§5. 同构. 等价关系. 例子

Review:

$$f: G \xrightarrow{Hom} G' \iff f(ab)=f(a)f(b) \ \forall a,b.$$

$$\ker f < G. \quad \mathrm{Im} f < G'$$

$$f: G \xrightarrow[inj]{Hom} G' \iff Hom \ \& \ \ker(f)=\{e\}$$

$$f: G \xrightarrow{Iso} G' \iff \begin{cases} f \text{ bij} \\ f \text{ hom} \\ (f^{-1} \text{ hom}). \end{cases}$$

$f: X \to Y$ homeo $\iff$
① f bij
② f conti.
③ f^{-1} conti.

Thm: $f: G \xrightarrow{Iso} G' \iff f: G \xrightarrow[bij.]{Hom} G'$

"$\Longrightarrow$" ✓

"$\Longleftarrow$" if $f: G \xrightarrow[bij]{Hom} G'$.

Check $f^{-1}: G' \xrightarrow{Hom} G$

$$f^{-1}(a'b') = \underset{\underset{a}{\|}}{f^{-1}(a')}\,\underset{\underset{b}{\|}}{f^{-1}(b')}$$

$$f(a) = a'. \quad f(b) = b'.$$

$$f(ab) = a'b'.$$

$$f^{-1}(a')f^{-1}(b') = ab = f^{-1}(a'b')$$

$\underline{Ex}.$ $(\mathbb{R}, +) \simeq (\mathbb{R}_{>0}, \cdot)$ exp. log.

$\underline{Ex}.$ $(G, \cdot) \simeq (G, \cdot)$ id_G.

等价关系 "$\simeq$"

① $G \simeq G$ ② $G \simeq G' \Rightarrow G' \simeq G$

③ $G \simeq G', G' \simeq G'' \Rightarrow G \simeq G''$.

id_G. ② $f : G \xrightarrow{iso} G' \Rightarrow f$ bij. f hom. f^{-1} hom

$\Downarrow$

$f^{-1} : G' \xrightarrow{iso} G \Leftarrow f^{-1}$ bij. f^{-1} hom. f hom

$\|$

(f^{-1})

③ $f : G \xrightarrow{iso} G'$. $g : G' \xrightarrow{iso} G''$

$g \circ f : G \xrightarrow[bij.]{Hom} G''$ $\quad (g \circ f)(a)\ (g \circ f)(b).$

$\|$

$$(g \circ f)(ab) = g(f(ab)) = g(f(a)f(b)) = g(f(a))\, g(f(b))$$ ✓

Thm: if $f : G \xrightarrow{\text{iso}} G'$. then

① $|G'| = |G|$ cardinality.

② $H < G \Leftrightarrow f(H) < G'$

③ if G'' group then,

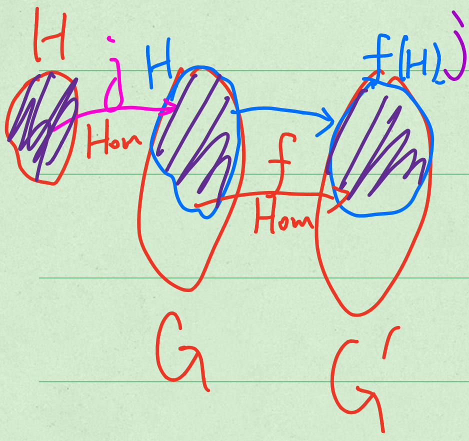
$\exists g : G' \xrightarrow{\text{Hom}} G'' \Longleftrightarrow \exists g' : G \xrightarrow{\text{Hom}} G''$

& $\exists g : G' \xrightarrow{\text{Iso}} G'' \Longleftrightarrow \exists g' : G \xrightarrow{\text{Iso}} G''$

①. $f : G \xrightarrow{\text{iso}} G'$. f bij. $|G'| = |G|$

②. if $H < G$ Check: $f(H) < G'$. "$\Rightarrow$"

$j : H \overset{\text{Hom}}{\hookrightarrow} G$ inclusion Hom.



$\forall a \in H.\ j(a) = a$ ✓

$$\begin{tikzcd} H \arrow[r, "j"', "\text{Hom}"] \arrow[rr, bend right, "f \circ j"', "\text{Hom}"] & G \arrow[r, "f"', "\text{Hom}"] & G' \end{tikzcd}$$

$f(H) = (f \circ j)(H) = \mathrm{Im}(f \circ j) < G'$

$\forall a \in H$ $j(a) = a$

$j(H) = H$.

抽象.

✓

"$\Leftarrow$" f^{-1} Hom. ✓.

③ if G'' group then,

$$\exists g : G' \xrightarrow{Hom} G'' \Longleftrightarrow \exists g' : G \xrightarrow{Hom} G''$$

$$\& \quad \exists g : G' \xrightarrow{Iso} G'' \Longleftrightarrow \exists g' : G \xrightarrow{Iso} G''$$

(1)

$$\begin{tikzcd} G \arrow[rr, "f", "iso"'] \arrow[dr, dashed, "Hom"', "g'"] & & G' \arrow[dl, "g"', "Hom"] \\ & G'' & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[rr, "f^{iso}", "iso"'] \arrow[dr, "Hom"', "g'"] & & G' \arrow[ll, bend right, "f^{-1}"'] \arrow[dl, dashed, "g"', "Hom"] \\ & G'' & \end{tikzcd}$$

$g' = g \circ f$ ✓ $\qquad g = g' \circ f^{-1}$.

(composition of hom is hom).

(1)

$$\begin{tikzcd} G \arrow[rr, "f", "iso"'] \arrow[dr, dashed, "Iso"', "g'"] & & G' \arrow[dl, "g"', "Iso"] \\ & G'' & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[rr, "f^{iso}", "iso"'] \arrow[dr, "Iso"', "g'"] & & G' \arrow[ll, bend right, "f^{-1}"'] \arrow[dl, dashed, "g"', "Iso"] \\ & G'' & \end{tikzcd}$$

$g' = g \circ f$ ✓ (composition of hom is hom)

$g = g' \circ f^{-1}$

Ex. Trivial Groups are isomorphic

def. $(\{e\}, \cdot)$

$\cdot$	e
e	e

if $(\{e\}, \cdot)$ & $(\{e'\}, *)$

$*$	e'
e'	e'

Check. " $\simeq$ "

$f : \{e\} \to \{e'\}$.

$f(e) = e'$

① Hom. $f(e\,e) = f(e) = e'$

$f(e)f(e) = e'$ ✓

② Bij $f^{-1} : \{e'\} \to \{e\}$

$f^{-1}(e') = e$. ✓