# Upper and lower limits

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**Abstract**

This note attempts to present the concepts of upper and lower limits in a lucid manner. It grew out of the unsatisfactorily terse treatment of this important topic in Rudin's Principles of Mathematical Analysis. Very basic familiarity with analysis is assumed.

## 1 Preliminaries

In the spirit of keeping this article fairly independent and establishing a common notation, let's review some preliminaries.

**Definition 1.1.** Suppose $S$ is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that $E$ is *bounded above*, and call $\beta$ an *upper bound* of $E$.

Lower bounds are defined in the same way with $\geq$ in place of $\leq$.

**Definition 1.2.** Suppose $S$ is an ordered set, $E \subset S$, and $E$ is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

1. $\alpha$ is an upper bound of $E$.
2. If $\gamma < \alpha$ then $\gamma$ is not an upper bound of $E$.

Then $\alpha$ is called the *least upper bound* of $E$ or the *supremum* of $E$, and we write

$$\alpha = \sup E$$

That there is at most one supremum is clear from the property 2 above. The *greatest lower bound*, or *infimum*, of a set $E$ which is bounded below is defined in the same manner, and we write it as $\inf E$. Note that if $x \in E$, $\alpha = \sup E$, and $x < \alpha$ then there exists $y \in E$ such that $x < y < \alpha$ because otherwise $x$ would a lower bound less than $\alpha$.

**Definition 1.3.** A *sequence* is function defined on the set $\mathbb{N}$ of positive integers. If $f(n) = x_n$, for $n \in \mathbb{N}$, we denote the sequence $f$ by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, \ldots$.

**Definition 1.4.** A sequence $\{p_n\}$ in a metric space $X$ is said to *converge* if there is a point $p \in X$ with the following property: For every $\epsilon > 0$ there is an integer $N$ such that $n \geq N$ implies that $d(p_n, p) < \epsilon$. We write this as $\lim_{n\to\infty} p_n = p$ or as $p_n \to p$.

**Definition 1.5.** Let $\{s_n\}$ be a sequence of real numbers with the following property: For every real $M$ there is an integer $N$ such that $n \geq N$ implies $s_n \geq M$. We then write $s_n \to +\infty$. Similarly for $s_n \to -\infty$.

**Remark.** *The symbol $\to$ is now used for certain type of divergent sequences as well.*

**Definition 1.6.** Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < \cdots$. Then the sequence $\{p_{n_k}\}$, which is a composition of the functions $\{n_k\}$ and $\{p_n\}$, is called a *subsequence* of $\{p_n\}$. If $\{p_{n_k}\}$ converges, its limit is called a *subsequential limit* of $\{p_n\}$.

**Lemma 1.** *The subsequential limits of a sequence $\{p_n\}$ in a metric space $X$ form a closed subset of $X$.*

*Proof.* Let $E$ be the set of all subsequential limits of $\{p_n\}$ and let $q$ be a limit point of $E$. We have to show that $q \in E$. To show this we will construct a subsequence of $\{p_n\}$ which converges to $q$.

Choose $n_1$ so that $p_{n_1} \neq q$. If no such $n_1$ exists, then $E$ has only one element, $q = p_1 = p_2 = \cdots$, and there is nothing to prove. Define $\delta = d(q, p_{n_1})$. Suppose $n_1, \ldots, n_{i-1}$ are chosen. Since $q$ is a limit point of $E$, there is an $x \in E$ with $d(q, x) < \frac{\delta}{2^i}$. Since $x \in E$, there is an $n_i > n_{i-1}$ such that $d(x, p_{n_i}) < \frac{\delta}{2^i}$. Thus

$$d(q, p_{n_i}) \leq d(q, x) + d(x, p_{n_i}) < \frac{\delta}{2^{i-1}}$$

for $i = 1, 2, \ldots$. This implies $\{p_{n_k}\} \to q$, and thus $q \in E$. $\square$

**Definition 1.7.** A sequence $\{s_n\}$ of real numbers is said to be

1. *monotonically increasing* if $s_n \leq s_{n+1}$ for $n \in \mathbb{N}$;

2. *monotonically decreasing* if $s_n \geq s_{n+1}$ for $n \in \mathbb{N}$.

