

FE 530 – Homework I

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All tables and plots should be generated by the attached Python scripts and referenced here.

1 A Simple Market Model

1.1 Conditional Expectation and Conditional Variance

We model the one-period return as

$$S_{t+1} = \begin{cases} S_t(1+u) & \text{with prob } \pi, \\ S_t(1+d) & \text{with prob } 1-\pi, \end{cases} \quad V_t = xS_t + yB_t.$$

Assuming that $x + y = 1$ and because $S_t = B_t = 100$, then $V_t = xS_t + yB_t = 100$. I start by deriving the conditional expectation equations for the risky and risk free assets.

$$\mathbb{E}[S_{t+1} | S_t] = \pi S_t(1+u) + (1-\pi)S_t(1+d)$$

$$\mathbb{E}[S_{t+1} | S_t] = S_t(1 + \pi u + (1-\pi)d)$$

And simply

$$\mathbb{E}[B_{t+1}] = B_t(1 + r_f).$$

Given that $V_{t+1} = xS_{t+1} + yB_{t+1} = xS_t(1 + \pi u + (1-\pi)d) + yB_t(1 + r_f)$,

$$\mathbb{E}[V_{t+1} | V_t] = x\mathbb{E}[S_{t+1} | S_t] + y\mathbb{E}[B_{t+1} | B_t].$$

I finally substitute the conditional expectations and V_t for S_t and B_t to get

$$\mathbb{E}[V_{t+1} | V_t] = V_t[x(1 + \pi u + (1-\pi)d) + y(1 + r_f)]$$

For the conditional variance, we first assume the variance of the risk-free position is zero and

$$\text{Var}(V_{t+1} | V_t) = \text{Var}(S_{t+1} | S_t)$$

Then, we start with

$$\text{Var}(S_{t+1} | S_t) = \mathbb{E}[S_{t+1}^2 | S_t] - \mathbb{E}[S_{t+1} | S_t]^2$$

Where

$$\mathbb{E}[S_{t+1}^2] = S_t(\pi u^2 + (1 - \pi)d^2)$$

And given above

$$\mathbb{E}[S_{t+1}] = S_t(1 + \pi u + (1 - \pi)d)$$

We then substitute and conclude

$$\text{Var}(S_{t+1} \mid S_t) = S_t(\pi u^2 + (1 - \pi)d^2) - (S_t(1 + \pi u + (1 - \pi)d))^2.$$

1.2 Estimate (π, u, d)

The pi estimate is calculated as (# of up months)/(total # of months observed). The u estimate is calculated as the average return of up months. The d estimate is calculated as the average retrrn of down months. All of these estimates are based on the SPY 2012-2022.

Table 1: Estimated binomial parameters (π, u, d) from SPY monthly returns (2012–2022).

