FE 530 – Homework I

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All tables and plots should be generated by the attached Python scripts and referenced here.

1 A Simple Market Model

1.1 Conditional Expectation and Conditional Variance

We model the one–period return as

$$S_{t+1} = \begin{cases} S_t(1+u) & \text{with prob } \pi, \\ S_t(1+d) & \text{with prob } 1-\pi, \end{cases} \qquad V_t = xS_t + yB_t.$$

Assuming that x + y = 1 and because $S_t = B_t = 100$, then $V_t = xS_t + yB_t = 100$. I start by deriving the conditional expectation equations for the risky and risk free assets.

$$\mathbb{E}[S_{t+1} \mid S_t] = \pi S_t(1+u) + (1-\pi)S_t(1+d)$$

$$\mathbb{E}[S_{t+1} \mid S_t] = S_t(1 + \pi u + (1-\pi)d)$$

And simply

$$\mathbb{E}[B_{t+1}] = B_t(1+r_f).$$

Given that $V_{t+1} = xS_{t+1} + yB_{t+1} = xS_t(1 + \pi u + (1-\pi)d) + yB_t(1+r_f)$,

$$\mathbb{E}[V_{t+1} \mid V_t] = x\mathbb{E}[S_{t+1}\ ] + y\mathbb{E}[B_{t+1}\ ].$$

I finally substitute the conditional expectations and V_t for S_t and B_t to get

$$\mathbb{E}[V_{t+1} \mid V_t] = V_t[x\ (1 + \pi u + (1-\pi)d) + y\ (1 + r_f\ )]$$

For the conditional variance, we first assume the variance of the risk-free position is zero and

$$\mathrm{Var}(V_{t+1} \mid V_t) = \mathrm{Var}(S_{t+1} \mid S_t)$$

Then, we start with

$$\mathrm{Var}(S_{t+1} \mid S_t) = \mathbb{E}[S_{t+1}^2] - \mathbb{E}[S_{t+1}]^2$$

Where

$$\mathbb{E}[S_{t+1}^2] = S_t(\pi u^2 + (1-\pi)d^2)$$

And given above

$$\mathbb{E}[S_{t+1}] = S_t(1 + \pi u + (1-\pi)d)$$

We then substitute and conclude

$$\text{Var}(S_{t+1} \mid S_t) = S_t(\pi u^2 + (1-\pi)d^2) - (S_t(1 + \pi u + (1-\pi)d))^2.$$

1.2 Estimate (π, u, d)

The pi estimate is calculated as (# of up months)/(total # of months observed). The u estimate is calculated as the average return of up months. The d estimate is calculated as the average retrrn of down months. All of these estimates are based on the SPY 2012-2022.

Table 1: Estimated binomial parameters (π, u, d) from SPY monthly returns (2012–2022).

