FE 530 – Homework I

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Source repository: Maknzs/FE530_HW1. github.com/Maknzs/FE530_HW1/tree/main/src holds the python scripts that generate the following tables: Table 1; Table 2; Table 3; Table 4; Table 5; Table 6; Table 7; Table 8; Table 9; Table 10; Table 11; Table 12; Table 13; Table 14; Table 15; Table 16; Table 17. They also produce the figures: Figure 1; Figure 2; Figure 3; Figure 4; Figure 5.

1 A Simple Market Model

1.1 Conditional Expectation and Conditional Variance

We model the one–period return as

$$S_{t+1} = \begin{cases} S_t(1+u) & \text{with prob } \pi, \\ S_t(1+d) & \text{with prob } 1-\pi, \end{cases} \qquad V_t = xS_t + yB_t.$$

Assuming that x + y = 1 and because $S_t = B_t = 100$, then $V_t = xS_t + yB_t = 100$. I start by deriving the conditional expectation equations for the risky and risk free assets.

$$\mathbb{E}[S_{t+1} \mid S_t] = \pi S_t(1+u) + (1-\pi)S_t(1+d)$$

$$\mathbb{E}[S_{t+1} \mid S_t] = S_t(1 + \pi u + (1-\pi)d)$$

And simply

$$\mathbb{E}[B_{t+1}] = B_t(1+r_f).$$

Given that $V_{t+1} = xS_{t+1} + yB_{t+1} = xS_t(1 + \pi u + (1-\pi)d) + yB_t(1+r_f)$,

$$\mathbb{E}[V_{t+1} \mid V_t] = x\mathbb{E}[S_{t+1}] + y\mathbb{E}[B_{t+1}].$$

I finally substitute the conditional expectations and V_t for S_t and B_t to get

$$\mathbb{E}[V_{t+1} \mid V_t] = V_t\left[x(1 + \pi u + (1-\pi)d) + y(1+r_f)\right]$$

For the conditional variance, we first assume the variance of the risk-free position is zero and

$$\mathrm{Var}(V_{t+1} \mid V_t) = \mathrm{Var}(S_{t+1} \mid S_t)$$

Then, we start with

$$\mathrm{Var}(S_{t+1} \mid S_t) = \mathbb{E}[S_{t+1}^2] - \mathbb{E}[S_{t+1}]^2$$

Where

$$\mathbb{E}[S_{t+1}^2] = S_t(\pi u^2 + (1-\pi)d^2)$$

And given above

$$\mathbb{E}[S_{t+1}] = S_t(1 + \pi u + (1-\pi)d)$$

We then substitute and conclude

$$\mathrm{Var}(S_{t+1} \mid S_t) = S_t(\pi u^2 + (1-\pi)d^2) - (S_t(1 + \pi u + (1-\pi)d))^2.$$

1.2 Estimate (π, u, d)

The pi estimate is calculated as (# of up months)/(total # of months observed). The u estimate is calculated as the average return of up months. The d estimate is calculated as the average return of down months. All of these estimates are based on the SPY 2012-2022.

Table 1: Estimated binomial parameters (π, u, d) from SPY monthly returns (2012–2022).

