A Polish insurance company is facing some rough economical times. The board of the company wants to get more insight in how initial capital or safety loading affect the probability that the company goes bankrupt. The board needs your help to get insight into the risk they are running and how things can be done better.

Historical data, provided by the insurance company, indicates that we may assume that the claim sizes follow a uniform distribution with mean 16 thousand and variance 12 million PLZ, and claims arrive according to a Poisson process with intensity of 4 per day. We shall next be interested in ruin probabilities:

- Use Monte Carlo simulation to investigate the influence of the premium c and the initial capital u on the ruin probability $\psi(u)$.

Instead of using simulation, the company board would like to see an explicit expression for the ruin probability. They wondered whether it would be a good idea to approximate the ruin probability by the same measure in the setting as if the claim sizes were Erlang-2 distributed with the same mean.

- Determine the distribution of the maximal aggregate loss in case of Erlang-2 distributed claim sizes with mean 16 thousand PLZ, and give an explicit expression for the ruin probability $\psi_{E2}(u)$.
- Compare the ruin probabilities $\psi_{E2}(u)$ and $\psi(u)$ for different values of c and u. Would $\psi_{E2}(u)$ serve as a good approximation for $\psi(u)$, and why?

We next aim for other approximations that may give more insight. For this, we shall use two approximations that are described at the end of this document (see page 3).

- Determine the quality of the approximations (i) and (ii) by comparing the approximations to the ruin probabilities estimated by Monte Carlo simulation under the influence of premium c and u. Note that in order to use approximation (i), we need to determine the adjustment coefficient R that corresponds to the particular situation of the insurance company with uniform distributed claims, and claims arriving according to a Poisson process.

In reality, the arrival process of claims might not be a Poisson process. Assume that the inter-claim times are still independent identically distributed random variables with mean $1/4$ day, but assume that they are not exponentially distributed.

- Use Monte Carlo simulation to investigate the influence of the *distribution* of the inter-claim times on the ruin probability $\psi(u)$. You are free to choose several probability distributions, but we ask you to always include the gamma(4, 16) and the gamma(0.2, 0.8) distributions.

Because of the potential bankruptcy, the insurance company wants to make an agreement with a local bank to receive a temporary loan at the time of ruin. Naturally, whether or not the bank provides the loan depends on the (expected) amount of the loan and the duration of repaying the loan. We next aim to provide more insight in these questions. Let $U(t)$ be the capital at time t, and let T be the moment of first ruin, if ruin occurs. The *capital at ruin* $U(T)$ is the capital at time T; the *deficit* at ruin is $D = -U(T)$. In case of a loan, the loan amount is exactly D PLZ, and the process $U(t)$ continues for $t > T$. At some time $t > T$, $U(t)$ may become positive again. Let S denote the *instant of recovery* and is defined as the first time

that the capital becomes positive again after the moment of ruin. We denote the *recovery time* by R, and it is defined as the duration the capital is negative (i.e. $R = S - T$). Figure 1 depicts a graphical representation of the process $U(t)$, the deficit at ruin, the instant of recovery and the recovery time. However, it may also happen that for some time t between T and S, $U(t) < U(T)$. Each time this happens, the bank immediately needs to provide a new loan of exactly $-U(t) - D$ (such that the total loan amount is always the maximum deficit before recovery).

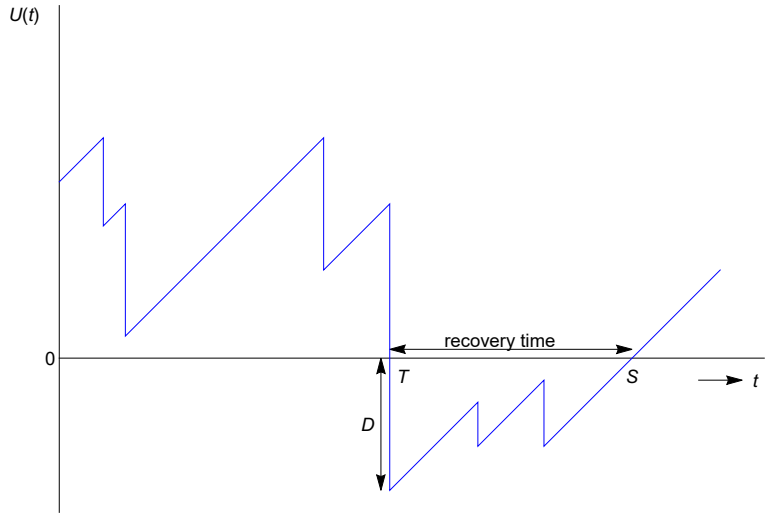

Figure 1: The capital $U(t)$, deficit at ruin D, and recovery time R and the instant of recovery S.

