

Solving linear systems II

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Last week

- ▶ Solving linear systems
 - ▶ Gaussian elimination
 - ▶ LU factorization

Review: Gaussian elimination

x_1	x_2	x_3	x_4	1
a_{11}	a_{12}	a_{13}	a_{14}	b_1
a_{21}	a_{22}	a_{23}	a_{24}	b_2
a_{31}	a_{32}	a_{33}	a_{34}	b_3
a_{41}	a_{42}	a_{43}	a_{44}	b_4

- ▶ A linear system with 4 unknowns and 4 equations
- ▶ Steps
 1. Subtract multiples $l_{i1} = a_{i1} / a_{11}$ of row 1 from row i , $i = 2, \dots, 4$.
 2. Set $a'_{ik} = a_{ik} - l_{i1} a_{1k}$, $i, k = 2, \dots, 4$.
 3. Set $b'_i = b_i - l_{i1} b_1$, $i = 2, \dots, 4$.

Review: Gaussian elimination

x_1	x_2	x_3	x_4	1
a_{11}	a_{12}	a_{13}	a_{14}	b_1
0	a'_{22}	a'_{23}	a'_{24}	b'_2
0	a'_{32}	a'_{33}	a'_{34}	b'_3
0	a'_{42}	a'_{43}	a'_{44}	b'_4

- ▶ A linear system with 4 unknowns and 4 equations
- ▶ Steps
 1. Subtract multiples $l'_{i2} = a'_{i2} / a'_{22}$ of row 2 from row $i, i = 3, \dots, 4$.
 2. Set $a''_{ik} = a'_{ik} - l'_{i2} a'_{2k}, i, k = 3, \dots, 4$.
 3. Set $b''_i = b'_i - l'_{i2} b'_2, i = 3, \dots, 4$.

Review: Gaussian elimination

x_1	x_2	x_3	x_4	1
a_{11}	a_{12}	a_{13}	a_{14}	b_1
0	a'_{22}	a'_{23}	a'_{24}	b'_2
0	0	a''_{33}	a''_{34}	b''_3
0	0	a''_{43}	a''_{44}	b''_4

- ▶ A linear system with 4 unknowns and 4 equations
- ▶ Steps
 1. Subtract multiples $l'_{i3} = a''_{i3} / a''_{33}$ of row 3 from row $i, i = 4$.
 2. Set $a'''_{ik} = a''_{ik} - l'_{i3} a''_{3k}, i, k = 4$.
 3. Set $b'''_i = b''_i - l'_{i3} b''_3, i = 4$.

x_1	x_2	x_3	x_4	1
a_{11}	a_{12}	a_{13}	a_{14}	b_1
0	a'_{22}	a'_{23}	a'_{24}	b'_2
0	0	a''_{33}	a''_{34}	b''_3
0	0	0	a'''_{44}	b'''_4

Review: $A = LU$

► Actual storage scheme

x_1	x_2	x_3	x_4	1
a_{11}	a_{12}	a_{13}	a_{14}	b_1
l_{21}	a'_{22}	a'_{23}	a'_{24}	b'_2
l_{31}	l'_{32}	a''_{33}	a''_{34}	b''_3
l_{41}	l'_{42}	l'_{43}	a'''_{44}	b'''_4

Review: Complexity of Gaussian Elimination

- ▶ Elimination: $O(n^3)$
- ▶ Substitution: $O(n^2)$

Today

- ▶ Error estimation
- ▶ Improving the naïve approach
 - ▶ Gaussian elimination with ***partial pivoting***
(PA = LU factorization)

Error estimation

true solution: \underline{x}

approximate solution: \underline{x}_a

- ▶ Two questions regarding the accuracy of \underline{x}_a as an approximation to the solution of the linear system of equations $A\underline{x} = \underline{b}$.
 1. First we investigate what we can derive from the residual $\underline{r}_a = \underline{b} - A\underline{x}_a$. Note that $\underline{r} = \underline{b} - A\underline{x} = \underline{0}$
 2. Then, how **sensitive** is the solution to the perturbations in the initial data? That is, what is the effect of errors in the initial data (\underline{b}, A) on the solution \underline{x} ?

