Denis Mitali

$$A=\begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ 1 & -13 & -14 & -13 \end{pmatrix}$$

$$\det(A-\lambda I)=0$$

$$A-\lambda I=\begin{vmatrix} 4-\lambda & 8 & -1 & -2 \\ 2 & -9-\lambda & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ 1 & -13 & -14 & -13-\lambda \end{vmatrix}$$

$$\det(A-\lambda I)=(4-\lambda)\cdot\det(M_{11})-8\det(M_{12})$$
$$-1\cdot\det(M_{13})+2\cdot\det(M_{14})$$

$$\det(M_{11})=\begin{vmatrix} -9-\lambda & -2 & -4 \\ 10 & 5-\lambda & -10 \\ -13 & -14 & -13-\lambda \end{vmatrix}$$

$$=(-9-\lambda)\left[(5-\lambda)(-13-\lambda)-(-10)(-14)\right]$$
$$-(-2)\left[(10)(-13-\lambda)-(-10)(-13)\right]$$
$$+(-4)\left[(10)(-14)-(5-\lambda)(-13)\right]$$

$$=(-9-\lambda)\left[-65-5\lambda+13\lambda+\lambda^2-140\right]$$
$$+2\left[-130-10\lambda-130\right]-140+65-13\lambda$$
$$=(-9-\lambda)\left[\lambda^2+8\lambda-205\right]+2\left[-260-10\lambda\right]$$
$$-4\left[-75-13\lambda\right]$$
$$=-9\lambda^2-72\lambda+1845-\lambda^3-8\lambda^2+205\lambda$$
$$-520-20\lambda+300+52\lambda$$
$$=-\lambda^3-17\lambda^2+165\lambda+1625$$

$$\det(M_{12})=\begin{vmatrix} -2 & -2 & -4 \\ 0 & 5-\lambda & -10 \\ -1 & -14 & -3-\lambda \end{vmatrix}$$

$$=(-2)\left[(5-\lambda)(-13-\lambda)-(-10)(-14)\right]$$
$$(-2)\left[0(-13-\lambda)-(-10)(-1)\right]$$
$$(-4)\left[0(-14)-(5-\lambda)(-1)\right]$$

$$=(-2)\left[(5-\lambda)(-13-\lambda)-140\right]$$
$$+2(0-10)-4(0+(5-\lambda))$$
$$=(-2)(-65-5\lambda+13\lambda+\lambda^2-140]+$$
$$2(-10)-4(5-\lambda)$$

$= (-2)(\lambda^2 + 8\lambda - 205] - 20 - 20 - 4\lambda$

$= -2\lambda^2 + 16\lambda + 410 - 40 - 4\lambda$

$= -2\lambda^2 - 12\lambda + 370$

$$\det(m_{13}) = \begin{vmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{vmatrix}$$

$= (-2)[10(-13-\lambda) - (-10)(-13)] -$

$(-9-\lambda)[0(-13-\lambda) - (-10)(-1) +$

$(-4)[0(-13) - 10(-1)]$

$= (-2)[-130 - 10\lambda - 130] + (9+\lambda)$

$[0-10] - 4(0+10)$

$= (-2)[-260 - 10\lambda] + (9+\lambda)(-10) - 40$

$= 520 + 20\lambda - 90 - 10\lambda - 40$

$= 10\lambda + 390$

$$\det(m_{14}) = \begin{vmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ -1 & -13 & -14 \end{vmatrix}$$

$= (-2)[10(5-\lambda) - (-10)(-13)] -$

$(-9-\lambda)[0(5-\lambda) - (-10)(-1)] +$

$(-2)[0(-13) - 10(-1)]$

$= (-2)[50 - 10\lambda - 130] + 9+\lambda[0-10]$

$-2[0+10]$

$= (-2)[-80 - 10\lambda] + (9+\lambda)[-10] - 20$

$= 160 + 20\lambda - 90 - 10\lambda - 20$

$= 10\lambda + 50$

$\det(m_{11}) = -\lambda^3 - 17\lambda^2 + 165\lambda + 1625$

$\det(m_{12}) = -2\lambda^2 - 12\lambda + 370$

$\det(m_{13}) = 10\lambda + 390$

$\det(m_{14}) = 10\lambda + 50$

$$\det(A-\lambda I) = (4-\lambda)(-\lambda^3-17\lambda^2+165\lambda+1625)-8(-2\lambda^2-12\lambda+370)$$
$$+(-1)(-10\lambda+390)-(-2)(10\lambda+50)$$

$$=-4\lambda^3-68\lambda^2+660\lambda+6500+\lambda^4+17\lambda^3-165\lambda^2$$
$$-1625\lambda+16\lambda^2+96\lambda-2960-10\lambda-390+20\lambda+100$$
$$=-4\lambda^3-68\lambda^2+660\lambda+6500+\lambda^4+17\lambda^3-165\lambda^2-1625\lambda$$
$$+16\lambda^2+96\lambda-2960-10\lambda-390+20\lambda+100$$

$$=\lambda^4+13\lambda^3-219\lambda^2-835\lambda+3500$$

Let's use $p(\lambda)=0$

$$p(\lambda)=\lambda^4+13\lambda^3-219\lambda^2-835\lambda+3500$$

$\lambda=2$, $P(2)=(2)^4+13(2)^3-219(2)^2-835+3500$
$$=1074\neq 0$$

$\lambda=3$, $P(3)=(3)^4+13(3)^3-219(3)^2-835(3)+3500$
$$=-544\neq 0$$

$\therefore$, λ is between 2 and 3

$\lambda=2.5$, $P(2.5)=(2.5)^4+13(2.5)^3-219(2.5)^2-835(2.5)$
$$+3500=286.94\neq 0$$

