

PCA - matrix      peer group 10

$$A = \begin{bmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & 2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 4 - \lambda & 8 & -1 & -2 \\ -2 & -9 + \lambda & 2 & -4 \\ 0 & 10 & 5 - \lambda & -10 \\ -1 & -13 & -14 & -13 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (4 - \lambda) \det(M_{11}) - 8 \det(M_{12}) + (-1) \det(M_{13}) + 2 \det(M_{14})$$

$$\det(M_{11}) = \begin{bmatrix} -9 + \lambda & -2 & -4 \\ 10 & 5 - \lambda & -10 \\ -13 & -14 & -13 - \lambda \end{bmatrix}$$

$$= (-9 + \lambda) [(5 - \lambda) - (-13 - \lambda) - (-10)(-14)] - (-2) [(10)(-13 - \lambda) - (-10)] + 4 [(-14)(5 - \lambda) - (-13)]$$

$$= (-9 + \lambda) (-65 - 5\lambda + 15\lambda + \lambda^2 - 140) + 2 [-130 - 10\lambda - 130] - 4 [-140 + 65 - 13\lambda]$$

$$= (-9 + \lambda) [\lambda^2 + 8\lambda - 205] + 2 [-260 - 10\lambda] - 4 [-75 - 13\lambda]$$

$$= -9\lambda^2 - 72\lambda + 1845 - \lambda^3 - 8\lambda^2 + 205\lambda - 520 - 20\lambda$$

$$= +300 + 52\lambda$$

$$= -\lambda^3 - 13\lambda^2 + 165\lambda + 1625$$

$$\det(M_{12}) = \begin{bmatrix} -2 & -9 & -4 \\ 0 & 5-\lambda & -10 \\ -1 & -14 & -3-\lambda \end{bmatrix}$$

$$= (-2) \begin{bmatrix} (5-\lambda)(-3-\lambda) - (-10)(-14) \end{bmatrix} - (-2) \begin{bmatrix} 0(-13-\lambda) \\ (-10)(-11) \end{bmatrix}$$

$$= (-2) \begin{bmatrix} (5-\lambda)(-13-\lambda) - 140 \end{bmatrix} + 2 \begin{bmatrix} 0(-10) \\ 0(-14) \end{bmatrix} - 4 \begin{bmatrix} 0 + (5-\lambda)(-1) \end{bmatrix}$$

$$= (-2) \begin{bmatrix} -65 - 5\lambda + 13\lambda - \lambda^2 - 140 \end{bmatrix} + 2 \begin{bmatrix} 10 \end{bmatrix} - 4 \begin{bmatrix} 5 - \lambda \end{bmatrix}$$

$$= (-2) \begin{bmatrix} \lambda^2 + 8\lambda - 205 \end{bmatrix} - 20 - 20 - 4\lambda$$

$$= -2\lambda^2 - 16\lambda - 40 - 4\lambda$$

$$= 2\lambda^2 - 12\lambda + 370$$

Hortance

Denis Mitali

$$= (-2) [(\lambda^2 + 8\lambda - 205)] - 20 - 20 - 4\lambda$$

$$= -2\lambda^2 + 16\lambda + 410 - 40 - 4\lambda$$

$$= -2\lambda^2 + 12\lambda + 370$$

$$\det(M_{13}) = \begin{vmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{vmatrix}$$

$$= (-2) [10(-13-\lambda) - (-10)(-13)] -$$

$$(-9-\lambda)[0(-13-\lambda) - (-10)(-1)] +$$

$$(-4)[0(-13) - 10(-1)]$$

$$= (-2) [-130 - 10\lambda - 130] + (9+\lambda)[-10] - 20$$

$$= 160 + 20\lambda - 90 - 10\lambda - 20$$

$$= 10\lambda + 50$$

$$= (-2) [-260 - 10\lambda] + (9+\lambda)(-10) - 40$$

$$= -520 + 20\lambda - 90 - 10\lambda - 40$$

$$= 10\lambda + 390$$

$$\det(M_{14}) = \begin{vmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{vmatrix}$$

