Homework 2.6

1(a) $y'' + y = \sec(\theta)\tan(\theta)$

$m^2 + 1 = 0$

$m = \pm i$

$y_c = C_1\cos(\theta) + C_2\sin(\theta)$ , $y_1 = \cos(\theta)$ $y_2 = \sin(\theta)$

$y_1' = -\sin(\theta)$ $y_2' = \cos(\theta)$

now, $-\sin(\theta)\cdot u_1' + \cos(\theta)\cdot u_2' = \sec(\theta)\tan(\theta)$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \sec(\theta)\tan(\theta) \end{pmatrix}$$

$\therefore W = \det\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \cos^2\theta + \sin^2\theta = 1$

$W_1 = \det\begin{pmatrix} 0 & \sin\theta \\ \sec\theta\tan\theta & \cos\theta \end{pmatrix} = -\sin\theta\cdot\frac{\sin\theta}{\cos^2\theta} = \frac{-\sin^2\theta}{\cos^2\theta}$

$W_2 = \begin{pmatrix} \cos\theta & 0 \\ -\sin\theta & \sec\theta\tan\theta \end{pmatrix} = \frac{\cos\theta\cdot\sin\theta}{\cos^2\theta} = \frac{\sin\theta}{\cos\theta}$

$\int u_1' = \int \frac{-\sin^2\theta}{\cos^2\theta}$ $\int u_2' = \frac{\sin\theta}{\cos\theta}$

$u_1 = \tan(\theta) - \theta + C$ $u_2 = -\ln|\cos(\theta)| + C$

$\therefore y_p = (\tan(\theta) - \theta)\cos\theta - \ln|\cos(\theta)|\cdot\sin(\theta)$

$\therefore y = C_1\cos(\theta) + C_2\sin(\theta) + (\tan(\theta) - \theta)\cos\theta - \ln|\cos(\theta)|\cdot\sin\theta$

b $4y'' - y = e^{\frac{x}{2}} + 6$.

$4m^2 - 1 = 0$

$(2m-1)(2m+1) = 0$

$m = \pm\frac{1}{2}$.

$y_c = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}$

$y_1 = e^{\frac{1}{2}x}$ $\quad y_2 = e^{-\frac{1}{2}x}$

$y_1' = \frac{1}{2}e^{\frac{1}{2}x}$ $\quad y_2' = -\frac{1}{2}e^{-\frac{1}{2}x}$

now, $\frac{1}{2}e^{\frac{1}{2}x}u_1' + \frac{1}{2}e^{-\frac{1}{2}x}u_2' = e^{\frac{x}{2}} + 6$.

$$\begin{pmatrix} e^{\frac{x}{2}} & e^{-\frac{x}{2}} \\ \frac{1}{2}e^{\frac{x}{2}} & -\frac{1}{2}e^{-\frac{x}{2}} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ e^{\frac{x}{2}} + 6 \end{pmatrix}.$$

$$W = \det\begin{pmatrix} e^{\frac{x}{2}} & e^{-\frac{x}{2}} \\ \frac{1}{2}e^{\frac{x}{2}} & -\frac{1}{2}e^{-\frac{x}{2}} \end{pmatrix} = -\frac{1}{2} - \left(\frac{1}{2}\right) = -1.$$

$$W_1 = \det\begin{pmatrix} 0 & e^{-\frac{x}{2}} \\ e^{\frac{x}{2}} + 6 & -\frac{1}{2}e^{-\frac{x}{2}} \end{pmatrix} = -\left(1 + 6e^{-\frac{x}{2}}\right)$$

$$W_2 = \det\begin{pmatrix} e^{\frac{x}{2}} & 0 \\ \frac{1}{2}e^{\frac{x}{2}} & e^{\frac{x}{2}} + 6 \end{pmatrix} = e^{x} + 6e^{\frac{x}{2}}.$$

$\int u_1' = \int 1 + 6e^{-\frac{x}{2}}$ $\qquad \int u_2' = \int e^{x} - 6e^{\frac{x}{2}}$

$u_1 = x + 12e^{-\frac{x}{2}}$ $\qquad u_2 = -e^{x} - 12e^{\frac{x}{2}}$.

$y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} + \left(x + 12e^{-\frac{x}{2}}\right)e^{\frac{x}{2}} - \left(e^{x} - 12e^{\frac{x}{2}}\right)e^{-\frac{x}{2}}$

$= c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} + xe^{\frac{x}{2}} - e^{\frac{3x}{2}}$.

(c) $2y'' + y' - y = x + 7$

$2m^2 + m - 1 = 0$

$(2m-1)(m+1) = 0 \quad , m = \frac{1}{2}, -1$

$\therefore y_c = c_1 e^{\frac{x}{2}} + c_2 e^{-x}$ , $y_1 = e^{\frac{x}{2}}$ $\quad y_2 = e^{-x}$

$y_1' = \frac{1}{2}e^{\frac{x}{2}}$ $\quad y_2' = -e^{-x}$

now, $\begin{pmatrix} e^{\frac{x}{2}} & e^{-x} \\ \frac{1}{2}e^{\frac{x}{2}} & -e^{-x} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ x+7 \end{pmatrix}$

$W = \det \begin{pmatrix} e^{\frac{x}{2}} & e^{-x} \\ \frac{1}{2}e^{\frac{x}{2}} & -e^{-x} \end{pmatrix} = -e^{-\frac{3x}{2}} - \left(\frac{1}{2}e^{-\frac{3x}{2}}\right) = \frac{3}{2}e^{-\frac{3x}{2}}.$

$W_1 = \begin{pmatrix} 0 & e^{-x} \\ x+7 & -e^{-x} \end{pmatrix} = -e^{-x}(x+7)$

$W_2 = \begin{pmatrix} e^{\frac{x}{2}} & 0 \\ \frac{1}{2}e^{\frac{x}{2}} & x+7 \end{pmatrix} = e^{\frac{x}{2}}(x+7)$

$\int u_1' = \int \frac{-e^{-x}(x+7)}{\frac{3}{2}e^{-\frac{3x}{2}}} = \int -\frac{3}{2} e^{\frac{x}{2}}(x+7) = -\frac{3}{2}\int 7e^{\frac{x}{2}} + xe^{\frac{x}{2}}$

$= -\frac{3}{2}\left(14e^{\frac{x}{2}} + 2e^{\frac{x}{2}}x - 4e^{\frac{x}{2}}\right) + C$

$= -\frac{3}{2}\left(10e^{\frac{x}{2}} + 2xe^{\frac{x}{2}}\right) + C$

$\int u_2' = \int \frac{e^{\frac{x}{2}}(x+7)}{\frac{3}{2}e^{-\frac{3x}{2}}} = \int \frac{3}{2}e^{2x}(x+7)$

$= \frac{3}{2}\int xe^{2x} + 7e^{2x}$

$= \frac{3}{2}\left(\frac{1}{2}xe^{2x} + \frac{13}{4}e^{2x}\right) + C$

$y = c_1 e^{\frac{x}{2}} + c_2 e^{-x} + \left(15e^{\frac{x}{2}} + 3xe^{\frac{x}{2}}\right)e^{\frac{x}{2}} + \left(\frac{3}{4}e^{2x} + \frac{39}{8}e^{2x}\right)e^{-x}$