	Unnamed: 0	pi	u	d
0	0	0.70229	0.031553	-0.038643

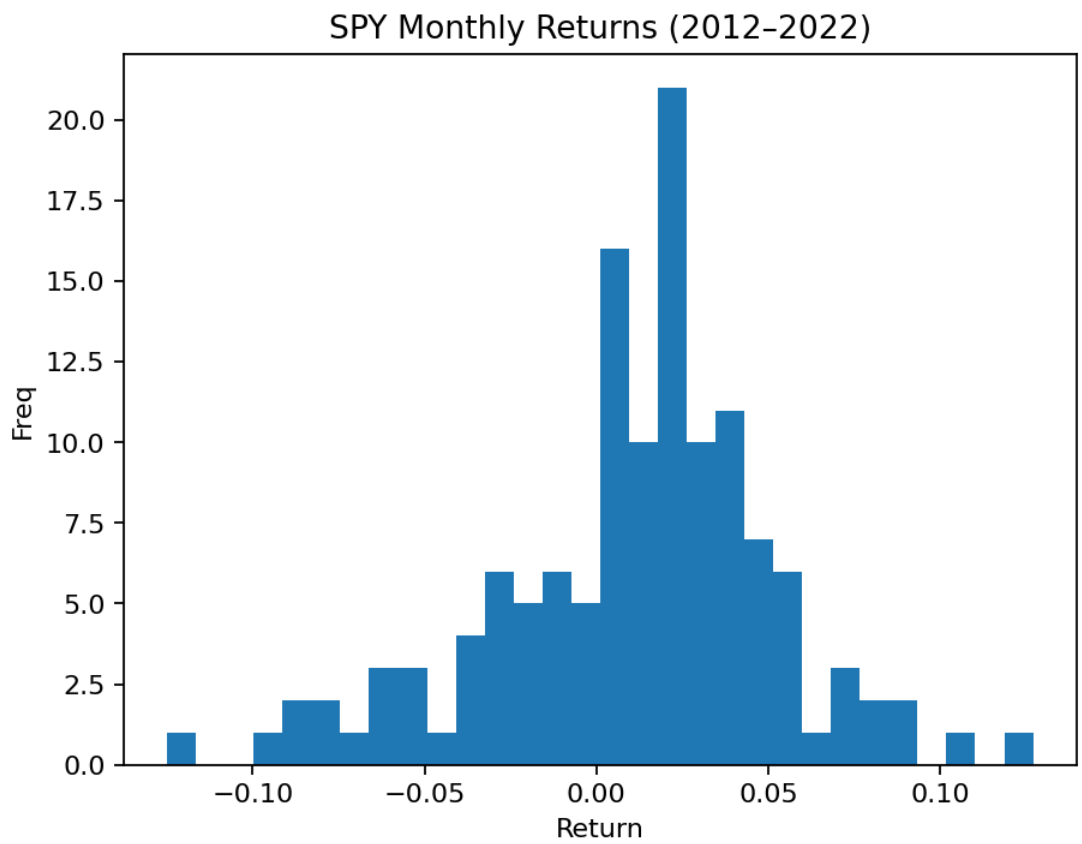

Figure 1: SPY monthly returns histogram (2012–2022).

1.3 Estimate (r_f) With SHY Between 2021 and 2022

The (r_f) estimate below is calculated by averaging the monthly returns of the SHY over 2021-2022.

Table 2: Estimated monthly risk-free rate from SHY (2021–2022).

	rf
0	-0.002029

1.4 Is the No-Arbitrage Condition Satisfied?

Yes, the no-arbitrage condition is satisfied as shown below.

Table 3: No-arbitrage test: check $d < r_f < u$.

	d	rf	u	no_arbitrage
0	-0.038643	-0.002029	0.031553	True

1.5 Minimize Portfolio Variance on $100

Given that we have no return target and the variance associated with the risk-free asset y is considered to be 0, placing all $100 in the risk-free asset y would result in 0 portfolio variance. ### Figure 1 – One-month portfolio variance as a function of risky weight w (0–1)