Infinity norm

- ▶ The **infinity norm**, or the **maximum norm**, of the vector $\underline{x} = [x_1, \dots, x_n]^T$ is

$$\|\underline{x}\|_{\infty} = \max |x_i|, \quad i = 1, \dots, n.$$

- ▶ The infinity norm of $x = [3, 2, -8, 1, 4, -2, -9, -4]^T$ is ?

Infinity norm

- ▶ The **matrix (absolute row sum) norm** of an $n \times n$ matrix A is

$$\|A\|_{\infty} = \max_i \sum_{j=1}^n |a_{ij}|.$$

- ▶ Example

$$A = \begin{bmatrix} 1 & 1 \\ 1.0001 & -1 \end{bmatrix}$$

$$\|A\|_{\infty} = 2.0001$$

More definitions

true solution: \underline{x}
approximate solution: \underline{x}_a

- ▶ **Residual** (note: it is a vector!)

$$\underline{b} - A\underline{x}_a$$

- ▶ **Backward error**

$$\|\underline{b} - A\underline{x}_a\|_{\infty}$$

- ▶ **Forward error**

$$\|\underline{x} - \underline{x}_a\|_{\infty}$$

Example

- ▶ Consider the linear system:

$$x_1 + x_2 = 2$$

$$1.0001x_1 + x_2 = 2.0001$$

- ▶ The solution $\underline{x} = [1, 1]^T$
- ▶ Consider the approximate solution $\underline{x}_a = [-1, 3.0001]^T$

$$\begin{aligned}x_1 + x_2 &= 2 \\1.0001x_1 + x_2 &= 2.0001\end{aligned}$$

▶ $\underline{x} = [1, 1]^\top$

▶ $\underline{x}_a = [-1, 3.0001]^\top$

▶ The **backward error** is:

$$\|\underline{r}_a\|_\infty = \|\underline{b} - A\underline{x}_a\|_\infty = \left\| \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3.0001 \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 2.0001 \\ 2 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} -0.0001 \\ 0.0001 \end{bmatrix} \right\|_\infty$$

$$= 0.0001$$



$$\begin{aligned}x_1 + x_2 &= 2 \\ 1.0001x_1 + x_2 &= 2.0001\end{aligned}$$

- ▶ $\underline{x} = [1, 1]^T$
- ▶ $\underline{x}_a = [-1, 3.0001]^T$
- ▶ The **forward error** is:

$$\begin{aligned}\|\underline{x} - \underline{x}_a\|_\infty &= \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3.0001 \end{bmatrix} \right\|_\infty \\ &= \left\| \begin{bmatrix} 2 \\ -2.0001 \end{bmatrix} \right\|_\infty \\ &= 2.0001\end{aligned}$$

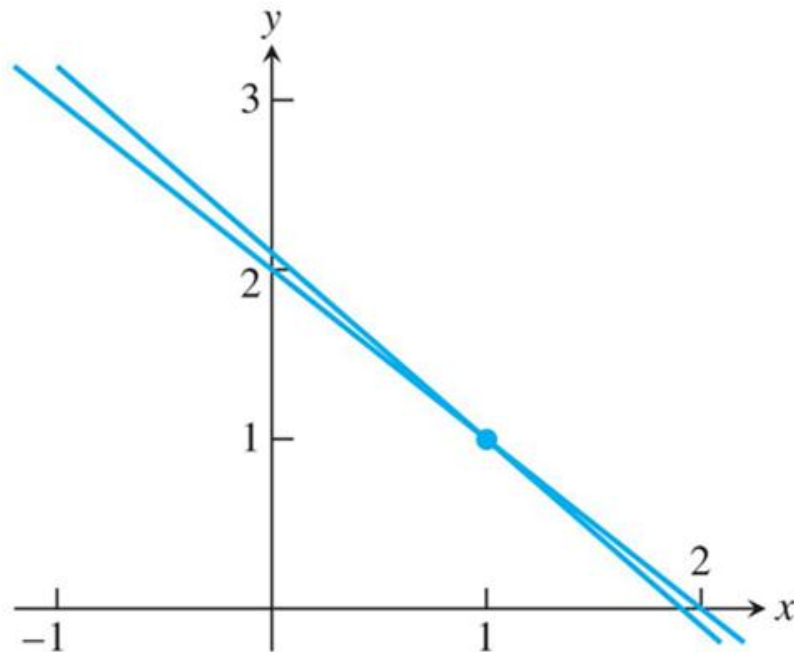
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What does this mean?

backward error 小 forward error 大



- ▶ Even though the “approximate solution” is relatively far from the exact solution, it nearly lies on both lines!



