This assignment has been inspired by the paper "Modelling Credit Risk: The Loss Distribution Of A Loan Portfolio" by Guillermo Magnou, FRM (2018) and Section 3.3.3 of the lecture notes on "Contagion Models in Credit Risk" by Mark H.A. Davis (Imperial College London).

In this credit risk assignment, we ask you to compare several risk measures for three Dutch banks:

- The ANB AMOR Bank has a portfolio of 8000 mortgages. The exposures are all 1 k€. The loss-given-default of all mortgages is assumed to be 0.25. All mortgages have a probability of default (PD) of PD $= 0.01$.
- The NSN Bank has the following portfolio: 1000 mortgages of 1 k€, 10 mortgages of 10 k€, 4 mortgages of 50 k€, 2 mortgages of 100 k€, and 1 mortgage of 500 k€. The loss-given-default of all mortgages is assumed to be 0.5. All mortgages have a default probability of PD $= 0.02$.
- The ROBA Bank has a similar portfolio as the NSN Bank, but the default probabilities are different: 1000 mortgages of 1 k€ with PD $= 0.0275$, 10 mortgages of 10 k€ with PD $= 0.02$, 4 mortgages of 50 k€ with PD $= 0.0175$, 2 mortgages of 100 k€ with PD $= 0.015$, and 1 mortgage of 500 k€ with PD $= 0.008$. The loss-given-default of all mortgages is assumed to be 0.5.

Each bank uses an internal credit risk system to calculate the required economic capital which is based on a one factor model as described in Section 4.1.3 in the lecture notes on Credit Risk. Let n denote the size of the portfolio and let X_i be the random variable indicating whether obligor i defaults ($X_i = 1$) or not ($X_i = 0$):

$$X_i = \begin{cases} 1 & \text{if } \tilde{r}_i < \tilde{c}_i, \\ 0 & \text{if } \tilde{r}_i \geq \tilde{c}_i, \end{cases} \tag{1}$$

where $\tilde{c}_i$ is the critical threshold for obligor i, chosen such that $\mathbb{P}(X_i = 1) = \text{PD}_i$. The one factor model gives the following expression for $\tilde{r}_i$:

$$\tilde{r}_i = R\tilde{Y} + \sqrt{1 - R^2}\tilde{\epsilon}_i,$$

where $\tilde{Y}$ and $\tilde{\epsilon}_i$ are i.i.d. $\mathcal{N}(0, 1)$ random variables, for $i = 1, \ldots, n$, and R denotes the correlation coefficient. Assume that the correlation coefficient in this model is $R = 0.25$. The portfolio loss is defined as

$$\tilde{L}_n = \sum_{i=1}^{n} \mathrm{EAD}_i \cdot \mathrm{LGD}_i \cdot X_i, \tag{2}$$

where EAD_i and LGD_i denote the exposure and loss-given-default of obligor i, respectively.

Question 1: Data analysis. Perform a short (statistical) data analysis for the three portfolios, by generating many samples of the portfolio loss of each portfolio. For each portfolio:

- Make a histogram of the portfolio loss;
- Provide the average, minimum, and maximum of the sampled values for the portfolio loss;
- Compute the theoretical expected, minimum and maximum portfolio loss for all three portfolios and compare to the corresponding values from the sample.

Interpret your results. Note: these portfolios are clearly not based on realistic data. This means that your data analysis is probably much shorter than usual. For example, you do *not* have to fit probability distributions.

Question 2: Stochastic simulation of the one factor model. We define the total number of defaults S_n as

$$S_n = \sum_{i=1}^{n} X_i.$$

Use *stochastic simulation* to determine the probability distributions of S_n and $\tilde{L}_n$, the expected loss (EL), the unexpected loss (UL), the value-at-risk (VaR_α), the economic capital (EC_α), and the tail conditional expectation (TCE_α), for $\alpha = 0.99$, for all three portfolios. Describe the similarities and the differences between the portfolios.

Question 3: Stochastic simulation of the one factor model with Infectious Defaults. In this question we introduce a credit risk model where there is contagion between the obligors, i.e. default of one party either directly causes default of other parties or (more commonly) changes other parties' risk of default. In more detail, we use the contagion model by Davis and Lo [DL2001] where an obligor that defaults may cause (randomly) other obligors to default as well. This is a two-step process, where the first step is equivalent to the one factor model above: denote by $X_i^{(1)}$ the random variable indicating whether obligor i defaults *in the first step* or not, according to Equation (1).

The second step is to determine which obligors are "infected" by the obligors that went bankrupt in the first step. Denote by $X_i^{(2)}$ the random variable indicating whether obligor i defaults *in the second step*. Davis and Lo propose the following model. For $i, j = 1, \ldots, n$ and $i \neq j$, let Q_{ij} be i.i.d. Bernoulli random variables with $\mathbb{P}(Q_{ij} = 1) = q$. Then

$$X_i^{(2)} = 1 - \prod_{j \neq i} (1 - X_j^{(1)} Q_{ji}). \tag{3}$$

The interpretation of (3) is as follows. Suppose that obligor j defaults in step 1 (i.e. $X_j^{(1)} = 1$). Then this obligor will "infect" each of the other obligors, say obligor i, if $Q_{ji} = 1$, which happens with probability q. If either $X_j^{(1)} = 0$ or $Q_{ji} = 0$, obligor i will *not* be infected by obligor j in the second step. However, obligor i could be infected by any of the other defaulting obligors. Equation (3) indicates that an obligor does *not* default in step 2, if it has not been infected by *any* of the other $n-1$ obligors. Obligors that have been infected (possibly by multiple other obligors) go bankrupt in step 2. Any obligor can either default in step 1, or (by infection) in step 2. This is expressed in the random variable $X_i^{(1+2)}$:

$$X_i^{(1+2)} = X_i^{(1)} + (1 - X_i^{(1)}) X_i^{(2)}.$$

Use *stochastic simulation* to determine the probability distributions of S_n and $\tilde{L}_n$, the expected loss (EL), the unexpected loss (UL), the value-at-risk (VaR_α), the economic capital (EC_α), and the tail conditional expectation (TCE_α), for $\alpha = 0.99$, for all three portfolios – but now in the contagion model with $q = 0.001$. Compare the results with those in the original one factor model. Interpret your results.

The assignment will be 10% of the final grade of the course 2DF30. The assignment can be made in groups of three. Each group should hand in a well-written report and the source code of their simulation programs. The report should contain a clear description of the problems, and extensive answers to the questions of the management in such a way that the management should be able to understand the information that is given. Tables and figures might help. More detailed guidelines can be found in Canvas. **Please check the rubric carefully, because it might contain additional information regarding the grading of this assignment.** Upload your report in Canvas, which is **due by 23:59 on January 10, 2025**.

References

[DL2001] Davis, M. and Lo, V., *Infectious defaults.*, Quantitative Finance 1, 382-387, 2001.