PCA - matrix   peer group 10

$$A = \begin{bmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -9\lambda I & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ -1 & -13 & -14 & -13-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (4-\lambda)\det(M11) - 8\det(M12) + (-1)\det(M13) + 2\det(M14)$$

$$\det(M_{11}) = \begin{bmatrix} -9\lambda & -2 & -4 \\ 10 & 5-\lambda & -10 \\ -13 & -14 & -13-\lambda \end{bmatrix}$$

$$= (-9-\lambda)\left[(5-\lambda)-(-13-\lambda)-(-10)(-14)\right] - (-2)\left[(10)(-13-\lambda)-(-10)\right] + (-4)\left[(10)(-14)-(5-\lambda)(-13)\right]$$

$$= (-9-\lambda)(-65-5\lambda+13\lambda+\lambda^2-140) + 2\left[-130-10\lambda-130\right] - 4\left[-140+65-13\lambda\right]$$

$$= (-9-\lambda)\left[\lambda^2+8\lambda-205\right] + 2\left[-260-10\lambda\right] - 4\left[-75-13\lambda\right]$$

$$= -9\lambda^2 - 72\lambda + 1845 - \lambda^3 - 8\lambda^2 + 205\lambda - 520 - 20\lambda + 300 + 52\lambda$$

$$= -\lambda^3 - 13\lambda^2 + 165\lambda + 1625$$

$$\det(M_{12}) = \begin{bmatrix} -2 & -2 & -4 \\ 0 & 5-\lambda & -10 \\ -1 & -14 & -13-\lambda \end{bmatrix}$$

$$= (-2)\left[(5-\lambda)(-13-\lambda) - (-10)(-14)\right] - (-2)\left[0(-13-\lambda) - (-10)(-1)\right] (-4)\left[0(-14) - (5-\lambda)(-1)\right]$$

$$= (-2)\left[(5-\lambda)(-13-\lambda) - 140\right] + 2\left[0 - 10\right] - 4\left[0 + (5-\lambda)\right]$$

$$= (-2)\left[-65 - 5\lambda + 13\lambda - \lambda^2 - 140\right] + 2\left[-10\right] - 4\left[5-\lambda\right]$$

$$= (-2)\left[\lambda^2 + 8\lambda - 205\right] - 20 - 20 - 4\lambda$$

$$= -2\lambda^2 - 16\lambda + 410 - 40 - 4\lambda$$

$$= -2\lambda^2 - 12\lambda + 370$$

Hortance Irakoze

Denis Mitali

$= (-2)(\lambda^2 + 8\lambda - 205] - 20 - 20 - 4\lambda$

$= -2\lambda^2 + 16\lambda + 410 - 40 - 4\lambda$

$= -2\lambda^2 - 12\lambda + 370$

$$\det(M_{13}) = \begin{vmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{vmatrix}$$

$= (-2)[10(-13-\lambda) - (-10)(-13)] -$

$(-9-\lambda)[0(-13-\lambda) - (-10)(-1) +$

$(-4)[0(-13) - 10(-1)]$

$= (-2)[-130 - 10\lambda - 130] + (9+\lambda)$

$[0-10] - 4(0+10)$

$= (-2)[-260 - 10\lambda] + (9+\lambda)(-10) - 40$

$= 520 + 20\lambda - 90 - 10\lambda - 40$

$= 10\lambda + 390$

$$\det(M_{14}) = \begin{vmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{vmatrix}$$

$= (-2)[10(5-\lambda) - (-10)(-13)] -$

$(-9-\lambda)[0(5-\lambda) - (-10)(-1)] +$

$(-2)[0(-13) - 10(-1)]$

$= (-2)[50 - 10\lambda - 130] + 9+\lambda[0-10]$

$-2[0+10]$

$= (-2)[-80 - 10\lambda] + (9+\lambda)[-10] - 20$

$= 160 + 20\lambda - 90 - 10\lambda - 20$

$= 10\lambda + 50$

$\det(M_{11}) = -\lambda^3 - 17\lambda^2 + 165\lambda + 1625$

$\det(M_{12}) = -2\lambda^2 - 12\lambda + 370$

$\det(M_{13}) = 10\lambda + 390$

$\det(M_{14}) = 10\lambda + 50$

$$\det(A-\lambda I) = (4-\lambda)(-\lambda^3-17\lambda^2+165\lambda+1625)-8(-2\lambda^2-12\lambda+370)$$
$$+(-1)(-10\lambda+390)-(-2)(10\lambda+50)$$