A sequence is *monotonic* if it is either monotonically increasing or monotonically decreasing.

**Lemma 2.** *Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.*

*Proof.* Suppose $\{s_n\}$ is monotonically increasing (the proof for monotonically decreasing is similar). Let $E$ be the range of $E$. If $\{s_n\}$ is bounded, let $s = \sup E$. Then

$$s_n \leq s, \quad n \in \mathbb{N}$$

For every $\epsilon > 0$, there is an integer $N$ such that

$$s - \epsilon < s_N \leq s$$

for otherwise $s - \epsilon$ would be an upper bound of $E$. Since $\{s_n\}$ is increasing,

$$s - \epsilon < s_n \leq s, \quad n \geq N$$

which shows $\{s_n\}$ converges to $s$.

For the converse let $s_n \to p$. There is an integer $N$ such that $n > N$ implies $|p - s_n| < 1$. Put

$$r = \max\{1, |p - s_1|, \ldots, |p - s_N|\}$$

Then $|s - s_n| \leq r$ for all $n \in \mathbb{N}$ and $\{s_n\}$ is bounded. $\square$

## 2 Upper and lower limits

**Definition 2.1.** Let $\{s_n\}$ be a sequence of real numbers. Let $E$ be the set of numbers $x$ (in the extended real number system, i.e. $x \in \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$) such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. Therefore, this set contains all the subsequential limits of $\{s_n\}$ plus possibly the numbers $+\infty$ and $-\infty$. We define

$$s^* = \sup E$$

$$s_* = \inf E$$

The numbers $s^*$ and $s_*$ are called the *upper* and *lower limits* of $\{s_n\}$. We use the notation

$$\limsup_{n \to \infty} s_n = s^*$$

$$\liminf_{n \to \infty} s_n = s_*$$

It immediately follows that $s_* \leq s^*$.

The fact that $E$ is non-empty (and thus taking sup or inf makes sense) follows from the observation that either $\{s_n\}$ is bounded or unbounded. If it is bounded then it must contain a convergent subsequence (Bolzano–Weierstrass theorem) and thus at least one element, or if it is unbounded then it must contain either $+\infty$ or $-\infty$.

**Lemma 3.** *Let $F$ be a nonempty closed set of real numbers which is bounded above. Let $\alpha = \sup F$. Then $\alpha \in F$.*

*Proof.* Assume for the sake of the contradiction that $\alpha \notin F$. Then since $F^{\mathsf{c}}$ is an open set (because $F$ is closed) there exists an $\epsilon > 0$ such that $(\alpha - \epsilon, \alpha + \epsilon) \subset F^{\mathsf{c}}$. But this implies $\alpha - \frac{\epsilon}{2}$ is an upper bound for $F$ which is lower that $\alpha$. This gives us our required contradiction. $\square$

**Theorem 4.** *Let $\{s_n\}$ be a sequence of real numbers. Let $E$ and $s^*$ have the same meaning as in Definition 2.1. Then $s^*$ has the following two properties:*

1. *$s^* \in E$.*

2. *If $x > s^*$, there is an integer $N$ such that $n \geq N$ implies $s_n < x$.*

*Moreover, $s^*$ is the only number with these two properties.*

Of course, an analogous result is true for $s_*$.

*Proof.* We start by showing the two properties.

1. If $s^* = +\infty$, then $E$ is not bounded above; hence $\{s_n\}$ is not bounded above, and thus there is a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \to +\infty$.

   If $s^*$ is real, then $E$ is bounded above, and at least one subsequential limit exists by the definition of sup. then $s^* \in E$ follows from the Lemmas 1 and 3.

   If $s^* = -\infty$, then $E$ contains only one element, namely $-\infty$, and there is no subsequential limit. Thus, $s^* \in E$.