$$= (-2) [10(5-\lambda) - (-10)(-13)] -$$

$$(-9-\lambda)[0(5-\lambda) - (-10)(-1)] +$$

$$(-2)[0(-13) - 10(-1)]$$

$$= (-2) [50 - 10\lambda - 130] + 9+\lambda[0 \cdot 10]$$

$$= -2[20+10]$$

$$= (-2) [-80 - 10\lambda] + (9+\lambda)[-10] - 20$$

$$= 160 + 20\lambda - 90 - 10\lambda - 20$$

$$= 10\lambda + 50$$

$$\det(M_{11}) = -\lambda^3 - 12\lambda^2 + 165 + 1625$$

$$\det(M_{12}) = -2\lambda^2 - 12\lambda + 370$$

$$\det(M_{13}) = 10\lambda + 390$$

$$\det(M_{14}) = 10\lambda + 50$$

$$\begin{aligned}
 \det(A - \lambda I) &= (4-\lambda)(-\lambda^3 - 17\lambda^2 + 165\lambda + 1625) - 8(-2\lambda^2 - 12\lambda + 370) \\
 &\quad + (-1)(-10\lambda + 390) - (-2)(10\lambda + 50) \\
 &= -4\lambda^3 - 68\lambda^2 + 660\lambda + 6500 + \lambda^4 + 17\lambda^3 - 165\lambda^2 \\
 &\quad - 1625\lambda + 16\lambda^2 + 96\lambda - 2960 - 10\lambda - 390 + 20\lambda + 100 \\
 &= -4\lambda^3 - 68\lambda^2 + 660\lambda + 6500 + \lambda^4 + 17\lambda^3 - 165\lambda^2 - 1625\lambda \\
 &\quad + 16\lambda^2 + 96\lambda - 2960 - 10\lambda - 390 + 20\lambda + 100 \\
 &= \lambda^4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500
 \end{aligned}$$

Let's use  $p(\lambda) = 0$

$$\begin{aligned}
 p(\lambda) &= \lambda^4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500 \\
 \lambda = 2, \quad p(2) &= (2)^4 + 13(2)^3 - 219(2)^2 - 835 + 3500 \\
 &= 1074 \neq 0
 \end{aligned}$$

$$\begin{aligned}
 \lambda = 3, \quad p(3) &= (3)^4 + 13(3)^3 - 219(3)^2 - 835(3) + 3500 \\
 &= -544 \neq 0
 \end{aligned}$$

$\therefore \lambda$  is between 2 and 3

$$\begin{aligned}
 \lambda = 2.5, \quad p(2.5) &= (2.5)^4 + 13(2.5)^3 - 219(2.5)^2 - 835(2.5) \\
 &+ 3500 = 286.94 \neq 0
 \end{aligned}$$

$$\lambda = 2.7, P(2.7) = (2.7)^4 + 13(2.7)^3 - 219(2.7)^2 - 835(2.7) + 3500 \\ = -51.13 \neq 0$$

$$\lambda = 2.675, P(2.675) = 51.18 + 248.36 - 1567.07 - 2235.13 \\ + 3500 = -6.66 \neq 0$$

$$\lambda = 2.67, P(2.67) = 50.79 + 246.56 - 1560.89 - 233.95 + 3500 \\ = -1.49$$

$$\lambda = 2.674, P(2.674) = 51.15 + 248.17 - 1566.08 - 2238.01 + 3500 \\ = -0.77$$

$$\lambda = 2.6746, P(2.6746) = 51.151 + 248.157 - 1566.288 \\ - 2232.191 + 3500 \\ = -0.0067 \approx 0$$

$$\lambda_2 = 2.6746$$

For the eigen vector,  $\lambda \approx 2.675$

$$(A - \lambda I) \vec{v} = 0, \vec{v} = [x, y, z, w]$$

$$A - 2.675 I = \begin{bmatrix} 1.325 & 8 & 2 \\ -2 & -11.675 & -4 \\ 0 & 10 & -10 \\ -1 & -15 & -15.67 \end{bmatrix}$$