	Unnamed: 0	pi	u	d
0	0	0.70229	0.031553	-0.038643

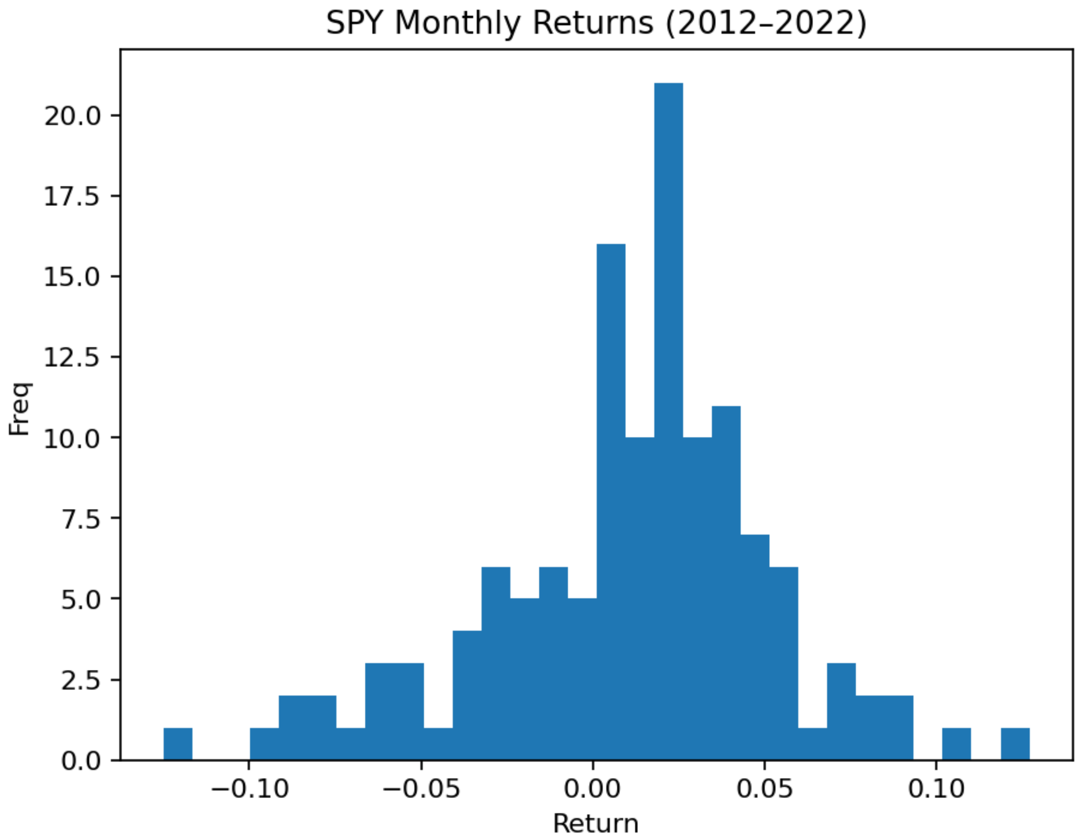

Figure 1: SPY monthly returns histogram (2012–2022).

1.3 Estimate (r_f) on SHY between 2021 and 2022

The (r_f) estimate below is calculated by averaging the monthly returns of the SHY over 2021-2022.

Table 2: Estimated monthly risk-free rate from SHY (2021–2022).

	rf
0	-0.002029

1.4 Is The No-Arbitrage Condition Satisfied?

Yes, The no arbitrage condition is satisfied as shown below.

Table 3: No-arbitrage test: check $d < r_f < u$.

	d	rf	u	no_arbitrage
0	-0.038643	-0.002029	0.031553	True

1.5 Minimize Portfolio Variance on $100

Given that we have no return target and the variance assiciated with risk-free assets y is considered to be 0, palcing all $100 in the risk-free asset y would result in 0 portfolio variance.