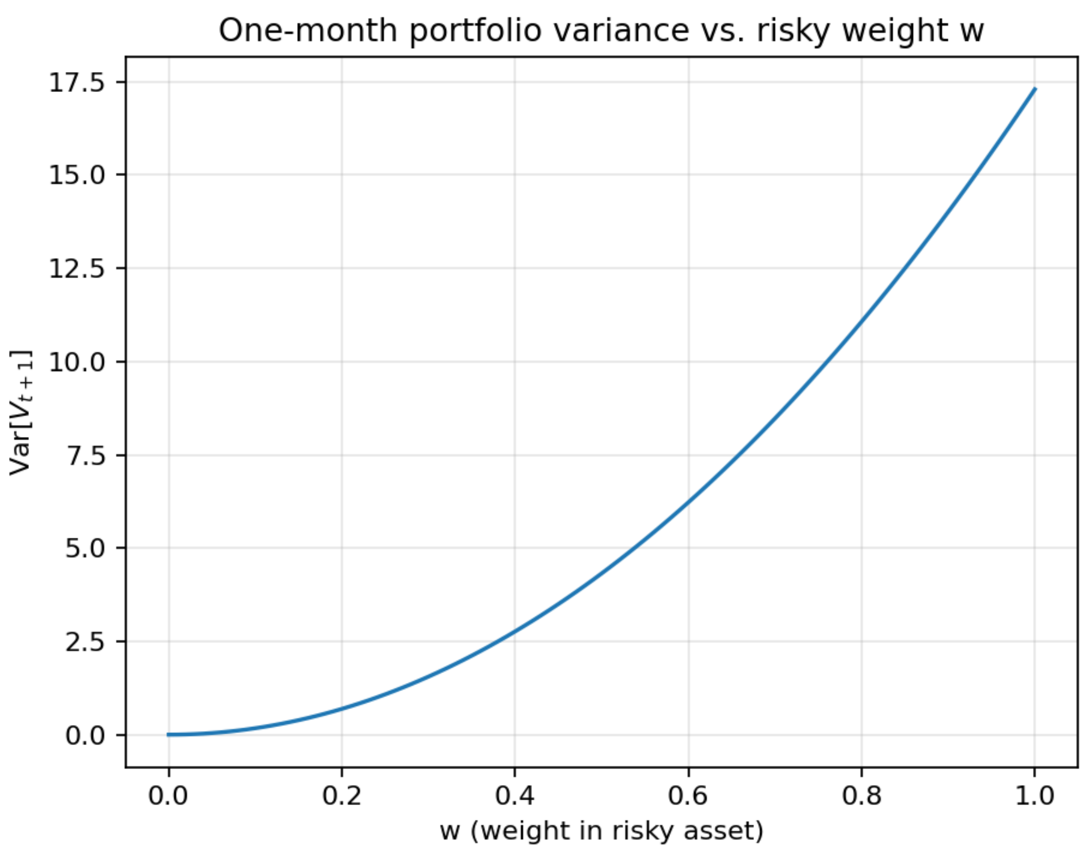

Figure 2: One-month portfolio variance as a function of risky weight w (0–1).

1.6 Allocation for \$102 Target at $t = 1$

Given the budget of $V_t = 100$ and the target of $E[V_{t+1}] = 102$, the weight is calculated

$$x = V_t \frac{\frac{V_{t+1}}{V_t} - (1 + r_f)}{\mu - r_f}, \quad \mu = \pi u + (1 - \pi)d$$

Table 4: Allocation (x, y) targeting $E[V_{t+1}] = 102$ with $V_t = 100$.

	pi	u	d	mu	rf	w	x	y	regime
0	0.7023	0.0316	-0.0386	0.0107	-0.002	1.7368	173.682	-73.682	levered long risky

From a trading perspective, this means that we need to borrow \$73.682 at the risk-free rate (or short the risk-free asset) and invest \$173.682 in the risky asset.

1.7 Option Pricing

Both methods assume that current prices are $S_0 = B_0 = 100$, with the same π, u, d from above.

1.7.1 Option Replicating Approach

To price the call option using the option replicating approach, first we define

$$S_{t+1} = \begin{cases} S_t(1+u) & \text{with prob } \pi, \\ S_t(1+d) & \text{with prob } 1 - \pi \end{cases}$$

and

$$C_{t+1} = \begin{cases} C_u = \max(S_t(1+u) - K, 0) & \text{with prob } \pi, \\ C_d = \max(S_t(1+d) - K, 0) & \text{with prob } 1 - \pi \end{cases}$$

and

$$xS_{t+1} + yB_{t+1} = \begin{cases} xS_t(1+u) + yB_t(1+r_f) & \text{with prob } \pi, \\ xS_t(1+d) + yB_t(1+r_f) & \text{with prob } 1 - \pi \end{cases}$$

solving for x we get

$$xS_t(1+u) + yB_t(1+r_f) - C_u = xS_t(1+d) + yB_t(1+r_f) - C_d,$$

$$x(S_t(1+u) - S_t(1+d)) = C_u - C_d,$$

$$x = \frac{C_u - C_d}{S_t(u-d)}$$

and plug our x in to solve for y

$$\frac{C_u - C_d}{S_t(u-d)} S_t(1+u) + yB_t(1+r_f) = C_u,$$

$$y = \frac{C_u - C_d - \frac{C_u}{S_t(u-d)} S_t(1+u)}{B_t(1+r_f)}$$

then we place our values for x and y into our value formula

$$C_t = xS_t + yB_t$$

and to check our work we also define

$$V_t = xS_t + yB_t - C_t$$

which implies

$$V_u = xS_t(1+u) + yB_t(1+r_f) - C_u$$

and

$$V_d = xS_t(1+d) + yB_t(1+r_f) - C_d$$

where want $V_u = V_d$

Table 5: One-step call via replicating portfolio (Delta and B0).

