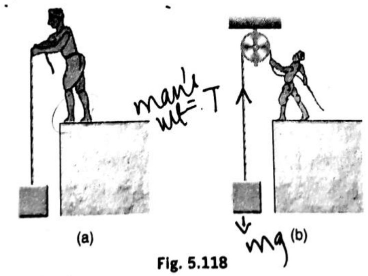


Fig. 5.118

$\therefore$ Total action on the floor by the man

$$= 50 \text{ kg-wt} + 25 \text{ kg-wt} = 75 \text{ kg-wt}$$

$$= 75 \times 9.8 \text{ N} = 735 \text{ N}$$

In mode (b), the man applies a downward force equal to 25 kg-wt. According to Newton's third law, the reaction will be in the upward direction.

$\therefore$ Total action on the floor by the man

$$= 50\text{ kg-wt} - 25\text{ kg-wt} = 25\text{ kg-wt}$$
$$= 25 \times 9.8\text{ N} = 245\text{ N}$$

As the floor yields to a downward force of 700 N, so the man should adopt mode (b).

Example 5 *A block of mass 200 kg is set into motion on a frictionless horizontal surface with the help of frictionless pulley and a rope system as shown in figure (a). What horizontal force F should be applied to produce in the block an acceleration of $1\ m/s^2$?*

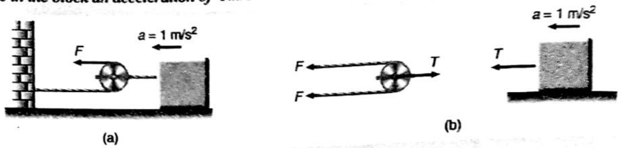


Fig. 5.119

Solution As shown in Fig. (b), when force F is applied at the end of the string, the tension in the lower part of the string is also F. If T is the tension in string connecting the pulley and the block, then,

$$T = 2F$$

But $\quad T = ma = (200)(1) = 200\text{ N}$

$\therefore \quad 2F = 200\text{ N}$

or $\quad F = 100\text{ N}$

Example 6 *A block of mass 1 kg is pushed against a rough vertical wall with a force of 20 N, coefficient of static friction being $\frac{1}{4}$. Another horizontal force of 10 N is applied on the block in a direction parallel to the wall. Will the block move? If yes, in which direction? If no, find the frictional force exerted by the wall on the block. ($g = 10\ m/s^2$)*

Solution Normal reaction on the block from the wall will be

$$N = F = 20\text{ N}$$

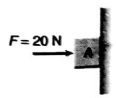


Fig. 5.120

Therefore, limiting friction

$$f_L = \mu N = \left(\frac{1}{4}\right)(20) = 5\text{ N}$$

Weight of the block is

$$W = mg = (1)(10) = 10\text{ N}$$

A horizontal force of 10 N is applied to the block. The resultant of these two forces will be $10\sqrt{2}$ N in the direction shown in figure. Since, this resultant is greater than the limiting friction. The block will move in the direction of $\vec{F}_{net}$ with acceleration

$$a = \frac{F_{net} - f_L}{m} = \frac{10\sqrt{2} - 5}{1} = 9.14\text{ m/s}^2$$

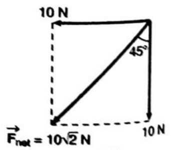


Fig. 5.121

Example 7 *Figure shows a man standing stationary with respect to a horizontal conveyor belt that is accelerating with $1\ ms^{-2}$. What is the net force on the man? If the coefficient of static friction between the man's shoes and the belt is 0.2, upto what acceleration of the belt can the man continue to be stationary relative to the belt? Mass of the man = 65 kg. ($g = 9.8\ m/s^2$)*

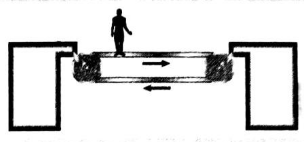


Fig. 5.122

Solution As the man is standing stationary w.r.t. the belt,

$\therefore \quad$ Acceleration of the man = Acceleration of the belt

$$= a = 1\text{ ms}^{-2}$$

Mass of the man, $\quad m = 65$ kg

Net force on the man $= ma = 65 \times 1 = 65$ N

Given coefficient of friction, $\quad \mu = 0.2$

$\therefore$ Limiting friction, $\quad f_L = \mu mg$

If the man remains stationary with respect to the maximum acceleration a_0 of the belt, then

$$ma_0 = f_L = \mu mg$$

$\therefore \quad a_0 = \mu g = 0.2 \times 9.8 = 1.96\text{ ms}^{-2}$

Example 8 *A block of mass m is at rest on a rough wedge as shown in figure. What is the force exerted by the wedge on the block?*

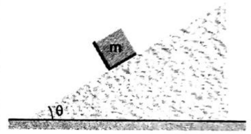


Fig. 5.123

Solution Since, the block is permanently at rest, it is in equilibrium. Net force on it should be zero. In this case only two forces are acting on the block.

(1) Weight = mg (downwards).

(2) Contact force (resultant of normal reaction and friction force) applied by the wedge on the block.

For the block to be in equilibrium these two forces should be equal and opposite.

Therefore, force exerted by the wedge on the block is mg (upwards).

Note *(i) From Newton's third law of motion-force exerted by the block on the wedge is also mg but downwards.*

(ii) The result can also be obtained in a different manner. The normal force on the block is $N = mg\cos\theta$ and the friction force on the block is $f = mg\sin\theta$ (not $\mu\ mg\cos\theta$)

These two forces are mutually perpendicular.

$\therefore$ Net contact force would be $\sqrt{N^2 + f^2}$ or $\sqrt{(mg\cos\theta)^2 + (mg\sin\theta)^2}$ which is equal to mg.

Problem 9 *A ball of mass 1 kg is at rest in position P by means of two light strings OP and RP. The string RP is now cut and the ball swings to position Q. If $\theta = 45°$. Find the tensions in the strings in positions OP (when RP was not cut) and OQ (when RP was cut). (Take $g = 10\ m/s^2$).*

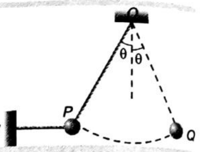


Fig. 5.124

Solution In the first case, ball is in equilibrium (permanent rest). Therefore, net force on the ball in any direction should be zero.

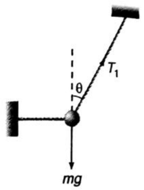


Fig. 5.125

$\therefore (\Sigma \vec{F})$ in vertical direction = 0

or $$T_1 \cos\theta = mg$$

or $$T_1 = \frac{mg}{\cos\theta}$$

Substituting $m_1 = 1\ \text{kg}, g = 10\ \text{m/s}^2$ and $\theta = 45°$.

we get, $$T_1 = 10\sqrt{2}\ \text{N}$$

Note *Here, we deliberately resolved all the forces in vertical direction because component of the tension in RP in vertical direction is zero. Although, since, the ball is in equilibrium, net force on it in any direction is zero. But in a direction other than vertical we will have to consider component of tension in RP also, which will unnecessarily increase the calculation.*

In the second case ball is not in equilibrium (temporary rest). After few seconds it will move in a direction perpendicular to OQ. Therefore, net force on the ball at Q is perpendicular to OQ, or net force along $OQ = 0$.

$\therefore$ $$T_2 = mg\cos\theta$$

Substituting the values, we get $$T_2 = 5\sqrt{2}\ \text{N}$$

Here, we can see that $$T_1 \neq T_2$$

Level 2

Example 1 *Two blocks of mass $m = 5$ kg and $M = 10$ kg are connected by a string passing over a pulley B as shown. Another string connects the centre of pulley B to the floor and passes over another pulley A as shown. An upward force F is applied at the centre of pulley A. Both the pulleys are massless. Find the acceleration of blocks m and M, if F is:*

(a) 100 N
(b) 300 N
(c) 500 N. (Take $g = 10\ m/s^2$)

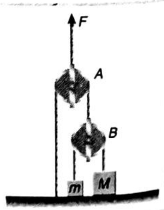


Fig. 5.126

Solution Let T_0 = tension in the string passing over A

T = tension in the string passing over B

$$2T_0 = F \quad \text{and} \quad 2T = T_0$$

$\Rightarrow$ $$T = F/4$$

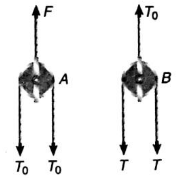


Fig. 5.127

(a) $$T = F/4 = 25\ \text{N}$$

weights of blocks are $mg = 50\ \text{N}$

$Mg = 100\ \text{N}$

As $T < mg$ and Mg both, the blocks will remain stationary on the floor.

(b) $$T = F/4 = 75\ \text{N}$$

As $T < Mg$ and $T > mg$, M will remain stationary on the floor, whereas m will move.

acceleration of m,

$$a = \frac{T - mg}{m} = \frac{75 - 50}{5} = 5\ \text{m/s}^2$$

(c) $$T = F/4 = 125\ \text{N}$$

As $T > mg$ and Mg, both the blocks will accelerate upwards.

Acceleration of m,

$$a_1 = \frac{T - mg}{m} = \frac{125 - 50}{5} = 15\ \text{m/s}^2$$

Acceleration of M,

$$a_2 = \frac{T - Mg}{M} = \frac{125 - 100}{10} = 2.5\ \text{m/s}^2$$

Example 2 *Consider the situation shown in figure the block B moves on a frictionless surface, while the coefficient of friction between A and the surface on which it moves is 0.2. Find the acceleration with which the masses move and also the tension in the strings. (Take $g = 10\ m/s^2$).*

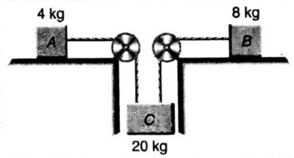


Fig. 5.128

Solution Let a be the acceleration with which the masses move and T_1 and T_2 be the tensions in left and right strings. Friction on mass A is $\mu mg = 8$ N. Then equations of motion of masses A, B and C are

For mass A $$T_1 - 8 = 4a \quad \text{...(i)}$$

For mass B $$T_2 = 8a \quad \text{...(ii)}$$

For mass C $$200 - T_1 - T_2 = 20a \quad \text{...(iii)}$$

Adding the above three equations, we get $32a = 192$

or $$a = 6\ \text{m/s}^2$$

From Eqs. (i) and (ii), we have $$T_2 = 48\ \text{N}$$

and $$T_1 = 32\ \text{N}$$

Example 3 *Three blocks of mass m_1, m_2 and m_3 are connected as shown in the figure. All the surfaces are frictionless and the string and the pulleys are light. Find the acceleration of m_1.*

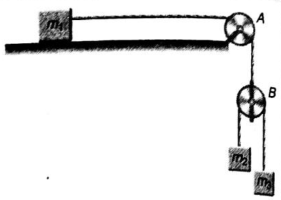


Fig. 5.129

Solution Suppose the acceleration of m_1 is a_0 towards right. The acceleration of pulley B will also be a_0 downwards because the string connecting m_1 and B is constant in length. Also the string connecting m_2 and m_3 has a constant length. This implies that the decrease in the separation between m_2 and B equals the increase in the separation between m_3 and B. So, the upward acceleration of m_2 with respect to B equals the downward acceleration of m_3 with respect to B. Let this acceleration be a_r.

The acceleration of m_2 with respect to the ground $= a_0 - a_r$ (downward) and the acceleration of m_3 with respect to the ground $= a_0 + a_r$ (downward).

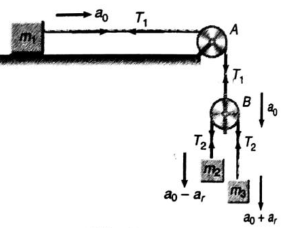


Fig. 5.130

Let the tension be T_1 in the upper string and T_2 in the lower string. Consider the motion of the pulley B.

The forces on this light pulley are

(a) T_1 upwards by the upper string and

(b) $2T_2$ downwards by the lower string.

As the mass of the pulley is negligible,

$$2T_2 - T_1 = 0 \quad \text{giving} \quad T_2 = \frac{T_1}{2} \qquad \text{...(i)}$$

Motion of m_1 : In the horizontal direction, the equation is

$$T_1 = m_1 a_0 \qquad \text{...(ii)}$$

Motion of m_2 : $\quad m_2 g - T_1/2 = m_2(a_0 - a_r) \qquad$ ...(iii)

Motion of m_3 : $\quad m_3 g - T_1/2 = m_3(a_0 + a_r) \qquad$ ...(iv)

Solving these four equations, we get

$$a_0 = \frac{g}{1 + \frac{m_1}{4}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)}$$

Example 4 *Two blocks A and B of mass 1 kg and 2 kg respectively are connected by a string, passing over a light frictionless pulley. Both the blocks are resting on a horizontal floor and the pulley is held such that string remains just taut.*

At moment $t = 0$, a force $F = 20t$ newton starts acting on the pulley along vertically upward direction as shown in figure. Calculate:

(a) velocity of A when B loses contact with the floor.

(b) height raised by the pulley upto that instant. (Take, $g = 10\ m/s^2$)

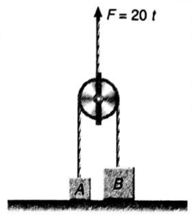


Fig. 5.131

Solution (a) Let T be the tension in the string. Then,

$$2T = 20t \quad \text{or} \quad T = 10t \text{ newton}$$

Let the block A loses its contact with the floor at time $t = t_1$. This happens when the tension in string becomes equal to the weight of A. Thus,

$$T = mg \quad \text{or} \quad 10t_1 = 1 \times 10 \quad \text{or} \quad t_1 = 1\text{s} \qquad \text{...(i)}$$

Similarly, for block B, we have

$$10t_2 = 2 \times 10 \quad \text{or} \quad t_2 = 2\text{s} \qquad \text{...(ii)}$$

i.e., the block B loses contact after 2 s. For block A, at time t such that $t \geq t_1$ let a be its acceleration in upward direction. Then,

$$10t - 1 \times 10 = 1 \times a = (dv/dt)$$

or $$dv = 10(t-1)\,dt \qquad \text{...(iii)}$$

Integrating this expression, we get

$$\int_0^v dv = 10\int_1^t (t-1)\,dt$$

or $$v = 5t^2 - 10t + 5 \qquad \text{...(iv)}$$

Substituting $t = t_2 = 2\text{s}$

$$v = 20 - 20 + 5 = 5\text{ m/s} \qquad \text{...(v)}$$

(b) From Eq. (iv), $$dy = (5t^2 - 10t + 5)\,dt \qquad \text{...(vi)}$$

Where y is the vertical displacement of block A at time t $(\geq t_1)$.

Integrating, we have

$$\int_{y=0}^{y=h} dy = \int_{t=1}^{t=2} (5t^2 - 10t + 5)\,dt$$

$$h = 5\left[\frac{t^3}{3}\right]_1^2 - 10\left[\frac{t^2}{2}\right]_1^2 + 5[t]_1^2 = \frac{5}{3}\text{ m}$$

$\therefore$ Height raised by pulley upto that instant $= \frac{h}{2} = \frac{5}{6}$ m

Example 5 *Find the acceleration of the body of mass m_2 in the arrangement shown in Fig. 5.132. If the mass m_2 is η times great as the mass m_1, and the angle that the inclined plane forms with the horizontal is equal to θ. The masses of the pulleys and threads, as well as the friction, are assumed to be negligible.*

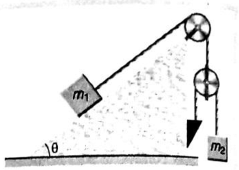


Fig. 5.132

Solution Here, by constraint relation we can see that the acceleration of m_2 is two times that of m_1. So, we assume if m_1 is moving up the inclined plane with an acceleration a, the acceleration of mass m_2 going down is $2a$. The tensions in different strings are shown in figure.

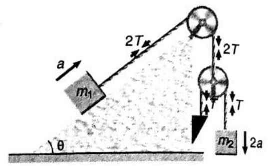


Fig. 5.133

The dynamic equations can be written as

For mass m_1 $\quad 2T - m_1 g \sin\theta = m_1 a \quad$ ...(i)

For mass m_2 $\quad m_2 g - T = m_2(2a) \quad$ ...(ii)

Substituting $m_2 = \eta m_1$ and solving Eqs. (i) and (ii), we get

Acceleration of $m_2 = 2a = \dfrac{2g(2\eta - \sin\theta)}{4\eta + 1}$

Example 6 *In the arrangement shown in Fig. 5.134 the mass of the ball is η times as great as that of the rod. The length of the rod is l, the masses of the pulleys and the threads, as well as the friction, are negligible. The ball is set on the same level as the lower end of the rod and then released. How soon will the ball be opposite the upper end of the rod?*

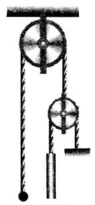


Fig. 5.134

Solution From constraint relation we can see that the acceleration of the rod is double than that of the acceleration of the ball. If ball is going up with an acceleration a, rod will be coming down with the acceleration $2a$, thus, the relative acceleration of the ball with respect to rod is $3a$ in upward direction. If it takes time t seconds to reach the upper end of the rod, we have

$$t = \sqrt{\frac{2l}{3a}} \qquad \text{...(i)}$$

Let mass of ball be m and that of rod is M, the dynamic equations of these are

For rod $\quad Mg - T = M(2a) \quad$ ...(ii)

For ball $\quad 2T - mg = ma \quad$ ...(iii)

Substituting $m = \eta M$ and solving Eqs. (ii) and (iii), we get

$$a = \left(\frac{2-\eta}{\eta+4}\right) g$$

From Eq. (i), we have $\quad t = \sqrt{\dfrac{2l(\eta+4)}{3g(2-\eta)}}$

Example 7 *Figure shows a small block A of mass m kept at the left end of a plank B of mass M = 2m and length l. The system can slide on a horizontal road. The system is started towards right with the initial velocity v. The friction coefficients between the road and the plank is 1/2 and that between the plank and the block is 1/4. Find:*

(a) the time elapsed before the block separates from the plank.

(b) displacement of block and plank relative to ground till that moment.

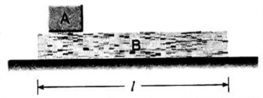


Fig. 5.135

Solution There will be relative motion between block and plank and plank and road. So at each surface limiting friction will act. The direction of friction forces at different surfaces are as shown in figure.

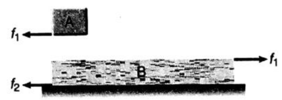


Fig. 5.136

Here, $\quad f_1 = \left(\dfrac{1}{4}\right)(mg)$

and $\quad f_2 = \left(\dfrac{1}{2}\right)(m + 2m)g = \left(\dfrac{3}{2}\right) mg$

Retardation of A is $\quad a_1 = \dfrac{f_1}{m} = \dfrac{g}{4}$

and retardation of B is $\quad a_2 = \dfrac{f_2 - f_1}{2m} = \dfrac{5}{8} g$

Since, $\quad a_2 > a_1$

Relative acceleration of A with respect to B is

$$a_r = a_2 - a_1 = \frac{3}{8} g$$

Initial velocity of both A and B is v. So, there is no relative initial velocity. Hence,

(a) Applying $\quad s = \dfrac{1}{2} at^2 \quad$ or $\quad l = \dfrac{1}{2} a_r t^2 = \dfrac{3}{16} gt^2$

$\therefore \quad t = 4\sqrt{\dfrac{l}{3g}}$

(b) Displacement of block $\quad s_A = u_A t - \dfrac{1}{2} a_A t^2$

or $\quad s_A = 4v\sqrt{\dfrac{l}{3g}} - \dfrac{1}{2} \cdot \dfrac{g}{4} \cdot \left(\dfrac{16l}{3g}\right) \qquad \left(a_A = a_1 = \dfrac{g}{4}\right)$

or $$s_A = 4v\sqrt{\frac{l}{3g}} - \frac{2}{3}l$$

Displacement of plank $$s_B = u_B t - \frac{1}{2}a_B t^2$$

or $$s_B = 4v\sqrt{\frac{l}{3g}} - \frac{1}{2}\left(\frac{5}{8}g\right)\left(\frac{16l}{3g}\right) \qquad \left(a_B = a_2 = \frac{5}{8}g\right)$$

or $$s_B = 4v\sqrt{\frac{l}{3g}} - \frac{5}{3}l$$ **Ans.**

Note *We can see that $s_A - s_B = l$. Which is quite obvious because block A has moved a distance l relative to plank.*

Example 8 *Two blocks A and B of mass 2 kg and 4 kg are placed one over the other as shown in figure. A time varying horizontal force $F = 2t$ is applied on the upper block as shown in figure. Here t is in second and F is in newton. Draw a graph showing accelerations of A and B on y-axis and time on x-axis. Coefficient of friction between A and B is $\mu = \frac{1}{2}$ and the horizontal surface over which B is placed is smooth. ($g = 10\ m/s^2$)*

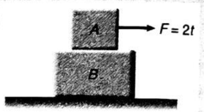


Fig. 5.137

Solution Limiting friction between A and B is

$$f_L = \mu m_A g = \left(\frac{1}{2}\right)(2)\ (10) = 10\,\text{N}$$

Block B moves due to friction only. Therefore, maximum acceleration of B can be

$$a_{\max} = \frac{f_L}{m_B} = \frac{10}{4} = 2.5\ \text{m/s}^2$$

Thus, both the blocks move together with same acceleration till the common acceleration becomes 2.5 m/s^2, after that acceleration of B will become constant while that of A will go on increasing. To find the time when the acceleration of both the blocks becomes 2.5 m/s^2 (or when slipping will start between A and B) we will write

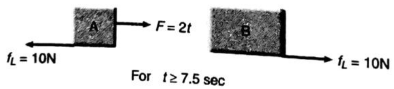


Fig. 5.138

$$2.5 = \frac{F}{(m_A + m_B)} = \frac{2t}{6}$$

$\therefore$ $$t = 7.5\text{s}$$

Hence, for $$t \leq 7.5\text{s}$$

$$a_A = a_B = \frac{F}{m_A + m_B} = \frac{2t}{6} = \frac{t}{3}$$

Thus, a_A *versus t* or a_B *versus t* graph is a straight line passing through origin of slope $\frac{1}{3}$.

For, $$t \geq 7.5\,\text{s}$$

$$a_B = 2.5\ \text{m/s}^2 = \text{constant}$$

and $$a_A = \frac{F - f_L}{m_A}$$

or $$a_A = \frac{2t - 10}{2} \quad \text{or} \quad a_A = t - 5$$

Thus, a_A *versus t* graph is a straight line of slope 1 and intercept –5. While a_B *versus t* graph is a straight line parallel to t axis. The corresponding graph is as shown in Fig. 5.139.

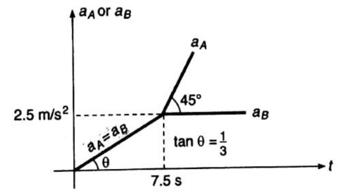


Fig. 5.139

EXERCISES

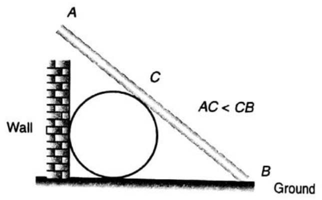
AIEEE Corner

Subjective Questions (Level 1)

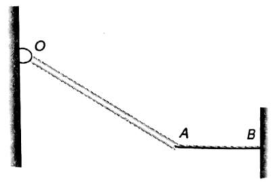
Free Body Diagram

1. A rod AB of weight W_1 is placed over a sphere of weight W_2 as shown in figure. Ground is rough and there is no friction between rod and sphere and sphere and wall. Draw free body diagrams of sphere and rod separately.

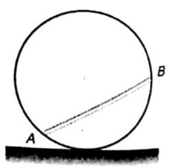
Note *No friction will act between sphere and ground, think why ?*



2. A rod OA is suspended with the help of a massless string AB as shown in figure. Rod is hinged at point O. Draw free body diagram of the rod.



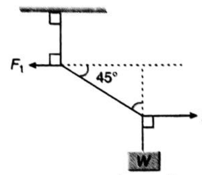
3. A rod AB is placed inside a rough spherical shell as shown in figure. Draw the free body diagram of the rod.



Equilibrium of Forces

(a) Concurrent forces

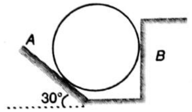
4. In figure the tension in the diagonal string is 60 N.



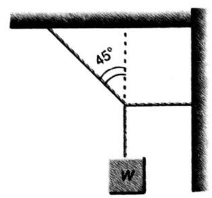
(a) Find the magnitude of the horizontal forces $\vec{F}_1$ and $\vec{F}_2$ that must be applied to hold the system in the position shown.

(b) What is the weight of the suspended block ?

5. The 50 kg homogeneous smooth sphere rests on the 30° incline A and against the smooth vertical wall B. Calculate the contact forces at A and B.

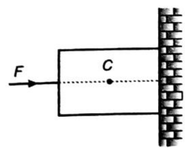


6. All the strings shown in figure are massless. Tension in the horizontal string is 30 N. Find W.

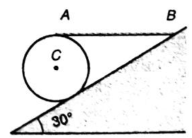


(b) Coplanar forces

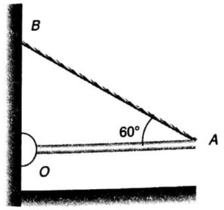
7. A cube of mass 2 kg is held stationary against a rough wall by a force $F = 40\text{N}$ passing through centre C. Find perpendicular distance of normal reaction between wall and cube from point C. Side of the cube is 20 cm. Take $g = 10\ \text{m/s}^2$.



8. A sphere of weight $W = 100\text{N}$ is kept stationary on a rough inclined plane by a horizontal string AB as shown in figure. Find :
(a) tension in the string,
(b) force of friction on the sphere and
(c) normal reaction on the sphere by the plane.



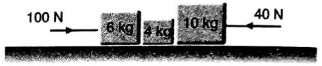
9. A rod OA of mass 4 kg is held in horizontal position by a massless string AB as shown in figure. Length of the rod is 2 m. Find :



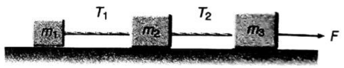
(a) tension in the string,
(b) net force exerted by hinge on the rod. ($g = 10\ \text{m/s}^2$)

Newton's Laws

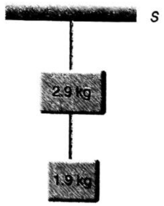
10. In the figure shown all surfaces are smooth. Find :
(a) acceleration of all the three blocks,
(b) net force on 6 kg, 4 kg and 10 kg blocks and
(c) force acting between 4 kg and 10 kg blocks.



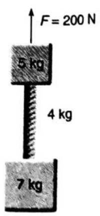
11. Three blocks $m_1 = 10\,\text{kg}$, $m_2 = 20\,\text{kg}$ and $m_3 = 30\,\text{kg}$ are on a smooth horizontal table, connected to each other by light horizontal strings. A horizontal force $F = 60$ N is applied to m_3, towards right. Find :
(a) tensions T_1 and T_2 and
(b) tension T_2 if all of a sudden the string between m_1 and m_2 snaps.



12. Two blocks of mass 2.9 kg and 1.9 kg are suspended from a rigid support S by two inextenible wires each of length 1 m, as shown in the figure. The upper wire has negligible mass and the lower wire has a uniformly distributed mass of 0.2 kg. The whole system of blocks, wires and support have an upward acceleration of $0.2\ \text{m/s}^2$. Acceleration due to gravity is $9.8\ \text{m/s}^2$.
(a) Find the tension at the mid-point of the lower wire.
(b) Find the tension at the mid-point of the upper wire.



13. Two blocks shown in figure are connected by a heavy uniform rope of mass 4 kg. An upward force of 200 N is applied as shown.
(a) What is the acceleration of the system ?
(b) What is the tension at the top of the rope ?
(c) What is the tension at the mid-point of the rope ?
(Take $g = 9.8\ \text{m/s}^2$)

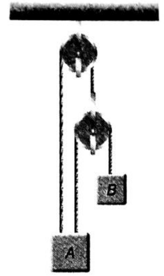


14. A 20 kg monkey has a firm hold on a light rope that passes over a frictionless pulley and is attached to a 20 kg bunch of bananas. The monkey looks upward, sees the bananas, and starts to climb the rope to get them.
(a) As the monkey climbs, do the bananas move up, move down, or remain at rest ?
(b) As the monkey climbs, does the distance between the monkey and the bananas decrease, increase, or remain constant ?
(c) The monkey releases her hold on the rope. What happens to the distance between the monkey and the bananas while she is falling ?
(d) Before reaching the ground, the monkey grabs the rope to stop her fall. What do the bananas do ?

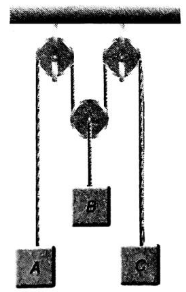
Constraint Equations

Note (Q. 15 to Q. 23) *Assume massless strings, massless and smooth pulleys.*

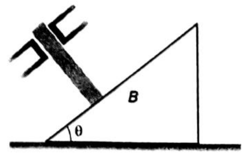
15. In the pulley-block arrangement shown in figure. Find the relation between acceleration of blocks A and B.



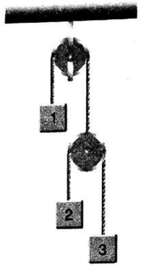
16. In the pulley-block arrangement shown in figure. Find relation between a_A, a_B and a_C.



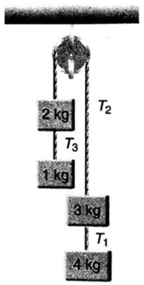
17. In the figure shown find relation between magnitudes of $\vec{a}_A$ and $\vec{a}_B$.



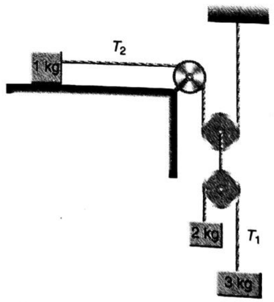
18. In the figure shown, $a_3 = 6\,\text{m/s}^2$ (downwards) and $a_2 = 4\,\text{m/s}^2$ (upwards). Find acceleration of 1.



19. In the figure shown, find acceleration of the system and tensions T_1, T_2 and T_3. (Take $g = 10\,\text{m/s}^2$)

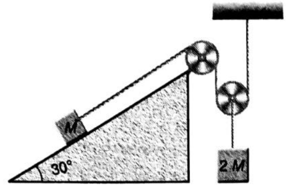


20. In the figure shown. Find : ($g = 10\,\text{m/s}^2$)

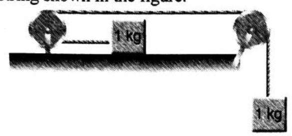


(a) acceleration of 1 kg, 2 kg and 3 kg blocks and
(b) tensions T_1 and T_2.

21. Find the acceleration of the block of mass M in the situation shown in the figure. All the surfaces are frictionless.

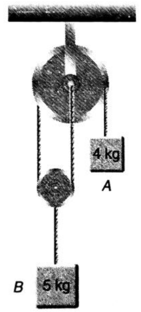


22. Calculate the tension in the string shown in the figure.



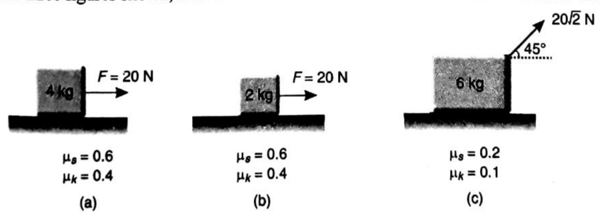
All the surfaces are frictionless. (Take $g = 10\,\text{m/s}^2$)

23. Find the acceleration of the blocks A and B in the situation shown in the figure.

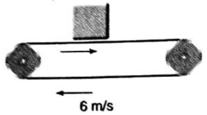


Friction

24. In the three figures shown, find acceleration of block and force of friction on it in each case.



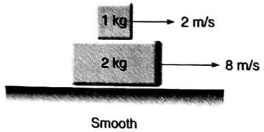
25. A conveyor belt is moving with constant speed of 6 m/s. A small block is just dropped on it. Coefficient of friction between the two is $\mu = 0.3$. Find :



(a) The time when relative motion between them will stop.
(b) Displacement of block upto that instant. ($g = 10 \text{m/s}^2$).

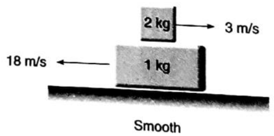
Note (*Q.* 26 *and Q.* 27) : *Assume that lower block is very long.*

26. Coefficient of friction between two blocks shown in figure is $\mu = 0.4$. The blocks are given velocities of 2 m/s and 8 m/s in the directions shown in figure. Find :



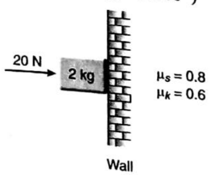
(a) The time when relative motion between them will stop.
(b) The common velocities of blocks upto that instant.
(c) Displacements of 1 kg and 2 kg blocks upto that instant. ($g = 10 \text{m/s}^2$)

27. Coefficient of friction between two blocks shown in figure is $\mu = 0.6$. The blocks are given velocities in the directions shown in figure. Find :

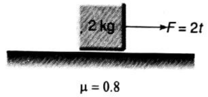


(a) Time when relative motion between them is stopped.
(b) The common velocity of the two blocks.
(c) The displacements of 1 kg and 2 kg blocks upto that instant. (Take $g = 10 \text{m/s}^2$)

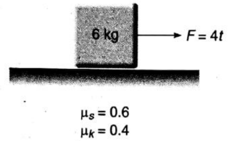
28. A 2 kg block is pressed against a rough wall by a force $F = 20$N as shown in figure. Find acceleration of the block and force of friction acting on it. (Take $g = 10 \text{m/s}^2$)



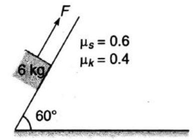
29. A 2 kg block is kept over a rough ground with coefficient of friction $\mu = 0.8$ as shown in figure. A time varying force $F = 2t$ (F in newton and t in second) is applied on the block. Plot a graph between acceleration of block *versus* time. ($g = 10 \text{m/s}^2$)



30. A 6 kg block is kept over a rough surface with coefficients of friction $\mu_s = 0.6$ and $\mu_k = 0.4$ as shown in figure. A time varying force $F = 4t$ (F in newton and t in second) is applied on the block as shown. Plot a graph between acceleration of block and time. (Take $g = 10 \text{m/s}^2$)



31. A 6 kg block is kept on an inclined rough surface as shown in figure. Find the force F required to :
(a) keep the block stationary,
(b) move the block downwards with constant velocity and
(c) move the block upwards with an acceleration of 4 m/s^2.
(Take $g = 10 \text{m/s}^2$)

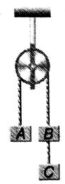


Objective Questions (Level 1)

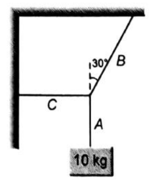
Single Correct Option

1. Two balls A and B of same size are dropped from the same point under gravity. The mass of A is greater than that of B. If the air resistance acting on each ball is same, then
(a) both the balls reach the ground simultaneously
(b) the ball A reaches earlier
(c) the ball B reaches earlier
(d) nothing can be said

2. Three equal weights A, B and C of mass 2 kg each are hanging on a string passing over a fixed frictionless pulley as shown in the figure. The tension in the string connecting weights B and C is
(a) zero (b) 13 N
(c) 3.3 N (d) 19.6 N



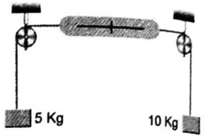
3. In a figure a block of mass 10 kg is in equilibrium. Identify the string in which the tension is zero.
(a) B
(b) C
(c) A
(d) None of the above



4. At what minimum acceleration should a monkey should slide a rope whose breaking strength $\frac{2}{3}$rd of its weight ?

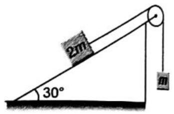
(a) $\frac{2g}{3}$ (b) g (c) $\frac{g}{3}$ (d) zero

5. For the arrangement shown in the figure the reading of spring balance is



(a) 50 N
(b) 100 N
(c) 150 N
(d) None of the above

6. For the arrangement shown in figure, the tension in the string is given by



(a) $\frac{mg}{2}$ (b) $\frac{3}{2}mg$
(c) mg (d) $2\,mg$

7. The time taken by a body to slide down a rough 45° inclined plane is twice that required to slide down a smooth 45° inclined plane. The coefficient of kinetic friction between the object and rough plane is given by

(a) $\frac{1}{3}$ (b) $\frac{3}{4}$ (c) $\sqrt{\frac{3}{4}}$ (d) $\sqrt{\frac{2}{3}}$

8. The force required to just move a body up the inclined plane is double the force required to just prevent the body from sliding down the plane. The coefficient of friction is μ. If θ is the angle of inclination of the plane than $\tan\theta$ is equal to

(a) μ (b) 3μ (c) 2μ (d) $0.5\,\mu$

9. A block of mass m is placed at rest on an inclined plane of inclination θ to the horizontal. If the coefficient of friction between the block and the plane is μ, then the total force the inclined plane exerts on the block is

(a) mg (b) $\mu\, mg\cos\theta$ (c) $mg\sin\theta$ (d) $\mu\, mg\tan\theta$

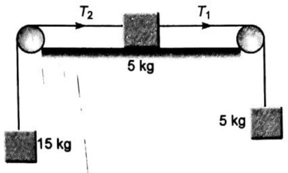
10. A force F_1 accelerates a particle from rest to a velocity v. Another force F_2 decelerates the same particle from v to rest, then

(a) F_1 is always equal to F_2
(b) F_2 is greater than F_1
(c) F_2 may be smaller than, greater than or equal to F_1
(d) F_2 cannot be equal to F_1

11. A particle is placed at rest inside a hollow hemisphere of radius R. The coefficient of friction between the particle and the hemisphere is $\mu = \frac{1}{\sqrt{3}}$. The maximum height up to which the particle can remain stationary is

(a) $\frac{R}{2}$ (b) $\left(1-\frac{\sqrt{3}}{2}\right)R$ (c) $\frac{\sqrt{3}}{2}R$ (d) $\frac{3R}{8}$

12. In the figure shown, the frictional coefficient between table and block is 0.2. Find the ratio of tensions in the right and left strings.



(a) 17 : 24 (b) 34 : 12 (c) 2 : 3 (d) 3 : 2

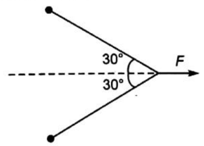
13. A smooth inclined plane of length L having inclination θ with the horizontal is inside a lift which is moving down with a retardation a. The time taken by a body to slide down the inclined plane from rest will be.

(a) $\sqrt{\frac{2L}{(g+a)\sin\theta}}$ (b) $\sqrt{\frac{2L}{(g-a)\sin\theta}}$ (c) $\sqrt{\frac{2L}{a\sin\theta}}$ (d) $\sqrt{\frac{2L}{g\sin\theta}}$

14. A block rests on a rough inclined plane making an angle of 30° with horizontal. The coefficient of static friction between the block and inclined plane is 0.8. If the frictional force on the block is 10N, the mass of the block in kg is ($g = 10\,\text{m/s}^2$)

(a) 2.0 (b) 4.0 (c) 1.6 (d) 2.5

15. In figure two identical particles each of mass m are tied together with an inextensible string. This is pulled at its centre with a constant force F. If the whole system lies on a smooth horizontal plane, then the acceleration of each particle towards each other is



(a) $\frac{\sqrt{3}}{2}\frac{F}{m}$ (b) $\frac{1}{2\sqrt{3}}\frac{F}{m}$ (c) $\frac{2}{\sqrt{3}}\frac{F}{m}$ (d) $\sqrt{3}\frac{F}{m}$

16. A block of mass m is placed at rest on a horizontal rough surface with angle of friction ϕ. The block is pulled with a force F at an angle θ with the horizontal. The minimum value of F required to move the block is

(a) $\frac{mg\sin\phi}{\cos(\theta-\phi)}$ (b) $\frac{mg\cos\phi}{\cos(\theta-\phi)}$ (c) $mg\tan\phi$ (d) $mg\sin\phi$

17. A block of mass 4 kg is placed on a rough horizontal plane. A time dependent horizontal force $F = kt$ acts on the block, $k = 2\,\text{Ns}^{-1}$. The frictional force between the block and plane at time $t = 2$ s is ($\mu = 0.2$)

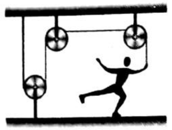
(a) 4 N (b) 8 N (c) 12 N (d) 10 N

18. A body takes times t to reach the bottom of an inclined plane of angle θ with the horizontal. If the plane is made rough, time taken now is $2t$. The coefficient of friction of the rough surface is

(a) $\frac{3}{4}\tan\theta$ (b) $\frac{2}{3}\tan\theta$ (c) $\frac{1}{4}\tan\theta$ (d) $\frac{1}{2}\tan\theta$

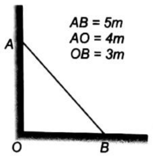
19. A man of mass m slides down along a rope which is connected to the ceiling of an elevator with deceleration a relative to the rope. If the elevator is going upward with an acceleration a relative to the ground, then tension in the rope is
(a) mg (b) $m(g+2a)$ (c) $m(g+a)$ (d) zero

20. A 50 kg person stands on a 25kg platform. He pulls on the rope which is attached to the platform *via* the frictionless pulleys as shown in the figure. The platform moves upwards at a steady rate if the force with which the person pulls the rope is



(a) 500 N (b) 250 N (c) 25 N (d) None of these

21. A ladder of length 5 m is placed against a smooth wall as shown in figure. The coefficient of friction is μ between ladder and ground. What is the minimum value of μ, if the ladder is not to slip?



(a) $\mu = \frac{1}{2}$ (b) $\mu = \frac{1}{4}$ (c) $\mu = \frac{3}{8}$ (d) $\mu = \frac{5}{8}$

22. If a ladder weighing 250 N is placed against a smooth vertical wall having coefficient of friction between it and floor 0.3, then what is the maximum force of friction available at the point of contact between the ladder and the floor?
(a) 75 N (b) 50 N (c) 35 N (d) 25 N

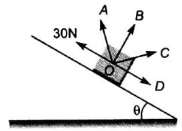
23. A rope of length L and mass M is being pulled on a rough horizontal floor by a constant horizontal force $F = Mg$. The force is acting at one end of the rope in the same direction as the length of the rope. The coefficient of kinetic friction between rope and floor is 1/2. Then, the tension at the midpoint of the rope is
(a) $\frac{Mg}{4}$ (b) $\frac{2Mg}{5}$ (c) $\frac{Mg}{8}$ (d) $\frac{Mg}{2}$

24. A heavy body of mass 25 kg is to be dragged along a horizontal plane $\left(\mu = \frac{1}{\sqrt{3}}\right)$. The least force required is
(a) 25 kgf (b) 2.5 kgf (c) 12.5 kgf (d) 6.25 kgf

25. A block A of mass 4 kg is kept on ground. The coefficient of friction between the block and the ground is 0.8. The external force of magnitude 30 N is applied parallel to the ground. The resultant force exerted by the ground on the block is
(a) 40 N (b) 30 N (c) zero (d) 50 N

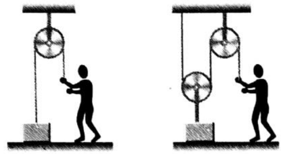
26. A block A of mass 2 kg rests on another block B of mass 8 kg which rests on a horizontal floor. The coefficient of friction between A and B is 0.2 while that between B and floor is 0.5. When a horizontal force F of 25 N is applied on the block B, the force of friction between A and B is
(a) 3 N (b) 4 N (c) 2 N (d) zero

27. A body of mass 10 kg lies on a rough inclined plane of inclination $\theta = \sin^{-1}\left(\frac{3}{5}\right)$ with the horizontal. When the force of 30 N is applied on the block parallel to and upward the plane, the total force by the plane on the block is nearly along



(a) OA (b) OB
(c) OC (d) OD

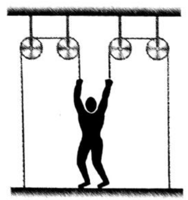
28. In the figure shown, a person wants to raise a block lying on the ground to a height h. In which case he has to exert more force. Assume pulleys and strings are light



(i) (ii)

(a) Fig. (i) (b) Fig. (ii)
(c) Same in both (d) Cannot be determined

29. A man of mass m stands on a platform of equal mass m and pulls himself by two ropes passing over pulleys as shown in figure. If he pulls each rope with a force equal to half his weight, his upward acceleration would be



(a) $\frac{g}{2}$
(b) $\frac{g}{4}$
(c) g
(d) zero

30. A varying horizontal force $F = at$ acts on a block of mass m kept on a smooth horizontal surface. An identical block is kept on the first block. The coefficient of friction between the blocks is μ. The time after which the relative sliding between the blocks prevails is
(a) $\frac{2mg}{a}$ (b) $\frac{2\mu mg}{a}$ (c) $\frac{\mu mg}{a}$ (d) $2\mu mga$

31. Two particles start together from a point O and slide down along straight smooth wires inclined at 30° and 60° to the vertical plane and on the same side of vertical through O. The relative acceleration of second with respect to first will be of magnitude
(a) $\frac{g}{2}$ (b) $\frac{\sqrt{3}g}{2}$ (c) $\frac{g}{\sqrt{3}}$ (d) g

JEE Corner

Assertion and Reason

Directions : *Choose the correct option.*

(a) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*

(b) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*

(c) *If **Assertion** is true, but the **Reason** is false.*

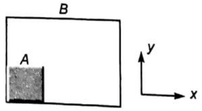
(d) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** If net force on a rigid body in zero, it is either at rest or moving with a constant linear velocity. Nothing else can happen.
Reason : Constant velocity means linear acceleration is zero.

2. **Assertion :** Three concurrent forces are $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$. Angle between $\vec{F}_1$ and $\vec{F}_2$ is 30° and between $\vec{F}_1$ and $\vec{F}_3$ is 120°. Under these conditions, forces cannot remain in equilibrium.
Reason : At least one angle should be greater than 180°.

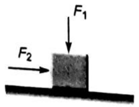
3. **Assertion :** Two identical blocks are placed over a rough inclined plane. One block is given an upward velocity to the block and the other in downward direction. If $\mu = \frac{1}{3}$ and $\theta = 45°$ the ratio of magnitudes of accelerations of two is 2 : 1.
Reason : The desired ratio is $\frac{1+\mu}{1-\mu}$.

4. **Assertion :** A block A is just placed inside a smooth box B as shown in figure. Now, the box is given an acceleration $\vec{a} = (3\hat{j} - 2\hat{i})\,ms^{-2}$. Under this acceleration block A can not remain in the position shown.

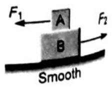


Reason : Block will require $m\vec{a}$ force for moving with acceleration $\vec{a}$.

5. **Assertion :** A block is kept at rest on a rough ground as shown. Two forces F_1 and F_2 are acting on it. If we increase either of the two forces F_1 or F_2, force of friction will increase.
Reason : By increasing F_1, normal reaction from ground will increase.



6. **Assertion :** In the figure shown force of friction on A from B will be right wards.
Reason : Friction always opposes the relative motion between two bodies in contact.



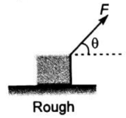
7. **Assertion :** In the figure shown tension in string AB always lies between $2m_1g$ and $2m_2g$. $(m_1 \neq m_2)$

Reason : Tension in massless string is uniform throughout.

8. **Assertion :** Two frames S_1 and S_2 are noninertial. Then frame S_2 when observed from S_1 is also noninertial.
Reason : A frame in motion is not necessarily a non-inertial frame.

9. **Assertion :** Moment of concurrent forces about any point is constant.
Reason : If vector sum of all the concurrent forces is zero, then moment of all the forces about any point is also zero.

10. **Assertion :** Minimum force is needed to move a block on rough surface, if θ = angle of friction.
Reason : Angle of friction and angle of repose are numerically same.

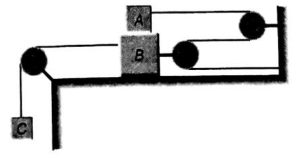


11. **Assertion :** When a person walks on a rough surface, the frictional force exerted by surface on the person is opposite to the direction of his motion.
Reason : It is the force exerted by the road on the person that causes the motion.

Objective Questions (Level 2)

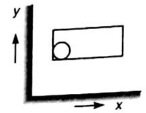
Single Correct Option

1. What is the largest mass of C in kg that can be suspended without moving blocks A and B ? The static coefficient of friction for all plane surface of contact is 0.3. Mass of block A is 50kg and block B is 70kg. Neglect friction in the pulleys



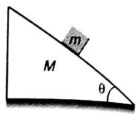
(a) 120 kg (b) 92 kg (c) 81 kg (d) None of these

2. A sphere of mass 1 kg rests at one corner of a cube. The cube is moved with a velocity $\vec{v} = 8t\,\hat{i} - 2t^2\,\hat{j}$. Where t is time in second. The force by sphere on the cube at $t = 1$ s is ($g = 10\,ms^{-2}$) [Figure shows vertical plane of the cube]



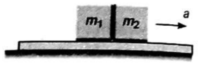
(a) 8 N (b) 10 N (c) 20 N (d) 6 N

3. A smooth block of mass m is held stationary on a smooth wedge of mass M and inclination θ as shown in figure. If the system is released from rest, then the normal reaction between the block and the wedge is



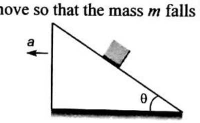
(a) $mg \cos \theta$
(b) less than $mg \cos \theta$
(c) greater than $mg \cos \theta$
(d) may be less or greater than $mg \cos \theta$ depending upon whether m is less or greater than m

4. Two blocks of masses m_1 and m_2 are placed in contact with each other on a horizontal platform as shown in figure. The coefficient of friction between m_1 and platform is 2μ and that between block m_2 and platform is μ. The platform moves with an acceleration a. The normal reaction between the blocks is



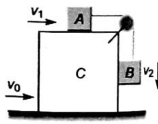
(a) zero in all cases (b) zero only if $m_1 = m_2$
(c) non zero only if $a > 2\mu g$ (d) non zero only if $a > \mu g$

5. A block of mass m is resting on a wedge of angle θ as shown in the figure. With what minimum acceleration a should the wedge move so that the mass m falls freely?



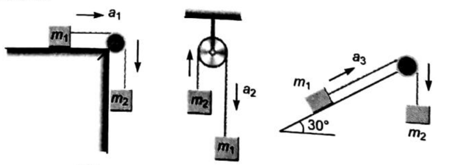
(a) g (b) $g \cos \theta$ (c) $g \cot \theta$ (d) $g \tan \theta$

6. To a ground observer the block C is moving with v_0 and the blocks A and B are moving with v_1 and v_2 relative to C as shown in the figure. Identify the correct statement.



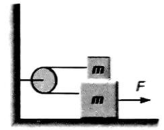
(a) $v_1 - v_2 = v_0$ (b) $v_1 = v_2$
(c) $v_1 + v_0 = v_2$ (d) None of these

7. In each case $m_1 = 4\,\text{kg}$ and $m_2 = 3\,\text{kg}$. If a_1, a_2 and a_3 are the respective accelerations of the block m_1 in given situations, then



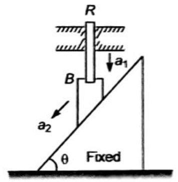
(a) $a_1 > a_2 > a_3$ (b) $a_1 > a_2 = a_3$ (c) $a_1 = a_2 = a_3$ (d) $a_1 > a_3 > a_2$

8. For the arrangement shown in figure the coefficient of friction between the two blocks is μ. If both the blocks are identical, then the acceleration of each block is



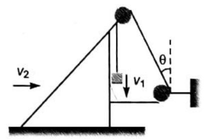
(a) $\dfrac{F}{2m} - 2\mu g$ (b) $\dfrac{F}{2m}$
(c) $\dfrac{F}{2m} - \mu g$ (d) zero

9. In the arrangement shown in the figure the rod R is restricted to move in the vertical direction with acceleration a_1, and the block B can slide down the fixed wedge with acceleration a_2. The correct relation between a_1 and a_2 is given by



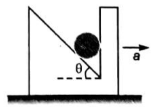
(a) $a_2 = a_1 \sin \theta$
(b) $a_2 \sin \theta = a_1$
(c) $a_2 \cos \theta = a_1$
(d) $a_2 = a_1 \cos \theta$

10. In figure the block moves downwards with velocity v_1, the wedge moves rightwards with velocity v_2. The correct relation between v_1 and v_2 is



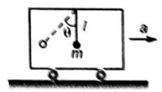
(a) $v_2 = v_1$ (b) $v_2 = v_1 \sin \theta$ (c) $2v_2 \sin \theta = v_1$ (d) $v_2 (1 + \sin \theta) = v_1$

11. In the figure, the minimum value of a at which the cylinder starts rising up the inclined surface is



(a) $g \tan \theta$ (b) $g \cot \theta$ (c) $g \sin \theta$ (d) $g \cos \theta$

12. When the trolley shown in figure is given a horizontal acceleration a, the pendulum bob of mass m gets deflected to a maximum angle θ with the vertical. At the position of maximum deflection, the net acceleration of the bob with respect to trolley is

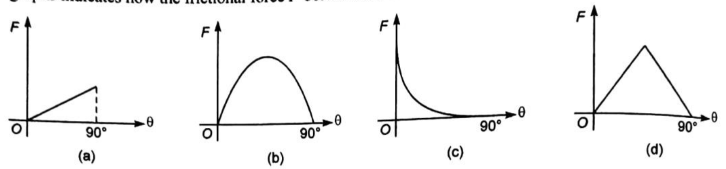


(a) $\sqrt{g^2 + a^2}$ (b) $a \cos \theta$
(c) $g \sin \theta - a \cos \theta$ (d) $a \sin \theta$

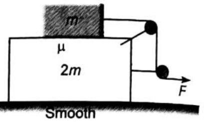
13. In the arrangement shown in figure the mass M is very heavy compared to m ($M >> m$). The tension T in the string suspended from the ceiling is

(a) 4 mg
(b) 2 mg
(c) zero
(d) None of the above

14. A block rests on a rough plane whose inclination θ to the horizontal can be varied. Which of the following graphs indicates how the frictional force F between the block and the plane varies as θ is increased?



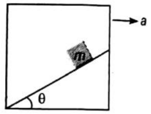
15. The minimum value of μ between the two blocks for no slipping is



(a) $\dfrac{F}{mg}$ (b) $\dfrac{F}{3mg}$

(c) $\dfrac{2F}{3mg}$ (d) $\dfrac{4F}{3mg}$

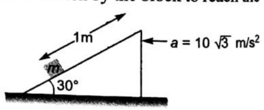
16. A block is sliding along incline as shown in figure. If the acceleration of chamber is a as shown in the figure. The time required to cover a distance L along incline is



(a) $\sqrt{\dfrac{2L}{g\sin\theta - a\cos\theta}}$ (b) $\sqrt{\dfrac{2L}{g\sin\theta + a\sin\theta}}$

(c) $\sqrt{\dfrac{2L}{g\sin\theta + a\cos\theta}}$ (d) $\sqrt{\dfrac{2L}{g\sin\theta}}$

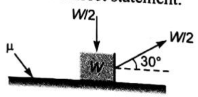
17. In the figure, the wedge is pushed with an acceleration of $10\sqrt{3}$ m/s^2. It is seen that the block starts climbing up on the smooth inclined face of wedge. What will be the time taken by the block to reach the top?



(a) $\dfrac{2}{\sqrt{5}}$ s (b) $\dfrac{1}{\sqrt{5}}$ s

(c) $\sqrt{5}$ s (d) $\dfrac{\sqrt{5}}{2}$ s

18. A block of weight W is kept on a rough horizontal surface (friction coefficient μ). Two forces $\dfrac{W}{2}$ each are applied as shown in the figure. Choose the correct statement.



(a) For $\mu > \dfrac{\sqrt{3}}{5}$ block will move

(b) For $\mu < \dfrac{\sqrt{3}}{5}$, work done by friction force is zero (in ground frame)

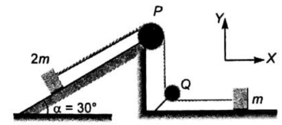
(c) For $\mu > \dfrac{\sqrt{3}}{5}$, friction force will do positive work (in ground frame)

(d) For $\mu < \dfrac{\sqrt{3}}{5}$ block will move

19. Two blocks A and B are separated by some distance and tied by a string as shown in the figure. The force of friction in both the blocks at $t = 2$ s is

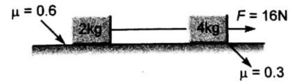
(a) 4 N (→), 5 N(←) (b) 2 N(→), 5 N(←) (c) 0 N(→), 10 N(←) (d) 1 N(←), 10 N(←)

20. All the surfaces and pulleys are frictionless in the shown arrangement. Pulleys P and Q are massless. The force applied by clamp on pulley P is



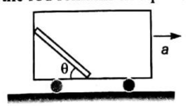
(a) $\dfrac{mg}{6}(-\sqrt{3}\,\hat{i} - 3\,\hat{j})$ (b) $\dfrac{mg}{6}(\sqrt{3}\,\hat{i} + 3\,\hat{j})$ (c) $\dfrac{mg}{6}\sqrt{2}$ (d) None of these

21. Two blocks of masses 2 kg and 4 kg are connected by a light string and kept on horizontal surface. A force of 16 N is acted on 4kg block horizontally as shown in figure. Besides it is given that coefficient of friction between 4 kg and ground is 0.3 and between 2kg block and ground is 0.6. Then frictional force between 2 kg block and ground is



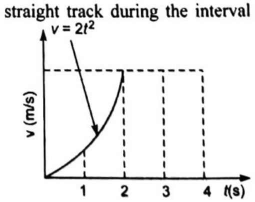
(a) 12 N (b) 6 N (c) 4 N (d) zero

22. A smooth rod of length l is kept inside a trolley at an angle θ as shown in the figure. What should be the acceleration a of the trolley so that the rod remains in equilibrium with respect to it?



(a) $g\tan\theta$ (b) $g\cos\theta$ (c) $g\sin\theta$ (d) $g\cot\theta$

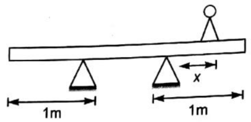
23. A car begins from rest at time $t = 0$, and then accelerates along a straight track during the interval $0 < t \le 2$s and thereafter with constant velocity as shown in the graph. A coin is initially at rest on the floor of the car. At $t = 1$ s, the coin begins to slip and its stops slipping at $t = 3$ s. The coefficient of static friction between the floor and the coin is ($g = 10$ m/s^2)



(a) 0.2 (b) 0.3

(c) 0.4 (d) 0.5

24. A horizontal plank is 10.0 m long with uniform density and mass 10kg. It rests on two supports which are placed 1.0 m from each end as shown in the figure. A man of mass 80kg can stand upto distance x on the plank without causing it to tip. The value of x is



(a) $\frac{1}{2}$ m (b) $\frac{1}{4}$ m (c) $\frac{3}{4}$ m (d) $\frac{1}{8}$ m

25. A block is kept on a smooth inclined plane of angle of inclination θ that moves with a constant acceleration so that the block does not slide relative to the inclined plane. If the inclined plane stops, the normal contact force offered by the plane on the block changes by a factor

(a) $\tan\theta$ (b) $\tan^2\theta$ (c) $\cos^2\theta$ (d) $\cot\theta$

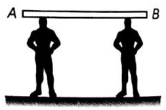
26. A uniform cube of mass m and side a is resting in equilibrium on a rough 45° inclined surface. The distance of the point of application of normal reaction measured from the lower edge of the cube is

(a) zero (b) $\frac{a}{3}$ (c) $\frac{a}{\sqrt{2}}$ (d) $\frac{a}{4}$

27. A horizontal force $F = \frac{mg}{3}$ is applied on the upper surface of a uniform cube of mass m and side a which is resting on a rough horizontal surface having $\mu = \frac{1}{2}$. The distance between lines of action of mg and normal reaction is

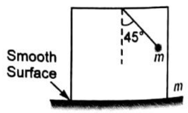
(a) $\frac{a}{2}$ (b) $\frac{a}{3}$ (c) $\frac{a}{4}$ (d) None of these

28. Two persons of equal height are carrying a long uniform wooden plank of length l. They are at distance $\frac{l}{4}$ and $\frac{l}{6}$ from nearest end of the rod. The ratio of normal reaction at their heads is



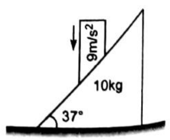
(a) 2 : 3 (b) 1 : 3
(c) 4 : 3 (d) 1 : 2

29. A ball connected with string is released at an angle 45° with the vertical as shown in the figure. Then the acceleration of the box at this instant will be (mass of the box is equal to mass of ball)



(a) $\frac{g}{4}$ (b) $\frac{g}{3}$
(c) $\frac{g}{2}$ (d) g

30. In the system shown in figure all surfaces are smooth. Rod is moved by external agent with acceleration 9 ms^{-2} vertically downwards. Force exerted on the rod by the wedge will be



(a) 120 N (b) 200 N
(c) 160 N (d) 180 N

31. A thin rod of length 1 m is fixed in a vertical position inside a train, which is moving horizontally with constant acceleration 4 ms^{-2}. A bead can slide on the rod and friction coefficient between them is 0.5. If the bead is released from rest at the top of the rod, it will reach the bottom in

(a) $\sqrt{2}$ s (b) 1 s (c) 2 s (d) 0.5 s

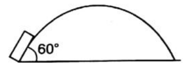
32. Three solid hemispheres of radii R each, are placed in contact with each other with their flat faces on a rough horizontal surface. A sphere of mass m and radius R is placed symmetrically on top of them. The normal reaction between the top sphere and any hemisphere assuming the system to be in static equilibrium is

(a) $\frac{mg}{3}$ (b) $\frac{mg}{\sqrt{6}}$ (c) $\frac{mg}{\sqrt{3}}$ (d) None of these

33. Mr. X of mass 80 kg enters a lift and selects the floor he wants. The lift now accelerates upwards at 2 ms^{-2} for 2 s and then moves with constant velocity. As the lift approaches his floor, it decelerates at the same rate as it previously accelerates. If the lift cables can safely withstand a tension of 2×10^4 N and the lift itself has a mass of 500 kg, how many Mr.X's could it safely carry at one time?

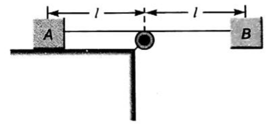
(a) 22 (b) 14 (c) 18 (d) 12

34. A particle when projected in vertical plane moves along smooth surface with initial velocity 20 ms^{-1} at an angle of 60°, so that its normal reaction on the surface remains zero throughout the motion. Then the slope of the tangent to the surface at height 5m from the point of projection will be



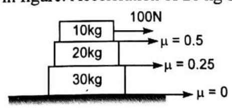
(a) 30° (b) 45° (c) $\tan^{-1} 2$ (d) None of these

35. Two blocks A and B, each of same mass are attached by a thin inextensible string through an ideal pulley. Initially block B is held in position as shown in figure. Now, the block B is released. Block A will slide to right and hit the pulley in time t_A. Block B will swing and hit the surface in time t_B. Assume the surface as frictionless, then



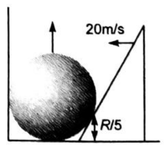
(a) $t_A > t_B$ (b) $t_A < t_B$
(c) $t_A = t_B$ (d) data insufficient

36. Three blocks are kept as shown in figure. Acceleration of 20 kg block with respect to ground is



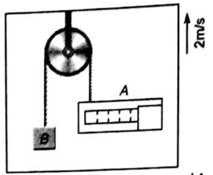
(a) 5 ms^{-2} (b) 2 ms^{-2} (c) 1 ms^{-2} (d) None of these

37. A sphere of radius R is in contact with a wedge. The point of contact is $\frac{R}{5}$ from the ground as shown in the figure. Wedge is moving with velocity 20 ms^{-1} towards left then the velocity of the sphere at this instant will be



(a) 20 ms^{-1} (b) 15 ms^{-1}
(c) 16 ms^{-1} (d) 12 ms^{-1}

38. In the figure it is shown that the velocity of lift is 2 ms^{-1} while string is winding on the motor shaft with velocity 2 ms^{-1} and shaft A is moving downward with velocity 2 ms^{-1} with respect to lift, then find out the velocity of block B

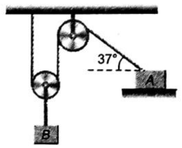


(a) 2 $ms^{-1}\uparrow$ (b) 2 $ms^{-1}\downarrow$ (c) 4 $ms^{-1}\uparrow$ (d) None of these

39. A monkey pulls the midpoint of a 10 cm long light inextensible string connecting two identical objects A and B lying on smooth table of masses 0.3 kg continuously along the perpendicular bisector of line joining the masses. The masses are found to approach each other at a relative acceleration of 5 ms^{-2} when they are 6 cm apart. The constant force applied by monkey is

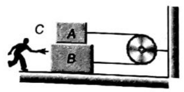
(a) 4 N (b) 2 N (c) 3 N (d) None of these

40. In the figure shown the block B moves with the velocity 10 ms^{-1}. The velocity of A in the position shown is



(a) 12.5 ms^{-1} (b) 25 ms^{-1} (c) 8 ms^{-1} (d) 16 ms^{-1}

41. In the figure $m_A = m_B = m_c = 60$ kg. The coefficient of friction between C and ground is 0.5, B and ground is 0.3, A and B is 0.4. C is pulling the string with the maximum possible force without moving. Then the tension in the string connected to A will be



(a) 120 N (b) 60 N (c) 100 N (d) zero

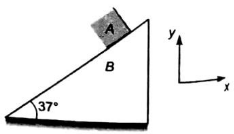
42. In the figure shown the acceleration of A is $\vec{a}_A = 15\hat{i} + 15\hat{j}$. Then the acceleration of B is (A remains in contact with B)

(a) $6\hat{i}$

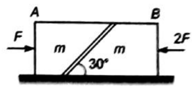
(b) $-15\hat{i}$

(c) $-10\hat{i}$

(d) $-5\hat{i}$

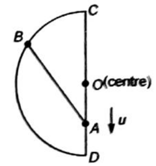


43. Two blocks A and B each of mass m are placed on a smooth horizontal surface. Two horizontal forces F and $2F$ are applied on the blocks A and B respectively as shown in figure. The block A does not slide on block B. Then the normal reaction acting between the two blocks is



(a) F (b) $\frac{F}{2}$ (c) $\frac{F}{\sqrt{3}}$ (d) $3F$

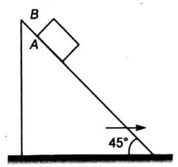
44. Two beads A and B move along a semicircular wire frame as shown in figure. The beads are connected by an inelastic string which always remains tight. At an instant the speed of A is u, $\angle BAC = 45°$ and $BOC = 75°$, where O is the centre of the semicircular arc. The speed of bead B at that instant is



(a) $\sqrt{2}u$ (b) u

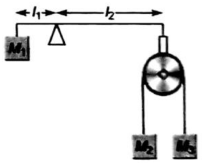
(c) $\frac{u}{2\sqrt{2}}$ (d) $\sqrt{\frac{2}{3}}u$

45. If the coefficient of friction between A and B is μ, the maximum acceleration of the wedge A for which B will remain at rest with respect to the wedge is



(a) μg (b) $g\left(\frac{1+\mu}{1-\mu}\right)$ (c) $g\left(\frac{1-\mu}{1+\mu}\right)$ (d) $\frac{g}{\mu}$

46. A pivoted beam of negligible mass has a mass suspended from one end and an Atwood's machine suspended from the other. The frictionless pulley has negligible mass and dimension. Gravity is directed downward and $M_2 = 3M_3$, $l_2 = 3l_1$. Find the ratio $\frac{M_1}{M_2}$ which will ensure that the beam has no tendency to rotate just after the masses are released



(a) $\frac{M_1}{M_2} = 2$ (b) $\frac{M_1}{M_2} = 3$

(c) $\frac{M_1}{M_2} = 4$ (d) None of these

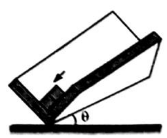
47. A block of mass m slides down an inclined right angled trough. If the coefficient of friction between block and the trough is μ_k, acceleration of the block down the plane is

(a) $g(\sin\theta + \sqrt{2}\mu_k \cos\theta)$

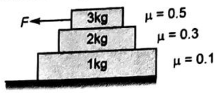
(b) $g(\sin\theta + \mu_k \cos\theta)$

(c) $g(\sin\theta - \sqrt{2}\mu_k \cos\theta)$

(d) $g(\sin\theta - \mu_k \cos\theta)$



48. If force F is increasing with time and at $t = 0$, $F = 0$, where will slipping first start?



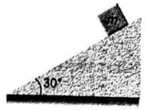
(a) between 3 kg and 2 kg
(b) between 2 kg and 1 kg
(c) between 1 kg and ground
(d) Both (a) and (b)

49. A plank of mass 2kg and length 1 m is placed on horizontal floor. A small block of mass 1kg is placed on top of the plank, at its right extreme end. The coefficient of friction between plank and floor is 0.5 and that between plank and block is 0.2. If a horizontal force = 30N starts acting on the plank to the right, the time after which the block will fall off the plank is
$(g = 10\,\text{ms}^{-2})$

(a) $\left(\frac{2}{3}\right)$ s (b) 1.5 s (c) 0.75 s (d) $\left(\frac{4}{3}\right)$ s

Passage 1 (Q. No. 50 to 54)

A man wants to slide down a block of mass m which is kept on a fixed inclined plane of inclination 30° as shown in the figure. Initially the block is not sliding.



To just start sliding the man pushes the block down the incline with a force F. Now, the block starts accelerating. To move it downwards with constant speed the man starts pulling the block with same force. Surfaces are such that ratio of maximum static friction to kinetic friction is 2. Now, answer the following questions.

50. What is the value of F?

(a) $\frac{mg}{4}$ (b) $\frac{mg}{6}$ (c) $\frac{mg\sqrt{3}}{4}$ (d) $\frac{mg}{2\sqrt{3}}$

51. What is the value of μ_s, the coefficient of static friction?

(a) $\frac{4}{3\sqrt{3}}$ (b) $\frac{2}{3\sqrt{3}}$ (c) $\frac{3}{3\sqrt{3}}$ (d) $\frac{1}{2\sqrt{3}}$

52. If the man continues pushing the block by force F, its acceleration would be

(a) $\frac{g}{6}$ (b) $\frac{g}{4}$ (c) $\frac{g}{2}$ (d) $\frac{g}{3}$

53. If the man wants to move the block up the incline, what minimum force is required to start the motion?

(a) $\frac{2}{3}mg$ (b) $\frac{mg}{2}$ (c) $\frac{7mg}{6}$ (d) $\frac{5mg}{6}$

54. What minimum force is required to move it up the incline with constant speed?

(a) $\frac{2}{3}mg$ (b) $\frac{mg}{2}$ (c) $\frac{7mg}{6}$ (d) $\frac{5mg}{6}$

Passage 2 (Q. No. 55 to 56)

A lift with a mass 1200 kg is raised from rest by a cable with a tension 1350 g newton. After some time the tension drops to 1000 g Newton and the lift comes to rest at a height of 25 m above its initial point. (1 g newton = 9.8 N)

55. What is the height at which the tension changes?

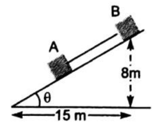
(a) 10.8 m (b) 12.5 m (c) 14.3 m (d) 16 m

56. What is greatest speed of lift?

(a) 9.8 ms^{-1} (b) 7.5 ms^{-1} (c) 5.92 ms^{-1} (d) None of these

Passage 3 (Q. No. 57 to 58)

Blocks A and B shown in the figure are connected with a bar of negligible weight. A and B each has mass 170 kg, the coefficient of friction between A and the plane is 0.2 and that between B and the plane is 0.4 ($g = 10\,\text{ms}^{-2}$)



57. What is the total force of friction between the blocks and the plane?

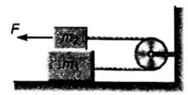
(a) 900 N (b) 700 N (c) 600 N (d) 300 N

58. What is the force acting on the connecting bar?

(a) 150 N (b) 100 N (c) 75 N (d) 125 N

More than One Correct Options

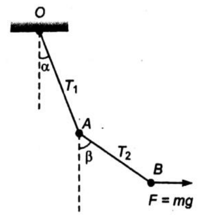
1. Two blocks each of mass 1 kg are placed as shown. They are connected by a string which passes over a smooth (massless) pulley.



There is no friction between m_1 and the ground. The coefficient of friction between m_1 and m_2 is 0.2. A force F is applied to m_2. Which of the following statements is/are correct

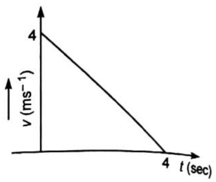
(a) The system will be in equilibrium if $F \leq 4$ N
(b) If $F > 4$ N tension in the string will be 4 N
(c) If $F > 4$ N the frictional force between the blocks will be 2 N
(d) If $F = 6$ N tension in the string will be 3 N

2. Two particles A and B, each of mass m are kept stationary by applying a horizontal force $F = mg$ on particle B as shown in figure. Then



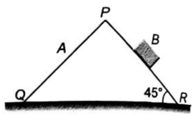
(a) $\tan\beta = 2\tan\alpha$ (b) $2T_1 = 5T_2$ (c) $\sqrt{2}\,T_1 = \sqrt{5}T_2$ (d) None of these

3. The velocity-time graph of the figure shows the motion of a wooden block of mass 1 kg which is given an initial push at $t = 0$ along a horizontal table.



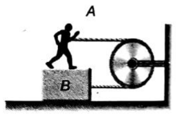
(a) The coefficient of friction between the block and the table is 0.1
(b) The coefficient of friction between the block and the table is 0.2
(c) If the table was half of its present roughness, the time taken by the block to complete the journey is 4 s
(d) If the table was half of its present roughness, the time taken by the block to complete the journey is 8 s

4. A block B of mass 0.6 kg slides down the smooth face PR of a wedge A of mass 1.7 kg which can move freely on a smooth horizontal surface. The inclination of the face PR to the horizontal is 45°. Then



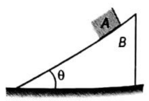
(a) the acceleration of A is $\frac{3g}{40}$
(b) the vertical component of the acceleration of B is $23\frac{g}{40}$
(c) the horizontal component of the acceleration of B is $17\frac{g}{40}$
(d) the acceleration of A is $\frac{g}{3}$

5. As shown in the figure, A is a man of mass 60kg standing on a block of mass 40 kg kept on ground. The coefficient of friction between the feet of the man and the block is 0.3 and that between B and the ground is 0.2. If the person pulls the string with 125 N force, then



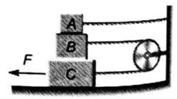
(a) B will slide on ground
(b) A and B will move with acceleration $0.5\ ms^{-2}$
(c) the force of friction acting between A and B will be 40 N
(d) the force of friction acting between A and B will be 180 N

6. In the figure shown A and B are free to move. All the surfaces are smooth. Mass of A is m. Then



(a) the acceleration of A will be more than g sin θ
(b) the acceleration of A will be less than $g \sin\theta$
(c) normal reaction on A due to B will be more than $mg\cos\theta$
(d) normal reaction on A due to B will be less than $mg\cos\theta$

7. $M_A = 3$ kg, $M_B = 4$ kg, and $M_C = 8$ kg. μ between any two surfaces is 0.25. Pulley is frictionless and string is massless. A is connected to wall through a massless rigid rod.