By setting up the augmented matrix.

$$\left[ \begin{array}{cccc|c} 1.325 & 8 & -1 & -2 & 0 \\ -2 & -11.675 & -2 & -4 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & -13 & -14 & -15.675 & 0 \end{array} \right]$$

By using Gaussian elimination,

$$1. R_1 \div 1.325 : \left[ \begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5044 & 0 \\ -2 & -11.675 & -2 & -4 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & -13 & -14 & -15.675 & 0 \end{array} \right]$$

2. We'll eliminate Row 1

$$\begin{array}{l} R_2 + 2R_1 \\ R_4 + R_1 \end{array} \left[ \begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5044 & 0 \\ 0 & 0.4004 & -3.5094 & -7.0183 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ 0 & -6.9623 & -14.7547 & -17.1844 & 0 \end{array} \right]$$

3. Divide R<sub>2</sub> by 0.4004

$$\begin{array}{l} R_2 \div 0.4004 \\ = \end{array} \left[ \begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5044 & 0 \\ 0 & 1 & -8.766 & -17.533 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ 0 & -11.2677 & -14.2512 & -12.1111 & 0 \end{array} \right]$$

4. We'll eliminate using  $R_2$

$$\begin{array}{l} R_3 - 10R_2 \\ R_4 + 6.9623 \cdot R_2 \end{array} \rightarrow \left| \begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5044 & 0 \\ 0 & 1 & -8.768 & -17.533 & 0 \\ 0 & 0 & 90.005 & 165.33 & 0 \\ 0 & 0 & 46.257 & 105.3 & 0 \end{array} \right.$$

Back Substitution, from the row reduced system;

$$\text{Row } 3: 90.005z + 165.33w = 0$$

$$z = \frac{-165.33w}{90.005}, z = -1.837w$$

$$\text{Row } 2: y - 8.768z - 17.533w = 0$$

$$y = 8.768z + 17.533w$$

$$\text{Since } z = -1.837w$$

$$y = 8.768(-1.837)w + 17.533w$$

$$\text{Row } 4: 46.257w + 105.3 = 0$$

$$w = \frac{105.3}{46.257} = -2.277$$

$$z = -1.837 \times -2.277 = 4.182$$

$$y = 8.768(4.18) + 17.533(-2.277)$$

$$y = 36.655 - 39.91 = -3.255$$

$$x + 6.0347y - 0.7547z - 1.5044w = 0$$

$$x = -6.0347(-3.755) + 0.7547(4.18) + 1.5044(-2.277)$$

$$x = 19.66 + 3.157 - 3.425 = 19.392$$

$$\vec{v} = \begin{bmatrix} 19.392 \\ -3.255 \\ 4.182 \\ -2.277 \end{bmatrix}$$

Now, let's normalize eigen vector  $\vec{v}$

$$\|\vec{v}\| = \sqrt{(19.392)^2 + (-3.255)^2 + (4.182)^2 + (-2.277)^2}$$

$$\|\vec{v}\| = \sqrt{376.05 + 10.59 + 17.49 + 5.18}$$

$$\|\vec{v}\| = \sqrt{409.35}$$

$$\|\vec{v}\| = 20.23$$

$$\vec{v}_{\text{norm.}} = \frac{1}{20.23} \begin{pmatrix} 19.392 \\ -3.255 \\ 4.182 \\ -2.277 \end{pmatrix} = \begin{pmatrix} 0.958 \\ -0.161 \\ 0.206 \\ -0.113 \end{pmatrix}$$

