

Solving Nonlinear Systems of Equations

Mei-Chen Yeh

Announcement

- Next week: midterm exam
 - starts at 14:20
 - about 100 minutes
 - range:
 - solving equations involving one variable
 - solving systems involving multiple variables
 - computing and written problems
 - use codes you submitted on moodle
 - (optional) a A4 cheat sheet, a USB disk, a laptop

Today

- Solving nonlinear systems of equations

$$\begin{aligned}x^2 - 2xy + y^2 &= 3 \\x^2 + xy + y^2 &= 12\end{aligned}$$

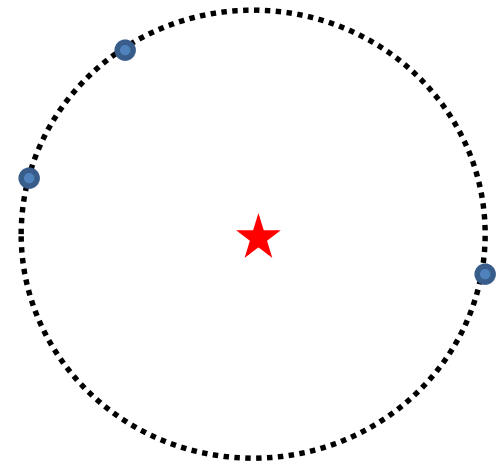
$$2u^2 - 4u + v^2 + 3w^2 + 6w + 2 = 0$$

$$u^2 + v^2 - 2v + 2w^2 - 5 = 0$$

$$3u^2 - 12u + v^2 + 3w^2 + 8 = 0$$

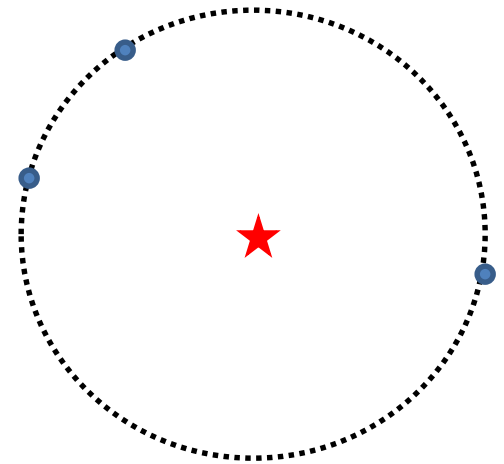
Today

- Solving nonlinear systems of equations
 - 代課老師小陳答應幫忙改100份數學考卷，題目為給定三個點，計算通過此三點的圓，求其圓心。但粗心的小陳忘記要參考答案！請你寫一個程式，輸入三個點(三點不會在同一直線上)，輸出通過此三點的圓之圓心位置。



Solving nonlinear systems

- 未知數?
 - x, y, R (圓心座標及半徑)
- 方程式?
 - 三個
 - 題目給的三個點到圓心的距離為 R
- Multivariate Newton's method



Review: One-variable Newton's method (HW#3)

Given a scalar differentiable function $f(x)$,

1. Start from an **initial guess** x_0

2. For $i = 0, 1, 2, \dots$ compute

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

until x_{i+1} satisfies **some termination criterion**

Extending to more variables

- Suppose we have 3 unknowns, 3 nonlinear equations:

$$f_1(u, v, w) = 0$$

$$f_2(u, v, w) = 0$$

$$f_3(u, v, w) = 0$$

$$2u^2 - 4u + v^2 + 3w^2 + 6w + 2 = 0$$

$$u^2 + v^2 - 2v + 2w^2 - 5 = 0$$

$$3u^2 - 12u + v^2 + 3w^2 + 8 = 0$$

- Define the vector-valued function:

$$F(\underline{x}) = F(u, v, w) = (f_1, f_2, f_3)$$

where

$$\underline{x} = (u, v, w).$$

Jacobian matrix

- 3 variables: $\underline{x} = (u, v, w)$
- 3 functions: f_1, f_2, f_3

$$D_F(\underline{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{bmatrix}$$

