ROTATIONAL DYNAMICS

Equation of Rotational Motion :

Derivation of first equation of motion

Angular Acceleration $(\alpha) = \frac{d\omega}{dt}$

$\therefore \quad d\omega = \alpha dt$

At $t=0$ let $\omega = \omega_0$ and at $t=t$, let $\omega = \omega$

on integrating both side

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha dt = \alpha \int_0^t dt$$

$$[\omega]_{\omega_0}^{\omega} = \alpha [t]_0^t$$

$$\therefore \quad \omega - \omega_0 = \alpha (t-0)$$

$$\therefore \quad \boxed{\omega = \omega_0 + \alpha t}$$

Derivation of second equation of motion.

Angular velocity $(\omega) = \frac{d\theta}{dt}$

$$d\theta = \omega dt$$

At $t=0$, let $\theta = 0$ and at $t=t$ let $\theta = \theta$

On integrating both side

$$\int_0^{\theta} d\theta = \int_0^t \omega dt = \int_0^t (\omega_0 + \alpha t) dt \qquad [As \ \omega = \omega_0 + \alpha t]$$

$$= \omega_0 \int_0^t dt + \alpha \int_0^t t \, dt$$

$$\therefore \quad [\theta]_0^{\theta} = \omega_0 [t]_0^t + \alpha \left[\frac{t^2}{2}\right]_0^t$$

$$\boxed{\theta = \omega_0 t + \frac{1}{2}\alpha t^2}$$

Derivation of third equation of motion.

Angular acceleration $(\alpha) = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \cdot \frac{d\theta}{dt} = \frac{d\omega}{d\theta} \cdot \omega$

or $\quad \omega d\omega = \alpha d\theta$

At $t=0$, $\theta = 0$ and $\omega = \omega_0$ and

at $t=t$, $\theta = \theta$ and $\omega = \omega$

on integrating both side

$$\int_{\omega_0}^{\omega} \omega d\omega = \int_0^{\theta} \alpha d\theta = \alpha \int_0^{\theta} d\theta$$

$$\left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega} = \alpha [\theta]_0^{\theta}$$

$$\frac{\omega^2}{2} - \frac{\omega_0^2}{2} = \alpha(\theta - 0)$$

$$\boxed{\omega^2 = \omega_0^2 + 2\alpha\theta}$$

Rotational Motion of a rigid body

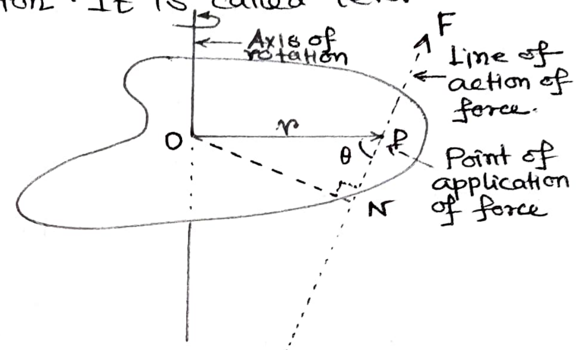
A body is said to possess rotational motion if all its particles move along circles in parallel planes. The centres of these circles lie on a fixed line perpendicular to the parallel planes and this line is called the axis of rotation.

Moment of force or torque

The turning effect of force is called moment of force or torque. It depends on two factors:

(i) the magnitude of the force

(ii) the perpendicular distance of the line of action of the force from the axis of rotation. It is called lever arm or moment arm.



$$\text{Torque } (\tau) = \text{Force} \times \text{moment arm}$$
$$= F \times ON = F \times OP \sin\theta$$
$$= Fr\sin\theta$$
$$= rF\sin\theta$$
$$\therefore \quad \vec{\tau} = \vec{r} \times \vec{F}$$

The direction of $\vec{\tau}$ is perpendicular to the plane containing Vector $\vec{r}$ and $\vec{F}$ and it is determined by right hand rule.

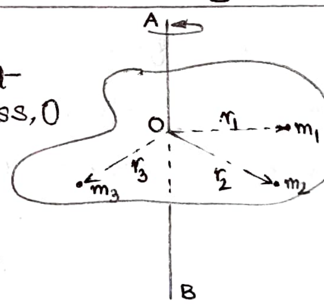
Special case

(i) when $\theta = 0^\circ$ or 180° $\quad \therefore \sin\theta = 0 \quad \therefore \vec{\tau}_{min} = 0$

(ii) when $\theta = \pi/2 \quad \sin \pi/2 = 1 \quad \vec{\tau}_{max} = Fr$

Torque acting on a rigid body: Relation between torque and angular acceleration:

Let us consider a rigid body rotating about a fixed axis AB through its centre of mass, O as shown in figure under the effect of external torque (τ).



Let $m_1, m_2, m_3, \dots$ = Masses of various particles consisting the body.

$r_1, r_2, r_3 \dots$ = perpendicular distance of these particles from the axis of rotation.