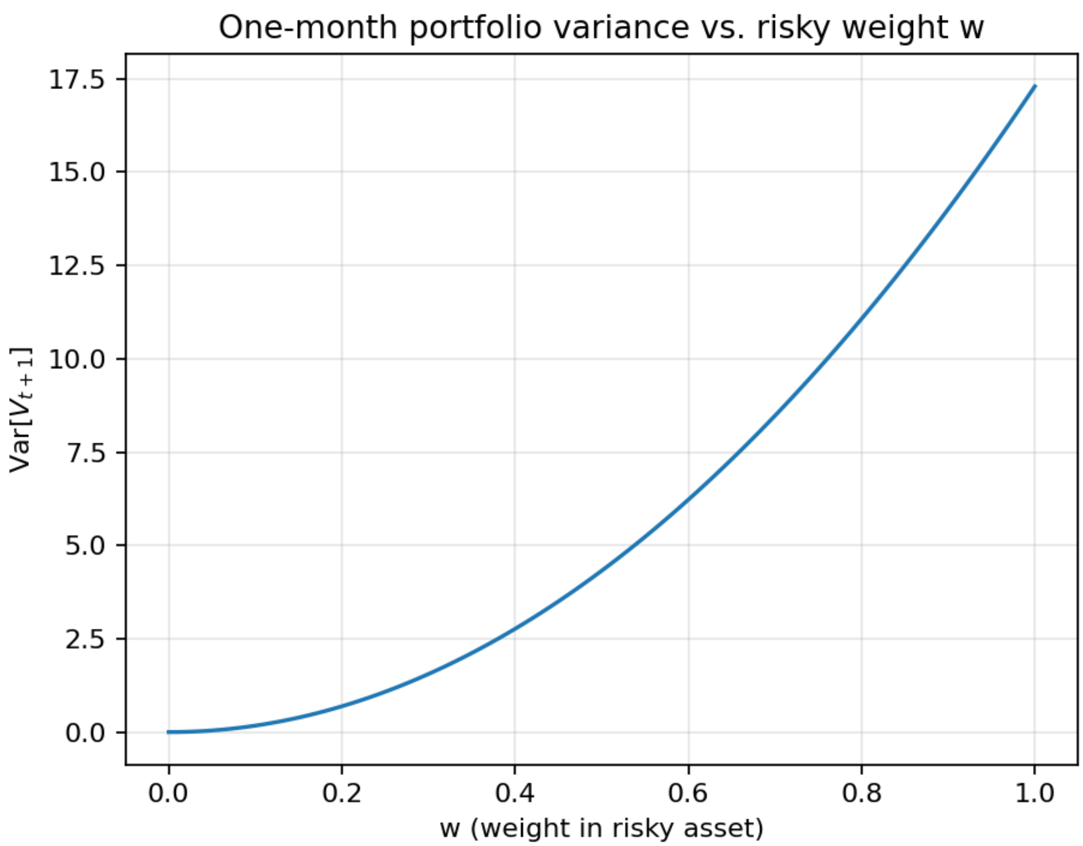

Figure 2: One-month portfolio variance as a function of risky weight w (0–1).

1.6 Allocation for $102 target

Given the budget of $V_t = 100$ and the target of $E[V_{t+1}] = 102$, the weight is calculated

$$x = V_t \frac{\frac{V_{t+1}}{V_t} - (1 + r_f)}{\mu - r_f}, \quad \mu = \pi u + (1 - \pi) d$$

Table 4: Allocation (x, y) targeting $E[V_{t+1}] = 102$ with $V_t = 100$.

	pi	u	d	mu	rf	w	x	y	regime
0	0.7023	0.0316	-0.0386	0.0107	-0.002	1.7368	173.682	-73.682	levered long risky

From a trading perspective, this means that we need to borrow $73.682 at the risk-free rate (or short the risk-free asset) and invest $173.682 in the risky asset.

1.7 Option Pricing (two methods)

Both methods assume that current prices are $S_0 = B_0 = 100$, with the same π, u, d from above.

1.7.1 Option Replicating Approach

To price the call option using the option replicating approach, first we define

$$S_{t+1} = \begin{cases} S_t(1+u) & \text{with prob } \pi, \\ S_t(1+d) & \text{with prob } 1 - \pi \end{cases}$$

and

$$C_{t+1} = \begin{cases} C_u = \max(S_t(1+u) - K, 0) & \text{with prob } \pi, \\ C_d = \max(S_t(1+d) - K, 0) & \text{with prob } 1 - \pi \end{cases}$$

and

$$xS_{t+1} + yB_{t+1} = \begin{cases} xS_t(1+u) + yB_t(1+r_f) & \text{with prob } \pi, \\ xS_t(1+d) + yB_t(1+r_f) & \text{with prob } 1 - \pi \end{cases}$$

solving for x we get

$$xS_t(1+u) + yB_t(1+r_f) - C_u = xS_t(1+d) + yB_t(1+r_f) - C_d,$$

$$x(S_t(1+u) - S_t(1+d)) = C_u - C_d,$$

$$x = \frac{C_u - C_d}{S_t(u-d)}$$

and plug our x in to solve for y

$$\frac{C_u - C_d}{S_t(u-d)} S_t(1+u) + yB_t(1+r_f) = C_u,$$

$$y = \frac{C_u - C_d - \frac{C_u}{S_t(u-d)} S_t(1+u)}{B_t(1+r_f)}$$

then we place our vlaues for x and y into our value formula

$$C_t = xS_t + yB_t$$

and to check our work we also define

$$V_t = xS_t + yB_t - C_t$$

which implies

$$V_u = xS_t(1+u) + yB_t(1+r_f) - C_u$$

and

$$V_d = xS_t(1+d) + yB_t(1+r_f) - C_d$$

where want $V_u = V_d$

Table 5: One-step call via replicating portfolio (Delta and B0).