Unnamed: 0		pi	u	d
0	0	0.70229	0.031553	-0.038643

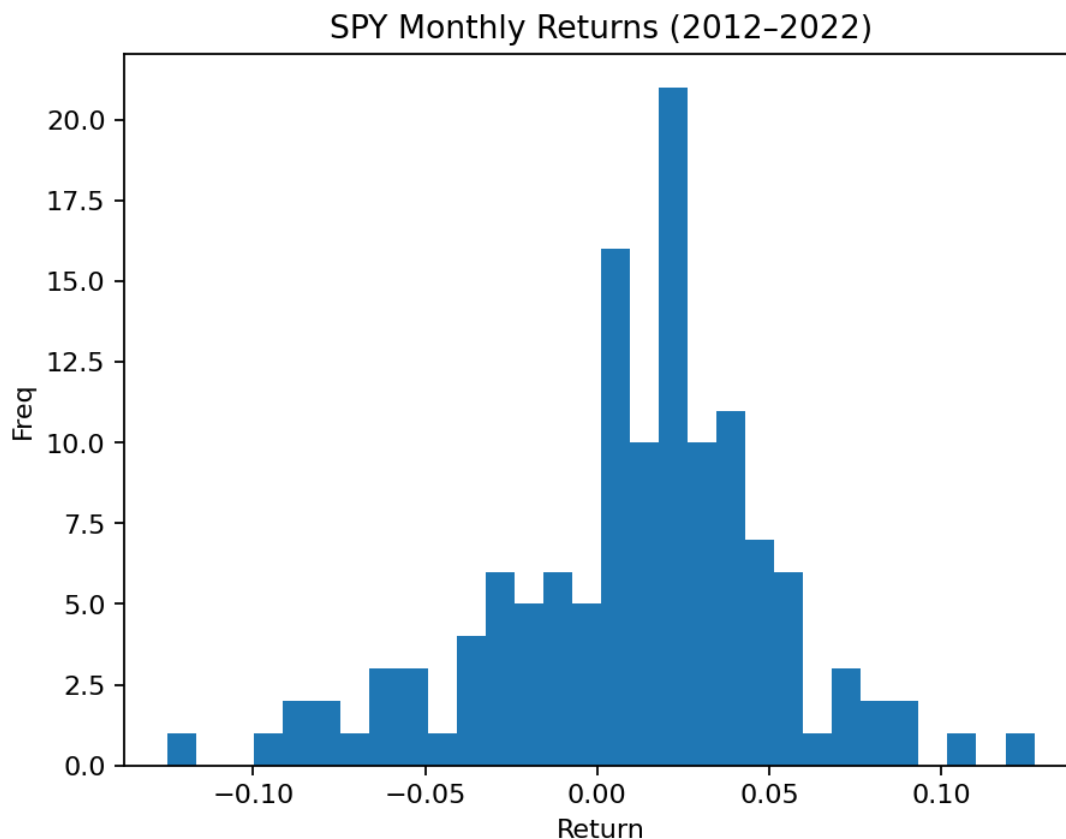


Figure 1: SPY monthly returns histogram (2012–2022).

1.3 Estimate (r_f) on SHY between 2021 and 2022

The (r_f) estimate below is calculated by averaging the monthly returns of the SHY over 2021-2022.

Table 2: Estimated monthly risk-free rate from SHY (2021–2022).

rf	
0	-0.002029

1.4 Is The No-Arbitrage Condition Satisfied?

Yes, The no arbitrage condition is satisfied as shown below.

Table 3: No-arbitrage test: check $d < r_f < u$.

	d	rf	u	no_arbitrage
0	-0.038643	-0.002029	0.031553	True

1.5 Minimize Portfolio Variance on \$100

Given that we have no return target and the variance associated with risk-free assets y is considered to be 0, placing all \$100 in the risk-free asset y would result in 0 portfolio variance.