	K	Cu	Cd	x_rep	y_rep	C0_rep	Vu	Vd	Vu=Vd
0	101	2.1553	0.0	0.307	-0.2958	1.1265	129.3144	129.3144	True

1.7.2 DCF Method

The risk-neutral probability is calculated as

$$\pi^* = \frac{r_f - d}{u - d}$$

And the value of the call option at $t = 0$ is defined as

$$C_t = \frac{\pi^* C_u + (1 - \pi^*) C_d}{1 + r_f}$$

Table 6: One-step call via risk-neutral expectation (DCF).

	pi_star	C0_dcf	C0_rep	C0_rep=C0_dcf
0	0.5216	1.1265	1.1265	True

2 Risk-Free Assets

2.1 Closed form solution for x as a function of α, r, g, n, τ

Let

$$\theta = \frac{1+g}{1+r}$$

Present values at t=0

$$\mathrm{PV}_{\mathrm{save}} = \sum_{t=1}^{n} \frac{x\,(1+g)^{\,t-1}}{(1+r)^{\,t}} = \frac{x}{(1+r)} \sum_{t=1}^{n} \theta^{\,t-1} = \frac{x}{(1+r)}\,\frac{1-\theta^{\,n}}{1-\theta},$$

$$\mathrm{PV}_{\mathrm{ret}} = \sum_{k=1}^{\tau} \frac{\alpha\,(1+g)^{\,n+k-1}}{(1+r)^{\,n+k}} = \alpha\,\frac{(1+g)^{\,n-1}}{(1+r)^{\,n}} \sum_{k=1}^{\tau} \theta^{\,k} = \alpha\,\frac{(1+g)^{\,n-1}}{(1+r)^{\,n}}\,\frac{\theta\,(1-\theta^{\,\tau})}{1-\theta}.$$

Equate and solve for x

$$\begin{aligned}
\mathrm{PV}_{\text{save}} = \mathrm{PV}_{\text{ret}} \quad &\implies \quad \frac{x}{(1+r)}\,\frac{1-\theta^{n}}{1-\theta} = \alpha\,\frac{(1+g)^{\,n-1}}{(1+r)^{\,n}}\,\frac{\theta\,(1-\theta^{\tau})}{1-\theta} \\
&\implies \quad x\,(1-\theta^{n}) = \alpha\,\frac{(1+g)^{\,n-1}}{(1+r)^{\,n-1}}\,\theta\,(1-\theta^{\tau}) \\
&\implies \quad x = \alpha\,\frac{(1+g)^{\,n-1}}{(1+r)^{\,n-1}}\,\frac{\theta\,(1-\theta^{\tau})}{1-\theta^{n}} = \alpha\theta^{n}\,\frac{1-\theta^{\tau}}{1-\theta^{n}}.
\end{aligned}$$

To get our final equation

$$\boxed{x_{\text{disc}}(\alpha, r, g, n, \tau) = \alpha\,\theta^{n}\,\frac{1-\theta^{\tau}}{1-\theta^{n}}, \quad \theta = \frac{1+g}{1+r}, \quad r \neq g,}$$

Special case, if $r = g$ ($\theta \to 1$)

$$x_{\text{disc}} = \alpha \lim_{\theta \to 1} \theta^{n}\,\frac{1-\theta^{\tau}}{1-\theta^{n}} = \alpha\,\frac{\tau}{n}.$$

2.2 Discrete Contribution Rate

Using our equation derived above, we compute

$$x_{\text{disc}}(\alpha = 0.5, r = 0.04, g = 0.01, n = 40, \tau = 20) = (0.5)\theta^{(40)}\,\frac{1-\theta^{(20)}}{1-\theta^{(40)}}, \quad \theta = \frac{1+0.01}{1+0.04},$$

Table 7: Discrete contribution rate and inputs.

	alpha	r	g	n	tau	x_discrete
0	0.5	0.04	0.01	40	20	0.099595

2.3 Continuous-Time Contribution Rate

$$x_{\text{disc}}(\alpha, r, g, n, \tau) = \alpha\,\theta^{n}\,\frac{1-\theta^{\tau}}{1-\theta^{n}}, \quad \theta = \frac{1+g/m}{1+r/m}, \quad r \neq g,$$

Define θ_m and the m-times-per-year version

$$\theta_m = \frac{1+g/m}{1+r/m}, \qquad x_{\text{disc}}^{(m)} = \alpha\,\theta_m^{mn}\,\frac{1-\theta_m^{m\tau}}{1-\theta_m^{mn}} \quad (r \neq g).$$