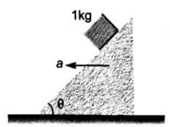


(a) the value of F to keep C moving with constant speed is 80 N.
(b) the value of F to keep C moving with constant speed is 120 N.
(c) if F is 200 N then acceleration of B is $10\ ms^{-2}$
(d) to slide C towards left, F should be at least 50 N. (take $g = 10\ ms^{-2}$)

8. A man pulls a block of mass equal to himself with a light string. The coefficient of friction between the man and the floor is greater than that between the block and the floor
(a) if the block does not move, then the man also does not move
(b) the block can move even when the man is stationary

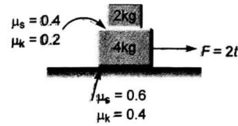
(c) if both move then the acceleration of the block is greater than the accéleration of man
(d) if both move then the acceleration of man is greater than the acceleration of block

9. A block of mass 1 kg is at rest relative to a smooth wedge moving leftwards with constant acceleration $a = 5\ ms^{-2}$. Let N be the normal reaction between the block and the wedge. Then ($g = 10\ ms^{-2}$)



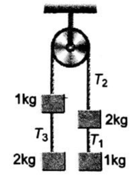
(a) $N = 5\sqrt{5}$ N
(b) $N = 15$ N
(c) $\tan\theta = \frac{1}{2}$
(d) $\tan\theta = 2$

10. For the given situation shown in figure, choose the correct options, ($g = 10\ ms^{-2}$)



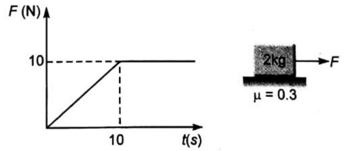
(a) at $t = 1$ s, force of friction between 2 kg and 4 kg is 2N
(b) At $t = 1$ s, force of friction between 2 kg and 4 kg is zero
(c) at $t = 4$ s force of friction between 4 kg and ground is 8 N
(d) At $t = 15$ s acceleration of 2kg is $1\ ms^{-2}$

11. In the figure shown, all the strings are massless and friction is absent everywhere. Choose the correct options.



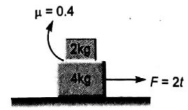
(a) $T_1 > T_3$
(b) $T_3 > T_1$
(c) $T_2 > T_1$
(d) $T_2 > T_3$

12. Force acting on a block *versus* time graph is as shown in figure. Choose the correct options. ($g = 10\ ms^{-2}$)



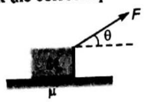
(a) At $t = 2$ s, force of friction is 2N
(b) At $t = 8$ s, force of friction is 6 N
(c) At $t = 10$ s, acceleration of block is $2\ ms^{-2}$
(d) At $t = 12$ s, velocity of block is $8\ ms^{-1}$

13. For the situation shown in figure, mark the correct options.



(a) At $t = 3$ s, pseudo force on 4 kg when applied from 2 kg is 4 N in forward direction
(b) At $t = 3$ s, pseudo force on 2 kg when applied from 4 kg is 2 N in backward direction
(c) Pseudo force does not make an equal and opposite pairs
(d) Pseudo force also makes a pair of equal and opposite forces.

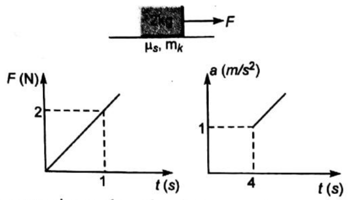
14. For the situation shown in figure, mark the correct options.



(a) Angle of friction is $\tan^{-1}(\mu)$
(b) Angle of repose is $\tan^{-1}(\mu)$
(c) At $\theta = \tan^{-1}(\mu)$, minimum force will be required to move the block
(d) Minimum froce required to move the block is $\frac{\mu Mg}{\sqrt{1+\mu^2}}$.

Match the Columns

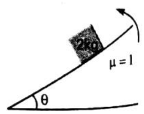
1.



Force acting on a block *versus* time and acceleration *versus* time graph are as shown in figure. Taking value of $g = 10\,\text{ms}^{-2}$, match the following two columns.

Column I	Column II
(a) Coefficient of static friction	(p) 0.2
(b) Coefficient of kinetic friction	(q) 0.3
(c) Force of friction at $t = 0.1$ s	(r) 0.4
(d) Value of $\frac{a}{10}$, where a is acceleration of block at $t = 8$ s	(s) 0.5

2. Angle θ is gradually increased as shown in figure. For the given situation match the following two columns. ($g = 10\,\text{ms}^{-2}$)

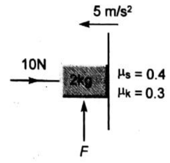


Column I	Column II
(a) Force of friction when $\theta = 0°$	(p) 10 N
(b) Force of friction when $\theta = 90°$	(q) $10\sqrt{3}$ N
(c) Force of friction when $\theta = 30°$	(r) $\frac{10}{\sqrt{3}}$ N
(d) Force of friction when $\theta = 60°$	(s) None

3. Match the following two columns regarding fundamental forces of nature.

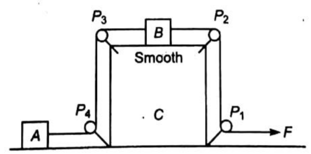
Column I	Column II
(a) Force of friction	(p) field force
(b) Normal reaction	(q) contact force
(c) Force between two neutrons	(r) electromagnetic force
(d) Force between two protons	(s) nuclear force

4. In the figure shown, match the following two columns. ($g = 10\,\text{ms}^{-2}$)



Column I	Column II
(a) Normal reactoin	(p) 5 N
(b) Force of friction when $F = 15$ N	(q) 10 N
(c) Minimum value of F for stopping the block moving down	(r) 15 N
(d) Minimum value of F for stopping the block moving up	(s) None

5. There is no friction between blocks B and C. But ground in rough. Pulleys are smooth and massless and strings are light. For $F = 10$ N, whole system remains stationary. Match the following two columns. ($m_B = m_C = 1$ kg and $g = 10\,\text{ms}^{-2}$)



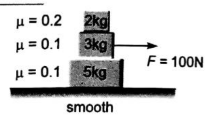
Column I	Column II
(a) Force of friction between A and ground	(p) 10 N
(b) Force of friction between C and ground	(q) 20 N
(c) Normal reaction on C from ground	(r) 5 N
(d) Tension in string between P_3 and P_4	(s) None

6. Match Column I with Column II.

Note *Applied force is parallel to plane.*

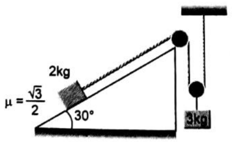
Column I	Column II
(a) If friction force is less than applied force then friction may be	(p) Static
(b) If friction force is equal to the force applied, then friction may be	(q) Kinetic
(c) If object is moving, then friction is	(r) Limiting
(d) If object is at rest, then friction may be	(s) No conclusion can be drawn

7. For the situation shown in figure, in Column I, the statements regarding friction forces are mentioned, while in Column II some information related to friction forces are given. Match the entries of Column I with the entries of Column II (Take $g = 10\,\text{ms}^{-2}$)



Column I	Column II
(a) Total friction force on 3 kg block is	(p) Towards right
(b) Total friction force on 5 kg block is	(q) Towards left
(c) Friction force on 2 kg block due to 3 kg block is	(r) Zero
(d) Friction force on 3 kg block due to 5 kg block is	(s) Non-zero

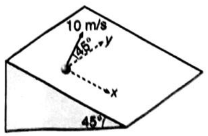
8. If the system is released from rest, then match the following columns.



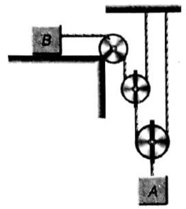
Column I	Column II
(a) Acceleration of 2 kg mass	(p) 2 SI unit
(b) Acceleration of 3 kg mass	(q) 5 SI unit
(c) Tension in the string connecting 2 kg mass	(r) Zero
(d) Frictional forces on 2 kg mass	(s) None of these

Subjective Questions (Level 2)

1. A small marble is projected with a velocity of 10 m/s in a direction 45° from the y-direction on the smooth inclined plane. Calculate the magnitude v of its velocity after 2s. (Take $g = 10\,\text{m/s}^2$)

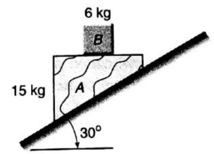


2. Determine the acceleration of the 5 kg block A. Neglect the mass of the pulley and cords. The block B has a mass of 10 kg. The coefficient of kinetic friction between block B and the surface is $\mu_k = 0.1$. (Take $g = 10\,\text{m/s}^2$)

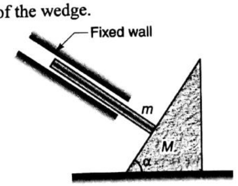


3. A 30 kg mass is initially at rest on the floor of a truck. The coefficient of static friction between the mass and the floor of truck in 0.3 and coefficient of kinetic friction is 0.2. Initially the truck is travelling due east at constant speed. Find the magnitude and direction of the friction force acting on the mass, if : (Take $g = 10\,\text{m/s}^2$)
   (a) The truck accelerates at 1.8 m/s^2 eastward,
   (b) The truck accelerates at 3.8 m/s^2 westward.

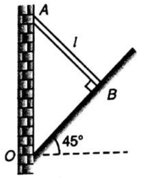
4. A 6 kg block B rests as shown on the upper surface of a 15 kg wedge A. Neglecting friction, determine immediately after the system is released from rest (a) the acceleration of A, (b) the acceleration of B relative to A. (Take $g = 10\,\text{m/s}^2$)



5. In the arrangement shown in the figure, the rod of mass m held by two smooth walls, remains always perpendicular to the surface of the wedge of mass M. Assuming all the surfaces are frictionless, find the acceleration of the rod and that of the wedge.



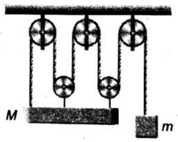
6. At the bottom edge of a smooth vertical wall, an inclined plane is kept at an angle of 45°. A uniform ladder of length l and mass M rests on the inclined plane against the wall such that it is perpendicular to the incline.



(a) If the plane is also smooth, which way will the ladder slide.
(b) What is the minimum coefficient of friction necessary so that the ladder does not slip on the incline?

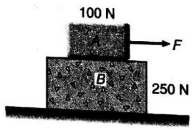
7. A plank of mass M is placed on a rough horizontal surface and a constant horizontal force F is applied on it. A man of mass m runs on the plank. Find the accelerations of the man so that the plank does not move on the surface. Coefficient of friction between the plank and the surface is μ. Assume that the man does not slip on the plank.

8. Find the acceleration of the two masses as shown in figure. The pulleys are light and frictionless and strings are light and inextensible.



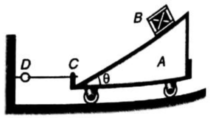
9. The upper portion of an inclined plane of inclination α is smooth and the lower portion is rough. A particle slides down from rest from the top and just comes to rest at the foot. If the ratio of smooth length to rough length is $m:n$, find the coefficient of friction.

10. Block B rests on a smooth surface. If the coefficient of static friction between A and B is $\mu = 0.4$. Determine the acceleration of each, if :

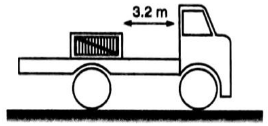


(a) $F = 30\,\text{N}$ and
(b) $F = 250\,\text{N}$. $(g = 10\,\text{m/s}^2)$

11. Block B has a mass m and is released from rest when it is on top of wedge A, which has a mass $3m$. Determine the tension in cord CD while B is sliding down A. Neglect friction.

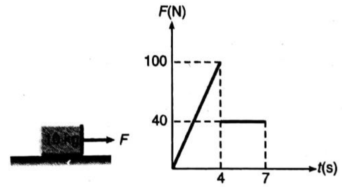


12. Coefficients of friction between the flat bed of the truck and crate are $\mu_s = 0.8$ and $\mu_k = 0.7$. The coefficient of kinetic friction between the truck tires and the road surface is 0.9. If the truck stops from an initial speed of 15 m/s with maximum braking (wheels skidding). Determine where on the bed the crate finally comes to rest. (Take $g = 10\,\text{m/s}^2$)

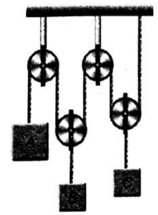


13. The 10 kg block is moving to the left with a speed of 1.2 m/s at time $t = 0$. A force F is applied as shown in the graph. After 0.2 s the force continues at the 10 N level. If the coefficient of kinetic friction is $\mu_k = 0.2$. Determine the time t at which the block comes to a stop. $(g = 10\,\text{m/s}^2)$

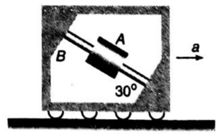
14. The 10 kg block is resting on the horizontal surface when the force F is applied to it for 7 s. The variation of F with time is shown. Calculate the maximum velocity reached by the block and the total time t during which the block is in motion. The coefficients of static and kinetic friction are both 0.50. $(g = 9.8\,\text{m/s}^2)$



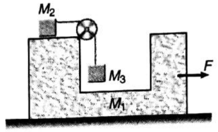
15. If block A of the pulley system is moving downward with a speed of 1 m/s while block C is moving up at 0.5 m/s, determine the speed of block B.



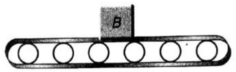
16. The collar A is free to slide along the smooth shaft B mounted in the frame. The plane of the frame is vertical. Determine the horizontal acceleration a of the frame necessary to maintain the collar in a fixed position on the shaft. $(g = 9.8\,\text{m/s}^2)$



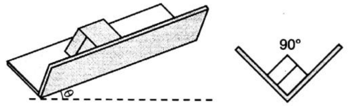
17. In the adjoining figure all surfaces are frictionless. What force F must by applied to M_1 to keep M_3 free from rising or falling?



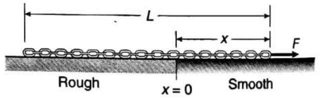
18. The conveyor belt is designed to transport packages of various weights. Each 10 kg package has a coefficient of kinetic friction $\mu_k = 0.15$. If the speed of the conveyor belt is 5 m/s, and then it suddenly stops, determine the distance the package will slide before coming to rest. $(g = 9.8\,\text{m/s}^2)$



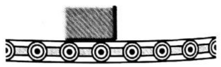
19. In figure, a crate slides down an inclined right-angled trough. The coefficient of kinetic friction between the crate and the trough is μ_k. What is the acceleration of the crate in terms of μ_k, θ and g ?



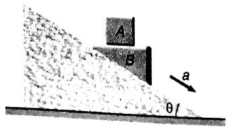
20. A heavy chain with a mass per unit length ρ is pulled by the constant force F along a horizontal surface consisting of a smooth section and a rough section. The chain is initially at rest on the rough surface with $x = 0$. If the coefficient of kinetic friction between the chain and the rough surface is μ_k, determine the velocity v of the chain when $x = L$. The force F is greater than $\mu_k \rho g L$ in order to initiate the motion.



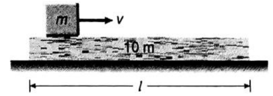
21. A package is at rest on a conveyor belt which is initially at rest. The belt is started and moves to the right for 1.3 s with a constant acceleration of 2 m/s². The belt then moves with a constant deceleration a_2 and comes to a stop after a total displacement of 2.2 m. Knowing that the coefficients of friction between the package and the belt are $\mu_s = 0.35$ and $\mu_k = 0.25$, determine (a) the deceleration a_2 of the belt, (b) the displacement of the package relative to the belt as the belt comes to a stop. $(g = 9.8\,\text{m/s}^2)$



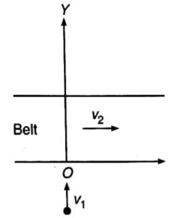
22. Determine the normal force the 10 kg crate A exerts on the smooth cart B, if the cart is given an acceleration of $a = 2\,\text{m/s}^2$ down the plane. Also, find the acceleration of the crate. Set $\theta = 30°$. $(g = 10\,\text{m/s}^2)$.



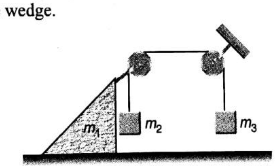
23. A small block of mass m is projected on a larger block of mass $10\,m$ and length l with a velocity v as shown in the figure. The coefficient of friction between the two blocks is μ_2 while that between the lower block and the ground is μ_1. Given that $\mu_2 > 11\mu_1$.
(a) Find the minimum value of v, such that the mass m falls off the block of mass $10\,m$.
(b) If v has this minimum value, find the time taken by block m to do so.



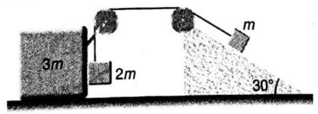
24. A particle of mass m and velocity v_1 in positive y direction is projected on to a belt that is moving with uniform velocity v_2 in x-direction as shown in figure. Coefficient of friction between particle and belt is μ. Assuming that the particle first touches the belt at the origin of fixed x-y co-ordinate system and remains on the belt, find the co-ordinates (x, y) of the point where sliding stops.



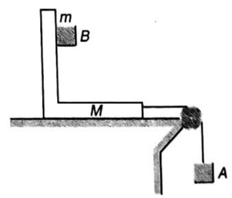
25. In the shown arrangement, both pulleys and the string are massless and all the surfaces are frictionless. Find the acceleration of the wedge.



26. Neglect friction. Find accelerations of m, $2m$ and $3m$ as shown in the figure. The wedge is fixed.



27. The figure shows an L shaped body of mass M placed on smooth horizontal surface. The block A is connected to the body by means of an inextensible string, which is passing over a smooth pulley of negligible mass. Another block B of mass m is placed against a vertical wall of the body. Find the minimum value of the mass of block A so that block B remains stationary relative to the wall. Coefficient of friction between the block B and the vertical wall is μ.



ANSWERS

Introductory Exercise 5.1

4. $F_{1x} = 2\sqrt{3}$ N, $F_2x = -2$ N, $F_3x = 0$, $F_4x = 4$ N, $F_1y = 2$N, $F_{2y} = 2\sqrt{3}$ N, $F_{3y} = -6$ N, $F_{4y} = 0$

5. W 6. $\frac{2}{\sqrt{3}}W$ 7. $F = 10.16$ newton, $N = 2.4$ newton 8. $\frac{mg}{\sqrt{2}}, \frac{g}{2}$

Introductory Exercise 5.2

1. (a) 10 ms^{-2} (b) 110 N (c) 20 N 2. zero 3. $\frac{3g}{4}$ 4. 30°, 10N

5. $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right) 5\sqrt{7}$ N 6. 4 N, 6 N

Introductory Exercise 5.3

1. 3 kg 2. 4 3. $\frac{1}{3}$ s 4. $\frac{10}{3}$ ms^{-2}

Introductory Exercise 5.4

1. (a) $\frac{2g}{3}$ (b) $\frac{10}{3}$ N 2. (a) $\frac{g}{2}, \frac{Mg}{2}$ 3. 4.8 kg 4. $\frac{12}{35}$ N, $\frac{2}{7}$ ms^{-2}

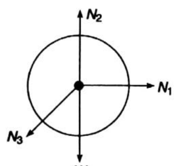
Introductory Exercise 5.5

1. 6.83 kg 2. (a) $x = x_0 + 10t - 2.5t^2, v = 10 - 5t$ (b) $t = 4$ s
3. (a) $x = x_0 - 2.5t^2, z = z_0 + 10t, v_x = -5t, v_z = 10\ ms^{-1}$ (b) $x = x_0, z = z_0 + 10t, v_x = 0, v_z = 10\ ms^{-1}$
4. $x = x_0 + 10t - 4t^2, V = 10 - 8t$ for $0 < t < 1.25$ s object stops at $t = 1.25$ s and remains at rest relative to car.
5. $\frac{9}{25}mg$

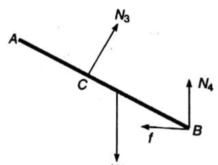
AIEEE Corner

Subjective Questions Level 1

1.



FBD of sphere
(i)

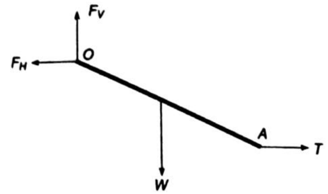


FBD of rod
(ii)

In the figure :

N_1 = normal reaction between sphere and wall,
N_2 = normal reaction between sphere and ground
N_3 = normal reaction between sphere and rod and
N_4 = normal reaction between rod and ground
f = force of friction between rod and ground

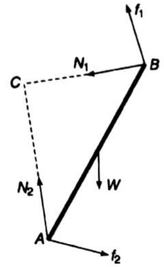
2.



FBD of rod

In the figure :
T = tension in the string, W = weight of the rod, F_v = vertical force exerted by hinge on the rod
F_H = horizontal force exerted by hinge on the rod

3.



FBD of the rod

In the figure :
N_1 = Normal reaction at B, f_1 = force of friction at B, N_2= normal reaction at A, f_2 = force of friction at A
W = weight of the rod.

4. (a) $F_1 = F_2 = 30\sqrt{2}$ N (b) $w = 30\sqrt{2}$ N 5. $N_A = \frac{1000}{\sqrt{3}}$ N, $N_B = \frac{500}{\sqrt{3}}$ N 6. 30 N 7. 5 cm

8. (a) 26.8 N (b) 26.8 N (c) 100 N 9. (a) $\frac{40}{\sqrt{3}}$ N (b) $\frac{40}{\sqrt{3}}$ N

10. (a) 3 ms^{-2} (b) 18 N, 12 N, 30 N (c) 70 N 11. (a) 10 N, 30 N (b) 24 N 12. (a) 20 N (b) 50 N
13. (a) 2.69 ms^{-2} (b) 137.5 N (c) 112.5 N 14. (a) move up (b) constant (c) constant (d) stop
15. $a_B = -3a_A$ 16. $a_A + 2a_B + a_C = 0$ 17. $a_A = a_B \sin\theta$ 18. 1 ms^{-2} (upwards)
19. 4 ms^{-2}, 24 N, 42 N, 14 N
20. (a) $a_1 = \frac{120}{11}$ ms^{-2}, $a_2 = \frac{50}{11}$ ms^{-2} (downwards), $a_3 = \frac{70}{11}$ ms^{-2} (downwards) (b) $T_1 = T_2 = \frac{120}{11}$ N

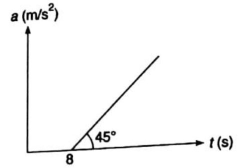
21. $\frac{g}{3}$ (up the plane) 22. 5 N 23. $\frac{2}{7}g$ (downwards), $\frac{g}{7}$ (upwards)

24. (a) zero, 20 N (b) 6 ms^{-2}, 8 N (c) $\frac{8}{3}$ ms^{-2}, 4 N 25. (a) 2s (b) 6 m

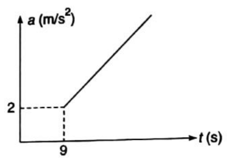
26. (a) 1 s (b) 6 ms^{-1} (c) 4 m, 7 m (both towards right)

27. (a) $\frac{7}{6}$ s (b) 4 ms^{-1} (c) 12.83 m (towards left), 0.58 m (towards left)

28. 4 ms^{-2} (downwards), 12 N (upwards) 29. $a = 0$ for $t \le 8$s, $a = t - 8$ for $t \ge 8$s



30. $a = 0$ for $t \le 9$s, $a = \left(\frac{2}{3}t - 4\right)$ for $t \ge 9$ s



31. (a) 34 N (b) 40 N (c) 88 N

Objective Questions (Level 1)

1. (b)	**2.** (b)	**3.** (d)	**4.** (c)	**5.** (d)	**6.** (c)	**7.** (b)	**8.** (b)	**9.** (a)	**10.** (c)
11. (b)	**12.** (a)	**13.** (a)	**14.** (a)	**15.** (b)	**16.** (a)	**17.** (a)	**18.** (a)	**19.** (b)	**20.** (b)
21. (c)	**22.** (a)	**23.** (d)	**24.** (c)	**25.** (d)	**26.** (d)	**27.** (a)	**28.** (a)	**29.** (d)	**30.** (b)
31. (a)									

JEE Corner

Assertion and Reason

1. (d)	**2.** (a)	**3.** (a)	**4.** (b)	**5.** (d)	**6.** (b)	**7.** (b)	**8.** (d)	**9.** (d)	**10.** (b)
11. (d)									

Objective Questions (Level 2)

1. (c)	**2.** (b)	**3.** (b)	**4.** (d)	**5.** (c)	**6.** (a)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (d)
11. (a)	**12.** (c)	**13.** (a)	**14.** (b)	**15.** (c)	**16.** (c)	**17.** (b)	**18.** (d)	**19.** (d)	**20.** (b)
21. (c)	**22.** (d)	**23.** (c)	**24.** (a)	**25.** (c)	**26.** (a)	**27.** (b)	**28.** (c)	**29.** (b)	**30.** (b)
31. (d)	**32.** (d)	**33.** (b)	**34.** (d)	**35.** (b)	**36.** (c)	**37.** (b)	**38.** (d)	**39.** (b)	**40.** (d)
41. (d)	**42.** (d)	**43.** (d)	**44.** (a)	**45.** (b)	**46.** (b)	**47.** (c)	**48.** (c)	**49.** (a)	**50.** (b)
51. (a)	**52.** (d)	**53.** (c)	**54.** (d)	**55.** (c)	**56.** (c)	**57.** (a)	**58.** (a)		

More than One Correct Options

1. (a,c,d)	**2.** (a,c)	**3.** (a,d)	**4.** (b,c)	**5.** (a,b)	**6.** (a,c)	**7.** (a,c)
8. (a,b,c)	**9.** (a,c)	**10.** (b,c,d)	**11.** (b,c,d)	**12.** (all)	**13.** (b,c)	**14.** (all)

Match the Columns

1. (a) → (r) (b)→ (q) (c)→ (p) (d)→ (s)
2. (a) → (s) (b) → (s) (c) → (p) (d) → (p)

3. (a) → (q,r) (b) → (q,r) (c) → (p,s) (d) → (s)
4. (a) → (s) (b)→ (p) (c)→ (s) (d)→ (s)
5. (a) → (p) (b) → (s) (c) → (q) (d) → (p)
6. (a) → (q) (b) → (p,r) (c) → (q) (d) → (p,r)
7. (a) → (q, s) (b)→ (p,s) (c)→ (p,s) (d)→ (q,s)
8. (a) → (r) (b)→ (r) (c)→ (s) (d)→ (q)

Subjective Questions (Level 2)

1. $10\ ms^{-1}$ **2.** $\frac{2}{33}\ ms^{-2}$ **3.** (a) 54 N (due east) (b) 60 N (due west) **4.** (a) $6.36\ ms^{-2}$ (b) $5.5\ ms^{-2}$

5. $\dfrac{mg\cos\alpha\sin\alpha}{m\sin\alpha + \frac{M}{\sin\alpha}}, \dfrac{mg\cos\alpha}{m\sin\alpha + \frac{M}{\sin\alpha}}$ **6.** (a) Clockwise (b) $\frac{1}{3}$ **7.** $\frac{F}{m} - \frac{\mu(M+m)g}{m} \le a \le \frac{F}{m} + \frac{\mu(M+m)g}{m}$

8. $a_M = \left(\frac{5m - M}{25m + M}\right) g$. (upwards) $a_m = 5a_M$ **9.** $\mu = \left(\frac{m+n}{m}\right)\tan\alpha$

10. (a) $a_A = a_B = 0.857\ m/s^2$ (b) $a_A = 21\ m/s^2$, $a_B = 1.6\ m/s^2$ **11.** $\frac{mg}{2}\sin 2\theta$ **12.** 2.77 m

13. $t = 0.33$ s **14.** 5.2 m/s , 5.55 s **15.** zero **16.** $5.66\ m/s^2$ **17.** $\frac{M_3}{M_2}(M_1 + M_2 + M_3)g$

18. 8.5 m **19.** $g(\sin\theta - \sqrt{2}\mu_k\cos\theta)$ **20.** $\sqrt{\frac{2F}{\rho} - \mu_k gL}$

21. (a) $6.63\ m/s^2$ (b) 0.33 m, 22.90 N, $1\ m/s^2$ **23.** (a) $v_{min} = \sqrt{\frac{22(\mu_2 - \mu_1)gl}{10}}$ (b) $t = \sqrt{\frac{20l}{11g(\mu_2 - \mu_1)}}$

24. $x = v_2\frac{\sqrt{v_1^2 + v_2^2}}{2\mu g}, y = \frac{v_1\sqrt{v_1^2 + v_2^2}}{2\mu g}$ **25.** $\dfrac{2m_1m_3g}{(m_2 + m_3)(m_1 + m_2) + m_2m_3}$

26. $a_m = \frac{13}{34}g$, $a_{2m} = \frac{\sqrt{397}}{34}g$, $a_{3m} = \frac{3}{17}g$ **27.** $m_A = \frac{M+m}{\mu - 1}$ but $\mu > 1$

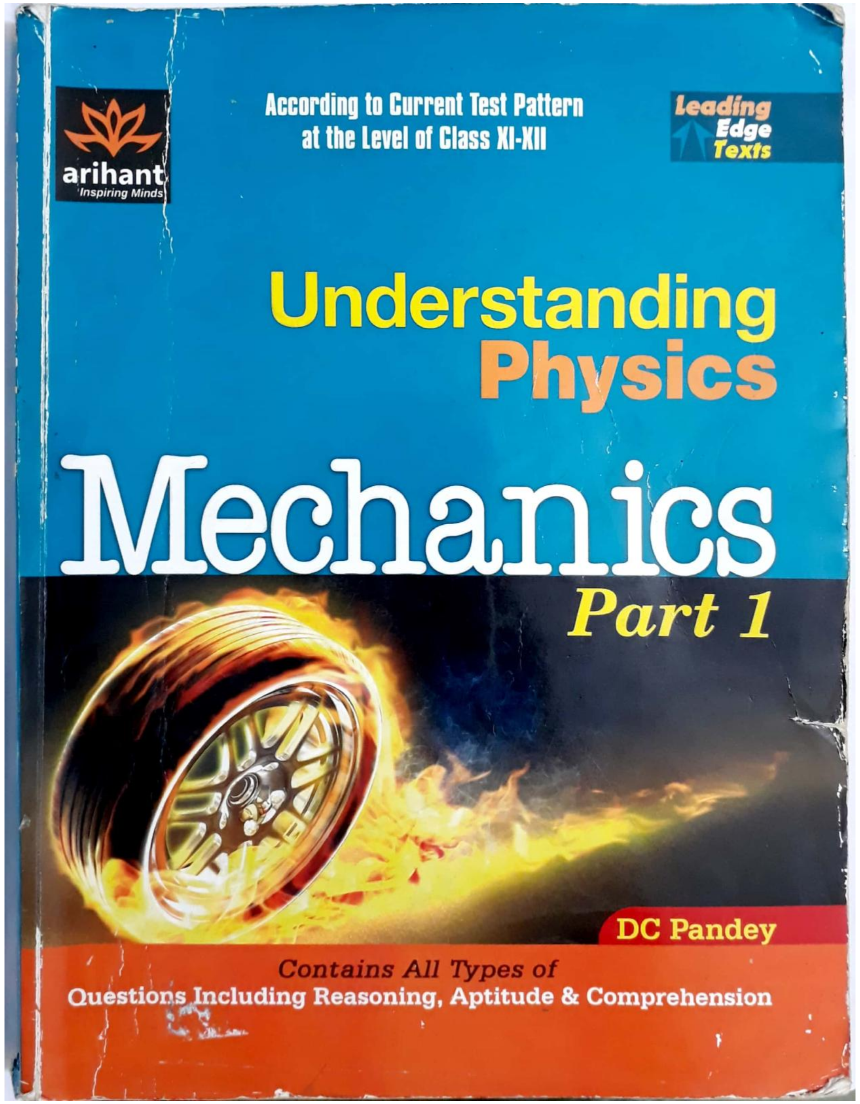
Arihant Mechanics (Chapter 6, 7 and 8) Two by Two

Part 1 by 2

Understanding Physics for IIT JEE

by

D C Pandey



Leading Edge Texts

According to Current Test Pattern
at the Level of Class XI-XII

Understanding Physics

Mechanics

Part 1

DC Pandey

arihant
Inspiring Minds.Inspiring Lives
Arihant Prakashan, Meerut

Contents

1. **Basic Mathematics & Measurements** 1-33

1.1 Basic Mathematics
1.2 Significant Figures
1.3 Error Analysis
1.4 Length Measuring Instruments

2. **Units & Dimensions, Vectors** 35-67

2.1 Units and Dimensions
2.2 Vector and Scalar Quantities
2.3 General Points Regarding Vectors
2.4 Addition & Subtraction of Two Vectors
2.5 The Components of a Vector
2.6 The Product of Two Vectors

3. **Motion In One Dimension** 69-136

3.1 Introduction to Mechanics & Kinematics
3.2 Basic Definitions
3.3 Motion in One, Two and Three Dimensions
3.4 Uniformly Accelerated Motion
3.5 Non-Uniformly Accelerated Motion
3.6 Graphs
3.7 Relative Motion

4. **Projectile Motion** 137-170

4.1 Projectile Motion
4.2 Projectile Motion in an Inclined Plane
4.3 Relative Motion Between Two Projectiles

5. **Laws of Motion** 171-257

5.1 Types of Forces
5.2 Free Body Diagram
5.3 Equilibrium
5.4 Newton's Laws of Motion
5.5 Pulleys
5.6 Constraint Equations
5.7 Pseudo Forces
5.8 Friction

6. **Work, Energy & Power** 259-320

6.1 Introduction to Work
6.2 The Work Done
6.3 Conservative & Non-Conservative Force Field
6.4 Kinetic Energy
6.5 Work-Energy Theorem
6.6 Potential Energy
6.7 Law of Conservation of Mechanical Energy
6.8 Three Types of Equilibria
6.9 Power

7. **Circular Motion** 321-361

7.1 Kinematics of Circular Motion
7.2 Dynamics of Circular Motion
7.3 Motion in a Vertical Circle

8. **Centre of Mass, Conservation of Linear Momentum, Impulse and Collision** 363-437

8.1 Centre of Mass
8.2 Law of Conservation of Linear Momentum
8.3 Variable Mass
8.4 Impulse
8.5 Collision

- **Hints & Solutions** 439-483

Chapter – 6 Work, Energy & Power

6

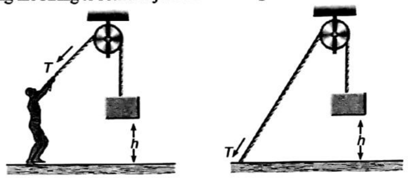
Work, Energy & Power

Chapter Contents

6.1 Introduction to Work
6.2 Work Done
6.3 Conservative & Non-Conservative Force Field
6.4 Kinetic Energy
6.5 Work-Energy Theorem
6.6 Potential Energy
6.7 Law of Conservation of Mechanical Energy
6.8 Three Types of Equilibrium
6.9 Power

6.1 Introduction to Work

In our daily life 'work' has many different meanings. For example, Ram is working in a factory. The machine is in working order. Let us work out a plan for the next year, etc. In physics however, the term 'work' has a special meaning. In physics, work is always associated with a force and a displacement. We note that for work to be done, the force must act through a distance. Consider a person holding a weight a distance 'h' off the floor as shown in figure. In everyday usage, we might say that the man is doing a work, but in our scientific definition, no work is done by a force acting on a stationary object. We could eliminate the effort of holding the weight by merely tying the string to some object and the weight could be supported with no help from us.



No work is done by the man holding the weight at a fixed position. The same task could be accomplished by tying the rope to a fixed point.

Fig. 6.1

Let us now see what does 'work' mean in the language of physics.

6.2 Work Done

There are mainly three methods of finding work done.

(i) Work done by a constant force ($W = \vec{F} \cdot \vec{S} = FS \cos \theta$).

(ii) Work done by a variable force ($W = \int \vec{F} \cdot d\vec{S}$).

(iii) Work done by area under F-S graph.

(i) Work done by a constant force

Let us first consider the simple case of a constant force $\vec{F}$ acting on a body. Further, let us also assume that the body moves in a straight line; in the direction of force. In this case we define the work done by the force on the body as the product of the magnitude of the force $\vec{F}$ and the distance S through which the body moves. That is, the work W is given by

$$W = F \cdot S$$

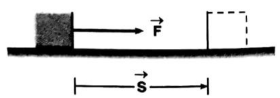


Fig. 6.2

On the other hand, in a situation when the constant force does not act along the same direction as the displacement of the body, the component of force $\vec{F}$ along the displacement $\vec{S}$ is effective in doing work.

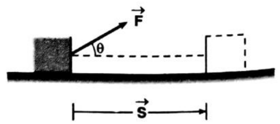


Fig. 6.3

Thus, in this case, work done by a constant force $\vec{F}$ is given by

$$W = (\text{component of force along the displacement}) \times (\text{displacement})$$

or $\quad W = (F \cos \theta)(S)$

or $\quad W = \vec{F} \cdot \vec{S} \quad$ (from the definition of dot product)

So, work done is a scalar or dot product of $\vec{F}$ and $\vec{S}$.

Regarding work it is worthnoting that :

1. Work can be positive, negative or even zero also, depending on the angle (θ) between the force vector $\vec{F}$ and displacement vector $\vec{S}$. Work done by a force is zero when $\theta = 90°$, it is positive when $\theta < 90°$ and negative when $\theta > 90°$. For example, when a person lifts a body, the work done by the lifting force is positive (as $\theta = 0°$) but work done by the force of gravity is negative (as $\theta = 180°$). Similarly work done by centripetal force is always zero (as $\theta = 90°$).
2. Work depends on frame of reference. With change of frame of reference inertial force does not change while displacement may change. So, the work done by a force will be different in different frames. For example, if a person is pushing a box inside a moving train, then work done as seen from the frame of reference of train is $\vec{F} \cdot \vec{S}$ while as seen from the ground it is $\vec{F} \cdot (\vec{S} + \vec{S}_0)$. Here $\vec{S}_0$, is the displacement of train relative to ground.
3. Suppose a body is displaced from point A to point B, then

$$\vec{S} = \vec{r}_B - \vec{r}_A$$
$$= (x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k}$$

Sample Example 6.1 *A body is displaced from $A = (2\,m, 4\,m, -6\,m)$ to $\vec{r}_B = (6\hat{i} - 4\hat{j} + 4\hat{k})\,m$ under a constant force $\vec{F} = (2\hat{i} + 3\hat{j} - \hat{k})\,N$. Find the work done.*

Solution $\quad \vec{r}_A = (2\hat{i} + 4\hat{j} - 6\hat{k})\,\text{m}$

$\therefore \quad \vec{S} = \vec{r}_B - \vec{r}_A$

$$= (6\hat{i} - 4\hat{j} + 2\hat{k}) - (2\hat{i} + 4\hat{j} - 6\hat{k})$$
$$= 4\hat{i} - 8\hat{j} + 8\hat{k}$$

$$W = \vec{F} \cdot \vec{S} = (2\hat{i} + 3\hat{j} - \hat{k}) \cdot (4\hat{i} - 8\hat{j} + 8\hat{k}) = 8 - 24 - 8 = -24\ \text{J}$$ **Ans.**

Sample Example 6.2 *A block of mass $m = 2\,kg$ is pulled by a force $F = 40\,N$ upwards through a height $h = 2\,m$. Find the work done on the block by the applied force F and its weight mg. ($g = 10\,m/s^2$)*

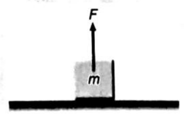


Fig. 6.4

Solution Weight $\quad mg = (2)(10) = 20\,\text{N}$

Work done by the applied force $W_F = Fh \cos 0°$

As the angle between force and displacement is 0°

or $$W_F = (40)(2)(1) = 80\text{ J}$$ **Ans.**

Similarly, work done by its weight

$$W_{mg} = (mg)(h)\cos 180°$$

or $$W_{mg} = (20)(2)(-1) = -40\text{ J}$$ **Ans.**

Sample Example 6.3 *Two unequal masses of 1 kg and 2 kg are attached at the two ends of a light inextensible string passing over a smooth pulley as shown in figure. If the system is released from rest, find the work done by string on both the blocks in 1 s. (Take $g = 10\text{ m/s}^2$).*

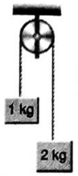


Fig. 6.5

Solution Net pulling force on the system is

$$F_{net} = 2g - 1g = 20 - 10 = 10\text{ N}$$

Total mass being pulled $$m = (1 + 2) = 3\text{ kg}$$

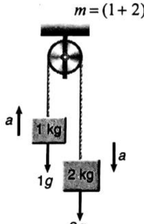


Fig. 6.6 (a)

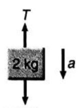


Fig. 6.6 (b)

Therefore, acceleration of the system will be

$$a = \frac{F_{net}}{m} = \frac{10}{3}\text{ m/s}^2$$

Displacement of both the blocks in 1 s is

$$S = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{10}{3}\right)(1)^2 = \frac{5}{3}\text{ m}$$

Free body diagram of 2 kg block is shown in Fig. 6.6 (b).

Using $\Sigma F = ma$, we get

$$20 - T = 2a = 2\left(\frac{10}{3}\right)$$

or $$T = 20 - \frac{20}{3} = \frac{40}{3}\text{ N}$$

$\therefore$ Work done by string (tension) on 1 kg block in 1 s is

$$W_1 = (T)(S)\cos 0°$$

$$= \left(\frac{40}{3}\right)\left(\frac{5}{3}\right)(1) = \frac{200}{9}\text{ J}$$ **Ans.**

Similarly, work done by string on 2 kg block in 1 s will be

$$W_2 = (T)(S)(\cos 180°)$$

$$= \left(\frac{40}{3}\right)\left(\frac{5}{3}\right)(-1) = -\frac{200}{9}\text{ J}$$ **Ans.**

(ii) Work done by a variable force

So far we have considered the work done by a force which is constant both in magnitude and direction. Let us now consider a force which acts always in one direction but whose magnitude may keep on varying. We can choose the direction of the force as x-axis. Further, let us assume that the magnitude of the force is also a function of x or say $F(x)$ is known to us. Now, we are interested in finding the work done by this force in moving a body from x_1 to x_2.

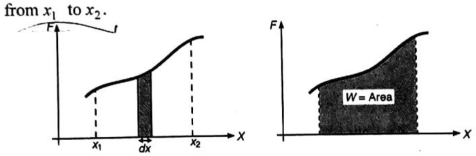


Fig. 6.7

Work done in a small displacement from x to $x + dx$ will be

$$dW = F \cdot dx$$

Now, the total work can be obtained by integration of the above elemental work from x_1 to x_2 or

$$W = \int_{x_1}^{x_2} dW = \int_{x_1}^{x_2} F \cdot dx$$

It is important to note that $\int_{x_1}^{x_2} F\,dx$ is also the area under F-x graph between $x = x_1$ to $x = x_2$.

Spring Force

An important example of the above idea is a spring that obeys Hooke's law. Consider the situation shown in figure. One end of a spring is attached to a fixed vertical support and the other end to a block which can move on a horizontal table. Let $x = 0$ denote the position of the block when the spring is in its natural length. When the block is displaced by an amount x (either compressed or elongated) a restoring force (F) is applied by the spring on the block. The direction of this force F is always towards its mean position ($x = 0$) and the magnitude is directly proportional to x or

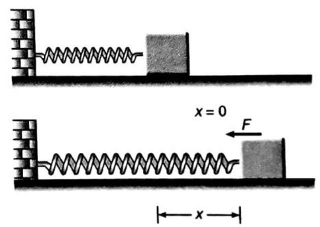


Fig. 6.8

$$F \propto x \qquad \text{(Hooke's law)}$$

$$\therefore \qquad F = -kx \qquad \ldots\text{(i)}$$

Here, k is a constant called force constant of spring and depends on the nature of spring. From Eq. (i) we see that F is a variable force and F-x graph is a straight line passing through origin with slope $= -k$. Negative sign in Eq. (i) implies that the spring force F is directed in a direction opposite to the displacement x of the block.

Let us now find the work done by this force F when the block is displaced from $x = 0$ to $x = x$. This can be obtained either by integration or the area under F-x graph.

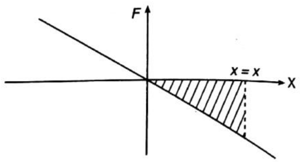


Fig. 6.9

Thus, $$W = \int dW = \int_0^x F dx = \int_0^x -kx\, dx = -\frac{1}{2}kx^2$$

Here, work done is negative because force is in opposite direction of displacement.

Similarly, if the block moves from $x = x_1$ to $x = x_2$. The limits of integration are x_1 and x_2 and the work done is

$$W = \int_{x_1}^{x_2} -kx\, dx = \frac{1}{2}k(x_1^2 - x_2^2)$$

Sample Example 6.4 *A force $F = (2 + x)$ acts on a particle in x-direction where F is in newton and x in metre. Find the work done by this force during a displacement from $x = 1.0\ m$ to $x = 2.0\ m$*

Solution As the force is variable, we shall find the work done in a small displacement from x to $x + dx$ and then integrate it to find the total work. The work done in this small displacement is

$$dW = F\, dx = (2 + x)\, dx$$

Thus, $$W = \int_{1.0}^{2.0} dW = \int_{1.0}^{2.0} (2 + x)\, dx$$

$$= \left[2x + \frac{x^2}{2}\right]_{1.0}^{2.0} = 3.5\ \text{J} \qquad \textbf{Ans.}$$

Sample Example 6.5 *A force $F = -\dfrac{k}{x^2}$ $(x \neq 0)$ acts on a particle in x-direction. Find the work done by this force in displacing the particle from. $x = +a$ to $x = +2a$. Here, k is a positive constant.*

Solution $$W = \int F\, dx = \int_{+a}^{+2a} \left(\frac{-k}{x^2}\right) dx = \left[\frac{k}{x}\right]_{+a}^{+2a} = -\frac{k}{2a} \qquad \textbf{Ans.}$$

Note *It is important to note that work comes out to be negative which is quite obvious as the force acting on the particle is in negative x-direction $\left(F = -\dfrac{k}{x^2}\right)$ while displacement is along positive x-direction. (from $x = a$ to $x = 2a$)*

(iii) Work done by area under F-S or F-x graph

This is applicable in one dimensional motion. When force and displacement are either parallel or antiparallel. Care has to be taken in the signs of F and x (force and displacement). If both have same signs work done will be + (Area) and if both have opposite signs, work done is – (Area). Let us take the following example.

Sample Example 6.6 *A force F acting on a particle varies with the position x as shown in figure. Find the work done by this force in displacing the particle from*

(a) $x = -2\,m$ to $x = 0$

(b) $x = 0$ to $x = 2\,m$

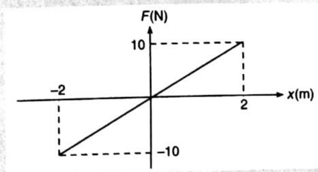


Fig. 6.10

Solution (a) From $x = -2\text{m}$ to $x = 0$, displacement of the particle is along positive x-direction while force acting on the particle is along negative x-direction. Therefore, work done is negative and given by the area under F-x graph.

$$\therefore \qquad W = -\frac{1}{2}(2)(10) = -10\ \text{J} \qquad \textbf{Ans.}$$

(b) From $x = 0$ to $x = 2$ m, displacement of particle and force acting on the particle both are along positive x-direction. Therefore, work done is positive and given by the area under F-x graph, or

$$W = \frac{1}{2}(2)(10) = 10\ \text{J} \qquad \textbf{Ans.}$$

6.3 Conservative & Non-Conservative Force Field

In the above article we considered the forces which were although variable but always directed in one direction. However, the most general expression for work done is

$$dW = \vec{F} \cdot d\vec{r} \quad \text{and} \quad W = \int_{\vec{r}_i}^{\vec{r}_f} dW = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

Here, $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\vec{r}_i$ = initial position vector and $\vec{r}_f$ = final position vector

Conservative and non-conservative forces can be better understood after going through the following two examples.

Sample Example 6.7 *An object is displaced from point $A(2\,m, 3\,m, 4\,m)$ to a point $B\,(1\,m, 2\,m, 3\,m)$ under a constant force $\vec{F} = (2\hat{i} + 3\hat{j} + 4\hat{k})\,N$. Find the work done by this force in this process.*

Solution $$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = \int_{(2\text{ m},\, 3\text{ m},\, 4\text{ m})}^{(1\text{ m},\, 2\text{ m},\, 3\text{ m})} (2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= [2x + 3y + 4z]_{(2\text{ m},\, 3\text{ m},\, 4\text{ m})}^{(1\text{ m},\, 2\text{ m},\, 3\text{ m})} = -9\ \text{J} \qquad \textbf{Ans.}$$

Alternate Solution

Since, $\vec{F}$ = constant, we can also use.

$$W = \vec{F} \cdot \vec{S}$$

Here, $\vec{S} = \vec{r}_f - \vec{r}_i = (\hat{i} + 2\hat{j} + 3\hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k})$

$$= (-\hat{i} - \hat{j} - \hat{k})$$

$\therefore$ $W = (2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (-\hat{i} - \hat{j} - \hat{k})$

$$= -2 - 3 - 4 = -9\text{J}$$ **Ans.**

Sample Example 6.8 *An object is displaced from position vector $\vec{r}_1 = (2\hat{i} + 3\hat{j})$ m to $\vec{r}_2 = (4\hat{i} + 6\hat{j})$ m under a force $\vec{F} = (3x^2\hat{i} + 2y\hat{j})$ N. Find the work done by this force.*

Solution $W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} (3x^2\hat{i} + 2y\hat{j}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$

$$= \int_{\vec{r}_1}^{\vec{r}_2} (3x^2\,dx + 2y\,dy) = [x^3 + y^2]_{(2,\,3)}^{(4,\,6)}$$

$$= 83\text{ J}$$ **Ans.**

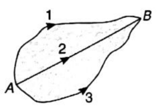


Fig. 6.11

In the above two examples, we saw that while calculating the work done we did not mention the path through which the object was displaced. Only initial and final coordinates were required. It shows that in both the examples, the work done is path independent or work done will be equal on whichever path we follow. Such forces in which work is path independent are known as **conservative forces.**

Thus, if a particle or an object is displaced from position A to position B through three different paths under a conservative force field. Then

$$W_1 = W_2 = W_3$$

Further, it can be shown that work done in a closed path is zero under a conservative force field. ($W_{AB} = -W_{BA}$ or $W_{AB} + W_{BA} = 0$). Gravitational force, Coulomb's force are few examples of conservative forces. On the other hand, if the work is path dependent or $W_1 \neq W_2 \neq W_3$, the force is called a **non-conservative.** Frictional forces, viscous forces are non-conservative in nature. Work done in a closed path is not zero in a non-conservative force field.

Introductory Exercise 6.1

1. A block is pulled a distance x along a rough horizontal table by a horizontal string. If the tension in the string is T, the weight of the block is W, the normal reaction is N and frictional force is F. Write down expressions for the work done by each of these forces.
2. A particle is pulled a distance l up a rough plane inclined at an angle α to the horizontal by a string inclined at an angle β to the plane ($\alpha + \beta < 90°$). If the tension in the string is T, the normal reaction between the particle and the plane is N, the frictional force is F and the weight of the particle is W. Write down expressions for the work done by each of these forces.

3. A bucket tied to a string is lowered at a constant acceleration of $g/4$. If the mass of the bucket is m and is lowered by a distance l then find the work done by the string on the bucket.
4. A 1.8 kg block is moved at constant speed over a surface for which coefficient of friction $\mu = \frac{1}{4}$. It is pulled by a force F acting at 45° with horizontal as shown in figure. The block is displaced by 2 m. Find the work done on the block by (a) the force F (b) friction (c) gravity.

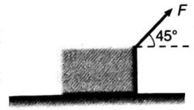


Fig. 6.12

5. A small block of mass 1 kg is kept on a rough inclined wedge of inclination 45° fixed in an elevator. The elevator goes up with a uniform velocity $v = 2$ m/s and the block does not slide on the wedge. Find the work done by the force of friction on the block in 1 s. ($g = 10\text{ m/s}^2$)
6. Two equal masses are attached to the two ends of a spring of force constant k. The masses are pulled out symmetrically to stretch the spring by a length $2x_0$ over its natural length. Find the work done by the spring on each mass.
7. Force acting on a particle varies with displacement as shown in figure. Find the work done by this force on the particle from $x = -4$ m to $x = +4$ m.

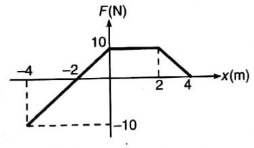


Fig. 6.13

6.4 Kinetic Energy

Kinetic energy (KE) is the capacity of a body to do work by virtue of its motion. If a body of mass m has a velocity v its kinetic energy is equivalent to the work which an external force would have to do to bring the body from rest upto its velocity v. The numerical value of the kinetic energy can be calculated from the formula.

$$\boxed{\text{KE} = \frac{1}{2}mv^2} \quad \frac{1}{2}mv^2$$

This can be derived as follows:

Consider a constant force F which acting on a mass m initially at rest, gives the mass a velocity v. If in reaching this velocity, the particle has been moving with an acceleration a and has been given a displacement s, then

$$F = ma \qquad \text{(Newton's law)}$$

$$v^2 = 2as$$

Work done by the constant force $= Fs$

or $$W = (ma)\left(\frac{v^2}{2a}\right) = \frac{1}{2}mv^2$$

But the kinetic energy of the body is equivalent to the work done in giving the body this velocity.

Hence, $$KE = \frac{1}{2} mv^2$$

Regarding the kinetic energy the following two points are important to note.

1. Since, both m and v^2 are always positive. KE is always positive and does not depend on the direction of motion of the body.
2. Kinetic energy depends on the frame of reference. For example, the kinetic energy of a person of mass m sitting in a train moving with speed v is zero in the frame of train but $\frac{1}{2} mv^2$ in the frame of earth.

6.5 Work Energy Theorem

This theorem is a very important tool that relates the works to kinetic energy. According to this theorem:

Work done by all the forces (conservative or nonconservative, external or internal) acting on a particle or an object is equal to the change in kinetic energy of it.

$$\therefore \qquad W_{net} = \Delta KE = K_f - K_i$$

Let, $\vec{F}_1, \vec{F}_2 \ldots$ be the individual forces acting on a particle. The resultant force is $\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots$ and the work done by the resultant force is

$$W = \int \vec{F} \cdot \vec{dr} = \int (\vec{F}_1 + \vec{F}_2 + \ldots) \cdot \vec{dr}$$

$$= \int \vec{F}_1 \cdot \vec{dr} + \int \vec{F}_2 \cdot \vec{dr} + \ldots$$

where $\int \vec{F}_1 \cdot \vec{dr}$ is the work done on the particle by $\vec{F}_1$ and so on. Thus, work energy theorem can also be written as: work done by the resultant force which is also equal to the sum of the work done by the individual forces is equal to change in kinetic energy.

Regarding the work-energy theorem it is worthnoting that:

(1) If W_{net} is positive then $K_f - K_i$ = positive,

i.e., $K_f > K_i$ or kinetic energy will increase and *vice-versa*.

(2) This theorem can be applied to non-inertial frames also. In a non-inertial frame it can be written as: work done by all the forces (including the pseudo forces) = change in kinetic energy in non-inertial frame. Let us take an example.

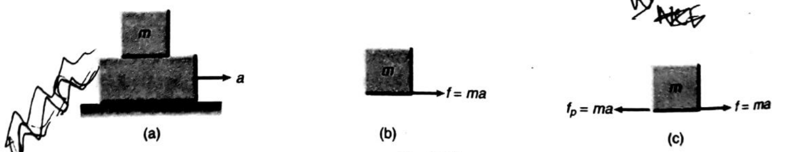


Fig. 6.14

Refer figure (a)

A block of mass m is kept on a rough plank moving with an acceleration a. There is no relative motion between block and plank. Hence, force of friction on block is $f = ma$ in forward direction.

Refer figure (b)

Horizontal forces on the block has been shown from ground (inertial) frame of reference.

If the plank moves a distance s on the ground the block will also move the same distance s as there is no slipping between the two. Hence, work done by friction on the block (w.r.t. ground) is

$$W_f = fs = mas$$

From work-energy principle if v is the speed of block (w.r.t. ground).

$$KE = W_f$$

or $$\frac{1}{2} mv^2 = mas \quad \text{or} \quad \boxed{v = \sqrt{2as}}$$

Thus velocity of block relative to ground is $\sqrt{2as}$.

Refer figure (c)

Free body diagram of the block has been shown from accelerating frame (plank).

Here, f_p = pseudo force = ma

Work done by all the forces,

$$W = W_f + W_{fp} = mas - mas = 0$$

From work-energy theorem,

$$\frac{1}{2} mv_r^2 = W = 0 \quad \text{or} \quad v_r = 0$$

Thus velocity of block relative to plank is zero.

Sample Example 6.9 *An object of mass m is tied to a string of length l and a variable force F is applied on it which brings the string gradually at angle θ with the vertical. Find the work done by the force F.*

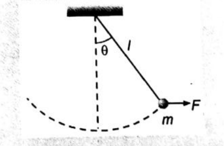


Fig. 6.15

Solution In this case three forces are acting on the object:

1. tension (T)
2. weight (mg) and
3. applied force (F)

Using work-energy theorem

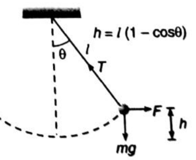


Fig. 6.16

$$W_{net} = \Delta KE$$

or $$W_T + W_{mg} + W_F = 0 \qquad \ldots(i)$$

as $$\Delta KE = 0$$

because $$K_i = K_f = 0$$

Further, $W_T = 0$, as tension is always perpendicular to displacement.

$$W_{mg} = -mgh \quad \text{or} \quad W_{mg} = -mgl\,(1 - \cos\theta)$$

Substituting these values in Eq. (i), we get

$$W_F = mgl\,(1 - \cos\theta)$$ **Ans.**

Note *Here, the applied force F is variable. So, if we do not apply the work energy theorem we will first find the magnitude of F at different locations and then integrate dW ($= \vec{F} \cdot d\vec{r}$) with proper limits.*

Sample Example 6.10 *A body of mass m was slowly hauled up the hill as shown in the figure by a force F which at each point was directed along a tangent to the trajectory. Find the work performed by this force, if the height of the hill is h, the length of its base is l and the coefficient of friction is μ.*

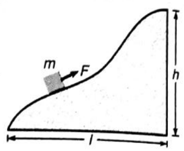


Fig. 6.17

Solution Four forces are acting on the body:

1. weight (mg)
2. normal reaction (N)
3. friction (f) and
4. the applied force (F)

Using work-energy theorem

$$W_{\text{net}} = \Delta KE$$

or
$$W_{mg} + W_N + W_f + W_F = 0 \qquad \ldots\text{(i)}$$

Here, $\Delta\text{KE} = 0$, because $K_i = 0 = K_f$

$$W_{mg} = -mgh$$
$$W_N = 0$$

(as normal reaction is perpendicular to displacement at all points)

W_f can be calculated as under:

$$f = \mu\, mg \cos\theta$$

$\therefore$
$$(dW_{AB})_f = -f\,ds$$
$$= -(\mu\, mg \cos\theta)\,ds$$
$$= -\mu\, mg\,(dl) \qquad (\text{as } ds \cos\theta = dl)$$

$\therefore$
$$_f = -\mu\, mg\, \Sigma dl$$
$$= -\mu\, mgl$$

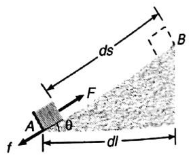


Fig. 6.18

Substituting these values in Eq. (i), we get

$$W_F = mgh + \mu mgl$$ **Ans.**

Note *Here again, if we want to solve this problem without using work-energy theorem we will first find magnitude of applied force $\vec{F}$ at different locations and then integrate dW (= $\vec{F} \cdot d\vec{r}$) with proper limits.*

Introductory Exercise 6.2

1. Velocity-time graph of a particle of mass 2 kg moving in a straight line is as shown in figure. Find the work done by all the forces acting on the particle.

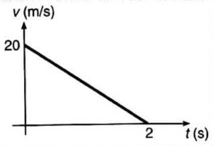


Fig. 6.19

2. Is work-energy theorem valid in a non-inertial frame?
3. A particle of mass m moves on a straight line with its velocity varying with the distance travelled according to the equation $v = \alpha\sqrt{x}$, where α is a constant. Find the total work done by all the forces during a displacement from $x = 0$ to $x = b$.
4. A 5 kg mass is raised a distance of 4 m by a vertical force of 80 N. Find the final kinetic energy of the mass if it was originally at rest. $g = 10\ \text{m/s}^2$.
5. A smooth sphere of radius R is made to translate in a straight line with a constant acceleration $a = g$. A particle kept on the top of the sphere is released from there at zero velocity with respect to the sphere. Find the speed of the particle with respect to the sphere as a function of angle θ as it slides down.
6. An object of mass m has a speed v_0 as it passes through the origin on its way out along the $+x$ axis. It is subjected to a retarding force given by $F_x = -Ax$. Here, A is a positive constant. Find its x-coordinate when it stops.
7. A block of mass M is hanging over a smooth and light pulley through a light string. The other end of the string is pulled by a constant force F. The kinetic energy of the block increases by 40J in 1s. State whether the following statements are true or false:
   (a) The tension in the string is Mg
   (b) The work done by the tension on the block is 40J
   (c) The tension in the string is F
   (d) The work done by the force of gravity is 40J in the above 1s

6.6 Potential Energy

The energy possessed by a body or system by virtue of its position or configuration is known as the potential energy. For example, a block attached to a compressed or elongated spring possesses some energy called elastic potential energy. This block has a capacity to do work. Similarly, a stone when released from a certain height also has energy in the form of gravitational potential energy. Two charged particles kept at certain distance has electric potential energy.

Regarding the potential energy it is important to note that it is defined for a conservative force field only. For non-conservative forces it has no meaning. The change in potential energy (dU) of a system corresponding to a conservative internal force is given by

$$dU = -\vec{F} \cdot d\vec{r} = -dW \qquad \left(F = -\frac{dU}{dr}\right)$$

or
$$\int_i^f dU = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

or
$$U_f - U_i = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

We generally choose the reference point at infinity and assume potential energy to be zero there, *i.e.*, if we take $r_i = \infty$ (infinite) and $U_i = 0$ then we can write

$$U = -\int_{\infty}^{\vec{r}} \vec{F} \cdot d\vec{r} = -W$$

or potential energy of a body or system is the negative of work done **by the conservative forces** in bringing it from infinity to the present position.

Regarding the potential energy it is worth noting that:

1. Potential energy can be defined only for conservative forces and it should be considered to be a property of the entire system rather than assigning it to any specific particle.
2. Potential energy depends on frame of reference.

Now, let us discuss three types of potential energies which we usually come across.

(a) Elastic Potential Energy

In Article 6.2, we have discussed the spring forces. We have seen there that the work done by the spring force (of course conservative for an ideal spring) is $-\frac{1}{2}kx^2$ when the spring is stretched or compressed by an amount x from its unstretched position. Thus,

$$U = -W = -\left(-\frac{1}{2}kx^2\right)$$

or
$$U = \frac{1}{2}kx^2 \qquad (k = \text{spring constant})$$

Note that elastic potential energy is always positive.

(b) Gravitational Potential Energy

The gravitational potential energy of two particles of masses m_1 and m_2 separated by a distance r is given by

$$U = -G\frac{m_1 m_2}{r}$$

Here, G = universal gravitation constant

$$= 6.67 \times 10^{-11} \frac{\text{N-m}^2}{\text{kg}^2}$$

If a body of mass m is raised to a height h from the surface of earth, the change in potential energy of the system (earth + body) comes out to be:

$$\Delta U = \frac{mgh}{\left(1 + \frac{h}{R}\right)} \qquad (R = \text{radius of earth})$$

or
$$\Delta U \approx mgh \quad \text{if} \quad h << R$$

Thus, the potential energy of a body at height h, *i.e.*, mgh is really the change in potential energy of the system for $h << R$. So, be careful while using $U = mgh$, that h should not be too large. This we will discuss in detail in the chapter of Gravitation.

(c) Electric Potential Energy

The electric potential energy of two point charges q_1 and q_2 separated by a distance r in vacuum is given by

$$U = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r}$$

Here,
$$\frac{1}{4\pi\varepsilon_0} = 9.0 \times 10^9 \frac{\text{N-m}^2}{\text{C}^2} = \text{constant}$$

6.7 Law of Conservation of Mechanical Energy

Suppose, only conservative forces operate on a system of particles and potential energy U is defined corresponding to these forces. There are either no other forces or the work done by them is zero. We have

$$U_f - U_i = -W$$

and
$$W = K_f - K_i \qquad \text{(from work energy theorem)}$$

then
$$U_f - U_i = -(K_f - K_i)$$

or
$$U_f + K_f = U_i + K_i \qquad \ldots\text{(i)}$$

The sum of the potential energy and the kinetic energy is called the total mechanical energy. We see from Eq. (i), that the **total mechanical energy of a system remains constant, if only conservative forces are acting on a system of particles and the work done by all other forces is zero.** This is called the conservation of mechanical energy.

The total mechanical energy is not constant, if non-conservative forces such as friction is acting between the parts of a system. However, the work energy theorem, is still valid. Thus, we can apply

$$W_c + W_{nc} + W_{\text{ext}} = K_f - K_i$$

Here,
$$W_c = -(U_f - U_i)$$

So, we get
$$W_{nc} + W_{\text{ext}} = (K_f + U_f) - (K_i + U_i)$$

or
$$W_{nc} + W_{\text{ext}} = E_f - E_i$$

Here, $E = K + U$ is the total mechanical energy.

• Problem Solving Technique

- If only conservative forces are acting on a system of particles and work done by any other external force is zero, then mechanical energy of the system will remain conserved. In this case some fraction of the mechanical energy will be decreasing while the other will be increasing. Problems can be solved by equating the magnitudes of the decrease and the increase. Let us see an example of this.

In the arrangement shown in figure string is light and inextensible and friction is absent everywhere. Find the speed of both the blocks after the block A has ascended a height of 1 m. Given that $m_A = 1$ kg and $m_B = 2$ kg. ($g = 10 \text{ m/s}^2$)

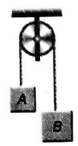


Fig. 6.20

Solution Friction is absent. Therefore, mechanical energy of the system will remain conserved. From constraint relations we see that speed of both the blocks will be same. Suppose it is v. Here gravitational potential energy of 2 kg block is decreasing while gravitational potential energy of 1 kg block is increasing. Similarly, kinetic energy of both the blocks is also increasing. So we can write:

Decrease in gravitational potential energy of 2 kg block = increase in gravitational potential energy of 1 kg block + increase in kinetic energy of 1 kg block + increase in kinetic energy of 2 kg block.

$\therefore \quad m_B gh = m_A gh + \frac{1}{2} m_A v^2 + \frac{1}{2} m_B v^2$

or $\quad (2)(10)(1) = (1)(10)(1) + \frac{1}{2}(1) v^2 + \frac{1}{2}(2) v^2$

or $\quad 20 = 10 + 0.5 v^2 + v^2$

or $\quad 1.5 v^2 = 10$

$\therefore \quad v^2 = 6.67 \text{ m}^2/\text{s}^2$

or $\quad v = 2.58$ m/s **Ans.**

- If some non-conservative forces such as friction are also acting on some parts of the system and work done by any other forces (excluding the conservative forces) is zero. Then we can apply

$$W_{nc} = E_f - E_i$$

or $\quad W_{nc} = (U_f - U_i) + (K_f - K_i) = \Delta U + \Delta K$

i.e., work done by non-conservative forces is equal to the change in mechanical (potential + kinetic) energy. But note that here all quantities are to be substituted with sign. Let us see an example of this.

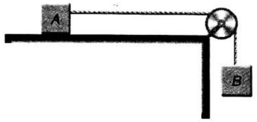


Fig. 6.21

In the arrangement shown in figure, $m_A = 1$ kg, $m_B = 4$ kg. String is light and inextensible while pulley is smooth. Coefficient of friction between block A and the table is $\mu = 0.2$. Find the speed of both the blocks when block B has descended a height $h = 1$ m. Take $g = 10 \text{ m/s}^2$.

Solution From constraint relation, we see that

$$v_A = v_B = v \text{ (say)}$$

Force of friction between block A and table will be

$$f = \mu m_A g = (0.2)(1)(10) = 2 \text{ N}$$

$\therefore \quad W_{nc} = \Delta U + \Delta K$

$\therefore \quad -fs = -m_B gh + \frac{1}{2}(m_A + m_B) v^2$

or $\quad (-2)(1) = -(4)(10)(1) + \frac{1}{2}(4+1) v^2$

$\quad -2 = -40 + 2.5 v^2$

or $\quad 2.5 v^2 = 38$

$\therefore \quad v^2 = 15.2 \text{ m}^2/\text{s}^2$

or $\quad v = 3.9$ m/s **Ans.**

Sample Example 6.11 *Consider the situation shown in figure. Mass of block A is m and that of block B is 2 m. The force constant of spring is K. Friction is absent everywhere. System is released from rest with the spring unstretched. Find :*

(a) the maximum extension of the spring x_m

(b) the speed of block A when the extension in the spring is $x = \frac{x_m}{2}$

(c) net acceleration of block B when extension in the spring is $x = \frac{x_m}{4}$

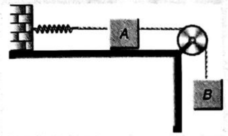


Fig. 6.22

Solution (a) At maximum extension in the spring

$$v_A = v_B = 0 \quad \text{(momentarily)}$$

Therefore, applying conservation of mechanical energy:

decrease in gravitational potential energy of block B = increase in elastic potential energy of spring.

or $\quad m_B g x_m = \frac{1}{2} K x_m^2$

or $\quad 2 mg x_m = \frac{1}{2} K x_m^2$

$\therefore \quad x_m = \frac{4 mg}{K}$ **Ans.**

(b) At $\quad x = \frac{x_m}{2} = \frac{2 mg}{K}$

Let $\quad v_A = v_B = v$ (say)

Then, decrease in gravitational potential energy of block B = increase in elastic potential energy of spring + increase in kinetic energy of both the blocks.

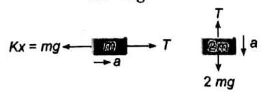
$\therefore \quad m_B g x = \frac{1}{2} K x^2 + \frac{1}{2}(m_A + m_B) v^2$

or $\quad (2m)(g)\left(\frac{2 mg}{K}\right) = \frac{1}{2} K \left(\frac{2 mg}{K}\right)^2 + \frac{1}{2}(m + 2m) v^2$

$\therefore \quad v = 2g \sqrt{\frac{m}{3K}}$ **Ans.**

(c) At $\quad x = \frac{x_m}{4} = \frac{mg}{K}$

or $\quad Kx = mg$



$$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{2 mg - mg}{3 m}$$

$= \frac{g}{3}$ (downwards) **Ans.**

Sample Example 6.12 *In the arrangement shown in figure $m_A = 4.0$ kg and $m_B = 1.0$ kg. The system is released from rest and block B is found to have a speed 0.3 m/s after it has descended through a distance of 1m. Find the coefficient of friction between the block and the table. Neglect friction elsewhere. (Take $g = 10$ m/s^2).*

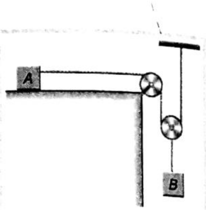


Fig. 6.23

Solution From constraint relations, we can see that

$$v_A = 2v_B$$

Therefore, $\quad v_A = 2(0.3) = 0.6$ m/s

as $\quad v_B = 0.3$ m/s (given)

Applying $\quad W_{nc} = \Delta U + \Delta K$

we get $\quad -\mu\, m_A g S_A = -m_B g S_B + \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2$

Here, $\quad S_A = 2S_B = 2$ m as $S_B = 1$ m (given)

$\therefore \quad -\mu(4.0)(10)(2) = -(1)(10)(1) + \frac{1}{2}(4)(0.6)^2 + \frac{1}{2}(1)(0.3)^2$

or $\quad -80\mu = -10 + 0.72 + 0.045$

or $\quad 80\mu = 9.235 \quad$ or $\quad \mu = 0.115$ **Ans.**

Introductory Exercise 6.3

1. In the figure block A is released from rest when the spring is in its natural length. For the block B of mass m to leave contact with the ground at some stage what should be the minimum mass of block A?

Fig. 6.24

2. A chain of mass m and length l lies on a horizontal table. The chain is allowed to slide down gently from the side of the table. Find the speed of the chain at the instant when last link of the chain slides from the table. Neglect friction everywhere.

3. As shown in figure a smooth rod is mounted just above a table top. A 10 kg collar, which is able to slide on the rod with negligible friction is fastened to a spring whose other end is attached to a pivot at O. The spring has negligible mass, a relaxed length of 10 cm and a spring constant of 500 N/m. The collar is released from rest at point A. (a) What is its velocity as it passes point B ? (b) Repeat for point C.

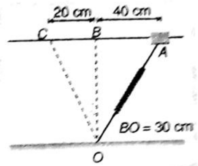


Fig. 6.25

4. A man pulls a bucket of water from a well of depth h. If mass of the rope and that of the bucket full of water are m and M respectively. Find the work done by the man.

5. A block of mass m is attached with a massless spring of force constant K. The block is placed over a rough inclined surface for which the coefficient of friction is $\mu = \frac{3}{4}$. Find the minimum value of M required to move the block up the plane. (Neglect mass of string and pulley. Ignore friction in pulley).

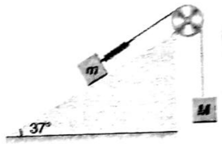


Fig. 6.26

6.8 Three Types of Equilibrium

A body is said to be in translatory equilibrium, if net force acting on the body is zero, *i.e.*,

$$\vec{\mathbf{F}}_{net} = 0$$

If the forces are conservative

$$F = -\frac{dU}{dr}$$

and for equilibrium $F = 0$.

So, $\quad -\frac{dU}{dr} = 0, \quad$ or $\quad \frac{dU}{dr} = 0$

i.e., at equilibrium position slope of U-r graph is zero or the potential energy is optimum (maximum or minimum or constant). Equilibrium are of three types, *i.e.*, the situation where $F = 0$ and $\frac{dU}{dr} = 0$ can be obtained under three conditions. These are stable equilibrium, unstable equilibrium and neutral equilibrium. These three types of equilibrium can be better understood from the given three figures:

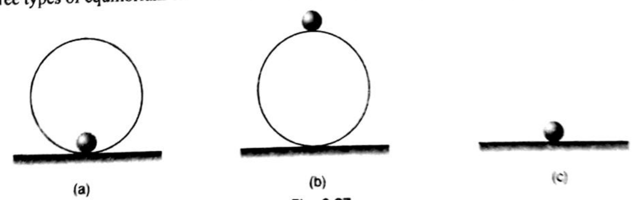


Fig. 6.27

Three identical balls are placed in equilibrium in positions as shown in figures (a), (b) and (c) respectively.

In Fig. (a) ball is placed inside a smooth spherical shell. This ball is in stable equilibrium position. In Fig. (b) the ball is placed over a smooth sphere. This is in unstable equilibrium position. In Fig. (c) the ball is placed on a smooth horizontal ground. This ball is in neutral equilibrium position.

The table given below explains what is the difference and what are the similarities between these three equilibrium positions in the language of physics.

Table 6.1

S. No.	Stable Equilibrium	Unstable Equilibrium	Neutral Equilibrium
1.	Net force is zero.	Net force is zero.	Net force is zero.
2.	$\frac{dU}{dr}=0$ or slope of U-r graph is zero.	$\frac{dU}{dr}=0$ or slope of U-r graph is zero.	$\frac{dU}{dr}=0$ or slope of U-r graph is zero.
3.	When displaced from its equilibrium position a net restoring force starts acting on the body which has a tendency to bring the body back to its equilibrium position.	When displaced from its equilibrium position, a net force starts acting on the body which moves the body in the direction of displacement or away from the equilibrium position.	When displaced from its equilibrium position the body has neither the tendency to come back nor to move away from the original position.
4.	Potential energy in equilibrium position is minimum as compared to its neighbouring points. or $\frac{d^2U}{dr^2}=$ positive	Potential energy in equilibrium position is maximum as compared to its neighbouring points. or $\frac{d^2U}{dr^2}=$ negative	Potential energy remains constant even if the body is displaced from its equilibrium position. or $\frac{d^2U}{dr^2}=0$
5.	When displaced from equilibrium position the centre of gravity of the body goes up.	When displaced from equilibrium position the centre of gravity of the body comes down.	When displaced from equilibrium position the centre of gravity of the body remains at the same level.

● *Important Points Regarding* Equilibrium

- If we plot graphs between F and r or U and r, F will be zero at equilibrium while U will be maximum, minimum or constant depending on the type of equilibrium. This all is shown in Fig. 6.28

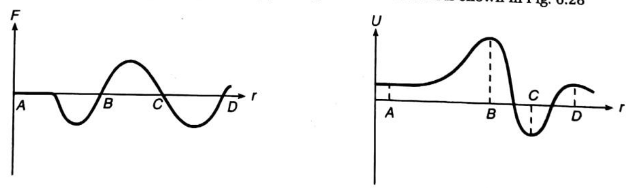


Fig. 6.28

At point A, $F=0$, $\frac{dU}{dr}=0$, but U is constant. Hence, A is neutral equilibrium position. At points B and D, $F=0$, $\frac{dU}{dr}=0$ but U is maximum. Thus, these are the points of unstable equilibrium.

At point C, $F=0$, $\frac{dU}{dr}=0$, but U is minimum. Hence, point C is in stable equilibrium position.