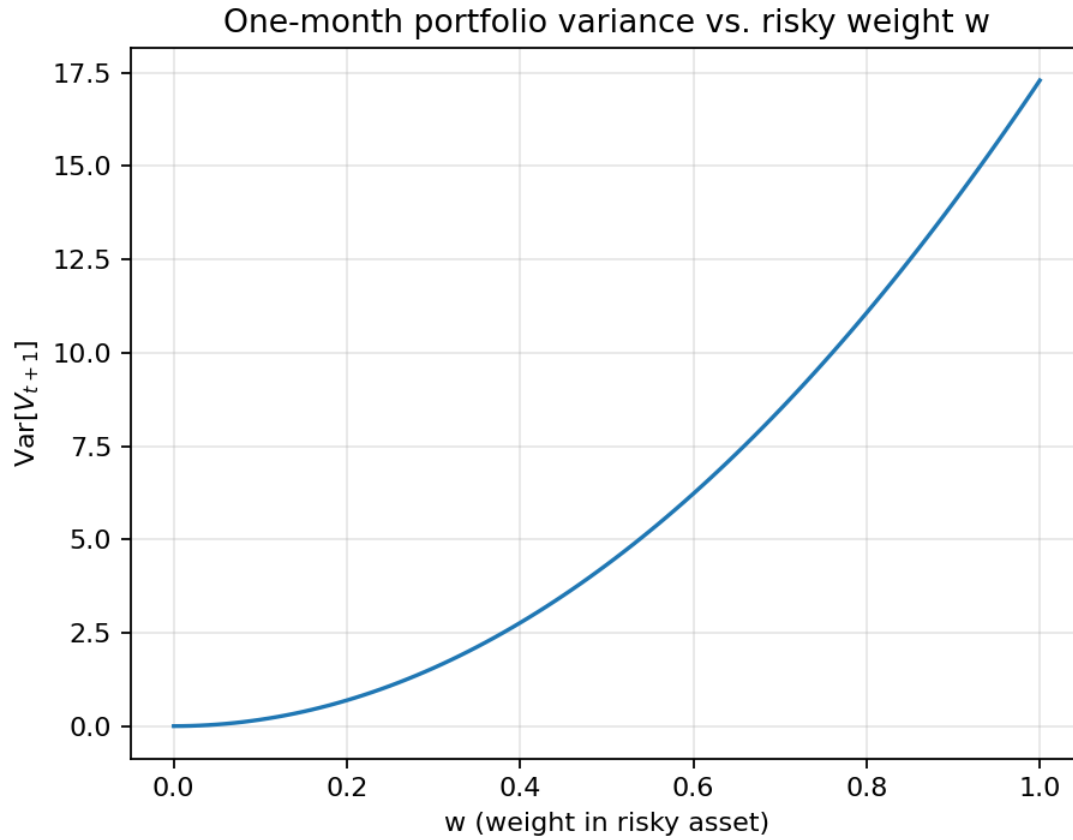


Figure 2: One-month portfolio variance as a function of risky weight w (0–1).

1.6 Allocation for \$102 target

Given the budget of $V_t = 100$ and the target of $E[V_{t+1}] = 102$, the weight is calculated

$$x = V_t \frac{\frac{V_{t+1}}{V_t} - (1 + r_f)}{\mu - r_f}, \quad \mu = \pi u + (1 - \pi)d$$

Table 4: Allocation (x, y) targeting $E[V_{t+1}] = 102$ with $V_t = 100$.

	pi	u	d	mu	rf	w	x	y	regime
0	0.7023	0.0316	-0.0386	0.0107	-0.002	1.7368	173.682	-73.682	levered long risky

From a trading perspective, this means that we need to borrow \$73.682 at the risk-free rate (or short the risk-free asset) and invest \$173.682 in the risky asset.

1.7 Option Pricing (two methods)

Both methods assume that current prices are $S_0 = B_0 = 100$, with the same π, u, d from above.

1.7.1 Option Replicating Approach

To price the call option using the option replicating approach, first we define

$$S_{t+1} = \begin{cases} S_t(1 + u) & \text{with prob } \pi, \\ S_t(1 + d) & \text{with prob } 1 - \pi \end{cases}$$

and

$$C_{t+1} = \begin{cases} C_u = \max(S_t(1 + u) - K, 0) & \text{with prob } \pi, \\ C_d = \max(S_t(1 + d) - K, 0) & \text{with prob } 1 - \pi \end{cases}$$

and

$$xS_{t+1} + yB_{t+1} = \begin{cases} xS_t(1 + u) + yB_t(1 + r_f) & \text{with prob } \pi, \\ xS_t(1 + d) + yB_t(1 + r_f) & \text{with prob } 1 - \pi \end{cases}$$

solving for x we get

$$xS_t(1+u) + yB_t(1+r_f) - C_u = xS_t(1+d) + yB_t(1+r_f) - C_d,$$

$$x(S_t(1+u) - S_t(1+d)) = C_u - C_d,$$

$$x = \frac{C_u - C_d}{S_t(u-d)}$$

and plug our x in to solve for y

$$\frac{C_u - C_d}{S_t(u-d)} S_t(1+u) + yB_t(1+r_f) = C_u,$$

$$y = \frac{C_u - C_d - \frac{C_u - C_d}{S_t(u-d)} S_t(1+u)}{B_t(1+r_f)}$$

then we place our vlaues for x and y into our value formula

$$C_t = xS_t + yB_t$$

and to check our work we also define

$$V_t = xS_t + yB_t - C_t$$

which implies

$$V_u = xS_t(1+u) + yB_t(1+r_f) - C_u$$

and

$$V_d = xS_t(1+d) + yB_t(1+r_f) - C_d$$

where want $V_u = V_d$

Table 5: One-step call via replicating portfolio (Delta and B0).

	K	Cu	Cd	x_rep	y_rep	C0_rep	Vu	Vd	Vu=Vd
0	101	2.1553	0.0	0.307	-0.2958	1.1265	129.3144	129.3144	True

1.7.2 DCF Method

The risk-neutral probability is calculates as

$$\pi^* = \frac{r_f - d}{u - d}$$

And the value of the call option at $t = 0$ is defined as

$$C_0 = \frac{\pi^* C_u + (1 - \pi^*) C_d}{1 + r_f}$$

Table 6: One-step call via risk-neutral expectation (DCF).