	K	Cu	Cd	x_rep	y_rep	C0_rep	Vu	Vd	Vu=Vd
0	101	2.1553	0.0	0.307	-0.2958	1.1265	129.3144	129.3144	True

1.7.2 DCF Method

The risk-neutral probability is calculates as

$$\pi^* = \frac{r_f - d}{u - d}$$

And the value of the call option at $t = 0$ is defined as

$$C_t = \frac{\pi^* C_u + (1 - \pi^*) C_d}{1 + r_f}$$

Table 6: One-step call via risk-neutral expectation (DCF).

	pi_star	C0_dcf	C0_rep	C0_rep=C0_dcf
0	0.5216	1.1265	1.1265	True

2 Risk-Free Assets

2.1 Closed form solution for x as a function of α, r, g, n, τ

Let

$$\theta = \frac{1+g}{1+r}$$

Present values at t=0

$$\mathrm{PV}_{\text{save}} = \sum_{t=1}^{n} \frac{x\,(1+g)^{\,t-1}}{(1+r)^{\,t}} = \frac{x}{(1+r)} \sum_{t=1}^{n} \theta^{\,t-1} = \frac{x}{(1+r)}\,\frac{1-\theta^{\,n}}{1-\theta},$$

$$\mathrm{PV}_{\text{ret}} = \sum_{k=1}^{\tau} \frac{\alpha\,(1+g)^{\,n+k-1}}{(1+r)^{\,n+k}} = \alpha\,\frac{(1+g)^{\,n-1}}{(1+r)^{\,n}} \sum_{k=1}^{\tau} \theta^{\,k} = \alpha\,\frac{(1+g)^{\,n-1}}{(1+r)^{\,n}}\,\frac{\theta\,(1-\theta^{\,\tau})}{1-\theta}.$$

Equate and solve for x

$$\begin{aligned}
\mathrm{PV}_{\text{save}} = \mathrm{PV}_{\text{ret}} \quad &\implies \quad \frac{x}{(1+r)}\,\frac{1-\theta^{\,n}}{1-\theta} = \alpha\,\frac{(1+g)^{\,n-1}}{(1+r)^{\,n}}\,\frac{\theta\,(1-\theta^{\,\tau})}{1-\theta} \\
&\implies \quad x\,(1-\theta^{\,n}) = \alpha\,\frac{(1+g)^{\,n-1}}{(1+r)^{\,n-1}}\,\theta\,(1-\theta^{\,\tau}) \\
&\implies \quad x = \alpha\,\frac{(1+g)^{\,n-1}}{(1+r)^{\,n-1}}\,\frac{\theta\,(1-\theta^{\,\tau})}{1-\theta^{\,n}} = \alpha\theta^{\,n}\,\frac{1-\theta^{\,\tau}}{1-\theta^{\,n}}.
\end{aligned}$$

To get our final equation

$$\boxed{x_{\text{disc}}(\alpha, r, g, n, \tau) = \alpha\,\theta^{\,n}\,\frac{1-\theta^{\,\tau}}{1-\theta^{\,n}}, \quad \theta = \frac{1+g}{1+r}, \quad r \neq g,}$$

Special case, if $r = g$ ($\theta \to 1$)

$$x_{\text{disc}} = \alpha\,\lim_{\theta\to 1}\,\theta^{\,n}\,\frac{1-\theta^{\,\tau}}{1-\theta^{\,n}} = \alpha\,\frac{\tau}{n}.$$

2.2 Discrete Contribution Rate

Using our equation derrived above, we compute

$$x_{\text{disc}}(\alpha = 0.5, r = 0.04, g = 0.01, n = 40, \tau = 20) = (0.5)\theta^{(40)}\frac{1-\theta^{(20)}}{1-\theta^{(40)}}, \quad \theta = \frac{1+0.01}{1+0.04},$$

Table 7: Discrete contribution rate and inputs.

	alpha	r	g	n	tau	x_discrete
0	0.5	0.04	0.01	40	20	0.099595

2.3 Continuous-Time Contribution Rate

$$x_{\text{disc}}(\alpha, r, g, n, \tau) = \alpha\,\theta^{\,n}\,\frac{1-\theta^{\,\tau}}{1-\theta^{\,n}}, \quad \theta = \frac{1+g/m}{1+r/m}, \quad r \neq g,$$