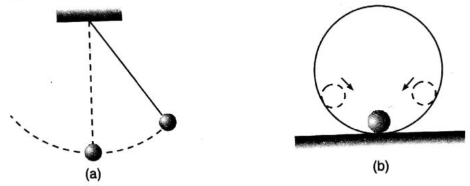


Fig. 6.29

- Oscillations of a body take place about stable equilibrium position. For example, bob of a pendulum oscillates about its lowest point which is also the stable equilibrium position of bob. Similarly, in Fig. 6.27 (b), the ball will oscillate about its stable equilibrium position.
- If a graph between F and r is as shown in figure, then $F=0$, at $r=r_1, r=r_2$ and $r=r_3$. Therefore, at these three points, body is in equilibrium. But these three positions are three different type of equilibriums. For example :
  at $r=r_1$, body is in unstable equilibrium. This is because, if we displace the body slightly rightwards (positive direction), force acting on the body is also positive, *i.e.*, away from $r=r_1$ position.
  at $r=r_2$, body is in stable equilibrium. Becaue if we displace the body rightwards (positive direction) force acting on the body is negative (or leftwards) or the force acting is restoring in nature.
  at $r=r_3$, equilibrium is neutral in nature. Because if we displace the body rightwards or leftwards force is again zero.

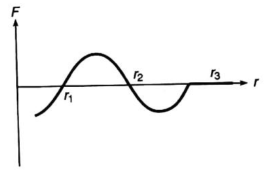


Fig. 6.30

Sample Example 6.13 *The potential energy of a conservative system is given by*

$$U=ax^2-bx$$

where a and b are positive constants. Find the equilibrium position and discuss whether the equilibrium is stable, unstable or neutral.

Solution In a conservative field

$$F=-\frac{dU}{dx}$$

$$\therefore \quad F=-\frac{d}{dx}(ax^2-bx)=b-2ax$$

For equilibrium $F=0$

or $$b-2ax=0 \quad \therefore \quad x=\frac{b}{2a}$$

From the given equation we can see that $\frac{d^2U}{dx^2}=2a$ (positive), *i.e.*, U is minimum.

Therefore, $x=\frac{b}{2a}$ is the stable equilibrium position. **Ans.**

6.9 Power

Power is the rate at which a force does work. If a force does 20 J of work in 10 s, the average rate at which it is working is 2 J/s or the average power is 2 W.

The work done by a force $\vec{F}$ in a small displacement $\vec{dr}$ is $dW = \vec{F} \cdot \vec{dr}$. Thus, the instantaneous power delivered by the force is

$$P = \frac{dW}{dt}$$

$$= \vec{F} \cdot \frac{\vec{dr}}{dt} = \vec{F} \cdot \vec{v} = Fv\cos\theta$$

Thus, power is equal to the scalar product of force with velocity. It is zero if force is perpendicular to velocity. For example, power of a centripetal force in a circular motion is zero.

Sample Example 6.14 *A train has a constant speed of 40 m/s on a level road against resistive force of magnitude 3×10^4 N. Find the power of the engine.*

Solution At constant speed, there is no acceleration, so the forces acting on the train are in equilibrium.

Therefore, $F = R$

$\therefore$ $F = 3 \times 10^4$ N

or $P = Fv$

We have, power $= 3 \times 10^4 \times 40 = 1.2 \times 10^6$ W **Ans.**

Sample Example 6.15 *A train of mass 2.0×10^5 kg has a constant speed of 20 m/s up a hill inclined at $\theta = \sin^{-1}\left(\frac{1}{50}\right)$ to the horizontal when the engine is working at 8.0×10^5 W. Find the resistance to motion of the train. ($g = 9.8$ m/s²)*

Solution Since, $P = Fv$

$$F = \frac{P}{v} = \frac{8.0 \times 10^5}{20} = 4.0 \times 10^4 \text{ N}$$

At constant speed, the forces acting on the train are in equilibrium. Resolving the forces parallel to the hill.

$$F = R + (2.0 \times 10^5)g \times \frac{1}{50}$$

$$4.0 \times 10^4 = R + 39200 \quad \text{or} \quad R = 800 \text{ N}$$

Therefore, the resistance is 800 N. **Ans.**

Sample Example Instance 6.16 *A block of mass m is pulled by a constant power P placed on a rough horizontal plane. The friction co-efficient between the block and surface is μ. Find the maximum velocity of the block.*

Solution Power $P = F \cdot v = \text{constant}$

$$F = \frac{P}{v} \quad \text{or} \quad F \propto \frac{1}{v}$$

as v increases, F decreases.

when $F = \mu mg$, net force on block becomes zero, *i.e.*, it has maximum or terminal velocity

$\therefore$ $P = (\mu mg)\, v_{max}$

or $v_{max} = \dfrac{P}{\mu mg}$ **Ans.**

Introductory Exercise 6.4

1. A ball of mass 1 kg is dropped from a tower. Find power of gravitational force at time $t = 2$ s. ($g = 10$ m/s²)
2. A particle of mass m is lying on smooth horizontal table. A constant force F tangential to the surface is applied on it. Find :
   (a) average power over a time interval from $t = 0$ to $t = t$,
   (b) instantaneous power as function of time t.
3. A constant power P is applied on a particle of mass m. Find kinetic energy, velocity and displacement of particle as function of time t.
4. A time varying power $P = 2t$ is applied on a particle of mass m. Find :
   (a) kinetic energy and velocity of particle as function of time,
   (b) average power over a time interval from $t = 0$ to $t = t$.
5. Potential energy of a particle along x-axis, varies as, $U = -20 + (x-2)^2$, where U is in joule and x in meter. Find the equilibrium position and state whether it is stable or unstable equilibrium.
6. Force acting on a particle constrained to move along x-axis is $F = (x - 4)$. Here, F is in newton and x in meter. Find the equilibrium position and state whether it is stable or unstable equilibrium.

Extra Points

- Work is a scalar quantity. It can be positive, negative or zero. The angle between $\vec{F}$ and $\vec{s}$ decides whether the work done is positive, negative or zero.
  If $0° \le \theta < 90°$, W = positive
  If $\theta = 90°$, $W = 0$ and
  If $90° < \theta \le 180°$, W = negative
- The CGS unit of work is erg.
  $$1 \text{ J} = 10^7 \text{ erg}$$
- Power is also measured in horse power (HP).
  $$1 \text{ HP} = 746 \text{ watt}$$
- If only conservative forces are acting on a system, the mechanical energy of the system remains constant. Mechanical energy comprises of kinetic and potential. In potential usually gravitational and elastic comes in the question (as far as problems of work, power and energy are concerned). Now it may happen that some part of the energy might be decreasing while other part might be increasing. Energy conservation equation now can be written in two ways.
  First method : Magnitude of decrease of energy = magnitude of increase of energy.

Second method : $E_i = E_f$ ($i \rightarrow$ initial and $f \rightarrow$ final)

i.e., write down total initial mechanical energy on one side and total final mechanical energy on the other side. While writing gravitational potential energy we choose some reference point (where $h = 0$), but throughout the question this reference point should not change. Let us take a simple example.

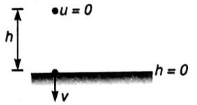


Fig. 6.31

A ball of mass 'm' is released from a height h as shown in figure. The velocity of particle at the instant when it strikes the ground can be found using energy conservation principle by following two methods.

Method 1 : Decrease in gravitational PE = increase in KE

or $$mgh = \frac{1}{2} mv^2$$

or $$v = \sqrt{2gh}$$

Method 2: $(\text{PE} + \text{KE})_i = (\text{PE} + \text{KE})_f$

For gravitational PE we take ground as the reference point.

$\therefore$ $$mgh + 0 = 0 + \frac{1}{2} mv^2$$

or $$v = \sqrt{2gh}$$

- If the system consists of frictional forces as well. Then, some mechanical energy will be lost in doing work against friction or
$$E_f < E_i$$
Now, suppose work done by friction is asked in the question, then find $E_f - E_i$ and if work done against friction is asked then write down $E_i - E_f$.
- Change in potential energy is equal to the negative of work done by the conservative force ($\Delta U = -\Delta W$). If work done by the conservative force is negative change in potential energy will be positive or potential energy of the system will increase and *vice-versa*.

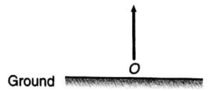


Fig. 6.32

This can be understood by a simple example. Suppose a ball is taken from the ground to some height, work done by gravity is negative, *i.e.*, change in potential energy should increase or potential energy of the ball will increase. Which happens so,

$$\Delta W_{\text{gravity}} = -\text{ ve}$$

$\therefore$ $$\Delta U = +\text{ ve} \qquad (\Delta U = -\Delta W)$$

or $$U_f - U_i = +\text{ ve}$$

- $F = -\dfrac{dU}{dr}$, *i.e.*, conservative forces always act in a direction where potential energy of the system is decreased. This can also be shown as in Fig 6.33.

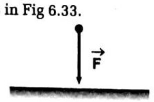


Fig. 6.33

If a ball is dropped from a certain height. The force on it (its weight) acts in a direction in which its potential energy decreases.

- Suppose a particle is released from point A with $u = 0$.

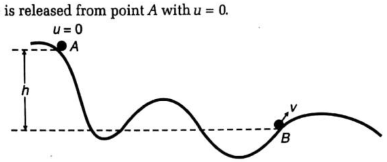


Fig. 6.34

Friction is absent everywhere. Then velocity at B will be
$$v = \sqrt{2gh}$$
(irrespective of the track it follows from A to B)

Here, $$h = h_A - h_B$$

- In circular motion, centripetal force acts towards centre. This force is perpendicular to small displacement $\vec{dS}$ and velocity $\vec{v}$. Hence, work done by it is zero and power of this force is also zero.
- For increase or decrease in gravitational potential energy of a particle (for small heights) we write,
$$\Delta U = mgh$$
Here, h is the change in height of particle. In case of a rigid body, h of centre of mass of the rigid body is seen.

Solved Examples

Level 1

Example 1 *An object of mass 5 kg falls from rest through a vertical distance of 20 m and attaches a velocity of 10 m/s. How much work is done by the resistance of the air on the object? ($g = 10\ m/s^2$)*

Solution Applying work-energy theorem,

work done by all the forces = change in kinetic energy

or
$$W_{mg} + W_{air} = \frac{1}{2}mv^2$$

$\therefore$
$$W_{air} = \frac{1}{2}mv^2 - W_{mg}$$
$$= \frac{1}{2}mv^2 - mgh$$
$$= \frac{1}{2}\times 5\times(10)^2 - (5)\times(10)\times(20)$$
$$= -750\ \text{J}$$ **Ans.**

Example 2 *A rod of length 1.0 m and mass 0.5 kg fixed at one end is initially hanging, vertical. The other end is now raised until it makes an angle 60° with the vertical. How much work is required?*

Solution For increase in gravitational potential energy of a rod we see the centre of the rod.

$$W = \text{change in potential energy}$$
$$= mg\frac{l}{2}(1-\cos\theta)$$

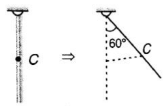


Fig. 6.35

Substituting the values, we have

$$W = (0.5)(9.8)\left(\frac{1.0}{2}\right)(1-\cos 60°)$$
$$= 1.225\ \text{J}$$ **Ans.**

Example 3 *A smooth narrow tube in the form of an arc AB of a circle of centre O and radius r is fixed so that A is vertically above O and OB is horizontal. Particles P of mass m and Q of mass 2 m with a light inextensible string of length ($\pi r/2$) connecting them are placed inside the tube with P at A and Q at B and released from rest. Assuming the string remains taut during motion, find the speed of particles when P reaches B.*

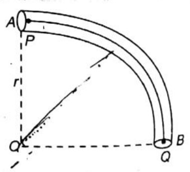


Fig. 6.36

Solution All surfaces are smooth. Therefore, mechanical energy of the system will remain conserved.

$\therefore$ Decrease in PE of both the blocks = increase in KE of both the blocks

$\therefore$
$$(mgr) + (2mg)\left(\frac{\pi r}{2}\right) = \frac{1}{2}(m+2m)v^2$$

or
$$v = \sqrt{\frac{2}{3}(1+\pi)gr}$$ **Ans.**

Example 4 *A small body of mass m is located on a horizontal plane at the point O. The body acquires a horizontal velocity v_0 due to friction. Find, the mean power developed by the friction force during the motion of the body, if the frictional coefficient $\mu = 0.27$, $m = 1.0$ kg and $v_0 = 1.5$ m/s.*

Solution The body gains velocity due to friction. The acceleration due to friction.

$$a = \frac{\text{force of friction}}{\text{mass}} = \frac{\mu mg}{m} = \mu g$$

Further,
$$v_0 = at$$

Therefore,
$$t = \frac{v_0}{a} = \frac{v_0}{\mu g} \quad \ldots\text{(i)}$$

From work energy theorem,

work done by force of friction = change in kinetic energy

or
$$W = \frac{1}{2}mv_0^2 \quad \ldots\text{(ii)}$$

$$\text{Mean power} = \frac{W}{t}$$

From Eqs. (i) and (ii)
$$P_{mean} = \frac{1}{2}\mu mg v_0$$

Substituting the values, we have
$$P_{mean} = \frac{1}{2}\times 0.27\times 1.0\times 9.8\times 1.5$$
$$\approx 2.0\ \text{W}$$ **Ans.**

Example 5 *A small mass m starts from rest and slides down the smooth spherical surface of R. Assume zero potential energy at the top. Find:*

(a) the change in potential energy,

(b) the kinetic energy,

(c) the speed of the mass as a function of the angle θ made by the radius through the mass with the vertical.

Solution In the figure $h = R(1-\cos\theta)$

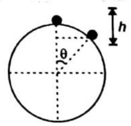


Fig. 6.37

(a) As the mass comes down, potential energy will decrease. Hence,
$$\Delta U = -mgh = -mgR(1-\cos\theta)$$

(b) Magnitude of decrease in potential energy = increase in kinetic energy

$\therefore$
$$\text{Kinetic energy} = mgh$$
$$= mgR(1-\cos\theta)$$ **Ans.**

(c) $$\frac{1}{2}mv^2 = mgR(1-\cos\theta)$$

$\therefore$ $$v = \sqrt{2gR(1-\cos\theta)}$$ **Ans.**

Example 6 *The displacement x of a particle moving in one dimension, under the action of a constant force is related to time t by the equation*

$$t = \sqrt{x} + 3$$

where x is in metre and t in second. Calculate: (a) the displacement of the particle when its velocity is zero, (b) the work done by the force in the first 6 s.

Solution As $$t = \sqrt{x} + 3 \quad i.e., \quad x = (t-3)^2 \qquad \text{...(i)}$$

So, $$v = (dx/dt) = 2(t-3) \qquad \text{...(ii)}$$

(a) v will be zero when $2(t-3) = 0$ *i.e.*, $t = 3$

Substituting this value of t in Eq. (i), $x = (3-3)^2 = 0$

i.e., when velocity is zero, displacement is also zero. **Ans.**

(b) From Eq. (ii),

$$(v)_{t=0} = 2(0-3) = -6 \text{ m/s}$$

and $$(v)_{t=6} = 2(6-3) = 6 \text{ m/s}$$

So, from work-energy theorem

$$W = \Delta\text{KE} = \frac{1}{2}m[v_f^2 - v_i^2] = \frac{1}{2}m[6^2 - (-6)^2] = 0$$

i.e., work done by the force in the first 6 s is zero. **Ans.**

Level 2

Example 1 *A smooth track in the form of a quarter-circle of radius 6 m lies in the vertical plane. A ring of weight 4 N moves from P_1 and P_2 under the action of forces F_1, F_2 and F_3. Force F_1 is always towards. P_2 and is always 20 N in magnitude; force $\vec{F}_2$ always acts horizontally and is always 30 N in magnitude; force $\vec{F}_3$ always acts tangentially to the track and is of magnitude $(15 - 10\,s)$ N, where s is in metre. If the particle has speed 4 m/s at P_1, what will its speed be at P_2?*

Solution The work done by $\vec{F}_1$ is

$$W_1 = \int_{P_1}^{P_2} F_1 \cos\theta\, ds \qquad (-2\theta)$$

From figure; $$s = R\left(\frac{\pi}{2} - 2\theta\right)$$

or $$ds = (6\text{ m})\,d(-2\theta) = -12\,d\theta$$

and $$F_1 = 20$$

Hence, $$W_1 = -240\int_{\pi/4}^{0} \cos\theta\, d\theta$$

$$= 240 \sin\frac{\pi}{4} = 120\sqrt{2} \text{ J}$$

$ds = R\,d\theta$

$= 6\,d(-2\theta)$

$= -12\,d\theta$

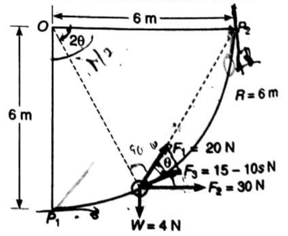


Fig. 6.38

The work done by $\vec{F}_3$ is

$$W_3 = \int F_3\, ds = \int_0^{6(\pi/2)} (15 - 10s)\, ds$$

$$= [15s - 5s^2]_0^{3\pi} = -302.8 \text{ J}$$

To calculate the work done by $\vec{F}_2$ and by W, it is convenient to take the projection of the path in the direction of the force, instead of *vice versa*. Thus,

$$W_2 = F_2(\overline{OP_2}) = 30(6) = 180 \text{ J}$$

$$W = (-W)(\overline{P_1O}) = (-4)(6) = -24 \text{ J}$$

The total work done is

$$W_1 + W_3 + W_2 + W = 23 \text{ J}$$

Then, by the work-energy principle.

$$K_{P_2} - K_{P_1} = 23 \text{ J}$$

$$= \frac{1}{2}\left(\frac{4}{9.8}\right)v_2^2 - \frac{1}{2}\left(\frac{4}{9.8}\right)(4)^2 = 23$$

$$v_2 = 11.3 \text{ m/s}$$ **Ans.**

Example 2 *One end of a light spring of natural length d and spring constant k is fixed on a rigid wall and the other is attached to a smooth ring of mass m which can slide without friction on a vertical rod fixed at a distance d from the wall. Initially the spring makes an angle of 37° with the horizontal as shown in Fig. 6.39. When the system is released from rest, find the speed of the ring when the spring becomes horizontal.* [sin 37° = 3/5]

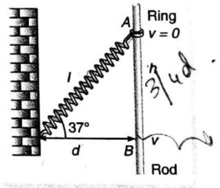


Fig. 6.39

Solution If l is the stretched length of the spring, then from figure

$$\frac{d}{l} = \cos 37° = \frac{4}{5}, \quad i.e., \quad l = \frac{5}{4}d$$

So, the stretch $$y = l - d = \frac{5}{4}d - d = \frac{d}{4}$$

and $$h = l\sin 37° = \frac{5}{4}d \times \frac{3}{5} = \frac{3}{4}d$$

Now, taking point B as reference level and applying law of conservation of mechanical energy between A and B,

$$E_A = E_B$$

or $$mgh + \frac{1}{2}ky^2 = \frac{1}{2}mv^2 \qquad [\text{as for } B,\ h = 0 \text{ and } y = 0]$$

or $$\frac{3}{4}mgd + \frac{1}{2}k\left(\frac{d}{4}\right)^2 = \frac{1}{2}mv^2 \qquad \left[\text{as for } A,\ h = \frac{3}{4}d \text{ and } y = \frac{1}{4}d\right]$$

or $$v = d\sqrt{\frac{3g}{2d} + \frac{k}{16m}}$$ **Ans.**

Example 3 *A single conservative force $F(x)$ acts on a 1.0 kg particle that moves along the x-axis. The potential energy $U(x)$ is given by:*

$$U(x) = 20 + (x-2)^2$$

where x is in meters. At $x = 5.0$ m the particle has a kinetic energy of 20 J.

(a) What is the mechanical energy of the system?

(b) Make a plot of $U(x)$ as a function of x for $-10\,m \le x \le 10\,m$, and on the same graph draw the line that represents the mechanical energy of the system.
Use part (b) to determine.

(c) The least value of x and

(d) The greatest value of x between which the particle can move.

(e) The maximum kinetic energy of the particle and

(f) The value of x at which it occurs.

(g) Determine the equation for $F(x)$ as a function of x.

(h) For what (finite) value of x does $F(x) = 0$?

Solution (a) Potential energy at $x = 5.0$ m is

$$U = 20 + (5-2)^2 = 29\text{ J}$$

$\therefore$ Mechanical energy

$$E = K + U = 20 + 29 = 49\text{ J}$$

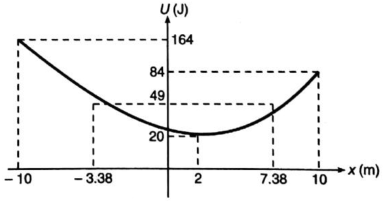


Fig. 6.40

(b) At $x = 10$ m, $U = 84$ J at $x = -10$ m, $U = 164$ J

and at $x = 2$ m, $U = \text{minimum} = 20$ J

(c) and (d) : Particle will move between the points where its kinetic energy becomes zero or its potential energy is equal to its mechanical energy.

Thus, $$49 = 20 + (x-2)^2$$

or $$(x-2)^2 = 29$$

or $$x - 2 = \pm\sqrt{29} = \pm 5.38\text{ m}$$

$\therefore$ $$x = 7.38\text{ m and } -3.38\text{ m}$$

or the particle will move between $x = -3.38$ m and $x = 7.38$ m

(e) and (f) : Maximum kinetic energy is at $x = 2$ m, where the potential energy is minimum and this maximum kinetic energy is,

$$K_{max} = E - U_{min} = 49 - 20$$
$$= 29\text{ J}$$

(g) $$F = -\frac{dU}{dx} = -2(x-2) = (2-x)$$

(h) $$F(x) = 0, \text{ at } x = 2.0\text{ m}$$

where potential energy is minimum (the position of stable equilibrium).

Example 4 *A small disc A slides down with initial velocity equal to zero from the top of a smooth hill of height H having a horizontal portion (Fig. 6.41). What must be the height of the horizontal portion h to ensure the maximum distance s covered by the disc? What is it equal to?*

Solution In order to obtain the velocity at point B, we apply the law of conservation of energy. So,

Loss in PE = Gain in KE

$$mg(H-h) = \frac{1}{2}mv^2$$

$\therefore$ $$v = \sqrt{[2g(H-h)]}$$

Further $$h = \frac{1}{2}gt^2$$

$\therefore$ $$t = \sqrt{(2h/g)}$$

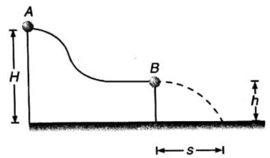


Fig. 6.41

Now, $$s = v \times t = \sqrt{[2g(H-h)]} \times \sqrt{(2h/g)}$$

or $$s = \sqrt{[4h(H-h)]} \quad \text{...(i)}$$

$\sqrt{\frac{2h}{g}}$

For maximum value of s $$ds/dh = 0$$

$\therefore$ $$\frac{1}{2\sqrt{[4h(H-h)]}} \times 4(H-2h) = 0$$

$h = \cancel{ut}^{0} + \frac{1}{2}gt^2$

or $$h = \frac{H}{2}$$ **Ans.**

Substituting $h = H/2$, in Eq. (i), we get

$$s = \sqrt{[4(H/2)(H-H/2)]} = \sqrt{H^2} = H$$ **Ans.**

Example 5 *A particle slides along a track with elevated ends and a flat central part as shown in Fig. 6.42. The flat portion BC has a length $l = 3.0$ m. The curved portions of the track are frictionless. For the flat part the coefficient of kinetic friction is $\mu_k = 0.20$, the particle is released at point A which is at height $h = 1.5$ m above the flat part of the track. Where does the particle finally comes to rest?*

Solution As initial mechanical energy of the particle is mgh and final is zero, so loss in mechanical energy = mgh. This mechanical energy is lost in doing work against friction in the flat part,

So, loss in mechanical energy = work done against friction

or $$mgh = \mu mgs \quad i.e., \quad s = \frac{h}{\mu} = \frac{1.5}{0.2} = 7.5\text{ m}$$

After starting from B the particle will reach C and then will rise up till the remaining KE at C is converted into potential energy. It will then again descend and at C will have the same value as it had when ascending, but now it will move from C to B. The same will be repeated and finally the particle will come to rest at E such that

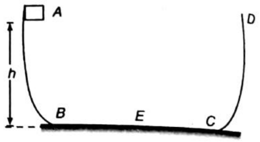


Fig. 6.42

$$BC + CB + BE = 7.5$$

or $$3 + 3 + BE = 7.5$$

i.e., $$BE = 1.5$$

So, the particle comes to rest at the centre of the flat part. **Ans.**

Example 6 *A 0.5 kg block slides from the point A on a horizontal track with an initial speed 3 m/s towards a weightless horizontal spring of length 1 m and force constant 2 N/m. The part AB of the track is frictionless and the part BC has the coefficient of static and kinetic friction as 0.22 and 0.20 respectively. If the distances AB and BD are 2 m and 2.14 m respectively, find the total distance through which the block moves before it comes to rest completely.* $[g = 10\ m/s^2]$

Solution As the track AB is frictionless, the block moves this distance without loss in its initial $\text{KE} = \frac{1}{2} mv^2 = \frac{1}{2} \times 0.5 \times 3^2 = 2.25$ J. In the path BD as friction is present, so work done against friction

$$= \mu_k mgs = 0.2 \times 0.5 \times 10 \times 2.14 = 2.14 \text{ J}$$

So, at D the KE of the block is $= 2.25 - 2.14 = 0.11$ J.

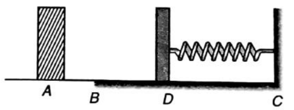


Fig. 6.43

Now, if the spring is compressed by x

$$0.11 = \frac{1}{2} \times k \times x^2 + \mu_k mgx$$

i.e., $$0.11 = \frac{1}{2} \times 2 \times x^2 + 0.2 \times 0.5 \times 10x$$

or $$x^2 + x - 0.11 = 0$$

which on solving gives positive value of $x = 0.1$ m

After moving the distance $x = 0.1$ m the block comes to rest. Now the compressed spring exerts a force:

$$F = kx = 2 \times 0.1 = 0.2 \text{ N}$$

on the block while limiting frictional force between block and track is $f_L = \mu_s mg = 0.22 \times 0.5 \times 10 = 1.1$ N. Since, $F < f_L$. The block will not move back. So, the total distance moved by the block

$$= AB + BD + 0.1$$
$$= 2 + 2.14 + 0.1$$
$$= 4.24 \text{ m}$$ **Ans.**

Work done = change in KE

Example 7 *A small disc of mass m slides down a smooth hill of height h without initial velocity and gets onto a plank of mass M lying on a smooth horizontal plane at the base of hill Fig. 6.44. Due to friction between the disc and the plank, disc slows down and finally moves as one piece with the plank. (a) Find out total work performed by the friction forces in this process. (b) can it be stated that the result obtained does not depend on the choice of the reference frame.*

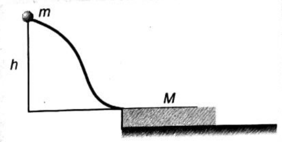


Fig. 6.44

Solution (a) When the disc slides down and comes onto the plank, then

$$mgh = \frac{1}{2} mv^2$$

$$\therefore \quad v = \sqrt{(2gh)} \quad \text{...(i)}$$

Let v_1 be the common velocity of both the disc and plank when they move together. From law of conservation of linear momentum,

$$mv = (M + m) v_1$$

$$\therefore \quad v_1 = \frac{mv}{(M + m)} \quad \text{...(ii)}$$

Now, change in KE $= (K)_f - (K)_i = (\text{work done})_{\text{friction}}$

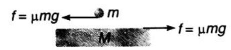
$$\therefore \quad \frac{1}{2}(M + m) v_1^2 - \frac{1}{2} mv^2 = (\text{work done})_{\text{friction}}$$

or $$W_{fr} = \frac{1}{2}(M + m)\left[\frac{mv}{M + m}\right]^2 - \frac{1}{2} mv^2$$

$$= \frac{1}{2} mv^2 \left[\frac{m}{M + m} - 1\right]$$

as $$\frac{1}{2} mv^2 = mgh \quad \therefore \quad W_{fr} = -mgh\left[\frac{M}{M + m}\right]$$ **Ans.**

(b) In part (a) we have calculated work done from the ground frame of reference. Now, let us take plank as the reference frame.



FBD with respect to ground.

Acceleration of plank $a_0 = \frac{f}{M} = \frac{\mu mg}{M}$

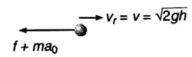


Fig. 6.45

Free body diagram of disc with respect to plank is shown in figure.

Here, ma_0 = pseudo force.

$\therefore$ Retardation of disc w.r.t. plank.

$$a_r = \frac{f + ma_0}{m} = \frac{\mu mg + \frac{\mu m^2 g}{M}}{m} = \mu g + \frac{\mu mg}{M}$$

$$= \left(\frac{M + m}{M}\right)\mu g$$

The disc will stop after travelling a distance S_r relative to plank, where

$$S_r = \frac{v_r^2}{2a_r} \qquad (0 = v_r^2 - 2a_r S_r)$$

$$= \frac{Mgh}{(M+m)\mu g}$$

$\therefore$ Work done by friction in this frame of reference

$$W_{fr} = -fS_r = -(\mu mg)\left[\frac{Mgh}{(M+m)\mu g}\right]$$

$$= -\frac{M\,mgh}{(M+m)} \qquad \text{which is same as part (a)}$$

Note *Work done by friction in this problem does not depend upon the frame of reference, otherwise in general work depends upon reference frame.*

Example 8 *Two blocks A and B are connected to each other by a string and a spring. The string passes over a frictionless pulley as shown in Fig. 6.46. Block B slides over the horizontal top surface of a stationary block C and the block A slides along the vertical side of C, both with the same uniform speed. The coefficient of friction between the surfaces of the blocks is 0.2. The force constant of the spring is* $1960\ Nm^{-1}$. *If the mass of block A is 2 kg, calculate the mass of block B and the energy stored in the spring.* ($g = 9.8\ m/s^2$)

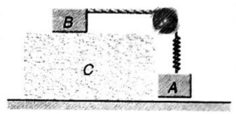


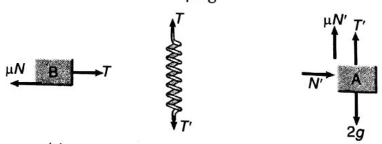
Fig. 6.46

Solution Let m be the mass of B. From its free-body diagram

$$T - \mu N = m \times 0 = 0$$

where T = tension of the string and $N = mg$

$$\therefore \qquad T = \mu mg$$



(a) (b) (c)

Fig. 6.47

From the free-body diagram of the spring

$$T - T' = 0$$

where T' is the force exerted by A on the spring

or
$$T = T' = \mu mg$$

From the free-body diagram of A

$$2g - (T' + \mu N') = 2 \times 0 = 0$$

where N' is the normal reaction of the vertical wall of C on A and $N' = 2 \times 0$ (as there is no horizontal acceleration of A)

$$\therefore \qquad 2g = T' = \mu mg \quad \text{or} \quad m = \frac{2g}{\mu g} = \frac{2}{0.2} = 10\ \text{kg} \qquad \textbf{Ans.}$$

Tensile force on the spring $= T$ or $T' = \mu mg = 0.2 \times 10 \times 9.8 = 19.6$ N

Now, in a spring tensile force = force constant × extension

$$\therefore \qquad 19.6 = 1960x \quad \text{or} \quad x = \frac{1}{100}\ \text{m}$$

or
$$U\ (\text{energy of a spring}) = \frac{1}{2}kx^2$$

$$= \frac{1}{2} \times 1960 \times \left(\frac{1}{100}\right)^2 = 0.098\ \text{J} \qquad \textbf{Ans.}$$

Example 9 *A particle of mass m is moving in a circular path of constant radius r, such that its centripetal acceleration* a_c *is varying with time t as* $a_c = k^2 r t^2$ *where k is a constant. What is the power delivered to the particle by the forces acting on it?* **[IIT JEE 1994]**

Solution As $a_c = (v^2/r)$ so $(v^2/r) = k^2 r t^2$

$$\therefore \qquad \text{Kinetic energy } K = \frac{1}{2}mv^2 = \frac{1}{2}mk^2r^2t^2$$

Now, from work-energy theorem $W = \Delta K = \frac{1}{2}mk^2r^2t^2 - 0$ [as at $t = 0$, $K = 0$]

So, $P = \dfrac{dW}{dt} = \dfrac{d}{dt}\left(\dfrac{1}{2}mk^2r^2t^2\right) = mk^2r^2t$ **Ans.**

Alternate solution : Given that $a_c = k^2rt^2$, so that

$$F_c = ma_c = mk^2rt^2$$

Now, as $a_c = (v^2/r)$, so $(v^2/r) = k^2rt^2$ or $v = krt$

So, that $a_t = (dv/dt) = kr$

i.e., $F_t = ma_t = mkr$

Now, as $\vec{\mathbf{F}} = \vec{\mathbf{F}}_c + \vec{\mathbf{F}}_t$

So, $P = \dfrac{dW}{dt} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} = (\vec{\mathbf{F}}_c + \vec{\mathbf{F}}_t) \cdot \vec{\mathbf{v}}$

Now, in circular motion $\vec{\mathbf{F}}_c$ is perpendicular to $\vec{\mathbf{v}}$ while $\vec{\mathbf{F}}_t$ parallel, so

$$P = F_t v \qquad [\text{as } \vec{\mathbf{F}}_c \cdot \vec{\mathbf{v}} = 0]$$

$$\therefore \qquad P = mk^2r^2t \qquad \textbf{Ans.}$$

EXERCISES

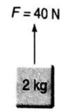
AIEEE Corner

Subjective Questions (Level 1)

Work Done

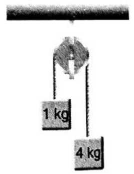
(a) By a constant force

1. A block of mass 2 kg is pulled upwards by a force $F = 40$ N as shown in figure. Block is displaced by 2 m. Find work done by the applied force and also due to the force of gravity on the block. (Take $g = 10 \text{ m/s}^2$)



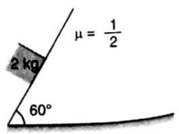
2. A block is displaced from (1m, 4m, 6m) to $(2\hat{i} + 3\hat{j} - 4\hat{k})$ m under a constant force $\vec{F} = (6\hat{i} - 2\hat{j} + \hat{k})$ N. Find the work done by this force.

3. The system shown in figure is released from rest. String is massless and pulley is smooth. Find :
(a) the work done by gravity on 4 kg block in 2 s,
(b) the work done by string on 1 kg block in the same interval of time.
(Take $g = 10 \text{ m/s}^2$)



4. A block of mass 2.5 kg is pushed 2.20 m along a frictionless horizontal table by a constant 16 N force directed 45° above the horizontal. Determine the work done by :
(a) the applied force,
(b) the normal force exerted by the table,
(c) the force of gravity and
(d) determine the total work done on the block.

5. A block of mass 2 kg is released from rest on a rough inclined ground as shown in figure. Find the work done on the block by :
(a) gravity,
(b) force of friction.
when the block is displaced downwards along the plane by 2 m.
(Take $g = 10 \text{ m/s}^2$)



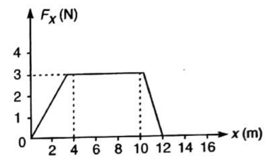
(b) By a variable force

6. A block is constrained to move along x-axis under a force $F = -2x$. Here, F is in newton and x in metre. Find the work done by this force when the block is displaced from $x = 2$ m to $x = -4$ m.

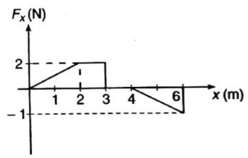
7. A block is constrained to move along x-axis under a force $F = \frac{4}{x^2}$ $(x \neq 0)$. Here, F is in newton and x in metre. Find the work done by this force when the block is displaced from $x = 4$ m to $x = 2$ m.

(c) By area under *F-x* graph

8. A particle is subjected to a force F_x that varies with position as shown in figure. Find the work done by the force on the body as it moves :
(a) from $x = 10.0$ m to $x = 5.0$ m,
(b) from $x = 5.0$ m to $x = 10.0$ m,
(c) from $x = 10.0$ m to $x = 15.0$ m,
(d) what is the total work done by the force over the distance $x = 0$ to $x = 15.0$ m ?



9. A child applies a force $\vec{F}$ parallel to the x-axis to a block moving on a horizontal surface. As the child controls the speed of the block, the x-component of the force varies with the x-coordinate of the block as shown in figure. Calculate the work done by the force $\vec{F}$ when the block moves :
(a) from $x = 0$ to $x = 3.0$ m
(b) from $x = 3.0$ m to $x = 4.0$ m
(c) from $x = 4.0$ m to $x = 7.0$ m
(d) from $x = 0$ to $x = 7.0$ m



Conservative Force Field and Potential Energy

10. The potential energy of a two particle system separated by a distance r is given by $U(r) = \frac{A}{r}$ where A is a constant. Find the radial force F_r, that each particle exerts on the other.

11. A single conservative force F_x acts on a 2 kg particle that moves along the x-axis. The potential energy is given by :
$$U = (x - 4)^2 - 16$$
Here, x is in metre and U in joule. At $x = 6.0$ m kinetic energy of particle is 8 J. Find :
(a) total mechanical energy
(b) maximum kinetic energy
(c) values of x between which particle moves
(d) the equation of F_x as a function of x.
(e) the value of x at which F_x is zero.

Kinetic Energy and Work-Energy Theorem

12. Momentum of a particle is increased by 50%. By how much percentage kinetic energy of particle will increase ?

13. Kinetic energy of a particle is increased by 1%. By how much percentage momentum of the particle will increase ?

14. Displacement of a particle of mass 2 kg varies with time as : $s = (2t^2 - 2t + 10)$ m. Find total work done on the particle in a time interval from $t = 0$ to $t = 2$ s.

15. A block of mass 30 kg is being brought down by a chain. If the block acquires a speed of 40 cm/s in dropping down 2 m. Find the work done by the chain during the process. $(g = 10\,\text{m/s}^2)$

16. An object of mass 5 kg falls from rest through a vertical distance of 20 m and reaches a velocity of 10 m/s. How much work is done by the push of the air on the object ? $(g = 10\,\text{m/s}^2)$

17. A 1.5 kg block is initially at rest on a horizontal frictionless surface when a horizontal force in the positive direction of x-axis is applied to the block. The force is given by $\vec{\mathbf{F}}(x) = (2.5 - x^2)\hat{\mathbf{i}}$ N, where x is in metre and the initial position of the block is $x = 0$.

(a) What is the kinetic energy of the block as it passes through $x = 2.0$ m ?

(b) What is the maximum kinetic energy of the block between $x = 0$ and $x = 2.0$ m ?

18. A helicopter lifts a 72 kg astronaut 15 m vertically from the ocean by means of a cable. The acceleration of the astronaut is $g/10$. How much work is done on the astronaut by $(g = 9.8\,\text{m/s}^2)$

(a) the force from the helicopter and

(b) the gravitational force on her ?

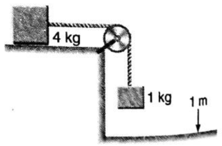
(c) What are the kinetic energy and

(d) the speed of the astronaut just before she reaches the helicopter ?

Mechanical Energy

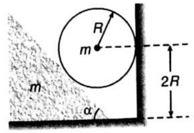
(a) Without friction (when mechanical energy is conserved)

19. A 4 kg block is on a smooth horizontal table. The block is connected to a second block of mass 1 kg by a massless flexible taut cord that passes over a frictionless pulley. The 1 kg block is 1 m above the floor. The two blocks are released from rest. With what speed does the 1 kg block hit the ground ?



20. Block A has a weight of 300 N and block B has a weight of 50 N. Determine the distance that A must descend from rest before it obtains a speed of 2.5 m/s. Neglect the mass of the cord and pulleys.

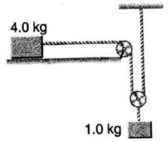
21. A sphere of mass m held at a height $2R$ between a wedge of same mass m and a rigid wall, is released from rest. Assuming that all the surfaces are frictionless. Find the speed of both the bodies when the sphere hits the ground.



22. The system is released from rest with the spring initially stretched 75 mm. Calculate the velocity v of the block after it has dropped 12 mm. The spring has a stiffness of 1050 N/m. Neglect the mass of the small pulley.

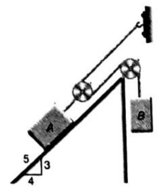
(b) With friction (when mechanical energy does not remain conserved)

23. Consider the situation shown in figure. The system is released from rest and the block of mass 1 kg is found to have a speed 0.3 m/s after it has descended through a distance of 1 m. Find the coefficient of kinetic friction between the block and the table.

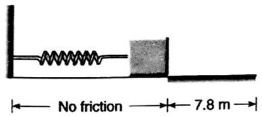


24. A disc of mass 50 g slides with zero initial velocity down an inclined plane set at an angle 30° to the horizontal. Having traversed a distance of 50 cm along the horizontal plane, the disc stops. Find the work performed by the friction forces over the whole distance, assuming the friction coefficient 0.15 for both inclined and horizontal planes. $(g = 10\,\text{m/s}^2)$

25. Block A has a weight of 300 N and block B has a weight of 50 N. If the coefficient of kinetic friction between the incline and block A is $\mu_k = 0.2$. Determine the speed of block A after it moves 1 m down the plane, starting from rest. Neglect the mass of the cord and pulleys.

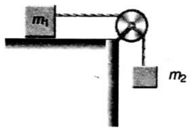


26. Figure shows, a 3.5 kg block accelerated by a compressed spring whose spring constant is 640 N/m. After leaving the spring at the spring's relaxed length, the block travels over a horizontal surface, with a coefficient of kinetic friction of 0.25, for a distance of 7.8 m before stopping. ($g = 9.8\,\text{m/s}^2$)



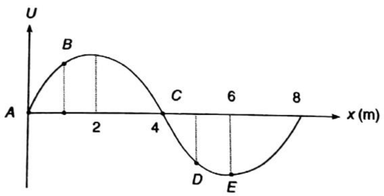
(a) What is the increase in the thermal energy of the block-floor system ?
(b) What is the maximum kinetic energy of the block ?
(c) Through what distance is the spring compressed before the block begins to move ?

27. Two masses $m_1 = 10$ kg and $m_2 = 5$ kg are connected by an ideal string as shown in the figure. The coefficient of friction between m_1 and the surface is $\mu = 0.2$. Assuming that the system is released from rest. Calculate the velocity of blocks when m_2 has descended by 4 m. ($g = 10\,\text{m/s}^2$)



Three Types of Equilibrium

28. For the potential energy curve shown in figure.



(a) Determine whether the force F is positive, negative or zero at the five points A, B, C, D and E.
(b) Indicate points of stable, unstable and neutral equilibrium.

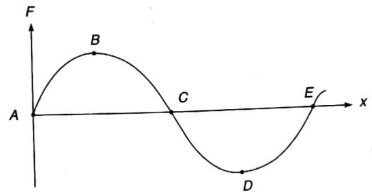
29. Potential energy of a particle moving along x-axis is given by :

$$U = \left(\frac{x^3}{3} - 4x + 6\right)$$

Here, U is in joule and x in metre. Find position of stable and unstable equilibrium.

30. Force acting on a particle moving along x-axis is as shown in figure. Find points of stable and unstable equilibrium.

31. Two point charges $+q$ and $+q$ are fixed at $(a, 0, 0)$ and $(-a, 0, 0)$. A third point charge $-q$ is at origin. State whether its equilibrium is stable, unstable or neutral if it is slightly displaced :



(a) along x-axis. (b) along y-axis.

Power

32. A block of mass 1 kg starts moving with constant acceleration $a = 4\,\text{m/s}^2$. Find :
(a) average power of the net force in a time interval from $t = 0$ to $t = 2$ s,
(b) instantaneous power of the net force at $t = 4$ s.

33. An engine working at a constant power P draws a load of mass m against a resistance r. Find the maximum speed of the load and the time taken to attain half this speed.

Objective Questions (Level 1)

Single Correct Option

1. Identify, which of the following energies can be positive only
(a) kinetic energy (b) potential energy
(c) mechanical energy (d) both kinetic and mechanical energies

2. The total work done on a particle is equal to the change in its kinetic energy
(a) always
(b) only if the force acting on the body are conservative
(c) only in the inertial frame
(d) only if no external force is acting

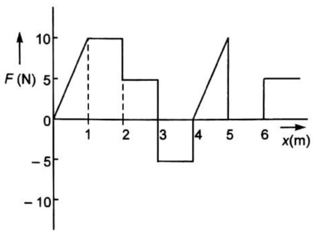
3. Work done by force of static friction
(a) can be positive (b) can be negative (c) can be zero (d) All of these

4. Work done when a force $\vec{F} = (\hat{i} + 2\hat{j} + 3\hat{k})$ N acting on a partricle takes it from the point $\vec{r}_1 = (\hat{i} + \hat{j} + \hat{k})$ to the point $\vec{r}_2 = (\hat{i} - \hat{j} + 2\hat{k})$ is
(a) -3 J (b) -1 J (c) zero (d) 2 J

5. A particle moves along the x-axis from $x = 0$ to $x = 5$ m under the influence of a force given by $F = 7 - 2x + 3x^2$. The work done in the process is
(a) 360 J (b) 85 J (c) 185 J (d) 135 J

6. A particle moves with a velocity $\vec{v} = (5\hat{i} - 3\hat{j} + 6\hat{k})\ ms^{-1}$ under the influence of a constant force $\vec{F} = (10\hat{i} + 10\hat{j} + 20\hat{k})$ N. The instantaneous power applied to the particle is
(a) 200 W (b) 320 W (c) 140 W (d) 170 W

7. The relationship between the force F and position x of a body is as shown in figure. The work done in displacing the body from $x = 1$ m to $x = 5$ m will be



(a) 30 J (b) 15 J (c) 25 J (d) 20 J

8. A pump is required to lift 800 kg of water per minute from a 10 m deep well and eject it with speed of 20 m/s. The required power in watts of the pump will be
(a) 6000 (b) 4000 (c) 5000 (d) 8000

9. Under the action of a force, a 2 kg body moves such that its position x as a function of time is given by $x = \frac{t^3}{3}$, where x is in metre and t in second. The work done by the force in the first two seconds is
(a) 1600 J (b) 160 J (c) 16 J (d) 1.6 J

10. The kinetic energy of a projectile at its highest position is K. If the range of the projectile is four times the height of the projectile, then the initial kinetic energy of the projectile is
(a) $\sqrt{2}K$ (b) $2K$ (c) $4K$ (d) $2\sqrt{2}K$

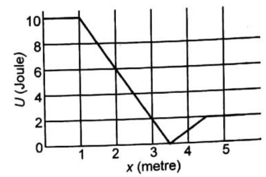
11. Power applied to a particle varies with time as $P = (3t^2 - 2t + 1)$ watt, where t is in second. Find the change in its kinetic energy between time $t = 2$ s and $t = 4$ s
(a) 32 J (b) 46 J (c) 61 J (d) 102 J

12. A block of mass 10 kg is moving in x-direction with a constant speed of 10 m/s. It is subjected to a retarding force $F = -0.1x$ J/m during its travel from $x = 20$ m to $x = 30$ m. Its final kinetic energy will be
(a) 475 J (b) 450 J (c) 275 J (d) 250 J

13. A ball of mass 12 kg and another of mass 6 kg are dropped from a 60 feet tall building. After a fall of 30 feet each, towards earth, their kinetic energies will be in the ratio of
(a) $\sqrt{2}:1$ (b) $1:4$ (c) $1:2$ (d) $1:\sqrt{2}$

14. A spring of spring constant 5×10^3 N/m is stretched initially by 5 cm from the unstretched position. The work required to further stretch the spring by another 5 cm is
(a) 6.25 N-m (b) 12.50 N-m (c) 18.75 N-m (d) 25.00 N-m

15. A body with mass 1 kg moves in one direction in the presence of a force which is described by the potential energy graph. If the body is released from rest at $x = 2$m, than its speed when it crosses $x = 5$ m is (Neglect dissipative forces)



(a) $2\sqrt{2}\ ms^{-1}$ (b) $1\ ms^{-1}$ (c) $2\ ms^{-1}$ (d) $3\ ms^{-1}$

16. A body has kinetic energy E when projected at angle of projection for maximum range. Its kinetic energy at the highest point of its path will be
(a) E (b) $\frac{E}{2}$ (c) $\frac{E}{\sqrt{2}}$ (d) zero

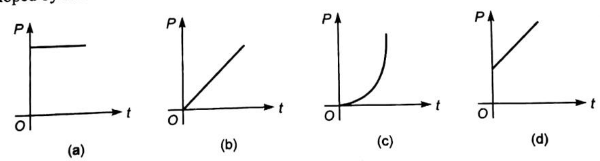
17. A person pulls a bucket of water from a well of depth h. If the mass of uniform rope is m and that of the bucket full of water is M, then work done by the person is
(a) $\left(M + \frac{m}{2}\right)gh$ (b) $\frac{1}{2}(M + m)gh$ (c) $(M + m)gh$ (d) $\left(\frac{M}{2} + m\right)gh$

18. The minimum stopping distance of a car moving with velocity v is x. If the car is moving with velocity $2v$, then the minimum stopping distance will be
(a) $2x$ (b) $4x$ (c) $3x$ (d) $8x$

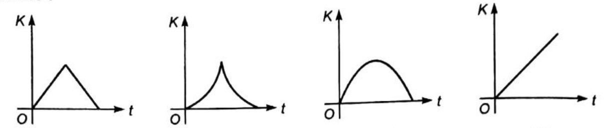
19. A projectile is fired from the origin with a velocity v_0 at an angle θ with the x-axis. The speed of the projectile at an altitude h is
(a) $v_0 \cos\theta$ (b) $\sqrt{v_0^2 - 2gh}$ (c) $\sqrt{v_0^2 \sin^2\theta - 2gh}$ (d) None of these

20. A particle of mass m moves from rest under the action of a constant force F which acts for two seconds. The maximum power attained is
(a) $2Fm$ (b) $\frac{F^2}{m}$ (c) $\frac{2F}{m}$ (d) $\frac{2F^2}{m}$

21. A body moves under the action of a constant force along a straight line. The instantaneous power developed by this force with time t is correctly represented by



22. A ball is dropped at $t = 0$ from a height on a smooth elastic surface. Identify the graph which correctly represents the variation of kinetic energy K with time t



(a) (b) (c) (d)

23. A block of mass 5 kg is raised from the bottom of the lake to a height of 3 m without change in kinetic energy. If the density of the block is 3000 kg m^{-3}, then the work done is equal to
(a) 100 J (b) 150 J (c) 50 J (d) 75 J

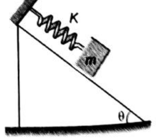
24. A body of mass m is projected at an angle θ with the horizontal with an initial velocity v_0. The average power of gravitational force over the whole time of flight is
(a) $mg \cos\theta$ (b) $\frac{1}{2} mg \sqrt{u \cos\theta}$
(c) $\frac{1}{2} mgu \sin\theta$ (d) zero

25. A spring of force constant k is cut in two parts at its one-third length. When both the parts are stretched by same amount. The work done in the two parts will be :

Note *Spring constant of a spring is inversely proportional to length of spring*
(a) equal in both (b) greater for the longer part
(c) greater for the shorter part (d) data insufficient

26. A particle moves under the action of a force $\vec{F} = 20\hat{i} + 15\hat{j}$ along a straight line $3y + \alpha x = 5$ where α is a constant. If the work done by the force F is zero, then the value of α is
(a) $\frac{4}{9}$ (b) $\frac{9}{4}$ (c) 3 (d) 4

27. A system of wedge and block as shown in figure, is released with the spring in its natural length. All surfaces are frictionless. Maximum elongation in the spring will be
(a) $\frac{2mg\sin\theta}{K}$
(b) $\frac{mg\sin\theta}{K}$
(c) $\frac{4mg\sin\theta}{K}$
(d) $\frac{mg\sin\theta}{2K}$



28. A force $\vec{F} = (3t\,\hat{i} + 5\hat{j})$N acts on a body due to which its displacement varies as $\vec{S} = (2t^2\,\hat{i} - 5\hat{j})$m. Work done by this force in 2 second is
(a) 23 J (b) 32 J (c) 46 J (d) 20 J

29. An open knife of mass m is dropped from a height h on a wooden floor. If the blade penetrates up to the depth d into the wood, the average resistance offered by the wood to the knife edge is
(a) $mg\left(1+\frac{h}{d}\right)$ (b) $mg\left(1+\frac{h}{d}\right)^2$ (c) $mg\left(1-\frac{h}{d}\right)$ (d) $mg\left(1+\frac{d}{h}\right)$

30. Two springs have force constants k_A and k_B such that $k_B = 2k_A$. The four ends of the springs are stretched by the same force. If energy stored in spring A is E, then energy stored in spring B is
(a) $\frac{E}{2}$ (b) $2E$ (c) E (d) $4E$

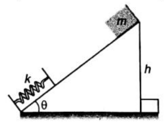
31. A mass of 0.5 kg moving with a speed of 1.5 m/s on a horizontal smooth surface, collides with a nearly weightless spring of force constant $k = 50$ N/m. The maximum compression of the spring would be
(a) 0.15 m (b) 0.12 m (c) 0.5 m (d) 0.25 m

32. A bullet moving with a speed of 100 ms^{-1} can just penetrate into two planks of equal thickness. Then the number of such planks, if speed is doubled will be
(a) 6 (b) 10 (c) 4 (d) 8

33. A body of mass 100 g is attached to a hanging spring whose force constant is 10 N/m. The body is lifted until the spring is in its unstretched state and then released. Calculate the speed of the body when it strikes the table 15 cm below the release point
(a) 1 m/s (b) 0.866 m/s (c) 0.225 m/s (d) 1.5 m/s

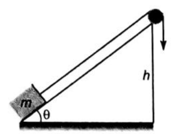
34. An ideal massless spring S can be compressed 1.0 m in equilibrium by a force of 100 N. This same spring is placed at the bottom of a friction less inclined plane which makes an angle $\theta = 30°$ with the horizontal. A 10 kg mass m is released from the rest at the top of the inclined plane and is brought to rest momentarily after compressing the spring by 2.0 m. The distance through which the mass moved before coming to rest is
(a) 8 m (b) 6 m (c) 4 m (d) 5 m

35. A body of mass m is released from a height h on a smooth inclined plane that is shown in the figure. The following can be true about the velocity of the block knowing that the wedge is fixed



(a) v is highest when it just touches the spring
(b) v is highest when it compresses the spring by some amount
(c) v is highest when the spring comes back to natural position
(d) data insufficient

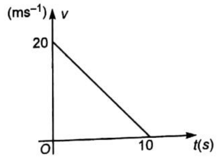
36. A block of mass m is directly pulled up slowly on a smooth inclined plane of height h and inclination θ with the help of a string parallel to the incline. Which of the following statement is incorrect for the block when it moves up from the bottom to the top of the incline?
(a) Work done by the normal reaction force is zero.
(b) Work done by the string is mgh
(c) Work done by gravity is mgh
(d) Net work done on the block is zero



37. A spring of natural length l is compressed vertically downward against the floor so that its compressed length becomes $\frac{l}{2}$. On releasing, the spring attains its natural length. If k is the stiffness constant of spring, then the work done by the spring on the floor is

(a) zero (b) $\frac{1}{2}kl^2$ (c) $\frac{1}{2}k\left(\frac{l}{2}\right)^2$ (d) kl^2

38. The velocity of a particle decreases uniformly from 20 ms^{-1} to zero in 10 s as shown in figure. If the mass of the particle is 2 kg, then identify the correct statement.



(a) The net force acting on the particle is opposite to the direction of motion
(b) The work done by friction force is −400 J
(c) The magnitude of friction force acting on the particle is 4 N
(d) All of the above

39. A ball is dropped onto a floor from a height of 10 m. If 20% of its initial energy is lost, then the height of bounce is
(a) 2 m (b) 4 m (c) 8 m (d) 6.4 m

JEE Corner

Assertion and Reason

Directions : *Choose the correct option.*

*(a) If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*
*(b) If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*
*(c) If **Assertion** is true, but the **Reason** is false.*
*(d) If **Assertion** is false but the **Reason** is true.*

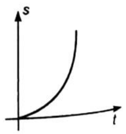
1. **Assertion :** Power of a constant force is also constant.
Reason : Net constant force will always produce a constant acceleration.

2. **Assertion :** A body is moved from $x = 2$ to $x = 1$, under a force $F = 4x$, the work done by this force is negative.
Reason : Force and displacement are in opposite directions.

3. **Assertion :** If work done by conservative forces is positive, kinetic energy will increase.
Reason : Because potential energy will decrease.

4. **Assertion :** In circular motion work done by all the forces acting on the body is zero.
Reason : Centripetal froce and velocity are mutually perpendicular.

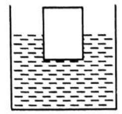
5. **Assertion :** Corresponding to displacement- time graph of a particle moving in a straight line we can say that total work done by all the forces acting on the body is positive.
Reason : Speed of particle is increasing.



6. **Assertion :** Work done by a constant force is path independent.
Reason : All constant forces are conservative in nature.

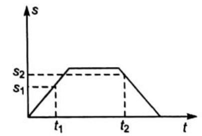
7. **Assertion :** Work-energy theorem can be applied for non-inertial frames also.
Reason : Earth is a non-inertial frame.

8. **Assertion :** A wooden block is floating in a liquid as shown in figure. In vertical direction equilibrium of block is stable.



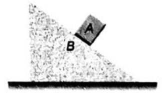
Reason : When depressed in downward direction is starts oscillating.

9. **Assertion :** Displacement-time graph of a particle moving in a straight line is shown in figure. Work done by all the forces between time interval t_1 and t_2 is definitely zero.

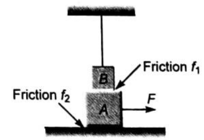


Reason : Work done by all the forces is equal to change in kinetic energy.

10. **Assertion :** All surfaces shown in figure are smooth. Block A comes down along the wedge B. Work done by normal reaction (between A and B) on B is positive while on A it is negative.
Reason : Angle between normal reaction and net displacement of A is greater than 90° while between normal reaction and net displacement of B is less than 90°.



11. **Assertion :** A plank A is placed on a rough surface over which a block B is placed. In the shown situation, elastic cord is unstretched. Now a gradually increasing force F is applied slowly on A until the moment relative motion between the block and plank starts.
At this moment cord is making an angle θ with the vertical. Work done by force F is equal to energy lost against friction f_2, plus potential energy stored in the cord.
Reason : Work done by static friction f_1 on the system as a whole is zero.

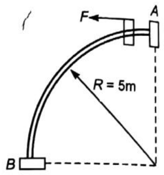


12. **Assertion :** A block of mass m starts moving on a rough horizontal surface with a velocity v. It stops due to friction between the block and the surface after moving through a ceratin distance. The surface is now tilted to an angle of 30° with the horizontal and the same block is made to go up on the surface with the same initial velocity v. The decrease in the mechanical energy in the second situation is smaller than that in the first situation.
Reason : The coefficient of friction between the block and the surface decreases with the increase in the angle of inclination.

Objective Questions (Level 2)

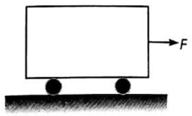
Single Correct Option

1. A bead of mass $\frac{1}{2}$ kg starts from rest from A to move in a vertical plane along a smooth fixed quarter ring of radius 5 m, under the action of a constant horizontal force $F = 5$ N as shown. The speed of bead as it reaches the point B is [Take $g = 10 \text{ ms}^{-2}$]



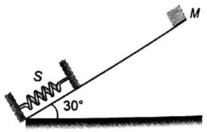
(a) 14.14 ms^{-1}
(b) 7.07 ms^{-1}
(c) 4 ms^{-1}
(d) 25 ms^{-1}

2. A car of mass m is accelerating on a level smooth road under the action of a single force F. The power delivered to the car is constant and equal to P. If the velocity of the car at an instant is v, then after travelling how much distance it becomes double?



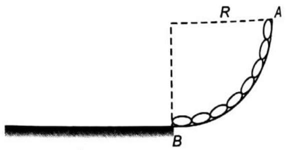
(a) $\frac{7mv^3}{3P}$ (b) $\frac{4mv^3}{3P}$
(c) $\frac{mv^3}{P}$ (d) $\frac{18mv^3}{7P}$

3. An ideal massless spring S can be compressed 1 m by a force of 100 N in equilibrium. The same spring is placed at the bottom of a frictionless inclined plane inclined at 30° to the horizontal. A 10 kg block M is released from rest at the top of the incline and is brought to rest momentarily after compressing the spring by 2 m. If $g = 10 \text{ ms}^{-2}$, what is the speed of mass just before it touches the spring?



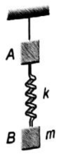
(a) $\sqrt{20} \text{ ms}^{-1}$ (b) $\sqrt{30} \text{ ms}^{-1}$
(c) $\sqrt{10} \text{ ms}^{-1}$ (d) $\sqrt{40} \text{ ms}^{-1}$

4. A smooth chain AB of mass m rests against a surface in the form of a quarter of a circle of radius R. If it is released from rest, the velocity of the chain after it comes over the horizontal part of the surface is



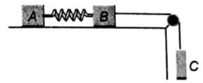
(a) $\sqrt{2gR}$ (b) $\sqrt{gR}$
(c) $\sqrt{2gR\left(1-\frac{2}{\pi}\right)}$ (d) $\sqrt{2gR\,(2-\pi)}$

5. Initially the system shown in figure is in equilibrium. At the moment, the string is cut the downward acceleration of blocks A and B are respectively a_1 and a_2. The magnitudes of a_1 and a_2 are



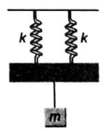
(a) zero and zero
(b) $2g$ and zero
(c) g and zero
(d) None of the above

6. In the diagram shown, the blocks A and B are of the same mass M and the mass of the block C is M_1. Friction is present only under the block A. The whole system is suddenly released from the state of rest. The minimum coefficient of friction to keep the block A in the state of rest is equal to



(a) $\frac{M_1}{M}$ (b) $\frac{2M_1}{M}$
(c) $\frac{M_1}{2M}$ (d) None of these

7. System shown in figure is in equilibrium. Find the magnitude of net change in the string tension between two masses just after, when one of the springs is cut. Mass of both the blocks is same and equal to m and spring constant of both the springs is k

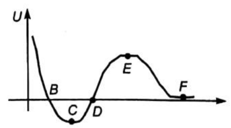


(a) $\frac{mg}{2}$ (b) $\frac{mg}{4}$
(c) $\frac{mg}{3}$ (d) $\frac{3mg}{2}$

8. A body is moving down an inclined plane of slope 37°. The coefficient of friction between the body and the plane varies as $\mu = 0.3\,x$, where x is the distance traveled down the plane by the body. The body will have maximum speed. $\left(\sin 37° = \frac{3}{5}\right)$

(a) at $x = 1.16$ m (b) at $x = 2$ m
(c) at bottommost point of the plane (d) at $x = 2.5$ m

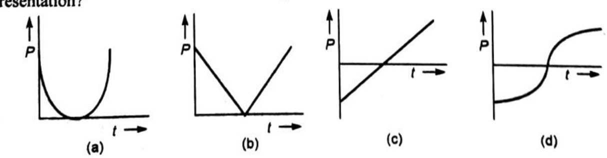
9. The given plot shows the variation of U, the potential energy of interaction between two particles with the distance separating them r.



1. B and D are equilibrium points
2. C is a point of stable equilibrium
3. The force of interaction between the two particles is attractive between points C and D and repulsive between D and E
4. The force of interaction between particles is repulsive between points E and F. Which of the above statements are correct?

(a) 1 and 2 (b) 1 and 4 (c) 2 and 4 (d) 2 and 3

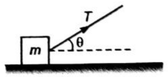
10. A particle is projected at $t = 0$ from a point on the ground with certain velocity at an angle with the horizontal. The power of gravitation force is plotted against time. Which of the following is the best representation?



11. A block of mass m is attached to one end of a mass less spring of spring constant k. The other end of spring is fixed to a wall. The block can move on a horizontal rough surface. The coefficient of friction between the block and the surface is μ. Then the compression of the spring for which maximum extension of the spring becomes half of maximum compression is

(a) $\frac{2mg\mu}{k}$ (b) $\frac{mg\mu}{k}$ (c) $\frac{4mg\mu}{k}$ (d) None of these

12. A block of mass m slides along the track with kinetic friction μ. A man pulls the block through a rope which makes an angle θ with the horizontal as shown in the figure. The block moves with constant speed V. Power delivered by man is

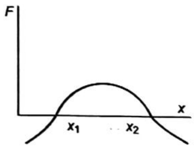


(a) TV (b) $TV\cos\theta$ (c) $(T\cos\theta - \mu mg)V$ (d) zero

13. The potential energy ϕ in joule of a particle of mass 1 kg moving in x-y plane obeys the law, $\phi = 3x + 4y$. Here x and y are in metres. If the particle is at rest at (6m, 8m) at time 0, then the work done by conservative force on the particle from the initial position to the instant when it crosses the x-axis is

(a) 25 J (b) – 25 J (c) 50 J (d) – 50 J

14. The force acting on a body moving along x-axis varies with the position of the particle shown in the figure. The body is in stable equilibirum at

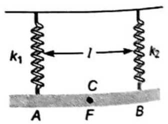


(a) $x = x_1$ (b) $x = x_2$ (c) both x_1 and x_2 (d) neither x_1 nor x_2

15. A small mass slides down an inclined plane of inclination θ with the horizontal. The co-efficient of friction is $\mu = \mu_0 x$ where x is the distance through which the mass slides down and μ_0 a positive constant. Then the distance covered by the mass before it stops is

(a) $\frac{2}{\mu_0}\tan\theta$ (b) $\frac{4}{\mu_0}\tan\theta$ (c) $\frac{1}{2\mu_0}\tan\theta$ (d) $\frac{1}{\mu_0}\tan\theta$

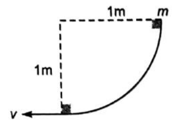
16. Two light vertical springs with spring constants k_1 and k_2 are separated by a distance l. Their upper ends are fixed to the ceiling and their lower ends to the ends A and B of a light horizontal rod AB. A vertical downward force F is applied at point C on the rod. AB will remain horizontal in equilibrium if the distance AC is



(a) $\frac{lk_1}{k_2}$ (b) $\frac{lk_1}{k_2 + k_1}$

(c) $\frac{lk_2}{k_1}$ (d) $\frac{lk_2}{k_1 + k_2}$

17. A block of mass 1 kg slides down a curved track which forms one quadrant of a circle of radius 1 m as shown in figure. The speed of block at the bottom of the track is $v = 2\,\text{ms}^{-1}$. The work done by the force of friction is

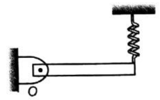


(a) + 4 J (b) – 4 J (c) – 8 J (d) + 8 J

18. The potential energy function for a diatomic molecule is $U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$. In stable equilibrium, the distance between the particles is

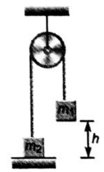
(a) $\left(\frac{2a}{b}\right)^{1/6}$ (b) $\left(\frac{a}{b}\right)^{1/6}$ (c) $\left(\frac{b}{2a}\right)^{1/6}$ (d) $\left(\frac{b}{a}\right)^{1/6}$

19. A rod of mass M hinged at O is kept in equilibrium with a spring of stiffness k as shown in figure. The potential energy stored in the spring is



(a) $\frac{(mg)^2}{4k}$ (b) $\frac{(mg)^2}{2k}$ (c) $\frac{(mg)^2}{8k}$ (d) $\frac{(mg)^2}{k}$

20. In the figure m_1 and m_2 $(< m_1)$ are joined together by a pulley. When the mass m_1 is released from the height h above the floor, it strikes the floor with a speed



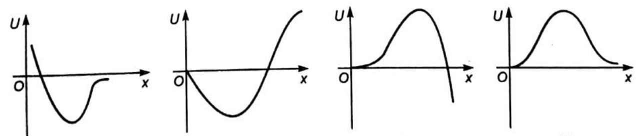
(a) $\sqrt{2gh\left(\frac{m_1 - m_2}{m_1 + m_2}\right)}$

(b) $\sqrt{2gh}$

(c) $\sqrt{\frac{2m_2 gh}{m_1 + m_2}}$

(d) $\sqrt{\frac{2m_1 gh}{m_1 + m_2}}$

21. A particle free to move along x- axis is acted upon by a force $F = -ax + bx^2$ where a and b are positive constants. For $x \geq 0$, the correct variation of potential energy function U(x) is best represented by

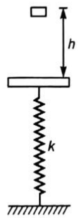


(a) (b) (c) (d)

22. Equal net forces act on two different blocks A and B of masses m and 4m respectively. For same displacement, identify the correct statement.
(a) Their kinetic energies are in the ratio $\frac{K_A}{K_B}=\frac{1}{4}$
(b) Their speeds are in the ratio $\frac{v_A}{v_B}=\frac{1}{1}$
(c) Work done on the blocks are in the ratio $\frac{W_A}{W_B}=\frac{1}{1}$
(d) All of the above

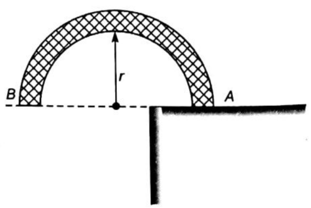
23. The potential energy function of a particle in the x-y plane is given by $U=k(x+y)$, where k is a constant. The work done by the conservative force in moving a particle from (1, 1) to (2, 3) is
(a) -3k (b) $+3$k (c) k (d) None of these

24. A vertical spring is fixed to one of its end and a massless plank fitted to the other end. A block is released from a height h as shown. Spring is in relaxed position. Then choose the correct statement.



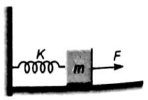
(a) The maximum compression of the spring does not depend on h.
(b) The maximum kinetic energy of the block does not depend on h.
(c) The compression of the spring at maximum KE of the block does not depend on h.
(d) The maximum compression of the spring does not depend on k.

25. A uniform chain of length πr lies inside a smooth semicircular tube AB of radius r. Assuming a slight disturbance to start the chain in motion, the velocity with which it will emerge from the end B of the tube will be



(a) $\sqrt{gr\left(1+\frac{2}{\pi}\right)}$
(b) $\sqrt{2gr\left(\frac{2}{\pi}+\frac{\pi}{2}\right)}$
(c) $\sqrt{gr\,(\pi+2)}$
(d) $\sqrt{\pi gr}$

26. A block of mass m is connected to a spring of force constant k. Initially the block is at rest and the spring has natural length. A constant force F is applied horizontally towards right. The maximum speed of the block will be (there is no friction between block and the surface)

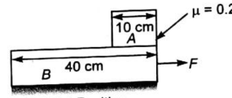


(a) $\frac{F}{\sqrt{2mk}}$ (b) $\frac{F}{\sqrt{mk}}$
(c) $\frac{\sqrt{2}F}{\sqrt{mk}}$ (d) $\frac{2F}{\sqrt{mk}}$

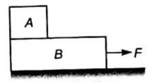
27. Two blocks are connected to an ideal spring of stiffness 200N/m. At a certain moment, the two blocks are moving in opposite directions with speeds 4 ms^{-1} and 6 ms^{-1}, and the instantaneous elongation of the spring is 10 cm. The rate at which the spring energy $\left(\frac{kx^2}{2}\right)$ is increasing is
(a) 500 J/s (b) 400 J/s (c) 200 J/s (d) 100 J/s

28. A mass less spring of stiffness k connects two blocks of masses m and $3m$. The system is lying on a frictionless horizontal surface. A constant horizontal force F starts acting on the block of mass m, directed towards the other block. Then the maximum compression of the spring will be
(a) $\frac{3F}{4k}$ (b) $\frac{9F}{8k}$ (c) $\frac{3F}{2k}$ (d) $\frac{4F}{3k}$

29. A block A of mass 45 kg is placed on another block B of mass 123 kg. Now block B is displaced by external agent by 50 cm horizontally towards right. During the same time block A just reaches to the left end of block B. Initial and final positions are shown in figures. The work done on block A in ground frame is



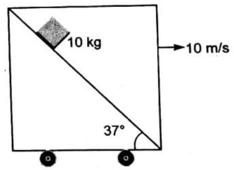
Intial Position



Final Position

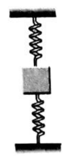
(a) -18 J
(b) 18 J
(c) 36 J
(d) -36 J

30. A block of mass 10 kg is released on a fixed wedge inside a cart which is moved with constant velocity 10 ms^{-1} towards right. Take initial velocity of block with respect to cart zero. Then work done by normal reaction on block in two seconds from ground frame will be ($g=10\,\text{ms}^{-2}$)



(a) 1320 J
(b) 960 J
(c) 1200 J
(d) 240 J

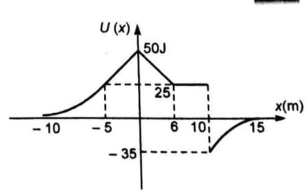
31. A block tied between two springs is in equilibrium. If upper spring is cut, then the acceleration of the block just after cut is 5 ms^{-2}. Now if instead of upper string lower spring is cut, then the acceleration of the block just after the cut will be : (Take $g=10\,\text{m/s}^2$)



(a) 1.25 ms^{-2} (b) 5 ms^{-2}
(c) 10 ms^{-2} (d) 2.5 ms^{-2}

Passage (Q. No. 32 to 33)

The figure shows the variation of potential energy of a particle as a function of x, the x-coordinate of the region. It has been assumed that potential energy depends only on x. For all other values of x, U is zero, *i.e.*, for $x<-10$ and $x>15$, $U=0$.
Based on above information answer the following questions.



32. If total mechanical energy of the particle is 25 J, then it can be found in the region
(a) $-10 < x < -5$ and $6 < x < 15$
(b) $-10 < x < 0$ and $6 < x < 10$
(c) $-5 < x < 6$
(d) $-10 < x < 10$

33. If total mechanical energy of the particle is -40 J, then it can be found in region
(a) $x < -10$ and $x > 15$
(b) $-10 < x < -5$ and $6 < x < 15$
(c) $10 < x < 15$
(d) It is not possible

More than One Correct Options

1. The potential energy of a particle of mass 5 kg moving in xy plane is given as $U = 7x + 24y$ joules, X and Y being in metre. Initially at $t = 0$ the particle is at the origin (0, 0) moving with a velocity of $(8.6\hat{i} + 23.2\hat{j})\ ms^{-1}$. Then
(a) The velocity of the particle at $t = 4$ s, is $5\ ms^{-1}$
(b) The acceleration of the particle is $5\ ms^{-2}$
(c) The direction of motion of the particle initially (at $t = 0$) is at right angles to the direction of acceleration
(d) The path of the particle is circle

2. The potential energy of a particle is given by formula $U = 100 - 5x + 100x^2$, U and x are in SI units. If mass of the particle is 0.1 kg then magnitude of it's acceleration
(a) At 0.05 m from the origin is $50\ ms^{-2}$
(b) At 0.05 m from the mean position is $100\ ms^{-2}$
(c) At 0.05 m from the origin is $150\ ms^{-2}$
(d) At 0.05 m from the mean position is $200\ ms^{-2}$

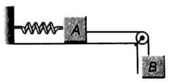
3. One end of a light spring of spring constant k is fixed to a wall and the other end is tied to a block placed on a smooth horizontal surface. In a displacement, the work done by the spring is $+\left(\frac{1}{2}\right)kx^2$. The possible cases are
(a) The spring was initially compressed by a distance x and was finally in its natural length
(b) It was initially stretched by a distance x and finally was in its natural length
(c) It was initially in its natural length and finally in a compressed position
(d) It was initially in its natural length and finally in a stretched position

4. Identify the correct statement about work energy theorem
(a) work done by all the conservative forces is equal to the decrease in potential energy.
(b) work done by all the forces except the conservative forces is equal to the change in mechanical energy.
(c) work done by all the forces is equal to the change in kinetic energy.
(d) work done by all the forces is equal to the change in potential energy.

5. A disc of mass 3m and a disc of mass m are connected by a mass less spring of stiffness k. The heavier disc is placed on the ground with the spring vertical and lighter disc on top. From its equilibrium position the upper disc is pushed down by a distance δ and released. Then
(a) if $\delta > \frac{3\,mg}{k}$, the lower disc will bounce up
(b) if $\delta = \frac{2\,mg}{k}$, maximum normal reaction from ground on lower disc = 6 mg.

(c) if $\delta = \frac{2\,mg}{k}$, maximum normal reaction from ground on lower disc = 4 mg.
(d) if $\delta > \frac{4\,mg}{k}$, the lower disc will bounce up

6. In the adjoining figure block A is of mass m and block B is of mass 2 m. The spring has force constant k. All the surfaces are smooth and the system is released from rest with spring unstretched.



