NATIONAL TAIWAN NORMAL UNIVERSITY
Department of Computer Science and Information Engineering

Numerical Methods

Final Examination
Friday 01/14/2022

Instructions:

- When the exam begins, write your name on every page of this exam booklet.
- This exam contains 3 programming problems and 3 written problems, some with multiple parts. You have 90 minutes.
- This exam is closed book. No internet or mobile phones are permitted.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it.
- **Good luck!**

Part 1: Computing problems (50). Round off to the 4th decimal place if the answer is not an integer.

(a) Please find the line ($y = ax + b$) that fits the following four points (1, 6), (2, 5), (3, 7), (4, 10). **(20 pts)**

(b) Let $P(x)$ be the degree 5 polynomial that takes the value 10 at $x = 1, 2, 3, 4, 5$ and the value 15 at $x = 6$. Find $P(7)$. **(10 pts)**

(c) Apply Gram-Schmidt orthogonalization to find the reduced QR factorization of the matrix:

$$\begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 0 \\ -3 & 2 & 1 \\ 1 & 1 & 5 \\ -2 & 0 & 3 \end{bmatrix}$$

Report Q and R. **(20 pts)**

Part 2: Written problems (50)

Problem #1 (15). Please answer the following questions regarding *polynomial interpolation*, along with a brief explanation of your answer.

(a) Let $(x_1, y_1), \ldots, (x_n, y_n)$ be n points in the plane with distinct x_i. How many polynomials of degree $n - 1$ or less that satisfies $P(x_i) = y_i$ for $i = 1, \ldots, n$?

(b) Following the previous question, how many degree n polynomials pass through n points?

(c) Let P_1, P_2, P_3 and P_4 be four different points lying on a parabola $y = ax^2 + bx + c$. How many cubic (degree 3) polynomials pass through those four points?

Problem #2 (15). Suppose we are given a set of n data points (t_1, y_1), … (t_n, y_n). We choose the exponential model $y = c_1 \cdot e^{c_2 \cdot t}$ to fit the data points. Below are the steps to compute the coefficients of the model using the least squares approach. Fill the '?' parts in steps 3, 4, and 5.

1. Apply the natural logarithm to the model: $\ln y = \ln(c_1 \cdot e^{c_2 \cdot t}) = \ln c_1 + c_2 \cdot t$
2. Rename $k = \ln c_1$, and therefore, $\ln y = k + c_2 \cdot t$.
3. Construct a matric $\mathrm{A} = [?]$
4. Construct a vector $b = [?]$
5. Let $x = [?]$. Solve the system of normal equations $A^T A x = A^T b$. Get $c_1 = e^k$.

Problem #3 (20). In the last lecture we introduced the Gauss-Newton method for solving nonlinear least squares problems. In particular, we used this method to find a point for which the sum of the squared distances to three given circles is minimized. Please find the Jacobian matrix D_r needed for applying Gauss-Newton iteration to the following modified problems.

(a) Now we are given four circles, with centers (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) and radii R_1, R_2, R_3, and R_4, respectively. Write down D_r.

(b) Continuing (a), instead of looking for a point as described above, we now expand the circles' radii by a common amount until they mostly have a common intersection. Please refer to the figure below. In other words, we want to find the point (x, y) and a constant K for which the sum of the squared distances from the point to the four circles with radii increased by K is minimized. Write down D_r.