Example: 2 unknowns, 2 equations

$$\begin{aligned} v - u^3 &= 0 \\ u^2 + v^2 - 1 &= 0 \end{aligned}$$

$$\underline{x} = (u, v)$$

- $f_1 = v - u^3$
- $f_2 = u^2 + v^2 - 1$
- $\partial f_1 / \partial u = -3u^2$
- $\partial f_1 / \partial v = 1$
- $\partial f_2 / \partial u = 2u$
- $\partial f_2 / \partial v = 2v$

$$D_F(\underline{x}) = \begin{bmatrix} -3u^2 & 1 \\ 2u & 2v \end{bmatrix}$$

Multivariate Newton's method

- The Taylor expansion for $F(\underline{x})$ around \underline{x}_0 is

$$F(\underline{x}) = F(\underline{x}_0) + D_F(\underline{x}_0) \cdot (\underline{x} - \underline{x}_0) + O(\underline{x} - \underline{x}_0)^2.$$

- Let \underline{r} be the root, \underline{x}_0 be the current guess,

$$\underline{0} = F(\underline{r}) \approx F(\underline{x}_0) + D_F(\underline{x}_0) \cdot (\underline{r} - \underline{x}_0)$$

$$-F(\underline{x}_0) \approx D_F(\underline{x}_0) \cdot (\underline{r} - \underline{x}_0)$$

$$-D_F(\underline{x}_0)^{-1} F(\underline{x}_0) \approx (\underline{r} - \underline{x}_0)$$



destination



current guess

Multivariate Newton's method: Algorithm

\underline{x}_0 = initial vector

$$\underline{x}_{k+1} = \underline{x}_k - \left(D_F(\underline{x}_k) \right)^{-1} F(\underline{x}_k) \text{ for } k = 0, 1, 2, \dots$$

$$\underline{s} = - \left(D_F(\underline{x}_k) \right)^{-1} F(\underline{x}_k)$$

$$D_F(\underline{x}_k) \underline{s} = -F(\underline{x}_k)$$



$$\underline{A} \underline{x} = \underline{b}$$

Solving $\underline{A} \underline{x} = \underline{b}$

\underline{s} is the solution
of $D_F(\underline{x}_k) \underline{s} = -F(\underline{x}_k)$

Multivariate Newton's method: Algorithm

$$\underline{x}_0 = \text{initial vector}$$
$$\underline{x}_{k+1} = \underline{x}_k - \left(D_F(\underline{x}_k)\right)^{-1} F(\underline{x}_k) \text{ for } k = 0, 1, 2, \dots$$




$$\underline{x}_0 = \text{initial vector}$$
$$\text{solve } D_F(\underline{x}_k) \underline{s} = -F(\underline{x}_k)$$
$$\underline{x}_{k+1} = \underline{x}_k + \underline{s} \text{ for } k = 0, 1, 2, \dots$$

Example

- Find a solution of the following system with starting guess (1, 2)

$$\begin{aligned}v - u^3 &= 0 \\ u^2 + v^2 - 1 &= 0\end{aligned}$$

- $f_1 = v - u^3$
 - $f_2 = u^2 + v^2 - 1$
 - $\partial f_1 / \partial u = -3u^2$
 - $\partial f_1 / \partial v = 1$
 - $\partial f_2 / \partial u = 2u$
 - $\partial f_2 / \partial v = 2v$
- $$D_F(u, v) = \begin{bmatrix} -3u^2 & 1 \\ 2u & 2v \end{bmatrix}$$

\underline{x}_0 = initial vector

solve $D_F(\underline{x}_k) \underline{s} = -F(\underline{x}_k)$

$\underline{x}_{k+1} = \underline{x}_k + \underline{s}$ for $k = 0, 1, 2, \dots$

$$F = \begin{bmatrix} v - u^3 \\ u^2 + v^2 - 1 \end{bmatrix}$$
$$D_F(u, v) = \begin{bmatrix} -3u^2 & 1 \\ 2u & 2v \end{bmatrix}$$

- Start point $\underline{x}_0 = (1, 2)$, solve
 $u \quad v$

$$\begin{bmatrix} -3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = - \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

- Get $\underline{s} = (0, -1)$
- Update \underline{x}

$$\underline{x}_1 = \underline{x}_0 + \underline{s} = (1, 2) + (0, -1) = (1, 1)$$

