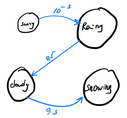
Week 6 Lecture

Markov Chain (MDP)



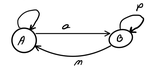
Properties:

1) $(t+1)$th state depends only on the tth state

2)

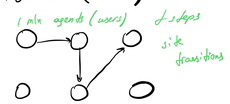
$\sum_{i}^{n} P_i = 1$

3)



$p > m$

Page Rank (Google)

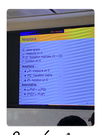


T - time (steps, transitions, epochs)

X_t = outcome of the tth step

States $= \{0, 1, \dots, N\} = S$

$\bigcup_{t=1}^{T} X_t \neq S$



$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & P_{00} & P_{01} & P_{02} \\ 1 & P_{10} & P_{11} & P_{12} \\ 2 & P_{20} & P_{21} & P_{22} \end{array} \quad \begin{array}{c} \\ = 1 \\ \\ \end{array}$$

$|S| \cdot |S| = N \times N$

$|S| = N$

$\neq 1.$

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & P_{00} & P_{01} & P_{02} \\ 1 & & & \\ 2 & & & \end{array} \qquad P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & P_0 & & \\ 1 & & P_1 & \\ 2 & & & P_2 \end{array}$$

$$P \cdot P\,(0,1) = P_0 P_{01} + P_1 P_{11} + P_{02} \cdot P_2$$

$$\sum P[i,j]\, P[:,j]$$

Gambler's Ruin Problem

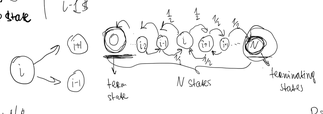
A (0) (N$) (i) $

stop — stop time

i+1 $

i-1 $

i^{th} step randomly

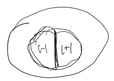


P_i = prob N$ / i

$P_i = P(\text{A wins the game} \mid \text{he has i\$ in his budget})$

$P(A) = P(A|B_1)\cdot P(B_1) + P(A|B_2)\cdot P(B_2)$ — total prob formula

$P_i = \frac{1}{2}$



total prob formula

P_{k+1}

$P(A \text{ wins} \mid k+1\$)$

$P_k = P(A \text{ wins the game} \mid A \text{ wins 1st round}) \cdot P(A \text{ wins 1st round}) + P(A \text{ wins} \mid A \text{ loses 1st round}) \cdot P(A \text{ loses 1st round})$

$P_k \triangleq P(A \text{ wins} \mid k\$)$

$P_{k+1} \triangleq P(A \text{ wins} \mid k+1\$)$

P_{k-1}

$\frac{1}{2}$

p

$\frac{1}{2}$

q

$p = 1 - q$

$P_0 = P_2 = P_0$

$P_N = 1$

P_1

$P_3 = P_2$

$\vdots$

$P_{k-1} = P_{k-2}\cdot\frac{1}{2} + P_k\cdot\frac{1}{2}$

$$\boxed{P_k = P_{k+1}\cdot\frac{1}{2} + P_{k-1}\cdot\frac{1}{2}}$$

$$P_{k+1} = P_{k+2}\cdot\frac{1}{2} + P_k\cdot\frac{1}{2}$$

P_k

$N = 3$, stop the game when we have 0 $

$k = 0$

$k = 1$

$k = 2$

$k = 3$

$\boxed{P_k = 1}$

$P_2 = P_1\cdot\frac{1}{2} + P_3\cdot\frac{1}{2}$

$P_{k+1} \quad P_k \quad P_{k+2}$

$P_1 = P_0\cdot\frac{1}{2} + P_2\cdot\frac{1}{2}$

$P_0 = 0$

$$\begin{cases} P_2 = \frac{1}{2}P_1 + \frac{1}{2} \\ P_1 = 0 + P_2\cdot\frac{1}{2} \end{cases} \Rightarrow \begin{cases} P_2 = \frac{1}{2}P_1 + \frac{1}{2} \\ P_2 + \frac{1}{2}P_2 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} P_2 = \frac{1}{2}P_1 + \frac{1}{2} \\ P_2 = \frac{2}{3} \end{cases} \Rightarrow \begin{cases} P_1 = \frac{2}{3} \\ P_2 = \frac{2}{3} \\ P_3 = 1 \\ P_0 = 0 = \frac{2}{3} \end{cases}$$