$a_1, a_2, a_3 \dots$ = Linear accelerations of the various particles

α = uniform angular acceleration, then

$$a_1 = r_1\alpha, \quad a_2 = r_2\alpha, \quad \dots \quad a_n = r_n\alpha$$

$\therefore$ Force acting on particles

$$F_1 = m_1a_1 = m_1r_1\alpha, \quad F_2 = m_2a_2 = m_2r_2\alpha \quad \dots$$

$\therefore$ Torque on the particles

$$\tau_1 = m_1r_1^2\alpha, \quad \tau_2 = m_2r_2^2\alpha, \quad \tau_3 = m_3r_3^2\alpha \quad \dots$$

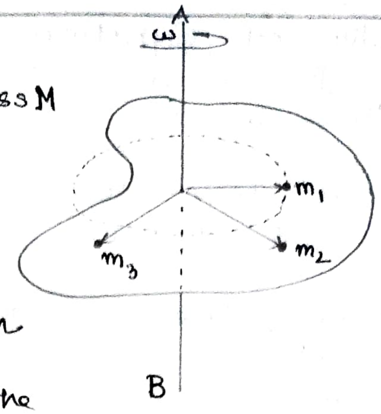
$\therefore$ Sum of the torques about the fixed axis

$$\tau = (m_1r_1^2\alpha + m_2r_2^2\alpha + m_3r_3^2\alpha + \dots m_nr_n^2\alpha) = \left(\sum m_ir_i^2\right)\alpha$$

$\tau = I\alpha \quad$ where $I = \sum m_ir_i^2$ is the moment of Inertia.

Kinetic Energy of rotation

Let us consider a rigid body of mass M rotating about a fixed axis AB.



$m_1, m_2, m_3 \dots m_n$ = Masses of the various particles.

$r_1, r_2, r_3 \dots r_n$ = Perpendicular distances of these particles from the axis of rotation.

$v_1, v_2, v_3 \dots v_n$ = Linear velocities of the various particles

ω = Uniform angular velocity

$\therefore$ $v_1 = r_1\omega$, $v_2 = r_2\omega$, $v_3 = r_3\omega$ ----

$\therefore$ Kinetic energy of the particle of mass $m_1 = \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1r_1^2\omega^2$

Similarly, Kinetic energy of the other particles are

$\frac{1}{2}m_2r_2^2\omega^2$, $\frac{1}{2}m_3r_3^2\omega^2$, ----- $\frac{1}{2}m_nr_n^2\omega^2$

$\therefore$ The sum of the Kinetic energy of all the particles

$$K = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \dots + \frac{1}{2}m_nr_n^2\omega^2$$

$$= \frac{1}{2}(m_1r_1^2 + m_2r_2^2 + \dots + m_nr_n^2)\omega^2$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n} m_ir_i^2\right)\omega^2 = I\omega^2$$

where $I = \sum_{i=1}^{n} m_ir_i^2$ is the moment of inertia of the body about the given axis.

$\therefore$ $K = \frac{1}{2}I\omega^2$.

Moment of Inertia and its physical significance

The property of a body by virtue of which it opposes the torque tending to change its state of rest or of uniform rotation about a given axis, is called moment of inertia or rotational inertia of the body about that axis.

The moment of inertia of a body about a given axis play the same role in rotational motion about that axis as the mass of a body does in translational motion. The moment of inertia of a body about a given axis depends upon:

(i) its mass

(ii) position and direction of the axis of rotation

(iii) Shape of the body.

$\therefore$ Moment of inertia of a body about a given axis is equal to the sum of the product of the masses of the constituent particles and square of their respective perpendicular distances from the axis of rotation.

$\therefore$

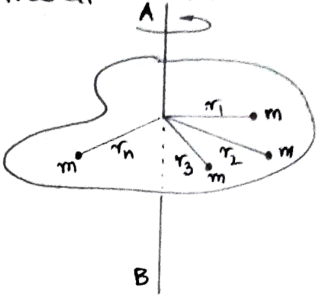
$$I = m_1r_1^2 + m_2r_2^2 + \dots + m_nr_n^2$$

$$I = \sum_{i=1}^{n} m_ir_i^2 \quad \text{[For discreat mass distribution]}$$

$$I = \int dm\, r^2 \quad \text{[for contineous mass distribution]}$$

Radius of Gyration

The radius of gyration of a body about a given axis is the distance of a point from the axis of rotation where if the whole of the mass of the body were concentrated, it would have the same moment of inertia as it has with the actual distribution of mass.



Moment of Inertia of the body about the axis AB.

$$I = mr_1^2 + mr_2^2 + \cdots + mr_n^2$$

$$= m\left(r_1^2 + r_2^2 + \cdots + r_n^2\right)$$

$$= mn\left(\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}\right)$$

$$I = M\left(\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}\right)$$

where mn = Mass of the body (M)

$\therefore$ $I = MK^2$

where $K = \sqrt{\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}}$ = Root mean square distance

The radius of gyration of a body about a given axis of rotation is thus the root mean square of the perpendicular distances of its constituent particles from the axis of rotation.

Its S.I unit is metre (m).

Angular momentum of a particle

The moment of momentum of a particle about an axis of rotation is called its angular momentum and is a measure of twisting or turning effect associated with the momentum of the particle.

Angular momentum of a moving particle about a point is thus defined as

$$\vec{L} = \vec{r} \times \vec{p}, \quad \therefore \; L = rp\sin\theta.$$

Where $\vec{p}$ is the linear momentum of the particle and $\vec{r}$ is its position vector from that point.

Its S.I unit $kg\,m^2 s^{-1}$

- If $r = 0$, $L = 0$. particle has no angular momentum about the origin.
- If $\theta = 0^\circ$ or 180°, $\sin\theta = 0$ and $L = 0$. It follows that L is zero when $\vec{r}$ is parallel or antiparallel to $\vec{p}$.
- If $\theta = 90^\circ$, $\sin\theta = 1$ and $L = rp = mvr$ = maximum