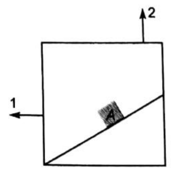
(a) The maximum extension of the spring is $\frac{4mg}{k}$
(b) The speed of block A when extension in spring is $\frac{2mg}{k}$, is $2g\sqrt{\frac{2m}{3k}}$
(c) Net acceleration of block B when the extension in the spring is maximum, is $\frac{2}{3}g$.
(d) Tension in the thread for extension of $\frac{2mg}{k}$ in spring is mg

7. Mark out the correct statement (s).
(a) Total work done by internal forces on a system is always zero
(b) Total work done by internal forces on a system may sometimes be zero
(c) Total work done by internal forces acting between the particles of a rigid body is always zero
(d) Total work done by internal forces acting between the particles of a rigid body may sometimes be zero

8. If kinetic energy of a body is increasing then :
(a) work done by conservative forces must be positive
(b) work done by conservative forces may be positive
(c) work done by conservative forces may be zero
(d) work done by non conservative forces may be zero

9. At two positions kinetic energy and potential energy of a particle are
$K_1 = 10\,J : U_1 = -20\,J, K_2 = 20\,J, U_2 = -10\,J$
In moving from 1 to 2
(a) work done by conservative forces is positive
(b) work done by conservative forces is negative
(c) work done by all the forces is positive
(d) work done by all the forces is negative

10. Block A has no relative motion with respect to wedge fixed to the lift as shown in figure during motion-1 or motion-2



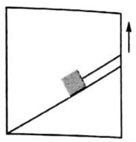
(a) work done by gravity on block A in motion-2 is less than in motion-1
(b) work done by normal reaction on block A in both the motions will be positive
(c) work done by force of friction in motion-1 may be positive
(d) work done by force of friction in motion-1 may be negative

Match the Columns

1. A body is displaced from $x = 4$ m to $x = 2$m along the x-axis. For the forces mentioned in column I, match the corresponding work done is column II.

Column I	Column II
(a) $\vec{F} = 4\hat{i}$	(p) positive
(b) $\vec{F} = (4\hat{i} - 4\hat{j})$	(q) negative
(c) $\vec{F} = -4\hat{i}$	(r) zero
(d) $\vec{F} = (-4\hat{i} - 4\hat{j})$	(s) $\lvert W \rvert = 8$ units

2. A block is placed on a rough wedge fixed on a lift as shown in figure. A string is also attached with the block. The whole system moves upwards. Block does not lose contact with wedge on the block. Match the following two columns regarding the work done.



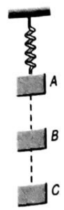
Column I	Column II
(a) Work done by normal reaction	(p) positive
(b) Work done by gravity	(q) negative
(c) Work done by friction	(r) zero
(d) Work done by tension	(s) Can't say anything

3. Two positive charges $+q$ each are fixed at points $(-a, 0)$ and $(a, 0)$. A third charge $+Q$ is placed at origin. Corresponding to small displacement of $+Q$ in the direction mentioned in column I, match the corresponding equilibrium of column II.

Column I	Column II
(a) Along positive x-axis	(p) stable equilibrium
(b) Along positive y-axis	(q) unstable equilibrium
(c) Along positive z-axis	(r) neutral equilibrium
(d) Along the line $x = y$	(s) no equilibrium

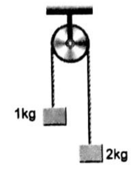
4. A block attached with a spring is released from A. Position-B is the mean position and the block moves to point C. Match the following two columns.

Column I	Column II
(a) From A to B decrease in gravitational potential energy is........ the increase in spring potential energy.	(p) less than
(b) From A to B increase in kinetic energy of block is......... the decrease in gravitational potential energy.	(q) more than
(c) From B to C decrease in kinetic energy of block is..... the increase in spring potential energy.	(r) equal to
(d) From B to C decrease in gravitational potential energy is .... the increase in spring potential energy.	



5. System shown in figure is released from rest. Friction is absent and string is massless. In time $t = 0.3$ s. Take $g = 10\,\text{ms}^{-2}$

Column I	Column II
(a) Work done by gravity on 2 kg block	(p) -1.5 J
(b) Work done by gravity on 1 kg block	(q) 2 J
(c) Work done by string on 2 kg block	(r) 3 J
(d) Work done by string on 1 kg block	(s) -2 J



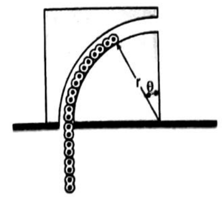
6. In column I some statements are given related to work done by a force on an object while in column II the sign and information about value of work done is given. Match the entries of Column I with the entries of Column II.

Column I	Column II
(a) Work done by friction force on the block as it slides down a rigid fixed incline with respect to ground.	(p) Positive
(b) In above case work done by friction force on incline with respect to ground.	(q) Negative
(c) Work done by a man in lifting a bucket out of a well by means of a rope tied to the bucket with respect to ground.	(r) Zero
(d) Total work done by friction force in (a) with respect to ground.	(s) may be positive, negative or zero.

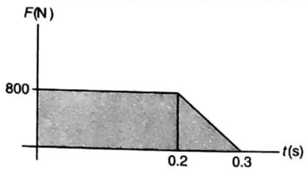
Subjective Questions (Level 2)

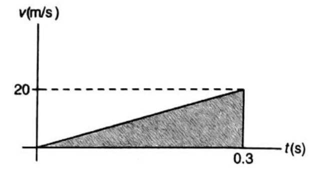
1. Two blocks of masses m_1 and m_2 connected by a light spring rest on a horizontal plane. The coefficient of friction between the blocks and the surface is equal to μ. What minimum constant force has to be applied in the horizontal direction to the block of mass m_1 in order to shift the other block?

2. The flexible bicycle type chain of length $\frac{\pi r}{2}$ and mass per unit length ρ is released from rest with $\theta = 0°$ in the smooth circular channel and falls through the hole in the supporting surface. Determine the velocity v of the chain as the last link leaves the slot.

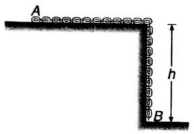


3. A baseball having a mass of 0.4 kg is thrown such that the force acting on it varies with time as shown in the first graph. The corresponding velocty time graph is shown in the second graph. Determine the power applied as a function of time and the work done till $t = 0.3$ s.

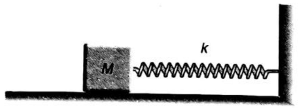




4. A chain AB of length l is loaded in a smooth horizontal table so that its fraction of length h hangs freely and touches the surface of the table with its end B. At a certain moment, the end A of the chain is set free, with what velocity will this end of the chain slip out of the table?

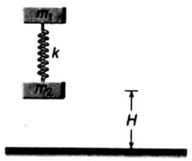


5. The block shown in the figure is acted on by a spring with spring constant k and a weak frictional force of constant magnitude f. The block is pulled a distance x_0 from equilibrium position and then released. It oscillates many times and ultimately comes to rest.



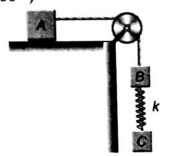
(a) Show that the decrease of amplitude is the same for each cycle of oscillation.
(b) Find the number of cycles the mass oscillates before coming to rest.

6. A spring mass system is held at rest with the spring relaxed at a height H above the ground. Determine the minimum value of H so that the system has a tendency to rebound after hitting the ground. Given that the coefficient of restitution between m_2 and ground is zero.



7. A block of mass m moving at a speed v compresses a spring through a distance x before its speed is halved. Find the spring constant of the spring.

8. In the figure shown masses of the blocks A, B and C are 6 kg, 2 kg and 1 kg respectively. Mass of the spring is negligibly small and its stiffness is 1000 N/m. The coefficient of friction between the block A and the table is $\mu = 0.8$. Initially block C is held such that spring is in relaxed position. The block is released from rest. Find : $(g = 10\,\text{m/s}^2)$



(a) the maximum distance moved by the block C.
(b) the acceleration of each block, when elongation in the spring is maximum.

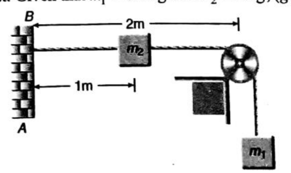
9. A body of mass m slides down a plane inclined at an angle α. The coefficient of friction is μ. Find the rate at which kinetic plus gravitational potential energy is dissipated at any time t.

10. A particle moving in a straight line is acted upon by a force which works at a constant rate and changes its velocity from u and v over a distance x. Prove that the time taken in it is

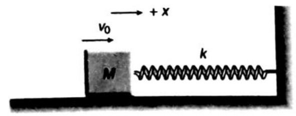
$$\frac{3}{2}\frac{(u+v)x}{u^2+v^2+uv}$$

11. A chain of length l and mass m lies on the surface of a smooth sphere of radius $R > l$ with one end tied to the top of the sphere. (a) Find the gravitational potential energy of the chain with reference level at the centre of the sphere. (b) Suppose the chain is released and slides down the sphere. Find the kinetic energy of the chain, when it has slid through an angle θ. (c) Find the tangential acceleration $\frac{dv}{dt}$ of the chain when the chain starts sliding down.

12. Find the speed of both the blocks arrangement at the moment the block m_2 hits the wall AB, after the blocks are released from rest. Given that $m_1 = 0.5$ kg and $m_2 = 2$ kg, $(g = 10\,\text{m/s}^2)$

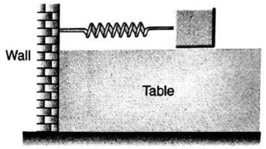


13. A block of mass M slides along a horizontal table with speed v_0. At $x = 0$ it hits a spring with spring constant k and begins to experience a friction force. The coefficient of friction is variable and is given by $\mu = bx$ where b is a positive constant. Find the loss in mechanical energy when the block has first come momentarily to rest.

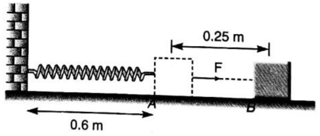


x = 0

14. A small block of ice with mass 0.120 kg is placed against a horizontal compressed spring mounted on a horizontal table top that is 1.90 m above the floor. The spring has a force constant $k = 2300$ N/m and is initially compressed 0.045 m. The mass of the spring is negligible. The spring is released, and the block slides along the table, goes off the edge, and travels to the floor. If there is negligible friction between the ice and the table, what is the speed of the block of ice when it reaches the floor. ($g = 9.8$ m/s^2)



15. A 0.500 kg block is attached to a spring with length 0.60 m and force constant $k = 40.0$N/m. The mass of the spring is negligible. You pull the block to the right along the surface with a constant horizontal force $F = 20.0$ N. (a) What is the block's speed when the block reaches point B, which is 0.25 m to the right of point A? (b) When the block reaches point B, you let go off the block. In the subsequent motion, how close does the block get to the wall where the left end of the spring is attached? Neglect size of block and friction.



ANSWERS

Introductory Exercise 6.1

1. $Tx, 0, 0, -Fx$ 2. $Tl \cos\beta,\ 0,\ -Fl,\ -Wl \sin\alpha$ 3. $-\frac{3}{4} mgl$ 4. (a) 7.2 J (b) − 7.2 J (c) zero
5. 10 J 6. $-Kx_0^2$ 7. 30 J

Introductory Exercise 6.2

1. − 400 J 2. Yes 3. $\frac{1}{2} m\alpha^2 b$ 4. 120 J 5. $\sqrt{2Rg(1 + \sin\theta - \cos\theta)}$ 6. $v_0\sqrt{\frac{m}{A}}$
7. (a) False (b) False (c) True (d) False

Introductory Exercise 6.3

1. $\frac{m}{2}$ 2. $\sqrt{gl}$ 3. (a) 2.45 m/s (b) 2.15 m/s 4. $\left(M + \frac{m}{2}\right) gh$ 5. $\frac{3}{5}$ m

Introductory Exercise 6.4

1. 200 W 2. (a) $\frac{F^2t}{2m}$ (b) $\frac{F^2t}{m}$ 3. $K = Pt,\ v = \sqrt{\frac{2Pt}{m}},\ S = \sqrt{\frac{8P}{9m}}\, t^{3/2}$
4. (a) $K = t^2,\ v = \sqrt{\frac{2}{m}}\, t$ (b) $P_{av} = t$ 5. $x = 2$ m, stable 6. $x = 4$ m, unstable

AIEEE Corner

Subjective Questions (Level 1)

1. 80 J, − 40 J 2. −2 J 3. (a) 480 J (b) 192 J 4. (a) 24.9 J (b) zero (c) zero (d) 24.9 J
5. (a) 34.6 J (b) −10 J 6. −12 J 7. −1 J 8. (a) −15 J (b) + 15 J (c) 3 J (d) 27 J
9. (a) 4.0 J (b) zero (c) −1.0 J (d) 3.0 J 10. $F_r = \frac{A}{r^2}$
11. (a) −4 J (b) 12 J (c) $x = (4 - 2\sqrt{3})$ m to $x = (4 + 2\sqrt{3})$ m (d) $F_x = 8 - 2x$ (e) $x = 4$ m
12. 125% 13. 0.5% 14. 32 J 15. −597.6 J 16. −750 J 17. (a) 2.33 J (b) 2.635 J
18. (a) 11642 J (b) −10584 J (c) 1058 J (d) 5.42 ms^{-1} 19. 2 ms^{-1} 20. 0.796 m
21. $v_w = \sqrt{2gR} \cos\alpha$, $v_s = \sqrt{2gR} \sin\alpha$ 22. $v = 0.371$ ms^{-1} 23. $\mu_k = 0.12$ 24. − 0.051 J
25. 1.12 ms^{-1} 26. (a) 66.88 J (b) 66.88 J (c) 45.7 cm 27. 4 m/s
28. (a) F_A and F_B are negative, F_C and F_D are positive and F_E is zero
(b) $x = 2$ m is unstable equilibrium position
$x = 6$ m is stable equilibrium position
There is no point of neutral equilibrium.
29. $x = 2$ m is position of stable equilibrium. $x = -2$ m is position of unstable equilibrium.
30. Points A and E are unstable equilibrium positions. Point C is stable equilibrium position.
31. (a) unstable (b) stable 32. (a) 16 W (b) 64 W 33. $\frac{P}{r}, \frac{Pm}{8r^2}$

Objective Questions (Level 1)

1. (a)	2. (a)	3. (d)	4. (b)	5. (d)	6. (c)	7. (b)	8. (b)	9. (c)	10. (b)
11. (b)	12. (a)	13. (c)	14. (c)	15. (a)	16. (b)	17. (a)	18. (b)	19. (b)	20. (d)
21. (b)	22. (b)	23. (a)	24. (d)	25. (c)	26. (d)	27. (a)	28. (b)	29. (a)	30. (a)
31. (a)	32. (d)	33. (b)	34. (c)	35. (b)	36. (c)	37. (a)	38. (a)	39. (c)	

JEE Corner

Assertion and Reason

1. (d) 2. (a) 3. (d) 4. (d) 5. (a) 6. (c) 7. (b) 8. (a) 9. (d) 10. (a)
11. (a) 12. (c)

Objective Questions (Level 2)

1. (a) 2. (a) 3. (a) 4. (c) 5. (b) 6. (b) 7. (a) 8. (d) 9. (c) 10. (c)
11. (c) 12. (b) 13. (c) 14. (b) 15. (a) 16. (d) 17. (c) 18. (a) 19. (c) 20. (a)
21. (c) 22. (c) 23. (a) 24. (c) 25. (b) 26. (b) 27. (c) 28. (c) 29. (b) 30. (b)
31. (b) 32. (a) 33. (d)

More than One Correct Options

1. (a,b) 2. (a,b,c) 3. (a,b) 4. (b,c) 5. (b,d) 6. (a) 7. (b,c)
8. (b,c,d) 9. (b,c) 10. (all)

Match the Columns

1. (a) → (q,s) (b) → (q,s) (c) → (p,s) (d) → (p,s)
2. (a) → (p) (b) → (q) (c) → (r) (d) → (p)
3. (a) → (p) (b) → (q) (c) → (q) (d) → (s)
4. (a) → (q) (b) → (p) (c) → (p) (d) → (p)
5. (a) → (r) (b) → (p) (c) → (s) (d) → (q)
6. (a) → (q) (b) → (r) (c) → (p) (d) → (q)

Subjective Questions (Level 2)

1. $\left(m_1 + \frac{m_2}{2}\right)\mu g$ 2. $\sqrt{gr\left(\frac{\pi}{2} + \frac{4}{\pi}\right)}$

3. For $t \le 0.2$ s, $P = (53.3\,t)$ kW, for $t > 0.2$ s, $P = (160t - 533t^2)$ kW, 1.69 kJ. 4. $\sqrt{2gh \ln\left(\frac{l}{h}\right)}$

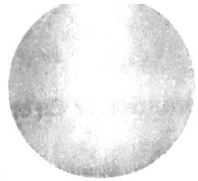
5. (b) $\frac{1}{4}\left[\frac{kx_0}{f} - 1\right]$ 6. $H_{min} = \frac{m_2 g}{k}\left[\frac{m_2 + 2m_1}{2m_1}\right]$ 7. $\frac{3mv^2}{4x^2}$

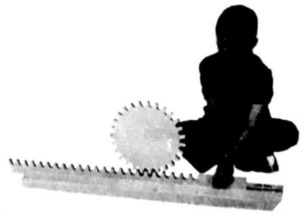
8. (a) 2×10^{-2}m (b) $a_A = a_B = 0$, $a_C = 10\ m/s^2$ 9. $\mu mg^2 \cos\alpha(\sin\alpha - \mu\cos\alpha)t$

11. (a) $\frac{mR^2 g}{l} \sin\left(\frac{l}{R}\right)$ (b) $\frac{mR^2 g}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]$ (c) $\frac{Rg}{l}\left[1 - \cos\left(\frac{l}{R}\right)\right]$

12. $v_1 = 3.03\ ms^{-1}$; $v_2 = 3.39\ ms^{-1}$ 13. $\frac{bgV_0^2M^2}{2(k + bMg)}$ 14. $8.72\ ms^{-1}$ 15. (a) $3.87\ ms^{-1}$ (b) 0.10 m

Chapter 7 – Circular Motion





7

Circular Motion

Chapter Contents

7.1 Kinematics of Circular Motion
7.2 Dynamics of Circular Motion
7.3 Motion in a Vertical Circle

7.1 Kinematics of Circular Motion

Angular Variables

Circular motion is a two dimensional motion or motion in a plane.

Suppose a particle P is moving in a circle of radius r and centre O.

The position of the particle P at a given instant may be described by the angle θ between OP and OX. This angle θ is called the **angular position** of the particle. As the particle moves on the circle its angular position θ changes. Suppose the point rotates an angle $\Delta\theta$ in time Δt. The rate of change of angular position is known as the **angular velocity** (ω). Thus,

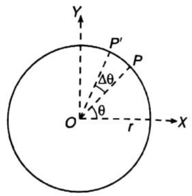


Fig. 7.1

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

The rate of change of angular velocity is called the **angular acceleration** (α). Thus,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

If angular acceleration is constant, we have

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega = \omega_0 + \alpha t \quad \text{and} \quad \omega^2 = \omega_0^2 + 2\alpha\theta$$

Here, ω_0 and ω are the angular velocities at time $t = 0$ and t and θ the angular position at time t. The linear distance PP' travelled by the particle in time Δt is

$$\Delta s = r\Delta\theta$$

or $$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = r \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

or $$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

or $$v = r\omega$$

Here, v is the linear speed of the particle.

Differentiating again with respect to time, we have

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} \quad \text{or} \quad a_t = r\alpha$$

Here, $a_t = \frac{dv}{dt}$ is the rate of change of speed (not the rate of change of velocity). This is called the tangential acceleration of the particle. Later, we will see that a_t is the component of net acceleration $\vec{a}$ of the particle moving in a circle along the tangent.

Unit vectors along the radius and the tangent

Consider a particle P moving in a circle of radius r and centre at origin O. The angular position of the particle at some instant is say θ. Let us here define two unit vectors, one is $\hat{e}_r$ (called radial unit vector) which is along OP and the other is $\hat{e}_t$ (called the tangential unit vector) which is perpendicular to OP. Now, since

$$|\hat{e}_r| = |\hat{e}_t| = 1$$

We can write these two vectors as $\quad \hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

and $\quad \hat{e}_t = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

Velocity and acceleration of particle in circular motion: The position vector of particle P at the instant shown in figure can be written as

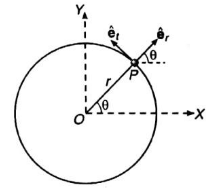


Fig. 7.2

$$\vec{r} = \vec{OP} = r\hat{e}_r$$

or $$\vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

The velocity of the particle can be obtained by differentiating $\vec{r}$ with respect to time t. Thus,

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}[r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})]$$

$$= r\left[\left(-\sin\theta\frac{d\theta}{dt}\right)\hat{i} + \left(\cos\theta\frac{d\theta}{dt}\right)\hat{j}\right]$$

or $$\vec{v} = r\omega[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}] \qquad \left(\because \frac{d\theta}{dt} = \omega\right) \quad \text{...(i)}$$

$\therefore$ $$\vec{v} = r\omega\,\hat{e}_t$$

Thus, we see that velocity of the particle is $r\omega$ along $\hat{e}_t$ or in tangent direction. Acceleration of the particle can be obtained by differentiating Eq. (i) with respect to time t. Thus,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}[r\omega(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})]$$

$$= r\left[\omega\frac{d}{dt}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) + \frac{d\omega}{dt}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\right]$$

$$= \omega r\left[-\cos\theta\frac{d\theta}{dt}\hat{i} - \sin\theta\frac{d\theta}{dt}\hat{j}\right] + r\frac{d\omega}{dt}\hat{e}_t$$

$$= -\omega^2 r[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}] + r\frac{d\omega}{dt}\hat{e}_t$$

$$\vec{a} = -\omega^2 r\hat{e}_r + \frac{dv}{dt}\hat{e}_t$$

Thus, acceleration of a particle moving in a circle has two components one is along $\hat{e}_t$ (along tangent) and the other along $-\hat{e}_r$ (or towards centre). Of these the first one is called the tangential acceleration. (a_t) and the other is called radial or centripetal acceleration (a_r). Thus,

$$a_t = \frac{dv}{dt} = \text{rate of change of speed}$$

and $$a_r = r\omega^2 = r\left(\frac{v}{r}\right)^2 = \frac{v^2}{r}$$

Here, the two components are mutually perpendicular. Therefore, net acceleration of the particle will be:

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(r\omega^2)^2 + \left(\frac{dv}{dt}\right)^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$$

Following three points are important regarding the above discussion:

1. In uniform circular motion, speed (v) of the particle is constant, *i.e.*, $\frac{dv}{dt}=0$. Thus,

$$a_t = 0 \quad \text{and} \quad a = a_r = r\omega^2$$

2. In accelerated circular motion, $\frac{dv}{dt}$ = positive, *i.e.*, a_t is along $\hat{\mathbf{e}}_t$ or tangential acceleration of particle is parallel to velocity $\vec{\mathbf{v}}$ because $\vec{\mathbf{v}} = r\omega\,\hat{\mathbf{e}}_t$ and $\vec{\mathbf{a}}_t = \frac{dv}{dt}\hat{\mathbf{e}}_t$.

3. In decelerated circular motion, $\frac{dv}{dt}$ = negative and hence, tangential acceleration is anti-parallel to velocity $\vec{\mathbf{v}}$.

Sample Example 7.1 *A particle moves in a circle of radius 0.5 m with a linear speed of 2 m/s. Find its angular speed.*

Solution The angular speed is

$$\omega = \frac{v}{r} = \frac{2}{0.5} = 4 \text{ rad/s}$$

Sample Example 7.2 *A particle moves in a circle of radius 0.5 m at a speed that uniformly increases. Find the angular acceleration of particle if its speed changes from 2.0 m/s to 4.0 m/s in 4.0 s.*

Solution The tangential acceleration of the particle is

$$a_t = \frac{dv}{dt} = \frac{4.0-2.0}{4.0} = 0.5 \text{ m/s}^2$$

The angular acceleration is :

$$\alpha = \frac{a_t}{r} = \frac{0.5}{0.5} = 1 \text{ rad/s}^2$$

Sample Example 7.3 *The speed of a particle moving in a circle of radius r = 2 m varies with time t as $v = t^2$ where t is in second and v in m/s. Find the radial, tangential and net acceleration at t = 2 s.*

Solution Linear speed of particle at $t = 2$ s is

$$v = (2)^2 = 4 \text{ m/s}$$

$\therefore$ Radial acceleration $\quad a_r = \frac{v^2}{r} = \frac{(4)^2}{2} = 8 \text{ m/s}^2$

The tangential acceleration is $\quad a_t = \frac{dv}{dt} = 2t$

$\therefore$ Tangential acceleration at $t = 2$ s is

$$a_t = (2)(2) = 4 \text{ m/s}^2$$

$\therefore$ Net acceleration of particle at $t = 2$ s is

$$a = \sqrt{(a_r)^2 + (a_t)^2} = \sqrt{(8)^2 + (4)^2}$$

or $\quad a = \sqrt{80} \text{ m/s}^2$

Note *On any curved path (not necessarily a circular one) the acceleration of the particle has two components a_t and a_n in two mutually perpendicular directions. Component of $\vec{\mathbf{a}}$ along $\vec{\mathbf{v}}$ is a_t and perpendicular to $\vec{\mathbf{v}}$ is a_n. Thus,*

$$|\vec{\mathbf{a}}| = \sqrt{a_t^2 + a_n^2}$$

Introductory Exercise 7.1

1. Is the acceleration of a particle in uniform circular motion constant or variable?
2. Is it necessary to express all angles in radian while using the equation $\omega = \omega_0 + \alpha t$?
3. Which of the following quantities may remain constant during the motion of an object along a curved path?
   (i) Velocity (ii) Speed (iii) Acceleration (iv) Magnitude of acceleration
4. A particle moves in a circle of radius 1.0 cm with a speed given by $v = 2t$ where v is in cm/s and t in seconds.
   (a) Find the radial acceleration of the particle at $t = 1$ s.
   (b) Find the tangential acceleration at $t = 1$ s.
   (c) Find the magnitude of net acceleration at $t = 1$ s.
5. A particle is moving with a constant speed in a circular path. Find the ratio of average velocity to its instantaneous velocity when the particle describes an angle $\theta = \left(\frac{\pi}{2}\right)$.
6. A particle is moving with a constant angular acceleration of 4 rad/s^2 in a circular path. At time $t = 0$, particle was at rest. Find the time at which the magnitudes of centripetal acceleration and tangential acceleration are equal.

7.2 Dynamics of Circular Motion

If a particle moves with constant speed in a circle, motion is called uniform circular. In uniform circular motion a resultant non-zero force acts on the particle. This is because a particle moving in a circle is accelerated even if speed of the particle is constant. This acceleration is due to the change in direction of the velocity vector. As we have seen in Article 7.1 that in uniform circular motion tangential acceleration (a_t) is zero. The acceleration of the particle is towards the centre and its magnitude is $\frac{v^2}{r}$. Here, v is the speed of the particle and r the radius of the circle. The direction of the resultant force F is, therefore, towards centre and its magnitude is

$$F = ma \quad \text{or} \quad F = \frac{mv^2}{r}$$

or $\quad F = mr\omega^2 \quad$ (as $v = r\omega$)

Here, ω is the angular speed of the particle. This force F is called the centripetal force. Thus, a centripetal force of magnitude $\frac{mv^2}{r}$ is needed to keep the particle moving in a circle with constant speed. This force is provided by some external source such as friction, magnetic force, Coulomb force, gravitation, tension, etc.

Note *I have found students often confused over the centripetal force. They think that this force acts on a particle moving in a circle. This force does not act but required for moving in a circle which is being provided by the other forces acting on the particle. Let, us take an example. Suppose a particle of mass 'm' is moving in a vertical circle with the help of a string of length l fixed at point O. Let v be the speed of the particle at lowest position. When I ask the students what forces are acting on the particle in this position? They immediately say, three forces are acting on the particle (1) tension T (2) weight mg and (3) centripetal force* $\frac{mv^2}{l}$ *(r = l). However, they are wrong. Only first two forces T and mg are acting on the particle. Third force* $\frac{mv^2}{l}$ *is required for circular motion which is being provided by T and mg. Thus, the resultant of these two forces is* $\frac{mv^2}{l}$ *towards O. Or we can write*

$$T - mg = \frac{mv^2}{l}$$

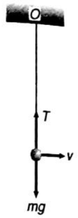


Fig. 7.3

Circular Turning of Roads

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which will produce the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by following three ways:

1. By friction only.
2. By banking of roads only.
3. By friction and banking of roads both.

In real life the necessary centripetal force is provided by friction and banking of roads both. Now, let us write equations of motion in each of the three cases separately and see what are the constraints in each case.

1. By Friction Only

Suppose a car of mass m is moving at a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force to the car will be provided by force of friction f acting towards centre.

Thus,
$$f = \frac{mv^2}{r}$$

Further, limiting value of f is μN.

or
$$f_L = \mu N = \mu mg \qquad (N = mg)$$

Therefore, for a safe turn without sliding

$$\frac{mv^2}{r} \le f_L \quad \text{or} \quad \frac{mv^2}{r} \le \mu mg$$

or
$$\mu \ge \frac{v^2}{rg} \quad \text{or} \quad v \le \sqrt{\mu rg}$$

Here, two situations may arise. If μ and r are known to us, the speed of the vehicle should not exceed $\sqrt{\mu rg}$ and if v and r are known to us, the coefficient of friction should be greater than $\frac{v^2}{rg}$.

Note *You might have seen that if the speed of the car is too high, car starts skidding outwards. With this radius of the circle increases or the necessary centripetal force is reduced (centripetal force* $\propto \frac{1}{r}$*).*

2. By Banking of Roads Only

Friction is not always reliable at circular turns if high speeds and sharp turns are involved. To avoid dependence on friction, the roads are banked at the turn so that the outer part of the road is some what lifted compared to the inner part.

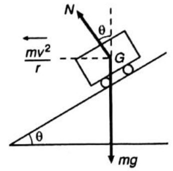


Fig. 7.4

Applying Newton's second law along the radius and the first law in the vertical direction.

$$N \sin\theta = \frac{mv^2}{r} \quad \text{and} \quad N\cos\theta = mg$$

From these two equations, we get

$$\tan\theta = \frac{v^2}{rg} \qquad \text{...(i)}$$

or
$$v = \sqrt{rg\tan\theta} \qquad \text{...(ii)}$$

Note *This is the speed at which car does not slide down even if track is smooth. If track is smooth and speed is less than* $\sqrt{rg\tan\theta}$, *vehicle will move down so that r gets decreased and if speed is more than this vehicle will move up.*

3. By Friction and Banking of Road Both

If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle, of these the first force, *i.e.*, weight (mg) is fixed both in magnitude and direction. The direction of

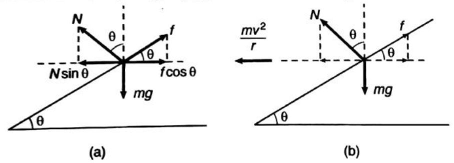


Fig. 7.5

second force, *i.e.*, normal reaction N is also fixed (perpendicular to road) while the direction of the third force, *i.e.*, friction f can be either inwards or outwards while its magnitude can be varied upto a maximum limit ($f_L = \mu N$). So, the magnitude of normal reaction N and direction plus magnitude of friction f are so adjusted that the resultant of the three forces mentioned above is $\frac{mv^2}{r}$ towards the centre. Of these m and r are also constant. Therefore, magnitude of N and direction plus magnitude of friction mainly depends on the speed of the vehicle v. Thus, situation varies from problem to problem. Even though we can see that:

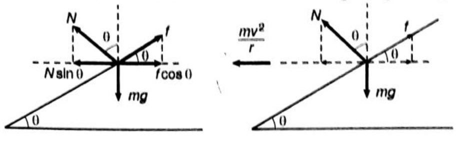
(i) Friction f is outwards if the vehicle is at rest or $v = 0$. Because in that case the component of weight $mg\sin\theta$ is balanced by f.

(ii) Friction f is inwards if $\quad v > \sqrt{rg\tan\theta}$

(iii) Friction f is outwards if $\quad v < \sqrt{rg\tan\theta}$

(iv) Friction f is zero if $\quad v = \sqrt{rg\tan\theta}$

Let us now see how the force of friction and normal reaction change as speed is gradually increased.



(a) (b)

Fig. 7.6

In figure (a): When the car is at rest force of friction is upwards. We can resolve the forces in any two mutually perpendicular directions. Let us resolve them in horizontal and vertical directions.

$$\Sigma F_{H} = 0 \quad \therefore \quad N \sin \theta - f \cos \theta = 0 \qquad \text{...(i)}$$

$$\Sigma F_{V} = 0 \quad \therefore \quad N \cos \theta + f \sin \theta = mg \qquad \text{...(ii)}$$

In figure (b): Now, the car is given a small speed v, so that a centripetal force $\frac{mv^2}{r}$ is now required in horizontal direction towards centre. So, Eq. (i) will now become,

$$N \sin \theta - f \cos \theta = \frac{mv^2}{r}$$

or we can say, in case (a) $N \sin \theta$ and $f \cos \theta$ were equal while in case (b) their difference is $\frac{mv^2}{r}$. This can occur in following three ways:

(i) N increases while f remains same.
(ii) N remains same while f decreases or
(iii) N increases and f decreases.

But only third case is possible, *i.e.*, N will increase and f will decrease. This is because equation number (ii), $N \cos \theta + f \sin \theta = mg$ is still has to be valid.

So, to keep $N \cos \theta + f \sin \theta$ to be constant $(= mg)$ N should increase and f should decrease (as θ = constant)

Now, as speed goes on increasing, force of friction first decreases. Becomes zero at $v = \sqrt{rg \tan \theta}$ and then reverses it direction.

Let us take an example which illustrates the theory.

Sample Example 7.4 *A turn of radius 20 m is banked for the vehicle of mass 200 kg going at a speed of 10 m/s. Find the direction and magnitude of frictional force acting on a vehicle if it moves with a speed*
(a) 5 m/s
(b) 15 m/s. Assume that friction is sufficient to prevent slipping. $(g = 10 \text{ m/s}^2)$

Solution (a) The turn is banked for speed $v = 10$ m/s

Therefore, $$\tan \theta = \frac{v^2}{rg} = \frac{(10)^2}{(20)(10)} = \frac{1}{2}$$

Now, as the speed is decreased, force of friction f acts upwards.

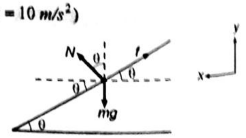


Fig. 7.7

Using the equations

$$\Sigma F_x = \frac{mv^2}{r}$$

and $\Sigma F_y = 0$, we get

$$N \sin \theta - f \cos \theta = \frac{mv^2}{r} \qquad \text{...(i)}$$

$$N \cos \theta + f \sin \theta = mg \qquad \text{...(ii)}$$

Substituting, $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, $v = 5$ m/s, $m = 200$ kg and $r = 20$ m, in the above equations, we get

$$f = 300\sqrt{5} \text{ N} \quad \text{(outwards)}$$

(b) In the second case force of friction f will act downwards.

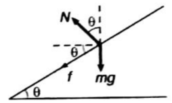


Fig. 7.8

Using $$\Sigma F_x = \frac{mv^2}{r}$$

and $\Sigma F_y = 0$, we get

$$N \sin \theta + f \cos \theta = \frac{mv^2}{r} \qquad \text{...(iii)}$$

$$N \cos \theta - f \sin \theta = mg \qquad \text{...(iv)}$$

Substituting $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, $v = 15$ m/s, $m = 200$ kg and $r = 20$ m in the above equations, we get

$$f = 500\sqrt{5} \text{ N} \quad \text{(downwards)}$$

Conical Pendulum

If a small particle of mass m tied to a string is whirled in a horizontal circle, as shown in Fig. 7.9 The arrangement is called the 'conical pendulum'. In case of conical pendulum the vertical component of tension balances the weight while its horizontal component provides the necessary centripetal force. Thus,

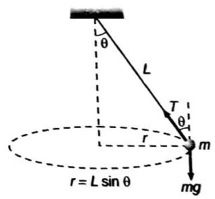


Fig. 7.9

$$T \sin \theta = \frac{mv^2}{r} \qquad \text{...(i)}$$

and $$T \cos \theta = mg \qquad \text{...(ii)}$$

From these two equations, we can find

$$v = \sqrt{rg \tan \theta}$$

$\therefore$ Angular speed $$\omega = \frac{v}{r} = \sqrt{\frac{g \tan \theta}{r}}$$

So, the time period of pendulum is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{r}{g \tan \theta}}$$

$$= 2\pi \sqrt{\frac{L \cos \theta}{g}}$$

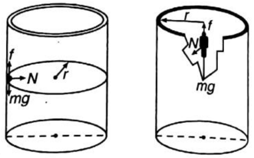
or
$$T = 2\pi\sqrt{\frac{L\cos\theta}{g}}$$

Note *This is similar to the case, when necessary centripetal force to vehicles is provided by banking. The only difference is that the normal reaction is being replaced by the tension.*

'Death Well' or Rotor

In case of 'death well' a person drives a bicycle on a vertical surface of a large wooden well while in case of a rotor at a certain angular speed of rotor a person hangs resting against the wall without any support from the bottom. In death well walls are at rest and person revolves while in case of rotor person is at rest and the walls rotate.

In both cases friction balances the weight of person while reaction provides the centripetal force for circular motion, *i.e.*,



(A) Death well (B) Rotor

Fig. 7.10

$$f = mg$$

and
$$N = \frac{mv^2}{r} = mr\omega^2 \qquad (v = r\omega)$$

A cyclist on the bend of a road

In figure,
$$F = \sqrt{N^2 + f^2}$$

When the cyclist is inclined to the centre of the rounding of its path, the resultant of N, f and mg is directed horizontally to the centre of the circular path of the cycle. This resultant force imparts a centripetal acceleration to the cyclist.

Resultant of N and f, *i.e.*, F should pass through G, the centre of gravity of cyclist (for complete equilibrium, rotational as well as translational). Hence,

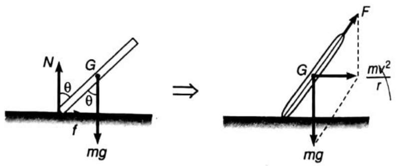


Fig. 7.11

$$\tan\theta = \frac{f}{N}$$

where
$$f = \frac{mv^2}{r} \quad \text{and} \quad N = mg$$

∴
$$\tan\theta = \frac{v^2}{rg}$$

Centrifugal Force

Newton's laws are valid only in inertial frames. In non-inertial frames a pseudo force $-m\vec{a}$ has to be applied on a particle of mass m ($\vec{a}$ = acceleration of frame of reference). After applying the pseudo force one can apply Newton's laws in their usual form. Now, suppose a frame of reference is rotating with constant angular velocity ω in a circle of radius 'r'. Then it will become a non-inertial frame of acceleration $r\omega^2$ towards the centre. Now, if we see an object of mass 'm' from this frame then obviously a pseudo force of magnitude

$mr\omega^2$ will have to be applied to this object in a direction away from the centre. This pseudo force is called the centrifugal force. After applying this force we can now apply Newton's laws in their usual form. Following example will illustrate the concept more clearly.

Sample Example 7.5 *A particle of mass m is placed over a horizontal circular table rotating with an angular velocity ω about a vertical axis passing through its centre. The distance of the object from the axis is r. Find the force of friction f between the particle and the table.*

Solution Let us solve this problem from both frames. The one is a frame fixed on ground and the other is a frame fixed on table itself.

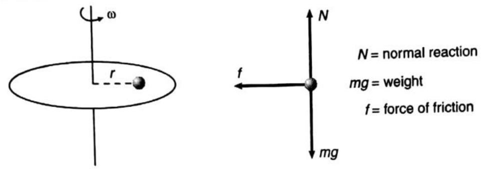


Fig. 7.12

From frame of reference fixed on ground (inertial)

Here, N will balance its weight and the force of friction f will provide the necessary centripetal force.

Thus,
$$f = mr\omega^2 \qquad \textbf{Ans.}$$

From frame of reference fixed on table itself (non-inertial)

In the free body diagram of particle with respect to table, in addition to above three forces (N, mg and f) a pseudo force of magnitude $mr\omega^2$ will have to be applied in a direction away from the centre. But one thing should be clear that in this frame the particle is in equilibrium, *i.e.*, N will balance its weight in vertical direction while f will balance the pseudo force in horizontal direction.

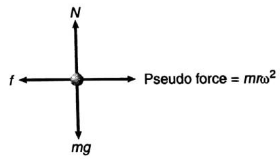


Fig. 7.13

or
$$f = mr\omega^2 \qquad \textbf{Ans.}$$

Thus, we see that 'f' comes out to be $mr\omega^2$ from both the frames.

Now, let us take few more examples of circular motion.

Sample Example 7.6 *A simple pendulum is constructed by attaching a bob of mass m to a string of length L fixed at its upper end. The bob oscillates in a vertical circle. It is found that the speed of the bob is v when the string makes an angle α with the vertical. Find the tension in the string and the magnitude of net force on the bob at that instant.*

Solution (i) The forces acting on the bob are :

(a) the tension T

(b) the weight mg

As the bob moves in a circle of radius L with centre at O. A centripetal force of magnitude $\frac{mv^2}{L}$ is required towards O. This force will be provided by the resultant of T and $mg\cos\alpha$. Thus,

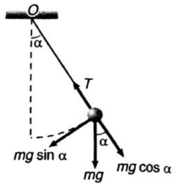


Fig. 7.14

$$T - mg\cos\alpha = \frac{mv^2}{L}$$

or

$$T = m\left(g\cos\alpha + \frac{v^2}{L}\right)$$

(ii)

$$|\vec{F}_{net}| = \sqrt{(mg\sin\alpha)^2 + \left(\frac{mv^2}{L}\right)^2} = m\sqrt{g^2\sin^2\alpha + \frac{v^4}{L^2}}$$

Sample Example 7.7 *A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping on its surface. If the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is α. Find the angular speed at which the bowl is rotating.*

Solution Let ω be the angular speed of rotation of the bowl. Two forces are acting on the ball.

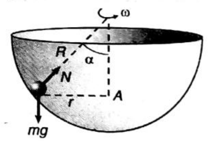


Fig. 7.15

1. normal reaction N
2. weight mg

The ball is rotating in a circle of radius $r (= R\sin\alpha)$ with centre at A at an angular speed ω. Thus,

$$N\sin\alpha = mr\omega^2$$

$$= mR\omega^2\sin\alpha \qquad \ldots(i)$$

and

$$N\cos\alpha = mg \qquad \ldots(ii)$$

Dividing Eq. (i) by (ii), we get

$$\frac{1}{\cos\alpha} = \frac{\omega^2 R}{g}$$

$\therefore$

$$\omega = \sqrt{\frac{g}{R\cos\alpha}}$$

Introductory Exercise 7.2

1. Is a body in uniform circular motion in equilibrium?
2. Find the maximum speed at which a truck can safely travel without toppling over, on a curve of radius 250 m. The height of the centre of gravity of the truck above the ground is 1.5 m and the distance between the wheels is 1.5 m, the truck being horizontal.
3. (a) How many revolutions per minute must the apparatus shown in figure make about a vertical axis so that the cord makes an angle of 45° with the vertical?
   (b) What is the tension in the cord then. Given, $l = \sqrt{2}$ m, $a = 20$ cm and $m = 5.0$ kg?

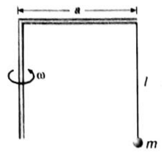


Fig. 7.16

4. A car moves at a constant speed on a straight but hilly road. One section has a crest and dip of the same 250 m radius.
   (a) As the car passes over the crest the normal force on the car is one half the 16 kN weight of the car. What will be the normal force on the car as its passes through the bottom of the dip?
   (b) What is the greatest speed at which the car can move without leaving the road at the top of the hill?
   (c) Moving at a speed found in part (b) what will be the normal force on the car as it moves through the bottom of the dip? (Take $g = 10\text{ m/s}^2$)
5. A car driver going at speed v suddenly finds a wide wall at a distance r. Should he apply brakes or turn the car in a circle of radius r to avoid hitting the wall.
6. Show that the angle made by the string with the vertical in a conical pendulum is given by $\cos\theta = \frac{g}{L\omega^2}$ where L is the length of the string and ω is the angular speed.

7.3 Motion in a Vertical Circle

Suppose a particle of mass m is attached to an inextensible light string of length R. The particle is moving in a vertical circle of radius R about a fixed point O. It is imparted a velocity u in horizontal direction at lowest point A. Let v be its velocity at point B of the circle as shown in figure. Here,

$$h = R(1 - \cos\theta) \qquad \ldots(i)$$

From conservation of mechanical energy

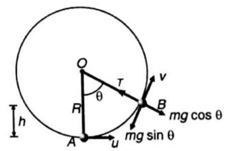


Fig. 7.17

$$\frac{1}{2}m(u^2 - v^2) = mgh$$

or

$$v^2 = u^2 - 2gh \qquad \ldots(ii)$$

The necessary centripetal force is provided by the resultant of tension T and $mg\cos\theta$

$\therefore$

$$T - mg\cos\theta = \frac{mv^2}{R} \qquad \ldots(iii)$$

Now, following three conditions arise depending on the value of u.

Condition of Looping the Loop ($u \geq \sqrt{5gR}$)

The particle will complete the circle if the string does not slack even at the highest point ($\theta = \pi$). Thus, tension in the string should be greater than or equal to zero ($T \geq 0$) at $\theta = \pi$. In critical case substituting $T = 0$ and $\theta = \pi$ in Eq. (iii), we get

$$mg = \frac{mv^2_{min}}{R}$$

$$v^2_{min} = gR$$

or

or $$v_{\min} = \sqrt{gR} \quad \text{(at highest point)}$$

Substituting $\theta = \pi$ in Eq. (i), $$h = 2R$$

Therefore, from Eq. (ii)

$$u_{\min}^2 = v_{\min}^2 + 2gh$$

or $$u_{\min}^2 = gR + 2g(2R) = 5\,gR$$

or $$u_{\min} = \sqrt{5\,gR}$$

Thus, if $u \geq \sqrt{5gR}$, the particle will complete the circle. At $u = \sqrt{5gR}$, velocity at highest point is $v = \sqrt{gR}$ and tension in the string is zero.

Substituting $\theta = 0°$ and $v = \sqrt{5gR}$ in Eq. (iii), we get $T = 6\,mg$ or in the critical condition tension in the string at lowest position is $6\,mg$. This is shown in figure.

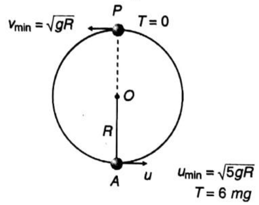


Fig. 7.18

If $u < \sqrt{5gR}$, following two cases are possible.

Condition of Leaving the Circle ($\sqrt{2gR} < u < \sqrt{5gR}$)

If $u < \sqrt{5gR}$, the tension in the string will become zero before reaching the highest point. From Eq. (iii), tension in the string becomes zero ($T = 0$)

where, $$\cos\theta = \frac{-v^2}{Rg}$$

or $$\cos\theta = \frac{2gh - u^2}{Rg}$$

Substituting, this value of $\cos\theta$ in Eq. (i), we get

$$\frac{2gh - u^2}{Rg} = 1 - \frac{h}{R}$$

or $$h = \frac{u^2 + Rg}{3g} = h_1 \text{ (say)} \qquad \ldots\text{(iv)}$$

or we can say that at height h_1 tension in the string becomes zero. Further, if $u < \sqrt{5gR}$, velocity of the particle becomes zero when

$$0 = u^2 - 2gh$$

or $$h = \frac{u^2}{2g} = h_2 \text{ (say)} \qquad \ldots\text{(v)}$$

i.e., at height h_2 velocity of particle becomes zero.

Now, the particle will leave the circle if tension in the string becomes zero but velocity is not zero. or $T = 0$ but $v \neq 0$. This is possible only when

$$h_1 < h_2$$

or $$\frac{u^2 + Rg}{3g} < \frac{u^2}{2g}$$

or $$2u^2 + 2Rg < 3u^2$$

or $$u^2 > 2Rg$$

or $$u > \sqrt{2Rg}$$

Therefore, if $\sqrt{2gR} < u < \sqrt{5gR}$, the particle leaves the circle.

From Eq. (iv), we can see that $h > R$ if $u^2 > 2gR$. Thus, the particle, will leave the circle when $h > R$ or $90° < \theta < 180°$. This situation is shown in the figure.

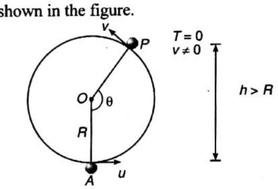


Fig. 7.19

$$\sqrt{2gR} < u < \sqrt{5gR} \quad \text{or} \quad 90° < \theta < 180°$$

Note *That after leaving the circle, the particle will follow a parabolic path.*

Condition of Oscillation ($0 < u \leq \sqrt{2gR}$)

The particle will oscillate, if velocity of the particle becomes zero but tension in the string is not zero. or $v = 0$, but $T \neq 0$. This is possible when

$$h_2 < h_1$$

or $$\frac{u^2}{2g} < \frac{u^2 + Rg}{3g} \quad \text{or} \quad 3u^2 < 2u^2 + 2Rg$$

or $$u^2 < 2Rg \quad \text{or} \quad u < \sqrt{2Rg}.$$

Moreover, if $h_1 = h_2$, $u = \sqrt{2Rg}$ and tension and velocity both becomes zero simultaneously.

Further, from Eq. (iv), we can see that $h \leq R$ if $u \leq \sqrt{2Rg}$. Thus, for $0 < u \leq \sqrt{2gR}$, particle oscillates in lower half of the circle ($0° < \theta \leq 90°$). This situation is shown in the figure.

$$0 < u \leq \sqrt{2gR} \quad \text{or} \quad 0° < \theta \leq 90°$$

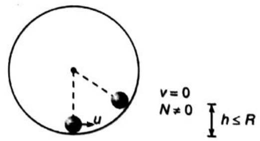


Fig. 7.20

Note *The above three conditions have been derived for a particle moving in a vertical circle attached to a string. The same conditions apply, if a particle moves inside a smooth spherical shell of radius R. The only difference is that the tension is replaced by the normal reaction N.*

Condition of looping the loop is $u \geq \sqrt{5\,gR}$

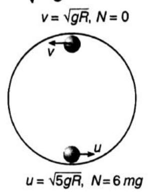


Fig. 7.21

Condition of leaving the circle $\sqrt{2gR} < u < \sqrt{5gR}$

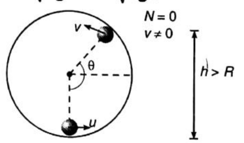


Fig. 7.22

Condition of oscillation is $0 < u \leq \sqrt{2\,gR}$

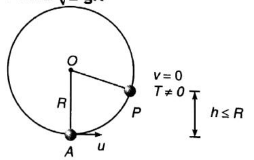


Fig. 7.23

Sample Example 7.8 *A heavy particle hanging from a fixed point by a light inextensible string of length l is projected horizontally with speed $\sqrt{gl}$. Find the speed of the particle and the inclination of the string to the vertical at the instant of the motion when the tension in the string is equal to the weight of the particle.*

Solution Let $T = mg$ at angle θ as shown in figure.

$$h = l(1 - \cos\theta) \qquad \ldots\text{(i)}$$

Applying conservation of mechanical energy between points A and B, we get

$$\frac{1}{2}m(u^2 - v^2) = mgh$$

Here, $$u^2 = gl \qquad \ldots\text{(ii)}$$

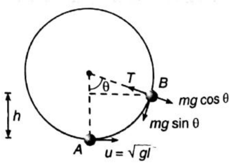


Fig. 7.24

and v = speed of particle in position B

$\therefore$ $$v^2 = u^2 - 2gh \qquad \ldots\text{(iii)}$$

Further, $$T - mg\cos\theta = \frac{mv^2}{l}$$

or $$mg - mg\cos\theta = \frac{mv^2}{l} \qquad (T = mg)$$

or $$v^2 = gl(1 - \cos\theta) \qquad \ldots\text{(iv)}$$

Substituting values of v^2, u^2 and h from Eqs. (iv), (ii) and (i) in Eq. (iii), we get

$$gl(1 - \cos\theta) = gl - 2gl(1 - \cos\theta)$$

or $$\cos\theta = \frac{2}{3}$$

or $$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

Substituting $\cos\theta = \frac{2}{3}$ in Eq. (iv), we get

$$v = \sqrt{\frac{gl}{3}}$$

Introductory Exercise 7.3

1. A stone tied to a string of length L is whirled in a vertical circle with the other end of the string at the centre. At a certain instant of time the stone is at it lowest position and has a speed u. Find the magnitude of the change in its velocity as it reaches a position, where the string is horizontal.
2. With what minimum speed v must a small ball should be pushed inside a smooth vertical tube from a height h so that it may reach the top of the tube? Radius of the tube is R.

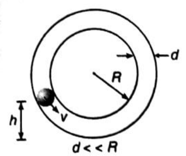


Fig. 7.25

3. A bob is suspended from a crane by a cable of length $l = 5$ m. The crane and load are moving at a constant speed v_0. The crane is stopped by a bumper and the bob on the cable swings out an angle of 60°. Find the initial speed v_0. (g = 9.8 m/s²)

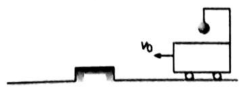


Fig. 7.26

Extra Points

- If a particle of mass m is connected to a light rod and whirled in a vertical circle of radius R, then to complete the circle, the minimum velocity of the particle at the bottommost point is not $\sqrt{5gR}$. Because in this case, velocity of the particle at the topmost point can be zero also. Using conservation of mechanical energy between points A and B as shown in Fig. 7.27, we get

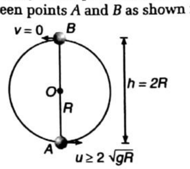


Fig. 7.27

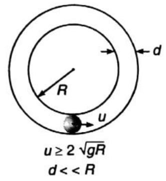


Fig. 7.28

$$\frac{1}{2} m(u^2 - v^2) = mgh$$

or $$\frac{1}{2} mu^2 = mg\,(2R) \qquad \text{(as } v = 0)$$

$\therefore$ $$u = 2\sqrt{gR}$$

Therefore, the minimum value of u in this case is $2\sqrt{gR}$.

Same is the case when a particle is compelled to move inside a smooth vertical tube as shown in Fig. 7.28.

- In uniform circular motion although the speed of the particle remains constant yet the particle is accelerated due to change in direction of velocity. Therefore, the forces acting on the particle in uniform circular motion can be resolved in two directions one along the radius (parallel to acceleration) and another perpendicular to radius (perpendicular to acceleration). Along the radius net force should be equal to $\frac{mv^2}{R}$ and perpendicular to it net force should be zero.
- Oscillation of a pendulum is part of a circular motion. At point A and C since velocity is zero, net centripetal force will be zero. Only tangential force is present. From A and B or C to B speed of the bob increases. Therefore, tangential force is parallel to velocity. From B to A or B to C speed of the bob decreases. Hence, tangential force is antiparallel to velocity.

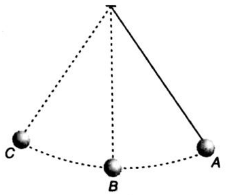


Fig. 7.29

- In circular motion a particle has two speeds:
  (i) angular speed $\omega = \frac{d\theta}{dt}$ and
  (ii) linear speed $v = \frac{ds}{dt}$
  They are related to each other by the relation $v = R\omega$. Here, R is the radius of the circle.

- In circular motion acceleration of the particle has two components :
  (i) tangential acceleration $a_t = \frac{dv}{dt} = R\alpha$
  (ii) normal or radial acceleration $a_n = \frac{v^2}{R} = R\omega^2$

a_t and a_n are two perpendicular components of $\mathbf{a}$. Hence, we can write $a = \sqrt{a_t^2 + a_n^2}$

Since, circular motion, is a 2-D motion we can write

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{r}\right)^2}$$

Here, $$v = \sqrt{v_x^2 + v_y^2} \quad \text{or} \quad v^2 = v_x^2 + v_y^2$$

- **Condition of toppling of a vehicle on circular tracks :** While moving in a circular track normal reaction on the outer wheels (N_1) is more than the normal reaction on inner wheels (N_2).

or $$N_1 > N_2$$

This can be shown as:

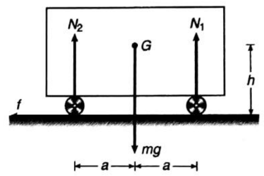


Fig. 7.30

Distance between two wheels = $2a$

Height of centre of gravity of car from road = h

For translational equilibrium of car:

$$N_1 + N_2 = mg \qquad \text{...(i)}$$

and $$f = \frac{mv^2}{r} \qquad \text{...(ii)}$$

and for rotational equilibrium of car, net torque about centre of gravity should be zero.

or $$N_1(a) = N_2(a) + f(h) \qquad \text{...(iii)}$$

From Eq. (iii), we can see that

$$N_2 = N_1 - \left(\frac{h}{a}\right) f = N_1 - \left(\frac{mv^2}{r}\right)\left(\frac{h}{a}\right) \qquad \text{...(iv)}$$

or $$N_2 < N_1$$

From Eq. (iv), we see that N_2 decreases as v is increased.

In critical case, $$N_2 = 0$$

or $$N_1 = mg \qquad \text{[From Eq. (i)]}$$

$\therefore$ $$N_1(a) = f(h) \qquad \text{[From Eq. (iii)]}$$

or $$(mg)(a) = \left(\frac{mv^2}{r}\right)(h)$$

or $$v = \frac{gra}{h}$$

Now, if $v > \sqrt{\frac{gra}{h}}$, $N_2 = 0$, and the car topples outwards.

Therefore, for a safe turn without toppling $v \le \sqrt{\frac{gra}{h}}$.

- From the above discussion we can conclude that while taking a turn on a level road there are two critical speeds, one is the maximum speed for sliding $(= \sqrt{\mu rg})$ and another is maximum speed for toppling $\left(= \sqrt{\frac{gra}{h}}\right)$. One should keep his car's speed less than both of them for neither to slide nor to overturn.

- **Motion of a ball over a smooth solid sphere**
  Suppose a small ball of mass m is given a velocity v over the top of a smooth sphere of radius R. The equation of motion for the ball at the topmost point will be.

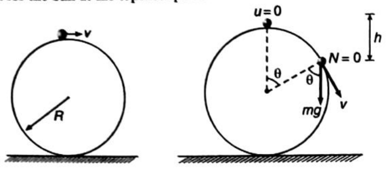


Fig. 7.31

$$mg - N = \frac{mv^2}{R}$$

or
$$N = mg - \frac{mv^2}{R}$$

From this equation we see that value of N decreases as v increases. Minimum value of N can be zero. Hence,

$$0 = mg - \frac{mv^2_{max}}{R}$$

or
$$v_{max} = \sqrt{Rg}$$

From here we can conclude that ball will lose contact with the sphere right from the beginning if velocity of ball at topmost point $v > \sqrt{Rg}$. If $v < \sqrt{Rg}$ it will definitely lose contact but after moving certain distance over the sphere. Now let us find the angle θ where the ball loses contact with the sphere if velocity at topmost point is just zero. Fig. 7.31 (b)

$$h = R(1 - \cos\theta) \qquad \text{...(i)}$$

$$v^2 = 2gh \qquad \text{...(ii)}$$

$$mg\cos\theta = \frac{mv^2}{R} \qquad (\text{as } N = 0) \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2°$$

Thus the ball can move on the sphere maximum upto $\theta = \cos^{-1}\left(\frac{2}{3}\right)$.

Exercise : Find angle θ where the ball will lose contact with the sphere, if velocity at topmost point is $u = \frac{v_{max}}{2} = \frac{\sqrt{gR}}{2}$.

$$\theta = \cos^{-1}\left(\frac{3}{4}\right) = 41.4° \qquad \textbf{Ans.}$$

Hint: Only equation (ii) will change as,

$$v^2 = u^2 + 2gh \quad (u \neq 0)$$

- **Relation between angular velocity vector $\vec{\omega}$, velocity vector $\vec{v}$ and position vector of the particle with respect to centre, $\vec{r}$ is given by:**

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Solved Examples

Example 1 *If a point moves along a circle with constant speed, prove that its angular speed about any point on the circle is half of that about the centre.*

Solution Let, O be a point on a circle and P be the position of the particle at any time t, such that

$$\angle POA = \theta. \quad \text{Then,} \quad \angle PCA = 2\theta$$

Here, C is the centre of the circle.

Angular velocity of P about O is

$$\omega_O = \frac{d\theta}{dt}$$

and angular velocity of P about C is,

$$\omega_C = \frac{d}{dt}(2\theta) = 2\frac{d\theta}{dt}$$

or
$$\omega_C = 2\omega_O \qquad \textbf{Proved.}$$

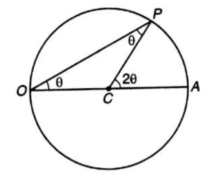


Fig. 7.32

Example 2 *A car is travelling along a circular curve that has a radius of 50 m. If its speed is 16 m/s and is increasing uniformly at 8 m/s², determine the magnitude of its acceleration at this instant.*

Solution
$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{R}\right)^2} = \sqrt{(8)^2 + \left(\frac{256}{50}\right)^2} = 9.5 \text{ m/s}^2$$

Example 3 *A particle moves in a circle of radius 2.0 cm at a speed given by $v = 4t$, where v is in cm/s and t in seconds.*

(a) Find the tangential acceleration at $t = 1$ s.

(b) Find total acceleration at $t = 1$ s.

Solution (a) Tangential acceleration

$$a_t = \frac{dv}{dt} \quad \text{or} \quad a_t = \frac{d}{dt}(4t) = 4 \text{ cm/s}^2$$

i.e., a_t is constant or tangential acceleration at $t = 1$ s is 4 cm/s².

(b) Normal acceleration

$$a_n = \frac{v^2}{R} = \frac{(4t)^2}{R}$$

or
$$a_n = \frac{16t^2}{2.0} = 8.0t^2$$

At $t = 1$ s, $\quad a_n = 8.0 \text{ cm/s}^2$

$\therefore$ Total acceleration $a = \sqrt{a_t^2 + a_n^2}$

or
$$a = \sqrt{(4)^2 + (8)^2} = \sqrt{80} = 4\sqrt{5} \text{ cm/s}^2$$

Example 4 *A boy whirls a stone in a horizontal circle of radius 1.5 m and at height 2.0 m above level ground. The string breaks, and the stone flies off horizontally and strikes the ground after travelling a horizontal distance of 10 m. What is the magnitude of the centripetal acceleration of the stone while in circular motion?*

Solution

$$t=\sqrt{\frac{2h}{g}}=\sqrt{\frac{2\times 2}{9.8}}=0.64\text{ s}$$

$$v=\frac{10}{t}=15.63\text{ m/s}$$

$$\therefore \quad a=\frac{v^2}{R}=163\text{ m/s}^2$$

Example 5 *Two particles A and B start at the origin O and travel in opposite directions along the circular path at constant speeds $v_A=0.7$ m/s and $v_B=1.5$ m/s, respectively. Determine the time when they collide and the magnitude of the acceleration of B just before this happens.*

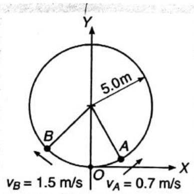


Fig. 7.33

Solution $1.5t+0.7t=2\pi R=10\pi$

$$\therefore \quad t=\frac{10\pi}{2.2}=14.3\text{ s}$$

$$a=\frac{v_B^{\,2}}{R}=0.45\text{ m/s}^2$$

Example 6 *A particle is projected with a speed u at an angle θ with the horizontal. What is the radius of curvature of the parabola traced out by the projectile at a point where the particle velocity makes an angle $\frac{\theta}{2}$ with the horizontal.*

Solution Let v be the velocity at the desired point. Horizontal component of velocity remains unchanged. Hence,

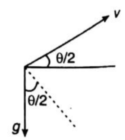


Fig. 7.34

$$v\cos\frac{\theta}{2}=u\cos\theta$$

$$\therefore \quad v=\frac{u\cos\theta}{\cos\frac{\theta}{2}} \quad \text{...(i)}$$

Radial acceleration is the component of acceleration perpendicular to velocity or

$$a_n=g\cos\left(\frac{\theta}{2}\right)$$

$$\therefore \quad \frac{v^2}{R}=g\cos\left(\frac{\theta}{2}\right) \quad \text{...(ii)}$$

Substituting value of v from Eq. (i) in Eq. (ii), we have radius of curvature

$$R=\frac{\left[\dfrac{u\cos\theta}{\cos\left(\frac{\theta}{2}\right)}\right]^2}{g\cos\left(\frac{\theta}{2}\right)}=\frac{u^2\cos^2\theta}{g\cos^3\left(\frac{\theta}{2}\right)}$$

Example 7 *A point moves along a circle with a velocity $v=kt$, where $k=0.5$ m/s². Find the total acceleration of the point at the moment when it has covered the n^{th} fraction of the circle after the beginning of motion, where $n=\frac{1}{10}$.*

Solution $v=\frac{ds}{dt}=kt$ or $\int_0^s ds=k\int_0^t t\,dt$

$$\therefore \quad s=\frac{1}{2}kt^2$$

For completion of nth fraction of circle, $s=2\pi rn$

or $$t^2=(4\pi nr)/k \quad \text{...(i)}$$

$$\text{Tangential acceleration}=a_T=\frac{dv}{dt}=k \quad \text{...(ii)}$$

$$\text{Normal acceleration}=a_N=\frac{v^2}{r}=\frac{k^2t^2}{r} \quad \text{...(iii)}$$

or $$a_N=4\pi nk$$

$$\therefore \quad a=\sqrt{(a_T^2+a_N^2)}=[k^2+16\pi^2n^2k^2]^{1/2}$$

$$=k[1+16\pi^2n^2]^{1/2}$$

$$=0.50\,[1+16\times(3.14)^2\times(0.10)^2]^{1/2}$$

$$=0.8\text{ m/s}^2$$

Example 8 *In a two dimensional motion of a body prove that tangential acceleration is nothing but component of acceleration along velocity.*

Solution Let velocity of the particle be,

$$\vec{\mathbf{v}}=v_x\hat{\mathbf{i}}+v_y\hat{\mathbf{j}}$$

Acceleration $$\vec{\mathbf{a}}=\frac{dv_x}{dt}\hat{\mathbf{i}}+\frac{dv_y}{dt}\hat{\mathbf{j}}$$

Component of $\vec{a}$ along $\vec{v}$ will be,

$$\frac{\vec{a}\cdot\vec{v}}{|\vec{v}|} = v_x \cdot \frac{dv_x}{dt} + v_y \cdot \frac{dv_y}{dt} \quad \text{...(i)}$$

Further, tangential acceleration of particle is rate of change of speed.

or $$a_t = \frac{dv}{dt} = \frac{d}{dt}\left(\sqrt{v_x^2 + v_y^2}\right)$$

or $$a_t = \frac{1}{2\sqrt{v_x^2 + v_y^2}}\left[2v_x \cdot \frac{dv_x}{dt} + 2v_y \frac{dv_y}{dt}\right]$$

or $$a_t = \frac{v_x \cdot \frac{dv_x}{dt} + v_y \cdot \frac{dv_y}{dt}}{\sqrt{v_x^2 + v_y^2}} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we can see that

$$a_t = \frac{\vec{a}\cdot\vec{v}}{|\vec{v}|}$$

or Tangential acceleration = component of acceleration along velocity. **Hence proved.**

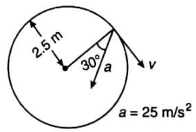
EXERCISES

AIEEE Corner

Subjective Questions (Level 1)

Kinematics of Circular Motion

1. A particle rotates in a circular path of radius 54 m with varying speed $v = 4t^2$. Here v is in m/s and t in second. Find angle between velocity and acceleration at $t = 3$ s.
2. A car is travelling along a circular curve that has a radius of 50 m. If its speed is 16 m/s and is increasing uniformly at 8 m/s^2. Determine the magnitude of its acceleration at this instant.
3. A particle is projected with a speed u at an angle θ with the horizontal. Consider a small part of its path near the highest position and take it approximately to be a circular arc. What is the radius of this circle? This radius is called the radius of curvature of the curve at the point.
4. Figure shows the total acceleration and velocity of a particle moving clockwise in a circle of radius 2.5 m at a given instant of time. At this instant, find :
(a) the radial acceleration,
(b) the speed of the particle and
(c) its tangential acceleration.



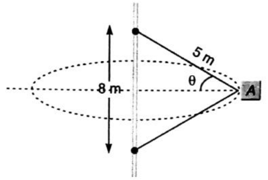
5. A particle moves in a circle of radius 1.0 cm at a speed given by $v = 2.0t$, where v is in cm/s and t in seconds.
(a) Find the radial acceleration of the particle at $t = 1$s.
(b) Find the tangential acceleration at $t = 1$s.
(c) Find the magnitude of the total acceleration at $t = 1$s.
6. A boy whirls a stone of small mass in a horizontal circle of radius 1.5 m and at height 2.9 m above level ground. The string breaks and the stone flies off horizontally and strikes the ground after traveling a horizontal distance of 10 m. What is the magnitude of the centripetal acceleration of the stone while in circular motion ?

Dynamics of Circular Motion

7. A turn has a radius of 10 m. If a vehicle goes round it at an average speed of 18 km/h, what should be the proper angle of banking?
8. If the road of the previous problem is horizontal (no banking), what should be the minimum friction coefficient so that a scooter going at 18 km/h does not skid?
9. A circular road of radius 50 m has the angle of banking equal to 30°. At what speed should a vehicle go on this road so that the friction is not used?

10. A 70 kg man stands in contact against the inner wall of a hollow cylindrical drum of radius 3 m rotating about its vertical axis. The coefficient of friction between the wall and his clothing is 0.15. What is the minimum rotational speed of the cylinder to enable the man to remain stuck to the wall (without falling) when the floor is suddenly removed?

11. A 4 kg block is attached to a vertical rod by means of two strings of equal length. When the system rotates about the axis of the rod, the strings are extended as shown in figure.



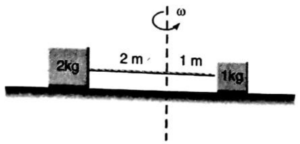
(a) How many revolutions per minute must the system make in order for the tension in the upper string to be 200 N?
(b) What is the tension in the lower string then ?

12. A block of mass m is kept on a horizontal ruler. The friction coefficient between the ruler and the block is μ. The ruler is fixed at one end and the block is at a distance L from the fixed end. The ruler is rotated about the fixed end in the horizontal plane through the fixed end.
(a) What can the maximum angular speed be for which the block does not slip ?
(b) If the angular speed of the ruler is uniformly increased from zero at an angular acceleration α, at what angular speed will the block slip ?

13. Three particles, each of mass m are situated at the vertices of an equilateral triangle of side a. The only forces acting on the particles are their mutual gravitational forces. It is desired that each particle moves in a circle while maintaining the original mutual separation a. Find the initial velocity that should be given to each particle and also the time period of the circular motion. $\left(F = \dfrac{Gm_1 m_2}{r^2}\right)$

14. A thin circular wire of radius R rotates about its vertical diameter with an angular frequency ω. Show that a small bead on the wire remains at its lowermost point for $\omega \le \sqrt{g/R}$. What is angle made by the radius vector joining the centre to the bead with the vertical downward direction for $\omega = \sqrt{2g/R}$? Neglect friction.

15. Two blocks tied with a massless string of length 3 m are placed on a rotating table as shown. The axis of rotation is 1 m from 1 kg mass and 2 m from 2 kg mass. The angular speed $\omega = 4$ rad/s. Ground below 2 kg block is smooth and below 1 kg block is rough. $(g = 10 \text{ m/s}^2)$



(a) Find tension in the string, force of friction on 1 kg block and its direction.

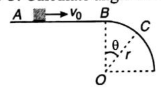
(b) If coefficient of friction between 1 kg block and ground is $\mu = 0.8$. Find maximum angular speed so that neither of the blocks slips.
(c) If maximum tension in the string can be 100 N, then find maximum angular speed so that neither of the blocks slips.

Note *Assume that in part (b) tension can take any value and in parts (a) and (c) friction can take any value.*

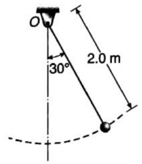
16. What is the maximum speed at which a railway carriage can move without toppling over along a curve of radius $R = 200$ m if the distance from the centre of gravity of the carriage to the level of the rails is $h = 1.5$ m, the distance between the rails is $l = 1.5$ m and the rails are laid horizontally ? (Take $g = 10 \text{ m/s}^2$)

Motion in Vertical Circle

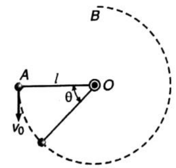
17. A small block slides with velocity $0.5\sqrt{gr}$ on the horizontal frictionless surface as shown in the figure. The block leaves the surface at point C. Calculate angle θ in the figure.



18. The bob of the pendulum shown in figure describes an arc of circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob for the position shown. Find the velocity and the acceleration of the bob in that position.

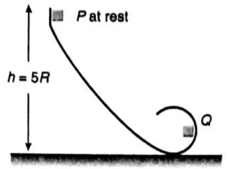


19. The sphere at A is given a downward velocity v_0 of magnitude 5 m/s and swings in a vertical plane at the end of a rope of length $l = 2$ m attached to a support at O. Determine the angle θ at which the rope will break, knowing that it can withstand a maximum tension equal to twice the weight of the sphere.



20. A particle is suspended from a fixed point by a string of length 5 m. It is projected from the equilibrium position with such a velocity that the string slackens after the particle has reached a height 8 m above the lowest point. Find the velocity of the particle, just before the string slackens. Find also, to what height the particle can rise further ?

21. A small block of mass m slides along a smooth circular track of radius R as shown in the figure.



(a) If it starts from rest at P, what is the resultant force acting on it at Q?
(b) At what height above the bottom of the loop should the block be released so that the force it exerts against the track at the top of the loop equals its weight?