$$= -4\lambda^3-68\lambda^2+660\lambda+6500+\lambda^4+17\lambda^3-165\lambda^2$$
$$-1625\lambda+16\lambda^2+96\lambda-2960-10\lambda-390+20\lambda+100$$
$$= -4\lambda^3-68\lambda^2+660\lambda+6500+\lambda^4+17\lambda^3-165\lambda^2-1625\lambda$$
$$+16\lambda^2+96\lambda-2960-10\lambda-390+20\lambda+100$$

$$= \lambda^4+13\lambda^3-219\lambda^2-835\lambda+3500$$

Let's use $p(\lambda)=0$

$$p(\lambda) = \lambda^4+13\lambda^3-219\lambda^2-835\lambda+3500$$

$\lambda=2$, $P(2) = (2)^4+13(2)^3-219(2)^2-835+3500$

$= 1074 \neq 0$

$\lambda=3$, $P(3) = (3)^4+13(3)^3-219(3)^2-835(3)+3500$

$= -544 \neq 0$

$\therefore$, λ is between 2 and 3

$\lambda = 2\cdot5$, $P(2\cdot5) = (2\cdot5)^4+13(2\cdot5)^3-219(2\cdot5)^2-835(2\cdot5)$

$+3500 = 286.94 \neq 0$

$\lambda = 2.7$, $p(2.7) = (2.7)^4 + 13(2.7)^3 - 219(2.7)^2 - 835(2.5) + 3500$

$= -51.13 \neq 0$

$\lambda = 2.675$, $p(2.675) = 51.18 + 248.36 - 1567.07 - 2239.13$

$+ 3500 = -6.66 \neq 0$

$\lambda = 2.67$, $p(2.67) = 50.79 + 246.56 - 1560.89 - 233.95 + 3500$

$= -1.49$

$\lambda = 2.674$, $p(2.674) = 51.15 + 248.17 - 1566.08 - 2238.01 + 3500$

$= -0.77$

$\lambda = 2.6746$, $p(2.6746) = 51.151 + 248.157 - 1566.288$

$- 2232191 + 3500$

$= -0.0067 \approx 0$

$$\underline{\underline{\lambda_2 = 2.6746}}$$

For the eigen vector, $\lambda \approx 2.675$

$(A - \lambda I)\vec{v} = 0$, $\vec{v} = [x, y, z, w]$

$$A - 2.675I = \begin{bmatrix} 1.325 & 8 & 2 \\ -2 & -11.675 & -4 \\ 0 & 10 & -10 \\ -1 & -15 & -15.67 \end{bmatrix}$$

By setting up the augmented matrix.

$$\left[\begin{array}{cccc|c} 1.325 & 8 & -1 & -2 & 0 \\ -2 & -11.675 & -2 & -4 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & -13 & -14 & -15.675 & 0 \end{array}\right]$$

By using Gaussian elimination,

1. $R_1 \div 1.325 = \left[\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5044 & 0 \\ -2 & -11.675 & -2 & -4 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & -13 & -14 & -15.675 & 0 \end{array}\right]$

2. We'll eliminate Row 1

$R_2 + 2R_1$

$R_4 + R_1$

$$\left|\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.05044 & 0 \\ 0 & 0.4004 & -3.5094 & -7.0189 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ 0 & -6.9623 & -14.7547 & -17.1844 & 0 \end{array}\right.$$

3. Divide R_2 by 0.4004

$R_2 \div 0.4004 =$

$$\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5094 & 0 \\ 0 & 1 & -8.766 & -17.533 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ 0 & -6.9623 & -14.7547 & -17.1844 & 0 \end{array}$$

4. We'll eliminate using R_2

$R_3 - 10 R_2$

$R_4 + 6.9623 \cdot R_2$

$$= \left[\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5044 & 0 \\ 0 & 1 & -8.768 & -17.533 & 0 \\ 0 & 0 & 90.005 & 165.33 & 0 \\ 0 & 0 & 46.257 & 105.3 & 0 \end{array}\right]$$

Back Substitution, from the row reduced system;