$\therefore$  The eigen vector for the eigen value,  $\lambda_2 = 2.675$  is  $\begin{pmatrix} 19.392 \\ -3.255 \\ 4.182 \\ -2.277 \end{pmatrix}$

$$\text{and the normalized vector is } \begin{pmatrix} 0.958 \\ -0.161 \\ 0.206 \\ -0.113 \end{pmatrix}$$

$$\text{Importance of } \lambda_2 = 2.675 \text{ in terms of } \gamma_i = |\lambda|$$
$$\frac{|\lambda_2|}{\sum |\lambda_i|} \times 100\%$$

$$\lambda_1 = 11.05, \lambda_2 = 2.675, \lambda_3 = -21.1, \lambda_4 = -5.60$$

$$= \frac{2.675}{|11.05| + |2.675| + |-21.1| + |-5.60|} \times 100\%$$

$$\text{Importance of } \lambda_2 = \underline{6.61\%}$$

Davy

$$\text{So } p(\lambda) = \lambda^4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500$$

By analyzing the polynomial's behavior.

① for  $\lambda = 0$  .  $p(\lambda) = 3500 > 0$

② for  $\lambda = -8$   $p(-8) = (-8)^4 + 13(-8)^3 - 219(-8)^2 - 835(-8) + 3500$   
 $= 4096 + 13(-512) - 219 \cdot 64 + 835 \cdot 8 + 3500$   
 $= 14276 - 20672 = -6396 < 0$

$$\text{for } \textcircled{3} \quad \lambda = -6 \quad (\Rightarrow) \quad p(-6) = (-6)^4 + 13(-6)^3 - 219(-6^2) - 835(-6) + 3500 \\ = 1296 - 2808 - 7884 + 5010 + 3500 \\ = -886 \neq 0$$

~~④~~

The nearest root is between  $(-6, -5)$ .

$$\textcircled{4} \quad \lambda = 5 \quad p(5) = (5)^4 + 13(5)^3 - 219(5^2) - 835(5) + 3500 \\ = 625 + 13(125) - 219(25) - 4175 + 3500 \\ = 625 + 715 - 5475 - 4175 + 3500 \\ = -51433 \neq 0$$

~~⑤~~ for  $\lambda = -5.603$

$$p(-5.603) = [-5.603]^4 + 13(-5.603)^3 - 219(-5.603)^2 - 835(-5.603) + 3500 \\ = 985.97 - 2291.97 - 6869.97 + 4685.51 + 3500 \\ = 0$$

~~⑥~~ for  $\lambda = 2.606$

$$p(2.606) = (2.606)^4 + 13(2.603)^3 - 219(2.606)^2 - 835(2.606) + 3500 \\ = 46.13 + 230.13 - 1487.13 - 2176.01 + 3500 \\ = 0$$

~~⑦~~ for  $\lambda = 11.058$

$$p(11.058) = (11.058)^4 + 13(11.058)^3 - 219(11.058)^2 - 835(11.058) + 3500 \\ = 14934.97 + 19589.97 - 26745.97 - 9333.43 + 3500 \\ = 0$$

~~⑧~~ for  $\lambda = -21.061$

$$p(-21.061) = (-21.061)^4 + 13(-21.061)^3 - 219(-21.061)^2 - 835(-21.061) + 3500 \\ = 196683.97 - 120853.97 - 97089.97 + 17585.94 + 3500 \\ = 0$$

(=) eigen values,  $\lambda_1 = -5.603, \lambda_2 = 2.606, \lambda_3 = 11.058, \lambda_4 = -21.061$

By replacing the eigen values in zero the matrix.

for  $\lambda = -5.603$

$$\begin{bmatrix} 9.603 & 8 & -1 & -2 \\ -2 & -3.393 & -2 & -4 \\ 0 & \lambda_1 & -5.603 & -10 \\ -1 & -13 & -14 & -7.397 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By solving this we get  $= (0.5, 1, 0.5, 1)$ .