- Use Monte Carlo simulation to investigate the influence of the premium c and the initial capital u on the deficit at ruin (D) and the recovery time (R). What is the relation between the mean deficit at ruin and the mean recovery time?
- Use Monte Carlo simulation to investigate the influence of the c and u on the probability that, in case the bank provides a loan, the bank also has to provide a second loan before the insurance company recovers. That is, determine the influence of c and u on $P(U(t) < U(T)$ for some t in $[T, S])$.

The assignment will be 20% of the final grade of the course 2DF30. The assignment can be made in groups of three. Each group should hand in a well-written report and the source code of their simulation programs. The report should contain a clear description of the problems, and extensive answers to the questions in such a way that the insurance company should be able to understand the information that is given. Tables and figures might help. We enforce a page limit of **10 pages** for your report. Please include your source code as an attachment. More detailed guidelines can be found in Canvas.

References

[1] Kaas R., Goovaerts M., Dhaene J., Denuit M., *Actuarial Risk Theory, 2nd edition*, Kluwer Academic Publishers, 2009.

Approximations of $\psi(\approxeq)$

In this section we will describe two approximations of the probability of ultimate ruin $\psi(u)$. They correspond, in this order, to approximations 1 and 4 in Section 4.9 of [1].

(i) Approximate $\psi(u)$ by $\frac{1}{1+\theta}\mathrm{e}^{-Ru}$.

(ii) Approximate $\psi(u)$ by a function of the form $\frac{1}{1+\theta}\mathrm{e}^{-ku}$.

We will first establish some notation. Denote the k-th moment of the claim size distribution by μ_k, that is, $\mu_k := \mathbb{E}X^k$. We know that $\psi(0) = 1/(1+\theta)$. We know that $\psi(u) = \mathbb{P}(L > u) = 1 - F_L(u)$, where L is the maximal aggregate loss. The first two moments of L_i follow from Corollary 4.7.2 in [1] and are given by

$$\mathbb{E}L_i = \frac{\mu_2}{2\mu_1}, \qquad \mathbb{E}L_i^2 = \frac{\mu_3}{3\mu_1}. \tag{1}$$

Since

$$L = L_1 + L_2 + ... + L_M \tag{2}$$

is a geometric sum, and $\mathbb{E}M = 1/\theta$, we find

$$\mathbb{E}L = \frac{\mu_2}{2\theta\mu_1}, \qquad \mathbb{E}L^2 = \frac{\mu_3}{3\theta\mu_1} + \frac{\mu_2^2}{2\theta^2\mu_1^2}. \tag{3}$$

Note that

$$\mathbb{E}L = \int_{u=0}^{\infty} uF_L'(u)\mathrm{d}u = \int_{u=0}^{\infty}(1 - F_L(u))\mathrm{d}u = \int_{u=0}^{\infty}\psi(u)\mathrm{d}u, \tag{4}$$

so

$$\int_{u=0}^{\infty}\psi(u)\mathrm{d}u = \frac{\mu_2}{2\theta\mu_1}. \tag{5}$$

In a similar way, one can show that

$$\int_{u=0}^{\infty} 2u\psi(u) = \mathbb{E}L^2. \tag{6}$$

Approximation (i)

For the approximation $\psi(u) \approx \psi(0)\mathrm{e}^{-Ru}$, we simply need to determine the adjustment coefficient R which is defined as the unique positive solution to

$$1 + (1+\theta)\mu_1 R = m_X(R), \tag{7}$$

where $m_X(t)$ is the moment generating function of the claim size distribution.

Approximation (ii)

In the approximation $\psi(u) \approx \psi(0)\mathrm{e}^{-ku}$, we can choose k such that Equation (5) is satisfied, or

$$\int_{u=0}^{\infty}\psi(0)\mathrm{e}^{-ku}\mathrm{d}u = \frac{\mu_2}{2\theta\mu_1}. \tag{8}$$

Now the left hand side of this equation is $(k(1+\theta))^{-1}$, from which it follows that $k = (2\theta\mu_1)/((1+\theta)\mu_2)$, and our approximation becomes

$$\psi(u) \approx \frac{1}{1+\theta} \exp\left(\frac{-2\theta\mu_1}{(1+\theta)\mu_2} u\right). \tag{9}$$

Note that this approximation is exact when the claim size distribution is exponential, since then $\mu_2 = 2\mu_1^2$ and we find

$$\frac{1}{1+\theta} \exp\left(\frac{-2\theta\mu_1}{(1+\theta)\mu_2} u\right) = \frac{1}{1+\theta} \exp\left(-\frac{\theta}{(1+\theta)\mu_1} u\right). \tag{10}$$