Objective Questions (Level 1)

Single Correct Option

1. A particle is revolving in a circle with increasing its speed uniformly. Which of the following is constant?
(a) Centripetal acceleration (b) Tangential acceleration
(c) Angular acceleration (d) None of these

2. A particle is moving in a circular path with a constant speed. If θ is the angular displacement, then starting from $\theta = 0$, the maximum and minimum change in the linear momentum will occur when value of θ is respectively
(a) 45° and 90° (b) 90° and 180° (c) 180° and 360° (d) 90° and 270°

3. A simple pendulum of length l has maximum angular displacement θ. Then maximum kinetic energy of a bob of mass m is
(a) $\frac{1}{2}mgl$ (b) $\frac{1}{2}mgl\cos\theta$ (c) $mgl(1-\cos\theta)$ (d) $\frac{1}{2}mgl\sin\theta$

4. A particle of mass m is fixed to one end of a light rigid rod of length l and rotated in a vertical circular path about its other end. The minimum speed of the particle at its highest point must be
(a) zero (b) $\sqrt{gl}$ (c) $\sqrt{1.5\,gl}$ (d) $\sqrt{2\,gl}$

5. A simple pendulum of length l and mass m is initially at its lowest position. It is given the minimum horizontal speed necessary to move in a circular path about the point of suspension. The tension in the string at the lowest position of the bob is
(a) 3 mg (b) 4 mg (c) 5 mg (d) 6 mg

6. A point moves along a circle having a radius 20 cm with a constant tangential acceleration 5 cms^{-2}. How much time is needed after motion begins for the normal acceleration of the point to be equal to tangential acceleration?
(a) 1 s (b) 2 s (c) 3 s (d) 4 s

7. A ring of mass 2π kg and of radius 0.25 m is making 300 rpm about an axis through its perpendicular to its plane. The tension in newton developed in ring is approximately
(a) 50 (b) 100 (c) 175 (d) 250

8. A car is moving on a circular level road of curvature 300 m. If the coefficient of friction is 0.3 and acceleration due to gravity is 10 m/s^2, the maximum speed of the car can be
(a) 90 km/h (b) 81 km/h (c) 108 km/h (d) 162 km/h

9. A string of length 1 m is fixed at one end with a bob of mass 100 g and the string makes $\frac{2}{\pi}$ rev s^{-1} around a vertical axis through a fixed point. The angle of inclination of the string with vertical is
(a) $\tan^{-1}\left(\frac{5}{8}\right)$ (b) $\tan^{-1}\left(\frac{3}{5}\right)$ (c) $\cos^{-1}\left(\frac{8}{5}\right)$ (d) $\cos^{-1}\left(\frac{5}{8}\right)$

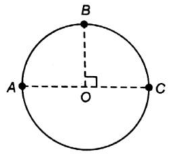
10. In the previous question, the tension in the string is
(a) $\frac{5}{8}$ N (b) $\frac{8}{5}$ N (c) $\frac{50}{8}$ N (d) $\frac{80}{5}$ N

11. A small particle of mass 0.36 g rests on a horizontal turntable at a distance 25 cm from the axis of spindle. The turntable is accelerated at a rate of $\alpha = \frac{1}{3}$ rad s^{-2}. The frictional force that the table exerts on the particle 2 s after the startup is
(a) 40 μN (b) 30 μN (c) 50 μN (d) 60 μN

12. A simple pendulum of length l and bob of mass m is displaced from its equilibrium position O to a position P so that height of P above O is h. It is then released. What is the tension in the string when the bob passes through the equilibrium position O? Neglect friction. v is the velocity of the bob at O.
(a) $m\left(g+\frac{v^2}{l}\right)$ (b) $\frac{2\,mgh}{l}$ (c) $mg\left(1+\frac{h}{l}\right)$ (d) $mg\left(1+\frac{2h}{l}\right)$

13. Two particles revolve concentrically in a horizontal plane in the same direction. The time required to complete one revolution for particle A is 3 min, while for particle B is 1 min. The time required for A to complete one revolution relative to B is
(a) 2 min (b) 1 min (c) 1.5 min (d) 1.25 min

14. Three particles A, B and C move in a circle in anticlockwise direction with speeds 1 ms^{-1}, 2.5 ms^{-1} and 2 ms^{-1} respectively. The initial positions of A, B and C are as shown in figure. The ratio of distance travelled by B and C by the instant A, B and C meet for the first time is
(a) 3 : 2 (b) 5 : 4
(c) 3 : 5 (d) data insufficient



JEE Corner

Assertion and Reason

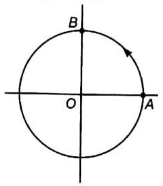
Directions : *Choose the correct option.*

*(a) If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*
*(b) If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*
*(c) If **Assertion** is true, but the **Reason** is false.*
*(d) If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** A car moving on a horizontal rough road with velocity v can be stopped in a minimum distance d. If the same car, moving with same speed v takes a circular turn, then minimum safe radius can be $2d$.

   Reason : $d = \dfrac{v^2}{2\mu g}$ and minimum safe radius $= \dfrac{v^2}{\mu g}$

2. **Assertion :** A particle is rotating in a circle with constant speed as shown. Between points A and B ratio of average acceleration and average velocity is angular velocity of particle about point O.



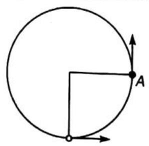
   Reason : Since speed is constant, angular velocity is also constant.

3. **Assertion :** A frame moving in a circle with constant speed can never be an inertial frame.
   Reason : It has a constant acceleration.

4. **Assertion :** In circular motion, dot product of velocity vector ($\vec{v}$) and acceleration vector ($\vec{a}$) may be positive, negative or zero.
   Reason : Dot product of angular velocity vector and linear velocity vector is always zero.

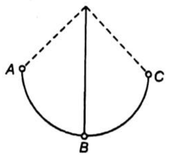
5. **Assertion :** Velocity and acceleration of a particle in circular motion at some instant are:
   $\vec{v} = (2\hat{i})\ \text{ms}^{-1}$ and $\vec{a} = (-\hat{i} + 2\hat{j})\ \text{ms}^{-2}$, then radius of circle is 2 m.
   Reason : Speed of particle is decreasing at a rate of 1 ms^{-2}.

6. **Assertion :** In vertical circular motion, acceleration of bob at position A is greater than 'g'.



   Reason : Net acceleration at A is resultant of tangential and radial components of acceleration.

7. **Assertion :** A pendulum is oscillating between points A, B and C. Acceleration of bob at points A or C is zero.

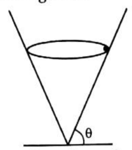


   Reason : Velocity at these points is zero.

8. **Assertion :** Speed of a particle moving in a circle varies with time as, $v = (4t - 12)$. Such type of circular motion is not possible.
   Reason : Speed cannot be change linearly with time.

9. **Assertion :** Circular and projectile both are two dimensional motion. But in circular motion we cannot apply $\vec{v} = \vec{u} + \vec{a}\,t$ directly, whereas in projectile motion we can.
   Reason : Projectile motion takes place under gravity, while in circular motion gravity has no role.

10. **Assertion :** A particle of mass m takes uniform horizontal circular motion inside a smooth funnel as shown. Normal reaction in this case is not $mg \cos \theta$.



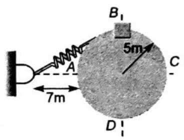
   Reason : Acceleration of particle is not along the surface of funnel.

11. **Assertion :** When water in a bucket is whirled fast overhead, the water does not fall out at the top of the circular path.
    Reason : The centripetal force in this position on water is more than the weight of water.

Objective Questions (Level 2)

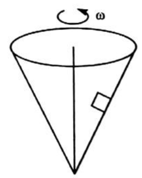
Single Correct Option

1. A collar B of mass 2 kg is constrained to move along a horizontal smooth and fixed circular track of radius 5 m. The spring lying in the plane of the circular track and having spring constant 200 Nm^{-1} is undeformed when the collar is at A. If the collar starts from rest at B, the normal reaction exerted by the track on the collar when it passes through A is



   (a) 360 N (b) 720 N
   (c) 1440 N (d) 2880 N

2. A particle is at rest with respect to the wall of an inverted cone rotating with uniform angular velocity ω about its central axis. The surface between the particle and the wall is smooth. Regarding the displacement of particle along the surface up or down, the equilibrium of particle is

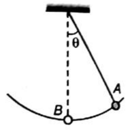


   (a) stable (b) unstable
   (c) neutral (d) None of these

3. A rough horizontal plate rotates with angular velocity ω about a fixed vertical axis. A particle of mass m lies on the plate at a distance $\dfrac{5a}{4}$ from this axis. The coefficient of friction between the plate and the particle is $\dfrac{1}{3}$. The largest value of ω^2 for which the particle will continue to be at rest on the revolving plate is

   (a) $\dfrac{g}{3a}$ (b) $\dfrac{4g}{5a}$ (c) $\dfrac{4g}{9a}$ (d) $\dfrac{4g}{15a}$

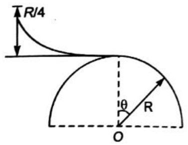
4. A ball attached to one end of a string swings in a vertical plane such that its acceleration at point A (extreme position) is equal to its acceleration at point B (mean position). The angle θ is



(a) $\cos^{-1}\left(\frac{2}{5}\right)$ (b) $\cos^{-1}\left(\frac{4}{5}\right)$

(c) $\cos^{-1}\left(\frac{3}{5}\right)$ (d) None of these

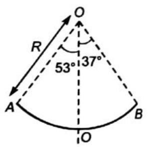
5. A skier plans to ski a smooth fixed hemisphere of radius R. He starts from rest from a curved smooth surface of height $\left(\frac{R}{4}\right)$. The angle θ at which he leaves the hemisphere is



(a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\frac{5}{\sqrt{3}}$

(c) $\cos^{-1}\left(\frac{5}{6}\right)$ (d) $\cos^{-1}\left[\frac{5}{2\sqrt{3}}\right]$

6. A section of fixed smooth circular track of radius R in vertical plane is shown in the figure. A block is released from position A and leaves the track at B. The radius of curvature of its trajectory just after it leaves the track at B is ?



(a) R (b) $\frac{R}{4}$

(c) $\frac{R}{2}$ (d) None of these

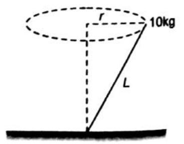
7. A particle is projected with velocity u horizontally from the top of a smooth sphere of radius a so that it slides down the outside of the sphere. If the particle leaves the sphere when it has fallen a height $\frac{a}{4}$, the value of u is

(a) $\sqrt{ag}$ (b) $\frac{\sqrt{ag}}{4}$ (c) $\frac{\sqrt{ag}}{2}$ (d) $\frac{\sqrt{ag}}{3}$

8. A particle of mass m describes a circle of radius r. The centripetal acceleration of the particle is $\frac{4}{r^2}$. What will be the momentum of the particle ?

(a) $2\frac{m}{r}$ (b) $2\frac{m}{\sqrt{r}}$ (c) $4\frac{m}{\sqrt{r}}$ (d) None of these

9. A 10 kg ball attached at the end of a rigid massless rod of length 1 m rotates at constant speed in a horizontal circle of radius 0.5 m and period of 1.58 s, as shown in the figure. The force exerted by the rod on the ball is ($g = 10\,\text{ms}^{-2}$)



(a) 158 N (b) 128 N

(c) 110 N (d) 98 N

10. A particle is moving in a circle of radius R with initial speed v. It starts retarding with constant retardation $\frac{v^2}{4\pi R}$. The number of revolutions it makes in time $\frac{8\pi R}{v}$ is

(a) 3 (b) 4 (c) 5 (d) 2

11. A disc is rotating in a room. A boy standing near the rim of the disc of radius R finds the water droplet falling from the ceiling is always falling on his head. As one drop hits his head, other one starts from the ceiling. If height of the roof above his head is H, then angular velocity of the disc is

(a) $\pi\sqrt{\frac{2gR}{H^2}}$ (b) $\pi\sqrt{\frac{2gH}{R^2}}$ (c) $\pi\sqrt{\frac{2g}{H}}$ (d) None of these

12. In a clock, what is the time period of meeting of the minute hand and the second hand?

(a) 59 s (b) $\frac{60}{59}$ s (c) $\frac{59}{60}$ s (d) $\frac{3600}{59}$ s

13. When a driver of car A sees a car B moving towards his car and at distance 30 m, driver of car A, takes a left turn of 30°. At the same instant the driver of the car B takes a turn to his right at an angle 60°. The two cars collide after two seconds, then the velocity (in ms^{-1}) of the car A and B respectively will be (assume both cars are moving along same line with constant speed)

(a) 7.5, $7.5\sqrt{3}$ (b) 7.5, 7.5 (c) $7.5\sqrt{3}$, 7.5 (d) None of these

14. A particle of mass m starts to slide down from the top of the fixed smooth sphere. What is the acceleration when it breaks off the sphere ?

(a) $\frac{2g}{3}$ (b) $\frac{\sqrt{5}g}{3}$ (c) g (d) $\frac{g}{3}$

15. An automobile enters a turn of radius r. If the road is banked at an angle of 45° and the coefficient of friction is 1, the minimum and maximum speed with which the automobile can negotiate the turn without skidding is?

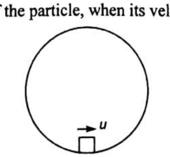
(a) $\sqrt{\frac{rg}{2}}$ and $\sqrt{rg}$

(b) $\frac{\sqrt{rg}}{2}$ and $\sqrt{rg}$

(c) $\frac{\sqrt{rg}}{2}$ and $2\sqrt{rg}$

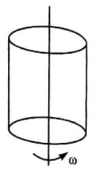
(d) zero and infinite provided plane does not break

16. A particle is given an initial speed u inside a smooth spherical shell of radius R so that it is just able to complete the circle. Acceleration of the particle, when its velocity is vertical, is



(a) $g\sqrt{10}$ (b) g (c) $g\sqrt{2}$ (d) $g\sqrt{6}$

17. An insect of mass $m = 3$ kg is inside a vertical drum of radius 2 m that is rotating with an angular velocity of 5 rad s^{-1}. The insect doesn't fall off. Then the minimum coefficient of friction required is



(a) 0.5
(b) 0.4
(c) 0.2
(d) None of the above

18. A rod is given an angular acceleration α from rest so that it rotates in horizontal plane about a vertical axis. It has a ring at a distance r from the axis of rotation. The friction coefficient between the ring and the rod is μ. Neglecting gravity find the time after witch the ring will start to slip on the rod is.
(Take $\alpha = 3$ rad s^{-2} and $\mu = 1/3$)

(a) 1 s (b) $\frac{1}{3}$ s (c) $\frac{1}{3\sqrt{3}}$ s (d) $3\sqrt{3}$ s

19. A simple pendulum is released from rest with the string in horizontal position. The vertical component of the velocity of the bob becomes maximum, when the string makes an angle θ with the vertical. The angle θ is equal to

(a) $\frac{\pi}{4}$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (c) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\frac{\pi}{3}$

20. A particle is moving in a circle of radius R in such a way that at any instant the normal and tangential component of its acceleration are equal. If its speed at $t = 0$ is v_0. The time taken to complete the first revolution is

(a) $\frac{R}{v_0}$ (b) $\frac{R}{v_0} e^{-2\pi}$ (c) $\frac{R}{v_0}(1 - e^{-2\pi})$ (d) $\frac{R}{v_0}(1 + e^{-2\pi})$

21. A particle is moving in a circular path in the vertical plane. It is attached at one end of a string of length l whose other end is fixed. The velocity at lowest point is u. The tension in the string is $\vec{T}$ and acceleration of the particle is $\vec{a}$ at any position. Then $\vec{T}\cdot\vec{a}$ is zero at highest point if ?

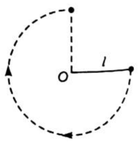
(a) $u > \sqrt{5gl}$ (b) $u = \sqrt{5gl}$
(c) Both (a) and (b) are correct (d) Both (a) and (b) are wrong

22. In the above question, $\vec{T}\cdot\vec{a}$ is positive at the lowest point for

(a) $u \le \sqrt{2gl}$ (b) $u = \sqrt{2gl}$ (c) $u < \sqrt{2gl}$ (d) any value of u

Passage-1 (Q.No. 23 to 24)

A ball with mass m is attached to the end of a rod of mass M and length l. The other end of the rod is pivoted so that the ball can move in a vertical circle. The rod is held in the horizontal position as shown in the figure and then given just enough a downward push so that the ball swings down and just reaches the vertical upward position having zero speed there. Now answer the following questions.



23. The change in potential energy of the system (ball + rod) is

(a) mgl (b) $(M + m)gl$ (c) $\left(\frac{M}{2} + m\right)gl$ (d) $\frac{(M + m)}{2}gl$

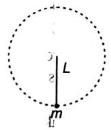
24. The initial speed given to the ball is

(a) $\sqrt{\frac{Mgl + 2mgl}{m}}$ (b) $\sqrt{2gl}$ (c) $\sqrt{\frac{2Mgl + mgl}{m}}$ (d) None of these

Note *Attempt the above question after studying chapter of rotational motion.*

Passage-2 (Q.No 25 to 27)

A small particle of mass m attached with a light inextensible thread of length L is moving in a vertical circle. In the given case particle is moving in complete vertical circle and ratio of its maximum to minimum velocity is 2 : 1.



25. Minimum velocity of the particle is

(a) $4\sqrt{\frac{gL}{3}}$ (b) $2\sqrt{\frac{gL}{3}}$ (c) $\sqrt{\frac{gL}{3}}$ (d) $3\sqrt{\frac{gL}{3}}$

26. The kinetic energy of particle at the lower most position is

(a) $\frac{4mgL}{3}$ (b) $2mgL$ (c) $\frac{8mgL}{3}$ (d) $\frac{2mgL}{3}$

27. Velocity of particle when it is moving vertically downward is

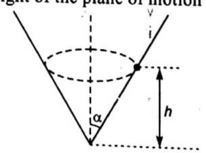
(a) $\sqrt{\frac{10gL}{3}}$ (b) $2\sqrt{\frac{gL}{3}}$ (c) $\sqrt{\frac{8gL}{3}}$ (d) $\sqrt{\frac{13gL}{3}}$

More than One Correct Options

1. A ball tied to the end of the string swings in a vertical circle under the influence of gravity.
(a) When the string makes an angle 90° with the vertical, the tangential acceleration is zero and radial acceleration is somewhere between minimum and maximum.
(b) When the string makes an angle 90° with the vertical, the tangential acceleration is maximum and radial acceleration is somewhere between maximum and minimum.
(c) At no place in circular motion, tangential acceleration is equal to radial acceleration.
(d) When radial acceleration has its maximum value, the tangential acceleration is zero.

2. A small spherical ball is suspended through a string of length l. The whole arrangement is placed in a vehicle which is moving with velocity v. Now, suddenly the vehicle stops and ball starts moving along a circular path. If tension in the string at the highest point is twice the weight of the ball then (Assume that the ball completes the vertical circle)
(a) $v = \sqrt{5gl}$
(b) $v = \sqrt{7gl}$
(c) velocity of the ball at highest point is $\sqrt{gl}$
(d) velocity of the ball at the highest point is $\sqrt{3gl}$

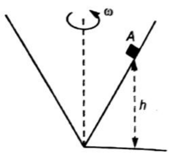
3. A particle is describing circular motion in a horizontal plane in contact with the smooth surface of a fixed right circular cone with its axis vertical and vertex down. The height of the plane of motion above the vertex is h and the semi-vertical angle of the cone is α. The period of revolution of the particle



(a) increases as h increases
(b) decreases as h decreases
(c) increases as α increases
(d) decreases as α increases

4. In circular motion of a particle,
(a) particle cannot have uniform motion
(b) particle cannot have uniformly accelerated motion
(c) particle cannot have net force equal to zero
(d) particle cannot have any force in tangential direction

5. A smooth cone is rotated with an angular velocity ω as shown. A block A is placed at height h. A block has no motion relative to cone. Choose the correct options, when ω is increased.
(a) net force acting on block will increase
(b) normal reaction acting on block will increase
(c) h will increase
(d) normal reaction will remain unchanged



Match the Columns

1. A bob of mass m is suspended from point O by a massless string of length l as shown. At the bottommost point it is given a velocity $u = \sqrt{12gl}$ for $l = 1$ m and $m = 1$ kg, match the following two columns when string becomes horizontal ($g = 10\,\text{ms}^{-2}$)

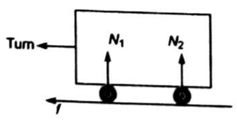
Column I	Column II (SI units)
(a) Speed of bob	(p) 10
(b) Acceleration of bob	(q) 20
(c) Tension in string	(r) 100
(d) Tangential acceleration of bob	(s) None

2. Speed of a particle moving in a circle of radius 2 m varies with time as $v = 2t$ (SI units). At $t = 1$ s match the following two columns.

Column I	Column II (SI units)
(a) $\vec{a}\cdot\vec{v}$	(p) $2\sqrt{2}$
(b) $\lvert\vec{a}\times\vec{\omega}\rvert$	(q) 2
(c) $\vec{v}\cdot\vec{\omega}$	(r) 4
(d) $\lvert\vec{v}\times\vec{a}\rvert$	(s) None

Here, symbols have their usual meanings.

3. A car is taking turn on a rough horizontal road without slipping as shown in figure. Let F is centripetal force : f the force of friction, N_1 and N_2 are two normal reactions. As the speed of car is increased. match the following two columns.



Column I	Column II
(a) N_1	(p) will increase
(b) N_2	(q) will decrease
(c) F/f	(r) will remain unchanged
(d) f	(s) cannot say anything

4. Position vector (with respect to centre) velocity vector and acceleration vector of a particle in circular motion are $\vec{r} = (3\hat{i} - 4\hat{j})$ m, $\vec{v} = (4\hat{i} - a\hat{j})\,\text{ms}^{-1}$ and $\vec{a} = (-6\hat{i} + b\hat{j})\,\text{ms}^{-2}$. Speed of particle is constant. Match the following two columns.

Column I	Column II (SI units)
(a) Value of a	(p) 8
(b) Value of b	(q) 3
(c) Radius of circle	(r) 5
(d) $\vec{r}\cdot(\vec{v}\times\vec{a})$	(s) None

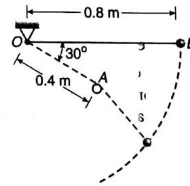
5. A particle is rotating in a circle of radius $R = \frac{2}{\pi}$ m, with constant speed 1 ms^{-1}. Match the following two columns for the time interval when it completes $\frac{1}{4}$th of the circle.

Column I	Column II (SI units)
(a) Average speed	(p) $\frac{\sqrt{2}}{\pi}$
(b) Average velocity	(q) $2\frac{\sqrt{2}}{\pi}$
(c) Average acceleration	(r) $\sqrt{2}$
(d) Displacement	(s) 1

Subjective Questions (Level 2)

1. Bob B of the pendulum AB is given an initial velocity $\sqrt{3Lg}$ in horizontal direction. Find the maximum height of the bob from the starting point :
(a) if AB is a massless rod,
(b) if AB is a massless string.

2. A small sphere B of mass m is released from rest in the position shown and swings freely in a vertical plane, first about O and then about the peg A after the cord comes in contact with the peg. Determine the tension in the cord :
(a) just before the sphere comes in contact with the peg.
(b) just after it comes in contact with the peg.



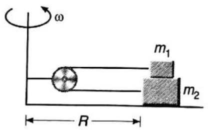
3. A particle of mass m is suspended by a string of length l from a fixed rigid support. A sufficient horizontal velocity $v_0 = \sqrt{3gl}$ is imparted to it suddenly. Calculate the angle made by the string with the vertical when the acceleration of the particle is inclined to the string by 45°.

4. A turn of radius 20 m is banked for the vehicles going at a speed of 36 km/h. If the coefficient of static friction between the road and the tyre is 0.4. What are the possible speeds of a vehicle so that it neither slips down nor skids up ? $(g = 9.8\,\text{m/s}^2)$

5. A particle is projected with a speed u at an angle θ with the horizontal. Find the radius of curvature of the parabola traced out by the projectile at a point, where the particle velocity makes an angle $\theta/2$ with the horizontal.

6. A particle is projected with velocity $20\sqrt{2}$ m/s at 45° with horizontal. After 1 s find tangential and normal acceleration of the particle. Also, find radius of curvature of the trajectory at that point. (Take $g = 10\,\text{m/s}^2$)

7. If the system shown in the figure is rotated in a horizontal circle with angular velocity ω. Find : $(g = 10\,\text{m/s}^2)$

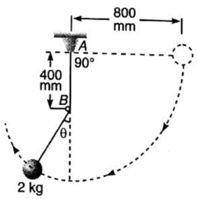


(a) the minimum value of ω to start relative motion between the two blocks.

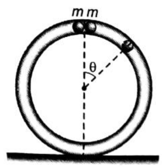
(b) tension in the string connecting m_1 and m_2 when slipping just starts between the blocks.

The coefficient of friction between the two masses is 0.5 and there is no friction between m_2 and ground. The dimensions of the masses can be neglected. (Take $R = 0.5$ m, $m_1 = 2$ kg, $m_2 = 1$ kg)

8. The simple 2 kg pendulum is released from rest in the horizontal position. As it reaches the bottom position, the cord wraps around the smooth fixed pin at B and continues in the smaller arc in the vertical plane. Calculate the magnitude of the force R supported by the pin at B when the pendulum passes the position $\theta = 30°$. $(g = 9.8\,\text{m/s}^2)$



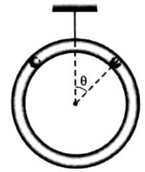
9. A circular tube of mass M is placed vertically on a horizontal surface as shown in the figure. Two small spheres, each of mass m, just fit in the tube, are released from the top. If θ gives the angle between radius vector of either ball with the vertical, obtain the value of the ratio M/m if the tube breaks its contact with ground when $\theta = 60°$. Neglect any friction.



10. A table with smooth horizontal surface is turning at an angular speed ω about its axis. A groove is made on the surface along a radius and a particle is gently placed inside the groove at a distance a from the centre. Find the speed of the particle with respect to the table as its distance from the centre becomes L.

11. A block of mass m slides on a frictionless table. It is constrained to move inside a ring of radius R. At time $t = 0$, block is moving along the inside of the ring (*i.e.*, in the tangential direction) with velocity v_0. The coefficient of friction between the block and the ring is μ. Find the speed of the block at time t.

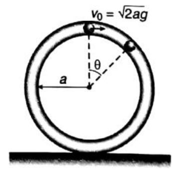
12. A ring of mass M hangs from a thread and two beads of mass m slides on it without friction. The beads are released simultaneously from the top of the ring and slides down in opposite sides. Show that the ring will start to rise, if $m > \dfrac{3M}{2}$.



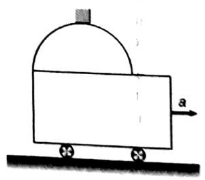
13. A smooth circular tube of radius R is fixed in a vertical plane. A particle is projected from its lowest point with a velocity just sufficient to carry it to the highest point. Show that the time taken by the particle to reach the end of the horizontal diameter is $\sqrt{\dfrac{R}{g}}\ln(1+\sqrt{2})$.

Hint : $\int \sec\theta \cdot d\theta = \ln(\sec\theta + \tan\theta)$

14. A heavy particle slides under gravity down the inside of a smooth vertical tube held in vertical plane. It starts from the highest point with velocity $\sqrt{2ag}$ where a is the radius of the circle. Find the angular position θ (as shown in figure) at which the vertical acceleration of the particle is maximum.



15. A vertical frictionless semicircular track of radius 1 m is fixed on the edge of a movable trolley (figure). Initially the system is rest and a mass m is kept at the top of the track. The trolley starts moving to the right with a uniform horizontal accelertion $a = 2g/9$. The mass slides down the track, eventually losing contact with it and dropping to the floor 1.3 m below the trolley. This 1.3 m is from the point where mass loses contact. $(g = 10\,\text{m/s}^2)$



(a) Calculate the angle θ at which it loses contact with the trolley and

(b) the time taken by the mass to drop on the floor, after losing contact.

ANSWERS

Introductory Exercise 7.1

1. Variable 2. No 3. speed, acceleration, magnitude of acceleration
4. (a) 4.0 cms^{-2} (b) 2.0 cms^{-2} (c) $2\sqrt{5}$ cms^{-2} 5. $\frac{2\sqrt{2}}{\pi}$ 6. $\frac{1}{2}$ s

Introductory Exercise 7.2

1. No 2. 35 ms^{-1} 3. (a) 27.6 (b) 69.3 N 4. (a) 24 kN (b) 50 ms^{-1} (c) 32 kN
5. He should apply the brakes

Introductory Exercise 7.3

1. $\sqrt{2(u^2 - gL)}$ 2. $\sqrt{2g(2R - h)}$ 3. 7 ms^{-1}

AIEEE Corner

Subjective Questions (Level 1)

1. 45° 2. 9.5 ms^{-2} 3. $\frac{u^2 \cos^2 \theta}{g}$ 4. (a) 21.65 ms^{-2} (b) 7.35 ms^{-1} (c) 12.5 ms^{-2}
5. (a) 4.0 cms^{-2} (b) 2.0 cms^{-2} (c) 4.47cms^{-2} 6. 113 ms^{-2} 7. $\tan^{-1}(1/4)$ 8. 0.25
9. 17 ms^{-1} 10. 4.7 $rads^{-1}$ 11. (a) 39.6 rpm (b) 150 N 12. (a) $\sqrt{\mu g/L}$ (b) $\left[\left(\frac{\mu g}{L}\right)^2 - \alpha^2\right]^{1/4}$
13. $v = \sqrt{\frac{Gm}{a}}$, $T = 2\pi\sqrt{\frac{a^3}{3Gm}}$ 14. 60°
15. (a) $T = 64$ N, $f = 48$ N (outwards) (b) 1.63 rad/s (c) 5 rad/s 16. 31.6 ms^{-1} 17. $\theta = \cos^{-1}\left(\frac{3}{4}\right)$
18. 5.66 ms^{-1}, 16.75 ms^{-2} 19. $\sin^{-1}\left(\frac{1}{4}\right)$ 20. 5.42 ms^{-1}, 0.96 m 21. (a) $F = \sqrt{65}\, mg$ (b) $h = 3R$

Objective Questions (Level 1)

1. (b) 2. (c) 3. (c) 4. (a) 5. (d) 6. (b) 7. (d) 8. (c) 9. (d) 10. (b)
11. (c) 12. (d) 13. (c) 14. (b)

JEE Corner

Assertion and Reason

1. (a) 2. (b) 3. (c) 4. (b) 5. (b) 6. (a) 7. (d) 8. (c) 9. (c) 10. (a)
11. (a)

Objective Questions (Level 2)

1. (c) 2. (b) 3. (d) 4. (c) 5. (c) 6. (c) 7. (c) 8. (b) 9. (b) 10. (d)
11. (c) 12. (d) 13. (c) 14. (b) 15. (d) 16. (a) 17. (c) 18. (b) 19. (b) 20. (c)
21. (b) 22. (d) 23. (c) 24. (d) 25. (b) 26. (c) 27. (a)

More than One Correct Options

1. (b,d) 2. (b,d) 3. (a,c) 4. (all) 5. (a,b,c)

Match the Columns

1. (a) → (p) (b) → (s) (c) → (r) (d) → (p)
2. (a) → (r) (b) → (p) (c) → (s) (d) → (r)
3. (a) → (q) (b) → (p) (c) → (r) (d) → (p)
4. (a) → (s) (b) → (s) (c) → (r) (d) → (s)
5. (a) → (s) (b) → (q) (c) → (r) (d) → (q)

Subjective Questions (Level 2)

1. (a) $\frac{3L}{2}$ (b) $\frac{40L}{27}$ 2. (a) $\frac{3mg}{2}$ (d) $\frac{5mg}{2}$ 3. $\theta = \frac{\pi}{2}$ 4. 4.2 $ms^{-1} \le v \le 15$ ms^{-1}
5. $\frac{u^2 \cos^2 \theta}{g \cos^3(\theta/2)}$ 6. $a_t = -2\sqrt{5}$ ms^{-2}, $a_n = 4\sqrt{5}$ ms^{-2}, $R = 25\sqrt{5}$ m 7. (a) $\omega_{min} = 6.32$ rad/s (b) $T = 30$ N
8. 45 N 9. $\frac{M}{m} = \frac{1}{2}$ 10. $v = \omega\sqrt{L^2 - a^2}$ 11. $\frac{v_0}{1 + \frac{\mu v_0 t}{R}}$ 14. $\theta = \cos^{-1}\left(\frac{2}{3}\right)$
15. (a) 37° (b) 0.38 s

Chapter 8 – Center of Mass Conservation of Linear Momentum Impulse and Collision

8

Centre of Mass, Conservation of Linear Momentum, Impulse and Collision

Chapter Contents

8.1 Centre of Mass
8.2 Law of Conservation of Linear Momentum
8.3 Variable Mass
8.4 Impulse
8.5 Collision

8.1 Centre of Mass

When we consider the motion of a system of particles, there is one point in it which behaves as though the entire mass of the system (*i.e.*, the sum of the masses of all the individual particles) is concentrated there and its motion is the same as would ensue if the resultant of all the forces acting on all the particles were applied directly to it. This point is called the centre of mass (COM) of the system. The concept of COM is very useful in solving many problems, in particular, those concerned with collision of particles.

Position of centre of mass

First of all we find the position of COM of a system of particles. Just to make the subject easy we classify a system of particles in three groups:

1. System of two particles.
2. System of a large number of particles and
3. Continuous bodies.

Now, let us take them separately.

Position of COM of Two Particles

Centre of mass of two particles of mass m_1 and m_2 separated by a distance of d lies in between the two particles. The distance of centre of mass from any of the particle (r) is inversely proportional to the mass of the particle (m).

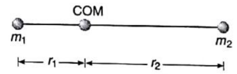


Fig. 8.1

i.e., $$r \propto \frac{1}{m}$$

or $$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

or $$m_1 r_1 = m_2 r_2$$

or $$r_1 = \left(\frac{m_2}{m_2 + m_1}\right) d \quad \text{and} \quad r_2 = \left(\frac{m_1}{m_1 + m_2}\right) d$$

Here, r_1 = distance of COM from m_1

and r_2 = distance of COM from m_2

From the above discussion, we see that

$r_1 = r_2 = \frac{d}{2}$ if $m_1 = m_2$, *i.e.*, COM lies midway between the two particles of equal masses.

Similarly, $r_1 > r_2$ if $m_1 < m_2$ and $r_1 < r_2$ if $m_1 > m_2$, *i.e.*, COM is nearer to the particle having larger mass.

Sample Example 8.1 *Two particles of mass 1 kg and 2 kg are located at $x = 0$ and $x = 3$ m. Find the position of their centre of mass.*

Solution Since, both the particles lie on x-axis, the COM will also lie on x-axis. Let the COM is located at $x = x$, then

r_1 = distance of COM from the particle of mass 1 kg = x

and r_2 = distance of COM from the particle of mass 2 kg

$= (3 - x)$

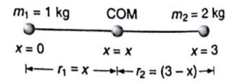


Fig. 8.2

Using $$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

or $$\frac{x}{3-x} = \frac{2}{1} \quad \text{or} \quad x = 2\text{ m}$$

Thus, the COM of the two particles is located at $x = 2$ m.

Position of COM of a Large Number of Particles

If we have a system consisting of n particles, of mass $m_1, m_2, \ldots, m_n$ with $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ as their position vectors at a given instant of time. The position vector $\vec{r}_{COM}$ of the COM of the system at that instant is given by:

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i}$$

or $$\vec{r}_{COM} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{M}$$

Here, $M = m_1 + m_2 + \ldots + m_n$ and $\Sigma m_i\vec{r}_i$ is called the first moment of the mass.

Further, $$\vec{r}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$$

and $$\vec{r}_{COM} = x_{COM}\hat{\mathbf{i}} + y_{COM}\hat{\mathbf{j}} + z_{COM}\hat{\mathbf{k}}$$

So, the cartesian co-ordinates of the COM will be

$$x_{COM} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_ix_i}{\Sigma m_i}$$

or $$x_{COM} = \frac{\sum_{i=1}^{n} m_ix_i}{M}$$

Similarly, $$y_{COM} = \frac{\sum_{i=1}^{n} m_iy_i}{M}$$

and $$z_{COM} = \frac{\sum_{i=1}^{n} m_iz_i}{M}$$

Sample Example 8.2 *The position vector of three particles of mass $m_1 = 1$ kg, $m_2 = 2$ kg and $m_3 = 3$ kg are $\vec{r}_1 = (\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}})$ m, $\vec{r}_2 = (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ m and $\vec{r}_3 = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ m respectively. Find the position vector of their centre of mass.*

Solution The position vector of COM of the three particles will be given by

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

Substituting the values, we get

$$\vec{r}_{COM} = \frac{(1)(\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}) + (2)(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) + 3(2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}})}{1+2+3}$$

$$= \frac{9\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}}{6}$$

$$\vec{r}_{COM} = \frac{1}{2}(3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})\text{ m}$$

Sample Example 8.3 *Four particles of mass 1 kg, 2 kg, 3 kg and 4 kg are placed at the four vertices A, B, C and D of a square of side 1 m. Find the position of centre of mass of the particles.*

Solution Assuming D as the origin, DC as x-axis and DA as y-axis, we have

$$m_1 = 1\text{ kg}, (x_1, y_1) = (0, 1\text{ m})$$
$$m_2 = 2\text{ kg}, (x_2, y_2) = (1\text{ m}, 1\text{ m})$$
$$m_3 = 3\text{ kg}, (x_3, y_3) = (1\text{ m}, 0)$$

and $$m_4 = 4\text{ kg}, (x_4, y_4) = (0, 0)$$

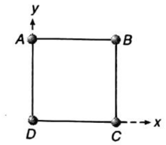


Fig. 8.3

Coordinates of their COM are

$$x_{COM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(1)(0) + 2(1) + 3(1) + 4(0)}{1+2+3+4}$$

$$= \frac{5}{10} = \frac{1}{2}\text{ m} = 0.5\text{ m}$$

Similarly, $$y_{COM} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(1)(1) + 2(1) + 3(0) + 4(0)}{1+2+3+4}$$

$$= \frac{3}{10}\text{ m} = 0.3\text{ m}$$

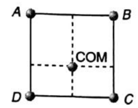


Fig. 8.4

$\therefore$ $$(x_{COM}, y_{COM}) = (0.5\text{ m}, 0.3\text{ m})$$

Thus, position of COM of the four particles is as shown in figure.

Position of COM of Continuous Bodies

If we consider the body to have continuous distribution of matter the summation in the formula of COM is replaced by integration. Suppose x, y and z are the co-ordinates of a small element of mass dm, we write the co-ordinates of COM as

$$x_{COM} = \frac{\int x\,dm}{\int dm} = \frac{\int x\,dm}{M}$$

$$y_{COM} = \frac{\int y\,dm}{\int dm} = \frac{\int y\,dm}{M}$$

and
$$z_{COM} = \frac{\int z\,dm}{\int dm} = \frac{\int z\,dm}{M}$$

Let us take an example.

Centre of Mass of a Uniform Rod

Suppose a rod of mass M and length L is lying along the x-axis with its one end at $x = 0$ and the other at $x = L$.

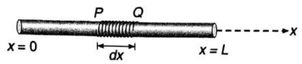


Fig. 8.5

Mass per unit length of the rod $= \dfrac{M}{L}$

Hence, the mass of the element PQ of length dx situated at $x = x$ is $dm = \dfrac{M}{L}\,dx$

The coordinates of the element PQ are $(x, 0, 0)$. Therefore, x-coordinate of COM of the rod will be

$$x_{COM} = \frac{\int_0^L x\,dm}{\int dm} = \frac{\int_0^L (x)\left(\frac{M}{L}\,dx\right)}{M} = \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

The y-coordinate of COM is

$$y_{COM} = \frac{\int y\,dm}{\int dm} = 0 \qquad \text{(as } y = 0)$$

Similarly, $z_{COM} = 0$

i.e., the coordinates of COM of the rod are $\left(\frac{L}{2}, 0, 0\right)$. Or it lies at the centre of the rod.

Proceeding in the similar manner, we can find the COM of certain rigid bodies. Centre of mass of some well known rigid bodies are given below:

1. Centre of mass of a uniform rectangular, square or circular plate lies at its centre.

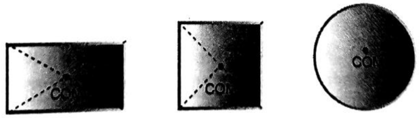


Fig. 8.6

2. Centre of mass of a uniform **semicircular ring** lies at a distance of $h = \dfrac{2R}{\pi}$ from its centre, on the axis of symmetry where R is the radius of the ring.

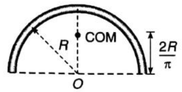


Fig. 8.7

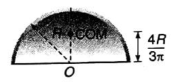


Fig. 8.8

3. Centre of mass of a uniform **semicircular disc** of radius R lies at a distance of $h = \dfrac{4R}{3\pi}$ from the centre on the axis of symmetry as shown in Fig. 8.8.

4. Centre of mass of a **hemispherical shell** of radius R lies at a distance of $h = \dfrac{R}{2}$ from its centre on the axis of symmetry as shown in figure.

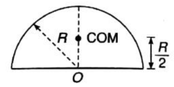


Fig. 8.9

5. Centre of mass of a **solid hemisphere** of radius R lies at a distance of $h = \dfrac{3R}{8}$ from its centre on the axis of symmetry.

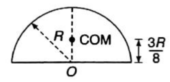


Fig. 8.10

Sample Example 8.4 *A rod of length L is placed along the x-axis between x = 0 and x = L. The linear density (mass/length) ρ of the rod varies with the distance x from the origin as ρ = a + bx. Here, a and b are constants. Find the position of centre of mass of this rod.*

Solution Mass of element PQ of length dx situated at $x = x$ is

$$dm = \rho\,dx = (a + bx)\,dx$$

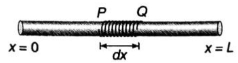


Fig. 8.11

The COM of the element has co-ordinates $(x, 0, 0)$. Therefore, x-coordinate of COM of the rod will be

$$x_{COM} = \frac{\int_0^L x\,dm}{\int_0^L dm} = \frac{\int_0^L (x)(a + bx)\,dx}{\int_0^L (a + bx)\,dx}$$

$$= \frac{\left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^L}{\left[ax + \frac{bx^2}{2}\right]_0^L}$$

or
$$x_{COM} = \frac{3aL + 2bL^2}{6a + 3bL}$$

The y-coordinate of COM of the rod is

$$y_{COM} = \frac{\int y\,dm}{\int dm} = 0 \qquad \text{(as } y = 0\text{)}$$

Similarly, $z_{COM} = 0$

Hence, the centre of mass of the rod lies at $\left[\frac{3aL + 2bL^2}{6a + 3bL}, 0, 0\right]$ **Ans.**

Important Points Regarding Position of Centre of Mass

- For a laminar type (2-dimensional) body the formulae for finding the position of centre of mass are as follows:

(i) $$\vec{r}_{COM} = \frac{A_1\vec{r}_1 + A_2\vec{r}_2 + \dots + A_n\vec{r}_n}{A_1 + A_2 + \dots + A_n}$$

(ii) $$x_{COM} = \frac{A_1x_1 + A_2x_2 + \dots + A_nx_n}{A_1 + A_2 + \dots + A_n}$$

$$y_{COM} = \frac{A_1y_1 + A_2y_2 + \dots + A_ny_n}{A_1 + A_2 + \dots + A_n}$$

and $$z_{COM} = \frac{A_1z_1 + A_2z_2 + \dots + A_nz_n}{A_1 + A_2 + \dots + A_n}$$

Here, A stands for the area.

- If some mass or area is removed from a rigid body, then the position of centre of mass of the remaining portion is obtained from the following formulae:

(i) $$\vec{r}_{COM} = \frac{m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 - m_2}$$

or $$\vec{r}_{COM} = \frac{A_1\vec{r}_1 - A_2\vec{r}_2}{A_1 - A_2}$$

(ii) $$x_{COM} = \frac{m_1x_1 - m_2x_2}{m_1 - m_2}$$

or $$x_{COM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

$$y_{COM} = \frac{m_1y_1 - m_2y_2}{m_1 - m_2}$$

or $$y_{COM} = \frac{A_1y_1 - A_2y_2}{A_1 - A_2}$$

and $$z_{COM} = \frac{m_1z_1 - m_2z_2}{m_1 - m_2}$$

or $$z_{COM} = \frac{A_1z_1 - A_2z_2}{A_1 - A_2}$$

Here, $m_1, A_1, \vec{r}_1, x_1, y_1$ and z_1 are the values for the whole mass while $m_2, A_2, \vec{r}_2, x_2, y_2$ and z_2 are the values for the mass which has been removed. Let us see two examples in support of the above theory.

Sample Example 8.5 *Find the position of centre of mass of the uniform lamina shown in figure.*

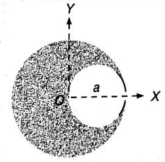


Fig. 8.12

Solution Here,

A_1 = area of complete circle $= \pi a^2$

A_2 = area of small circle

$$= \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$$

(x_1, y_1) = coordinates of centre of mass of large circle $= (0, 0)$

(x_2, y_2) = coordinates of centre of mass of small circle $= \left(\frac{a}{2}, 0\right)$

Using $$x_{COM} = \frac{A_1x_1 - A_2x_2}{A_1 - A_2}$$

we get $$x_{COM} = \frac{-\frac{\pi a^2}{4}\left(\frac{a}{2}\right)}{\pi a^2 - \frac{\pi a^2}{4}} = \frac{-\left(\frac{1}{8}\right)}{\left(\frac{3}{4}\right)}a = -\frac{a}{6}$$

and $y_{COM} = 0$ as y_1 and y_2 both are zero.

Therefore, coordinates of COM of the lamina shown in figure are $\left(-\frac{a}{6}, 0\right)$.

Introductory Exercise 8.1

1. What is the difference between centre of mass and centre of gravity?
2. The centre of mass of a rigid body always lies inside the body. Is this statement true or false ?
3. The centre of mass always lies on the axis of symmetry if it exists. Is this statement true or false?
4. If all the particles of a system lie in y-z plane, the x-coordinate of the centre of mass will be zero. Is this statement true or false?
5. What can be said about the centre of mass of a solid hemisphere of radius r without making any calculation. Will its distance from the centre be more than $r/2$ or less than $r/2$?
6. All the particles of a body are situated at a distance R from the origin. The distance of the centre of mass of the body from the origin is also R. Is this statement true or false?

7. Three particles of mass 1 kg, 2 kg and 3 kg are placed at the corners A, B and C respectively of an equilateral triangle ABC of edge 1 m. Find the distance of their centre of mass from A.
8. Find the distance of centre of mass of a uniform plate having semicircular inner and outer boundaries of radii a and b from the centre O.

Hint : Distance of COM of semicircular plate from centre is $\frac{4r}{3\pi}$.

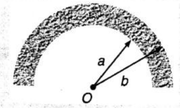


Fig. 8.13

9. Find the position of centre of mass of the section shown in figure.

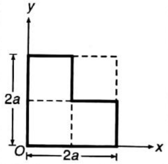


Fig. 8.14

Note *Solve the problem by using both the formulae:*

(i) $x_{COM} = \frac{A_1 x_1 + A_2 x_2}{A_1 + A_2}$ *and*

(ii) $x_{COM} = \frac{A_1 x_1 - A_2 x_2}{A_1 - A_2}$

Motion of the Centre of Mass

Let us consider the motion of a system of n particles of individual masses $m_1, m_2, \dots, m_n$ and total mass M. It is assumed that no mass enters or leaves the system during its motion, so that M remains constant. Then, as we have seen, we have the relation

$$\vec{r}_{COM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_n \vec{r}_n}{m_1 + m_2 + \dots + m_n}$$

$$= \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_n \vec{r}_n}{M}$$

or $$M \vec{r}_{COM} = m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_n \vec{r}_n$$

Differentiating this expression with respect to time t, we have

$$M \frac{d\vec{r}_{COM}}{dt} = m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + \dots + m_n \frac{d\vec{r}_n}{dt}$$

Since, $$\frac{d\vec{r}}{dt} = \text{velocity}$$

Therefore, $$M \vec{v}_{COM} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + \dots + m_n \vec{v}_n \quad \dots\text{(i)}$$

or velocity of the COM is

$$\vec{v}_{COM} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + \dots + m_n \vec{v}_n}{M}$$

or $$\vec{v}_{COM} = \frac{\sum_{i=1}^{n} m_i \vec{v}_i}{M}$$

Further, $m\vec{v}$ = momentum of a particle $\vec{p}$. Therefore, Eq. (i) can be written as

$$\vec{p}_{COM} = \vec{p}_1 + \vec{p}_1 + \dots + \vec{p}_n$$

or $$\vec{p}_{COM} = \sum_{i=1}^{n} \vec{p}_i$$

Differentiating Eq. (i) with respect to time t, we get

$$M \frac{d\vec{v}_{COM}}{dt} = m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} + \dots + m_n \frac{d\vec{v}_n}{dt}$$

or $$M \vec{a}_{COM} = m_1 \vec{a}_1 + m_2 \vec{a}_2 + \dots + m_n \vec{a}_n \quad \dots\text{(ii)}$$

or $$\vec{a}_{COM} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2 + \dots + m_n \vec{a}_n}{M}$$

or $$\vec{a}_{COM} = \frac{\sum_{i=1}^{n} m_i \vec{a}_i}{M}$$

Further, in accordance with Newton's second law of motion $\vec{F} = m\vec{a}$. Hence, Eq. (ii) can be written as

$$\vec{F}_{COM} = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n$$

or $$\vec{F}_{COM} = \sum_{i=1}^{n} \vec{F}_i$$

Thus, as pointed out earlier also, the centre of mass of a system of particles moves as though it were a particle of mass equal to that of the whole system with all the external forces acting directly on it.

Important Points Regarding Motion of Centre of Mass

- Students are often confused over the problems of centre of mass. They cannot answer even the basic problems of COM. For example, let us take a simple problem: two particles one of mass 1 kg and the other of 2 kg are projected simultaneously with the same speed from the roof of a tower, the one of mass 1 kg vertically upwards and the other vertically downwards. What is the acceleration of centre of mass of these two particles? When I ask this question in my first class of centre of mass, three answers normally come among the students g, $\frac{g}{3}$ and zero. The correct answer is g. Because

$$\vec{a}_{COM} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2}{m_1 + m_2}$$

Here, $$\vec{a}_1 = \vec{a}_2 = g \quad \text{(downwards)}$$

$\therefore$ $$\vec{a}_{COM} = \frac{(1)(g) + (2)(g)}{1+2} = g \quad \text{(downwards)}$$

The idea behind this is that apply the basic equations when asked anything about centre of mass. Just as a revision I am writing below all the basic equations of COM at one place.

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots + m_n}$$

$$x_{COM} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n}$$

$$y_{COM} = \frac{m_1y_1 + m_2y_2 + \ldots + m_ny_n}{m_1 + m_2 + \ldots + m_n}$$

$$z_{COM} = \frac{m_1z_1 + m_2z_2 + \ldots + m_nz_n}{m_1 + m_2 + \ldots + m_n}$$

$$\vec{v}_{COM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n}{m_1 + m_2 + \ldots + m_n}$$

$$\vec{P}_{COM} = \vec{P}_1 + \vec{P}_2 + \ldots + \vec{P}_n$$

$$\vec{a}_{COM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots + m_n\vec{a}_n}{m_1 + m_2 + \ldots + m_n}$$

and $$\vec{F}_{COM} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n$$

Sample Example 8.6 *Two particles A and B of mass 1 kg and 2 kg respectively are projected in the directions shown in figure with speeds $u_A = 200$ m/s and $u_B = 50$ m/s. Initially they were 90 m apart. Find the maximum height attained by the centre of mass of the particles. Assume acceleration due to gravity to be constant. ($g = 10\ m/s^2$)*

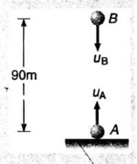


Fig. 8.15

Solution Using $$m_Ar_A = m_Br_B$$

or $$(1)(r_A) = (2)(r_B)$$

or $$r_A = 2r_B \quad \ldots\text{(i)}$$

and $$r_A + r_B = 90\ \text{m} \quad \ldots\text{(ii)}$$

Solving these two equations, we get

$$r_A = 60\ \text{m} \quad \text{and} \quad r_B = 30\ \text{m}$$

i.e., COM is at height 60 m from the ground at time $t = 0$.

Further, $$\vec{a}_{COM} = \frac{m_A\vec{a}_A + m_B\vec{a}_B}{m_A + m_B}$$

$$= g = 10\ \text{m/s}^2 \quad \text{(downwards)}$$

as $$\vec{a}_A = \vec{a}_B = g \quad \text{(downwards)}$$

$$\vec{u}_{COM} = \frac{m_A\vec{u}_A + m_B\vec{u}_B}{m_A + m_B}$$

$$= \frac{(1)(200) - (2)(50)}{1+2} = \frac{100}{3}\ \text{m/s} \quad \text{(upwards)}$$

Let, h be the height attained by COM beyond 60 m. Using,

$$v_{COM}^2 = u_{COM}^2 + 2a_{COM}h$$

or $$0 = \left(\frac{100}{3}\right)^2 - (2)(10)h$$

or $$h = \frac{(100)^2}{180} = 55.55\ \text{m}$$

Therefore, maximum height attained by the centre of mass is

$$H = 60 + 55.55$$
$$= 115.55\ \text{m}$$

Sample Example 8.7 *In the arrangement shown in figure, $m_A = 2$ kg and $m_B = 1$ kg. String is light and inextensible. Find the acceleration of centre of mass of both the blocks. Neglect friction everywhere.*

Fig. 8.16

Solution Net pulling force on the system is $(m_A - m_B)g$

or $$(2-1)g = g$$

Total mass being pulled is $m_A + m_B$ or 3 kg

$\therefore$ $$a = \frac{\text{Net pulling force}}{\text{Total mass}} = \frac{g}{3}$$

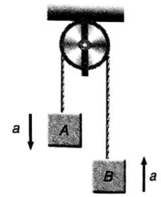


Fig. 8.17

Now, $$\vec{a}_{COM} = \frac{m_A\vec{a}_A + m_B\vec{a}_B}{m_A + m_B}$$

$$= \frac{(2)(a) - (1)(a)}{1+2} = \frac{a}{3}$$

$$= \frac{g}{9} \text{ downwards}$$

Alternate Method

Free body diagram of block A is shown in figure.

$$2g - T = m_A(a)$$

or $$T = 2g - m_Aa$$

$$= 2g - (2)\left(\frac{g}{3}\right) = \frac{4g}{3}$$

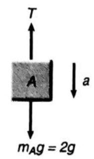


Fig. 8.18

Free body diagrams of A and B both are as shown in Fig. 8.19.

$$\vec{a}_{COM} = \frac{\text{Net force on both the blocks}}{m_A + m_B}$$

$$= \frac{(m_A + m_B)g - 2T}{2+1}$$

$$= \frac{3g - \frac{8g}{3}}{3}$$

$$= \frac{g}{9} \text{ downwards}$$

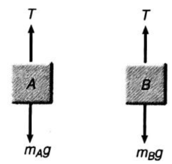


Fig. 8.19

Introductory Exercise 8.2

1. Two particles of mass 1 kg and 2 kg respectively are initially 10 m apart. At time $t = 0$, they start moving towards each other with uniform speeds 2 m/s and 1 m/s respectively. Find the displacement of their centre of mass at $t = 1$ s.
2. Two blocks A and B of equal masses are attached to a string passing over a smooth pulley fixed to a wedge as shown in figure. Find the magnitude of acceleration of centre of mass of the two blocks when they are released from rest. Neglect friction.

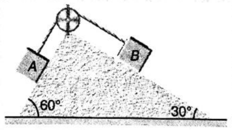


Fig. 8.20

8.2 Law of Conservation of Linear Momentum

The product of mass and the velocity of a particle is defined as its linear momentum ($\vec{p}$). So,

$$\vec{p} = m\vec{v}$$

The magnitude of linear momentum may be written as

$$p = mv$$

or $$p^2 = m^2v^2 = 2m\left(\frac{1}{2}mv^2\right) = 2mK$$

Thus, $$p = \sqrt{2Km} \quad \text{or} \quad K = \frac{p^2}{2m}$$

Here, K is the kinetic energy of the particle. In accordance with Newton's second law,

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$$

Thus, $$\vec{F} = \frac{d\vec{p}}{dt}$$

In case the external force applied to a particle (or a body) be zero, we have

$$\vec{F} = \frac{d\vec{p}}{dt} = 0 \quad \text{or} \quad \vec{p} = \text{constant}$$

showing that in the absence of an external force, the linear momentum of a particle (or the body) remains constant. This is called the law of conservation of linear momentum. The law may be extended to a system of particles or to the centre of mass of a system of particles. For example, for a system of particles it takes the form.

If net force (or the vector sum of all the forces) on a system of particles is zero, the vector sum of linear momentum of all the particles remain conserved, or

If $$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n = 0$$

Then, $$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots + \vec{p}_n = \text{constant}$$

The same is the case for the centre of mass of a system of particles, *i.e.*, if

$$\vec{F}_{COM} = 0, \; \vec{p}_{COM} = \text{constant}.$$

Thus, the law of conservation of linear momentum can be applied to a single particle, to a system of particles or even to the centre of mass of the particles.

The law of conservation of linear momentum enables us to solve a number of problems which can not be solved by a straight application of the relation $\vec{F} = m\vec{a}$.

For example, suppose a particle of mass m intially at rest, suddenly explodes into two fragments of masses m_1 and m_2 which fly apart with velocities $\vec{v}_1$ and $\vec{v}_2$ respectively. Obviously, the forces resulting in the explosion of the particle must be internal forces, since no external force has been applied. In the absence of the external forces, therefore, the momentum must remain conserved and we should have

$$m\vec{v} = m_1\vec{v}_1 + m_2\vec{v}_2$$

Since, the particle was initially at rest, $\vec{v} = 0$ and therefore,

$$m_1\vec{v}_1 + m_2\vec{v}_2 = 0$$

$$\vec{v}_1 = -\frac{m_2}{m_1}\vec{v}_2 \quad \text{or} \quad \frac{|\vec{v}_1|}{|\vec{v}_2|} = \frac{m_2}{m_1}$$

Showing at once that the velocities of the two fragments must be inversely proportional to their masses and in opposite directions along the same line. This result could not possibly be arrived at from the relation $\vec{F} = m\vec{a}$, since we know nothing about the forces that were acting during the explosion. Nor, could we derive it from the law of conservation of energy.

Important Points Regarding Net Force Equal to Zero on Centre of Mass

- If a projectile explodes in air in different parts, the path of the centre of mass remains unchanged. This is because during explosion no external force (except gravity) acts on the centre of mass. The situation is as shown in figure.
Path of COM is ABC, even though the different parts travel in different directions after explosion.

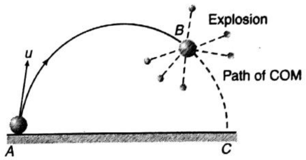


Fig. 8.21

- Suppose a system consists of more than one particle (or bodies). Net external force on the system in a particular direction is zero. Initially the centre of mass of the system is at rest, then obviously the centre of mass will not move along that particular direction even though some particles (or bodies) of the system may move along that direction. The following example will illustrate the above theory.

Sample Example 8.8 *A projectile of mass 3 m is projected from ground with velocity $20\sqrt{2}$ m/s at 45°. At highest point it explodes into two pieces. One of mass 2 m and the other of mass m. Both the pieces fly off horizontally in opposite directions. Mass 2 m falls at a distance of 100 m from point of projection. Find the distance of second mass from point of projection where it strikes the ground. ($g = 10\ m/s^2$)*

Solution Range of the projectile in the absence of explosion

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$= \frac{(20\sqrt{2})^2 \sin 90°}{10}$$

$$= 80\ \text{m}$$

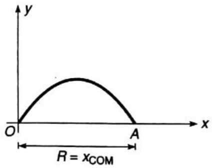


Fig. 8.22

The path of centre of mass of projectile will not change, *i.e.*, x_{COM} is still 80 m. Now, from the definition of centre of mass

$$x_{COM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

or
$$80 = \frac{(m)(x_1) + (2m)(100)}{m + 2m}$$

Solving this equation, we get

$$x_1 = 40\ \text{m}$$

Therefore, the mass m will fall at a distance $x_1 = 40$ cm from point of projection. **Ans.**

Sample Example 8.9 *A wooden plank of mass 20 kg is resting on a smooth horizontal floor. A man of mass 60 kg starts moving from one end of the plank to the other end. The length of the plank is 10 m. Find the displacement of the plank over the floor when the man reaches the other end of the plank.*

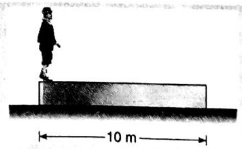


Fig. 8.23

Solution Here, the system is man + plank. Net force on this system in horizontal direction is zero and initially the centre of mass of the system is at rest. Therefore, the centre of mass does not move in horizontal direction.

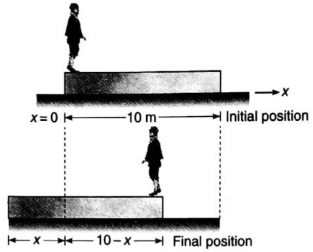


Fig. 8.24

Let x be the displacement of the Plank. Assuming the origin, *i.e.*, $x = 0$ at the position shown in figure.

As we said earlier also, the centre of mass will not move in horizontal direction (x-axis). Therefore, for centre of mass to remain stationary,

$$x_i = x_f$$

$$\frac{(60)(0) + 20\left(\frac{10}{2}\right)}{60 + 20} = \frac{(60)(10 - x) + 20\left(\frac{10}{2} - x\right)}{60 + 20}$$

or
$$\frac{5}{4} = \frac{6(10 - x) + 2\left(\frac{10}{2} - x\right)}{8} = \frac{60 - 6x + 10 - 2x}{8}$$

or $5 = 30 - 3x + 5 - x$ or $4x = 30$

or $x = \frac{30}{4}$ m or $x = 7.5$ m

Note *The centre of mass of the plank lies at its centre.*

Sample Example 8.10 *A man of mass m_1 is standing on a platform of mass m_2 kept on a smooth horizontal surface. The man starts moving on the platform with a velocity v_r relative to the platform. Find the recoil velocity of platform.*

Solution Absolute velocity of man $= v_r - v$ where $v =$ recoil velocity of platform. Taking the platform and the man as a system, net external force on the system in horizontal direction is zero. The linear momentum of the system remains constant. Initially both the man and the platform were at rest.

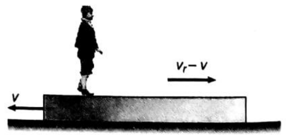


Fig. 8.25

Hence, $0 = m_1 (v_r - v) - m_2 v$

$\therefore$ $$v = \frac{m_1 v_r}{m_1 + m_2}$$

Sample Example 8.11 *A gun (mass = M) fires a bullet (mass = m) with speed v_r relative to barrel of the gun which is inclined at an angle of 60° with horizontal. The gun is placed over a smooth horizontal surface. Find the recoil speed of gun.*

Solution Let the recoil speed of gun is v. Taking gun + bullet as the system. Net external force on the system in horizontal direction is zero. Initially the system was at rest. Therefore, applying the principle of conservation of linear momentum in horizontal direction, we get

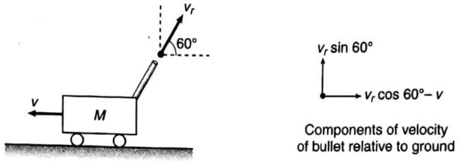


Fig. 8.26

$$Mv - m(v_r \cos 60° - v) = 0$$

$\therefore$
$$v = \frac{mv_r \cos 60°}{M + m}$$

or
$$v = \frac{mv_r}{2(M + m)}$$

Introductory Exercise 8.3

1. A man of mass 60 kg jumps from a trolley of mass 20 kg standing on smooth surface with absolute velocity 3 m/s. Find velocity of trolley and total energy produced by man.
2. Two blocks A and B of mass m_A and m_B are connected together by means of a spring and are resting on a horizontal frictionless table. The blocks are then pulled apart so as to stretch the spring and then released. Show that the kinetic energies of the blocks are, at any instant inversely proportional to their masses.
3. Three particles of mass 20 g, 30 g and 40 g are initially moving along the positive direction of the three coordinate axes respectively with the same velocity of 20 cm/s. When due to their mutual interaction, the first particle comes to rest, the second acquires a velocity $10\hat{i} + 20\hat{k}$. What is then the velocity of the third particle?
4. A projectile is fired from a gun at an angle of 45° with the horizontal and with a speed of 20 m/s relative to ground. At the highest point in its flight the projectile explodes into two fragments of equal mass. One fragment, whose initial speed is zero falls vertically. How far from the gun does the other fragment land, assuming a level terrain? Take $g = 10\ m/s^2$?
5. A particle of mass 2 m is projected at an angle of 45° with horizontal with a velocity of $20\sqrt{2}$ m/s. After 1 s explosion takes place and the particle is broken into two equal pieces. As a result of explosion one part comes to rest. Find the maximum height attained by the other part. Take $g = 10\ m/s^2$.
6. A boy of mass 60 kg is standing over a platform of mass 40 kg placed over a smooth horizontal surface. He throws a stone of mass 1 kg with velocity $v = 10$ m/s at an angle of 45° with respect to the ground. Find the displacement of the platform (with boy) on the horizontal surface when the stone lands on the ground. Take $g = 10\ m/s^2$.

7. A gun fires a bullet. The barrel of the gun is inclined at an angle of 45° with horizontal. When the bullet leaves the barrel it will be travelling at an angle greater than 45° with the horizontal. Is this statement true or false?

8.3 Variable Mass

In our discussion of the conservation of linear momentum, we have so far dealt with systems whose mass remains constant. We now consider those systems whose mass is variable, *i.e.,* those in which mass enters or leaves the system. A typical case is that of the rocket from which hot gases keep on escaping, thereby continuously decreasing its mass.

In such problems you have nothing to do but apply a thrust force ($\vec{F}_t$) to the main mass in addition to the all other forces acting on it. This thrust force is given by,

$$\vec{F}_t = \vec{v}_{rel}\left(\frac{dm}{dt}\right)$$

Here, $\vec{v}_{rel}$ is the velocity of the mass gained or mass ejected relative to the main mass. In case of rocket this is sometimes called the exhaust velocity of the gases. $\frac{dm}{dt}$ is the rate at which mass is increasing or decreasing.

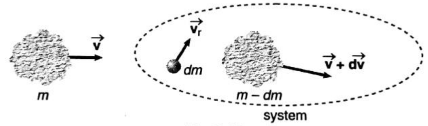


Fig. 8.27

The expression for the thrust force can be derived from the conservation of linear momentum in the absence of any external forces on a system as follows:

Suppose at some moment $t = t$ mass of a body is m and its velocity is $\vec{v}$. After some time at $t = t + dt$ its mass becomes $(m - dm)$ and velocity becomes $\vec{v} + d\vec{v}$. The mass dm is ejected with relative velocity $\vec{v}_r$. Absolute velocity of mass 'dm' is therefore $(\vec{v}_r + \vec{v} + d\vec{v})$. If no external forces are acting on the system, the linear momentum of the system will remain conserved, or

$$\vec{P}_i = \vec{P}_f$$

or
$$m\vec{v} = (m - dm)(\vec{v} + d\vec{v}) + dm(\vec{v}_r + \vec{v} + d\vec{v})$$

or
$$m\vec{v} = m\vec{v} + md\vec{v} - dm\vec{v} - (dm)(d\vec{v}) + dm\vec{v} + \vec{v}_r\,dm + (dm)(d\vec{v})$$

$\therefore$
$$m\,d\vec{v} = -\vec{v}_r\,dm$$

or
$$m\left(\frac{d\vec{v}}{dt}\right) = \vec{v}_r\left(-\frac{dm}{dt}\right)$$

Here,
$$m\left(\frac{d\vec{v}}{dt}\right) = \text{thrust force } (\vec{F}_t)$$

and
$$-\frac{dm}{dt} = \text{rate at which mass is ejecting}$$

Problems Related to Variable Mass can be Solved in Following Three Steps

1. Make a list of all the forces acting on the main mass and apply them on it.
2. Apply an additional thrust force $\vec{F}_t$ on the mass, the magnitude of which is $\left|\vec{v}_r\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by the direction of $\vec{v}_r$ in case the mass is increasing and otherwise the direction of $-\vec{v}_r$ if it is decreasing.
3. Find net force on the mass and apply

$$\vec{F}_{\text{net}} = m\frac{d\vec{v}}{dt} \qquad (m = \text{mass at that particular instant})$$

Rocket Propulsion

Let m_0 be the mass of the rocket at time $t = 0$. m its mass at any time t and v its velocity at that moment. Initially let us suppose that the velocity of the rocket is u.

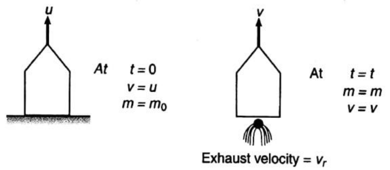


Fig. 8.28

Further, let $\left(\frac{-dm}{dt}\right)$ be the mass of the gas ejected per unit time and v_r the exhaust velocity of the gases. Usually $\left(\frac{-dm}{dt}\right)$ and v_r are kept constant throughout the journey of the rocket. Now, let us write few equations which can be used in the problems of rocket propulsion. At time $t = t$,

1. Thrust force on the rocket

$$F_t = v_r\left(-\frac{dm}{dt}\right) \qquad \text{(upwards)}$$

2. Weight of the rocket

$$W = mg \qquad \text{(downwards)}$$

3. Net force on the rocket

$$F_{\text{net}} = F_t - W \qquad \text{(upwards)}$$

or $$F_{\text{net}} = v_r\left(\frac{-dm}{dt}\right) - mg$$

4. Net acceleration of the rocket $a = \frac{F}{m}$

or $$\frac{dv}{dt} = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$$

or $$dv = v_r\left(\frac{-dm}{m}\right) - g\,dt$$

or $$\int_u^v dv = v_r\int_{m_0}^{m}\frac{-dm}{m} - g\int_0^t dt$$

or $$v - u = v_r \ln\left(\frac{m_0}{m}\right) - gt$$

Thus, $$\boldsymbol{v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)} \qquad \text{...(i)}$$

Note 1. *$F_t = v_r\left(-\frac{dm}{dt}\right)$ is upwards, as v_r is downwards and $\frac{dm}{dt}$ is negative.*

2. *If gravity is ignored and initial velocity of the rocket $u = 0$, Eq. (i) reduces to $v = v_r \ln\left(\frac{m_0}{m}\right)$.*

Sample Example 8.12 *(a) A rocket set for vertical firing weighs 50 kg and contains 450 kg of fuel. It can have a maximum exhaust velocity of 2 km/s. What should be its minimum rate of fuel consumption*

(i) to just lift it off the launching pad?

(ii) to give it an acceleration of 20 m/s²?

(b) What will be the speed of the rocket when the rate of consumption of fuel is 10 kg/s after whole of the fuel is consumed? (Take $g = 9.8$ m/s²)

Solution (a) (i) To just lift it off the launching pad

$$\text{weight} = \text{thrust force}$$

or $$mg = v_r\left(\frac{-dm}{dt}\right)$$

or $$\left(\frac{-dm}{dt}\right) = \frac{mg}{v_r}$$

Substituting the values, we get

$$\left(\frac{-dm}{dt}\right) = \frac{(450+50)(9.8)}{2\times 10^3} = 2.45 \text{ kg/s}$$

(ii) Net acceleration $a = 20$ m/s²

$\therefore$ $$ma = F_t - mg$$

or $$a = \frac{F_t}{m} - g$$

or $$a = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$$

This gives $$\left(-\frac{dm}{dt}\right) = \frac{m(g+a)}{v_r}$$

Substituting the values, we get

$$\left(-\frac{dm}{dt}\right)=\frac{(450+50)(9.8+20)}{2\times 10^3}$$

$$=7.45 \text{ kg/s}$$

(b) The rate of fuel consumption is 10 kg/s. So, the time for the consumption of entire fuel is

$$t=\frac{450}{10}=45 \text{ s}$$

Using Eq. (i), *i.e.*, $v=u-gt+v_r \ln\left(\frac{m_0}{m}\right)$

Here, $u=0$, $v_r=2\times 10^3$ m/s, $m_0=500$ kg and $m=50$ kg

Substituting the values, we get

$$v=0-(9.8)(45)+(2\times 10^3)\ln\left(\frac{500}{50}\right)$$

or $$v=-441+4605.17$$

or $$v=4164.17 \text{ m/s}$$

or $$v=4.164 \text{ km/s}$$

Introductory Exercise 8.4

1. A rocket of mass 20 kg has 180 kg fuel. The exhaust velocity of the fuel is 1.6 km/s. Calculate the minimum rate of consumption of fuel so that the rocket may rise from the ground. Also, calculate the ultimate vertical speed gained by the rocket when the rate of consumption of fuel is (g = 9.8 m/s^2)
   (i) 2 kg/s (ii) 20 kg/s
2. A rocket is moving vertically upward against gravity. Its mass at time t is $m=m_0-\mu t$ and it expels burnt fuel at a speed u vertically downward relative to the rocket. Derive the equation of motion of the rocket but do not solve it. Here, μ is constant.
3. A rocket of initial mass m_0 has a mass $m_0(1-t/3)$ at time t. The rocket is launched from rest vertically upwards under gravity and expels burnt fuel at a speed u relative to the rocket vertically downward. Find the speed of rocket at $t=1$.

8.4 Impulse

Consider a constant force $\vec{F}$ which acts for a time t on a body of mass m, thus, changing its velocity from $\vec{u}$ to $\vec{v}$. Because the force is constant, the body will travel with constant acceleration $\vec{a}$ where

$$\vec{F}=m\vec{a}$$

and $$\vec{a}t=\vec{v}-\vec{u}$$

hence, $$\frac{\vec{F}}{m}t=\vec{v}-\vec{u}$$

or $$\vec{F}t=m\vec{v}-m\vec{u}$$

The product of constant force $\vec{F}$ and the time t for which it acts is called the **impulse** ($\vec{J}$) of the force and this is equal to the change in linear momentum which it produces.

Thus, $$\text{Impulse }(\vec{J})=\vec{F}t=\Delta\vec{p}=\vec{p}_f-\vec{p}_i$$

Instantaneous Impulse : There are many occasions when a force acts for such a short time that the effect is instantaneous, *e.g.*, a bat striking a ball. In such cases, although the magnitude of the force and the time for which it acts may each be unknown but the value of their product (*i.e.*, impulse) can be known by measuring the initial and final momenta. Thus, we can write

$$\vec{J}=\int\vec{F}\,dt=\Delta\vec{p}=\vec{p}_f-\vec{p}_i$$

Regarding the impulse it is important to note that impulse applied to an object in a given time interval can also be calculated from the area under force-time (F-t) graph in the same time interval.

Sample Example 8.13 *A truck of mass 2×10^3 kg travelling at 4 m/s is brought to rest in 2 s when it strikes a wall. What force (assume constant) is exerted by the wall?*

Solution Using impulse = change in linear momentum

We have, $$F\cdot t=mv_f-mv_i=m(v_f-v_i)$$

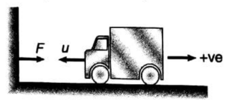


Fig. 8.29

or $$F(2)=2\times 10^3\,[0-(-4)]$$

or $$2F=8\times 10^3$$

or $$F=4\times 10^3 \text{ N}$$

Sample Example 8.14 *A ball of mass m, travelling with velocity $2\hat{i}+3\hat{j}$ receives an impulse $-3m\hat{i}$. What is the velocity of the ball immediately afterwards?*

Solution Using $$\vec{J}=m(\vec{v}_f-\vec{v}_i)$$

$$-3m\hat{i}=m[\vec{v}_f-(2\hat{i}+3\hat{j})]$$

or $$\vec{v}_f=-3\hat{i}+(2\hat{i}+3\hat{j})$$

or $$\vec{v}_f=-\hat{i}+3\hat{j}$$

Note *The velocity component in the direction of $\hat{j}$ is unchanged. This is because there is no impulse component in this direction.*

Sample Example 8.15 *A bullet of mass 10^{-3} kg strikes an obstacle and moves at 60° to its original direction. If its speed also changes from 20 m/s to 10 m/s. Find the magnitude of impulse acting on the bullet.*

Solution Mass of the bullet $m = 10^{-3}$ kg

Consider components parallel to J_1.

$$J_1 = 10^{-3}\,[-10\cos 60° - (-20)]$$

or $$J_1 = 15 \times 10^{-3} \text{ N-s}$$

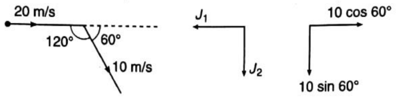


Fig. 8.30

Similarly, praralel to J_2, we have

$$J_2 = 10^{-3}[10\sin 60° - 0] = 5\sqrt{3} \times 10^{-3} \text{ N-s}$$

The magnitude of resultant impulse is given by

$$J = \sqrt{J_1^2 + J_2^2} = 10^{-3}\sqrt{(15)^2 + (5\sqrt{3})^2}$$

or $$J = \sqrt{3} \times 10^{-2} \text{ N-s}$$

Sample Example 8.16 *A particle of mass 2 kg is initially at rest. A force starts acting on it in one direction whose magnitude changes with time. The force time graph is shown in figure.*

Find the velocity of the particle at the end of 10 s.

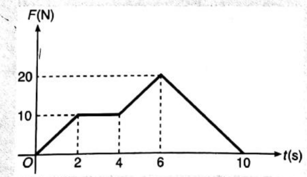


Fig. 8.31

Solution Using impulse = Change in linear momentum (or area under F-t graph)

We have, $$m(v_f - v_i) = \text{Area}$$

or $$2(v_f - 0) = \frac{1}{2} \times 2 \times 10 + 2 \times 10 + \frac{1}{2} \times 2 \times (10 + 20) + \frac{1}{2} \times 4 \times 20$$

$$= 10 + 20 + 30 + 40$$

or $$2v_f = 100$$

$\therefore$ $$v_f = 50 \text{ m/s}$$ **Ans.**

Introductory Exercise 8.5

1. A particle of mass 1 kg is projected from the ground at an angle of 60° with horizontal at a velocity of 20 m/s. Find the magnitude of change in its velocity in 1 s. ($g = 10 \text{ m/s}^2$)

2. Velocity of a particle of mass 2 kg varies with time t according to the equation $\vec{v} = (2t\hat{i} + 4\hat{j})$ m/s. Here, t is in seconds. Find the impulse imparted to the particle in the time interval from $t = 0$ to $t = 2$ s.