2. Suppose for the sake of contradiction that there is a number $x > s^*$ such that $s_n \geq x$ for infinitely many values of $n$. Let's denote this set of $n$'s with $K$ and let $\{s_k\}_{k \in K}$ be this subsequence. If $\{s_k\}_{k \in K}$ is unbounded then $s^* = +\infty$ contradicting the fact that there exists an $x > s^*$. And if $\{s_k\}_{k \in K}$ is bounded that it contains a convergent subsequence (Bolzano–Weierstrass theorem). Suppose this convergent subsequence converges to $y$. Then $y \geq x > s^*$. This contradicts the definition of $s^*$.

To show the uniqueness, suppose there are two numbers, $p$ and $q$, which satisfy the two properties, and suppose $p < q$. Choose $x$ such that $p < x < q$. Since $p$ satisfies the second property, we have $s_n < x$ for $n \geq N$. But then $q$ cannot satisfy the second property. $\square$

**Theorem 5.** *If $s_n \leq t_n$ for $n \geq N$, where $N$ is fixed, then*

$$\liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} t_n,$$
$$\limsup_{n \to \infty} s_n \leq \limsup_{n \to \infty} t_n$$

*Proof.* Let $s^* = \limsup_{n \to \infty} s_n$ and $t^* = \limsup_{n \to \infty} t_n$. Suppose for the sake of contradiction $t^* < s^*$. Choose $x$ such that $t^* < x < s^*$. Then by the second property of Theorem 4 there is an integer $N_1$ such that $n \geq N_1$ implies $t_n < x$. Also by the first property there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \to s^*$. This implies that there exists an integer $N_2$ such that $n \geq N_2$ implies $x < s_n$. But then for $n \geq \max\{N_1, N_2\}$ we have $t_n < x < s_n$. This gives us our required contradiction. A similar argument can be made for the $\liminf$ case. $\square$

**Theorem 6.** *For a real-valued sequence $\{s_n\}$, $\lim_{n \to \infty} s_n = s \in \overline{\mathbb{R}}$ if and only if*

$$\limsup_{n \to \infty} s_n = \liminf_{n \to \infty} s_n = s$$

*Proof.* We divide the analysis into three cases.

First, let $s \in \mathbb{R}$. Then if $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$, Theorem 4 implies that for any $\epsilon > 0$ we have $s_n \in (s-\epsilon, s+\epsilon)$ for all but finitely many $n$, which means $s_n \to s$. On the other hand if $s_n \to s$ then every subsequence $\{s_{n_k}\}$ must converge to $s$ and hence $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$.

Now let $s = +\infty$. Then $s_n \to s$, i.e., for every $M \in \mathbb{R}$ there is an integer $N$ such that $n \geq N$ implies $s_n \geq M$ if and only $\liminf_{n\to\infty} s_n = +\infty$, and then $\limsup_{n\to\infty} s_n = +\infty$ since $\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n$.

Lastly, let $s = -\infty$. Then $s_n \to s$, i.e., for every $M \in \mathbb{R}$ there is an integer $N$ such that $n \geq N$ implies $s_n \leq M$ if and only $\limsup_{n\to\infty} s_n = -\infty$, and then $\liminf_{n\to\infty} s_n = -\infty$ since $\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n$. □

## 3 Upper and lower limits - a reprise

There is an equivalent way to express upper and lower limits.

**Definition 3.1.** Let $\{s_n\}$ be a sequence of real numbers. We let

$$\sup_{k\geq n} s_k = \sup\{s_k : k \geq n\}$$

$$\inf_{k\geq n} s_k = \inf\{s_k : k \geq n\}$$

**Remark.** *The sequence* $\{\sup_{k\geq n} s_k\}$ *is monotonically decreasing and the sequence* $\{\inf_{k\geq n} s_k\}$ *is monotonically increasing.*

**Theorem 7.** *Let* $\{s_n\}$ *be a sequence of real numbers. Then*

$$\lim_{n\to\infty} \sup_{k\geq n} s_k = \limsup_{n\to\infty} s_n$$

$$\lim_{n\to\infty} \inf_{k\geq n} s_k = \liminf_{n\to\infty} s_n$$

*Proof.* We will prove the first equation. The proof for the second is similar. We prove the equation in two steps.

