Week 4 Tutorial

Suppose that we have observed the random sample $X_1, X_2, X_3, \ldots, X_n$, where $X_i \sim N(\theta_1, \theta_2)$, so

$$f_{X_i}(x_i; \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}.$$

Find the maximum likelihood estimators for θ_1 and θ_2.

1) $X_1, X_2, X_3, \ldots, X_4$

2) $X_i \sim N(\underset{\mu}{\theta_1}, \underset{\sigma}{\theta_2})$

Important!

$X \sim N(\mu, \sigma)$

1. MLE: $\mathcal{L}(\theta / X) = P(X_i / \theta) = P(\underbrace{X_1, X_2, X_3 \ldots}_{i.i.d}) = P(X_1) \cdot P(X_2) \ldots P(X_n)$

$$\frac{1}{\sqrt{2\pi\sigma}} e^{-\left(\frac{X_2 - \mu}{2\sigma}\right)}$$

2. $\mathcal{L} \to \max$

$\nabla \vec{\mathcal{L}} = \vec{0}$

$\nabla \mathcal{L} = \left(\frac{\partial L}{\partial \theta_1} ; \frac{\partial L}{\partial \theta_2}\right)$ $\quad \frac{\partial L}{\partial \theta_1} = 0$

$\frac{\partial L}{\partial \theta_2} = 0$

$X_i \sim exp(\lambda) \quad P(x) = \lambda e^{-\lambda x} \Rightarrow L(\lambda / X_n) = \lambda^n e^{-\lambda \sum_i X_i} \Rightarrow \nabla L = 0 = \lambda_{MLE} = \frac{n}{\sum X_i}$

$X_n \sim exp(\lambda)$

pdfs for various distributions

1) Bern(n, p)

$p(x) = p^x (1-p)^{n-x}$

x: # success

2) Binom(p)

$P(x) = p^x (1-p)$

I have a bag that contains 3 balls. Each ball is either red or blue, but I have no information in addition to this. Thus, the number of blue balls, call it θ, might be 0, 1, 2, or 3. I am allowed to choose 4 balls at random from the bag with replacement. We define the random variables X_1, X_2, X_3, and X_4 as follows

$$X_i = \begin{cases} 1 & \text{if the } i\text{th chosen ball is blue} \\ 0 & \text{if the } i\text{th chosen ball is red} \end{cases}$$

Note that X_i's are i.i.d. and $X_i \sim Bernoulli(\frac{\theta}{3})$. After doing my experiment, I observe the following values for X_i's.

$$x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 1.$$

Thus, I observe 3 blue balls and 1 red balls.

1. For each possible value of θ, find the probability of the observed sample, $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$.
2. For which value of θ is the probability of the observed sample is the largest?

probability of getting blue ball $\to \frac{\theta}{3}$

$\frac{\theta}{3} \quad \left(1 - \frac{\theta}{3}\right) \quad \frac{\theta}{3} \quad \frac{\theta}{3}$

$P_{\theta_1} = 0$

getting 1 blue ball $\quad P_{\theta_2} = \left(\frac{1}{3}\right)^3 \cdot \frac{2}{3} = \frac{1}{27} \cdot \frac{2}{3} = \frac{2}{81}$

2 balls $\quad P_{\theta_3} = \left(\frac{2}{3}\right)^3 \cdot \frac{1}{3} = \frac{8}{27} \cdot \frac{1}{3} = \frac{8}{81}$

3 balls $\quad P_{\theta_4} = \left(\frac{3}{3}\right)^3 \cdot \frac{3-3}{3} = 0$