	pi_star	C0_dcf	C0_rep	C0_rep=C0_dcf
0	0.5216	1.1265	1.1265	True

2 Risk-Free Assets

2.1 Closed form solution for x as a function of α, r, g, n, τ

Let

$$\theta = \frac{1 + g}{1 + r}$$

Present values at $t=0$

$$\begin{aligned} PV_{\text{save}} &= \sum_{t=1}^n \frac{x(1+g)^{t-1}}{(1+r)^t} = \frac{x}{(1+r)} \sum_{t=1}^n \theta^{t-1} = \frac{x}{(1+r)} \frac{1 - \theta^n}{1 - \theta}, \\ PV_{\text{ret}} &= \sum_{k=1}^{\tau} \frac{\alpha(1+g)^{n+k-1}}{(1+r)^{n+k}} = \alpha \frac{(1+g)^{n-1}}{(1+r)^n} \sum_{k=1}^{\tau} \theta^k = \alpha \frac{(1+g)^{n-1}}{(1+r)^n} \frac{\theta(1 - \theta^{\tau})}{1 - \theta}. \end{aligned}$$

Equate and solve for x

$$\begin{aligned}
PV_{\text{save}} = PV_{\text{ret}} &\implies \frac{x}{(1+r)} \frac{1-\theta^n}{1-\theta} = \alpha \frac{(1+g)^{n-1}}{(1+r)^n} \frac{\theta(1-\theta^\tau)}{1-\theta} \\
&\implies x(1-\theta^n) = \alpha \frac{(1+g)^{n-1}}{(1+r)^{n-1}} \theta(1-\theta^\tau) \\
&\implies x = \alpha \frac{(1+g)^{n-1}}{(1+r)^{n-1}} \frac{\theta(1-\theta^\tau)}{1-\theta^n} = \alpha \theta^n \frac{1-\theta^\tau}{1-\theta^n}.
\end{aligned}$$

To get our final equation

$$x_{\text{disc}}(\alpha, r, g, n, \tau) = \alpha \theta^n \frac{1-\theta^\tau}{1-\theta^n}, \quad \theta = \frac{1+g}{1+r}, \quad r \neq g,$$

Special case, if $r = g$ ($\theta \rightarrow 1$)

$$x_{\text{disc}} = \alpha \lim_{\theta \rightarrow 1} \theta^n \frac{1-\theta^\tau}{1-\theta^n} = \alpha \frac{\tau}{n}.$$

2.2 Discrete Contribution Rate

Using our equation derived above, we compute

$$x_{\text{disc}}(\alpha = 0.5, r = 0.04, g = 0.01, n = 40, \tau = 20) = (0.5)\theta^{(40)} \frac{1-\theta^{(20)}}{1-\theta^{(40)}}, \quad \theta = \frac{1+0.01}{1+0.04},$$

Table 7: Discrete contribution rate and inputs.

	alpha	r	g	n	tau	x_discrete
0	0.5	0.04	0.01	40	20	0.099595

2.3 Continuous-Time Contribution Rate

$$x_{\text{disc}}(\alpha, r, g, n, \tau) = \alpha \theta^n \frac{1-\theta^\tau}{1-\theta^n}, \quad \theta = \frac{1+g/m}{1+r/m}, \quad r \neq g,$$

Define θ_m and the m-times-per-year version

$$\theta_m = \frac{1 + g/m}{1 + r/m}, \quad x_{\text{disc}}^{(m)} = \alpha \theta_m^{mn} \frac{1 - \theta_m^{m\tau}}{1 - \theta_m^{mn}} \quad (r \neq g).$$

Key limit: $\theta_m^m \rightarrow e^{g-r}$

$$\ln \theta_m = \ln\left(1 + \frac{g}{m}\right) - \ln\left(1 + \frac{r}{m}\right) = \frac{g-r}{m} + O\left(\frac{1}{m^2}\right), \quad \Rightarrow \quad \lim_{m \rightarrow \infty} \theta_m^m = \exp\left(\lim_{m \rightarrow \infty} m \ln \theta_m\right) = e^{g-r}.$$

Therefore powers scale cleanly

$$\lim_{m \rightarrow \infty} \theta_m^{mn} = e^{(g-r)n}, \quad \lim_{m \rightarrow \infty} \theta_m^{m\tau} = e^{(g-r)\tau}.$$

Continuous-time limit

$$\boxed{\lim_{m \rightarrow \infty} x_{\text{disc}}^{(m)} = \alpha e^{(g-r)n} \frac{1 - e^{(g-r)\tau}}{1 - e^{(g-r)n}}}$$

Table 8: Continuous contribution rate and inputs.

	alpha	r	g	n	tau	x_continuous
0	0.5	0.04	0.01	40	20	0.097234

Table 9: Discrete vs continuous: percent difference.

	x_discrete	x_continuous	pct_diff	discrete_<_cont
0	0.099595	0.097234	-2.370414	True

2.4 Sensitivity of x to r and g

We evaluate x_{cont} on a grid of salary growth g and interest rate r to visualize how funding needs change. As expected, higher r reduces the required contribution rate, while higher g increases it.