Row 3 = $90.005z + 165.33w = 0$

$$z = \frac{-165.33w}{90.005} \quad , z = -1.837w$$

Row 2 = $y - 8.768z - 17.533w = 0$

$$y = 8.768z + 17.533w$$

Since $z = -1.837w$

$$y = 8.768(-1.837)w + 17.533w$$

Row 4 = $46.257w + 105.3 = 0$

$$w = \frac{-105.3}{46.257} = \underline{-2.277}$$

$z = -1.837 \times -2.277 = 4.1\underline{82}$

$y = 8.768(4.18) + 17.533(-2.277)$

$y = 36.655 - 39.91 = -3.2\underline{55}$

$$x + 6.0377y - 0.7547z - 1.5044w = 0$$

$$x = -6.0377(-3.255) + 0.7547(4.18) + 1.5044(-2.277)$$

$$x = 19.66 + 3.157 - 3.425 = 19.392$$

$$\vec{V} = \begin{bmatrix} 19.392 \\ -3.255 \\ 4.182 \\ -2.277 \end{bmatrix}$$

Now, Let's normalize eigen vector $\vec{V}$

$$\|\vec{V}\| = \sqrt{(19.392)^2 + (-3.255)^2 + (4.182)^2 + (-2.277)^2}$$

$$\|\vec{V}\| = \sqrt{376.05 + 10.59 + 17.49 + 5.18}$$

$$\|\vec{V}\| = \sqrt{409.31}$$

$$\|\vec{V}\| = 20.23$$

$$\vec{V}_{norm} = \frac{1}{20.23} \begin{pmatrix} 19.392 \\ -3.255 \\ 4.182 \\ -2.277 \end{pmatrix} = \begin{pmatrix} 0.958 \\ -0.161 \\ 0.206 \\ -0.113 \end{pmatrix}$$

$\therefore$, The eigen vector for the eigen value, $\lambda_2 = 2.675$ is $\begin{pmatrix} 19.392 \\ -3.255 \\ 4.182 \\ -2.277 \end{pmatrix}$,

and the normalized vector is $\begin{pmatrix} 0.958 \\ -0.161 \\ 0.206 \\ -0.113 \end{pmatrix}$.

Importance of $\lambda_2 = 2.675$ in terms of % $= \dfrac{|\lambda|}{\Sigma|\lambda_i|} \times 100\%$

$\lambda_1 = 11.05$, $\lambda_2 = 2.675$, $\lambda_3 = -21.1$, $\lambda_4 = -5.60$

$$= \frac{2.675}{|11.05| + |2.675| + |-21.12| + |5.60|} \times 100\%$$

Importance of $\lambda_2 = 6.\underline{6}1\%$

Davy Ngamije

So $p(\lambda) = \lambda^4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500$

By analyzing the polynomial's behavior.

* for $\lambda = 0$. $\quad p(\lambda) = 3500 > 0$.

* for $\lambda = -8$ $\quad p(-8) = (-8)^4 + 13(-8)^3 - 219(-8)^2 - 835(-8) + 3500$

$$= 4096 + 13(-512) - 219 \cdot 64 + 835 \cdot 8 + 3500$$

$$= 14276 - 20672 = -6396 < 0$$

for ⊛ $\lambda = -6$ $\Leftrightarrow$ $P(-6) = (-6)^4 + 13(-6)^3 - 219(-6^2) - 835(-6) + 3500$

$= 1296 - 2808 - 7884 + 5010 + 3500$

$= -886 < 0$

~~⊛ λ = 5~~

The nearest root is between $(-6, -5)$.

⊛ $\lambda = 5$ $\quad P(5) = (5)^4 + 13(5)^3 - 219(5^2) - 835(5) + 3500$

$= 625 + 13(55) - 219(25) - 4175 + 3500$

$= 625 + 715 - 5475 - 4175 + 3500$

$= -5,433 < 0$

⊛ for $\lambda = -5.603$

$P(-5.603) = (-5.603)^4 + 13(-5.603)^3 - 219(-5.603)^2 - 835(-5.603) + 3500$

$= 985.97 - 2291.97 - 6869.97 + 4685.51 + 3500$

$= 0$

⊛ for $\lambda = 2.606$

$P(2.606) = (2.606)^4 + 13(2.603)^3 - 219(2.606)^2 - 835(2.606) + 3500$

$= 46.13 + 230.13 - 1487.13 - 2176.01 + 3500$

$= 0$

⊛ for $\lambda = 11.058$

$P(11.058) = (11.058)^4 + (13)(11.058)^3 - 219(11.058)^2 - 835(11.058)^2 + 3500$

$= 14934.97 + 17589.97 - 26745.97 - 9233.43 + 3500$

$= 0$

⊛ for $\lambda = -21.061$

$P(-21.061) = (-21.061)^4 + (13)(-21.061)^3 - 219(-21.061)^2 - (835)(-21.061) + 3500$

$= 196683.97 - 120853.97 - 97089.97 + 17585.94 + 3500$

$= 0$

$\Rightarrow$ eigen values, $\lambda_1 = -5.603$, $\lambda_2 = 2.606$, $\lambda_3 = 11.058$, $\lambda_4 = -21.061$