Define θ_m and the m-times-per-year version

$$\theta_m = \frac{1+g/m}{1+r/m}, \qquad x_{\text{disc}}^{(m)} = \alpha\, \theta_m^{mn} \frac{1-\theta_m^{m\tau}}{1-\theta_m^{mn}} \quad (r \neq g).$$

Key limit: $\theta_m^m \to e^{g-r}$

$$\ln\theta_m = \ln\Big(1+\frac{g}{m}\Big) - \ln\Big(1+\frac{r}{m}\Big) = \frac{g-r}{m} + O\Big(\frac{1}{m^2}\Big), \qquad \Rightarrow \quad \lim_{m\to\infty} \theta_m^m = \exp\Big(\lim_{m\to\infty} m \ln\theta_m\Big) = e^{\,g-r}.$$

Therefore powers scale cleanly

$$\lim_{m\to\infty} \theta_m^{mn} = e^{(g-r)n}, \qquad \lim_{m\to\infty} \theta_m^{m\tau} = e^{(g-r)\tau}.$$

Continuous-time limit

$$\boxed{\lim_{m\to\infty} x_{\text{disc}}^{(m)} = \alpha\, e^{(g-r)n} \frac{1-e^{(g-r)\tau}}{1-e^{(g-r)n}}}$$

Table 8: Continuous contribution rate and inputs.

	alpha	r	g	n	tau	x_continuous
0	0.5	0.04	0.01	40	20	0.097234

Table 9: Discrete vs continuous: percent difference.

	x_discrete	x_continuous	pct_diff	discrete_<_cont
0	0.099595	0.097234	-2.370414	True

2.4 Sensitivity of x to r and g

We evaluate x_{cont} on a grid of salary growth g and interest rate r to visualize how funding needs change. As expected, higher r reduces the required contribution rate, while higher g increases it.

