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1. Recall the definition of a rational number, denoted as $\mathbb{Q}$. Prove that the Euler's number $e = \Sigma_{k=0}^{\infty} \frac{1}{k!} \notin \mathbb{Q}$. A factorial is defined as $k! = (k)(k-1)(k-2)(k-3)..., \forall k \in \mathbb{Z}^{+}$, note that $0! = 1$. Furthermore, a sum notation $\Sigma_{k=0}^{\infty} k = 0 + 1 + 2 + 3 + ... + ...$

   Suppose e is rational, such that $e = \frac{n}{d}$, where n and d are positive integers.

$$\frac{n}{d} = e = \sum_{k=0}^{\infty} \frac{1}{k!} \tag{1}$$

   Multiply both sides by $d!$, we get the following equation

$$\begin{aligned}
\frac{n}{d} d! &= d! \sum_{k=0}^{\infty} \frac{1}{k!} \\
n(d-1)! &= d! \left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{d!} + \frac{1}{(d+1)!} + \frac{1}{(d+2)!} + \frac{1}{(d+3)!} + \ldots \right] \\
&= \left(\frac{d!}{0!} + \frac{d!}{1!} + \frac{d!}{2!} + \cdots + \frac{d!}{d!} \right) + \left[\frac{d!}{(d+1)!} + \frac{d!}{(d+2)!} + \frac{d!}{(d+3)!} + \ldots \right] \\
&= \sum_{k=0}^{d} \frac{d!}{k!} + \sum_{k=d+1}^{\infty} \frac{d!}{k!}
\end{aligned} \tag{2}$$

   The term $n(d-1)!$ is clearly an integer. The first sum is also an integer, since $k \leq d$. The second sum we can simplify as follows

$$\frac{1}{d+1} + \frac{1}{(d+1)(d+2)} + \frac{1}{(d+1)(d+2)(d+3)} + \cdots \tag{3}$$

   The second sum is clearly greater than 0, and we can see (by considering respective terms) that

$$\left[\frac{1}{d+1} + \frac{1}{(d+1)(d+2)} + \frac{1}{(d+1)(d+2)(d+3)} + \ldots \right] < \sum_{k=0}^{\infty} \left(\frac{1}{d+1} \right)^{(k+1)} \tag{4}$$

   The right-hand side of the inequality is a geometric series, and we used the formula to show that the second sum is going to be less than 1

$$\begin{aligned}
\sum_{k=0}^{\infty} \left(\frac{1}{d+1} \right)^{(k+1)} &= \left[\frac{1}{d+1} + \frac{1}{(d+1)^2} + \frac{1}{(d+1)^3} + \ldots \right] \\
&= \frac{\frac{1}{d+1}}{1 - \frac{1}{d+1}} = \frac{1}{d} \leq 1
\end{aligned} \tag{5}$$

   With this logical statement, the second sum is greater than 0 and less than 1, which is not an integer. This is a contradiction, since $n(d-1)!$ is an integer, and the second sum is not an integer

$$n(d-1)! \neq \sum_{k=0}^{d} \frac{d!}{k!} + \sum_{k=d+1}^{\infty} \frac{d!}{k!} \tag{6}$$

   Therefore, e is irrational since $\mathbb{Z} = \mathbb{Z} + \overline{\mathbb{Z}}$ is simply impossible.

2. Prove Minkowski's Inequality for sums, $\forall (p > 1, (a_k, b_k) > 0)$:

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p \right)^{\frac{1}{p}} \tag{1}$$

To prove Minkowski's inequality, we can use Hölder's inequality.

(Hölder) if $p > 1$ and $q > 1$ are such that

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{2}$$

then for all x and $y \in \mathbb{R}^n$ we have

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}} \tag{3}$$

Proof of Minkowski Inequality. Observe that (1) holds for the case $p = 1$ by summing both sides of the simple triangle inequality.

$$|a_k + b_k| \leq |a_k| + |b_k| \tag{4}$$

We now concentrate on the case $p > 1$. For this case we show

$$\left(\sum_{k=1}^{n} (|a_k| + |b_k|)^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}} \tag{5}$$

so that the simple triangle inequality (4) will imply (1). For all a_k and b_k, we have

$$\begin{aligned} (|a_k| + |b_k|)^p &= (|a_k| + |b_k|)(|a_k| + |b_k|)^{p-1} \\ &= |a_k|(|a_k| + |b_k|)^{p-1} + |b_k|(|a_k| + |b_k|)^{p-1} \end{aligned} \tag{6}$$

Summing both sides of (6) we obtain

$$\sum_{k=1}^{n} (|a_k| + |b_k|)^p = \sum_{k=1}^{n} |a_k|(|a_k| + |b_k|)^{p-1} + \sum_{k=1}^{n} |b_k|(|a_k| + |b_k|)^{p-1} \tag{7}$$