$\lambda = 2.7$, $p(2.7) = (2.7)^4 + 13(2.7)^3 - 219(2.7)^2 - 835(2.5) + 3500$

$= -51.13 \neq 0$

$\lambda = 2.675$, $p(2.675) = 51.18 + 248.36 - 1567.07 - 2235.13$

$+ 3500 = -6.66 \neq 0$

$\lambda = 2.67$, $P(2.67) = 50.79 + 246.56 - 1560.89 - 235.95 + 3500$

$= -1.49$

$\lambda = 2.674$, $P(2.674) = 51.15 + 248.17 - 1566.08 - 2238.01 + 3500$

$= -0.77$

$\lambda = 2.6746$, $P(2.6746) = 51.151 + 248.157 - 1566.288$

$- 2232.191 + 3500$

$= -0.0067 \approx 0$

$\lambda_2 = 2.\underline{6}746$

For the eigen vector, $\lambda \approx 2.675$

$(A - \lambda I)\vec{v} = 0$, $\vec{v} = [x, y, z, w]$

$$A - 2.675 I = \begin{bmatrix} 1.325 & 8 & 2 \\ -2 & -11.675 & -4 \\ 0 & 10 & -10 \\ -1 & -15 & -15.67 \end{bmatrix}$$

By setting up the augmented matrix.

$$\left[\begin{array}{cccc|c} 1.325 & 8 & -1 & -2 & 0 \\ -2 & -11.675 & -2 & -4 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & -13 & -14 & -15.675 & 0 \end{array}\right]$$

By using Gaussian elimination,

1. $R_1 \div 1.325 = \left[\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5044 & 0 \\ -2 & -11.675 & -2 & -4 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & -13 & -14 & -15.675 & 0 \end{array}\right]$

2. We'll eliminate Row 1

$R_2 + 2R_1$

$R_4 + R_1$

$$\left|\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.05044 & 0 \\ 0 & 0.4004 & -3.5094 & -7.0189 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ 0 & -6.9623 & -14.7547 & -17.1844 & 0 \end{array}\right.$$

3. Divide R_2 by 0.4004

$R_2 \div 0.4004 =$

$$\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5094 & 0 \\ 0 & 1 & -8.766 & -17.533 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ 0 & -6.9623 & -14.7547 & -17.1844 & 0 \end{array}$$

4. We'll eliminate using R_2

$$\begin{array}{l} R_3 - 10R_2 \\ R_4 + 6.9623 \cdot R_2 \end{array} = \left[\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5044 & 0 \\ 0 & 1 & -8.768 & -17.533 & 0 \\ 0 & 0 & 90.005 & 165.33 & 0 \\ 0 & 0 & 46.257 & 105.3 & 0 \end{array}\right]$$

Back substitution, from the row reduced system;

Row 3 = $90.005z + 165.33w = 0$

$$z = \frac{-165.33w}{90.005}, \quad z = -1.837w$$

Row 2 = $y - 8.768z - 17.533w = 0$

$$y = 8.768z + 17.533w$$

Since $z = -1.837w$

$$y = 8.768(-1.837)w + 17.533w$$

Row 4 = $46.257w + 105.3 = 0$

$$w = \frac{-105.3}{46.257} = -2.277$$

$$z = -1.837 \times -2.277 = 4.182$$

$$y = 8.768(4.18) + 17.533(-2.277)$$

$$y = 36.655 - 39.91 = -3.255$$

$$x + 6.0377y - 0.7547z - 1.5044w = 0$$

$$x = -6.0377(-3.255) + 0.7547(4.18) + 1.5044(-2.2771)$$

$$x = 19.66 + 3.157 - 3.425 = 19.392$$

$$\vec{V} = \begin{bmatrix} 19.392 \\ -3.255 \\ 4.182 \\ -2.277 \end{bmatrix}$$

Now, let's normalize eigen vector $\vec{V}$

$$\|\vec{V}\| = \sqrt{(19.392)^2 + (-3.255)^2 + (4.182)^2 + (-2.277)^2}$$

$$\|\vec{V}\| = \sqrt{376.05 + 10.59 + 17.49 + 5.18}$$

$$\|\vec{V}\| = \sqrt{409.31}$$

$$\|\vec{V}\| = 20.23$$

$$\vec{V}_{norm} = \frac{1}{20.23}\begin{pmatrix} 19.392 \\ -3.255 \\ 4.182 \\ -2.277 \end{pmatrix} = \begin{pmatrix} 0.958 \\ -0.161 \\ 0.206 \\ -0.113 \end{pmatrix}$$

$\therefore$, The eigen vector for the eigen value, $\lambda_2 = 2.675$ is $\begin{pmatrix} 19.392 \\ -3.255 \\ 4.182 \\ -2.277 \end{pmatrix}$,

and the normalized vector is $\begin{pmatrix} 0.958 \\ -0.161 \\ 0.206 \\ -0.113 \end{pmatrix}$

Importance of $\lambda_2 = 2.675$ in terms of % $= \dfrac{|\lambda|}{\Sigma|\lambda_i|} \times 100\%$

$\lambda_1 = 11.05$, $\lambda_2 = 2.675$, $\lambda_3 = -21.1$, $\lambda_4 = -5.60$

$$= \frac{2.675}{|11.05| + |2.675| + |-21.12| + |5.60|} \times 100\%$$

Importance of $\lambda_2 = \underline{\underline{6.61}}\%$