Key limit: $\theta_m^m \to e^{g-r}$

$$\ln\theta_m = \ln\Big(1+\frac{g}{m}\Big)-\ln\Big(1+\frac{r}{m}\Big) = \frac{g-r}{m}+O\Big(\frac{1}{m^2}\Big), \qquad \Rightarrow \quad \lim_{m\to\infty}\theta_m^m = \exp\Big(\lim_{m\to\infty} m\ln\theta_m\Big) = e^{\,g-r}.$$

Therefore powers scale cleanly

$$\lim_{m\to\infty}\theta_m^{mn} = e^{(g-r)n}, \qquad \lim_{m\to\infty}\theta_m^{m\tau} = e^{(g-r)\tau}.$$

Continuous-time limit

$$\boxed{\lim_{m\to\infty} x_{\text{disc}}^{(m)} = \alpha\,e^{(g-r)n}\,\frac{1-e^{(g-r)\tau}}{1-e^{(g-r)n}}}$$

Table 8: Continuous contribution rate and inputs.

	alpha	r	g	n	tau	x_continuous
0	0.5	0.04	0.01	40	20	0.097234

Table 9: Discrete vs continuous: percent difference.

	x_discrete	x_continuous	pct_diff	discrete_<_cont
0	0.099595	0.097234	-2.370414	True

2.4 Sensitivity of x to r and g

We evaluate x_{cont} on a grid of salary growth g and interest rate r to visualize how funding needs change. As expected, higher r reduces the required contribution rate, while higher g increases it.

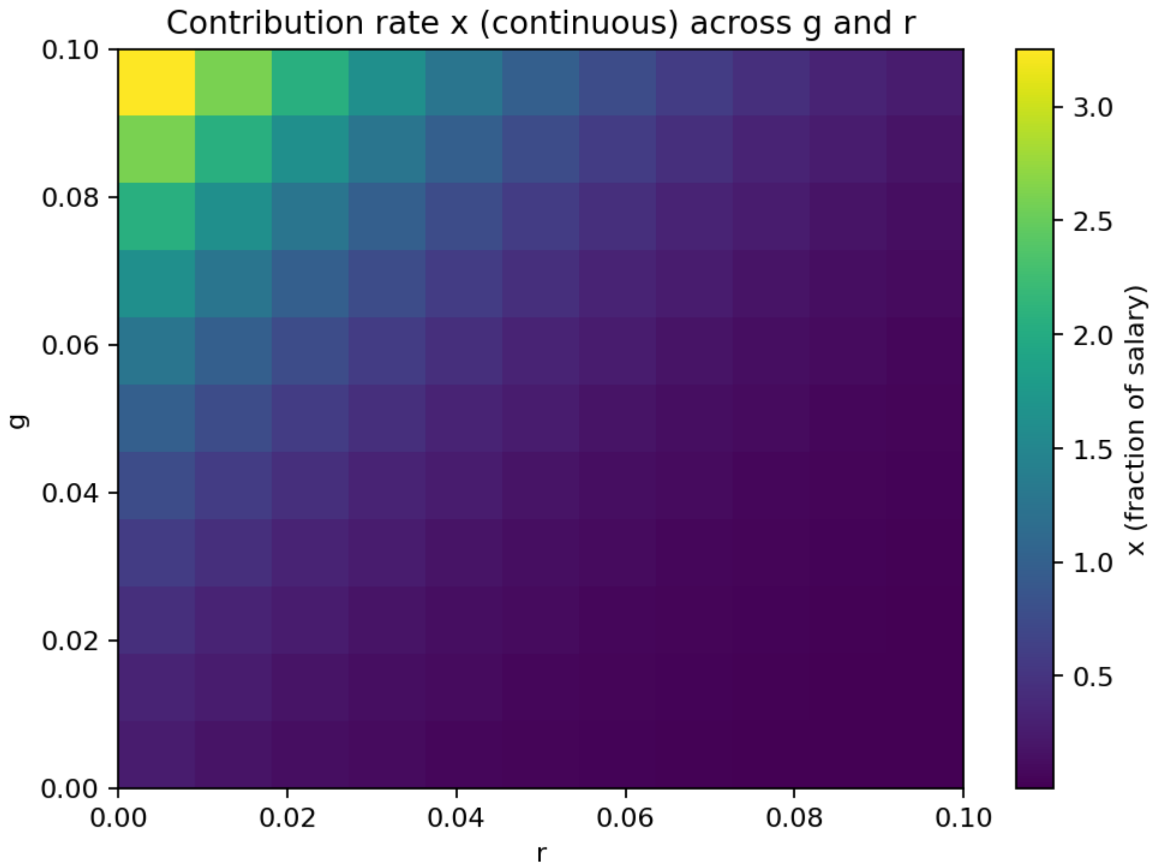

Figure 3: Contribution rate x (continuous) across r and g.