The error magnification factor

$$\text{error magnification factor} = \frac{\text{relative forward error}}{\text{relative backward error}} = \frac{\frac{\|x - x_a\|_\infty}{\|x\|_\infty}}{\frac{\|r\|_\infty}{\|b\|_\infty}}$$

► Example:

$$\begin{aligned}\underline{x} &= [1, 1]^T \\ \underline{x}_a &= [-1, 3.0001]^T \\ \underline{b} &= [2, 2.0001]^T\end{aligned}$$

relative forward error: $2.0001/1 = 2.0001$

relative backward error: $0.0001/2.0001 = 0.00005$

error magnification factor : $2.0001/0.00005 = \mathbf{40004.0001}$

The **condition number**

- ▶ The condition number of a square matrix A , **cond**(A), is the maximum possible error magnification factor for solving $A\underline{x} = \underline{b}$, over all right-hand sides \underline{b} .
- ▶ The condition number of the $n \times n$ matrix A is

$$\text{cond}(A) = \|A\| \cdot \|A^{-1}\|.$$

$$A \times A^{-1} = A^{-1} \times A = \mathbf{I}$$

Example

$$\begin{aligned}x_1 + x_2 &= 2 \\ 1.0001x_1 + x_2 &= 2.0001\end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \quad \|A\|_{\infty} = 2.0001$$

$$A^{-1} = \begin{bmatrix} -10000 & 10000 \\ 10001 & -10000 \end{bmatrix} \quad \|A^{-1}\|_{\infty} = 20001$$

The condition number of A is

$$\text{cond}(A) = 2.0001 * 20001 = \mathbf{40004.0001}$$

So, how does the **residual** $\hat{\mathbf{r}} := \mathbf{b} - A\hat{\mathbf{x}}$ affect the **error** $\mathbf{z} := \hat{\mathbf{x}} - \mathbf{x}$?

$$A\mathbf{z} = A(\hat{\mathbf{x}} - \mathbf{x}) = A\hat{\mathbf{x}} - \mathbf{b} = -\hat{\mathbf{r}}.$$

$$\|\mathbf{b}\| = \|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|, \rightarrow \frac{\|\mathbf{b}\|}{\|A\|} \leq \|\mathbf{x}\|$$

$$\|\mathbf{z}\| = \|-A^{-1}\hat{\mathbf{r}}\| \leq \|A^{-1}\|\|\hat{\mathbf{r}}\|$$

$$\frac{\|\mathbf{z}\|}{\|\mathbf{x}\|} \leq \frac{\|A^{-1}\|\|\hat{\mathbf{r}}\|}{\|\mathbf{b}\|/\|A\|} \leq \|A\|\|A^{-1}\| \frac{\|\hat{\mathbf{r}}\|}{\|\mathbf{b}\|}$$

relative forward error

relative backward error

Summary of error estimation

$$\textcolor{blue}{cond(A)} \cdot \epsilon$$

$$\begin{aligned} \text{error magnification factor} &= \frac{\text{relative forward error}}{\text{relative backward error}} = \text{cond}(A) \\ &\quad \textcolor{blue}{\epsilon} \\ &= \|A\| \times \|A^{-1}\| \end{aligned}$$

Today

- ▶ Error estimation
- ▶ Improving the naïve approach
 - ▶ Gaussian elimination with partial pivoting
(PA = LU factorization)


Cases where the naïve Gaussian elimination may fail?

- ▶ When meeting a zero multiple...

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

- ▶ Could be solved by exchanging row 1 and row 2

x_1	x_2	1
0	1	4
1	1	7



x_1	x_2	1
1	1	7
0	1	4

- ▶ It is rare to hit a precisely zero pivot, but common to hit a very small one.