3. A ball of mass 1 kg is attached to an inextensible string. The ball is released from the position shown in figure. Find the impulse imparted by the string to the ball immediately after the string becomes taut. (Take $g = 10 \text{ m/s}^2$)

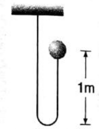


Fig. 8.32

8.5 Collision

Contrary to the meaning of the term 'collision' in our everyday life, in physics it does not necessarily mean one particle 'striking' against other. Indeed two particles may not even touch each other and may still be said to collide. All that is implied is that as the particles approach each other,

(i) an impulse (a large force for a relatively short time) acts on each colliding particles.

(ii) the total momentum of the particles remain conserved.

The collision is in fact a redistribution of total momentum of the particles. Thus, law of conservation of linear momentum is indispensible in dealing with the phenomenon of collision between particles. Consider a situation shown in figure.

Two blocks of masses m_1 and m_2 are moving with velocities v_1 and $v_2 (< v_1)$ along the same straight line in a smooth horizontal surface. A spring is attached to the block of mass m_2. Now, let us see what happens during the collision between two particles.

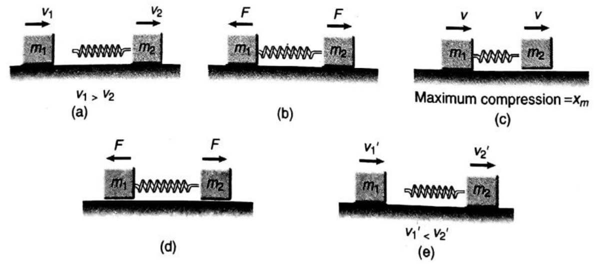


Fig. 8.33

Figure (a) Block of mass m_1 is behind m_2. Since, $v_1 > v_2$, the blocks will collide after some time.

Figure (b) The spring is compressed. The spring force $F\,(= kx)$ acts on the two blocks in the directions shown in figure. This force decreases the velocity of m_1 and increases the velocity of m_2.

Figure (c) The spring will compress till velocity of both the blocks become equal. So, at maximum compression (say x_m) velocities of both the blocks are equal (say v).

Figure (d) Spring force is still in the directions shown in figure, *i.e.*, velocity of block m_1 is further decreased and that of m_2 is increased. The spring now starts relaxing.

Figure (e) The two blocks are separated from one another. Velocity of block m_2 becomes more than the velocity of block m_1, *i.e.*, $v_2' > v_1'$.

Equations Which can be Used in the Above Situation

Assuming spring to be perfectly elastic following two equations can be applied in the above situation.

(i) In the absence of any external force on the system the linear momentum of the system will remain conserved before, during and after collision, *i.e.*,

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v = m_1 v_1' + m_2 v_2' \qquad \text{...(i)}$$

(ii) In the absence of any dissipative forces, the mechanical energy of the system will also remain conserved, *i.e.*,

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2}(m_1 + m_2) v^2 + \frac{1}{2} kx_m^2$$

$$= \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \qquad \text{...(ii)}$$

Note *In the above situation we have assumed spring to be perfectly elastic, i.e., it regains its original shape and size after the two blocks are separated. In actual practice there is no such spring between the two blocks. During collision both the blocks (or bodies) are a little bit deformed. This situation is similar to the compression of the spring. Due to deformation two equal and opposite forces act on both the blocks. These two forces redistribute their linear momentum in such a manner that both the blocks are separated from one another. The collision is said to be elastic if both the blocks regain their original shape and size completely after they are separated. On the other hand if the blocks do not return to their original form the collision is said to be inelastic. If the deformation is permanent and the blocks move together with same velocity after the collision, the collision is said to be perfectly inelastic.*

Sample Example 8.17 *Two blocks A and B of equal mass $m = 1.0$ kg are lying on a smooth horizontal surface as shown in figure. A spring of force constant $k = 200$ N/m is fixed at one end of block A. Block B collides with block A with velocity $v_0 = 2.0$ m/s. Find the maximum compression of the spring.*

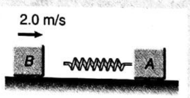


Fig. 8.34

Solution At maximum compression (x_m) velocity of both the blocks is same, say it is v. Applying conservation of linear momentum, we have

$$(m_A + m_B) v = m_B v_0$$

or $$(1.0 + 1.0) v = (1.0) v_0$$

or $$v = \frac{v_0}{2} = \frac{2.0}{2} = 1.0 \text{ m/s}$$

Using conservation of mechanical energy, we have

$$\frac{1}{2} m_B v_0^2 = \frac{1}{2}(m_A + m_B) v^2 + \frac{1}{2} kx_m^2$$

Substituting the values, we get

$$\frac{1}{2} \times (1) \times (2.0)^2 = \frac{1}{2} \times (1.0 + 1.0) \times (1.0)^2 + \frac{1}{2} \times (200) \times x_m^2$$

or $$2 = 1.0 + 100 x_m^2$$

or $$x_m = 0.1 \text{ m} = 10.0 \text{ cm}$$

Types of Collision

Collision between two bodies may be classified in two ways:

1. Elastic collision and inelastic collision.
2. Head on collision or oblique collision.

As discussed earlier also collision between two bodies is said to be **elastic** if both the bodies come to their original shape and size after the collision, *i.e.*, no fraction of mechanical energy remains stored as deformation potential energy in the bodies. Thus in addition to the linear momentum, kinetic energy also remains conserved before and after collision. On the other hand, in an **inelastic** collision, the colliding bodies do not return to their original shape and size completely after collision and some part of the mechanical energy of the system goes to the deformation potential energy. Thus, only linear momentum remains conserved in case of an inelastic collision.

Further, a collision is said to be **head on (or direct)** if the directions of the velocity of colliding objects are along the line of action of the impulses, acting at the instant of collision. If just before collision, at least one of the colliding objects was moving in a direction different from the line of action of the impulses, the collision is called **oblique or indirect.**

Problems related to oblique collision are usually not asked in any medical entrance test. Hence, only head on collision is discussed below.

(i) Head on Elastic Collision

Let the two balls of mass m_1 and m_2 collide each other elastically with velocities v_1 and v_2 in the directions shown in Fig. 8.35(a). Their velocities become v_1' and v_2' after the collision along the same line. Applying conservation of linear momentum, we get

(a) Before collision (b) After collision

Fig. 8.35

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \qquad \text{...(i)}$$

In an elastic collision kinetic energy before and after collision is also conserved. Hence,

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii) for v_1' and v_2', we get

$$v_1' = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2 \qquad \text{...(iii)}$$

and $$v_2' = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_2 + \left(\frac{2m_1}{m_1 + m_2}\right) v_1 \qquad \text{...(iv)}$$

Special Cases

1. If $m_1 = m_2$, then from Eqs. (iii) and (iv), we can see that

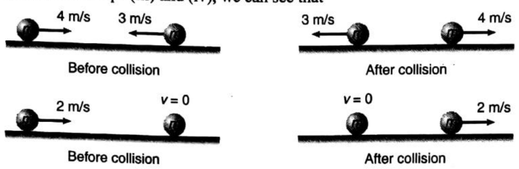


Fig. 8.36

$$v_1' = v_2 \text{ and } v_2' = v_1$$

i.e., when two particles of equal mass collide elastically and the collision is head on, they exchange their velocities., *e.g.,*

2. If $m_1 >> m_2$ and $v_1 = 0$.

Fig. 8.37

Then $$\frac{m_2}{m_1} \approx 0$$

with these two substitutions $\left(v_1 = 0 \text{ and } \frac{m_2}{m_1} = 0\right)$

we get the following two results:

$$v_1' \approx 0 \quad \text{and} \quad v_2' \approx -v_2$$

i.e., the particle of mass m_1 remains at rest while the particle of mass m_2 bounces back with same speed v_2.

3. If $\quad m_2 >> m_1 \quad$ and $\quad v_1 = 0$

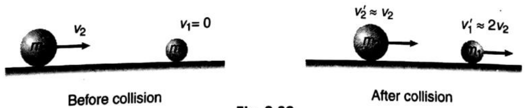


Fig. 8.38

with the substitution $\frac{m_1}{m_2} \approx 0$ and $v_1 = 0$, we get the results

$$v_1' \approx 2v_2 \quad \text{and} \quad v_2' \approx v_2$$

i.e., the mass m_1 moves with velocity $2v_2$ while the velocity of mass m_2 remains unchanged.

Note *It is important to note that Eqs. (iii) and (iv) and their three special cases can be used only in case of a head on elastic collision between two particles. I have found that many students apply these two equations even if the collision is inelastic and do not apply these relations where clearly a head on elastic collision is given in the problem.*

Sample Example 8.18 *Two particles of mass m and 2 m moving in opposite directions collide elastically with velocities v and 2v. Find their velocities after collision.*

Solution Here, $v_1 = -v, v_2 = 2v, m_1 = m$ and $m_2 = 2m$.

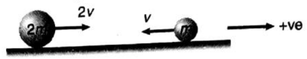


Fig. 8.39

Substituting these values in Eqs. (iii) and (iv), we get

$$v_1' = \left(\frac{m-2m}{m+2m}\right)(-v) + \left(\frac{4m}{m+2m}\right)(2v)$$

or $$v_1' = \frac{v}{3} + \frac{8v}{3} = 3v$$

and $$v_2' = \left(\frac{2m-m}{m+2m}\right)(2v) + \left(\frac{2m}{m+2m}\right)(-v)$$

or $$v_2' = \frac{2}{3}v - \frac{2}{3}v = 0$$

i.e., the second particle (of mass $2m$) comes to a rest while the first (of mass m) moves with velocity $3v$ in the direction shown in Fig. 8.40.

Fig. 8.40

Sample Example 8.19 *Two pendulum bobs of mass m and 2m collide elastically at the lowest point in their motion. If both the balls are released from a height H above the lowest point, to what heights do they rise for the first time after collision?*

Solution Given, $m_1 = m, m_2 = 2m, v_1 = -\sqrt{2gH}$ and $v_2 = \sqrt{2gH}$

Since, the collision is elastic. Using Eqs. (iii) and (iv) discussed in the theory the velocities after collision are

$$v_1' = \left(\frac{m-2m}{m+2m}\right)(-\sqrt{2gH}) + \left(\frac{4m}{m+2m}\right)\sqrt{2gH}$$

$$= \frac{\sqrt{2gH}}{3} + \frac{4\sqrt{2gH}}{3} = \frac{5}{3}\sqrt{2gH}$$

and $$v_2' = \left(\frac{2m-m}{m+2m}\right)(\sqrt{2gH}) + \left(\frac{2m}{m+2m}\right)(-\sqrt{2gH})$$

$$= \frac{\sqrt{2gH}}{3} - \frac{2\sqrt{2gH}}{3} = -\frac{\sqrt{2gH}}{3}$$

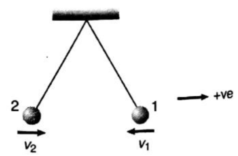


Fig. 8.41

i.e., the velocities of the balls after the collision are as shown in figure.

Therefore, the heights to which the balls rise after the collision are:

$$h_1 = \frac{(v_1')^2}{2g} \quad \text{(using } v^2 = u^2 - 2gh\text{)}$$

or $$h_1 = \frac{\left(\frac{5}{3}\sqrt{2gH}\right)^2}{2g} \quad \text{or} \quad h_1 = \frac{25}{9}H$$

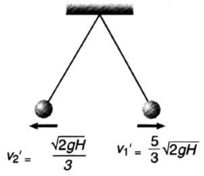


Fig. 8.42

and $$h_2 = \frac{(v_2')^2}{2g} \quad \text{or} \quad h_2 = \frac{\left(\frac{\sqrt{2gH}}{3}\right)^2}{2g}$$

or $$h_2 = \frac{H}{9}$$

Note *Since the collision is elastic, mechanical energy of both the balls will remain conserved, or*

$$E_i = E_f$$

$\Rightarrow$ $$(m + 2m)\,gH = mgh_1 + 2mgh_2$$

$\Rightarrow$ $$3mgH = (mg)\left(\frac{25}{9}H\right) + (2mg)\left(\frac{H}{9}\right)$$

$\Rightarrow$ $$3mgH = 3mgH$$

Introductory Exercise 8.6

1. Two blocks of mass 3 kg and 6 kg respectively are placed on a smooth horizontal surface. They are connected by a light spring of force constant $k = 200$ N/m. Initially the spring is unstretched. The indicated velocities are imparted to the blocks. Find the maximum extension of the spring.

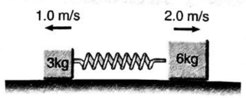


Fig. 8.43

2. A particle moving with kinetic energy K makes a head on elastic collision with an identical particle at rest. Find the maximum elastic potential energy of the system during collision.
3. Show that in a head on elastic collision between two particles, the transference of energy is maximum when their mass ratio is unity.
4. What is the fractional decrease in kinetic energy of a particle of mass m_1 when it makes a head on elastic collision with a particle of mass m_2 kept at rest?
5. A moving particle of mass m makes a head on elastic collision with a particle of mass $2m$ which is initially at rest. Find the fraction of kinetic energy lost by the colliding particle after collision.
6. Three balls A, B and C are placed on a smooth horizontal surface. Given that $m_A = m_C = 4m_B$. Ball B collides with ball C with an initial velocity v as shown in figure. Find the total number of collisions between the balls. All collisions are elastic.

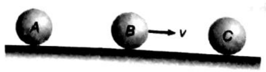


Fig. 8.44

7. In one dimensional elastic collision of equal masses, the velocities are interchanged. Can velocities in a one dimensional collision be interchanged if the masses are not equal.

8. Two balls shown in figure are identical. Ball A is moving towards right with a speed v and the second ball is at rest. Assume all collisions to be elastic. Show that the speeds of the balls remain unchanged after all the collisions have taken place.

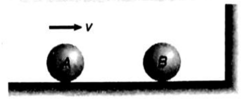


Fig. 8.45

(ii) Head on Inelastic Collision

As we have discussed earlier also, in an inelastic collision, the particles do not regain their shape and size completely after collision. Some fraction of mechanical energy is retained by the colliding particles in the form of deformation potential energy. Thus, the kinetic energy of the particles no longer remains conserved. However, in the absence of external forces, law of conservation of linear momentum still holds good.

Fig. 8.46

Suppose the velocities of two particles of mass m_1 and m_2 before collision be v_1 and v_2 in the directions shown in figure. Let v_1' and v_2' be their velocities after collision. The law of conservation of linear momentum gives

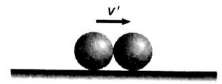


Fig. 8.47

$$\mathbf{m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'} \qquad \text{...(v)}$$

Collision is said to be **perfectly inelastic** if both the particles stick together after collision and move with same velocity, say v' as shown in figure. In this case, Eq. (v) can be written as

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2)v'$$

or $$\mathbf{v' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}} \qquad \text{...(vi)}$$

Newton's Law of Restitution

When two objects are in direct (head on) impact, the speed with which they separate after impact is usually less than or equal to their speed of approach before impact.

Experimental evidence suggests that the ratio of these relative speeds is constant for two given set of objects. This property formulated by Newton, is known as the law of restitution and can be written in the form

$$\frac{\textbf{separation speed}}{\textbf{approach speed}} = e \qquad \text{...(vii)}$$

The ratio e is called the coefficient of restitution and is constant for two particular objects.

In general $$0 \le e \le 1$$

$e = 0$, for completely inelastic collision, as both the objects stick together. So, their separation speed is zero or $e = 0$ from Eq. (vii).

$e = 1$, for an elastic collision, as we can show from Eq. (iii) and (iv), that

$$v_1' - v_2' = v_2 - v_1$$

or separation speed = approach speed

or $e = 1$

Fig. 8.48

Let us now find the velocities of two particles after collision if they collide directly and the coefficient of restitution between them is given as e.

Applying conservation of linear momentum

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \quad \text{...(viii)}$$

Further, separation speed = e(approach speed)

or $$v_1' - v_2' = e(v_2 - v_1) \quad \text{...(ix)}$$

Solving Eqs. (viii) and (ix), we get

$$v_1' = \left(\frac{m_1 - em_2}{m_1 + m_2}\right) v_1 + \left(\frac{m_2 + em_2}{m_1 + m_2}\right) v_2 \quad \text{...(x)}$$

and $$v_2' = \left(\frac{m_2 - em_1}{m_1 + m_2}\right) v_2 + \left(\frac{m_1 + em_1}{m_1 + m_2}\right) v_1 \quad \text{...(xi)}$$

Special Cases

1. If collision is elastic, *i.e.*, $e = 1$, then

$$v_1' = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_1 + \left(\frac{2m_2}{m_1 + m_2}\right) v_2$$

and $$v_2' = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_2 + \left(\frac{2m_1}{m_1 + m_2}\right) v_1$$

which are same as Eqs. (iii) and (iv).

2. If collision is perfectly inelastic, *i.e.*, $e = 0$, then

$$v_1' = v_2' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = v' \text{ (say)}$$

which is same as Eq. (vi).

3. If $m_1 = m_2$ and $v_1 = 0$, then

Fig. 8.49

$$v_1' = \left(\frac{1+e}{2}\right) v_2 \quad \text{and} \quad v_2' = \left(\frac{1-e}{2}\right) v_2$$

Note (i) *If mass of one body is very-very greater than that of the other, then after collision velocity of heavy body does not change appreciably. (Whether the collision is elastic or inelastic).*

(ii) *In the situation shown in figure if e is the coefficient of restitution between the ball and the ground, than after nth collision with the floor the speed of ball will remain $e^n v_0$ and it will go upto a height $e^{2n} h$ or,*

$$v_n = e^n v_0 = e^n \sqrt{2gh} \quad \text{and} \quad h_n = e^{2n} h$$

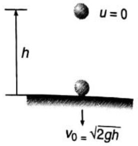


Fig. 8.50

EXERCISE *Derive the above two relations.*

Sample Example 8.20 *A ball of mass m moving at a speed v makes a head on collision with an identical ball at rest. The kinetic energy of the balls after the collision is 3/4th of the original. Find the coefficient of restitution.*

Fig. 8.51

Solution As we have seen in the above discussion, that under the given conditions:

$$v_1' = \left(\frac{1+e}{2}\right) v \quad \text{and} \quad v_2' = \left(\frac{1-e}{2}\right) v$$

Given that $$K_f = \frac{3}{4} K_i$$

or $$\frac{1}{2} m v_1'^2 + \frac{1}{2} m v_2'^2 = \frac{3}{4}\left(\frac{1}{2} m v^2\right)$$

Substituting the value, we get

$$\left(\frac{1+e}{2}\right)^2 + \left(\frac{1-e}{2}\right)^2 = \frac{3}{4}$$

or $$(1+e)^2 + (1-e)^2 = 3$$

or $$2 + 2e^2 = 3$$

or $$e^2 = \frac{1}{2}$$

or $$e = \frac{1}{\sqrt{2}}$$

Sample Example 8.21 *A ball is moving with velocity 2 m/s towards a heavy wall moving towards the ball with speed 1 m/s as shown in figure. Assuming collision to be elastic, find the velocity of ball immediately after the collision.*

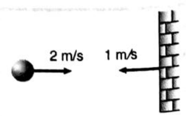


Fig. 8.52

Solution The speed of wall will not change after the collision. So, let v be the velocity of the ball after collision in the direction shown in figure. Since, collision is elastic ($e = 1$).

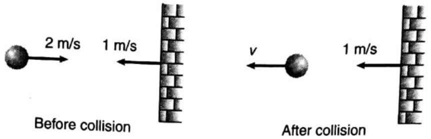


Fig. 8.53

$$\text{separation speed} = \text{approach speed}$$

or $$v - 1 = 2 + 1$$

or $$v = 4 \text{ m/s}$$

Sample Example 8.22 *After perfectly inelastic collision between two identical particles moving with same speed in different directions, the speed of the particles becomes half the initial speed. Find the angle between the two before collision.*

Solution Let θ be the desired angle. Linear momentum of the system will remain conserved. Hence

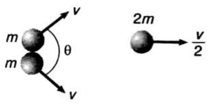


Fig. 8.54

$$P^2 = P_1^2 + P_2^2 + 2P_2P_2 \cos\theta$$

or $$\left\{2m\left(\frac{v}{2}\right)\right\}^2 = (mv)^2 + (mv)^2 + 2(mv)(mv)\cos\theta$$

or $$1 = 1 + 1 + 2\cos\theta$$

or $$\cos\theta = -\frac{1}{2}$$

$\therefore$ $$\theta = 120°$$

Introductory Exercise 8.7

1. Ball 1 collides directly with another identical ball 2 at rest. Velocity of second ball becomes two times that of 1 after collision. Find the coefficient of restitution between the two balls?
2. A particle of mass 0.1 kg moving at an initial speed v collides with another particle of same mass kept initially at rest. If the total energy becomes 0.2 J after the collision, what would be the minimum and maximum values of v ?
3. A particle of mass m moving with a speed v hits elastically another stationary particle of mass $2m$ on a smooth horizontal circular tube or radius r. Find the time when the next collision will take place?
4. In a one-dimensional collision between two identical particles A and B, B is stationary and A has momentum p before impact. During impact B gives an impulse J to A. Find the coefficient of restitution between A and B ?

Oblique Collision

During collision between two objects a pair of equal and opposite impulses act at the moment of impact. If just before impact at least one of the objects was moving in a direction different from the line of action of these impulses the collision is said to be oblique.

In the figure, two balls collide obliquely. During collision impulses act in the direction xx. Henceforth, we will call this direction as common normal direction and a direction perpendicular to it (*i.e.*, yy) as common tangent. Following four points are important regarding an oblique collision.

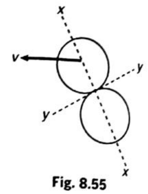


Fig. 8.55

1. A pair of equal and opposite impulses act along common normal direction. Hence, linear momentum of individual particles do change along common normal direction. If mass of the colliding particles remain constant during collision, then we can say that linear velocity of the individual particles change during collision in this direction.

2. No component of impulse act along common tangent direction. Hence, linear momentum or linear velocity of individual particles (if mass is constant) remain unchanged along this direction.

3. Net impulse on both the particles is zero during collision. Hence, net momentum of both the particles remain conserved before and after collision in any direction.

4. Definition of coefficient of restitution can be applied along common normal direction, *i.e.*, along common normal direction we can apply

$$\text{Relative speed of separation} = e\,(\text{relative speed of approach})$$

Here, e is the coefficient of restitution between the particles. Here, are few examples in support of the above theory.

Sample Example 8.23 *A ball of mass m hits a floor with a speed v_0 making an angle of incidence α with the normal. The coefficient of restitution is e. Find the speed of the reflected ball and the angle of reflection of the ball.*

Solution The component of velocity v_0 along common tangent direction $v_0 \sin\alpha$ will remain unchanged. Let v be the component along common normal direction after collision. Applying

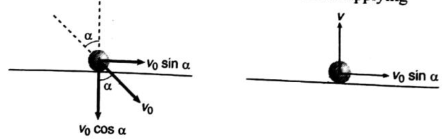


Fig. 8.56

$$\text{Relative speed of separation} = e\,(\text{relative speed of approach})$$

along common normal direction, we get

$$v = ev_0 \cos\alpha$$

Thus, after collision components of velocity v' are $v_0 \sin\alpha$ and $ev_0 \cos\alpha$

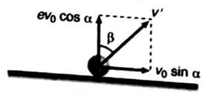


Fig. 8.57

$\therefore$ $$v' = \sqrt{(v_0 \sin\alpha)^2 + (ev_0 \cos\alpha)^2}$$

and $$\tan\beta = \frac{v_0 \sin\alpha}{e v_0 \cos\alpha}$$

or $$\tan\beta = \frac{\tan\alpha}{e}$$

Note *For elastic collision, $e = 1$*

$\therefore$ $v' = v_0$ *and* $\beta = \alpha$.

Introductory Exercise 8.8

1. A ball falls vertically on an inclined plane of inclination α with speed v_0 and makes a perfectly elastic collision. What is angle of velocity vector with horizontal after collision.
2. A ball falls on the ground from a height h. The coefficient of restitution between the ball and the ground is e. Find the speed and height upto which it rebounds after nth impact of the ball with the ground?
3. A sphere A of mass m, travelling with speed v, collides directly with a stationary sphere B. If A is brought to rest and B is given a speed V, find (a) the mass of B (b) the coefficient of restitution between A and B?
4. Two billiard balls of same size and mass are in contact on a billiard table. A third ball of same mass and size strikes them symmetrically and remains at rest after the impact. Find the coefficient of restitution between the balls?
5. A smooth sphere is moving on a horizontal surface with velocity vector $2\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ immediately before it hits a vertical wall. The wall is parallel to $\hat{\mathbf{j}}$ and the coefficient of restitution of the sphere and the wall is $e = \frac{1}{2}$. Find the velocity of the sphere after it hits the wall?
6. A ball is projected from the ground at some angle with horizontal. Coefficient of restitution between the ball and the ground is e. Let a, b and c be the ratio of times of flight, horizontal range and maximum height in two successive paths. Find a, b and c in terms of e?

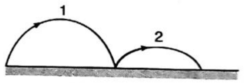


Fig. 8.58

Extra Points

- During collision if mass of one body is very much greater than the mass of the other body then the velocity of heavy body remains almost unchanged after collision, whether the collision is elastic or inelastic.
- Net force on a system is zero, it does not mean that centre of mass is at rest. It might be moving with constant velocity.
- Centre of mass of a rigid body is not necessarily the geometric centre of the rigid body.
- If two many particles are in air, then relative acceleration between any two is zero but acceleration of their centre of mass is g downwards.
- Coefficient of restitution is the mutual intrinsic properties of two bodies. Its value varies from 0 to 1.
- **Centre of mass frame of reference or C-frame of reference or zero momentum frame :** A frame of reference carried by the centre of mass of an isolated system of particles (*i.e.*, a system not subjected to any external forces) is called the centre of mass or C-frame of reference. In this frame of reference.
  (i) Position vector of centre of mass is zero.
  (ii) Velocity and hence momentum of centre of mass is also zero.
- In the situation discussed above we can also apply

$$\Sigma m_R x_R = \Sigma m_L x_L$$

Here, $\Sigma m_R x_R$ is the summation of product of x and m of the particles (or bodies) which are moving towards right and $\Sigma m_L x_L$ is the summation of product of x and m of the particles (or bodies) which are moving towards left. But remember the following three conditions while using the above equation.

(i) This equation can be applied when centre of mass does not move in x-direction.
(ii) In the above equation x is the displacement of particle relative to ground.
(iii) Apply the above equation while solving the objective problems only. Solve the subjective problems by the method discussed earlier.

Let us solve example 5.10 using the above method.

Here, x_L = displacement of plank towards left = x

m_L = mass of plank = 20 kg

x_R = displacement of man relative to ground towards right = $10 - x$

and m_R = mass of man = 60 kg

Applying $x_R m_R = x_L m_L$, we get

$$(10 - x)(60) = 20x$$

or $$x = 30 - 3x$$

or $$4x = 30$$

$$\therefore \quad x = \frac{30}{4} = 7.5 \text{ m}$$

- A liquid of density ρ is filled in a container as shown in figure. The liquid comes out from the container through a orifice of area 'a' at a depth 'h' below the free surface of the liquid with a velocity v. This exerts a thrust force in the container in the backward direction. This thrust force is given by

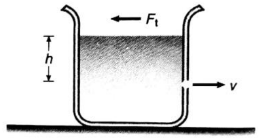


Fig. 8.59

$$F_t = v_r\left(-\frac{dm}{dt}\right)$$

Here, $v_r = v$ (in forward direction)

and $$\left(-\frac{dm}{dt}\right) = \rho a v$$

as $\left(\frac{dV}{dt}\right)$ = Volume of liquid flowing per second

$$= av$$

$$\therefore \quad \left(-\frac{dm}{dt}\right) = \rho\left(\frac{dV}{dt}\right) = \rho a v$$

$$\therefore \quad F_t = v(\rho a v)$$

or $F_t = \rho a v^2$ (in backward direction)

Further, we will see in the chapter of fluid mechanics that $v = \sqrt{2gh}$.

- Suppose, a chain of mass per unit length λ begins to fall through a hole in the ceiling as shown in Fig. 8.60(a) or the end of the chain piled on the platform is lifted vertically as in Fig. 8.60(b). In both the cases, due to

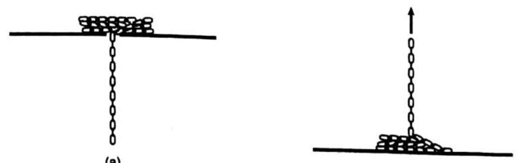


Fig. 8.60

increase of mass in the portion of the chain which is moving with a velocity v at certain moment of time a thrust force acts on this part of the chain which is given by

$$F_t = v_r\left(\frac{dm}{dt}\right)$$

Here, $\quad v_r = v \quad$ and $\quad \frac{dm}{dt} = \lambda v$

Here, v_r is upwards in case (a) and downwards in case (b). Thus,

$$F_t = \lambda v^2$$

The direction of F_t is upwards in case (a) and downwards in case (b).

- Suppose a ball is a projected with speed u at an angle θ with horizontal. It collides at some distance with a wall parallel to y-axis as shown in figure. Let v_x and v_y be the components of its velocity along x and y-directions at the time of impact with wall. Coefficient of restitution between the ball and the wall is e. Component of its velocity along y-direction (common tangent) v_y will remain unchanged while component of its velocity along x-direction (common normal) v_x will become ev_x is opposite direction.

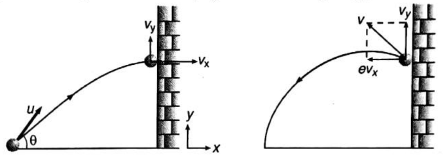


Fig. 8.61

Further, since v_y does not change due to collision, the time of flight (time taken by the ball to return to the same level) and maximum height attained by the ball will remain same as it would had been in the absence of collision with the wall. Thus,

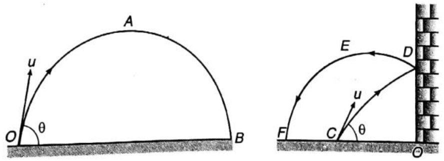


Fig. 8.62

$$t_{OAB} = t_{CD} + t_{DEF} = T = \frac{2u\sin\theta}{g}$$

and $\quad h_A = h_E = \frac{u^2\sin^2\theta}{2g}$

Further, $\quad CO + OF \le \text{Range or } OB$

It collision is elastic, then $\quad CO + OF = \text{Range} = \frac{u^2\sin 2\theta}{g}$

and if it is inelastic, $\quad CO + OF < \text{Range}$

Solved Examples

Level 1

Example 1 *The friction coefficient between the horizontal surface and each of the block shown in the figure is 0.2. The collision between the blocks is perfectly elastic. Find the separation between them when they come to rest.*
(Take $g = 10\,m/s^2$).

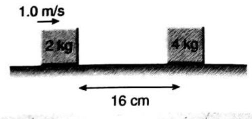


Fig. 8.63

Solution Velocity of first block before collision,

$$v_1^2 = 1^2 - 2(2)\times 0.16 = 1 - 0.64$$

$$v_1 = 0.6\text{ m/s}$$

By conservation of momentum, $2\times 0.6 = 2v_1' + 4v_2'$

also $v_2' - v_1' = v_1$ for elastic collision

It gives $\quad v_2' = 0.4\text{ m/s}$

$\quad v_1' = -0.2\text{ m/s}$

Now distance moved after collision

$$s_1 = \frac{(0.4)^2}{2\times 2} \quad\text{and}\quad s_2 = \frac{(0.2)^2}{2\times 2}$$

$\therefore \quad s = s_1 + s_2 = 0.05\text{ m} = 5\text{ cm}$

Example 2 *A pendulum bob of mass 10^{-2} kg is raised to a height 5×10^{-2} m and then released. At the bottom of its swing, it picks up a mass 10^{-3} kg. To what height will the combined mass rise ?*

Solution Velocity of pendulum bob in mean position

$$v_1 = \sqrt{2gh} = \sqrt{2\times 10\times 5\times 10^{-2}} = 1\text{ m/s}$$

When the bob picks up a mass 10^{-3} kg at the bottom, then by conservation of linear momentum the velocity of coalesced mass is given by

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v$$

$$10^{-2} + 10^{-3}\times 0 = (10^{-2} + 10^{-3})v$$

or $\quad v = \frac{10^{-2}}{1.1\times 10^{-2}} = \frac{10}{11}\text{ m/s}$

Now, $\quad h = \frac{v^2}{2g} = \frac{(10/11)^2}{2\times 10} = 4.1\times 10^{-2}\text{ m}$

Example 3 *Three identical balls, ball I, ball II and ball III are placed on a smooth floor on a straight line at the separation of 10 m between balls as shown in figure. Initially balls are stationary. Ball I is given velocity of 10 m/s towards ball II, collision between ball I and II is inelastic with coefficient of restitution 0.5 but collision between ball II and III is perfectly elastic. What is the time interval between two consecutive collisions between ball I and II ?*

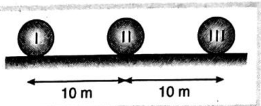


Fig. 8.64

Solution Let velocity of I ball and II ball after collision be v_1 and v_2

$$v_2 - v_1 = 0.5 \times 10 \quad \dots(i)$$

$$mv_2 + mv_1 = m \times 10 \quad \dots(ii)$$

$\Rightarrow$ $$v_2 + v_1 = 10$$

Solving Eqs. (i) and (ii)

$$v_1 = 2.5 \text{ m/s}, \qquad v_2 = 7.5 \text{ m/s}$$

Ball II after moving 10 m collides with ball III elastically and stops. But ball I moves towards ball II. Time taken between two consecutive collisions

$$\frac{10}{7.5} = \frac{10 - 10 \times \frac{2.5}{7.5}}{2.5} = 4 \text{ s}$$

Example 4 *A plank of mass 5 kg is placed on a frictionless horizontal plane. Further a block of mass 1 kg is placed over the plank. A massless spring of natural length 2 m is fixed to the plank by its one end. The other end of spring is compressed by the block by half of spring's natural length. They system is now released from the rest. What is the velocity of the plank when block leaves the plank ? (The stiffness constant of spring is 100 N/m)*

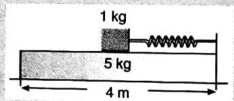


Fig. 8.65

Solution Let the velocity of the block and the plank, when the block leaves the spring be u and v respectively.

By conservation of energy $\frac{1}{2}kx^2 = \frac{1}{2}mu^2 + \frac{1}{2}Mv^2$

[M = mass of the plank, m = mass of the block]

$\Rightarrow$ $$100 = u^2 + 5v^2 \quad \dots(i)$$

By conservation of momentum

$$mu + Mv = 0$$

$\Rightarrow$ $$u = -5v \quad \dots(ii)$$

Solving Eqs (i) and (ii)

$$30v^2 = 100 \Rightarrow v = \sqrt{\frac{10}{3}} \text{ m/s}$$

From this moment until block falls, both plank and block keep their velocity constant.

Thus, when block falls, velocity of plank $= \sqrt{\frac{10}{3}}$ m/s.

Example 5 *Two identical blocks each of mass $M = 9$ kg are placed on a rough horizontal surface of frictional coefficient $\mu = 0.1$. The two blocks are joined by a light spring and block B is in contact with a vertical fixed wall as shown in figure. A bullet of mass $m = 1$ kg and $v_0 = 10$ m/s hits block A and gets embedded in it.*

Find the maximum compression of spring. (Spring constant = 240 N/m, $g = 10$ m/s^2)

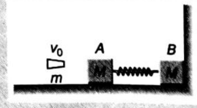


Fig. 8.66

Solution For the collision

$$1 \times 10 = 10 \times v \Rightarrow v = 1 \text{ m/s}$$

If x be the maximum compression

$$\frac{1}{2} \times 10 \times 1^2 = \mu(m + M)gx + \frac{1}{2}kx^2$$

$$5 = 10x + 120x^2 \Rightarrow x = \frac{1}{6} m$$

Example 6 *A particle of mass 2 kg moving with a velocity $5\hat{i}$ m/s collides head-on with another particle of mass 3 kg moving with a velocity $-2\hat{i}$ m/s. After the collision the first particle has speed of 1.6 m/s in negative x direction. Find :*

(a) velocity of the centre of mass after the collision,
(b) velocity of the second particle after the collision,
(c) coefficient of restitution.

Solution (a) $\vec{v}_c = \dfrac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2} = 0.8\hat{i}$ m/s

(b) $\vec{v}_1 = -1.6\hat{i}$ m/s

From COM, $$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$$

$\Rightarrow$ $$v_2 = 2.4\hat{i} \text{ m/s}$$

(c) $e = \dfrac{v_2 - v_1}{u_1 - u_2} = \dfrac{4}{7}$

Example 7 *Two blocks A and B of equal mass are released on two sides of a fixed wedge C as shown in figure. Find the acceleration of centre of mass of blocks A and B. Neglect friction.*

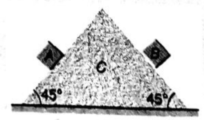


Fig. 8.67

Solution Acceleration of both the blocks will be $g \sin 45°$ or $\frac{g}{\sqrt{2}}$ at right angles to each other. Now,

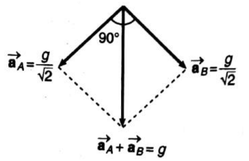


Fig. 8.68

$$\vec{a}_{COM} = \frac{m_A \vec{a}_A + m_B \vec{a}_B}{m_A + m_B}$$

Here,
$$m_A = m_B$$

$\therefore$
$$\vec{a}_{COM} = \frac{1}{2}(\vec{a}_A + \vec{a}_B) = \frac{1}{2} g \qquad \textbf{(downwards)}$$

Example 8 *A block of mass m is released from the top of a wedge of mass M as shown in figure. Find the displacement of wedge on the horizontal ground when the block reaches the bottom of the wedge. Neglect friction everywhere.*

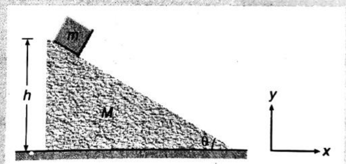


Fig. 8.69

Solution Here, the system is wedge + block. Net force on the system in horizontal direction (*x*-direction) is zero, therefore, the centre of mass of the system will not move in *x*-direction so we can apply,

$$x_R m_R = x_L m_L \qquad \text{...(i)}$$

Let x be the displacement of wedge. Then,

$$x_L = \text{displacement of wedge towards left} = x$$
$$m_L = \text{mass of wedge} = M$$
$$x_R = \text{displacement of block with respect to ground towards right} = h \cot \theta - x$$

and
$$m_R = \text{mass of block} = m$$

Substituting in Eq. (i), we get
$$m(h \cot \theta - x) = xM$$

$\therefore$
$$x = \frac{mh \cot \theta}{M + m}$$

Example 9 *A uniform chain of mass m and length l hangs on a thread and touches the surface of a table by its lower end. Find the force exerted by the table on the chain when half of its length has fallen on the table. The fallen part does not form heap.*

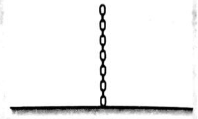


Fig. 8.70

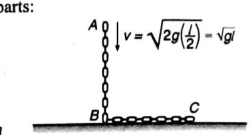


Fig. 8.71

Solution **Force exerted by the chain on the table.** It consists of two parts:

1. Weight of the portion *BC* of the chain lying on the table,
$$W = \frac{mg}{2} \quad \text{(downwards)}$$

2. Thrust force $F_t = \lambda v^2$

Here,
$$\lambda = \text{mass per unit length of chain} = \frac{m}{l}$$
$$v^2 = (\sqrt{gl})^2 = gl$$

$\therefore$
$$F_t = \left(\frac{m}{l}\right)(gl) = mg \qquad \text{(downwards)}$$

$\therefore$ Net force exerted by the chain on the table is
$$F = W + F_t = \frac{mg}{2} + mg = \frac{3}{2} mg \qquad \text{(downwards)}$$

So, from Newton's third law the force exerted by the table on the chain will be $\frac{3}{2} mg$ (vertically upwards).

Note *Here, the thrust force (F_t) applied by the chain on the table will be vertically downwards, as $F_t = v_r \left(\frac{dm}{dt}\right)$ and in this expression v_r is downwards plus $\frac{dm}{dt}$ is positive. So, F_t will be downwards.*

Example 10 *A ball of mass m makes an elastic collision with another identical ball at rest. Show that if the collision is oblique, the bodies go at right angles to each other after collision.*

Solution In head on elastic collision between two particles, they exchange their velocities. In this case, the component of ball 1 along common normal direction, $v \cos \theta$ becomes zero after collision, while that of 2 becomes $v \cos \theta$. While the components along common tangent direction of both the particles remain

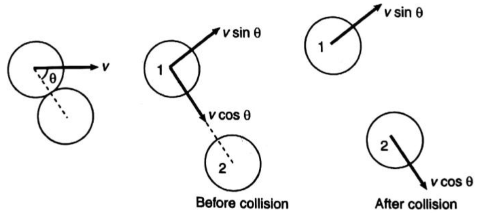


Fig. 8.72

unchanged. Thus, the components along common tangent and common normal direction of both the balls in tabular form are given a head :

Ball	Component along common tangent direction		Component along common normal direction	
	Before collision	After collision	Before collision	After collision
1	$v \sin\theta$	$v\sin\theta$	$v\cos\theta$	0
2	0	0	0	$v\cos\theta$

From the above table and figure, we see that both the balls move at right angles after collision with velocities $v\sin\theta$ and $v\cos\theta$.

Example 11 *A rocket, with an initial mass of 1000 kg, is launched vertically upwards from rest under gravity. The rocket burns fuel at the rate of 10 kg per second. The burnt matter is ejected vertically downwards with a speed of 2000 ms^{-1} relative to the rocket. If burning ceases after one minute, find the maximum velocity of the rocket. (Take g as constant at 10 ms^{-2})*

Solution Using the velocity equation as derived in theory.

$$v = u - gt + v_r \ln\left(\frac{m_0}{m}\right)$$

Here, $u = 0$, $t = 60$ s, $g = 10$ m/s^2, $v_r = 2000$ m/s, $m_0 = 1000$ kg

and $$m = 1000 - 10 \times 60 = 400 \text{ kg}$$

we get, $$v = 0 - 600 + 2000 \ln\left(\frac{1000}{400}\right)$$

or $$v = 2000 \ln 2.5 - 600$$

The maximum velocity of the rocket is 200 (10 ln 2.5 – 3) = 1232.6 ms^{-1}.

Level 2

Example 1 *A ball is projected from the ground with speed u at an angle α with horizontal. It collides with a wall at a distance a from the point of projection and returns to its original position. Find the coefficient of restitution between the ball and the wall.*

Solution As we have discussed in the theory, the horizontal component of the velocity of ball during the path *OAB* is $u\cos\alpha$ while in its return journey *BCO* it is $eu\cos\alpha$. The time of flight *T* also remains unchanged. Hence,

$$T = t_{OAB} + t_{BCO}$$

or $$\frac{2u\sin\alpha}{g} = \frac{a}{u\cos\alpha} + \frac{a}{eu\cos\alpha}$$

or $$\frac{a}{eu\cos\alpha} = \frac{2u\sin\alpha}{g} - \frac{a}{u\cos\alpha}$$

or $$\frac{a}{eu\cos\alpha} = \frac{2u^2\sin\alpha\cos\alpha - ag}{gu\cos\alpha}$$

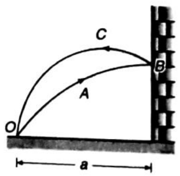


Fig. 8.73

∴ $$e = \frac{ag}{2u^2\sin\alpha\cos\alpha - ag}$$

or $$e = \frac{1}{\left(\frac{u^2\sin 2\alpha}{ag} - 1\right)}$$

Note *The concept which we have discussed in the oblique collision can also be applied when a ball collides with a wedge. This can be understood with the help of an Problem given below.*

Example 2 *A ball of mass $m = 1$ kg falling vertically with a velocity $v_0 = 2$ m/s strikes a wedge of mass $M = 2$ kg kept on a smooth, horizontal surface as shown in figure. The coefficient of restitution between the ball and the wedge is $e = \frac{1}{2}$. Find the velocity of the wedge and the ball immediately after collision.*

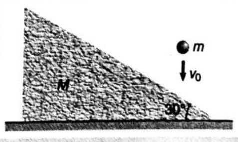


Fig. 8.74

Solution Given $M = 2$ kg and $m = 1$ kg

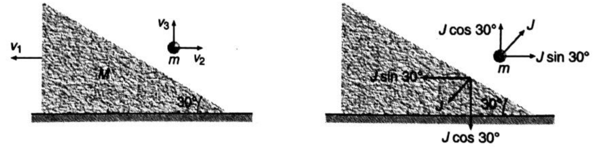


Fig. 8.75

Let, J be the impulse between ball and wedge during collision and v_1, v_2 and v_3 be the components of velocity of the wedge and the ball in horizontal and vertical directions respectively.

Applying impulse = change in momentum

we get $$J \sin 30° = Mv_1 = mv_2$$

or $$\frac{J}{2} = 2v_1 = v_2 \qquad \text{...(i)}$$

$$J\cos 30° = m(v_3 + v_0)$$

or $$\frac{\sqrt{3}}{2}J = (v_3 + 2) \qquad \text{...(ii)}$$

Applying, relative speed of separation = e (relative speed of approach) in common normal direction, we get

$$(v_1 + v_2)\sin 30° + v_3\cos 30° = \frac{1}{2}(v_0\cos 30°)$$

or $$v_1 + v_2 + \sqrt{3}v_3 = \sqrt{3} \qquad \text{...(iii)}$$

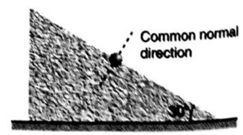


Fig. 8.76

Solving Eqs. (i), (ii) and (iii), we get

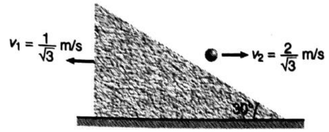


Fig. 8.77

$$v_1 = \frac{1}{\sqrt{3}} \text{ m/s}$$

$$v_2 = \frac{2}{\sqrt{3}} \text{ m/s} \quad \text{and} \quad v_3 = 0$$

Thus, velocities of wedge and ball are $v_1 = \frac{1}{\sqrt{3}}$ m/s and $v_2 = \frac{2}{\sqrt{3}}$ m/s in horizontal direction as shown in figure.

Note *If a particle (or a body) can move in a straight line and we want to find its velocity from the given conditions we take only one unknown v. If the particle can move in a plane we take two unknowns v_x and v_y (with $x \perp y$). Similarly if it can move in space we take three unknowns v_x, v_y and v_z.*
For instance in the above Problem, the wedge can move only in horizontal line, so we took only one unknown v_1. The ball can move in a plane, so we took two unknowns v_2 and v_3. Further, note that x and y axis should be perpendicular to each other. They may be along horizontal and vertical or along common tangent (along the plane in this case) and common normal (perpendicular to plane).

Example 3 *Two blocks of equal mass m are connected by an unstretched spring and the system is kept at rest on a frictionless horizontal surface. A constant force F is applied on one of the blocks pulling it away from the other as shown in figure. (a) Find the displacement of the centre of mass at time t. (b) If the extension of the spring is x_0 at time t, find the displacement of the two blocks at this instant.*

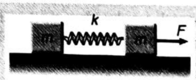


Fig. 8.78

Solution (a) The acceleration of the centre of mass is

$$a_{COM} = \frac{F}{2m}$$

The displacement of the centre of mass at time t will be

$$x = \frac{1}{2} a_{COM}\, t^2 = \frac{Ft^2}{4m}$$

(b) Suppose the displacement of the first block is x_1 and that of the second is x_2. Then,

$$x = \frac{mx_1 + mx_2}{2m}$$

or

$$\frac{Ft^2}{4m} = \frac{x_1 + x_2}{2}$$

or

$$x_1 + x_2 = \frac{Ft^2}{2m} \qquad \text{...(i)}$$

Further, the extension of the spring is $x_1 - x_2$. Therefore,

$$x_1 - x_2 = x_0 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$x_1 = \frac{1}{2}\left(\frac{Ft^2}{2m} + x_0\right)$$

and

$$x_2 = \frac{1}{2}\left(\frac{Ft^2}{2m} - x_0\right).$$

Example 4 *A block of mass m is connected to another block of mass M by a massless spring of spring constant k. The blocks are kept on a smooth horizontal plane. Initially, the blocks are at rest and the spring is unstretched when a constant force F starts acting on the block of mass M to pull it. Find the maximum extension of the spring.*

Fig. 8.79

Solution The centre of mass of the system (two blocks + spring) moves with an acceleration $a = \frac{F}{m+M}$. Let us solve the problem in a frame of reference fixed to the centre of mass of the system. As this frame is accelerated with respect to the ground, we have to apply a pseudo force ma towards left on the block of mass m and Ma towards left on the block of mass M. The net external force on m is

$$F_1 = ma = \frac{mF}{m+M} \qquad \text{(towards left)}$$

and the net external force on M is

$$F_2 = F - Ma = F - \frac{MF}{m+M} = \frac{mF}{m+M} \qquad \text{(towards right)}$$

As the centre of mass is at rest in this frame, the blocks move in opposite directions and come to instantaneous rest at some instant. The extension of the spring will be maximum at this instant. Suppose, the left block is displaced through a distance x_1 and the right block through a distance x_2 from the initial positions. The total work done by the external forces F_1 and F_2 in this period are

$$W = F_1 x_1 + F_2 x_2 = \frac{mF}{m+M}(x_1 + x_2)$$

This should be equal to the increase in the potential energy of the spring as there is no change in the kinetic energy. Thus,

$$\frac{mF}{m+M}(x_1 + x_2) = \frac{1}{2} k (x_1 + x_2)^2$$

or
$$x_1 + x_2 = \frac{2mF}{k(m+M)}$$

This is the maximum extension of the spring.

Example 5 *The coefficient of restitution between a snooker ball and the side cushion is $\frac{1}{3}$. If the ball hits the cushion and then rebounds at right angles to its original direction, show that the angles made with the side cushion by the direction of motion before and after impact are 60° and 30° respectively.*

Solution Let the original speed be u, in a direction making an angle θ with the side cushion.
Using the law of restitution

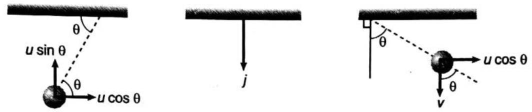


Fig. 8.80

$$v = \frac{1}{3}(u \sin \theta)$$

After impact,
$$\tan \theta = \frac{u \cos \theta}{v} = \frac{3 \cos \theta}{\sin \theta}$$

$\Rightarrow$ $$\tan^2 \theta = 3$$

$\Rightarrow$ $$\tan \theta = \sqrt{3}$$

$\Rightarrow$ $$\theta = 60°$$

Therefore, the directions of motion before and after impact are at 60° and 30° to the cushion.

Example 6 *Two blocks A and B of masses m and 2m respectively are placed on a smooth floor. They are connected by a spring. A third block C of mass m moves with a velocity v_0 along the line joining A and B and collides elastically with A, as shown in figure. At a certain instant of time t_0 after collision, it is found that the instantaneous velocities of A and B are the same. Further, at this instant the compression of the spring is found to be x_0. Determine (i) the common velocity of A and B at time t_0, and (ii) the spring constant.*

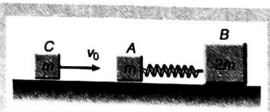


Fig. 8.81

Solution Initially, the blocks A and B are at rest and C is moving with velocity v_0 to the right.

As masses of C and A are same and the collision is elastic the body C transfers its whole momentum mv_0 to body A and as a result the body C stops and A starts moving with velocity v_0 to the right. At this instant the spring is uncompressed and the body B is still at rest.

The momentum of the system at this instant $= mv_0$

Now, the spring is compressed and the body B comes in motion. After time t_0, the compression of the spring is x_0 and common velocity of A and B is v (say).

As external force on the system is zero, the law of conservation of linear momentum gives
$$mv_0 = mv + (2m)v$$

or
$$v = \frac{v_0}{3}$$

The law of conservation of energy gives
$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}(2m)v^2 + \frac{1}{2}kx_0^2$$

or
$$\frac{1}{2}mv_0^2 = \frac{3}{2}mv^2 + \frac{1}{2}kx_0^2 \quad \text{...(ii)}$$

$$\frac{1}{2}mv_0^2 = \frac{3}{2}m\left(\frac{v_0}{3}\right)^2 + \frac{1}{2}kx_0^2$$

$\therefore$
$$\frac{1}{2}kx_0^2 = \frac{1}{2}mv_0^2 - \frac{1}{6}mv_0^2$$

or
$$\frac{1}{2}kx_0^2 = \frac{1}{3}mv_0^2$$

$\therefore$
$$k = \frac{2}{3}\frac{mv_0^2}{x_0^2}$$

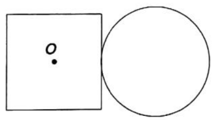
EXERCISES

AIEEE Corner

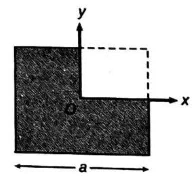
Subjective Questions (Level 1)

Centre of Mass

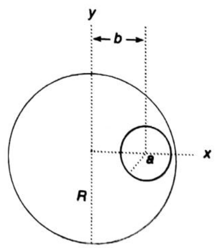
1. Four particles of mass 1 kg, 2 kg, 3 kg and 4 kg are placed at the four vertices A, B, C and D of a square of side 1 m. Find square of distance of their centre of mass from A.
2. A square lamina of side a and a circular lamina of diameter a are placed touching each other as shown in figure. Find distance of their centre of mass from point O, the centre of square.



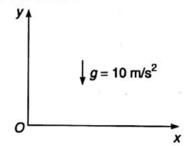
3. Consider a rectangular plate of dimensions $a \times b$. If this plate is considered to be made up of four rectangles of dimensions $\frac{a}{2} \times \frac{b}{2}$ and we now remove one out of four rectangles. Find the position where the centre of mass of the remaining system will be.



4. The uniform solid sphere shown in the figure has a spherical hole in it. Find the position of its centre of mass.



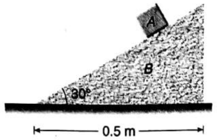
5. There are two masses m_1 and m_2 placed at a distance l apart. Let the centre of mass of this system is at a point named C. If m_1 is displaced by l_1 towards C and m_2 is displaced by l_2 away from C. Find the distance, from C where new centre of mass will be located.
6. The density of a thin rod of length l varies with the distance x from one end as $\rho = \rho_0 \frac{x^2}{l^2}$. Find the position of centre of mass of rod.
7. A block of mass 1 kg is at $x = 10$ m and moving towards negative x-axis with velocity 6 m/s. Another block of mass 2 kg is at $x = 12$ m and moving towards positive x-axis with velocity 4 m/s at the same instant. Find position of their centre of mass after 2 s.
8. Two particles of mass 1 kg and 2 kg respectively are initially 10 m apart. At time $t = 0$, they start moving towards each other with uniform speeds 2 m/s and 1 m/s respectively. Find the displacement of their centre of mass at $t = 1$ s.
9. x-y is the vertical plane as shown in figure. A particle of mass 1 kg is at (10 m, 20 m) at time $t = 0$. It is released from rest. Another particle of mass 2 kg is at (20 m, 40 m) at the same instant. It is projected with velocity $(10\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$ m/s. After 1 s. Find :



(a) acceleration, (b) velocity and (c) position of their centre of mass.

10. At one instant, the centre of mass of a system of two particles is located on the x-axis at $x = 3.0$ m and has a velocity of $(6.0\text{ m/s})\hat{\mathbf{j}}$. One of the particles is at the origin, the other particle has a mass of 0.10 kg and is at rest on the x-axis at $x = 12.0$ m.
(a) What is the mass of the particle at the origin ?
(b) Calculate the total momentum of this system.
(c) What is the velocity of the particle at the origin ?
11. A stone is dropped at $t = 0$. A second stone, with twice the mass of the first, is dropped from the same point at $t = 100$ ms.
(a) How far below the release point is the centre of mass of the two stones at $t = 300$ ms ? (Neither stone has yet reached the ground).
(b) How fast is the centre of mass of the two-stone system moving at that time ?
12. A system consists of two particles. At $t = 0$, one particle is at the origin; the other, which has a mass of 0.60 kg, is on the y-axis at $y = 80$ m. At $t = 0$ the centre of mass of the system is on the y-axis at $y = 24$ m and has a velocity given by $(6.0\text{ m/s}^3)t^2\hat{\mathbf{j}}$.
(a) Find the total mass of the system.
(b) Find the acceleration of the centre of mass at any time t.
(c) Find the net external force acting on the system at $t = 3.0$ s.
13. A straight rod of length L has one of its end at the origin and the other at $x = L$. If the mass per unit length of the rod is given by Ax where A is a constant, where is its centre of mass?

14. Block A has a mass of 5 kg and is placed on top of a smooth triangular block, B having a mass of 30 kg. If the system is released from rest, determine the distance, B moves when A reaches the bottom. Neglect the size of block A.



Conservation of Linear Momentum

15. Two blocks A and B of mass 1 kg and 2 kg are connected together by means of a spring and are resting on a horizontal frictionless table. The blocks are then pulled apart so as to stretch the spring and then released. Find the ratio of their :
(a) speed (b) momentum and (c) kinetic energy at any instant.

16. A trolley was moving horizontally on a smooth ground with velocity v with respect to the earth. Suddenly a man starts running from rear end of the trolley with a velocity $\frac{3}{2}v$ with respect to the trolley. After reaching the other end, the man turns back and continues running with a velocity $\frac{3}{2}v$ with respect to trolley in opposite direction. If the length of the trolley is L, find the displacement of the man with respect to earth when he reaches the starting point on the trolley. Mass of the trolley is equal to the mass of the man.

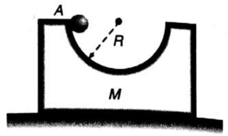
17. A man of mass m climbs to a rope ladder suspended below a balloon of mass M. The balloon is stationary with respect to the ground.
(a) If the man begins to climb the ladder at speed v (with respect to the ladder), in what direction and with what speed (with respect to the ground) will the balloon move ?
(b) What is the state of the motion after the man stops climbing ?

18. A 4.00 g bullet travelling horizontally with a velocity of magnitude 500 m/s is fired into a wooden block with a mass of 1.00 kg, initially at rest on a level surface. The bullet passes through the block and emerges with speed 100 m/s. The block slides a distance of 0.30 m along the surface from its initial position.
(a) What is the coefficient of kinetic friction between block and surface ?
(b) What is the decrease in kinetic energy of the bullet ?
(c) What is the kinetic energy of the block at the instant after the bullet has passed through it ?
Neglect friction during collision of bullet with the block.

19. A bullet of mass 0.25 kg is fired with velocity 302 m/s into a block of wood of mass $m_1 = 37.5$ kg. It gets embedded into it. The block m_1 is resting on a long block m_2 and the horizontal surface on which it is placed is smooth. The coefficient of friction between m_1 and m_2 is 0.5. Find the displacement of m_1 on m_2 and the common velocity of m_1 and m_2. Mass $m_2 = 1.25$ kg.

20. A wagon of mass M can move without friction along horizontal rails. A simple pendulum consisting of a sphere of mass m is suspended from the ceiling of the wagon by a string of length l. At the initial moment the wagon and the pendulum are at rest and the string is deflected through an angle α from the vertical. Find the velocity of the wagon when the pendulum passes through its mean position.

21. A block of mass M with a semicircular track of radius R rests on a horizontal frictionless surface shown in figure. A uniform cylinder of radius r and mass m is released from rest at the point A. The cylinder slips on the semicircular frictionless track. How far has the block moved when the cylinder reaches the bottom of the track ? How fast is the block moving when the cylinder reaches the bottom of the track ?

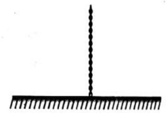


22. A ball of mass 50 g moving with a speed 2 m/s strikes a plane surface at an angle of incidence 45°. The ball is reflected by the plane at equal angle of reflection with the same speed. Calculate :
(a) The magnitude of the change in momentum of the ball.
(b) The change in the magnitude of the momentum of the wall.

Variable Mass

23. A rocket of mass 40 kg has 160 kg fuel. The exhaust velocity of the fuel is 2.0 km/s. The rate of consumption of fuel is 4 kg/s. Calculate the ultimate vertical speed gained by the rocket. ($g = 10\,\text{m/s}^2$)

24. A uniform rope of mass m per unit length, hangs vertically from a support so that the lower end just touches the tabletop shown in figure. If it is released, show that at the time a length y of the rope has fallen, the force on the table is equivalent to the weight of a length $3y$ of the rope.



25. Sand drops from a stationary hopper at the rate of 5 kg/s on to a conveyor belt moving with a constant speed of 2 m/s. What is the force required to keep the belt moving and what is the power delivered by the motor, moving the belt ?

26. Find the mass of the rocket as a function of time, if it moves with a constant acceleration a, in absence of external forces. The gas escapes with a constant velocity u relative to the rocket and its mass initially was m_0.

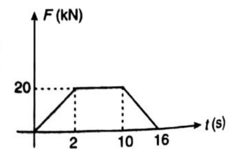
Impulse

27. A 5.0 g bullet moving at 100 m/s strikes a log. Assume that the bullet undergoes uniform deceleration and stops in 6.0 cm. Find (a) the time taken for the bullet to stop, (b) the impulse on the log and (c) the average force experienced by the log.

28. A 3.0 kg block slides on a frictionless horizontal surface, first moving to the left at 50 m/s. It collides with a spring as it moves left, compresses the spring and is brought to rest momentarily. The body continues to be accelerated to the right by the force of the compressed spring. Finally, the body moves to the right at 40 m/s. The block remains in contact with the spring for 0.020 s. What were the magnitude and direction of the impulse of the spring on the block? What was the spring's average force on the block?

29. Velocity of a particle of mass 2 kg varies with time t according to the equation $\vec{v} = (2t\,\hat{i} - 4\,\hat{j})$ m/s. Here, t is in seconds. Find the impulse imparted to the particle in the time interval from $t = 0$ to $t = 2$ s.

30. The net force *versus* time graph of a rocket is shown in figure. The mass of the rocket is 1200 kg. Calculate velocity of rocket, 16 seconds after starting from rest. Neglect gravity



Collision

31. After an elastic collision between two balls of equal masses, one is observed to have a speed of 3 m/s along the positive x-axis and the other has a speed of 2 m/s along the negative x-axis. What were the original velocities of the balls ?

32. A ball of mass 1 kg moving with 4 ms^{-1} along $+x$-axis collides elastically with an another ball of mass 2 kg moving with 6 m/s is opposite direction. Find their velocities after collision.

33. Ball 1 collides directly with an another identical ball 2 at rest. Velocity of second ball becomes two times that of 1 after collision. Find the coefficient of restitution between the two balls ?

34. A ball of mass m moving at a speed v makes a head on collision with an identical ball at rest. The kinetic energy of the balls after the collision is 3/4 of the original KE. Calculate the coefficient of restitution.

35. Block A has a mass 3 kg and is sliding on a rough horizontal surface with a velocity $v_A = 2$ m/s when it makes a direct collision with block B, which has a mass of 2 kg and is originally at rest. The collision is perfectly elastic. Determine the velocity of each block just after collision and the distance between the blocks when they stop sliding. The coefficient of kinetic friction between the blocks and the plane is $\mu_k = 0.3$. (Take $g = 10 \text{m/s}^2$)

Objective Questions (Level 1)

Single Correct Option

1. A ball is dropped from a height of 10 m. Ball is embedded in sand through 1 m and stops.
(a) only momentum remains conserved
(b) only kinetic energy remains conserved
(c) both momentum and kinetic energy are conserved
(d) neither kinetic energy nor momentum is conserved

2. If no external force acts on a system
(a) velocity of centre of mass remains constant
(b) position of centre of mass remains constant
(c) acceleration of centre of mass remains non-zero and constant
(d) All of the above

3. When two blocks connected by a spring move towards each other under mutual interaction
(a) their velocities are equal
(b) their accelerations are equal
(c) the force acting on them are equal and opposite
(d) All of the above

4. If two balls collide in air while moving vertically, then momentum of the system is conserved because
(a) gravity does not affect the momentum of the system
(b) force of gravity is very less compared to the impulsive force
(c) impulsive force is very less than the gravity
(d) gravity is not acting during collision

5. When a cannon shell explodes in mid air, then identify the incorrect statement
(a) the momentum of the system is conserved at the time of explosion
(b) the kinetic energy of the system always increases
(c) the trajectory of centre of mass remains unchanged
(d) None of the above

6. In an inelastic collision
(a) momentum of the system is always conserved.

(b) velocity of separation is less than the velocity of approach
(c) the coefficient of restitution can be zero.
(d) All of the above.

7. The momentum of a system is defined
(a) as the product of mass of the system and the velocity of centre of mass.
(b) as the vector sum of the momentum of individual particles.
(c) for bodies undergoing translational, rotational and oscillatory motion.
(d) All of the above

8. The momentum of a system with respect to centre of mass
(a) is zero only if the system is moving uniformly.
(b) is zero only if no external force acts on the system.
(c) is always zero
(d) can be zero in certain conditions

9. Three identical particles are located at the vertices of an equilateral triangle. Each particle moves along a meridian with equal speed towards the centroid and collides inelastically.
(a) all the three particles will bounce back along the meridians with lesser speed.
(b) all the three particles will become stationary
(c) all the particles will continue to move in their original directions but with lesser speed
(d) nothing can be said

10. The average resisting force that must act on a 5 kg mass to reduce its speed from 65 to 15 ms^{-1} in 2s is
(a) 12.5 N (b) 125 N (c) 1250 N (d) None of these

11. In a carbon monoxide molecule, the carbon and the oxygen atoms are separated by a distance 1.2×10^{-10} m. The distance of the centre of mass from the carbon atom is
(a) 0.48×10^{-10} m (b) 0.51×10^{-10} m (c) 0.74×10^{-10} m (d) 0.64×10^{-10} m

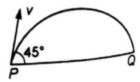
12. A bomb of mass 9 kg explodes into two pieces of masses 3 kg and 6 kg. The velocity of mass 3 kg is 16 ms^{-1}. The kinetic energy of mass 6 kg is
(a) 96 J (b) 384 J (c) 192 J (d) 768 J

13. A heavy ball moving with speed v collides with a tiny ball. The collision is elastic, then immediately after the impact, the second ball will move with a speed approximately equal to
(a) v (b) $2v$ (c) $\frac{v}{2}$ (d) $\frac{v}{3}$

14. A loaded 20, 000 kg coal wagon is moving on a level track at 6 ms^{-1}. Suddenly 5000 kg of coal is dropped out of the wagon. The final speed of the wagon is
(a) 6 ms^{-1} (b) 8 ms^{-1} (c) 4.8 ms^{-1} (d) 4.5 ms^{-1}

15. A machine gun shoots a 40 g bullet at a speed of 1200 ms^{-1}. The man operating the gun can bear a maximum force of 144 N. The maximum number of bullets shot per second is
(a) 3 (b) 5 (c) 6 (d) 9

16. A projectile of mass m is fired with a velocity v from point P at an angle 45°. Neglecting air resistance, the magnitude of the change in momentum leaving the point P and arriving at Q is



(a) $mv\sqrt{2}$ (b) $2mv$
(c) $\frac{mv}{2}$ (d) $\frac{mv}{\sqrt{2}}$

17. A ball after freely falling from a height of 4.9 m strikes a horizontal plane. If the coefficient of restitution is 3/4, the ball will strike second time with the plane after

(a) $\frac{1}{2}$ s (b) 1 s (c) $\frac{3}{2}$ s (d) $\frac{3}{4}$ s

18. The centre of mass of a non uniform rod of length L, whose mass per unit length varies as $\rho = \frac{k \cdot x^2}{L}$ where k is a constant and x is the distance of any point from one end is (from the same end)

(a) $\left(\frac{3}{4}\right)L$ (b) $\left(\frac{1}{4}\right)L$ (c) $\left(\frac{1}{6}\right)L$ (d) $\left(\frac{2}{3}\right)L$

19. A boat of length 10 m and mass 450 kg is floating without motion in still water. A man of mass 50 kg standing at one end of it walks to the other end of it and stops. The magnitude of the displacement of the boat in metres relative to ground is

(a) zero (b) 1 m (c) 2 m (d) 5 m

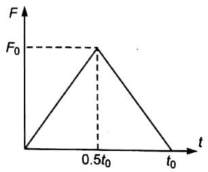
20. A man of mass M stands at one end of a stationary plank of length L, lying on a smooth surface. The man walks to the other end of the plank. If the mass of the plank is $\frac{M}{3}$, the distance that the man moves relative to the ground is

(a) $\frac{3L}{4}$ (b) $\frac{L}{4}$ (c) $\frac{4L}{5}$ (d) $\frac{L}{3}$

21. A ball of mass m moving at a speed v collides with another ball of mass $3m$ at rest. The lighter block comes to rest after collision. The coefficient of restitution is

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{4}$ (d) None of these

22. A particle of mass m moving with velocity u makes an elastic one-dimensional collision with a stationary particle of mass m. They come in contact for a very small time t_0. Their force of interaction increases from zero to F_0 linearly in time $0.5t_0$, and decreases linearly to zero in further time $0.5\,t_0$ as shown in figure. The magnitude of F_0 is



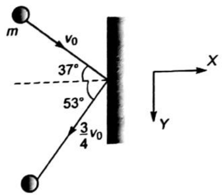
(a) $\frac{mu}{t_0}$ (b) $\frac{2mu}{t_0}$

(c) $\frac{mu}{2t_0}$ (d) None of these

23. Two identical blocks A and B of mass m joined together with a massless spring as shown in figure are placed on a smooth surface. If the block A moves with an acceleration a_0, then the acceleration of the block B is

(a) a_0 (b) $-a_0$ (c) $\frac{F}{m} - a_0$ (d) $\frac{F}{m}$

24. A ball of mass m moving with velocity v_0 collides a wall as shown in figure. After impact it rebounds with a velocity $\frac{3}{4}v_0$. The impulse acting on ball during impact is

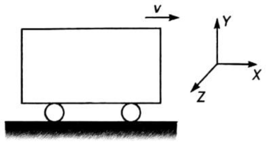


(a) $-\frac{m}{2}v_0\,\hat{\mathbf{j}}$ (b) $-\frac{3}{4}mv_0\,\hat{\mathbf{i}}$ (c) $\frac{-5}{4}mv_0\,\hat{\mathbf{i}}$ (d) None of these

25. A steel ball is dropped on a hard surface from a height of 1 m and rebounds to a height of 64 cm. The maximum height attained by the ball after n^{th} bounce is (in m)

(a) $(0.64)^{2n}$ (b) $(0.8)^{2n}$ (c) $(0.5)^{2n}$ (d) $(0.8)^n$

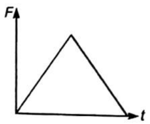
26. A car of mass 500 kg (including the mass of a block) is moving on a smooth road with velocity 1.0 ms^{-1} along positive x-axis. Now a block of mass 25 kg is thrown outside with absolute velocity of 20 ms^{-1} along positive z-axis. The new velocity of the car is (ms^{-1})



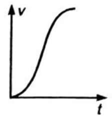
(a) $10\hat{\mathbf{i}} + 20\hat{\mathbf{k}}$ (b) $10\hat{\mathbf{i}} - 20\hat{\mathbf{k}}$

(c) $\frac{20}{19}\hat{\mathbf{i}} - \frac{20}{19}\hat{\mathbf{k}}$ (d) $10\hat{\mathbf{i}} - \frac{20}{19}\hat{\mathbf{k}}$

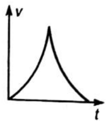
27. The net force acting on a particle moving along a straight line varies with time as shown in the diagram. Force is parallel to velocity. Which of the following graph is best representative of its speed with time? (Initial velocity of the particle is zero)



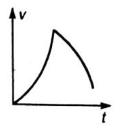
(a)



(b)

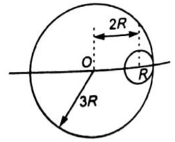


(c)



(d)

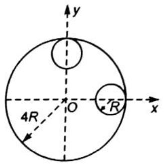
28. In the figure shown, find out centre of mass of a system of a uniform circular plate of radius $3R$ from O in which a hole of radius R is cut whose centre is at $2R$ distance from the centre of large circular plate



(a) $\frac{R}{2}$ (b) $\frac{R}{5}$

(c) $\frac{R}{4}$ (d) None of these

29. From the circular disc of radius $4R$ two small discs of radius R are cut off. The centre of mass of the new structure will be at



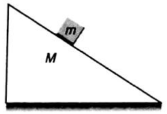
(a) $\hat{\mathbf{i}}\frac{R}{5}+\hat{\mathbf{j}}\frac{R}{5}$

(b) $-\hat{\mathbf{i}}\frac{R}{5}+\hat{\mathbf{j}}\frac{R}{5}$

(c) $-\hat{\mathbf{i}}\frac{R}{5}-\hat{\mathbf{j}}\frac{R}{5}$

(d) None of the above

30. A block of mass m rests on a stationary wedge of mass M. The wedge can slide freely on a smooth horizontal surface as shown in figure. If the block starts from rest



(a) the position of the centre of mass of the system will change
(b) the position of the centre of mass of the system will change along the vertical but not along the horizontal
(c) the total energy of the system will remain constant.
(d) All of the above

31. A bullet of mass m hits a target of mass M hanging by a string and gets embedded in it. If the block rises to a height h as a result of this collision, the velocity of the bullet before collision is

(a) $v=\sqrt{2gh}$ (b) $v=\sqrt{2gh}\left[1+\frac{m}{M}\right]$ (c) $v=\sqrt{2gh}\left[1+\frac{M}{m}\right]$ (d) $v=\sqrt{2gh}\left[1-\frac{m}{M}\right]$

32. A loaded spring gun of mass M fires a bullet of mass m with a velocity v at an angle of elevation θ. The gun is initially at rest on a horizontal smooth surface. After firing, the centre of mass of the gun and bullet system

(a) moves with velocity $\frac{v}{M}m$
(b) moves with velocity $\frac{vm}{M\cos\theta}$ in the horizontal direction
(c) remains at rest
(d) moves with velocity $\frac{v(M-m)}{M+m}$ in the horizontal direction

33. Two bodies with masses m_1 and m_2 $(m_1 > m_2)$ are joined by a string passing over fixed pulley. Assume masses of the pulley and thread negligible. Then the acceleration of the centre of mass of the system (m_1+m_2) is

(a) $\left(\frac{m_1-m_2}{m_1+m_2}\right)g$ (b) $\left(\frac{m_1-m_2}{m_1+m_2}\right)^2 g$ (c) $\frac{m_1 g}{m_1+m_2}$ (d) $\frac{m_2 g}{m_1+m_2}$

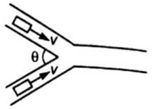
34. A rocket of mass m_0 has attained a speed equal to its exhaust speed and at that time the mass of the rocket is m. Then the ratio $\frac{m_0}{m}$ is (neglect gravity)

(a) 2.718 (b) 7.8 (c) 3.14 (d) 4

35. A jet of water hits a flat stationary plate perpendicular to its motion. The jet ejects 500g of water per second with a speed of $1\,\mathrm{ms}^{-1}$. Assuming that after striking, the water flows parallel to the plate, then the force exerted on the plate is

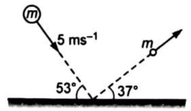
(a) 500 N (b) 1.0 N (c) 0.5 N (d) 1000 N

36. Two identical vehicles are moving with same velocity v towards an intersection as shown in figure. If the collision is completely inelastic, then



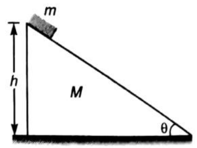
(a) the velocity of separation is zero
(b) the velocity of approach is $2v\sin\frac{\theta}{2}$
(c) the common velocity after collision is $v\cos\frac{\theta}{2}$
(d) All of the above

37. A ball of mass $m = 1$ kg strikes smooth horizontal floor as shown in figure. The impulse exerted on the floor is



(a) 6.25 Ns (b) 1.76 Ns (c) 7.8 Ns (d) 2.2 Ns

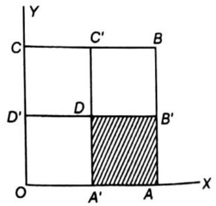
38. A small block of mass m is placed at rest on the top of a smooth wedge of mass M, which in turn is placed at rest on a smooth horizontal surface as shown in figure. If h be the height of wedge and θ is the inclination, then the distance moved by the wedge as the block reaches the foot of the wedge is



(a) $\frac{Mh\cot\theta}{M+m}$ (b) $\frac{mh\cot\theta}{M+m}$

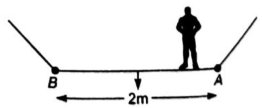
(c) $\frac{Mh\operatorname{cosec}\theta}{M+m}$ (d) $\frac{mh\operatorname{cosec}\theta}{M+m}$

39. A square of side 4 cm and uniform thickness is divided into four squares. The square portion $A'AB'D$ is removed and the removed portion is placed over the portion $DB'BC'$. The new position of centre of mass is



(a) (2 cm, 2 cm)
(b) (2 cm, 3 cm)
(c) (2 cm, 2.5 cm)
(d) (3 cm, 3 cm)

40. A boy having a mass of 40 kg stands at one end A of a boat of length 2 m at rest. The boy walks to the other end B of the boat and stops. What is the distance moved by the boat? Friction exists between the feet of the boy and the surface of the boat. But the friction between the boat and the water surface may be neglected. Mass of the boat is 15 kg.



