22/10/24

APL-410 — 2022AM11223

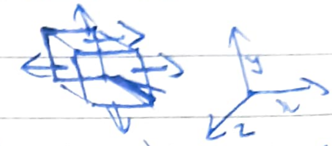
Assignment-1 — Sparsh Agarwal

(a) Write the weak form of the governing equation, the shape functions and the gaussian Integration scheme used.

The strong Form of Momentum conservation for solid body is

$$\sigma_{ij,j} + b_i = 0 \quad \text{(Index Notation)}$$

$$\nabla \cdot \sigma_{ij} + b_i = 0$$



To Formulate weak form multiply by a test function δv_i

$$\delta v_i \sigma_{ij,j} + \delta v_i b_i = 0$$

$$\Rightarrow \int_B (\delta v_i \sigma_{ij,j})\, dV + \int_B \delta v_i b_i \, dV = 0$$

Using Integration by Parts $\left[\int uv = u\int v - \int \left(u' \int v\right) dv \right]$

$$\Rightarrow \underbrace{\int_B (\delta v_i \sigma_{ij})_{,j}\, dV}_{(a)} - \int_B \delta v_{i,j} \sigma_{ij}\, dV + \int_B \delta v_i b_i \, dV = 0 \quad -(1)$$

Using divergence Theorem on (a)

$$\int_{\partial B} \delta v_i \sigma_{ij} n_j \, dA + \int_B \delta v_i b_i \, dV = \int_B \delta v_{i,j} \sigma_{ij}\, dV$$

$$\Rightarrow \int_B \delta v_{i,j} \sigma_{ij}\, dV = \int_B \delta v_i b_i \, dV + \int_{\partial B} \delta v_i t_i \, dA \quad -(2)$$

By defn of (1) & (2)

$$\boxed{\int_B \delta v_{i,j} \sigma_{ij}\, dV = \int_B \delta v_i b_i \, dV + \int_{\partial B_t} \delta v_i t_i \, dA}$$

Weak Form's derivation From Strong Form

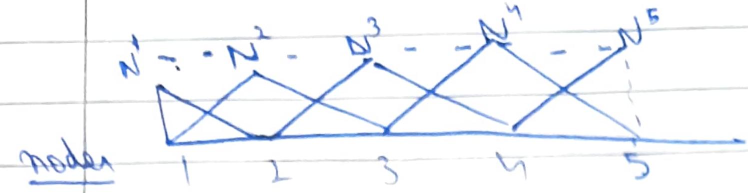
b_i are Body Forces (External) $\quad \sigma_{ij} \to$ stress

t_i are traction force (External)

→ Shape functions are used for discretization of the weak form

→ Shape function $N^a(\underline{x})$ corresponding to x^a is given by

$$N^a(\underline{x}) = \begin{cases} 1 & \text{if } \underline{x} = x^a \\ 0 & \text{if } \underline{x} = x^b \text{ where } b \neq a \text{ at some other node} \end{cases}$$



Linear Shape function

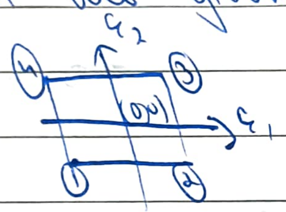
Shape functions are used to calculate displacement at any abitary point as :->

$$u_i(\underline{x}) = \sum_{a=1}^{n} N^a(\underline{x})\, u_i^a \quad ; \quad n \to \text{number of nodes}$$

$$\delta v_i(\underline{x}) = \sum_{a=1}^{n} N^a(\underline{x})\, \delta v_i^a$$

test function at node a

① For a 2-D mesh with 4 node points the shape functions are given by

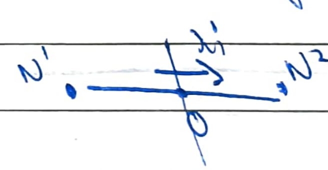


$$N^1 = \frac{(1-\xi_1)(1-\xi_2)}{4} \qquad N^4 = \frac{(1-\xi_1)(1+\xi_2)}{4}$$

$$N^2 = \frac{(1+\xi_1)(1-\xi_2)}{4}$$

$$N^3 = \frac{(1+\xi_1)(1+\xi_2)}{4}$$

② Linear shape functions were used for 1-D line elements also since the traction integral occurs on the boundary edge.

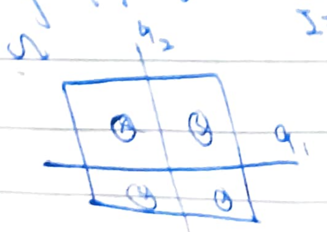


$$N^1 = 0.5(1-x_i)$$
$$N^2 = 0.5(1+x_i)$$

(#) Gaussian Integration Scheme

It is used to convert volume integral to discrete sum over integration points (Gaussian/ Quadrature points)

$$\int f(\xi)\, dV_\xi = \sum_{I=1}^{4} f(\xi^I)\, W_I \quad \{2\times 2 \text{ Integration scheme}\}$$



Gauss pt	ξ_1	ξ_2	W
1	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	1
2	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	1
3	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	1
4	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	1

(B) Write the expressions for the elemental stiffness matrix and force vector. Also mention how you have numbered the elements and nodes.