Swamping

- ▶ Consider the system:

$$\begin{aligned}10^{-20}x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 4.\end{aligned}$$

- ▶ Exact solution

$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array} \right] \xrightarrow{(2) - (1) * 10^{20}} \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & 2 - 10^{20} & 4 - 10^{20} \end{array} \right]$$

$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array} \right] \xrightarrow{(2) - (1) \cdot 10^{20}} \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & 2 - 10^{20} & 4 - 10^{20} \end{array} \right]$$

$$(2 - 10^{20})x_2 = 4 - 10^{20} \longrightarrow x_2 = \frac{4 - 10^{20}}{2 - 10^{20}},$$

$$10^{-20}x_1 + \frac{4 - 10^{20}}{2 - 10^{20}} = 1$$

$$x_1 = 10^{20} \left(1 - \frac{4 - 10^{20}}{2 - 10^{20}} \right)$$

$$x_1 = \frac{-2 \times 10^{20}}{2 - 10^{20}}.$$

$$[x_1, x_2] = \left[\frac{2 \times 10^{20}}{10^{20} - 2}, \frac{4 - 10^{20}}{2 - 10^{20}} \right] \approx [2, 1].$$

$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array} \right] \xrightarrow{(2) - (1) * 10^{20}} \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & 2 - 10^{20} & 4 - 10^{20} \end{array} \right]$$

► IEEE double precision

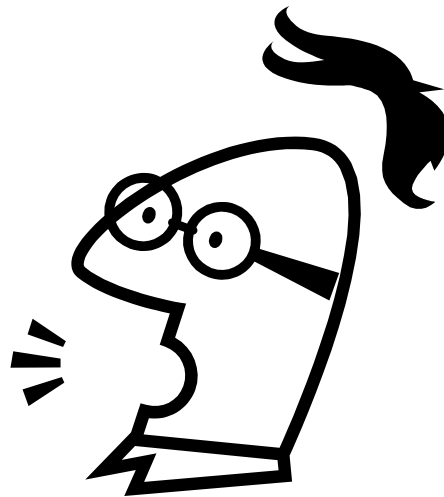
► $2 - 10^{20} = -10^{20}$

► $4 - 10^{20} = -10^{20}$

$$-10^{20}x_2 = -10^{20} \longrightarrow x_2 = 1.$$

$$10^{-20}x_1 + 1 = 1,$$

$$[x_1, x_2] = [0, 1].$$



$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 10^{-20} & 1 & 1 \end{array} \right] \xrightarrow{(2) - (1) * 10^{-20}} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 - 2 \times 10^{-20} & 1 - 4 \times 10^{-20} \end{array} \right]$$

► IEEE double precision, after row exchange

► $2 - 10^{20} = -10^{20}$

► $4 - 10^{20} = -10^{20}$

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_2 &= 1 \end{aligned}$$

$$[x_1, x_2] = [2, 1]$$

The difference?

$$\left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 1 & 2 & 4 \end{array} \right] \xrightarrow{(2) - (1) * 10^{20}} \left[\begin{array}{cc|c} 10^{-20} & 1 & 1 \\ 0 & 2 - 10^{20} & 4 - 10^{20} \end{array} \right]$$

“Swamp” the bottom equation!

Two independent equations → two copies of the top equation

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 10^{-20} & 1 & 1 \end{array} \right] \xrightarrow{(2) - (1) * 10^{-20}} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 - 2 \times 10^{-20} & 1 - 4 \times 10^{-20} \end{array} \right]$$

Remedy for swamping (and zero pivoting)

- ▶ Multiples in Gaussian elimination should be kept as **small** as possible to avoid swamping.
- ▶ **Partial pivoting**
 - ▶ Forces the absolute value of multiples to be **no larger than 1**

Partial pivoting

x_1	x_2	x_3	x_4	1
a_{11}	a_{12}	a_{13}	a_{14}	b_1
a_{21}	a_{22}	a_{23}	a_{24}	b_2
a_{31}	a_{32}	a_{33}	a_{34}	b_3
a_{41}	a_{42}	a_{43}	a_{44}	b_4

- ▶ Searches for the maximal element in modulus:
The index p of the pivot in the k -th step of GE is determined by

$$|a_{pk}^{(k-1)}| = \max_{i \geq k} |a_{ik}^{(k-1)}|$$

- ▶ If $p > k$ then rows p and k are exchanged.
- ▶ This strategy implies that $|l_{ik}| \leq 1$.