(a) 0.49 m (b) 2.46 m (c) 1.46 m (d) 3.2 m

41. Three identical particle with velocities $v_0\,\hat{\mathbf{i}}$, $-3v_0\,\hat{\mathbf{j}}$ and $5v_0\,\hat{\mathbf{k}}$ collide successively with each other in such a way that they form a single particle. The velocity vector of resultant particle is

(a) $\frac{v_0}{3}(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (b) $\frac{v_0}{3}(\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (c) $\frac{v_0}{3}(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+\hat{\mathbf{k}})$ (d) $\frac{v_0}{3}(\hat{\mathbf{i}}-3\hat{\mathbf{j}}+5\hat{\mathbf{k}})$

42. A mortar fires a shell of mass M which explodes into two pieces of mass $\frac{M}{5}$ and $\frac{4M}{5}$ at the top of the trajectory. The smaller mass falls very close to the mortar. In the same time the bigger piece lands a distance D from the mortar. The shell would have fallen at a distance R from the mortar if there was no explosion. The value of D is (neglect air resistance)
(a) $\frac{3R}{2}$ (b) $\frac{4R}{3}$ (c) $\frac{5R}{4}$ (d) None of these

43. A moving particle of mass m makes a head on elastic collision with a particle of mass $2m$ which is initially at rest. The fraction of the kinetic energy lost by the colliding particle is
(a) $\frac{1}{9}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{8}{9}$

JEE Corner

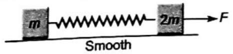
Assertion and Reason

Directions : *Choose the correct option.*

(a) *If both **Assertion** and **Reason** are true and the **Reason** is correct explanation of the **Assertion**.*
(b) *If both **Assertion** and **Reason** are true but **Reason** is not the correct explanation of **Assertion**.*
(c) *If **Assertion** is true, but the **Reason** is false.*
(d) *If **Assertion** is false but the **Reason** is true.*

1. **Assertion :** Centre of mass of a rigid body always lies inside the body.
Reason : Centre of mass and centre of gravity coincide if gravity is unifrom.

2. **Assertion :** A constant force F is applied on two blocks and one spring system as shown in figure. Velocity of centre of mass increases linearly with time.



Reason : Acceleration of centre of mass is constant.

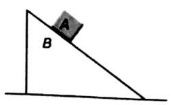
3. **Assertion :** To conserve linear momentum of a system, no force should act on the system.
Reason : If net force on a system is zero, its linear momentum should remain constant.

4. **Assertion :** A rocket moves forward by pushing the surrounding air backwards.
Reason : It derives the necessary thrust to move forward according to Newton's third law of motion.

5. **Assertion :** Internal forces cannot change linear momentum.
Reason : Internal forces can change the kinetic energy of a system.

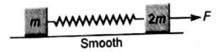
6. **Assertion :** In case of bullet fired from gun, the ratio of kinetic energy of gun and bullet is equal to ratio of mass of bullet and gun.
Reason : Kinetic energy $\propto \frac{1}{\text{mass}}$; if momentum is constant.

7. **Assertion :** All surfaces shown in figure are smooth. System is released from rest. Momentum of system in horizontal direction is constant but overall momentum is not constant.
Reason : A net vertically upward force is acting on the system.



8. **Assertion :** During head on collision between two bodies let Δp_1 is change in momentum of first body and Δp_2 the change in momentum of the other body, then $\Delta p_1 = \Delta p_2$.
Reason : Total momentum of the system should remain constant.

9. **Assertion :** In the system shown in figure spring is first stretched then left to oscillate. At some instant kinetic energy of mass m is K. At the same instant kinetic energy of mass $2m$ should be $\frac{K}{2}$.



Reason : Their linear momenta are equal and opposite and $K = \frac{p^2}{2m}$ or $K \propto \frac{1}{m}$.

10. **Assertion :** Energy can not be given to a system without giving it momentum.
Reason : If kinetic energy is given to a body it means it has acquired momentum.

11. **Assertion :** The centre mass of an electron and proton, when released moves faster towards proton.
Reason : Proton is heavier than electron.

12. **Assertion :** The relative velocity of the two particles in head-on elastic collision is unchanged both in magnitude and direction.
Reason : The relative velocity is unchanged in magnitude but gets reversed in direction.

13. **Assertion :** An object of mass m_1 and another of mass m_2 ($m_2 > m_1$) are released from certain distance. The objects move towards each other under the gravitational force between them. In this motion, centre of mass of their system will continuously move towards the heavier mass m_2.
Reason : In a system of a heavier and a lighter mass, centre of mass lies closer to the heavier mass.

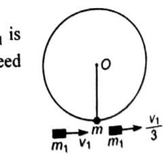
14. **Assertion :** A given force applied in turn to a number of different masses may cause the same rate of change in momentum in each but not the same acceleration to all.
Reason : $\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}$ and $\vec{\mathbf{a}} = \frac{\vec{\mathbf{F}}}{m}$

15. **Assertion :** In an elastic collision between two bodies, the relative speed of the bodies after collision is equal to the relative speed before the collision.
Reason : In an elastic collision, the linear momentum of the system is conserved.

Objective Questions (Level 2)

Single Correct Option

1. A pendulum consists of a wooden bob of mass m and length l. A bullet of mass m_1 is fired towards the pendulum with a speed v_1 and it emerges from the bob with speed $\frac{v_1}{3}$. The bob just completes motion along a vertical circle. Then v_1 is



(a) $\frac{m}{m_1}\sqrt{5gl}$ (b) $\frac{3m}{2m_1}\sqrt{5gl}$
(c) $\frac{2}{3}\left(\frac{m}{m_1}\right)\sqrt{5gl}$ (d) $\left(\frac{m_1}{m}\right)\sqrt{gl}$

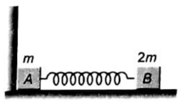
2. A bob of mass m attached with a string of length l tied to a point on ceiling is released from a position when its string is horizontal. At the bottom most point of its motion, an identical mass m gently stuck to it. Find the maximum angle from the vertical to which it rotates

(a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\left(\frac{3}{4}\right)$ (c) $\cos^{-1}\left(\frac{1}{4}\right)$ (d) $60°$

3. A train of mass M is moving on a circular track of radius R with constant speed v. The length of the train is half of the perimeter of the track. The linear momentum of the train will be

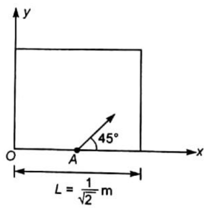
(a) zero (b) $\frac{2Mv}{\pi}$ (c) MvR (d) Mv

4. Two blocks A and B of mass m and $2m$ are connected together by a light spring of stiffness k. The system is lying on a smooth horizontal surface with the block A in contact with a fixed vertical wall as shown in the figure. The block B is pressed towards the wall by a distance x_0 and then released. There is no friction anywhere. If spring takes time Δt to acquire its natural length then average force on the block A by the wall is



(a) zero (b) $\frac{\sqrt{2mk}}{\Delta t}x_0$ (c) $\frac{\sqrt{mk}}{\Delta t}x_0$ (d) $\frac{\sqrt{3mk}}{\Delta t}x_0$

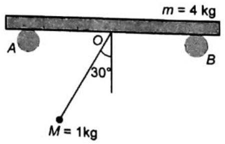
5. A striker is shot from a square carom board from a point A exactly at midpoint of one of the walls with a speed of 2 ms^{-1} at an angle of 45° with the x-axis as shown in the figure. The collisions of the striker with the walls of the fixed carom are perfectly elastic. The coefficient of kinetic friction between the striker and board is 0.2. The coordinate of the striker when it stops (taking point O to be the origin) is (in SI units)



(a) $\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}$ (b) $0, \frac{1}{2\sqrt{2}}$

(c) $\frac{1}{2\sqrt{2}}, 0$ (d) $\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}$

6. A ball of mass 1 kg is suspended by an inextensible string 1 m long attached to a point O of a smooth horizontal bar resting on fixed smooth supports A and B. The ball is released from rest from the position when the string makes an angle 30° with the vertical. The mass of the bar is 4 kg. The displacement of bar when ball reaches the other extreme position (in m) is



(a) 0.4 (b) 0.2

(c) 0.25 (d) 0.5

7. A ball falls vertically onto a floor with momentum p and then bounces repeatedly. If coefficient of restitution is e, then the total momentum imparted by the ball to the floor is

(a) $p(1+e)$ (b) $\frac{p}{1-e}$ (c) $p\left(\frac{1-e}{1+e}\right)$ (d) $p\left(\frac{1+e}{1-e}\right)$

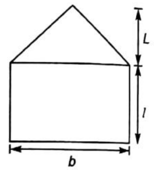
8. A bullet of mass m penetrates a thickness h of a fixed plate of mass M. If the plate was free to move, then the thickness penetrated will be

(a) $\frac{Mh}{M+m}$ (b) $\frac{2Mh}{M+m}$ (c) $\frac{mh}{2(M+m)}$ (d) $\frac{Mh}{2(M+m)}$

9. Two identical balls of equal masses A and B, are lying on a smooth surface as shown in the figure. Ball A hits the ball B (which is at rest) with a velocity $v = 16\,ms^{-1}$. What should be the minimum value of coefficient of restitution e between A and B so that B just reaches the highest point of inclined plane. ($g = 10\,ms^{-2}$)

(a) $\frac{2}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

10. The figure shows a metallic plate of uniform thickness and density. The value of l in terms of L so that the centre of mass of the system lies at the interface of the triangular and rectangular portion is



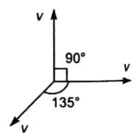
(a) $l = \frac{L}{3}$ (b) $l = \frac{L}{2}$

(c) $l = \frac{L}{\sqrt{3}}$ (d) $l = \sqrt{\frac{2}{3}}L$

11. Particle A makes a head on elastic collision with another stationary particle B. They fly apart in opposite directions with equal velocities. The mass ratio will be

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{2}{3}$

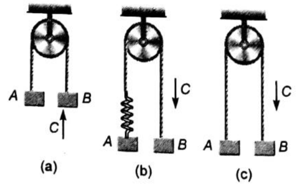
12. A particle of mass $4m$ which is at rest explodes into four equal fragments. All four fragments scattered in the same horizontal plane. Three fragments are found to move with velocity v as shown in the figure. The total energy released in the process is



(a) $mv^2(3-\sqrt{2})$ (b) $\frac{1}{2}mv^2(3-\sqrt{2})$

(c) $2mv^2$ (d) $\frac{1}{2}mv^2(1+\sqrt{2})$

13. In figures (a), (b) and (c) shown, the objects A, B and C are of same mass. String, spring and pulley are massless. C strikes B with velocity u in each case and sticks it. The ratio of velocity of B in case (a) to (b) to (c) is



(a) 1 : 1 : 1 (b) 3 : 3 : 2 (c) 3 : 2 : 2 (d) 1 : 2 : 3

14. A ladder of length L is slipping with its ends against a vertical wall and a horizontal floor. At a certain moment, the speed of the end in contact with the horizontal floor is v and the ladder makes an angle $\theta = 30°$ with horizontal. Then the speed of the ladder's centre of mass must be

(a) $\frac{\sqrt{3}}{2}v$ (b) $\frac{v}{2}$ (c) v (d) $2v$

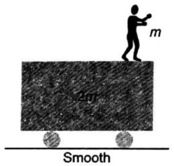
15. A body of mass 2 g, moving along the positive x-axis in gravity free space with velocity $20\,\text{cms}^{-1}$ explodes at $x = 1\,\text{m}$, $t = 0$ into two pieces of masses 2/3 g and 4/3 g. After 5s, the lighter piece is at the point (3m, 2m, – 4 m). Then the position of the heavier piece at this moment, in metres is

(a) (1.5, – 1, – 2) (b) (1.5, – 2, – 2) (c) (1.5, – 1, – 1) (d) None of these

16. A body of mass m is dropped from a height of h. Simultaneously another body of mass $2m$ is thrown up vertically with such a velocity v that they collide at height $\frac{h}{2}$. If the collision is perfectly inelastic, the velocity of combined mass at the time of collision with the ground will be

(a) $\sqrt{\frac{5gh}{4}}$ (b) $\sqrt{gh}$ (c) $\sqrt{\frac{gh}{4}}$ (d) None of these

17. A man is standing on a cart of mass double the mass of man. Initially cart is at rest. Now, man jumps horizontally with velocity u relative to cart. Then work done by man during the process of jumping will be



(a) $\frac{mu^2}{2}$

(b) $\frac{3mu^2}{4}$

(c) mu^2

(d) None of the above

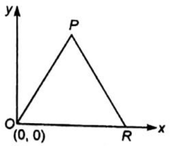
18. Two balls of equal mass are projected upwards simultaneously, one from the ground with initial velocity $50\,\text{ms}^{-1}$ and the other from a 40m tower with initial velocity of $30\,\text{ms}^{-1}$. The maximum height attained by their COM will be

(a) 80 m (b) 60 m (c) 100 m (d) 120 m

19. A particle of mass m and momentum $\vec{p}$ moves on a smooth horizontal table and collides directly and elastically with a similar particle (of mass m) having momentum $-2\vec{p}$. The loss (–) or gain (+) in the kinetic energy of the first particle in the collision is

(a) $+\frac{p^2}{2m}$ (b) $-\frac{p^2}{4m}$ (c) $+\frac{3p^2}{2m}$ (d) zero

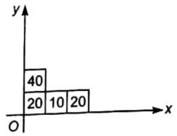
20. An equilateral triangular plate of mass $4m$ of side a is kept as shown. Consider two cases : (i) a point mass $4m$ is placed at the vertex P of the plate (ii) a point mass m is placed at the vertex R of the plate. In both cases the x coordinate of centre of mass remains the same. Then x coordinate of centre of mass of the plate is



(a) $\frac{a}{3}$ (b) $\frac{a}{6}$

(c) $\frac{6a}{7}$ (d) $\frac{2a}{3}$

21. Four cubes of side a each of mass 40 g, 20 g, 10 g and 20 g are arranged in XY plane as shown in the figure. The coordinates of COM of the combination with respect to point O is

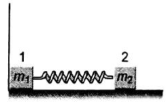


(a) $\frac{19a}{18}, \frac{17a}{18}$ (b) $\frac{17a}{18}, \frac{11a}{18}$ (c) $\frac{17a}{18}, \frac{13a}{18}$ (d) $\frac{13a}{18}, \frac{17a}{18}$

22. A particle of mass m_0, travelling at speed v_0, strikes a stationary particle of mass $2m_0$. As a result, the particle of mass m_0 is deflected through 45° and has a final speed of $\frac{v_0}{\sqrt{2}}$. Then the speed of the particle of mass $2m_0$ after this collision is

(a) $\frac{v_0}{2}$ (b) $\frac{v_0}{2\sqrt{2}}$ (c) $\sqrt{2}v_0$ (d) $\frac{v_0}{\sqrt{2}}$

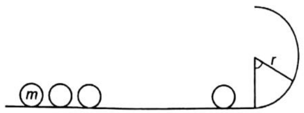
23. Two bars of masses m_1 and m_2, connected by a weightless spring of stiffness k, rest on a smooth horizontal plane. Bar 2 is shifted by a small distance x_0 to the left and released. The velocity of the centre of mass of the system when bar 1 breaks off the wall is



(a) $x_0\sqrt{\frac{km_2}{m_1 + m_2}}$ (b) $\frac{x_0}{m_1 + m_2}\sqrt{km_2}$

(c) $x_0 k\frac{m_1 + m_2}{m_2}$ (d) $x_0\frac{\sqrt{km_1}}{(m_1 + m_2)}$

24. n elastic balls are placed at rest on a smooth horizontal plane which is circular at the ends with radius r as shown in the figure. The masses of the balls are m, $\frac{m}{2}, \frac{m}{2^2}$............. $\frac{m}{2^{n-1}}$ respectively. What is the minimum velocity which should be imparted to the first ball of mass m such that n^{th} ball completes the vertical circle



(a) $\left(\frac{3}{4}\right)^{n-1}\sqrt{5gr}$ (b) $\left(\frac{4}{3}\right)^{n-1}\sqrt{5gr}$

(c) $\left(\frac{3}{2}\right)^{n-1}\sqrt{5gr}$ (d) $\left(\frac{2}{3}\right)^{n-1}\sqrt{5gr}$

Passage-1 (Q. No. 25 to 26)

A block of mass 2 kg is attached with a spring of spring constant $4000\,\text{Nm}^{-1}$ and the system is kept on smooth horizontal table. The other end of the spring is attached with a wall. Initially spring is stretched by 5 cm from its natural position and the block is at rest. Now suddenly an impulse of $4\,\text{kg-ms}^{-1}$ is given to the block towards the wall.

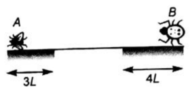
25. Find the velocity of the block when spring acquires its natural length

(a) $5\,\text{ms}^{-1}$ (b) $3\,\text{ms}^{-1}$ (c) $6\,\text{ms}^{-1}$ (d) None of these

26. Approximate distance travelled by the block when it comes to rest for a second time (not including the initial one) will be (Take $\sqrt{45} = 6.70$)
(a) 30 cm (b) 25 cm (c) 40 cm (d) 20 cm

Passage-2 (Q. No. 27 to 31)

A uniform bar of length $12L$ and mass $48\,m$ is supported horizontally on two smooth tables as shown in figure. A small moth (an insect) of mass $8m$ is sitting on end A of the rod and a spider (an insect) of mass $16m$ is sitting on the other end B. Both the insects moving towards each other along the rod with moth moving at speed $2v$ and the spider at half this speed (absolute). They meet at a point P on the rod and the spider eats the moth. After this the spider moves with a velocity $\frac{v}{2}$ relative to the rod towards the end A. The spider takes negligible time in eating on the other insect. Also, let $v = \frac{L}{T}$ where T is a constant having value 4 s.



27. Displacement of the rod by the time the insect meet the moth is
(a) $\frac{L}{2}$ (b) L (c) $\frac{3L}{4}$ (d) zero

28. The point P is at
(a) the centre of the rod (b) the edge of the table supporting the end B
(c) the edge of the table supporting end A (d) None of these

29. The speed of the bar after the spider eats up the moth and moves towards A is
(a) $\frac{v}{2}$ (b) v (c) $\frac{v}{6}$ (d) $2v$

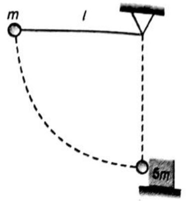
30. After starting from end B of the rod the spider reaches the end A at a time
(a) 40 s (b) 30 s (c) 80 s (d) 10 s

31. By what distance the centre of mass of the rod shifts during this time?
(a) $\frac{8L}{3}$ (b) $\frac{4L}{3}$ (c) L (d) $\frac{L}{3}$

More than One Correct Options

1. A particle of mass m, moving with velocity v collides a stationary particle of mass $2m$. As a result of collision, the particle of mass m deviates by 45° and has final speed of $\frac{v}{2}$. For this situation mark out the correct statement (s).
(a) The angle of divergence between particles after collision is $\frac{\pi}{2}$
(b) The angle of divergence between particles after collision is less than $\frac{\pi}{2}$
(c) Collision is elastic
(d) Collision is inelastic

2. A pendulum bob of mass m connected to the end of an ideal string of length l is released from rest from horizontal position as shown in the figure. At the lowest point the bob makes an elastic collision with a stationary block of mass $5m$, which is kept on a frictionless surface. Mark out the correct statement(s) for the instant just after the impact.

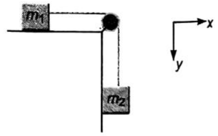


(a) tension in the string is $\frac{17}{9}\,mg$
(b) tension in the string is 3 mg..
(c) the velocity of the block is $\frac{\sqrt{2gl}}{3}$
(d) the maximum height attained by the pendulum bob after impact is (measured from the lowest position) $\frac{4l}{9}$

3. A particle of mass m strikes a horizontal smooth floor with a velocity u making an angle θ with the floor and rebound with velocity v making an angle ϕ with the floor. The coefficient of restitution between the particle and the floor is e. Then
(a) the impulse delivered by the floor to the body is $mv(1+e)\sin\theta$
(b) $\tan\phi = e\tan\theta$
(c) $v = u\sqrt{1-(1-e^2)\sin^2\theta}$
(d) the ratio of the final kinetic energy to the initial kinetic energy is $\cos^2\theta + e^2\sin^2\theta$

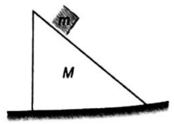
4. A particle of mass m moving with a velocity $(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}})\,\text{ms}^{-1}$ collides with another body of mass M and finally moves with velocity $(-2\hat{\mathbf{i}} + \hat{\mathbf{j}})\,\text{ms}^{-1}$. Then during the collision
(a) impulse received by m is $m(5\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (b) impulse received by m is $m(-5\hat{\mathbf{i}} - \hat{\mathbf{j}})$
(c) impulse received M is $m(-5\hat{\mathbf{i}} - \hat{\mathbf{j}})$ (d) impulse received by M is $m(5\hat{\mathbf{i}} + \hat{\mathbf{j}})$

5. All surfaces shown in figure are smooth. System is released from rest. X and Y components of acceleration of COM are

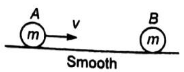


(a) $(a_{cm})_x = \frac{m_1 m_2 g}{m_1 + m_2}$
(b) $(a_{cm})_x = \frac{m_1 m_2 g}{(m_1 + m_2)^2}$
(c) $(a_{cm})_y = \left(\frac{m_2}{m_1 + m_2}\right)^2 g$
(d) $(a_{cm})_y = \left(\frac{m_2}{m_1 + m_2}\right) g$

6. A block of mass m is placed at rest on a smooth wedge of mass M placed at rest on a smooth horizontal surface. As the system is released
(a) the COM of the system remains stationary
(b) the COM of the system has an acceleration g vertically downward
(c) momentum of the system is conserved along the horizontal direction
(d) acceleration of COM is vertically downward ($a < g$)



7. In the figure shown, coefficient of restitution between A and B is $e = \frac{1}{2}$, then



(a) velocity of B after collision is $\frac{v}{2}$

(b) impulse between two during collision is $\frac{3}{4}mv$

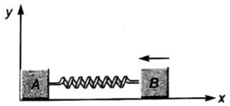
(c) loss of kinetic energy during the collision is $\frac{3}{16}mv^2$

(d) loss of kinetic energy during the collision is $\frac{1}{4}mv^2$

8. In case of rocket propulsion choose the correct options.
(a) Momentum of system always remains constant
(b) Newton's third law is applied
(c) If exhaust velocity and rate of burning of mass is kept constant, then acceleration of rocket will go on increasing
(d) Newton's second law can be applied

Match the Columns

1. Two identical blocks A and B are connected by a spring as shown in figure. Block A is not connected to the wall parallel to y-axis. B is compressed from the natural length of spring and then left. Neglect friction. Match the following two columns.



Column I	Column II
(a) Acceleration of centre of mass of two blocks	(p) remains constant
(b) Velocity of centre of mass of two blocks	(q) first increases then becomes constant
(c) x-coordinate of centre of mass of two blocks	(r) first decreases then becomes zero
(d) y-coordinate of centre of mass of two blocks	(s) continuously increases

2. One particle is projected from ground upwards with velocity 20 ms^{-1}. At the same time another indentical particle is dropped from a height of 180 m but not along the same vertical line. Assume that collision of first particle with ground is perfectly inelastic. Match the following two columns for centre of mass of the two particles ($g = 10\,\text{ms}^{-2}$)

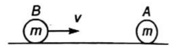
Column I	Column II
(a) Initial acceleration	(p) 5 SI units
(b) Initial velocity	(q) 10 SI units
(c) Acceleration at $t = 5$ s	(r) 20 SI units
(d) Velocity at $t = 5$ s	(s) 25 SI units

Note *Only magnitudes are given in column-II.*

3. Two identical blocks of mass 0.5 kg each are shown in figure. A massless elastic spring is connected with A. B is moving towards A with kinetic energy of 4 J. Match following two columns. Neglect friction.

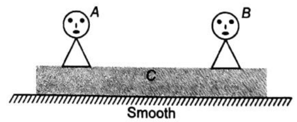
Column I	Column II
(a) Initial momentum of B	(p) zero
(b) Momentum of centre of mass of two blocks	(q) 1 kg-ms^{-1}
(c) Momentum of A at maximum compression	(r) 2 kg-ms^{-1}
(d) Momentum of B when spring is relaxed after compression	(s) 4 kg-ms^{-1}

4. Two identical balls A and B are kept on a smooth table as shown. B collides with A with speed v. For different conditions mentioned in column I, match with speed of A after collision given in column II.



Column I	Column II
(a) Elastic collision	(p) $\frac{3}{4}v$
(b) Perfectly inelastic collision	(q) $\frac{5}{8}v$
(c) Inelastic collision with $e = \frac{1}{2}$	(r) v
(d) Inelastic collision with $e = \frac{1}{4}$	(s) $\frac{v}{2}$

5. Two boys A and B of masses 30 kg and 60 kg are standing over a plank C of mass 30 kg as shown. Ground is smooth. Match the displacement of plank of column II with the conditions given in column I.

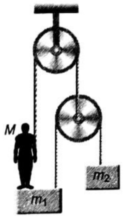


Column I	Column II
(a) A moves x towards right	(p) x towards right
(b) B moves x towards left	(q) $2x$ towards left
(c) A moves x towards right and B moves x towards left	(r) $\frac{x}{3}$ towards left
(d) A and B both move x towards right	(s) None

Note *All displacements mentioned in two columns are with respect to ground.*

6. A man of mass M is standing on a platform of mass m_1 and holding a string passing over a system of ideal pulleys. Another mass m_2 is hanging as shown.
($m_2 = 20\,\text{kg}$, $m_1 = 10\,\text{kg}$, $g = 10\,\text{ms}^{-2}$)

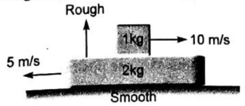
Column I	Column II
(a) Weight of man for equilibrium	(p) 100 N
(b) Force exerted by man on string to accelerate the COM of system upwards	(q) 150 N
(c) Force exerted by man on string to accelerate the COM of system downwards	(r) 500 N
(d) Normal reaction of platform on man in equilibrium	(s) 600 N



7. Two blocks of masses 3 kg and 6 kg are connected by an ideal spring and are placed on a frictionless horizontal surface. The 3 kg block is imparted a speed of 2 ms^{-1} towards left. (consider left as positive direction)

Column I	Column II
(a) When the velocity of 3 kg block is $\frac{2}{3}$ ms^{-1}	(p) velocity of centre of mass is $\frac{2}{3}$ ms^{-1}
(b) When the velocity of 6 kg block is $\frac{2}{3}$ ms^{-1}	(q) deformation of the spring is zero
(c) When the speed of 3 kg block is minimum	(r) deformation of the spring is maximum
(d) When the speed of 6 kg block is maximum	(s) both the blocks are at rest with respect to each other

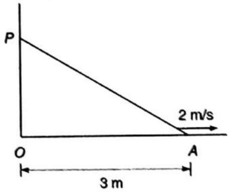
8. In a two block system shown in figure match the following



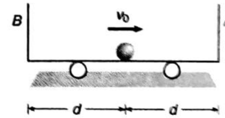
Column I	Column II
(a) Velocity of centre of mass	(p) Keep on changing all the time
(b) Momentum of centre of mass	(q) First decreases then become zero
(c) Momentum of 1 kg block	(r) Zero
(d) Kinetic energy of 2 kg block	(s) Constant

Subjective Questions (Level 2)

1. A ladder AP of length 5 m inclined to a vertical wall is slipping over a horizontal surface with velocity of 2 m/s, when A is at distance 3 m from ground. What is the velocity of COM at this moment?



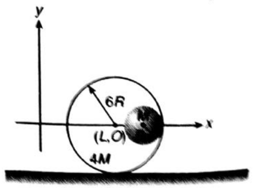
2. A ball of negligible size and mass m is given a velocity v_0 on the centre of the cart which has a mass M and is originally at rest. If the coefficient of restitution between the ball and walls A and B is e. Determine the velocity of the ball and the cart just after the ball strikes A. Also, determine the total time needed for the ball to strike A, rebound, then strike B, and rebound and then return to the centre of the cart. Neglect friction.



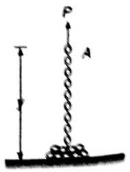
3. Two point masses m_1 and m_2 are connected by a spring of natural length l_0. The spring is compressed such that the two point masses touch each other and then they are fastened by a string. Then the system is moved with a velocity v_0 along positive x-axis. When the system reached the origin the string breaks ($t = 0$). The position of the point mass m_1 is given by $x_1 = v_0 t - A(1 - \cos \omega t)$ where A and ω are constants.

Find the position of the second block as a function of time. Also, find the relation between A and l_0.

4. A small sphere of radius R is held against the inner surface of larger sphere of radius $6R$ (as shown in figure). The masses of large and small spheres are $4M$ and M respectively. This arrangement is placed on a horizontal table. There is no friction between any surfaces of contact. The small sphere is now released. Find the coordinates of the centre of the large sphere, when the smaller sphere reaches the other extreme position.

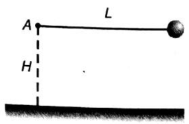


5. A chain of length l and mass m lies in a pile on the floor. If its end A is raised vertically at a constant speed v_0, express in terms of the length y of chain which is off the floor at any given instant.
(a) The magnitude of the force P applied to end A.
(b) Energy lost during the lifting of the chain.



6. A is a fixed point at a height H above a perfectly inelastic smooth horizontal plane. A light inextensible string of length $L(> H)$ has one end attached to A and other to a heavy particle. The particle is held at the

level of A with string just taut and released from rest. Find the height of the particle above the plane when it is next instantaneously at rest.



7. A particle of mass $2\,m$ is projected at an angle of 45° with horizontal with a velocity of $20\sqrt{2}$ m/s. After 1 s explosion takes place and the particle is broken into two equal pieces.

   As a result of explosion one part comes to rest. Find the maximum height attained by the other part. (Take $g = 10\,\text{m/s}^2$)

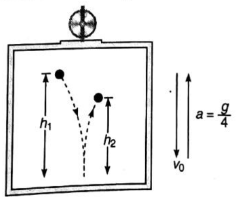
8. A sphere of mass m, impinges obliquely on a sphere, of mass M, which is at rest. Show that, if $m = eM$, the directions of motion of the spheres after impact are at right angles.

9. A gun of mass M (including the carriage) fires a shot of mass m. The gun along with the carriage is kept on a smooth horizontal surface. The muzzle speed of the bullet v_r is constant. Find :

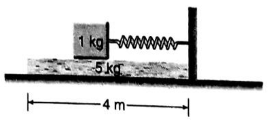
   (a) The elevation of the gun with horizontal at which maximum range of bullet with respect to the ground is obtained.

   (b) The maximum range of the bullet.

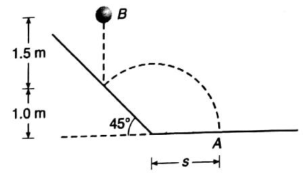
10. A ball is released from rest relative to the elevator at a distance h_1 above the floor. The speed of the elevator at the time of ball release is v_0. Determine the bounce height h_2 relative to elevator of the ball (a) if v_0 is constant and (b) if an upward elevator acceleration $a = g/4$ begins at the instant the ball is released. The coefficient of restitution for the impact is e.



11. A plank of mass 5 kg is placed on a frictionless horizontal plane. Further a block of mass 1 kg is placed over the plank. A massless spring of natural length 2 m is fixed to the plank by its one end. The other end of spring is compressed by the block by half of spring's natural length. The system is now released from the rest. What is the velocity of the plank when block leaves the plank? (The stiffness constant of spring is 100 N/m).

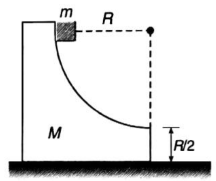


12. To test the manufactured properties of 10 N steel balls, each ball is released from rest as shown and strikes a 45° inclined surface. If the coefficient of restitution is to be $e = 0.8$, determine the distance s to where the ball must strike the horizontal plane at A. At what speed does the ball strike at A? ($g = 9.8\,\text{m/s}^2$)

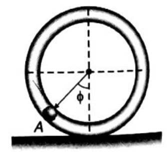


13. Two particles A and B of equal masses lie close together on a horizontal table and are connected by a light inextensible string of length l. A is projected vertically upwards with a velocity $\sqrt{10gl}$. Find the velocity with which it reaches the table again.

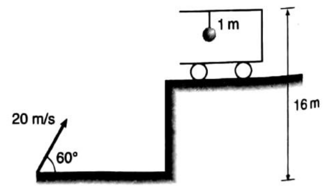
14. A small cube of mass m slides down a circular path of radius R cut into a large block of mass M, as shown in figure. M rests on a table, and both blocks move without friction. The blocks are initially at rest, and m starts from the top of the path. Find the horizontal distance from the bottom of block when cube hits the table.



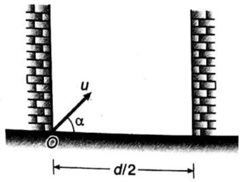
15. A thin hoop of mass M and radius r is placed on a horizontal plane. At the initial instant, the hoop is at rest. A small washer of mass m with zero initial velocity slides from the upper point of the hoop along a smooth groove in the inner surface of the hoop. Determine the velocity u of the centre of the hoop at the moment when the washer is at a certain point A of the hoop, whose radius vector forms an angle ϕ with the vertical (figure). The friction between the hoop and the plane should be neglected.



16. A shell of mass 1 kg is projected with velocity 20 m/s at an angle 60° with horizontal. It collides inelastically with a ball of mass 1 kg which is suspended through a thread of length 1 m. The other end of the thread is attached to the ceiling of a trolley of mass $\frac{4}{3}$ kg as shown in figure. Initially the trolley is stationary and it is free to move along horizontal rails without any friction. What is the maximum deflection of the thread with vertical? String does not slack. Take $g = 10\,\text{m/s}^2$.

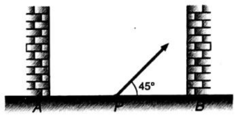


17. A small ball is projected at an angle α between two vertical walls such that in the absence of the wall its range would have been $5d$. Given that all the collisions are perfectly elastic, find



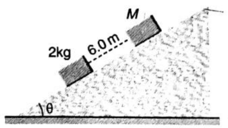
(a) maximum height attained by the ball.
(b) total number of collisions with the walls before the ball comes back to the ground, and
(c) point at which the ball finally falls. The walls are supposed to be very tall.

18. Two large rigid vertical walls A and B are parallel to each other and separated by 10 metres. A particle of mass 10 g is projected with an initial velocity of 20 m/s at 45° to the horizontal from point P on the ground, such that $AP = 5$ m. The plane of motion of the particle is vertical and perpendicular to the walls. Assuming that all the collisions are perfectly elastic, find the maximum height attained by the particle and the total number of collisions suffered by the particle with the walls before it hits ground. Take $g = 10\ \text{m/s}^2$.

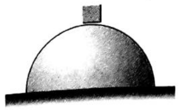


19. Two blocks of mass 2 kg and M are at rest on an inclined plane and are separated by a distance of 6.0 m as shown. The coefficient of friction between each block and the inclined plane is 0.25. The 2 kg block is given a velocity of 10.0 m/s up the inclined plane. It collides with M, comes back and has a velocity of 1.0 m/s when it reaches its initial position. The other block M after the collision moves 0.5 m up and comes to rest. Calculate the coefficient of restitution between the blocks and the mass of the block M.

[Take $\sin\theta \approx \tan\theta = 0.05$ and $g = 10\ \text{m/s}^2$]



20. A small block of mass m is placed on top of a smooth hemisphere also of mass m which is placed on a smooth horizontal surface. If the block begins to slide down due to a negligible small impulse, show that it will loose contact with the hemisphere when the radial line through vertical makes an angle θ given by the equation $\cos^3\theta - 6\cos\theta + 4 = 0$.



21. A ball is projected from a given point with velocity u at some angle with the horizontal and after hitting a vertical wall returns to the same point. Show that the distance of the point from the wall must be less than $\dfrac{eu^2}{(1+e)g}$, where e is the coefficient of restitution.

ANSWERS

Introductory Exercise 8.1

1. $\vec{r}_{COM} = \dfrac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$ while $\vec{r}_{CG} = \dfrac{\sum_{i=1}^{n} w_i \vec{r}_i}{\sum_{i=1}^{n} w_i}$ Here, w = weight (mg), $\vec{r}_{COM} = \vec{r}_{CG}$ when $\vec{g}$ = constant **2.** False

3. True **4.** True **5.** less than $\dfrac{r}{2}$ **6.** False **7.** $\dfrac{\sqrt{19}}{6}$ m **8.** $\dfrac{4}{3\pi}\left\{\dfrac{a^2+ab+b^2}{a+b}\right\}$ **9.** $\left(\dfrac{5a}{6}, \dfrac{5a}{6}\right)$

Introductory Exercise 8.2

1. zero **2.** $\left(\dfrac{\sqrt{3}-1}{4\sqrt{2}}\right)g$

Introductory Exercise 8.3

1. 9 ms^{-1}, 1.08 kJ **3.** $2.5\,\hat{i} + 15\hat{j} + 5\hat{k}$ **4.** 60 m **5.** 35 m **6.** 10 cm **7.** True

Introductory Exercise 8.4

1. 1.225 kgs^{-1} (i) 2.8 kms^{-1} (ii) 3.6 kms^{-1} **2.** $(m_0 - \mu t)\dfrac{d^2x}{dt^2} = \mu u - (m_0 - \mu t)\,g$ **3.** $u\ln\left(\dfrac{3}{2}\right) - g$,

Introductory Exercise 8.5

1. 10 m/s (downwards) **2.** $(8\,\hat{i})$ N-s **3.** $2\sqrt{10}$ N-s

Introductory Exercise 8.6

1. 30 cm **2.** $\dfrac{K}{2}$ **4.** $\dfrac{4m_1m_2}{(m_1+m_2)^2}$ **5.** $\dfrac{8}{9}$ **6.** Two **7.** No

Introductory Exercise 8.7

1. $\dfrac{1}{3}$ **2.** 2 ms^{-1}, $2\sqrt{2}$ ms^{-1} **3.** $\dfrac{2\pi r}{v}$ **4.** $\dfrac{2J}{P} - 1$

Introductory Exercise 8.8

1. $90° - 2\alpha$ **2.** $e^n\sqrt{2gh}, e^{2n}h$ **3.** (a) $\dfrac{mv}{V}$ (b) $\dfrac{V}{v}$ **4.** $\dfrac{2}{3}$ **5.** $-\hat{i} + 2\hat{j}$ **6.** $\dfrac{1}{e}, \dfrac{1}{e}, \dfrac{1}{e^2}$

AIEEE Corner

Subjective Questions (Level 1)

1. 0.74 m^2 **2.** $\left(\dfrac{\pi}{\pi+4}\right)a$ **3.** $\left(-\dfrac{a}{12}, -\dfrac{b}{12}\right)$ **4.** $x_{COM} = -\dfrac{a^3b}{R^3 - a^3}$ **5.** $\dfrac{m_1l_1 + m_2l_2}{m_1 + m_2}$

6. $x_{COM} = \dfrac{3l}{4}$ **7.** $x_{CM} = 12.67$ m **8.** zero **9.** (a) $(-10\hat{j})\,\text{ms}^{-2}$ (b) $\dfrac{10}{3}(2\hat{i} - \hat{j})\,\text{ms}^{-1}$ (c) $\left(\dfrac{70}{3}\hat{i} + 35\hat{j}\right)$ m

10. (a) 0.30 kg (b) $(2.4\ \text{kg-ms}^{-1})\hat{j}$ (c) $(8.0\ \text{ms}^{-1})\hat{j}$ **11.** (a) 28 cm (b) 2.3 ms^{-1}

12. (a) 2.0 kg (b) $(12.0\ \text{ms}^{-2})t\,\hat{j}$ (c) $(72.0\ \text{N})\hat{j}$ **13.** $\dfrac{2}{3}L$ **14.** 71.4 mm **15.** (a) 2 (b) 1 (c) 2

16. $\dfrac{4L}{3}$ **17.** (a) $\dfrac{mv}{M+m}$ (b) balloon will also stop moving **18.** (a) 0.435 (b) 480 J (c) 1.28 J

19. 0.013, 1.94 m/s **20.** $2m\sin\left(\dfrac{\alpha}{2}\right)\sqrt{\dfrac{gl}{M(M+m)}}$ **21.** $\dfrac{m(R-r)}{M+m}, m\sqrt{\dfrac{2g(R-r)}{M(M+m)}}$

22. (a) 0.14 kg-ms^{-1} (b) zero **23.** 2.82 kms^{-1} **25.** 10 N, 20 W **26.** $m = m_0 e^{-(a/u)t}$

27. (a) 1.2×10^{-3} s (b) 0.5 N-s (c) 417 N **28.** 270 N-s (to the right), 13.5 kN (to the right)

29. $(8\,\hat{i})$kg-ms^{-1} **30.** 200 ms^{-1} **31.** 2 ms^{-1} in negative x-axis, 3 m/s in positive x-axis.

32. $v_1 = \frac{28}{3}\ ms^{-1}$ (in negative x-direction) and $v_2 = \frac{2}{3}\ ms^{-1}$ (in positive x-direction) **33.** $e = \frac{1}{3}$ **34.** $e = \frac{1}{\sqrt{2}}$

35. 0.4 ms^{-1}, 2.4 ms^{-1}, 0.933 m

Objective Questions (Level 1)

1. (a)	**2.** (a)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (d)	**7.** (d)	**8.** (c)	**9.** (d)	**10.** (b)
11. (d)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (a)	**16.** (a)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (b)
21. (d)	**22.** (b)	**23.** (c)	**24.** (c)	**25.** (b)	**26.** (c)	**27.** (a)	**28.** (c)	**29.** (d)	**30.** (d)
31. (c)	**32.** (c)	**33.** (b)	**34.** (a)	**35.** (c)	**36.** (d)	**37.** (a)	**38.** (b)	**39.** (c)	**40.** (c)
41. (d)	**42.** (c)	**43.** (d)							

JEE Corner

Assertion and Reason

1. (d)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (b)	**6.** (a)	**7.** (c)	**8.** (d)	**9.** (a)	**10.** (d)
11. (d)	**12.** (d)	**13.** (d)	**14.** (a)	**15.** (d)					

Objective Questions (Level 2)

1. (b)	**2.** (b)	**3.** (b)	**4.** (b)	**5.** (a)	**6.** (b)	**7.** (d)	**8.** (a)	**9.** (b)	**10.** (c)
11. (a)	**12.** (a)	**13.** (b)	**14.** (c)	**15.** (d)	**16.** (d)	**17.** (d)	**18.** (c)	**19.** (c)	**20.** (b)
21. (a)	**22.** (b)	**23.** (b)	**24.** (a)	**25.** (b)	**26.** (b)	**27.** (d)	**28.** (b)	**29.** (c)	**30.** (c)
31. (a)									

More than One Correct Options

1. (b,d) **2.** (a,c,d) **3.** (all) **4.** (b,d) **5.** (b,c) **6.** (c,d) **7.** (b,c) **8.** (b,c,d)

Match the Columns

1. (a) → (r) (b) → (q) (c) → (s) (d) → (p)

2. (a) → (q) (b) → (q) (c) → (p) (d) → (s)

3. (a) → (r) (b) → (q) (c) → (q) (d) → (p)

4. (a) → (r) (b) → (s) (c) → (p) (d) → (q)

5. (a) → (r) (b) → (p) (c) → (p) (d) → (s)

6. (a) → (r) (b) → (r,s) (c) → (p,q) (d) → (p)

7. (a) → (p,r,s) (b) → (p,r,s) (c) → (p) (d) → (p,q)

8. (a) → (r,s) (b) → (r,s) (c) → (q) (d) → (q)

Subjective Questions (Level 2)

1. 1.25 ms^{-1} **2.** $v_{ball} = \left(\frac{eM - m}{M + m}\right) v_0$ (leftwards), $v_{cart} = \left(\frac{e + 1}{M + m}\right) mv_0$ (rightwards); $t = \frac{d}{v_0}\left(1 + \frac{2}{e} + \frac{1}{e^2}\right)$

3. $x_2 = v_0 t + \frac{m_1}{m_2} A(1 - \cos\omega t)$, $l_0 = \left(\frac{m_1}{m_2} + 1\right) A$ **4.** $(L + 2R, 0)$ **5.** (a) $\frac{m}{l}(gy + v_0^2)$ (b) $\frac{myv_0^2}{2l}$

6. $\frac{H^5}{L^4}$ **7.** 35 m **9.** (a) 45° (b) $\left(\frac{M}{M + m}\right)\frac{v_r^2}{g}$ **10.** (a) $e^2 h_1$ (b) $e^2 h_1$ **11.** $\sqrt{\frac{10}{3}}\ ms^{-1}$

12. 0.93 m, 6.6 ms^{-1} **13.** $2\sqrt{gl}$ **14.** $R\sqrt{\frac{2(M + m)}{M}}$ **15.** $v = m\cos\phi\sqrt{\frac{2gr(1 + \cos\phi)}{(M + m)(M + m\sin^2\phi)}}$

16. 60° **17.** (a) $\frac{u^2 \sin^2\alpha}{2g}$ (b) nine (c) point O **18.** 10 m, Four **19.** 0.84, 15.12 kg

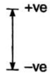
Hints & Solutions

(Subjective Questions : Level 2)

Chapter 3 Motion in One Dimension

Level 2

1. $$a_{av} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{\Delta t} = \frac{\sqrt{2g\,h_f} + \sqrt{2g\,h_i}}{\Delta t}$$
$$= \frac{\sqrt{2 \times 9.8 \times 2} + \sqrt{2 \times 9.8 \times 4}}{12 \times 10^{-3}} = 1.26 \times 10^3 \text{ m/s}^2$$ **Ans.**



Note : v_f is upwards (+ve) and v_i is downwards (– ve).

2. $v\,dv = a ds$

$\therefore$ $$\int_0^v v dv = \int_0^{12\text{ m}} a\, ds$$

$\therefore$ $$\frac{v^2}{2} = \text{area under } a\text{-}s \text{ graph from } s = 0 \text{ to } s = 12 \text{ m.}$$
$$= 2 + 12 + 6 + 4$$
$$= 24 \text{ m}^2/\text{s}^2$$

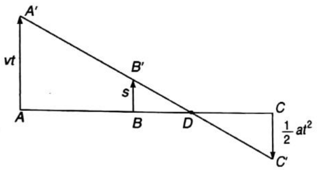
or $$v = \sqrt{48} \text{ m/s} = 4\sqrt{4} \text{ m/s}$$ **Ans.**

3. Let $AB = BC = d$

$BD = x$

and $BB' = s$ = displacement of point B.

From similar triangles we can write,



$$\frac{vt}{d+x} = \frac{s}{x} = \frac{\frac{1}{2}at^2}{d-x}$$

From first two equations we have,

$$1 + \frac{d}{x} = \frac{vt}{s}$$

or $$\frac{d}{x} = \frac{vt}{s} - 1 \quad \text{...(i)}$$

From last two equations we have,

$$\frac{d}{x} - 1 = \frac{\frac{1}{2}at^2}{s}$$

or $$\frac{d}{x} = \frac{\frac{1}{2}at^2}{s} + 1 \quad \text{...(ii)}$$

Equating (i) and (ii) we have,

$$\frac{vt}{s} - 1 = \frac{\frac{1}{2}at^2}{s} + 1$$

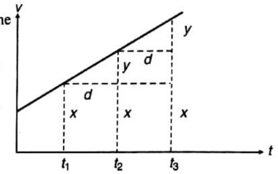
or $$\frac{vt - \frac{1}{2}at^2}{s} = 2$$

$\therefore$ $$s = \left(\frac{v}{2}\right)t - \frac{1}{2}\left(\frac{a}{2}\right)t^2$$

Comparing with $s = ut + \frac{1}{2}at^2$ we have,

Initial velocity of B is $+\frac{v}{2}$ and acceleration $-\frac{a}{2}$.

4. Let us draw v-t graph of the given situation, area of which will give the displacement and slope the acceleration.



$$s_2 - s_1 = x\,d + \frac{1}{2}yd \quad \text{...(i)}$$
$$s_3 - s_2 = xd + yd + \frac{1}{2}yd \quad \text{...(ii)}$$

Subtracting Eq. (i) from Eq. (ii), we have

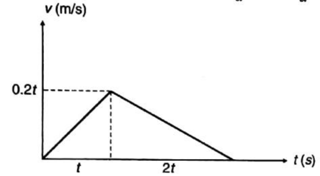
$$s_3 + s_1 - 2s_2 = yd$$

or $$s_3 + s_1 - 2\sqrt{s_1 s_3} = yd \qquad (s_2 = \sqrt{s_1 s_3})$$

Dividing by d^2 both sides we have,

$$\frac{(\sqrt{s_1} - \sqrt{s_3})^2}{d^2} = \frac{y}{d} = \text{slope of } v\text{-}t \text{ graph} = a.$$ **Hence proved.**

5.



Area of $v - t$ graph = displacement

$\therefore$ $$\frac{1}{2}(3t)(0.2\,t) = 14$$ **Ans.**

Solving this equation we get $3t = 20.5$ sec.

Note *Maximum speed 0.2t is less than 2.5 m/s.*

6. Let t_0 be the breaking time and a the magnitude of deceleration.

80.5 km/h = 22.36 m/s, 48.3 km/h = 13.42 m/s.

In the first case,
$$56.7 = (22.36 \times t_0) + \frac{(22.36)^2}{2a} \quad \text{...(i)}$$

and
$$24.4 = (13.42\,t_0) + \frac{(13.42)^2}{2a} \quad \text{...(ii)}$$

Solving these two equation we get $t_0 = 0.74$ s and $a = 6.2$ m/s^2 **Ans.**

7. Absolute velocity of ball = 30 m/s

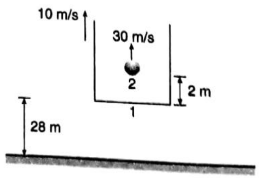
(a) Maximum height of ball from ground $= 28 + 2 + \frac{(30)^2}{2 \times 9.8} = 76$ m

(b) Ball will return to the elevator floor when,

$$s_1 = s_2 + 2$$

or $$10t = (30t - 4.9\,t^2) + 2$$

Solving we get $t = 4.2$ s **Ans.**



8. (a) Average velocity $= \dfrac{\text{displacement}}{\text{time}} = \dfrac{\int_0^5 v\,dt}{5} = \dfrac{\int_0^5 (3t - t^2)\,dt}{5}$

$= -0.833$ m/s

(b) Velocity of particle = 0 at $t = 3$ s

i.e., at 3 s, particle changes its direction of motion.

Average speed $= \dfrac{\text{total distance}}{\text{total time}} = \dfrac{(\text{distance from 0 to 3 s}) + (\text{distance from 3 s to 5 s})}{\text{time}}$

$$d_{0\text{-}3} = \int_0^3 (3t - t^2)\,dt = 4.5 \text{ m}$$

$$d_{3\text{-}5} = \int_3^5 (t^2 - 3t)\,dt = 8.67 \text{ m}$$

$\therefore$ Average speed $= \dfrac{4.5 + 8.67}{5} = 2.63$ m/s **Ans.**

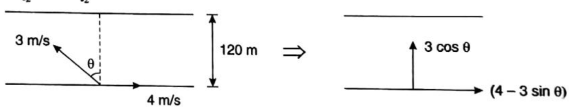
9. (a) $a = 2t - 2$ (from the graph)

Now $\int_0^v dv = \int_0^t a\,dt = \int_0^t (2t - 2)\,dt$

$\therefore$ $v = t^2 - 2t$

(b) $s = \int_2^4 v\,dt = \int_2^4 (t^2 - 2t)\,dt = 6.67$ m

10. (a)



Time to cross the river $t_1 = \dfrac{120}{3\cos\theta} = \dfrac{40}{\cos\theta} = 40 \sec\theta$

Drift along the river $x = (4 - 3\sin\theta)\left(\dfrac{40}{\cos\theta}\right)$

$= (160\sec\theta - 120\tan\theta)$

To reach directly opposite, this drift will be covered by walking speed. Time taken in this,

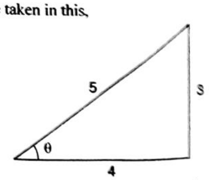
$$t_2 = \frac{160\sec\theta - 120\tan\theta}{1} = 160\sec\theta - 120\tan\theta$$

$\therefore$ Total time taken $t = t_1 + t_2 = (200\sec\theta - 120\tan\theta)$

For t to be minimum, $\dfrac{dt}{d\theta} = 0$

or $200\sec\theta\tan\theta - 120\sec^2\theta = 0$

or $\theta = \sin^{-1}(3/5)$



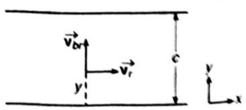
(b) $t_{min} = 200\sec\theta - 120\tan\theta$ (where $\sin\theta = \frac{3}{5}$)

$= 200 \times \dfrac{5}{4} - 120 \times \dfrac{3}{4}$

$= 250 - 90 = 160$ s $= 2$ min 40 s **Ans.**

11. Given that $|\vec{v}_{br}| = v_y = \dfrac{dy}{dt} = u$ ...(i)

$|\vec{v}_r| = v_x = \dfrac{dx}{dt} = \left(\dfrac{2v_0}{c}\right) y$ ...(ii)



From Eqs. (i) and (ii) we have, $\dfrac{dy}{dx} = \dfrac{uc}{2v_0 y}$

or $\int_0^y y\,dy = \dfrac{uc}{2v_0}\int_0^x dx$ or $y^2 = \dfrac{ucx}{v_0}$ **Ans.**

At $y = \dfrac{c}{2}, x = \dfrac{cv_0}{4u}$ or $x_{net} = 2x = \dfrac{cv_0}{2u}$ **Ans.**

12. $a = v\dfrac{dv}{ds} = v$ (slope of *v-s* graph)

At $s = 50$ m : $v = 20$ m/s and $\dfrac{dv}{ds} = \dfrac{40}{100} = 0.4$ per sec

$\therefore$ $a = 20 \times 0.4 = 8$ m/s^2

At $s = 150$ m : $v = (40 + 5) = 45$ m/s and $\dfrac{dv}{ds} = \dfrac{10}{100} = 0.1$ per sec

$\therefore$ $a = 45 \times 0.1 = 4.5$ m/s^2

***a-s* graph :**

From $s = 0$ to $s = 100$ m : $v = 0.4\,s$ and $\dfrac{dv}{ds} = 0.4$

$\therefore$ $a = v.\dfrac{dv}{ds} = 0.16\,s$

i.e., *a-s* graph is a straight line passing though origin of slope 0.16 per (sec)^2.

At $s = 100$ m, $a = 0.16 \times 100 = 16$ m/s^2

From $s = 100$ m to $s = 200$ m :

$$v = 0.1\,s + 30$$

$$\frac{dv}{ds} = 0.1$$

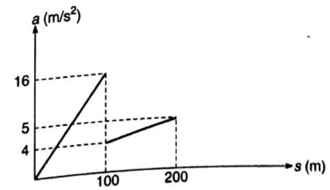
$\therefore$ $a = v\dfrac{dv}{ds} = (0.1\,s + 30)(0.1) = (0.01\,s + 3)$

i.e., *a-s* graph is straight line of slope 0.01 (sec)$^{-2}$ and intercept 3 m/s^2.

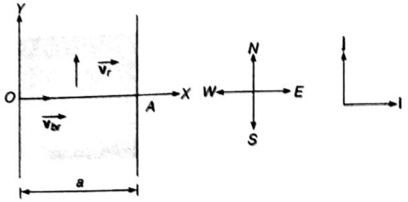
At $s = 100$ m, $a = 4$ m/s^2

and at $s = 200$ m, $a = 5$ m/s^2

Corresponding *a-s* graph is as shown in figure



13. (a) Let $\vec{v}_{br}$ be the velocity of boatman relative to river, $\vec{v}_r$ the velocity of river and $\vec{v}_b$ is the absolute velocity of boatman. Then



$$\vec{v}_b = \vec{v}_{br} + \vec{v}_r$$

Given, $\quad |\vec{v}_{br}| = v$ and $|\vec{v}_r| = u$

Now $\quad u = v_y = \dfrac{dy}{dt} = x(a - x)\dfrac{v}{a^2}$ ...(i)

and $\quad v = v_x = \dfrac{dx}{dt} = v$ ...(ii)

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{dy}{dx} = \frac{x(a-x)}{a^2} \quad \text{or} \quad dy = \frac{x(a-x)}{a^2}\,dx$$

or $\quad \displaystyle\int_0^y dy = \int_0^x \frac{x(a-x)}{a^2}\,dx$

or $\quad y = \dfrac{x^2}{2a} - \dfrac{x^3}{3a^2}$ ...(iii)

This is the desired equation of trajectory.

(b) Time taken to cross the river is

$$t = \frac{a}{v_x} = \frac{a}{v}$$

(c) When the boatman reaches the opposite side, $x = a$ or $v_y = 0$ [from Eq. (i)]
Hence, resultant velocity of boatman is v along positive x-axis or due east.

(d) From Eq. (iii)

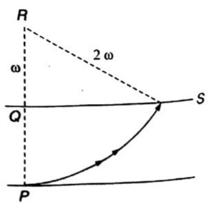
$$y = \frac{a^2}{2a} - \frac{a^3}{3a^2} = \frac{a}{6}$$

At $x = a$ (at opposite bank).
Hence, displacement of boatman will be

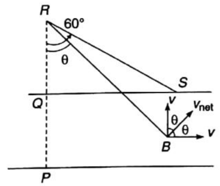
$$\vec{s} = x\hat{i} + y\hat{j} \quad \text{or} \quad \vec{s} = a\hat{i} + \frac{a}{6}\hat{j}$$

14. (a) Since the resultant velocity is always perpendicular to the line joining boat and R, the boat is moving in a circle of redius 2ω and centre at R.

(b) Drifting $= QS = \sqrt{4\omega^2 - \omega^2} = \sqrt{3}\omega$.



(c) Suppose at any arbitrary time, the boat is at point B.



$$V_{net} = 2v\cos\theta$$

$$\frac{d\theta}{dt} = \frac{V_{net}}{2\omega} = \frac{v\cos\theta}{\omega}$$

or $\quad \dfrac{\omega}{v}\sec\theta\,d\theta = dt$

$\therefore \quad \displaystyle\int_0^t dt = \frac{\omega}{v}\int_0^{60°}\sec\theta\,d\theta$

$\therefore \quad t = \dfrac{\omega}{v}[\ln(\sec\theta + \tan\theta)]_0^{60°}$

or $\quad t = \dfrac{1.317\omega}{v}$ **Ans.**

15. For $0 < s \le 60$ m

$$v = \frac{12}{60}s + 3 = 3 + \frac{s}{5}$$

$$\frac{dv}{dt} = \left(\frac{1}{5}\right)\cdot\frac{ds}{dt} = \frac{1}{5}(v) = \frac{1}{5}\left(3 + \frac{s}{5}\right) = \frac{3}{5} + \frac{s}{25} \qquad \text{...(i)}$$

or $\quad a = \dfrac{3}{5} + \dfrac{s}{25}$

i.e., a-s graph is a straight line.

At $s = 0$, $\quad a = \dfrac{3}{5}\ \text{m/s}^2 = 0.6\ \text{m/s}^2$

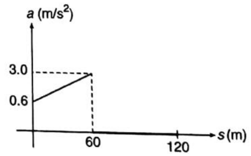
and $\quad$ at $s = 60$ m, $\quad a = 3.0\ \text{m/s}^2$

For $\quad s > 60$ m

v = constant

$\therefore \quad a = 0$

Therefore the corresponding a-s graph is shown in figure.



From Eq. (i), $\quad \dfrac{dv}{dt} = \dfrac{v}{5}$ or $\displaystyle\int_3^v \frac{dv}{v} = \frac{1}{5}\int_0^t dt$

$$\ln\left(\frac{v}{3}\right) = \frac{t}{5}$$

$\therefore \quad v = 3e^{t/5}$ or $\displaystyle\int_0^{60} ds = 3\int_0^{t_1} e^{t/5}\,dt$

$60 = 15(e^{t_1/5} - 1)$ or $t_1 = 8.0$ s

Time taken to travel next 60 m with speed 15m/s will be $\dfrac{60}{15} = 4$ s

$\therefore$ Total time = 12.0 s **Ans.**

16. From the graph, $\quad a = 22.5 - \dfrac{22.5}{150}\cdot s$ or $\displaystyle\int_0^v v\cdot dv = \int_0^{60}\left(22.5 - \frac{22.5}{150}\times s\right)ds$

$\therefore \quad \dfrac{v^2}{2} = 22.5 \times 60 - \dfrac{22.5}{150} \times \dfrac{(60)^2}{2}$

$\therefore \quad v = 46.47$ m/s **Ans.**

17. (a) $\quad u_x = 3$ m/s

$a_x = -1.0\ \text{m/s}^2$

Maximum x coordinate is attained after time $t = \left|\frac{u_x}{a_x}\right| = 3$ s

At this instant $v_x = 0$ and $v_y = u_y + a_y t = 0 - 0.5 \times 3 = -1.5$ m/s

∴ $\vec{v} = (-1.5\,\hat{j})$ m/s **Ans.**

(b) $x = u_x t + \frac{1}{2} a_x t^2 = 3 \times 3 + \frac{1}{2}(-1.0)(3)^2 = 4.5$ m

$y = u_y t + \frac{1}{2} a_y t^2 = 0 - \frac{1}{2}(0.5)(3)^2 = -2.25$ m

∴ $\vec{r} = (4.5\,\hat{i} - 2.25\,\hat{j})$ m **Ans.**

18. (b) $\vec{v} = v_x\,\hat{i} + v_y\,\hat{j}$ and $\vec{a} = a_x\,\hat{i} + a_y\,\hat{j}$

$$\vec{v} \cdot \vec{a} = v_x a_x + v_y a_y$$

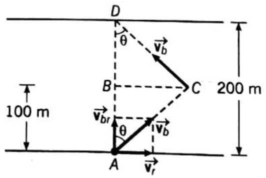
Further $v = \sqrt{v_x^2 + v_y^2}$

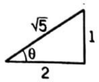
∴ $\frac{\vec{v} \cdot \vec{a}}{v} = \frac{v_x a_x + v_y a_y}{\sqrt{v_x^2 + v_y^2}} = \frac{dv}{dt} = a_t$

or component of $\vec{a}$ parallel to $\vec{v}$ = tangential acceleration.

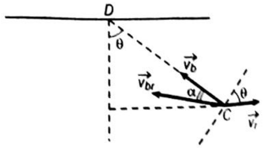
19. (a) $v_{br} = 4$ m/s, $v_r = 2$ m/s

$$\tan\theta = \frac{BC}{AB} = \frac{|\vec{v}_r|}{|\vec{v}_{br}|} = \frac{2}{4} = \frac{1}{2}$$





In this case $\vec{v}_b$ should be along CD



∴ $v_r \cos\theta = v_{br} \sin\alpha$

$2\left(\frac{2}{\sqrt{5}}\right) = 4 \sin\alpha$ or $\sin\alpha = \frac{1}{\sqrt{5}}$

∴ $\alpha = \theta = \tan^{-1}\left(\frac{1}{2}\right)$

(b) $t_1 = \frac{200}{|\vec{v}_{br}|} = \frac{200}{4} = 50$ s

$$DC = DB \sec\theta = (100)\frac{\sqrt{5}}{2} = 50\sqrt{5} \text{ m}$$

$$|\vec{v}_b| = |\vec{v}_{br}| \cos\alpha - |\vec{v}_r| \sin\theta = 4\left(\frac{2}{\sqrt{5}}\right) - 2\left(\frac{1}{\sqrt{5}}\right) = \frac{6}{\sqrt{5}} \text{ m/s}$$

∴ $t_2 = \frac{t_1}{2} + \frac{DC}{|\vec{v}_b|} = 25 + \frac{50\sqrt{5}}{\frac{6}{\sqrt{5}}} = \frac{200}{3}$ s

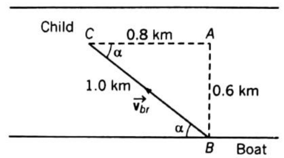
or $\frac{t_2}{t_1} = \frac{4}{3}$ **Ans.**

20. $\vec{v}_b$ = velocity of boatman = $\vec{v}_{br} + \vec{v}_r$

And $\vec{v}_c$ = velocity of child = $\vec{v}_r$

∴ $\vec{v}_{bc} = \vec{v}_b - \vec{v}_c = \vec{v}_{br}$

$\vec{v}_{bc}$ should be along BC

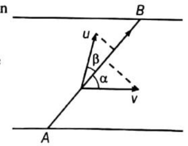


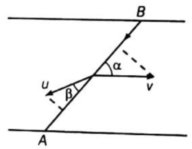
i.e., $\vec{v}_{br}$ should be along BC, where $\tan\alpha = \frac{0.6}{0.8} = \frac{3}{4}$

or $\alpha = 37°$ **Ans.**

Further $t = \frac{BC}{|\vec{v}_{br}|} = \frac{1}{20}$ h = 3 min **Ans.**

21. In order that the moving launch is always on the straight line AB, the components of velocity of the current and of the launch in the direction perpendicular to AB should be equal, *i.e.*,





$u \sin\beta = v \sin\alpha$ ...(i)

$S = AB = (u \cos\beta + v \cos\alpha)t_1$ ...(ii)

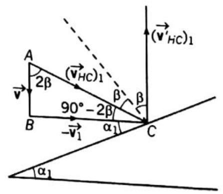
Further $BA = (u \cos\beta - v \cos\alpha)t_2$ ...(iii)

$t_1 + t_2 = t$ ...(iv)

Solving these equations after proper substitution, we get

$u = 8$ m/s and $\beta = 12°$ **Ans.**

22. Here, absolute velocity of hail stones $\vec{v}$ before colliding with wind screens is vertically downwards and velocity of hail stones with respect to cars after collision $\vec{v}'_{HC}$ is vertically upwards. Collision is elastic, hence, velocity of hail stones with respect to cars before collision $\vec{v}_{HC}$ and after collision $\vec{v}'_{HC}$ will make equal angles with the normal to the wind screen.



$(\vec{v}_{HC})_1$ = velocity of hail stones − velocity of car 1

$= \vec{v} - \vec{v}_1$

From the figure, we can see that

$$\beta + 90° - 2\beta + \alpha_1 = 90°$$

or $\alpha_1 = \beta$

or $2\beta = 2\alpha_1$

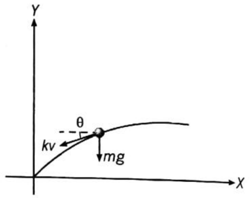
In ΔABC, $$\tan 2\beta = \tan 2\alpha_1 = \frac{v_1}{v} \qquad \text{...(i)}$$

Similarly, we can show that $$\tan 2\alpha_2 = \frac{v_2}{v} \qquad \text{...(ii)}$$

From Eq. (i) and (ii), we get $$\frac{v_1}{v_2} = \frac{\tan 2\alpha_1}{\tan 2\alpha_2} = \frac{\tan 60°}{\tan 30°} = \frac{\sqrt{3}}{1/\sqrt{3}} = 3$$

∴ $$\frac{v_1}{v_2} = 3 \qquad \textbf{Ans.}$$

23 $$a_x = \frac{dv_x}{dt} = -\frac{kv\cos\theta}{m} = -\frac{k}{m}v_x$$



∴ $$\frac{dv_x}{v_x} = -\frac{k}{m}dt \quad \text{or} \quad \int_{v_0\cos\theta_0}^{v_x}\frac{dv_x}{v_x} = -\frac{k}{m}\int_0^t dt$$

or $$v_x = v_0\cos\theta_0\, e^{-\frac{k}{m}t} \qquad \text{...(i)}$$

Similarly $$a_y = \frac{dv_y}{dt} = -\frac{kv\sin\theta}{m} - g = -\left(\frac{k}{m}v_y + g\right)$$

or $$\int_{v_0\sin\theta_0}^{v_y}\frac{dv_y}{\frac{k}{m}v_y + g} = -\int_0^t dt \quad \text{or} \quad \frac{m}{k}\left[\ln\left(\frac{k}{m}v_y + g\right)\right]_{v_0\sin\theta_0}^{v_y} = -t$$

or $$\frac{\left(\frac{k}{m}v_y + g\right)}{\left(\frac{k}{m}v_0\sin\theta_0 + g\right)} = e^{-\frac{k}{m}t}$$

or $$v_y = \frac{m}{k}\left[\left(\frac{k}{m}v_0\sin\theta_0 + g\right)e^{-\frac{k}{m}t} - g\right] \qquad \text{...(ii)}$$

(b) Eq. (i) can be written as: $$\frac{dx}{dt} = v_0\cos\theta_0\, e^{-\frac{k}{m}t}$$

or $$\int_0^x dx = v_0\cos\theta_0\int_0^t e^{-\frac{k}{m}t}dt \quad \text{or} \quad x = \frac{mv_0\cos\theta_0}{k}\left[1 - e^{-\frac{k}{m}t}\right]$$

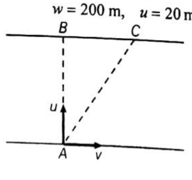
$$x_m = \frac{mv_0\cos\theta_0}{k} \quad \text{at} \quad t = \infty. \qquad \textbf{Ans.}$$

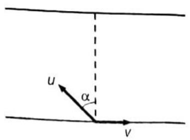
24. In the first case $BC = vt_1$ and $w = ut_1$

In the second case $u\sin\alpha = v$ and $w = (u\cos\alpha)\,t_2$

Solving these four equations with proper substitution, we get

$$w = 200\text{ m}, \quad u = 20\text{ m/min}, \quad v = 12\text{ m/min} \quad \text{and} \quad \alpha = 36°50' \qquad \textbf{Ans.}$$



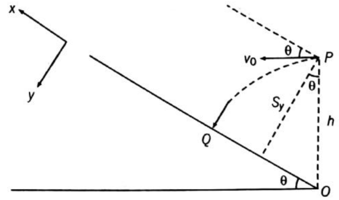


Chapter 4 Projectile Motion

Level 2

1. $u_x = v_0\cos\theta$, $u_y = v_0\sin\theta$, $a_x = -g\sin\theta$, $a_y = g\cos\theta$

At $Q : v_x = 0$ ∴ $u_x + a_x t = 0$



Or $$t = \frac{v_0\cos\theta}{g\sin\theta} \qquad \text{...(i)}$$

$$S_y = h\cos\theta$$

∴ $$u_y t + \frac{1}{2}a_y t^2 = h\cos\theta$$

∴ $$(v_0\sin\theta)\left(\frac{v_0\cos\theta}{g\sin\theta}\right) + \frac{1}{2}(g\cos\theta)\left(\frac{v_0\cos\theta}{g\sin\theta}\right)^2 = h\cos\theta$$

Solving this equation we get,

$$v_0 = \sqrt{\frac{2gh}{2+\cot^2\theta}} \qquad \textbf{Ans.}$$

2. Let v_x and v_y be the compounts of v_0 along x and y directions.

$$(v_x)(2) = 2$$

∴ $$v_x = 1\text{ m/s}$$

$$v_y(2) = 10 \quad \text{Or} \quad v_y = 5\text{ m/s}$$

$$v_0 = \sqrt{v_x^2 + v_y^2} = \sqrt{26}\text{ m/s}$$

$$\tan\theta = v_y/v_x = 5/1$$

∴ $$\theta = \tan^{-1}(5) \qquad \textbf{Ans.}$$

Note *We have seen relative motion between two particles.*

Relative acceleration between them is zero.

3. $$\vec{v}_1 = (u\cos\alpha)\,\hat{i} + (u\sin\alpha - gt)\,\hat{j}$$

$$\vec{v}_2 = (v\cos\beta)\,\hat{i} + (v\sin\beta - gt)\,\hat{j}$$

These two velocity vectors will be parallel when, the ratio of coefficients of $\hat{i}$ and $\hat{j}$ are equal.

$\therefore \quad \dfrac{u\cos\alpha}{v\cos\beta}=\dfrac{u\sin\alpha-gt}{v\sin\beta-gt}$

Solving we get, $\quad t=\dfrac{uv\sin(\alpha-\beta)}{g(v\cos\beta-u\cos\alpha)}$ **Ans.**

4. At height 2 m, projectile will be at two times, which are obtained from the equation,

$$+2=(10\sin 45^\circ)t+\frac{1}{2}(-10)t^2$$

or $\quad 2=5\sqrt{2}t-5t^2$

or $\quad 5t^2-5\sqrt{2}t+2=0$

or $\quad t_1=\dfrac{5\sqrt{2}-\sqrt{50-40}}{10}=\dfrac{5\sqrt{2}-\sqrt{10}}{10}$

and $\quad t_2=\dfrac{5\sqrt{2}+\sqrt{10}}{10}$

Now $\quad d=(10\cos 45^\circ)(t_2-t_1)$

$$=\frac{10}{\sqrt{2}}\left(\frac{2\sqrt{10}}{10}\right)=4.47\text{ m}$$

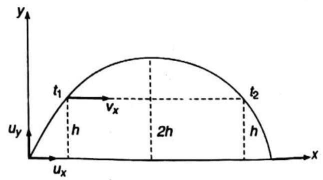
Distance of point of projection from first hurdle $=(10\cos 45^\circ)t_1$

$$=\frac{10}{\sqrt{2}}\left(\frac{5\sqrt{2}-\sqrt{10}}{10}\right)$$

$$=5-\sqrt{5}$$

$$=2.75\text{ m}$$ **Ans.**

5. $\quad 2h=\dfrac{u_y^2}{2g}$



or $\quad u_y=2\sqrt{gh}$

Now $\quad (t_2-t_1)u_x=t_2v_x$ or $\dfrac{v_x}{u_x}=\dfrac{t_2-t_1}{t_2}$ ...(i)

Further $\quad h=u_yt-\dfrac{1}{2}gt^2$

or $\quad gt^2-2u_yt+h=0$

or $\quad gt^2-4\sqrt{gh}\,t+2h=0$

$$t_1=\frac{4\sqrt{gh}-\sqrt{16gh-8gh}}{2g}=(2-\sqrt{2})\sqrt{\frac{h}{g}}\quad\text{and}\quad t_2=(2+\sqrt{2})\sqrt{\frac{h}{g}}$$

Substituting in Eq. (i) we have,

$$\frac{v_x}{u_x}=\frac{2}{\sqrt{2}+1}$$ **Ans.**

6. (a) Time of descent $t=\sqrt{\dfrac{2H}{g}}=\sqrt{\dfrac{2\times 400}{10}}=8.94$ s

Now $\quad v_x=ay=\sqrt{5}y$

or $\quad \dfrac{dx}{dt}=\sqrt{5}\left(\dfrac{1}{2}gt^2\right)=5\sqrt{5}t^2$

$\therefore \quad \int_0^x dx=5\sqrt{5}\int_0^t t^2\,dt$

or horizontal drift $x=\dfrac{5\sqrt{5}}{3}(8.94)^3=2663\text{ m}\approx 2.67$ km.