Let $S = \{s_n : n \in \mathbb{N}\}$. Let's show that $\sup S = +\infty$ if and only if $\limsup_{n\to\infty} s_n = +\infty$. Suppose first that $\sup S = +\infty$. Then we construct a subsequence $\{s_{n_k}\}$ as follows. We let $n_1 = 1$. Suppose $n_1, \dots, n_k$ are chosen and let

$$S_k = \{n \in \mathbb{N} : s_n \geq \max\{s_{n_1}, \dots, s_{n_k}, k\} + 1\}$$

Notice that $S_k$ is infinite as otherwise we can find an $M \in \mathbb{R}$ such that $s_n \leq M$ for all $n \geq 1$, contradicting the fact that $\sup S = +\infty$. We pick $n_{k+1}$ to be the smallest element of $S_k$ which is bigger than $n_k$. The resulting subsequence satisfies the condition that $s_{n_k} \geq k$ for $k \geq 2$ and thus we conclude that $s_{n_k} \to +\infty$ which gives $\limsup_{n\to\infty} s_n = +\infty$. Now suppose that $\limsup_{n\to\infty} s_n = +\infty$.

From Theorem 4 we can conclude that there exists a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \to +\infty$. This immediately implies $\sup S = +\infty$.

For the second step, suppose $\sup S < +\infty$. Let

$$a_n = \sup_{k \geq n} s_k$$

Notice that $a_1 < +\infty$ and $\{a_n\}$ is a monotonically decreasing sequence. Therefore, by Lemma 2 we have that either $\{a_n\}$ is lower bounded in which case it converges to, say, $a$ or it is not in which case $\lim_{n\to\infty} a_n = -\infty$. Since $s_n \leq a_n$, by Theorem 5 we can conclude

$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n$$

where the last equality follows from Theorem 6. We will now show that

$$\limsup_{n\to\infty} s_n \geq \lim_{n\to\infty} a_n$$

which will give us our required equality. If $\lim_{n\to\infty} a_n = -\infty$ there is nothing to prove and so we assume that $a > -\infty$. Let $\epsilon > 0$ be given and let

$$B = \{n \in \mathbb{N} : s_n \geq a - \epsilon\}$$

We claim that $B$ is infinite. Indeed, if $B$ were finite we can find $N \in \mathbb{N}$ so that $N \geq \max(B)$. This will imply that $s_n \leq a - \epsilon$ for all $n \geq N$ and so $a_n \leq a - \epsilon$ for $n \geq N$. But then by Theorem 5 we would conclude $a = \lim_{n\to\infty} a_n \leq a - \epsilon$, which is absurd. Thus $B$ is infinite and we let $\{s_{n_k}\}$ be a subsequence of $\{s_n\}$ with $n_k \in B$. Notice that $\limsup_{n\to\infty} s_{n_k} \leq \limsup_{n\to\infty} s_n$, since any subsequential limit of $\{s_{n_k}\}$ is also a subsequential limit of $\{s_n\}$. This along with Theorem 5 give

$$a - \epsilon \leq \limsup_{n\to\infty} s_{n_k} \leq \limsup_{n\to\infty} s_n$$

Since $\epsilon$ was arbitrary we conclude that $\limsup_{n\to\infty} s_n \geq a$, which is what we wanted.

□

## 4 Properties of upper and lower limits

**Theorem 8.** *Let $\{s_n\}$ be a sequence of real numbers. Then*

$$\liminf_{n\to\infty} s_n = \limsup_{n\to\infty}(-s_n)$$

*Proof.* TODO □

**Theorem 9.** *For any two real sequences $\{a_n\}$ and $\{b_n\}$,*

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

*provided the sum on the right is not of the form $\infty - \infty$.*

*Proof.* If $\limsup_{n\to\infty} a_n = \infty$ then as by assumption the right side is not $\infty - \infty$, it is $\infty$ and there is nothing to prove. Similarly for the case $\limsup_{n\to\infty} b_n = \infty$. We may thus assume that

$$\limsup_{n\to\infty} a_n = A < \infty \text{ and } \limsup_{n\to\infty} b_n = B < \infty$$

We note that $\sup_{k\geq 1} a_k < \infty$, $\sup_{k\geq 1} b_k < \infty$, and so $\square$