Now, applying Hölder's inequality to both terms in the right-hand side of (7), we have

$$\sum_{k=1}^{n} |a_k|(|a_k| + |b_k|)^{p-1} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} [(|a_k| + |b_k|)^{p-1}]^q\right)^{\frac{1}{q}} \tag{8}$$

$$\sum_{k=1}^{n} |b_k|(|a_k| + |b_k|)^{p-1} \leq \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} [(|a_k| + |b_k|)^{p-1}]^q\right)^{\frac{1}{q}} \tag{9}$$

Thus, combining (7) with (8) and (9), we obtain

$$\begin{aligned} \sum_{k=1}^{n} (|a_k| + |b_k|)^p &\leq \left[\left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}}\right] \left(\sum_{k=1}^{n} (|a_k| + |b_k|)^{(p-1)q}\right)^{\frac{1}{q}} \\ &= \left[\left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}}\right] \left(\sum_{k=1}^{n} (|a_k| + |b_k|^p)\right)^{\frac{1}{q}} \end{aligned} \tag{10}$$

where we have used (2). Rearanging (10), we have

$$\frac{\sum_{k=1}^{n} (|a_k| + |b_k|)^p}{\left(\sum_{k=1}^{n} (|a_k| + |b_k|)^p\right)^{\frac{1}{q}}} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}} \tag{11}$$

Applying (2) once more we have (5) and so (1) follows.

3. Prove the triangle inequality $|x+y| \leq |x|+|y|, \forall(x, y) \in \mathbb{R}$.

   Case 1: Both x and y are nonnegative:

   - If $x \geq 0$ and $y \geq 0$, then $|x+y| = x+y$ and $|x|+|y| = x+y$. Since $x+y \leq x+y$, the inequality holds.

   Case 2: Both x and y are nonpositive:

   - If $x \leq 0$ and $y \leq 0$, then $|x+y| = -(x+y)$ and $|x|+|y| = (-x)+(-y)$. Since $-(x+y) \leq (-x)+(-y)$, the inequality holds.

   Case 3: One of x or y is nonnegative and the other is nonpositive:

   - If $x \geq 0$ and $y \leq 0$, then $|x+y| = |x-(-y)|$ and $|x|+|y| = x+(-y)$. Since $|x-(-y)| \leq x+(-y)$, the inequality holds.
   - This method also applies to $x \leq 0$ and $y \geq 0$.

   By examining all possible cases, we have shown that the triangle inequality holds $\forall(x, y) \in \mathbb{R}$. Therefore,the inequality $|x+y| \leq |x|+|y|$ is proven.

4. Prove Sedrakyan's Lemma $\forall u_i, v_i \in \mathbb{R}^+$:

$$\frac{\left(\sum_{i=1}^{n} u_i\right)^2}{\sum_{i=1}^{n} v_i} \leq \sum_{i=1}^{n} \frac{(u_i)^2}{v_i} \tag{1}$$

   To prove Sedrakyan's Lemma, we can use Cauchy-Schwarz Inequality.
   Let's denote $x_i = \sqrt{vi}$ and $y_i = u_i$. Then the given inequality becomes

$$\left(\frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} v_i}}\right)^2 \leq \sum_{i=1}^{n} \frac{(x_i)^2 (y_i)^2}{v_i} \tag{2}$$

   This is essentially the squared form of the Cauchy-Schwarz Inequality, which states

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \tag{3}$$

   where

$$a_i = x_i \sqrt{\frac{v_i}{\sum_{i=1}^{n} v_i}} \tag{4}$$

$$b_i = \frac{y_i}{\sqrt{\frac{v_i}{\sum_{i=1}^{n} v_i}}} \tag{5}$$

   By applying the Cauchy-Schwarz Inequality to **(4)** and **(5)**, we get

$$\left(\sum_{i=1}^{n} x_i y_i \sqrt{\frac{v_i}{\sum_{i=1}^{n} v_i}}\right)^2 \leq \left(\sum_{i=1}^{n} (x_i)^2 \frac{v_i}{\sum_{i=1}^{n} v_i}\right)\left(\sum_{i=1}^{n} (y_i)^2 \frac{1}{\frac{v_i}{\sum_{i=1}^{n} v_i}}\right) \tag{6}$$

   Simplify

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} (x_i)^2\right)\left(\sum_{i=1}^{n} (y_i)^2\right) \tag{7}$$

   Which is equivalent to

$$\left(\frac{\sum_{i=1}^{n} u_i}{\sqrt{\sum_{i=1}^{n} v_i}}\right)^2 \leq \sum_{i=1}^{n} \frac{(u_i)^2}{v_i} \tag{8}$$

   Hence, Sedrakyan's Lemma is proved.