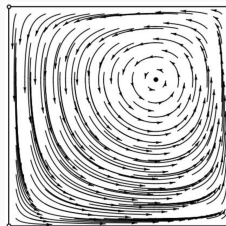# 1. Game Dynamics as the Meaning of a Game (2018)
2. From Nash Equilibria to Chain Recurrent Sets: An Algorithmic Solution Concept for Game Theory (2018)

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The setting is standard: $(K, \mathsf{S} = \bigtimes_{k \in [K]} \mathsf{S}_k, (u_k)_{k \in [K]})$ denotes a K player game, with strategy of player k chosen from S_k and $u_k \colon \mathsf{S} \to \mathbb{R}$ denoting its utility function.

A mixed strategy profile $x = (x_k)_{k \in [K]}$ is a (mixed) *Nash equilibrium* if no player has an incentive to deviate, i.e., $u_k(x) \geq u_k(x'_k, x_{-k})$ for all x'_k.

Nash Equilibrium

There are many problems with Nash equilibria:

1. **Computing** a Nash equilibrium is PPAD-complete.
2. There are often many Nash equilibria, leading to intractable equilibrium **selection** issues even if Nash equilibria can be computed.
3. It is not guaranteed that dynamical systems (dynamics caused by learning) will **converge** to a Nash equilibrium.
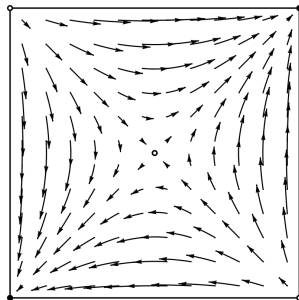
Game Dynamics

	Player 2	
	H	T
Player 1 H	(1,−1)	(−1,1)
T	(−1,1)	(1,−1)

	Player 2	
	A	B
Player 1 A	(1,1)	(−1,−1)
B	(−1,−1)	(1,1)
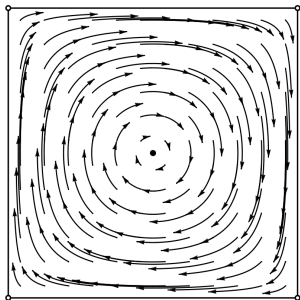
**(a)** Matching Pennies game.   **(b)** Partnership, coordination game.

Figure 1: Replicator dynamics trajectories for two games. Each point (x, y) in the unit square encodes a mixed strategy and gives the probability assigned by the players to their first strategy. Image source: [Omidshafiei et al. 2019]

Game Dynamics

- Since no dynamics can converge to a Nash equilibrium in every game, to which object do natural dynamics (FTRL, for example) converge?
- An answer comes from applying deep results from dynamical systems theory to game dynamics.
- A beautiful theory involving analysis, topology and differential equations.
- An algorithmic theory, rather than the non-constructive notion of Nash equilibria. See [Daskalakis et al. 2009] for more details.

Game Dynamics

- Let X denote the space of all mixed strategy profiles of a game.
- Focusing on discrete time, let $\phi\colon X \to X$ denote the **game dynamics**, i.e., $\phi(x)$ denotes the evolution of mixed strategy x after one time step.
- A finite sequence of points $x_0, x_1, \ldots, x_n$ is called an ε-**chain** if for all $i < n$, $|x_{i+1} - \phi(x_i)| < \varepsilon$.

Game Dynamics
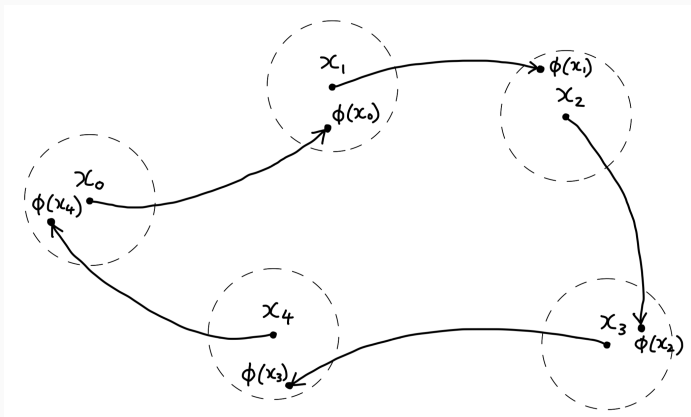
- Two points $x, y \in \mathsf{X}$ are **chain equivalent**, written $x \sim y$, if for every $\varepsilon > 0$ there is an ε-chain from x to y and an ε-chain from y to x.

- The relation $\sim$ is an equivalence relation. The equivalence classes are called **chain components**.

- The union of all chain components is called the **chain recurrent** part $\mathcal{R}(\mathsf{X}, \phi)$ of the dynamics.

- Conley (1978) showed that any dynamical system will eventually converge to its recurrent part.

- **Zero-Sum Games**: Under replicator dynamics for a zero-sum game with a fully mixed Nash equilibrium, the whole space X is a chain component.

- **Weighted Potential Games**: Under replicator dynamics for a weighted potential game, the set of chain recurrent points coincides with its set of Nash equilibria.

Markov-Conley Chains of a Game

- **Response Graph of a Game**: A directed graph whose vertices are pure strategy profiles, i.e., S, and there is an edge from s to s' if $u_k(s') \geq u_k(s)$ for some player $k \in [K]$.

- **Markov-Conley Chain (MCC)**: The state space is a sink strongly connected component. The transition probabilities are uniform.

- There are non-zero polynomially many sink strongly connected components. Therefore, same for MCCs.

- MCCs can be computed in polynomial time.

- Pure Nash equilibria are MCCs.

- There is a class of natural dynamics that converges to the MCCs almost surely.

- MCCs are tangible surrogates of the game dynamics' sink chain components. Therefore, MCCs can be thought of as discretizations of chain components.

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