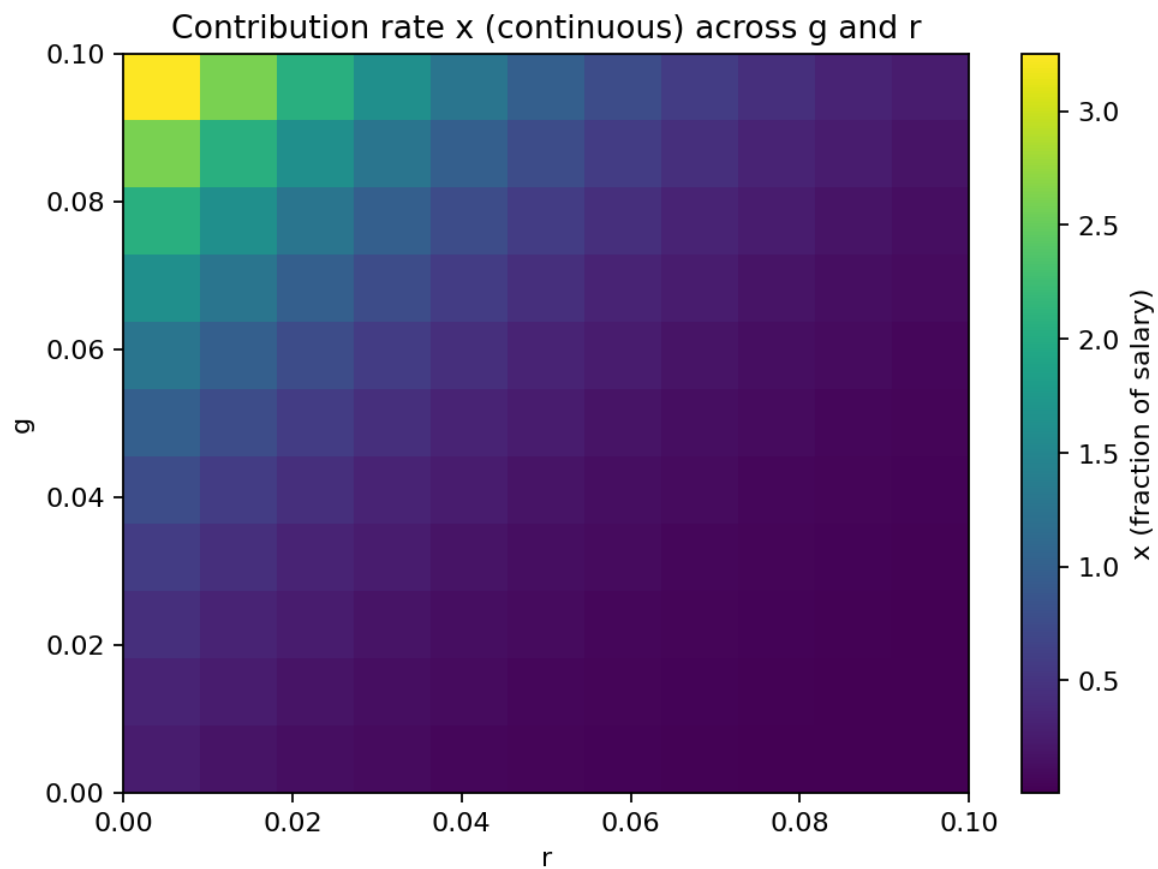


Figure 3: Contribution rate x (continuous) across r and g .

3 Portfolio Management

Show GMV & tangency portfolios and efficient frontier plot.

3.1

3.2

3.3

3.4

3.5

3.5.1

3.5.2

4 Forward Contracts

Prove pricing relation, payoff table, and arbitrage cases.

4.1

4.2

4.3

4.4

4.4.1

4.4.2

4.4.3