By replacing the eigen values in the matrix

for $\lambda = -5.603$

$$\begin{bmatrix} 9.603 & 8 & -1 & -2 \\ -2 & -3.393 & -2 & -4 \\ 0 & 10 & 10.603 & -10 \\ -1 & -13 & -14 & -7.397 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By solving this we get $= (0.5, 1, 0.5, 1)$.

* for $\lambda_2 = 2.606$

$$\begin{pmatrix} 1.394 & 8 & -1 & -2 \\ -2 & -11.606 & -2 & -4 \\ 0 & 10 & 2.394 & -10 \\ -1 & -13 & -14 & -15.606 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By solving that we get $\lambda_2 = [-2, 1, -1, 1]$

* for $\lambda_3 = 11.058$

$$\begin{pmatrix} -7.058 & 8 & -1 & -2 \\ -2 & -20.058 & -2 & -4 \\ 0 & 10 & -6.058 & -10 \\ -1 & -13 & -14 & -24.058 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By solving that $\lambda_3 = [-1, 1, 0, 1]$

* for $\lambda_4 = -21.061$

$$\begin{pmatrix} 25.061 & 8 & -1 & -2 \\ -2 & -20.058 & -2 & -4 \\ 0 & 10 & -6.058 & -10 \\ -1 & -13 & -14 & \cancel{-24.058}\ 8.061 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 25.061 & 8 & -1 & -2 \\ -2 & -20.058 & -2 & -4 \\ 0 & 10 & 26.061 & -10 \\ -1 & -13 & -14 & 8.061 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By solving that we get $\lambda_4 = [1, 0, 0, 1]$

Makuochi Prince

Page 1

$$A = \begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ 1 & -13 & 14 & -13 \end{pmatrix}$$

Set up to find the eigenvalues,

$$\det(A - \lambda I) = 0$$

That is

$$\begin{vmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -9-\lambda & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ 1 & -13 & -14 & -13-\lambda \end{vmatrix} = 0$$

Determinant

The determinant of a 4x4 Matrix Can be expand as:

$$\det(A-\lambda I) = (4-\lambda)\begin{vmatrix} -9-\lambda & -2 & -4 \\ 10 & 5-\lambda & -10 \\ -13 & -14 & -13-\lambda \end{vmatrix} - 8\begin{vmatrix} -2 & -2 & -4 \\ 0 & 5-\lambda & -10 \\ -1 & -14 & -13-\lambda \end{vmatrix}$$

$$+(-1)\begin{vmatrix} -2 & -9-\lambda & -4 \\ 0 & 10 & -10 \\ -1 & -13 & -13-\lambda \end{vmatrix} - (-2)\begin{vmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{vmatrix}$$

Let expand along the first row

$$M_{11} = \begin{vmatrix} -9-\lambda & -2 & -4 \\ 10 & 5-\lambda & -10 \\ -13 & -14 & -13-\lambda \end{vmatrix}$$

Expand along the first row

$$= (-9-\lambda)\begin{vmatrix} 5-\lambda & -10 \\ -14 & -13-\lambda \end{vmatrix} - (-2)\begin{vmatrix} 10 & -10 \\ -13 & -13-\lambda \end{vmatrix}$$

$$+(-4)\begin{vmatrix} 10 & 5-\lambda \\ -13 & -14 \end{vmatrix}$$

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Now, expand each 2x2 minor

$$\begin{vmatrix} 5-\lambda & -10 \\ -14 & -13-\lambda \end{vmatrix} = |5-\lambda|\,|(-13-14) - (-10)(-14)$$

$$= -(5-\lambda)(13+\lambda) - 140$$

$$\begin{vmatrix} 10 & -10 \\ -13 & -13-\lambda \end{vmatrix} = 10(-13-\lambda) - (-10)(-13) = -130-$$

$$10\lambda + 130 = -10\lambda$$

$$\begin{vmatrix} 10 & 5-\lambda \\ -13 & -14 \end{vmatrix} = 10\cdot(-14) - (-13)(5-\lambda) = -140 +$$