$P_k = \frac{k}{N}$

N variables

N equations $P_k = \frac{k}{N}$

$$P_k = \frac{1 - \left(\frac{1}{p}\right)^k}{1 - \left(\frac{1}{p}\right)^N} \quad \frac{0}{0} \quad \boxed{q = p}$$

$q = p \quad P_k = \frac{k}{N}$

$P_k = \frac{k}{N}$

$P(A \text{ loses the game}) = 1 - \frac{k}{N} = \boxed{\frac{N-k}{N}}$

Math. Trend

1) $P_3 = 1$ — trend

2) $\boxed{N = k_{\text{stop}}}$ — trend $\boxed{P_N = 1}$

3) $N = k+1$ — trend $\left(\frac{P_{k+1}}{P_k}\right)$

wealth at the end

$k \le N$

$k \le t \le N$

$\boxed{P_k = \frac{k}{N}}$

$EM = k + \left(\frac{k}{N} - \frac{N-k}{N}\right)N = 2k - N + k = 2k - N$

M - total wealth

$EM = \sum_i p_i X_i$

$X_i = 0, \frac{t}{N}$

$X_2 = N, \frac{t}{N}$

$\boxed{EM = 0\cdot\frac{N-k}{N} + N\cdot\frac{k}{N} = k}$

$N = 3$

$\mu_1 = 1 + \sum_j p_{1j}\mu_j$

$\mu_2 = 1 + \sum_j p_{2j}\mu_j$

$\mu_3 = 0$

$\mu_1 = 1 + \frac{1}{2}\mu_2$

$\mu_2 = 1 + \frac{1}{2}\mu_1$

$\mu_3 = 0$

i=0	$\mu_0 = 0$
i=1	$\mu_1 = 1 + \sum_j p_{1j}\mu_j = 1 + 0 + \frac{1}{2}\mu_2$
i=2	$\mu_2 = 1 + \sum_j p_{2j}\mu_j = 1 + \frac{1}{2}\mu_1 + 0$
i=3	$\mu_3 = 0$

$\mu_1 = 2$

$\mu_2 = 1 + \frac{1}{2}(2)$

$\mu_2 = 2$

$\mu_1 = 1 + \frac{1}{2}\mu_2$

$\mu_2 = 1 + \frac{1}{2}\mu_1$

$\mu_3 = 0$

$\mu_1 = 1 + \frac{1}{2}\mu_2$

$\mu_2 = 1 + \frac{1}{2}\mu_1$

$\mu_1 = (\mu_2 - 1)2$

$(\mu_2 - 1)2 = 1 + \frac{1}{2}\mu_2$

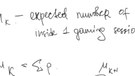
$2\mu_2 - 2 = 1 + \frac{1}{2}\mu_2$

$\frac{3}{2}\mu_2 = 3$

$\mu_2 = 2$; $\mu_1 = 2$, $\mu_0 = 0$

μ_k — expected number of plays (rounds) inside 1 gaming session, with 0 or N

$\mu_k = \sum p$



$\mu_k = 1 + 0.5\mu_{k-1} + 0.5\mu_{k+1}$

Finishing the game from μ_k

$\mu_k = 1 + 0.5\mu_{k-1} + 0.5\mu_{k+1}$

$P(k,k+1) \quad P(k,k-1)$

$N = 300$

$k = 100$

$$\boxed{\mu_k = k(N-k)}$$

$$\boxed{\mu_k = 1 + \sum_j P_{ij}\mu_j}$$

$\mu_{100} = 100 \cdot 200 = 20\,000$