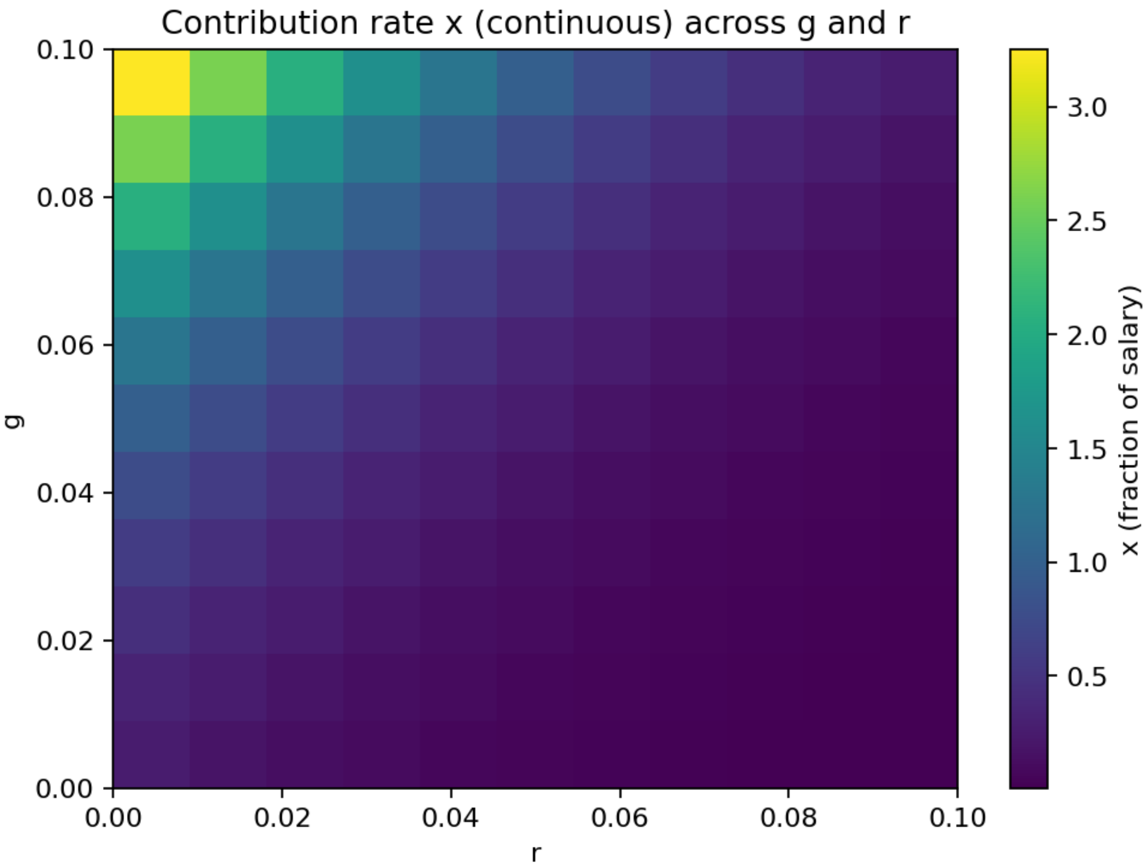

Figure 3: Contribution rate x (continuous) across r and g.

3 Portfolio Management

3.1 Define Mean Vector (μ)

$$R_i = \begin{cases} u_i & \text{with prob } \pi_i, \\ d_i & \text{with prob } 1 - \pi_i. \end{cases}$$

Starting with the mean return for asset i, $\mu_i = p_i u_i + (1 - p_i) d_i$, for $i \in \{1, 2\}$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} \pi_1 u_1 + (1 - \pi_1) d_1 \\ \pi_2 u_2 + (1 - \pi_2) d_2 \end{bmatrix}.$$

3.2 Define Covariance Matrix (Σ)

$$\mu = \mathbb{E}[R_i] = \pi_i u_i + (1-\pi_i)d_i, \qquad \mathbb{E}[R_i^2] = \pi_i u_i^2 + (1-\pi_i)d_i^2.$$

$$\sigma_i^2 = \mathrm{Var}(R_i) = \mathbb{E}[R_i^2] - \big(\mu\big)^2 = \pi_i u_i^2 + (1-\pi_i)d_i^2 - \big(\pi_i u_i + (1-\pi_i)d_i\big)^2.$$

$$\sigma_i^2 = \mathrm{Var}(R_i) = \pi_i(1-\pi_i)\left(u_i^2 + d_i^2 - 2u_i d_i\right) = \boxed{\pi_i(1-\pi_i)\,(u_i - d_i)^2}.$$

$$\sigma_i = \sqrt{\pi_i(1-\pi_i)}\,|u_1 - d_1|;$$

$$\sigma_{12} = \mathrm{Cov}(R_{12}) = \rho\,\sigma_1\,\sigma_2$$

$$\Sigma = \begin{bmatrix} \mathrm{Var}(R_1) & \mathrm{Cov}(R_{1,2}) \\ \mathrm{Cov}(R_{1,2}) & \mathrm{Var}(R_2) \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\,\sigma_1\sigma_2 \\ \rho\,\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} \pi_1(1-\pi_1)(u_1-d_1)^2 & \rho\,\sqrt{\pi_1(1-\pi_1)}\,\sqrt{\pi_2(1-\pi_2)}\,|u_1-d_1||u_2-d_2| \\ \rho\,\sqrt{\pi_1(1-\pi_1)}\,\sqrt{\pi_2(1-\pi_2)}\,|u_1-d_1||u_2-d_2| & \pi_2(1-\pi_2)(u_2-d_2)^2 \end{bmatrix}.$$

Table 10: Effective inputs used in Q3.

	mu1	mu2	sigma1	sigma2
0	0.012	0.07	0.107778	0.220454

3.3 Global minimum-variance (GMV) and Sharpe portfolios

3.3.1 GMV

For two assets, the closed form weights are

$$w_{\mathrm{GMV},1} = \frac{\sigma_2^2 - \rho\,\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\sigma_2}, \qquad w_{\mathrm{GMV},2} = 1 - w_{\mathrm{GMV},1}.$$

Table 11: GMV Portfolio Weights at $\rho = 0.5, 0$

	rho	w1_GMV	w2_GMV
0	0.5	1.007242	-0.007242
1	0.0	0.807094	0.192906

3.3.2 Sharpe:

With risk-free rate r_f, the tangency portfolio maximizes

$$\mathrm{SR}(w) = \frac{w^\top \mu - r_f}{\sqrt{w^\top \Sigma w}} \quad \text{subject to} \quad \mathbf{1}^\top w = 1.$$

A standard result gives

$$\boxed{w_{\mathrm{SR}} = \frac{\Sigma^{-1}(\mu - r_f\,\mathbf{1})}{\mathbf{1}^\top \Sigma^{-1}(\mu - r_f\,\mathbf{1})}}.$$

Table 12: Sharpe Portfolio Weights at $\rho = 0.5, 0$

	rho	w1_SR	w2_SR
0	0.5	-0.350552	1.350552
1	0.0	0.448996	0.551004

3.4 Diversification

Given on the above results, how do you justify the changes in the portfolio weights relative to changes in the correlation coefficient? Recall the wisdom of diversification.