$$13(5-\lambda) = -140 + 65 - 13\lambda = -75$$

So

$$M_{11} = (-9-\lambda)\,[-(5-\lambda)(13+\lambda) - 140] + 2$$

$$[-10\lambda] - 4\,(-75 - 13\lambda)$$

Now expand

$$M_{11} = (-9-\lambda)\,[-(5-\lambda)(13+\lambda) - 140]$$

$$20\lambda + 300 + 52\lambda$$

$$P(\lambda) = a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$

$$\lambda = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7,$$
$$\pm 8, \pm 9, \pm 10,$$

Evaluate $P(\lambda)$ at Integers

$$P(1) = a_4(1)^4 + a_3(1)^3 + a_3(1)^3 + a_2(1)^2$$
$$+ a_1(1)^2 + a_1(1) + a_0$$

$\lambda = 1$ is a root.

$$\lambda_4 = -21.125$$

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Set up the System for Eigenvector
plug λ_4 into $A-\lambda_4 I$:

$$A-\lambda_4 I = \begin{pmatrix} 4-(-21.125) & 8 & -1 & -2 \\ -2 & -9-(-21.125) & -2 & -4 \\ 0 & 10 & 5-(-21.125) & -10 \\ -1 & -13 & -14 & -13-(-21.125) \end{pmatrix}$$

$$25.125x_1 + 8x_2 - x_3 - 2x_4 = 0$$

$$-2x_1 + 12.125x_2 - 2x_3 - 4x_4 = 0$$
$$10x_2 + 26.125x_3 - 10x_4 = 0$$
$$-x_1 - 13x_2 - 14x_3 + 8.125x_4 = 0$$

Row3 gives

$$10x_2 + 26.125x_3 - 10x_4 = 0 \Rightarrow x_2 = \frac{10x_4 - 26.125x_3}{10}$$

Row 1:

$$25.125x_1 + 8x_2 - x_3 - 2x_4 = 0$$

$$25.125x_1 + 8\left(\frac{10x_4 - 26.125x_3}{10}\right) - x_3 - 2x_4 = 0$$

$$25.125x_1 + 8x_4 - 20.9x_3 - x_3 - 2x_4 = 0$$
$$25.125x_1 + 6x_4 - 21.9x_3 = 0$$
$$x_1 = \frac{21.9x_3 - 6x_4}{25.125}$$

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Plug x_3 into Row 2:

$$-2x_1 + 12.125x_2 - 2x_3 - 4x_4 = 0$$

$$-2x_1 + 12.125\left(\frac{10x_4 - 26.125x_3}{10}\right) - 2x_3 - 4x_4$$

$$= 0$$

$$-2x_1 + 8.125x_4 - 33.926x_3 = 0$$

$$2x_1 = 8.125x_4 - 33.726x_3 = 0$$

$$2x_1 = 8.125x_4 - 33.926x_3$$

$$x_1 = \frac{8.125x_4 - 33.726x_3}{2}$$

Set equal to the previous expression for x_1:

$$\frac{21.9x_3 - 6x_4}{25.125} = \frac{8.125x_4 - 33.726x_3}{2}$$

eigenvectors

$$v_1 \begin{pmatrix} -0.036 \\ 0.958 \\ 0.025 \\ 0.563 \end{pmatrix} \qquad v_4 = \begin{pmatrix} 0.509 \\ -0.113 \\ -0.915 \\ -0.053 \end{pmatrix}$$

$$v_2 \begin{pmatrix} -0.012 \\ -0.161 \\ -0.335 \\ -0.616 \end{pmatrix}$$

$$v_3 \begin{pmatrix} -0.860 \\ 0.207 \\ -0.222 \\ 0.546 \end{pmatrix}$$

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Importance Of Each Eigenvalue

$$\text{Improtance} = \frac{|\lambda_i|}{\sum_{j=1}^{4} |\lambda_j|} \times 100\%$$

$$S = |11.054| + |2.675| + |5.604| + |21.125| = 40.458$$

$$\lambda_4 : \frac{21.125}{40.458} \times 100\%$$

$$= 52.21\%$$

Eigenvalue : $\lambda_4 = -21.125$

Eigenvector : $V_4 \begin{pmatrix} 0.509 \\ -0.113 \\ -0.915 \\ -0.033 \end{pmatrix}$

Importance : 52.21%