Sol^n

$$k_{aibk} = \int_B C_{ijkl} N^a_{,j} N^b_{,l} \, dV$$

$$F_i^a = \int_B b_i N^a dV + \int_{\partial B_t} t_i N^a dA \quad \{\text{Traction Boundary cond}^n\}$$

(#) Using Gaussian Integration scheme

$$k_{aibk} = \int C_{ijkl} \frac{\partial N^a(\xi^I)}{\partial \xi_p} \frac{\partial \xi_p}{\partial x_j} \frac{\partial N^b(\xi)}{\partial \xi_q} \frac{\partial \xi_q}{\partial x_l} J(\xi) \, dV_\xi$$

$$\Rightarrow \sum_{I=1}^{4} w_I C_{ijkl} \frac{\partial N^a(\xi^I)}{\partial \xi_p} \frac{\partial \xi_p}{\partial x_j} \frac{\partial N^b(\xi)}{\partial \xi_q} \frac{\partial \xi_q}{\partial x_l} J(\xi^I)$$

$$F_i^a = \int_B b_i N^a(\xi) J \, dV_\xi + \int_{\partial B_t} t_i N^a(\xi) J \, dA_\xi$$

$$\sum_{I=1}^{4} b_i w_I N^a(\xi^I) J(\xi^I) + \sum_{I=1}^{4} t_i N^a(\xi^I) w_I J(\xi^I)$$

where $J(\xi_I) = \det\left(\frac{\partial x_i}{\partial \xi_j}\right)\Big|_{\xi=\xi_I}$ $\qquad \frac{\partial \xi_j}{\partial x_i} = \left[\frac{\partial x_i}{\partial \xi_j}\right]^{-1}$

$$\frac{\partial x_i(\xi)}{\partial \xi_j} = \sum_{a=1}^{4} \frac{\partial N^a(\xi)}{\partial \xi_j} x_i^a$$

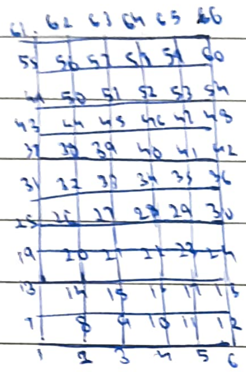
Python Script could be run in interactive mode to see generated plots

size of elemental stiffness Matrix = n dof_ele × n dof_ele = $\boxed{8\times8}$

elemental force = n_dof × 1 = $\boxed{8\times1}$

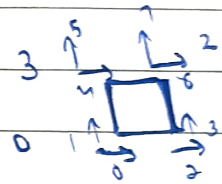
Global stiffness Matrix = ndof × ndof $\boxed{132\times132}$

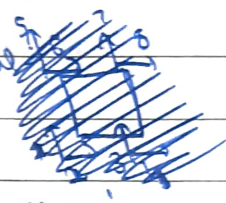
Global force vector ndof × 1 = $\boxed{132\times1}$



These are the node markings

For each node





Traction Fe[5] Fe[7] are updated therefore.

$u_x = 2i$

$v_y = 2i+1$

To determine displacement using FEM analysis we need to solve for KU=F where K is Global stiffness matrix obtained in separate function and later assembled using element connectivity the dof where u=0 is also put in ~~cond^n similar~~.

For calculation of K using gauss points the used shape functions are 2 dimensional linear whereas for fe the integral is line hence the shape functions used for traction at edge are univariate linear.

After K&F assembling for 2 t values 25 & 50 separately F is evaluated. K remains same using KU=F U is solved.

<u>Now</u> U_x & U_y for each node are extracted. And contour plots interpolating the the node displacement parameters are plotted.

(d) For stress strain calculation

For each element using the $\bar{u}(x,y) = \sum_{i=a} N^a(\xi)\, u^a$ we evaluated strain-displacement matrix B at gauss points yielding strain at gauss points and using linear isotropicity stress is also evaluated now avg of evaluated strain is used for making contour plots.

(Hooke's law)

The evaluation is done wrt (ξ_1, ξ_2) coordinates specific to element and $\det(J)$ and J^{-1} are separately evaluated and used as stated in continuum mechanics.

(e) Required plotting the load vs displacement curve and the stress vs strain. The average displacement was calculated from nodal displacement at the top edge of the plate over a range of time steps. The ~~total~~ applied force was computed by summing forces in the y direction across all nodes at the upper traction edge.

Stress vs strain was plotted a bit differently with evaluating avg strain using top edge strain during loading and stress was averaged over the area.

$$\sigma_{av} = \frac{\int \sigma \, dA}{A} \rightarrow \text{Gaussian Rule applied}$$

σ_{av}

ϵ_{av} $\rightarrow \frac{\Sigma u}{N}$ for all top edge nodes

(f) The $\frac{\text{Stress(av)}}{\text{Strain(av)}}$ ratios at different time instants were plotted depicting the material's stiffness during deformation. Overall this comprehensive analysis utilized FEM principles to evaluate mechanical responses.