③ for  $\lambda_2 = 2.606$

$$\begin{pmatrix} 1.394 & 8 & -1 & -2 \\ -2 & -11.606 & -2 & -4 \\ 0 & 10 & 2.394 & -10 \\ -1 & -13 & -14 & -15.606 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By solving that we get  $\lambda_2 = [-2, 1, -1, 1]$

④ for  $\lambda_3 = 11.058$

$$\begin{pmatrix} -7.058 & 8 & -1 & -2 \\ -2 & -20.058 & -2 & -4 \\ 0 & 10 & -6.058 & -10 \\ -1 & -13 & -14 & -24.058 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By solving that  $\lambda_3 = [-1, 1, 0, 1]$

⑤ for  $\lambda_4 = -21.061$

$$\begin{pmatrix} 25.061 & 8 & -1 & -2 \\ -2 & -20.058 & -2 & -4 \\ 0 & 10 & -6.058 & -10 \\ -1 & -13 & -14 & -24.058 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 25.061 & 8 & -1 & -2 \\ -2 & -20.058 & -2 & -4 \\ 0 & 10 & +25.061 & -10 \\ -1 & -13 & -14 & 8.061 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By solving that we get  $\lambda_4 = [1, 0, 0, 1]$

## Page 3

Set up the System for Eigenvector  
Plug  $\lambda_4$  into  $A - \lambda_4 I$ :

$$A - \lambda_4 I = \begin{pmatrix} 4 - (-21.125) & 8 & -1 & -2 \\ -2 & -9 - (-21.125) & -2 & -4 \\ 0 & 10 & 5 - (-21.125) & 0 \\ -1 & -13 & -14 & -13 - (-21.125) \end{pmatrix}$$

$$25.125x_1 + 8x_2 - x_3 - 2x_4 = 0$$

$$-2x_1 + 12.125x_2 - 2x_3 - 4x_4 = 0$$

$$10x_2 + 26.125x_3 = 10x_4 = 0$$

$$-x_1 - 13x_2 - 14x_3 + 8.125x_4 = 0$$

Row 3 gives

$$10x_2 + 26.125x_3 - 10x_4 = 0 \Rightarrow x_2$$

$$= \frac{10x_4 - 26.125x_3}{10}$$

Row 1:

$$25.125x_1 + 8x_2 - x_3 - 2x_4 = 0$$

$$25.125x_1 + 8\left(\frac{10x_4 - 26.125x_3}{10}\right) - x_3 - 2x_4 = 0$$

$$2x_4 = 0$$

$$25.125x_1 + 8x_4 - 20.9x_3 - x_3 - 2x_4 = 0$$

$$25.125x_1 + 6x_4 - 21.9x_3 = 0$$

$$\frac{x_1 = 21.9x_3 - 6x_4}{25.125}$$

Page 4

Plug  $x_3$  into Row 2:

$$-2x_1 + 12.125x_2 - 2x_3 - 4x_4 = 0$$

$$\rightarrow 2x_1 + 12.125 \left( \frac{10x_4 - 26.125x_5}{10} \right) - 2x_3 - 4x_4$$

$$= 0$$

$$-2x_1 + 8.125x_4 - 53.926x_5 = 0$$

$$2x_1 = 8.125x_4 - 53.926x_5 = 0$$

$$2x_1 = 8.125x_4 - 53.926x_5$$

$$x_1 = \frac{8.125x_4 - 53.926x_5}{2}$$

Set equal to the previous expression for  $x_1$ :

$$\frac{21.9x_3 - 6x_4}{25.125} = \frac{8.125x_4 - 53.926x_5}{2}$$

eigen vectors

$$v_1 = \begin{pmatrix} -0.036 \\ 0.958 \\ 0.025 \\ 0.563 \end{pmatrix} \quad v_f = \begin{pmatrix} 0.509 \\ -0.113 \\ -0.915 \\ -0.053 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -0.012 \\ -0.161 \\ -0.335 \\ -0.616 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} -0.860 \\ 0.207 \\ -0.222 \\ 0.546 \end{pmatrix}$$