Example: Partial pivoting

- Solve the linear system:
- $$\begin{aligned}x_1 - x_2 + 3x_3 &= -3 \\ -x_1 - 2x_3 &= 1 \\ 2x_1 + 2x_2 + 4x_3 &= 0.\end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 2 & 4 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & -1 & 3 & -3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 3 & -3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array} \right]$$

Example: Partial pivoting (cont.)

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & -2 & 1 & -3 \\ 0 & 1 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

$$\frac{1}{2}x_3 = -\frac{1}{2}$$

$$-2x_2 + x_3 = -3$$

$$2x_1 + 2x_2 + 4x_3 = 0,$$

$$x = [1, 1, -1]$$

Permutation matrix

- ▶ A permutation matrix is a $n \times n$ matrix consisting of all zeros, except for a single 1 in every row and column.
- ▶ Is the identity matrix a permutation matrix?
- ▶ How many 3×3 permutation matrices?



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Fundamental theorem of Permutation matrices

- ▶ Let P be the $n \times n$ permutation matrix formed by a particular set of row exchanges applied to the identity matrix. Then, for any $n \times n$ matrix A , PA is the matrix obtained by applying exactly the same set of row exchanges to A .
- ▶ Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

The international journal of science / 6 October 2022

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PA = LU factorization

► Find the PA=LU factorization of $A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow{\text{exchange rows 1 and 2}} \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix}$

$\xrightarrow{\text{subtract } \frac{1}{2} \times \text{row 1 from row 2}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ 1 & 3 & 1 \end{bmatrix} \xrightarrow{\text{subtract } \frac{1}{4} \times \text{row 1 from row 3}} \begin{bmatrix} 4 & 4 & -4 \\ \frac{1}{2} & -1 & 7 \\ \frac{1}{4} & 2 & 2 \end{bmatrix}$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

→ exchange rows 2 and 3 →

$$\begin{bmatrix} 4 & 4 & -4 \\ \left(\frac{1}{4}\right) & 2 & 2 \\ \left(\frac{1}{2}\right) & -1 & 7 \end{bmatrix}$$

subtract $-\frac{1}{2} \times$ row 2
from row 3 →

$$\begin{bmatrix} 4 & 4 & -4 \\ \left(\frac{1}{4}\right) & 2 & 2 \\ \left(\frac{1}{2}\right) & \left(-\frac{1}{2}\right) & 8 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

$P \qquad \qquad A \qquad \qquad L \qquad \qquad U$

Solving $A\underline{x} = \underline{b}$ (using $PA = LU$)

- ▶ $A\underline{x} = \underline{b}$
- ▶ $PA\underline{x} = P\underline{b}$
- ▶ $LU\underline{x} = P\underline{b}$
- ▶ $L(U\underline{x}) = P\underline{b}$, $U\underline{x} = \underline{c}$
- ▶ Solve
 - $L\underline{c} = P\underline{b}$ for \underline{c}
 - $U\underline{x} = \underline{c}$ for \underline{x}

Example: $PA = LU$ for solving $A\underline{x} = \underline{b}$

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

$P \qquad A \qquad L \qquad U$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

$P \qquad A \qquad L \qquad U$

Solve

$$\underline{L}\underline{c} = \underline{P}\underline{b} \text{ for } \underline{c}$$

$$\underline{U}\underline{x} = \underline{c} \text{ for } \underline{x}$$

1. $\underline{L}\underline{c} = \underline{P}\underline{b}$ for \underline{c}

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 5 \end{bmatrix}$$

get $\underline{c} = [0, 6, 8]^T$

2. $\underline{U}\underline{x} = \underline{c}$ for \underline{x}

$$\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 8 \end{bmatrix}$$

get $\underline{x} = [-1, 2, 1]^T$

程式練習

And, please upload your code on moodle.

- ▶ Return the P , L and U of the $PA = LU$ factorization of the matrix:

$$A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}$$