3.5 $\rho = 0.5$

3.5.1 Mean-variance efficient frontier

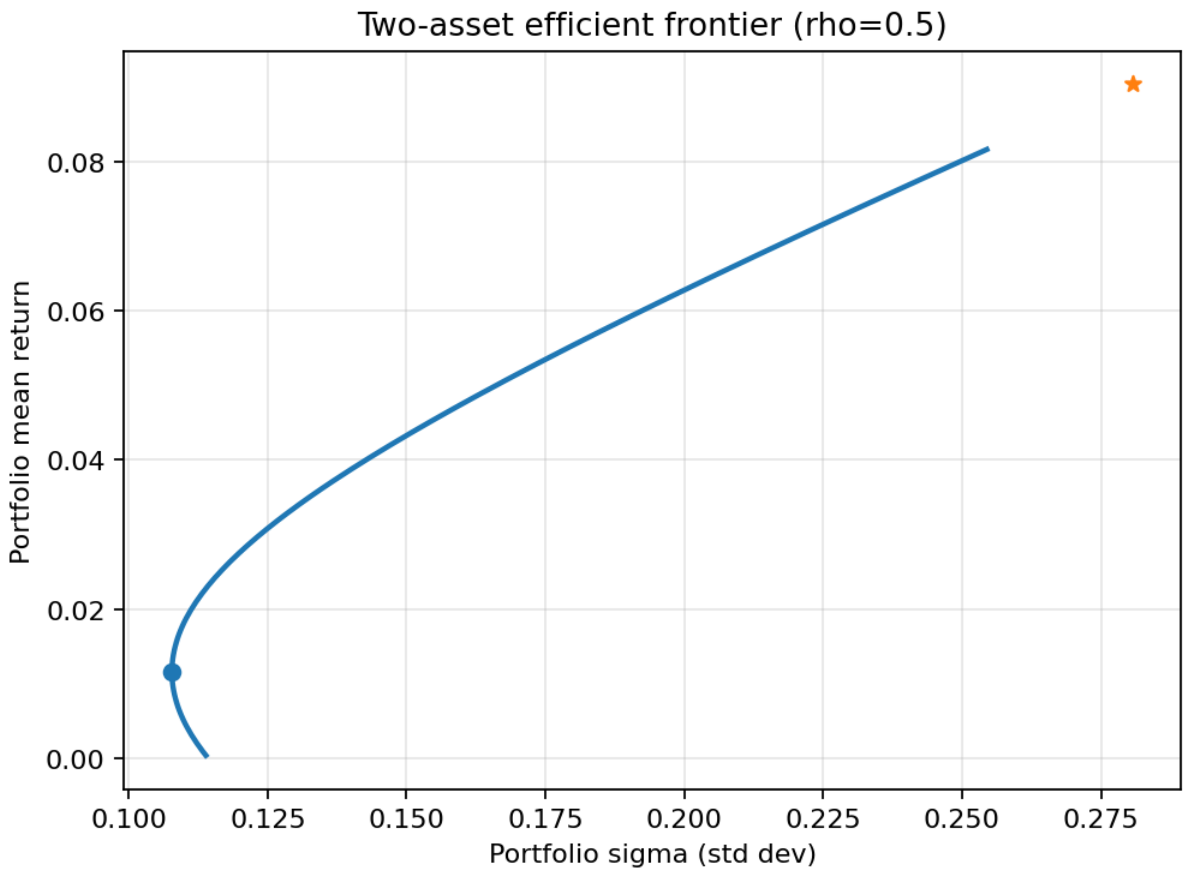

Figure 4: Two-asset efficient frontier (rho = 0.5); GMV (o) and Tangency (*).