\underline{x}_0 = initial vector

solve $D_F(\underline{x}_k) \underline{s} = -F(\underline{x}_k)$

$\underline{x}_{k+1} = \underline{x}_k + \underline{s}$ for $k = 0, 1, 2, \dots$

$$F = \begin{bmatrix} v - u^3 \\ u^2 + v^2 - 1 \end{bmatrix}$$
$$D_F(u, v) = \begin{bmatrix} -3u^2 & 1 \\ 2u & 2v \end{bmatrix}$$

- $\underline{x}_1 = (1, 1)$, solve
 $u \quad v$

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

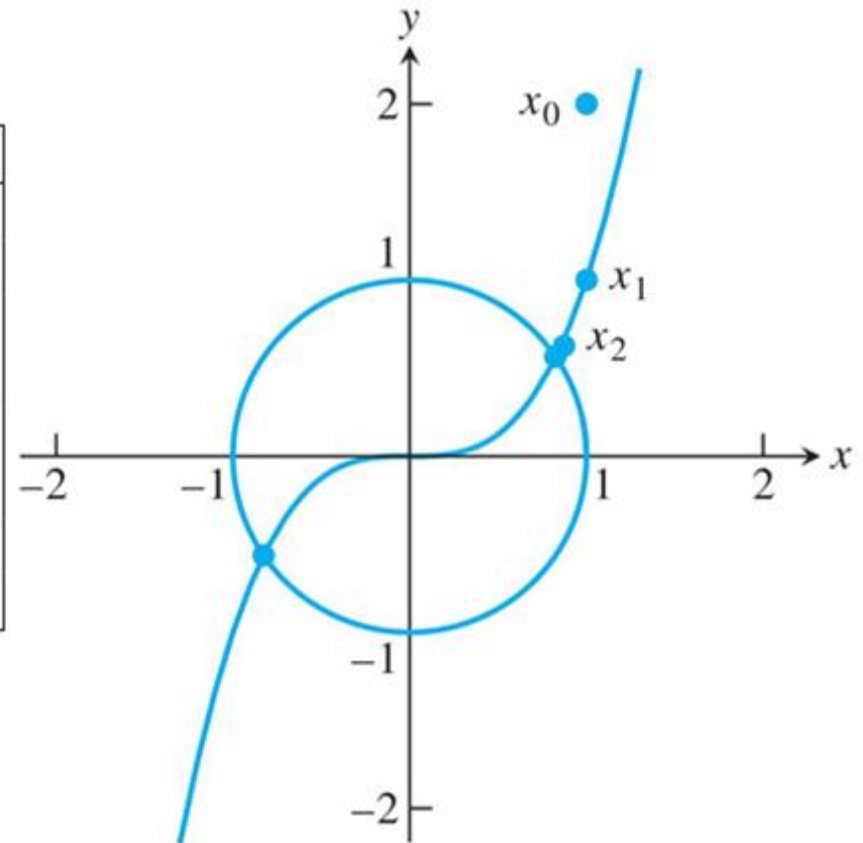
- Get $\underline{s} = (-1/8, -3/8)$
- Update \underline{x}

$$\underline{x}_2 = \underline{x}_1 + \underline{s} = (1, 1) + (-1/8, -3/8) = (7/8, 5/8)$$

$$v - u^3 = 0$$

$$u^2 + v^2 - 1 = 0$$

step	u	v
0	1.0000000000000000	2.0000000000000000
1	1.0000000000000000	1.0000000000000000
2	0.8750000000000000	0.6250000000000000
3	0.82903634826712	0.56434911242604
4	0.82604010817065	0.56361977350284
5	0.82603135773241	0.56362416213163
6	0.82603135765419	0.56362416216126
7	0.82603135765419	0.56362416216126



程式練習

And, please upload your program on moodle.

- 代課老師小陳答應幫忙改100份數學考卷，題目為給定三個點，計算通過此三點的圓，求其圓心。但粗心的小陳忘記要參考答案！請你寫一個程式，輸入三個點(三點不會在同一直線上)，輸出通過此三點的圓之圓心位置。
- 圓方程式： $(x - h)^2 + (y - k)^2 = r^2$ ，圓心 (h, k) ，半徑 r

test 1
(-8, -4)
(6, 9)
(4, -9)

test 2
(-1, 6)
(-2, -6)
(5, 0)