Game Dynamics as the Meaning of a Game

Christos Papadimitriou and Georgios Piliouras

Aditya Makkar

10 April 2024

Nash Equilibrium

The setting is standard: $(K, \mathsf{S} = \bigtimes_{k\in[K]} \mathsf{S}_k, (u_k)_{k\in[K]})$ denotes a K player game, with strategy of player k chosen from S_k and $u_k \colon \mathsf{S} \to \mathbb{R}$ denoting its utility function.

A mixed strategy profile $x = (x_k)_{k\in[K]}$ is a (mixed) *Nash equilibrium* if no player has an incentive to deviate, i.e., $u_k(x) \geq u_k(x'_k, x_{-k})$ for all x'_k.

Nash Equilibrium

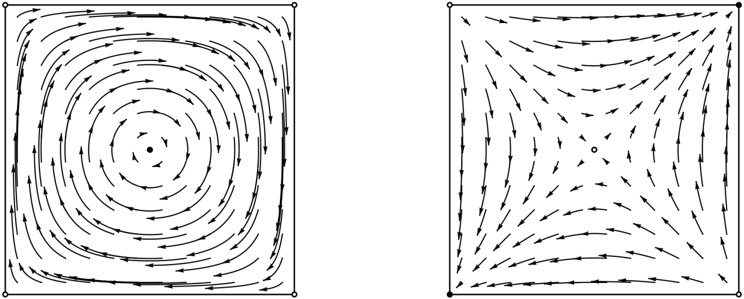
There are many problems with Nash equilibria:

1. **Computing** a Nash equilibrium is PPAD-complete.
2. There are often many Nash equilibria, leading to intractable equilibrium **selection** issues even if Nash equilibria can be computed.
3. It is not guaranteed that dynamical systems (dynamics caused by learning) will **converge** to a Nash equilibrium.

Game Dynamics

		Player 2	
		H	T
Player 1	H	$(1,-1)$	$(-1,1)$
	T	$(-1,1)$	$(1,-1)$

		Player 2	
		A	B
Player 1	A	$(1,1)$	$(-1,-1)$
	B	$(-1,-1)$	$(1,1)$



(a) Matching Pennies game.

(b) Partnership, coordination game.

Figure 1: Replicator dynamics trajectories for two games. Each point (x, y) in the unit square encodes a mixed strategy and gives the probability assigned by the players to their first strategy. Image source: [Omidshafiei et al. 2019]

Game Dynamics

- Since no dynamics can converge to a Nash equilibrium in every game, to which object do natural dynamics (FTRL, for example) converge?
- An answer comes from applying deep results from dynamical systems theory to game dynamics.
- A beautiful theory involving analysis, topology and differential equations.
- An algorithmic theory, rather than the non-constructive notion of Nash equilibria. See [Daskalakis et al. 2009] for more details.

Game Dynamics

- Let X denote the space of all mixed strategy profiles of a game.
- Focusing on discrete time, let $\phi\colon \mathsf{X} \to \mathsf{X}$ denote the **game dynamics**, i.e., $\phi(x)$ denotes the evolution of mixed strategy x after one time step.
- A finite sequence of points $x_0, x_1, \ldots, x_n$ is called an **ε-chain** if for all $i < n$, $|x_{i+1} - \phi(x_i)| < \varepsilon$.
- Two points $x, y \in \mathsf{X}$ are **chain equivalent**, written $x \sim y$, if for every $\varepsilon > 0$ there is an ε-chain from x to y and an ε-chain from y to x.
- The relation $\sim$ is an equivalence relation. The equivalence classes are called **chain components**.
- The union of all chain components is called the **chain recurrent** part $\mathcal{R}(\mathsf{X}, \phi)$ of the dynamics.
- Conley (1978) showed that any dynamical system will eventually converge to its recurrent part.

Game Dynamics – Results

- **Zero-Sum Games**: Under replicator dynamics for a zero-sum game with a fully mixed Nash equilibrium, the whole space X is a chain component.
- **Weighted Potential Games**: Under replicator dynamics for a weighted potential game, the set of chain recurrent points coincides with its set of Nash equilibria.

Markov-Conley Chains of a Game

- **Response Graph of a Game**: A directed graph whose vertices are pure strategy profiles, i.e., S, and there is an edge from s to s' if $u_k(s') \geq u_k(s)$ for some player $k \in [K]$.
- **Markov-Conley Chain (MCC)**: The state space is a sink strongly connected component. The transition probabilities are uniform.
- There are non-zero polynomially many sink strongly connected components. Therefore, same for MCCs.
- MCCs can be computed in polynomial time.
- Pure Nash equilibria are MCCs.
- There is a class of natural dynamics that converges to the MCCs almost surely.
- MCCs are tangible surrogates of the game dynamics' sink chain components. Therefore, MCCs can be thought of as discretizations of chain components.

References

1. Christos Papadimitriou and Georgios Piliouras. *Game Dynamics as the Meaning of a Game*. ACM SIGecom Exchanges, Vol. 16, No. 2, June 2018, Pages 53–63.
2. Christos Papadimitriou and Georgios Piliouras. *From Nash Equilibria to Chain Recurrent Sets: An Algorithmic Solution Concept for Game Theory*. Entropy 2018, 20, 782; doi:10.3390/e20100782.
3. Constantinos Daskalakis, Paul W. Goldberg, and Christos Papadimitriou. *The Complexity of Computing a Nash Equilibrium*. SIAM Journal of Computing, Vol. 39, No. 1, pp. 195-259, 2009.
4. Shayegan Omidshafiei, Christos Papadimitriou, Georgios Piliouras, Karl Tuyls, Mark Rowland, Jean-Baptiste Lespiau, Wojciech M. Czarnecki, Marc Lanctot, Julien Perolat, and Remi Munos. *α-Rank: Multi-Agent Evaluation by Evolution*. Nature, Scientific Reports, volume 9, Article number: 9937 (2019).
5. Charles Conley. *Isolated invariant sets and the morse index*. CBMS Regional Conference Series in Mathematics, Volume 38, 1978. American Mathematical Society, Providence, RI 16.