(b) When particle strikes the ground :

$$v_x=\sqrt{5}y=(\sqrt{5})(400)=400\sqrt{5}\text{ m/s}$$

$$v_y=gt=89.4\text{ m/s}$$

$\therefore \quad$ Speed $=\sqrt{v_x^2+v_y^2}=899\text{ m/s}\approx 0.9$ km/s **Ans.**

7. At $t=0$, $\vec{v}_T=(10\hat{j})$ m/s $\quad \vec{v}_{ST}=10\cos 37^\circ\hat{k}-10\sin 37^\circ\hat{i}=(8\hat{k}-6\hat{i})$ m/s

$\therefore \quad \vec{v}_S=\vec{v}_{ST}+\vec{v}_T=(-6\hat{i}+10\hat{j}+8\hat{k})$ m/s

(a) At highest point vertical component $(\hat{k})$ of $\vec{v}_S$ will become zero. Hence, velocity of particle at highest point will become $(-6\hat{i}+10\hat{j})$ m/s .

(b) Time of flight, $\quad T=\dfrac{2v_Z}{g}=\dfrac{2\times 8}{10}=1.6$ s

$$x=x_i+v_xT$$

$$=\frac{16}{\pi}-6\times 1.6=-4.5\text{ m}$$

$$y=(10)(1.6)=16\text{ m}\quad\text{and}\quad z=0$$

Therefore coordinates of particle where it finally lands on the ground are (– 4.5 m, 16 m , 0).

At highest point $\quad t=\dfrac{T}{2}=0.8$ s

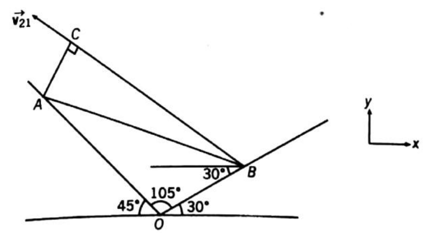
$\therefore \quad x=\dfrac{16}{\pi}-(6)(0.8)=0.3$ m

$$y=(10)(0.8)=8.0\text{ m}$$

and $\quad z=\dfrac{v_Z^2}{2g}=\dfrac{(8)^2}{20}=3.2$ m

Therefore, coordinates at highest point are, (0.3 m, 8.0 m, 3.2 m) **Ans.**

8. $|v_{21x}|=(v_1+v_2)\cos 60^\circ=12$ m/s



$|v_{21y}|=(v_2-v_1)\sin 60^\circ=4\sqrt{3}$ m/s

$\therefore \quad v_{21}=\sqrt{(12)^2+(4\sqrt{3})^2}=\sqrt{192}$ m/s

$BC=(v_{21})\,t=240$ m

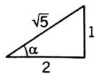
$AC=70$ m $\quad$ (Given)

Hence, $AB=\sqrt{(240)^2+(70)^2}=250$ m **Ans.**

9. (a) Let, (x, y) be the coordinates of point C.

$$x = OD = OA + AD$$

$$\therefore \quad x = \frac{10}{3} + y \cot 37° = \frac{10 + 4y}{3} \quad \ldots \text{(i)}$$



As point C lies on the trajectory of a parabola, we have

$$y = x \tan \alpha - \frac{gx^2}{2u^2}(1 + \tan^2 \alpha) \quad \ldots \text{(ii)}$$

Given that, $\tan \alpha = 0.5 = \frac{1}{2}$

Solving Eqs. (i) and (ii), we get $x = 5$ m and $y = 1.25$ m.

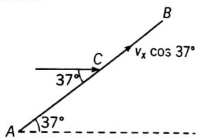
Hence, the coordinates of point C are (5 m, 1.25 m). **Ans.**

(b) Let v_y be the vertical component of velocity of the particle just before collision at C.

Using $v_y = u_y + a_y t$, we have

$$v_y = u \sin \alpha - g\,(x/u \cos \alpha) \qquad (\because t = x/u \cos \alpha)$$

$$= \frac{5\sqrt{5}}{\sqrt{5}} - \frac{10 \times 5}{(5\sqrt{5} \times 2/\sqrt{5})} = 0$$



Thus, at C, the particle has only horizontal component of velocity

$$v_x = u \cos \alpha = 5\sqrt{5} \times (2/\sqrt{5}) = 10 \text{ m/s}$$

Given, that the particle does not rebound after collision. So, the normal component of velocity (normal to the plane AB) becomes zero. Now, the particle slides up the plane due to tangential component

$$v_x \cos 37° = (10)\left(\frac{4}{5}\right) = 8 \text{ m/s}$$

Let h be the further height raised by the particle. Then

$$mgh = \frac{1}{2} m (8)^2 \quad \text{or} \quad h = 3.2 \text{ m}$$

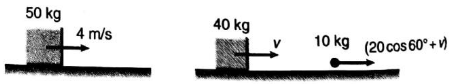
Height of the particle from the ground $= y + h$

$$\therefore \quad H = 1.25 + 3.2 = 4.45 \text{ m}$$ **Ans.**

10. For shell $u_z = 20 \sin 60° = 17.32$ m/s

$$\therefore \quad z = u_z t - \frac{1}{2} g t^2$$

$$= (17.32 \times 2) - \left(\frac{1}{2} \times 9.8 \times 4\right)$$



or $\quad z = 15$ m

$$u_y = 0 \quad \therefore \quad y = 0$$

For u_x : conservation of linear momentum gives,

$$50 \times 4 = (40)(v) + 10(20 \cos 60° + v) \quad \text{or} \quad v = 2 \text{ m/s}$$

$$\therefore \quad u_x = (20 \cos 60°) + 2 = 12 \text{ m/s}$$

$$\therefore \quad x = u_x t = (12)(2) = 24 \text{ m}$$

$$\therefore \quad \vec{r} = (24\hat{i} + 15\hat{k}) \text{ m}$$ **Ans.**

Chapter 5 Laws of Motion

Level 2

1. It is just like a projectile motion with g to be replaced by $g \sin 45°$.

After 2 seconds,

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(10 \sin 45° - \frac{g}{\sqrt{2}} \times 2\right) + (10 \cos 45°)^2}$$

$$= 10 \text{ m/s}$$ **Ans.**

2. Suppose T be the tension in the string attached to block B. Then tension in the string connected to block A would be $4T$.

Similarly, if a be the acceleration of block A (downwards), then acceleration of block B towards right will be $4a$.

Equations of motion are:

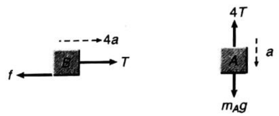
For block A, $\quad m_A g - 4T = m_A a$

or $\quad 50 - 4T = 5a \quad \ldots$(i)

For block B, $\quad T - f = 10(4a)$

or $\quad T - (0.1)(10)(10) = 40a$

or $\quad T - 10 = 40a \quad \ldots$(ii)



Solving Eqs. (i) and (ii), we get $\quad a = \frac{2}{33}$ m/s^2 **Ans.**

3. (a) When the truck accelerates eastward force of friction on mass is eastwards.

$$f_{\text{required}} = \text{mass} \times \text{acceleration} = (30 \times 1.8)$$
$$= 54 \text{ N}$$

Since it is less than $\mu_s mg \quad \therefore f = 54$ N (eastwards)

(b) When the truck accelerates westwards, force of friction is westwards.

$$f_{\text{required}} = \text{mass} \times \text{acceleration} = 30 \times 3.8$$
$$= 114 \text{ N}$$

Since it is greater than $\mu_s mg$. Hence

$$f = f_k = \mu_k mg = 60 \text{ N (westwards)}$$ **Ans.**

4. Block B will fall vertically downwards and A along the plane.

Writing the equations of motion.

For block B, $\quad m_B g - N = m_B a_B$

or $\quad 60 - N = 6a_B \quad \ldots$(i)

$$(N + m_A g) \sin 30° = m_A a_A$$

or $\quad (N + 150) = 30\, a_A \quad \ldots$(ii)

Further $\quad a_B = a_A \sin 30°$

or $\quad a_A = 2a_B \quad \ldots$(iii)

Solving these three equations, we get

(a) $\quad a_A = 6.36$ m/s^2 **Ans.**

(b) $\quad a_{BA} = a_A \cos 30° = 5.5$ m/s^2 **Ans.**

5. Let acceleration of m be a_1 (absolute) and that of M be a_2 (absolute).
Writing equations of motion.

For m: $\quad mg\cos\alpha - N = ma_1 \quad$ ...(i)

For M: $\quad N\sin\alpha = Ma_2 \quad$ ...(ii)

Constraint equation can be written as,

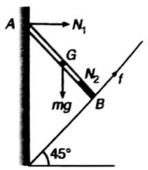
$$a_1 = a_2\sin\alpha \qquad (a_2 \neq a_1\sin\alpha\text{, think why?})$$

Solving above three equations, we get

acceleration of rod, $\quad a_1 = \dfrac{mg\cos\alpha\sin\alpha}{\left(m\sin\alpha + \dfrac{M}{\sin\alpha}\right)}$ **Ans.**

and acceleration of wedge $\quad a_2 = \dfrac{mg\cos\alpha}{m\sin\alpha + \dfrac{M}{\sin\alpha}}$ **Ans.**

6. (a) N_2 and mg pass through G. N_1 has clockwise moment about G, so the ladder has a tendency to slip by rotating clockwise and the force of friction (f) at B is then up the plane.



(b) $\quad \Sigma M_A = 0$

$\therefore \quad fl = mg\left(\dfrac{l}{2}\sin 45°\right) \quad$ ...(i)

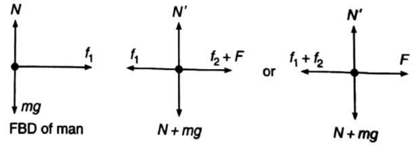
$\Sigma F_V = 0$

$\therefore \quad mg = N_2\cos 45° + f\sin 45° \quad$ ...(ii)

From Eqs. (i) and (ii), $\quad N_2 = \dfrac{3}{2\sqrt{2}}mg \quad$ and $\quad f = \dfrac{mg}{2\sqrt{2}}$

or $\quad \mu_{min} = \dfrac{f}{N_2} = \dfrac{1}{3}$ **Ans.**

7.



FBD of man

FBD of plank

Here f_1 = force of friction between man and plank and f_2 = force of friction between plank and surface.
For the plank not to move

$$F - (f_2)_{max} \le f_1 \le F + (f_2)_{max}$$

or $\quad F - \mu(M+m)g \le ma \le F + \mu(M+m)g$

or $\quad a$ should lie between $\dfrac{F}{m} - \dfrac{\mu(M+m)g}{m}$

and $\quad \dfrac{F}{m} + \dfrac{\mu(M+m)g}{m}$ **Ans.**

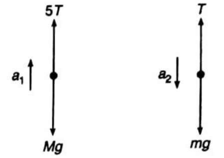
8. Writing equations of motion:

For M: $\quad 5T - Mg = Ma_1 \quad$ ...(i)

For m: $\quad mg - T = ma_2 \quad$ ...(ii)

From constraint equation, $\quad a_2 = 5a_1 \quad$ ...(iii)

Solving these equations, we get



FBD of M

FBD of m

acceleration of M, $\quad a_1 = \left(\dfrac{5m - M}{25m + M}\right)g$

and of m, $\quad a_2 = 5\left(\dfrac{5m - M}{25m + M}\right)g$

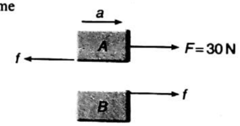
9. $2a_1s_1 = 2a_2s_2$

or $\quad \dfrac{a_1}{a_2} = \dfrac{s_2}{s_1} = \dfrac{n}{m} \quad$ or $\quad \dfrac{g\sin\alpha}{\mu g\cos\alpha - g\sin\alpha} = \dfrac{m}{n}$

Solving it, we get $\quad \mu = \left(\dfrac{m+n}{m}\right)\tan\alpha$ **Ans.**

10. Limiting friction between A and B $\quad f_L = \mu N = 0.4 \times 100 = 40$ N

(a) Both the blocks will have a tendency to move together with same acceleration (say a). So, the force diagram is as shown.



Equations of motion are,

$$30 - f = 10 \times a \quad \text{...(i)}$$
$$f = 25 \times a \quad \text{...(ii)}$$

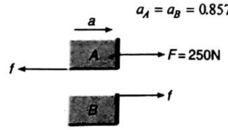
Solving these two equations, we get

$$a = 0.857\ \text{m/s}^2 \quad \text{and} \quad f = 21.42\ \text{N}$$

As this force is less than f_L, both the blocks will move together with same acceleration,

$$a_A = a_B = 0.857\ \text{m/s}^2$$

(b)

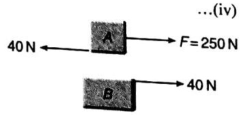


$$250 - f = 10a \quad \text{...(iii)}$$
$$f = 25a \quad \text{...(iv)}$$

Solving Eqs. (iii) and (iv), we get $\quad f = 178.6$ N

As $f > f_L$, slipping will take place between the two blocks and

$$f = f_L = 40\ \text{N}$$



$$a_A = \frac{250 - 40}{10} = 21.0\ \text{m/s}^2$$

$$a_B = \frac{40}{25} = 1.6\ \text{m/s}^2$$ **Ans.**

11. Normal reaction between A and B would be $N = mg\cos\theta$. Its horizontal component is $N\sin\theta$. Therefore, tension in cord CD is equal to this horizontal component.

Hence, $\quad T = N\sin\theta = (mg\cos\theta)(\sin\theta)$

$$= \frac{mg}{2}\sin 2\theta$$ **Ans.**

12. Assuming that mass of truck >> mass of crate.

Retardation of truck $a_1 = (0.9)\,g = 9\text{ m/s}^2$, retardation of crate $a_2 = (0.7)\,g = 7\text{ m/s}^2$

or relative acceleration of crate $a_r = 2\text{m/s}^2$.

Truck will stop after time $t_1 = \frac{15}{9} = 1.67$ sec and crate will strike the wall at

$$t_2 = \sqrt{\frac{2s}{a_r}} = \sqrt{\frac{2 \times 3.2}{2}} = 1.78\text{ s}$$

As $t_2 > t_1$, crate will come to rest after travelling a distance

$$s = \frac{1}{2}a_r t_1^2 = \frac{1}{2} \times 2.0 \times \left(\frac{15}{9}\right)^2 = 2.77\text{ m}$$ **Ans.**

13. $\mu_k mg = 0.2 \times 10 \times 10 = 20$ N

For $t \le 0.2$ sec :

Retardation $$a_1 = \frac{F + \mu_k mg}{m} = \frac{20 + 20}{10} = 4\text{ m/s}^2$$

At the end of 0.2 sec,

$$v = u - a_1 t = 1.2 - 4 \times 0.2 = 0.4\text{ m/s}$$

For $t > 0.2$ sec :

Retardation $$a_2 = \frac{10 + 20}{10} = 3\text{ m/s}^2$$

Block will come to rest after time $t_0 = \frac{v}{a_2} = \frac{0.4}{3} = 0.13$ s

$\therefore$ Total time $= 0.2 + 0.13 = 0.33$ s **Ans.**

14. Block will start moving at, $F = \mu\, mg$

or $$25t = (0.5)(10)(9.8) = 49\text{ N}$$

$\therefore$ $$t = 1.96\text{ s}$$

Velocity is maximum at the end of 4 second.

$\therefore$ $$\frac{dv}{dt} = \frac{25t - 49}{10} = 2.5t - 4.9$$

$\therefore$ $$\int_0^{v_{max}} dv = \int_{1.96}^{4} (2.5\,t - 4.9)\,dt$$

$\therefore$ $$v_{max} = 5.2\text{ m/s}$$ **Ans.**

For $4\text{ s} < t < 7\text{ s}$

Net retardation $$a_1 = \frac{49 - 40}{10} = 0.9\text{ m/s}^2$$

$\therefore$ $$v = v_{max} - a_1 t_1 = 5.2 - 0.9 \times 3 = 2.5\text{ m/s}$$

For $t > 7$ s

$$\text{Retardation } a_2 = \frac{49}{10} = 4.9\text{ m/s}^2$$

$\therefore$ $$t = \frac{v}{a_2} = \frac{2.5}{4.9} = 0.51\text{ s}$$

$\therefore$ $$\text{Total time} = (4 - 1.96) + (7 - 4) + (0.51) = 5.55\text{ s}$$ **Ans.**

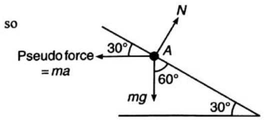
15. Let B and C both move upwards (alongwith their pulleys) with speeds v_B and v_C then we can see that, A will move downward with speed, $2v_B + 2v_C$. So, with sign we can write,

$\therefore$ $$v_B = \frac{v_A}{2} - v_c$$

Substituting the values we have, $v_B = 0$ **Ans.**

16. FBD of A with respect to frame is shown in figure..

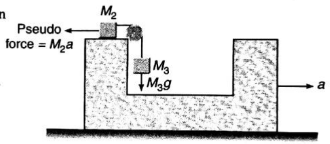
A is in equilibrium under three concurrent forces shown in figure, so applying Lami's theorem



$$\frac{ma}{\sin(90 + 60)} = \frac{mg}{\sin(90 + 30)}$$

$\therefore$ $$a = \frac{g\cos 60°}{\cos 30°} = 5.66\text{ m/s}^2$$

17. FBD of M_2 and M_3 in accelerated frame of reference is shown in figure.



Note: Only the necessary forces have been shown.

Mass M_3 will neither rise nor fall if net pulling force is zero.

i.e., $$M_2 a = M_3 g$$

or $$a = \frac{M_3}{M_2}g$$

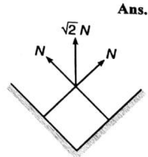
$\therefore$ $$F = (M_1 + M_2 + M_3)\,a = (M_1 + M_2 + M_3)\frac{M_3}{M_2}g$$ **Ans.**

18. Retardation $a = \mu_k g = 0.15 \times 9.8 = 1.47\text{ m/s}^2$

Distance travelled before sliding stops is, $s = \frac{v^2}{2a} = \frac{(5)^2}{2 \times 1.47}$

$$\approx 8.5\text{ m}$$ **Ans.**

19. $\sqrt{2}\,N = mg\cos\theta$



$\therefore$ $$N = \frac{mg\cos\theta}{\sqrt{2}}$$

$$a = \frac{mg\sin\theta - 2\mu_k N}{m} = g\sin\theta - \sqrt{2}\,\mu_k g\cos\theta$$

$$= g(\sin\theta - \sqrt{2}\mu_k\cos\theta)$$ **Ans.**

20. $$v\cdot\frac{dv}{dx} = \frac{\text{Net force}}{\text{mass}} = \frac{F - \mu_k\rho(L - x)g}{\rho L}$$

$\therefore$ $$\int_0^v v\,dv = \int_0^L \frac{F - \mu_k\rho(L - x)g}{\rho L}dx$$

$\therefore$ $$\frac{v^2}{2} = \frac{F}{\rho} - \mu_k gL + \frac{\mu_k gL}{2}$$

$\therefore$ $$v = \sqrt{\frac{2F}{\rho} - \mu_k gL}$$ **Ans.**

21. (a) $v = a_1 t_1 = 2.6$ m/s

$$s_1 = \frac{1}{2} a_1 t_1^2 = \frac{1}{2} \times 2 \times (1.3)^2 = 1.69 \text{ m}$$

$$s_2 = (2.2 - 1.69) = 0.51 \text{ m}$$

Now, $$s_2 = \frac{v^2}{2a_2}$$

$\therefore$ $$a_2 = \frac{v^2}{2s_1} = \frac{(2.6)^2}{2 \times 0.51} = 6.63 \text{ m/s}^2$$

and $$t_2 = \frac{v}{a_2} = 0.4 \text{ s}$$

(b) Acceleration of package will be 2 m/s^2 while retardation will be $\mu_k g$ or 2.5 m/s^2 not 6.63 m/s^2.

For the package,

$$v = a_1 t_1 = 2.6 \text{ m/s}$$

$$s_1 = \frac{1}{2} a_1 t_1^2 = 1.69 \text{ m}$$

$$s_2 = vt_2 - \frac{1}{2} a_2 t_2^2 = 2.6 \times 0.4 - \frac{1}{2} \times 2.5 \times (0.4)^2$$

$$= 0.84 \text{ m}$$

$\therefore$ Displacement of package w.r.t. belt $= (0.84 - 0.51)$ m

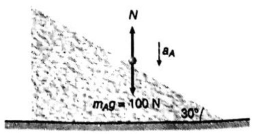
$= 0.33$ m **Ans.**

Alternate Sol : For last 0.4 seconds

$$|a_r| = 6.63 - 2.5 = 4.13 \text{ m/s}^2$$

$\therefore$ $$s_r = \frac{1}{2}|a_r|t^2 = \frac{1}{2} \times 4.13 \times (0.4)^2 = 0.33 \text{ m}$$

22. Free body diagram of crate A w.r.t ground is shown in figure.



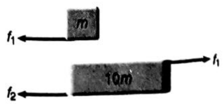
Equation of motion is: $100 - N = 10a_A$ ...(i)

$$a_A = a \sin 30° = (2)\left(\frac{1}{2}\right)$$

or $$a_A = 1 \text{m/s}^2$$

Substituting in Eq. (i), we get $N = 90$ newton.

23. (a) Force of friction at different contacts are shown in figure.



Here, $f_1 = \mu_2 mg$ and $f_2 = \mu_1 (11\, mg)$

Given that $\mu_2 > 11\mu_1$

$\therefore$ $f_1 > f_2$

Retardation of upper block $a_1 = \frac{f_1}{m} = \mu_2 g$

Acceleration of lower block $a_2 = \frac{f_1 - f_2}{m} = \frac{(\mu_2 - 11\mu_1)g}{10}$

Relative retardation of upper block $a_r = a_1 + a_2$

or $$a_r = \frac{11}{10}(\mu_2 - \mu_1)g$$

Now, $$0 = v_{\min}^2 - 2a_r l$$

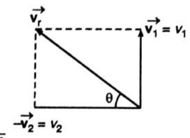
$\therefore$ $$v_{\min} = \sqrt{2a_r l} = \sqrt{\frac{22(\mu_2 - \mu_1)gl}{10}}$$ **Ans.**

(b) $$0 = v_{\min} - a_r t$$

or $$t = \frac{v_{\min}}{a_r} = \sqrt{\frac{20l}{11(\mu_2 - \mu_1)g}}$$ **Ans.**

24. $v_r = \sqrt{v_1^2 + v_2^2}$

Retardation $a = \mu g$



$\therefore$ Time when slipping will stop is $t = \frac{v_r}{a}$ or $t = \frac{\sqrt{v_1^2 + v_2^2}}{\mu g}$ ...(i)

$$s_r = \frac{v_r^2}{2a} = \frac{v_1^2 + v_2^2}{2\,\mu g}$$

$$x_r = -s_r \cos\theta = -\left(\frac{v_1^2 + v_2^2}{2\mu g}\right)\left(\frac{v_2}{\sqrt{v_1^2 + v_2^2}}\right) = \frac{-v_2\sqrt{v_1^2 + v_2^2}}{2\,\mu g}$$

$$y_r = s_r \sin\theta = \left(\frac{v_1^2 + v_2^2}{2\,\mu g}\right)\left(\frac{v_1}{\sqrt{v_1^2 + v_2^2}}\right) = \frac{v_1\sqrt{v_1^2 + v_2^2}}{2\,\mu g}$$

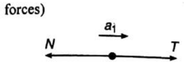
In time t, belt will move a distance

$s = v_2 t$ or $\frac{v_2\sqrt{v_1^2 + v_2^2}}{\mu g}$ in x-direction. Hence, coordinate of particle,

$$x = x_r + s = \frac{v_2\sqrt{v_1^2 + v_2^2}}{2\mu g}$$

and $$y = y_r = \frac{v_1\sqrt{v_1^2 + v_2^2}}{2\,\mu g}$$ **Ans.**

25. FBD of m_1 (showing only the horizontal forces)

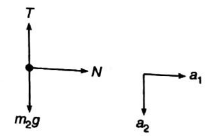


Equation of motion for m_1 is

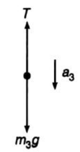
$$T - N = m_1 a_1 \quad \text{...(i)}$$

Equations of motion for m_2 are $$N = m_2 a_1 \quad \text{...(ii)}$$

and $$m_2 g - T = m_2 a_2 \quad \text{...(iii)}$$



FBD of m_2



Equation of motion for m_3 are

$$m_3 g - T = m_3 a_3 \qquad \text{...(iv)}$$

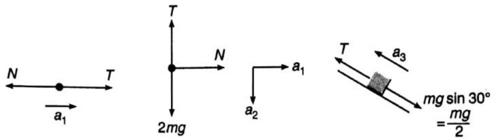
Further from constraint equation we can find the relation,

$$a_1 = a_2 + a_3 \qquad \text{...(v)}$$

We have five unknowns a_1, a_2, a_3, T and N solving, we get

$$a_1 = \frac{2m_1 m_3\, g}{(m_2 + m_3)(m_1 + m_2) + m_2 m_3} \qquad \textbf{Ans.}$$

26.



FBD of $3m$ FBD of $2m$ FBD of m

Writing equations of motion,

$$T - N = 3ma_1 \qquad \text{...(i)}$$
$$N = 2ma_1 \qquad \text{...(ii)}$$
$$2mg - T = 2ma_2 \qquad \text{...(iii)}$$
$$T - \frac{mg}{2} = ma_3 \qquad \text{...(iv)}$$

From constraint equation,

$$a_1 = a_2 - a_3 \qquad \text{...(v)}$$

We have five unknowns. Solving the above five equations, we get

$$a_1 = \frac{3}{17}g, \quad a_2 = \frac{19}{34}g \quad \text{and} \quad a_3 = \frac{13}{34}g$$

Acceleration of $m = a_3 = \frac{13}{34}g$, acceleration of $2m = \sqrt{a_1^2 + a_2^2} = \frac{\sqrt{397}}{34}g$

and acceleration of $3m = a_1 = \frac{3}{17}g$ **Ans.**

27. $a = \dfrac{m_A g}{m_A + M + m}$

For the equilibrium of B,

$$mg = \mu N = \mu(ma) = \frac{\mu m m_A g}{m_A + M + m}$$

$$\therefore \quad m_A = \frac{(M+m)m}{(\mu - 1)m}$$

$$m_A = \frac{(M+m)}{\mu - 1} \qquad \textbf{Ans.}$$

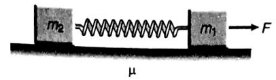
Note $m_A > 0 \;\therefore\; \mu > 1$

Chapter 6

Work, Energy & Power

Level 2

1.



$$(F - \mu m_1 g)x_m = \frac{1}{2}kx_m^2$$

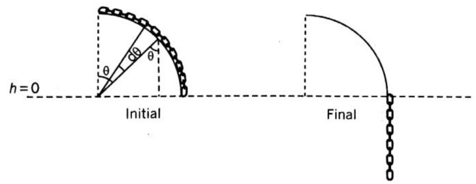
or $$kx_m = 2(F - \mu m_1 g)$$

Second block will shift if $kx_m \geq \mu m_2 g$

$\therefore$ $$2(F - \mu m_1 g) > \mu m_2 g$$

or $$F > \left(m_1 + \frac{m_2}{2}\right)\mu g \qquad \textbf{Ans.}$$

2.



Initial PE, $$U_i = \int_{\theta=0°}^{\theta=\pi/2} (r\,d\theta)(\rho)(g)(r\cos\theta)$$

$$= (\rho g r^2)[\sin\theta]_0^{\pi/2} = \rho g r^2$$

Final PE $$U_f = \left(\frac{\pi r}{2} \times \rho\right)(g)\left(-\frac{\pi r/2}{2}\right) = -\frac{\pi^2 r^2 \rho g}{8}$$

$$\Delta U = r^2 \rho g\left(1 + \frac{\pi^2}{8}\right)$$

$$\Delta U = \text{KE}$$

or $$r^2 \rho g\left(1 + \frac{\pi^2}{8}\right) = \frac{1}{2}\left(\frac{\pi r}{2}\right)(\rho)v^2$$

$\Rightarrow$ $$v = \sqrt{4rg\left(\frac{1}{\pi} + \frac{\pi}{8}\right)}$$

or $$v = \sqrt{rg\left(\frac{\pi}{2} + \frac{4}{\pi}\right)} \qquad \textbf{Ans.}$$

3. For $t \le 0.2$ second

$$F = 800\ \text{N} \quad \text{and} \quad v = \left(\frac{20}{0.3}\right) t$$

$\therefore$ $$P = Fv = (53.3\, t)\ \text{kW}$$ **Ans.**

For $t > 0.2$ second

$$F = 800 - \left(\frac{800}{0.1}\right)(t - 0.2)$$

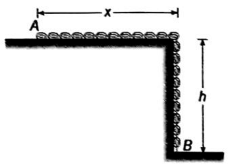
and $$v = \frac{20}{0.3} t$$

$\therefore$ $$P = Fv = (160\, t - 533\, t^2)\ \text{kW}$$

$$W = \int_0^{0.2} (53.3\, t)\, dt + \int_{0.2}^{0.3} (160\, t - 533\, t^2)\, dt$$

$$= 1.69\ \text{kJ}$$ **Ans.**

4. At the instant shown in figure, net pulling force $= \frac{m}{l} gh$



Total mass being pulled $$= \frac{m}{l}(x + h)$$

$\therefore$ Acceleration $$a = \frac{\text{net pulling force}}{\text{total mass being pulled}}$$

$$= \frac{gh}{x + h}$$

$\therefore$ $$v\left(-\frac{dv}{dx}\right) = \frac{gh}{x + h}$$

or $$\int_0^v v \cdot dv = -gh \int_{(l-h)}^0 \frac{dx}{x + h}$$

$\therefore$ $$\frac{v^2}{2} = gh\, [\ln(x + h)]_0^{l-h}$$

or $$\frac{v^2}{2} = gh \ln\left(\frac{l}{h}\right)$$

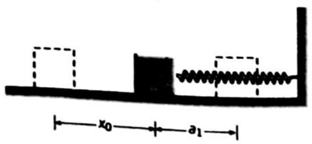
$\therefore$ $$v = \sqrt{2gh \ln(l/h)}$$ **Ans.**

5. (a) From energy conservation principle:

Work done against friction = decrease in elastic P.E.

or $$f(x_0 + a_1) = \frac{1}{2} k(x_0^2 - a_1^2)$$

or $$x_0 - a_1 = \frac{2f}{k} \quad \text{...(i)}$$



From Eq. (i), we see that decrease of amplitude $(x_0 - a_1)$ is $\frac{2f}{k}$, which is constant and same for each cycle of oscillation

(b) The block will come to rest when $$ka = f$$

or $$a = \frac{k}{f} \quad \text{...(A)}$$

In the similar manner we can write

$$a_1 - a_2 = \frac{2f}{k} \quad \text{...(ii)}$$

$$a_2 - a_3 = \frac{2f}{k} \quad \text{...(iii)}$$

$$\dots \dots$$

$$a_{n-1} - a_n = \frac{2f}{k} \quad \text{...(n)}$$

Adding Eqs. (i), (ii),... etc., we get $$x_0 - a_n = n\left(\frac{2f}{k}\right)$$

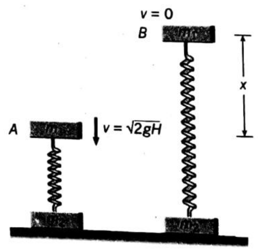
or $$a_n = x_0 - n\left(\frac{2f}{k}\right) \quad \text{...(B)}$$

Equating Eq. (A) and (B), we get $$\frac{k}{f} = x_0 - n\left(\frac{2f}{k}\right)$$

or $$n = \frac{x_0 - \frac{k}{f}}{\frac{2f}{k}} = \frac{kx_0}{2f} - \frac{1}{2}$$

Number of cycles, $$m = \frac{n}{2} = \frac{kx_0}{4f} - \frac{1}{4} = \frac{1}{4}\left(\frac{kx_0}{f} - 1\right)$$ **Ans.**

6. Conservation of mechanical energy gives,



$$E_A = E_B$$

or $$\frac{1}{2} m_1 v^2 = \frac{1}{2} kx^2 + m_1 gx$$

or $$2m_1 gH = kx^2 + 2m_1 gx \quad \text{...(i)}$$

The lower block will rebounce when

$$x > \frac{m_2 g}{k} \qquad (kx = m_2 g)$$

Substituting, $x = \frac{m_2 g}{k}$ in Eq. (i), we get

$$2m_1 gH = k\left(\frac{m_2 g}{k}\right)^2 + 2m_1 g\left(\frac{m_2 g}{k}\right)$$

or $$H = \frac{m_2 g}{k}\left(\frac{m_2 + 2m_1}{2m_1}\right)$$

Thus, $$H_{\min} = \frac{m_2 g}{k}\left(\frac{m_2 + 2m_1}{2m_1}\right)$$ **Ans.**

7. $E_i = E_f$

$\therefore$ $$\frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}kx^2$$

or $$k = \frac{3v^2m}{4x^2}$$ **Ans.**

8. $$\mu m_A g = 0.8 \times 6 \times 10 = 48 \text{ N}$$

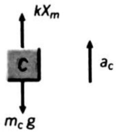
$$(m_B + m_C)g = (1 + 2) \times 10 = 30 \text{ N}$$

Since $$(m_B + m_C)g > \mu m_A g, a_A = a_B = 0.$$

From conservation of energy principle we can prove that maximum distance moved by C or maximum extension in the spring would be :

$$X_m = \frac{2m_C g}{k} = \frac{2 \times 1 \times 10}{1000} = 0.02 \text{ m}$$

At maximum extension :



$$a_C = \frac{kX_m - m_C g}{m_C}$$

Substituting the values we have, $a_C = 10 \text{ m/s}^2$. **Ans.**

9. Rate at which kinetic plus gravitational potential energy is dissipated at time t is actually the magnitude of power of frictional force at time t.

$$|P_f| = f.v = (\mu mg \cos\alpha)(at)$$

$$= (\mu mg \cos\alpha)[(g \sin\alpha - \mu g \cos\alpha)t]$$

$$= \mu mg^2 \cos\alpha(\sin\alpha - \mu \cos\alpha)t$$ **Ans.**

10. From work-energy principle, $$W = \Delta KE$$

$\therefore$ $$Pt = \frac{1}{2}m(v^2 - u^2)$$ (P = power)

or $$t = \frac{m}{2P}(v^2 - u^2) \quad \text{...(i)}$$

Further

$$F \cdot v = P$$

$\therefore$ $$m \cdot \frac{dv}{ds} \cdot v^2 = P$$

or $$\int_u^v v^2\, dv = \frac{P}{m}\int_0^x ds$$

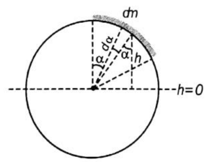
$\therefore$ $$(v^3 - u^3) = \frac{3P}{m} \cdot x$$

or $$\frac{m}{P} = \frac{3x}{v^3 - u^3}$$

Substituting in Eq. (i)

$$t = \frac{3x(u + v)}{2(u^2 + v^2 + uv)}$$ **Hence proved.**

11. (a) Mass per unit length $= \dfrac{m}{l}$



$$dm = \frac{m}{l}R\, d\alpha$$

$$h = R\cos\alpha$$

$$dU = (dm)\, gh = \frac{mgR^2}{l}\cos\alpha \cdot d\alpha$$

$\therefore$ $$U = \int_0^{l/R} dU = \frac{mgR^2}{l}\sin\left(\frac{l}{R}\right)$$

(b) $$\text{KE} = U_i - U_f$$

Here $$U_i = \frac{mgR^2}{l}\sin\left(\frac{l}{R}\right)$$

and $$U_f = \int_\theta^{l/R+\theta} dU = \frac{mgR^2}{l}\left[\sin\left(\frac{l}{R} + \theta\right) - \sin\theta\right]$$

$\therefore$ $$\text{KE} = \frac{mgR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]$$ **Ans.**

(c) $$\frac{1}{2}mv^2 = \frac{mgR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]$$

or $$v = \sqrt{\frac{2gR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]}$$

$$v^2 = \frac{2gR^2}{l}\left[\sin\left(\frac{l}{R}\right) + \sin\theta - \sin\left(\theta + \frac{l}{R}\right)\right]$$

$\therefore$ $$2v \cdot \frac{dv}{dt} = \frac{2gR^2}{l}\left[\cos\theta - \cos\left(\theta + \frac{l}{R}\right)\right] \cdot \frac{d\theta}{dt}$$

$$\frac{dv}{dt} = \frac{\frac{2gR^2}{l}\left[\cos\theta - \cos\left(\theta + \frac{l}{R}\right)\right]}{2v}\left(\frac{d\theta}{dt}\right) \quad \text{...(i)}$$

Here $$\frac{\left(\frac{d\theta}{dt}\right)}{v} = \frac{\omega}{v} = \frac{1}{R}$$

Substituting in Eq. (i) $$\frac{dv}{dt} = \frac{gR}{l}\left[\cos\theta - \cos\left(\theta + \frac{l}{R}\right)\right]$$

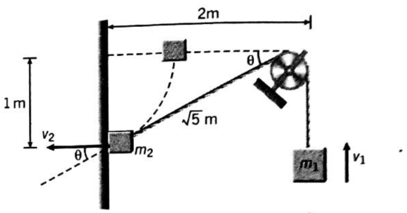
At $$t = 0,\ \theta = 0°$$

Hence, $$\frac{dv}{dt} = \frac{gR}{l}\left[1 - \cos\left(\frac{l}{R}\right)\right]$$ **Ans.**

12. From conservation of energy,

$$m_2 g h_2 = m_1 g h_1 + \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

Here, $$v_1 = v_2\cos\theta$$



$\therefore$ $$2 \times 10 \times 1 = (0.5)(10)(\sqrt{5} - 1) + \frac{1}{2} \times 0.5 \times v_2^2 \times \left(\frac{2}{\sqrt{5}}\right)^2 + \frac{1}{2} \times 2 \times v_2^2$$

$\therefore$ $$20 = 6.18 + 0.2\, v_2^2 + v_2^2$$

$$v_2 = 3.39 \text{ m/s}$$ **Ans.**

and $$v_1 = v_2 \cos\theta = \frac{2}{\sqrt{5}} \times 3.39$$

or $$v_1 = 3.03 \text{ m/s}$$ **Ans.**

13. Net retarding force $= kx + bMgx$

$\therefore$ $$\text{Net retardation} = \left(\frac{k + bMg}{M}\right) \cdot x$$

So, we can write $$v \cdot \frac{dv}{dx} = -\left(\frac{k + bMg}{M}\right) \cdot x$$

or $$\int_{v_0}^{0} v \cdot dv = -\left(\frac{k + bMg}{M}\right) \int_0^x x\, dx$$

or $$x = \sqrt{\frac{M}{k + bMg}}\, v_0$$

Loss in mechanical energy $$\Delta E = \frac{1}{2} M v_0^2 - \frac{1}{2} k x^2$$

or $$\Delta E = \frac{1}{2} M v_0^2 - \frac{k}{2}\left(\frac{M}{k + bMg}\right) v_0^2$$

or $$\Delta E = \frac{v_0^2}{2}\left[M - k\left(\frac{M}{k + bMg}\right)\right] = \frac{v_0^2 b M^2 g}{2(k + bMg)}$$ **Ans.**

14. From conservation of mechanical energy,

$$E_i = E_f$$

or $$\frac{1}{2} k x_i^2 + m g h_i = \frac{1}{2} m v_f^2$$

$\therefore$ $$v_f = \sqrt{2 g h_i + \frac{k}{m} x_i^2}$$

Substituting the values we have,

$$v_f = \sqrt{2 \times 9.8 \times 1.9 + \frac{2300}{0.12}(0.045)^2}$$

$$= 8.72 \text{ m/s}$$ **Ans.**

15. (a) From work energy theorem,
Work done by all forces = change in kinetic energy

$\therefore$ $$Fx - \frac{1}{2} k x^2 = \frac{1}{2} m v^2$$

$\therefore$ $$v = \sqrt{\frac{2Fx - kx^2}{m}}$$

Substituting the values we have,

$$v = \sqrt{\frac{2 \times 20 \times 0.25 - 40 \times 0.25 \times 0.25}{0.5}}$$

$$= \sqrt{15} \text{ m/s} = 3.87 \text{ m/s}$$ **Ans.**

(b) From conservation of mechanical energy,

$$E_i = E_f$$

or $$\frac{1}{2} m v_i^2 + \frac{1}{2} k x_i^2 = \frac{1}{2} k x_f^2$$

or $$x_f = \sqrt{\frac{m v_i^2}{k} + x_i^2}$$

$$= \sqrt{\frac{0.5 \times 15}{40} + (0.25)^2}$$

$$= 0.5 \text{ m (compression)}$$

$\therefore$ Distance of block from the wall $= (0.6 - 0.5)$ m $= 0.1$ m **Ans.**

Chapter 7 Circular Motion

Level 2

1. (a) Applying conservation of energy

$$mgh = \frac{1}{2} m (\sqrt{3Lg})^2$$

$\therefore$ $$h = \frac{3L}{2}$$ **Ans.**

(b) Since $\sqrt{3Lg}$ lies between $\sqrt{2Lg}$ and $\sqrt{5Lg}$, the string will slack in upper half of the circle. Assuming that string slacks when it makes an angle θ with horizontal. We have

$$mg \sin\theta = \frac{mv^2}{L} \quad \text{...(i)}$$

$$v^2 = (\sqrt{3gL})^2 - 2gL(1 + \sin\theta) \quad \text{...(ii)}$$

Solving Eq. (i) and (ii), we get

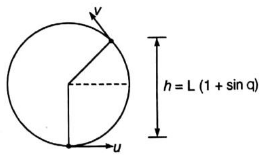
$$\sin\theta = \frac{1}{3} \quad \text{and} \quad v^2 = \frac{gL}{3}$$

Maximum height of the bob from starting point,

$$H = L(1 + \sin\theta) + \frac{v^2 \sin^2(90° - \theta)}{2g}$$

$$= \frac{4L}{3} + \left(\frac{gL}{6g}\right)\cos^2\theta = \frac{4L}{3} + \frac{4L}{27}$$

$$= \frac{40}{27}L$$ **Ans.**



Note *Maximum height in part (b) is less than that in part (a), think why?*

2. $$h = 0.8 \sin 30° = 0.4 \text{ m}$$

$\therefore$ $$v^2 = 2gh$$

(a) Just before,

$$T_1 - mg \sin 30° = \frac{mv^2}{R_1} \quad (R_1 = 0.8 \text{ m})$$

$$T_1 = \frac{mg}{2} + \frac{m(2g)(0.4)}{0.8} = \frac{3mg}{2}$$ **Ans.**

(b) Just after,

$$T_2 - mg \sin 30° = \frac{mv^2}{R_2} (R_2 = 0.4 \text{ m})$$

$$T_2 = \frac{mg}{2} + \frac{m(2g)(0.4)}{0.4}$$

or $$T_2 = \frac{5mg}{2}$$ **Ans.**

3. $h = l(1 - \cos\theta)$

$$v^2 = v_0^2 - 2gh = 3gl - 2gl(1 - \cos\theta) = gl(1 + 2\cos\theta)$$

At 45° means radial and tangential components of acceleration are equal.

$\therefore$ $$\frac{v^2}{l} = g \sin\theta$$

or $$1 + 2\cos\theta = \sin\theta$$

Solving the equation we get, $\theta = 90°$ or $\frac{\pi}{2}$ **Ans.**

4. Banking angle, $\theta = \tan^{-1}\left(\frac{v^2}{Rg}\right)$

$$36 \frac{\text{km}}{\text{h}} = 10 \text{ m/s}$$

$\therefore$ $$\theta = \tan^{-1}\left(\frac{100}{20 \times 9.8}\right) = 27°$$

Angle of repose, $\theta_r = \tan^{-1}(\mu) = \tan^{-1}(0.4) = 21.8°$

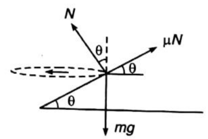
Since $\theta > \theta_r$, vehicle can not remain in the given position with $v = 0$. At rest it will slide down. To find minimum speed, so that vehicle does not slip down, maximum friction will act up the plane. To find maximum speed, so that the vehicle does not skid up, maximum friction will act down the plane.

Minimum Speed :

Equation of motion are,

$$N\cos\theta + \mu N \sin\theta = mg \quad \text{...(i)}$$

$$N\sin\theta - \mu N\cos\theta = \frac{m}{R}v_{min}^2 \quad \text{...(ii)}$$



Solving these two equations we get

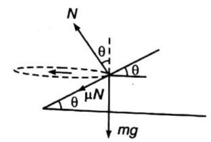
$$v_{min} = 4.2 \text{ m/s}$$ **Ans.**

Maximum speed :

Equations of motion are,

$$N\cos\theta - \mu N\sin\theta = mg \quad \text{...(iii)}$$

$$N\sin\theta + \mu N\cos\theta = \frac{m}{R}v_{max}^2 \quad \text{...(iv)}$$



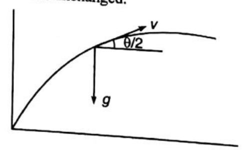
Solving these two equations, we have,

$$v_{max} = 15 \text{ m/s}$$ **Ans.**

5. Let v be the velocity at that instant. Then, horizontal component of velocity remains unchanged.

$\therefore$ $$v\cos\frac{\theta}{2} = u\cos\theta$$

or $$v = \frac{u\cos\theta}{\cos\frac{\theta}{2}}$$



Tangential component of acceleration of this instant will be,

$$a_t = g\cos(\pi/2 + \theta/2) = -g\sin\theta/2$$

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{g^2 - g^2\sin^2\frac{\theta}{2}} = g\cos\frac{\theta}{2}$$

Since, $a_n = \frac{v^2}{R}$

or $R = \frac{v^2}{a_n} = \frac{\left(\frac{u\cos\theta}{\cos\frac{\theta}{2}}\right)^2}{g\cos\frac{\theta}{2}} = \frac{u^2\cos^2\theta}{g\cos^3\left(\frac{\theta}{2}\right)}$ **Ans.**

6. After 1 sec : $\vec{v} = \vec{u} + \vec{a}t = 20\hat{i} + 10\hat{j}$, $v = \sqrt{500}$ m/s $= 10\sqrt{5}$ m/s

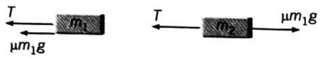
$$\vec{a} = -10\hat{j}$$

$$a_t = a\cos\theta = \frac{\vec{a}.\vec{v}}{v} = \frac{-100}{10\sqrt{5}} = -2\sqrt{5}\ \text{m/s}^2$$ **Ans.**

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{(10)^2 - (2\sqrt{5})^3} = \sqrt{80}\ \text{m/s}^2 = 4\sqrt{5}\ \text{m/s}^2$$ **Ans.**

$$R = \frac{v^2}{a_n} = \frac{(10\sqrt{5})^2}{4\sqrt{5}} = 25\sqrt{5}\ \text{m.}$$ **Ans.**

7. (a) Force diagrams of m_1 and m_2 are as shown below:



(Only horizontal forces have been shown)
Equations of motion are :

$$T + \mu m_1 g = m_1 R\omega^2 \quad \text{...(i)}$$
$$T - \mu m_1 g = m_2 R\omega^2 \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we have

$$\omega = \sqrt{\frac{2m_1 \mu g}{(m_1 - m_2)R}}$$

Substituting the values, we have

$$\omega_{min} = 6.32\ \text{rad/s}$$ **Ans.**

(b) $T = m_2 R\omega^2 + \mu m_1 g$

$= (1)(0.5)(6.32)^2 + (0.5)(2)(10)$

≈ 30 N **Ans.**

8. Speed of bob in the given position,

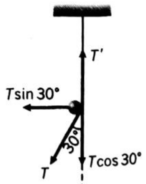
$$v = \sqrt{2gh}$$

Here, $h = (400 + 400\cos 30°)$ mm

$= 746$ mm $= 0.746$ m

∴ $v = \sqrt{2 \times 9.8 \times 0.746} = 3.82$ m/s

Now $T - mg\cos\theta = \frac{mv^2}{r}$



$$T = 2 \times 9.8 \times \cos 30° + \frac{2 \times (3.82)^2}{(0.4)}$$

or $T = 90$ N

∴ $R = T\sin 30° = 45$ N

$T' = T\cos 30°$ **Ans.**

9. Speed of each particle at angle θ is,

$v = \sqrt{2gh}$ (from energy conservation)

where $h = R(1 - \cos\theta)$

∴ $v = \sqrt{2gR(1 - \cos\theta)}$

$N + mg\cos\theta = \frac{mv^2}{R}$

or $N + mg\cos\theta = 2mg(1 - \cos\theta)$

or $N = 2mg - 3mg\cos\theta$ ...(i)

The tube breaks its contact with ground when $2N\cos\theta > Mg$

Substituting, $2N\cos\theta = Mg$

or $4mg\cos\theta - 6mg\cos^2\theta = Mg$

Substituting, $\theta = 60°$

$2mg - \frac{3mg}{2} = Mg$

or $\frac{M}{m} = \frac{1}{2}$ **Ans.**

Note *Initially normal reaction on each ball will be radially outward and later it will be radially inward, so that normal reactions on tube is radially outward to break it off from the ground.*

10. At distance x from centre,
Centrifugal force $= mx\omega^2$
∴ acceleration $a = x\omega^2$

or $v.\frac{dv}{dx} = x\omega^2$

or $\int_0^v v\,dv = \omega^2 \int_a^L x\,dx$

or $\frac{v^2}{2} = \frac{\omega^2}{2}(L^2 - a^2)$

or $v = \omega\sqrt{L^2 - a^2}$ **Ans**

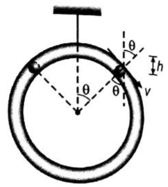
11. $N = \frac{mv^2}{R}$

$f_{max} = \mu N = \frac{\mu m v^2}{R}$ ∴ Retardation $a = \frac{f_{max}}{m} = \frac{\mu v^2}{R}$

∴ $\left(-\frac{dv}{dt}\right) = \frac{\mu v^2}{R}$ or $\int_{v_0}^{v}\frac{dv}{v^2} = -\frac{\mu}{R}\int_0^t dt$

or $v = \frac{v_0}{1 + \frac{\mu v_0 t}{R}}$ **Ans.**

12. Let R be the radius of the ring



$$h = R(1 - \cos\theta)$$
$$v^2 = 2gh = 2gR(1 - \cos\theta)$$
$$\frac{mv^2}{R} = N + mg\cos\theta$$

or $$N = 2mg(1 - \cos\theta) - mg\cos\theta$$
$$N = 2mg - 3mg\cos\theta$$

In the critical condition, tension in the string is zero and net upward force on the ring :

$$F = 2N\cos\theta = 2mg(2\cos\theta - 3\cos^2\theta) \quad \ldots(i)$$

F is maximum when $\frac{dF}{d\theta} = 0$

or $$-2\sin\theta + 6\sin\theta\cos\theta = 0$$

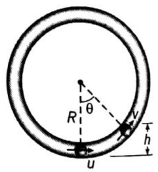
or $$\cos\theta = \frac{1}{3}$$

Substituting in Eq. (i) $$F_{max} = 2mg\left(2 \times \frac{1}{3} - 3 \times \frac{1}{9}\right) = \frac{2}{3}mg$$
$$F_{max} > Mg$$

or $$\frac{2}{3}mg > Mg$$

or $$m > \frac{3}{2}M$$ **Proved.**

13. Minimum velocity of particle at the lowest position to complete the circle should be $\sqrt{4gR}$ inside a tube.



So, $$u = \sqrt{4gR}$$
$$h = R(1 - \cos\theta)$$

$\therefore$ $$v^2 = u^2 - 2gh$$

or $$v^2 = 4gR - 2gR(1 - \cos\theta)$$
$$= 2gR(1 + \cos\theta)$$

or $$v^2 = 2gR\left(2\cos^2\frac{\theta}{2}\right)$$

or $$v = 2\sqrt{gR}\cos\frac{\theta}{2}$$

From $$ds = v \cdot dt$$

We get $$R\,d\theta = 2\sqrt{gR}\cos\frac{\theta}{2} \cdot dt$$

or $$\int_0^t dt = \frac{1}{2}\sqrt{\frac{R}{g}}\int_0^{\pi/2}\sec\left(\frac{\theta}{2}\right)d\theta \quad \text{or} \quad t = \sqrt{\frac{R}{g}}\left[\ln\left(\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right)\right]_0^{\pi/2}$$

or $$t = \sqrt{\frac{R}{g}}\ln(1 + \sqrt{2})$$ **Proved.**

14 At position θ,

$$v^2 = v_0^2 + 2gh$$

where $$h = a(1 - \cos\theta)$$

$\therefore$ $$v^2 = (\sqrt{2ag})^2 + 2ag(1 - \cos\theta)$$

or $$v^2 = 2ag(2 - \cos\theta) \quad \ldots(i)$$
$$N + mg\cos\theta = \frac{mv^2}{a}$$

or $$N + mg\cos\theta = 2mg(2 - \cos\theta)$$

or $$N = mg(4 - 3\cos\theta)$$

Net vertical force, $$F = N\cos\theta + mg = mg(4\cos\theta - 3\cos^2\theta + 1)$$

This force (or acceleration) will be maximum when $\frac{dF}{d\theta} = 0$

or $$-4\sin\theta + 6\sin\theta\cos\theta = 0$$

So, either $$\sin\theta = 0, \quad \theta = 0°$$

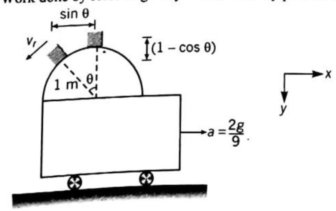
or $$\cos\theta = \frac{2}{3}, \quad \theta = \cos^{-1}\left(\frac{2}{3}\right)$$
$$\theta = 0° \text{ is unacceptable}$$

Therefore, the desired position is at $$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$ **Ans.**

15. (a) Let v_r be the velocity of mass relative to track at angular position θ.

From work energy theorem, KE of particle relative to track:

= Work done by force of gravity + work done by pseudo force



$\therefore$ $$\frac{1}{2}mv_r^2 = mg(1 - \cos\theta) + m\left(\frac{2g}{9}\right)\sin\theta$$

or $$v_r^2 = 2g(1 - \cos\theta) + \frac{4g}{9}\sin\theta \quad \ldots(i)$$

Particle leaves contact with the track where $N = 0$

or $$mg\cos\theta - m\left(\frac{2g}{9}\right)\sin\theta = mv_r^2$$

or $$g\cos\theta - \frac{2g}{9}\sin\theta = 2g(1 - \cos\theta) + \frac{4g}{9}\sin\theta$$

or $$3 \cos \theta - \frac{6}{9} \sin \theta = 2$$

Solving this, we get $$\theta \approx 37°$$ **Ans.**

(b) From Eq. (i), $$v_r = \sqrt{2g(1 - \cos \theta) + \frac{4g}{9} \sin \theta}$$

or $$v_r = 2.58 \text{ m/s at } \theta = 37°$$

Vertical component of its velocity is $$v_y = v_r \sin \theta = 2.58 \times \frac{3}{5}$$

$$= 1.55 \text{ m/s}$$

Now, $$1.3 = 1.55t + 5t^2 \qquad \left(s = ut + \frac{1}{2} gt^2\right)$$

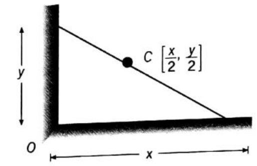
or $$5t^2 + 1.55t - 1.3 = 0$$

or $$t = 0.38 \text{ s}$$ **Ans.**

Chapter 8 Centre of Mass, Conservation of Linear Momentum, Impulse and Collision

Level 2

1. $$y = \sqrt{L^2 - x^2}$$



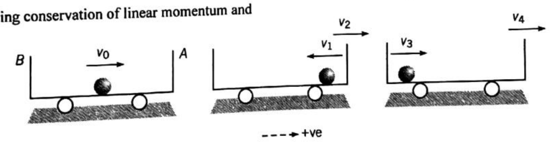
$\Rightarrow$ $$\frac{dy}{dt} = -\frac{x}{\sqrt{L^2 - x^2}} \frac{dx}{dt}$$

$$= -\frac{3 \times 2}{4} = -\frac{3}{2} \text{ m/s}$$

$$v_{CM} = \sqrt{\left(\frac{1}{2}\frac{dx}{dt}\right)^2 + \left(\frac{1}{2}\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(1)^2 + \left(\frac{3}{4}\right)^2} = 1.25 \text{ m/s}$$ **Ans.**

2. Applying conservation of linear momentum and



$$e = \frac{\text{relative speed of separation}}{\text{relative speed of approach}},$$ we get

$$mv_0 = Mv_2 - mv_1 \quad \ldots\text{(i)}$$

$$v_1 + v_2 = ev_0 \quad \ldots\text{(ii)}$$

Solving these two equations, we get, $$v_1 = \left(\frac{eM - m}{M + m}\right) v_0, \quad v_2 = m\left(\frac{e + 1}{M + m}\right) v_0$$ **Ans.**

The desired time is: $$t = \frac{d}{v_0} + \frac{2d}{ev_0} + \frac{d}{e^2 v_0}$$

or $$t = \frac{d}{v_0}\left(1 + \frac{2}{e} + \frac{1}{e^2}\right)$$ **Ans.**

3. (i) $x_1 = v_0 t - A(1 - \cos \omega t)$ $$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = v_0 t$$

$\therefore$ $$x_2 = v_0 t + \frac{m_1}{m_2} A(1 - \cos \omega t)$$

(ii) $$a_1 = \frac{d^2 x_1}{dt^2} = -\omega^2 A \cos \omega t$$

The separation $x_2 - x_1$ between the two blocks will be equal to l_0 when $a_1 = 0$ or $\cos \omega t = 0$

$$x_2 - x_1 = \frac{m_1}{m_2} A(1 - \cos \omega t) + A(1 - \cos \omega t)$$

or
$$l_0 = \left(\frac{m_1}{m_2} + 1\right) A \qquad (\cos \omega t = 0)$$

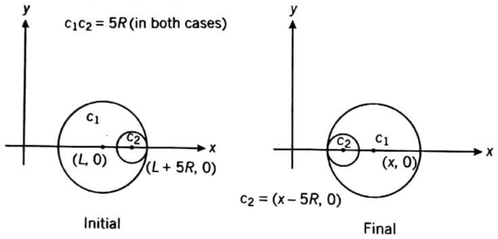
Thus, the relation between l_0 and A is,

$$l_0 = \left(\frac{m_1}{m_2} + 1\right) A$$

Ans.

4. Since, all the surfaces are smooth, no external force is acting on the system in horizontal direction. Therefore, the centre of mass of the system in horizontal direction remains stationary.

x-coordinate of COM initially will given by



$$x_i = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$= \frac{(4M)(L) + M(L + 5R)}{4M + M} = (L + R) \qquad \text{...(i)}$$

Let $(x, 0)$ be the coordinates of the centre of large sphere in final position. Then, x-coordinate of COM finally will be

$$x_f = \frac{(4M)(x) + M(x - 5R)}{4M + M} = (x - R) \qquad \text{...(ii)}$$

Equating Eqs. (i) and (ii), we have

$$x = L + 2R$$

Therefore, coordinates of large sphere, when the smaller sphere reaches the other extreme position are $(L + 2R, 0)$.

Ans.

5. (a) Chain has a constant speed. Therefore, net force on it should be zero. Thus,

$$P = \text{Weight of length } y \text{ of chain} + \text{thrust force}$$

$$= \frac{m}{l} yg + \rho v_0^2 \qquad \left(\text{here } \rho = \frac{m}{l}\right)$$

$$= \frac{m}{l}(gy + v_0^2)$$

Ans.

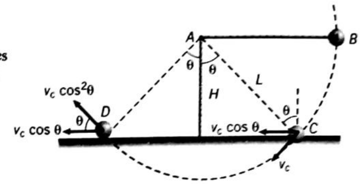
(b) Energy lost during the lifting = work done by applied force – increase in mechanical energy of chain

$$= \int_0^y P \cdot dy - \left(\frac{m}{l} y\right) g \left(\frac{y}{2}\right) - \frac{1}{2}\left(\frac{m}{l} \cdot y\right) v_0^2$$

$$= \frac{myv_0^2}{2l}$$

Ans.

6. In perfectly inelastic collision with the horizontal surface the component parallel to the surface will remain unchanged. Similarly when the string becomes taut again, the component perpendicular to its length will remain unchanged.



$$\cos \theta = \frac{H}{L}$$

$$v_c = \sqrt{2gH}$$

$$v_c \cos^2 \theta = (\sqrt{2gH}) \frac{H^2}{L^2} = v \text{ (say)}$$

$$\therefore \quad h = \frac{v^2}{2g} = \frac{(2gH)\frac{H^4}{L^4}}{2g} = \frac{H^5}{L^4}$$

Ans.

7. Applying conservation of linear momentum at the time of collision, or at $t = 1$ s,

$$m\vec{\mathbf{v}} + m(0) = 2m(20\hat{\mathbf{i}} + 10\hat{\mathbf{j}})$$

$$\therefore \quad \vec{\mathbf{v}} = 40\hat{\mathbf{i}} + 20\hat{\mathbf{j}}$$

At 1 sec, masses will be at height :

$$h_1 = u_y t + \frac{1}{2} v_y t^2 = (20)(1) + \frac{1}{2}(-10)(1)^2 = 15 \text{ m}$$

After explosion other mass will further rise to a height :

$$h_2 = \frac{u_y^2}{2g} = \frac{(20)^2}{2 \times 10} = 20 \text{ m} : u_y = 20 \text{ m/s just after collisiion.}$$

$\therefore$ Total height $h = h_1 + h_2 = 35$ m

Ans.

8. Let CT stands for common tangent direction and CN for common normal directions.

	Mass $m = eM$		Mass M	
	CT	**CN**	**CT**	**CN**
Befor collision	v_1 (let)	v_2 (let)	Zero (given)	Zero (given)
After collision	v_1	v_3 (suppose)	Zero	v_4 (suppose)

In the common tangent directions velocity components remain unchanged.

In common normal direction applying conservation of linear momentum and definition of e.

$$eMv_2 = eMv_3 + Mv_4 \qquad \text{...(i)}$$

From the definition of coefficient of restitution :

$$e = \frac{v_4 - v_3}{v_2} \qquad \text{...(ii)}$$

Solving these two equations we get,

$$v_3 = 0 \text{ but } v_4 \neq 0$$

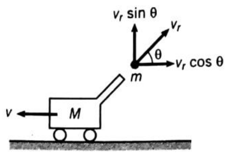
So, after collision velocity of m is along CT while that of M along CN or they are moving at right angles.

9. Muzzle velocity v_r is given to be constant

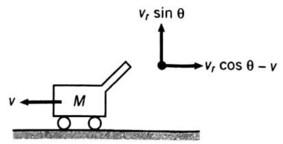
From conservation of linear momentum in horizontal direction we have,

$$Mv = m(v_r \cos \theta - v)$$

or
$$v = \frac{mv_r \cos \theta}{M + m} \qquad \text{...(i)}$$



Components of velocity of bullet with respect to gun



Components of velocity of bullet with respect to ground

Further, range of bullet on horizontal ground

$$R = \frac{2v_r \sin\theta}{g}(v_r \cos\theta - v)$$

$$= \frac{2v_r \sin\theta}{g}\left(v_r \cos\theta - \frac{mv_r \cos\theta}{M+m}\right)$$

$$= \frac{2Mv_r^2 \sin\theta \cos\theta}{(M+m)g}$$

or

$$R = \left(\frac{M}{M+m}\right)\frac{v_r^2 \sin 2\theta}{g} \qquad \text{...(ii)}$$

(a) From Eq. (ii) we see that maximum range is at $\theta = 45°$ **Ans.**

(b) At $\theta = 45°$,

$$R_{max} = \left(\frac{M}{M+m}\right)\frac{v_r^2}{g}$$ **Ans.**

10. (a) $u_r = 0, \quad a_r = g$

$\therefore$ $$v_r = \sqrt{2gh_1}$$

After collision relative velocity $v_r' = e\sqrt{2gh_1}$ and relative retardation is still g (downwards). Hence,

$$h_2 = \frac{(v_r')^2}{2g} = e^2 h_1$$ **Ans.**

(b) $$u_r = 0, \quad a_r = g + \frac{g}{4} = \frac{5g}{4}$$

$\therefore$ Just before collision $v_r = \sqrt{2\left(\frac{5g}{4}\right)h_1}$

Just after collision $v_r' = ev_r$.

Relative retardation is still $\frac{5g}{4}$.

Hence, $$h_2 = \frac{(v_r')^2}{2\left(\frac{5g}{4}\right)} = e^2 h_1$$ **Ans.**

11. Let the velocity of the block and the plank, when the block leaves the spring be u and v respectively.

By conservation of energy $\frac{1}{2}kx^2 = \frac{1}{2}mu^2 + \frac{1}{2}Mv^2$

[M = mass of the plank, m = mass of the block]

$\therefore$ $$100 = u^2 + 5v^2 \qquad \text{...(i)}$$

By conservation of momentum

$$mu + Mv = 0$$

$\Rightarrow$ $$u = -5v \qquad \text{...(ii)}$$

Solving Eqs. (i) and (ii)

$$30v^2 = 100$$

$\therefore$ $$v = \sqrt{\frac{10}{3}} \text{ m/s}$$

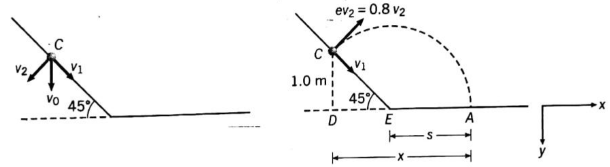
From this moment until block falls, both plank and block keep their velocity constant.

Thus, when block falls velocity of plank $= \sqrt{\frac{10}{3}}$ m/s. **Ans.**

12. $v_0 = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 1.5} = 5.42$ m/s

Component of velocity parallel and perpendicular to plane at the time of collision.

$$v_1 = v_2 = \frac{v_0}{\sqrt{2}} = 3.83 \text{ m/s}$$



Component parallel to plane (v_1) remains unchanged, while component perpendicular to plane becomes ev_2, where

$$ev_2 = 0.8 \times 3.83 = 3.0 \text{ m/s}$$

$\therefore$ Component of velocity in horizontal direction after collision

$$v_x = \frac{(v_1 + ev_2)}{\sqrt{2}} = \frac{(3.83 + 3.0)}{\sqrt{2}} = 4.83 \text{ m/s}$$

While component of velocity in vertical direction after collision.

$$v_y = \frac{v_1 - ev_2}{\sqrt{2}} = \frac{3.83 - 3.0}{\sqrt{2}} = 0.59 \text{ m/s}$$

Let t be the time, the particle takes from point C to A, then

$$1.0 = 0.59t + \frac{1}{2} \times 9.8 \times t^2$$

Solving this we get,

$$t = 0.4 \text{ s} \qquad \text{(Positive value)}$$

$\therefore$ $$DA = v_x t = (4.83)(0.4) = 1.93 \text{ m}$$

$\therefore$ $$S = DA - DE$$

$$= 1.93 - 1.0$$

$$S = 0.93 \text{m}$$ **Ans.**

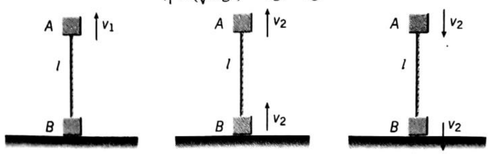
$$v_{yA} = v_{yc} + gt = (0.59) + (9.8)(0.4) = 4.51 \text{ m/s}$$

$$v_{xA} = v_{xC} = 4.83 \text{ m/s}$$

$\therefore$ $$v_A = \sqrt{(v_{xA})^2 + (v_{yA})^2} = 6.6 \text{ m/s}$$ **Ans.**

13. String becomes tight when A moves upwards by a distance l. Let v_1 be the velocity of A at this moment, then

$$v_1^2 = (\sqrt{10gl})^2 - 2gl = 8gl$$



or
$$v_1 = \sqrt{8gl}$$

Let v_2 be the common velocities of both A and B just after string becomes tight. Then from conservation of linear momentum.

$$v_2 = \frac{v_1}{2} = \frac{\sqrt{8gl}}{2}$$

Both particles return to their original height with same speed v_2. String becomes loose after B strikes the ground and the speed v with which A strikes the ground is,

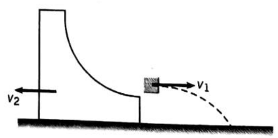
$$v^2 = v_2^2 + 2gl = \frac{8gl}{4} + 2gl$$

or
$$v^2 = 4gl$$

or
$$v = 2\sqrt{gl}$$ **Ans.**

14.
$$mv_1 = Mv_2 \qquad \text{...(i)}$$

$$mgR = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2 \qquad \text{...(ii)}$$



$$t = \sqrt{\frac{2(R/2)}{g}} = \sqrt{\frac{R}{g}} \qquad \text{...(iii)}$$

The desired distance is
$$S = (v_1 + v_2)t \qquad \text{...(iv)}$$

Solving Eqs. (i) and (ii) for v_1 and v_2 and substituting in Eq. (iv), we get

$$S = R\sqrt{\frac{2(M+m)}{M}}$$ **Ans.**

15. Let v_r be the velocity of washer relative to centre of hoop and v the velocity of centre of hoop. Applying conservation of linear momentum and mechanical energy we have,

$$m(v_r \cos\phi - v) = Mv \qquad \text{...(i)}$$

$$mgr(1 + \cos\phi) = \frac{1}{2}Mv^2 + \frac{1}{2}m(v_r^2 + v^2 - 2vv_r \cos\phi) \qquad \text{...(ii)}$$

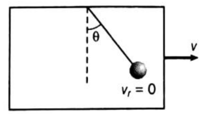
Solving Eqs. (i) and (ii), we have,
$$v = m\cos\phi\sqrt{\frac{2gr(1+\cos\phi)}{(M+m)(M+m\sin^2\phi)}}$$ **Ans.**

16. $H = \dfrac{u^2\sin^2\theta}{2g} = \dfrac{(20)^2 \times \frac{3}{4}}{2\times 10} = 15\text{ m}$

i.e., the shell strikes the ball at highest point of its trajectory. Velocity of (ball + shell) just after collision,

$$v = \frac{u\cos 60°}{2} \qquad \text{(from conservation of linear momentum)}$$

$$= \frac{20}{2\times 2} = 5\text{ m/s}$$



At highest point combined mass is at rest relative to the trolley. Let v be the velocity of trolley at this instant. From conservation of linear momentum we have,

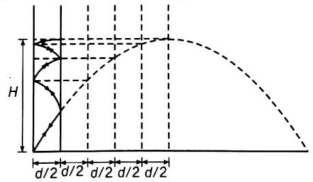
$$2\times 5 = \left(2 + \frac{4}{3}\right)v \quad \text{or} \quad v = 3\text{ m/s}$$

From conservation of energy, we have

$$\frac{1}{2}\times 2\times(5)^2 - \frac{1}{2}\left(2+\frac{4}{3}\right)(3)^2 = 2\times 10(1-\cos\theta)$$

solving we get $\cos\theta = \dfrac{1}{2}$ $\quad\therefore\ \theta = 60°$ **Ans.**

17. While colliding with the wall its vertical component (v_y) of velocity will remain unchanged (component along common tangent direction remains unchanged) while horizontal component (v_x) is reversed remaining same in magnitude. Thus, path of the particle will be as shown in figure.



(a) $H = \dfrac{u^2\sin^2\alpha}{2g}$

(b) Total number of collisions with the walls before the ball comes back to the ground are nine.

(c) Ball will return to point O (the starting point)

18. As the collisions are perfectly elastic, collision of the ball will not affect the vertical component of its velocity while the horizontal component will be simply reversed.

Hence,
$$H_{max} = \frac{v_y^2}{2g} = \frac{[20\times\sin 45°]^2}{2\times 10} = 10\text{ m}$$

Total time of flight

$$T = \frac{2v_y}{g} = \frac{2\times 20\times\left(\frac{1}{\sqrt{2}}\right)}{10} = 2\sqrt{2}\text{ s}$$

Total horizontal distance travelled before striking the ground $x = v_xT = 40$ m

$$PB + BA + AB + BA + AB = 45\text{ m}$$

Hence, total number of collision suffered by the particle with the walls before it hits ground = 4. **Ans.**

19. Let v_1 = velocity of block 2 kg just before collision
v_2 = velocity of block 2 kg just after collision
and v_3 = velocity of block M just after collision.

Applying work energy theorem

(change in kinetic energy = work done by all the forces) at different stages as shown in figure.

Figure 1.

$$\Delta KE = W_{\text{friction}} + W_{\text{gravity}}$$

$$\left[\frac{1}{2}m\{v_1^2 - (10)^2\}\right] = -6\mu\, mg\cos\theta - mgh_1 \qquad (m = 2\text{ kg})$$

or $$v_1^2 - 100 = 2[6\mu g\cos\theta + gh_1]$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-(0.05)^2} \approx 0.99$$

$\therefore$ $$v_1^2 = 100 - 2[(6)(0.25)(10)(0.99) + (10)(0.3)]$$

$\Rightarrow$ $$v_1 \approx 8\text{ m/s}$$

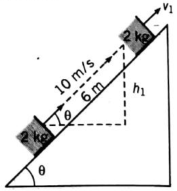


Fig. 1

Figure 2.

$$\Delta KE = W_{\text{friction}} + W_{\text{gravity}}$$

$$\frac{1}{2}m[(1)^2 - (v_2^2)] = -6\mu\, mg\cos\theta + mgh_1$$

or $$1 - v_2^2 = 2[-6\mu\, g\cos\theta + gh_1]$$

$$= 2[(-6)(0.25)(10)(0.99) + (10)(0.3)]$$

$$= -23.7$$

$\therefore$ $$v_2^2 = 24.7 \quad \text{or} \quad v_2 \approx 5\text{ m/s}$$

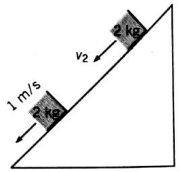


Fig. 2

Figure 3.

$$\Delta KE = W_{\text{friction}} + W_{\text{gravity}}$$

$$\frac{1}{2}M[0 - v_3^2] = -(0.5)(\mu)(M)\,g\cos\theta - Mgh_2$$

or $$-v_3^2 = -\mu\, g\cos\theta - 2gh_2$$

or $$v_3^2 = (0.25)(10)(0.99) + 2(10)(0.025)$$

or $$v_3^2 = 2.975$$

$\therefore$ $$v_3 \approx 1.72\text{ m/s}$$

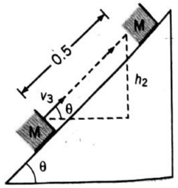


Fig. 3

Now

(i) $$\text{Coefficient of restitution} = \frac{\text{Relative velocity of separation}}{\text{Relative velocity of approach}}$$

$$= \frac{v_2 + v_3}{v_1}$$

$$= \frac{5 + 1.72}{8} = \frac{6.72}{8}$$

or $$e \approx 0.84$$

(ii) Applying conservation of linear momentum before and after collision

$$2v_1 = Mv_3 - 2v_2$$

$\therefore$ $$M = \frac{2(v_1 + v_2)}{v_3} = \frac{2(8+5)}{1.72} = \frac{26}{1.72}$$

$$M \approx 15.12\text{ kg}$$ **Ans.**

20. Let v_r be the relative velocity of block as it leaves contact with the sphere ($N = 0$) and v the horizontal velocity of sphere at this instant.

Absolute components of velocity of block

Applying conservation of linear momentum in horizontal direction, we get

$$mv = m(v_r\cos\theta - v)$$

or $$2v = v_r\cos\theta \qquad \text{...(i)}$$

Conservation of mechanical energy gives,

$$mgr(1-\cos\theta) = \frac{1}{2}mv^2 + \frac{1}{2}m(v_r^2 + v^2 - 2vv_r\cos\theta)$$

or $$gr(1-\cos\theta) = v^2 + \frac{v_r^2}{2} - vv_r\cos\theta \qquad \text{...(ii)}$$

Equation of laws of motion gives,

$$mg\cos\theta = \frac{mv_r^2}{r}$$

or $$gr = \frac{v_r^2}{\cos\theta} \qquad \text{...(iii)}$$

Solving Eqs. (i), (ii) and (iii), we get

$$\cos^3\theta - 6\cos\theta + 4 = 0$$ **Ans.**

Note *We have not considered pseudo force while writing the equation of motion. Think why?*

21. $$\frac{x}{u\cos\alpha} + \frac{x}{eu\cos\alpha} = T = \frac{2u\sin\alpha}{g}$$

or $$x = \frac{eu^2\sin 2\alpha}{(1+e)g}$$

$$x_{\max} = \frac{eu^2}{(1+e)g} \quad \text{at} \quad 2\alpha = 90^\circ$$ **Ans.**