3.5.2 Compare arbitrary weights for w_1 and w_2 to the previous frontier

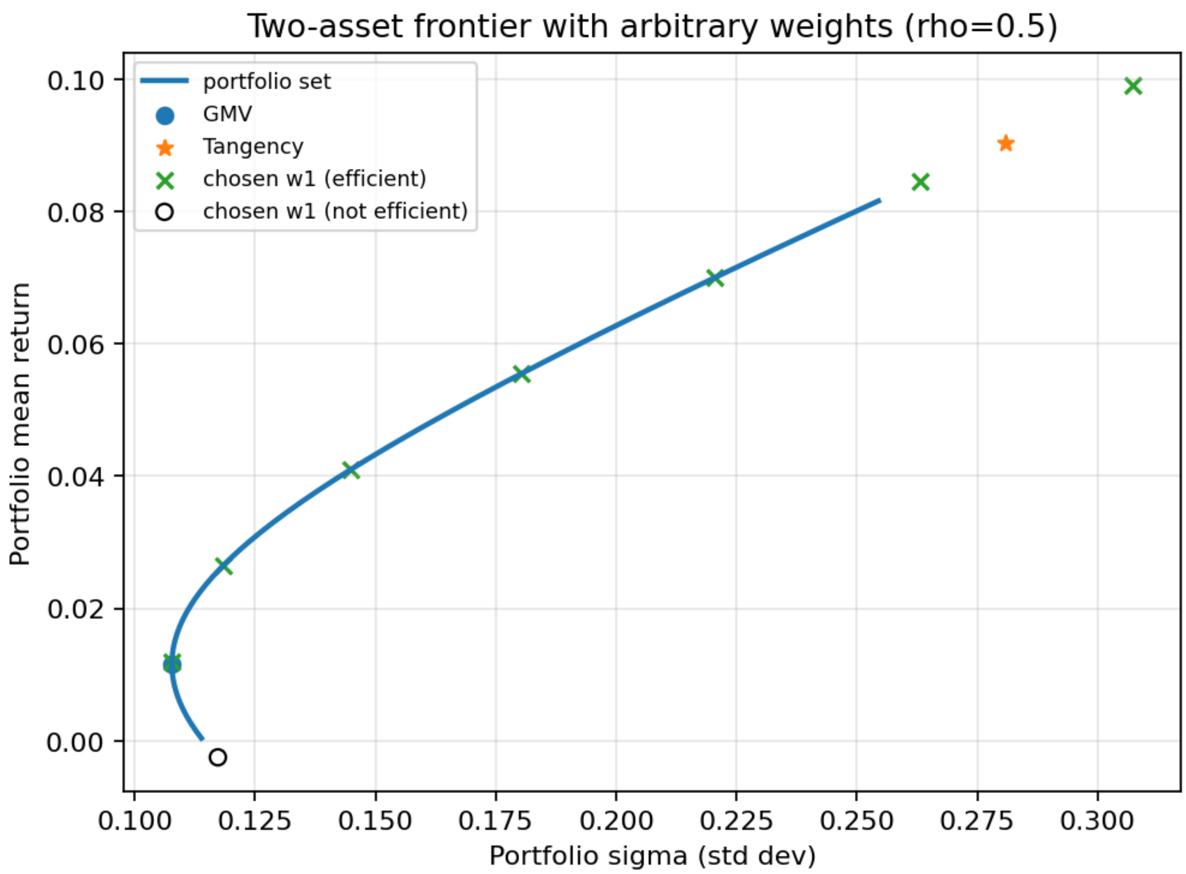

Figure 5: Frontier with arbitrary weights overlaid (rho = 0.5).

Table 13: Selected weights and their mean/sigma (rho = 0.5).

	w1	w2	mean	sigma
0	-0.50	1.50	0.0990	0.307301
1	-0.25	1.25	0.0845	0.263132
2	0.00	1.00	0.0700	0.220454
3	0.25	0.75	0.0555	0.180329
4	0.50	0.50	0.0410	0.144893
5	0.75	0.25	0.0265	0.118434
6	1.00	0.00	0.0120	0.107778
7	1.25	-0.25	-0.0025	0.117314

4 Forward Contracts

Prove pricing relation, payoff table, and arbitrage cases.

4.1

4.2

4.3

4.4

4.4.1

4.4.2

4.4.3