3 Portfolio Management

3.1 Define Mean Vector (μ)

$$R_i = \begin{cases} u_i & \text{with prob } \pi_i, \\ d_i & \text{with prob } 1 - \pi_i. \end{cases}$$

Starting with the mean return for asset i, $\mu_i = p_i u_i + (1 - p_i) d_i$, for $i \in \{1, 2\}$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} \pi_1 u_1 + (1 - \pi_1) d_1 \\ \pi_2 u_2 + (1 - \pi_2) d_2 \end{bmatrix}.$$

3.2 Define Covariance Matrix (Σ)

$$\mu = \mathbb{E}[R_i] = \pi_i u_i + (1-\pi_i)d_i, \qquad \mathbb{E}[R_i^2] = \pi_i u_i^2 + (1-\pi_i)d_i^2.$$

$$\sigma_i^2 = \mathrm{Var}(R_i) = \mathbb{E}[R_i^2] - \big(\mu\big)^2 = \pi_i u_i^2 + (1-\pi_i)d_i^2 - \big(\pi_i u_i + (1-\pi_i)d_i\big)^2.$$

$$\sigma_i^2 = \mathrm{Var}(R_i) = \pi_i(1-\pi_i)\left(u_i^2 + d_i^2 - 2u_i d_i\right) = \boxed{\pi_i(1-\pi_i)\,(u_i - d_i)^2}.$$

$$\sigma_i = \sqrt{\pi_i(1-\pi_i)}\,|u_1 - d_1|;$$

$$\sigma_{12} = \mathrm{Cov}(R_{12}) = \rho\,\sigma_1\,\sigma_2$$

$$\Sigma = \begin{bmatrix} \mathrm{Var}(R_1) & \mathrm{Cov}(R_{1,2}) \\ \mathrm{Cov}(R_{1,2}) & \mathrm{Var}(R_2) \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\,\sigma_1\sigma_2 \\ \rho\,\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} \pi_1(1-\pi_1)(u_1-d_1)^2 & \rho\sqrt{\pi_1(1-\pi_1)}\sqrt{\pi_2(1-\pi_2)}\,|u_1-d_1||u_2-d_2| \\ \rho\sqrt{\pi_1(1-\pi_1)}\sqrt{\pi_2(1-\pi_2)}\,|u_1-d_1||u_2-d_2| & \pi_2(1-\pi_2)(u_2-d_2)^2 \end{bmatrix}.$$

Table 10: Effective inputs used in Q3.

	mu1	mu2	sigma1	sigma2
0	0.012	0.07	0.107778	0.220454

3.3 Global Minimum-Variance (GMV) and Sharpe Portfolios

3.3.1 GMV

For two assets, the closed form weights are

$$w_{\mathrm{GMV},1} = \frac{\sigma_2^2 - \rho\,\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\sigma_2}, \qquad w_{\mathrm{GMV},2} = 1 - w_{\mathrm{GMV},1}.$$

Table 11: GMV Portfolio Weights at $\rho = 0.5, 0$

	rho	w1_GMV	w2_GMV
0	0.5	1.007242	-0.007242
1	0.0	0.807094	0.192906

3.3.2 Sharpe

With risk-free rate r_f, the tangency portfolio maximizes

$$\mathrm{SR}(w) = \frac{w^\top \mu - r_f}{\sqrt{w^\top \Sigma w}} \quad \text{subject to} \quad \mathbf{1}^\top w = 1.$$

A standard result gives

$$\boxed{w_{\mathrm{SR}} = \frac{\Sigma^{-1}(\mu - r_f\,\mathbf{1})}{\mathbf{1}^\top \Sigma^{-1}(\mu - r_f\,\mathbf{1})}}.$$

Table 12: Sharpe Portfolio Weights at $\rho = 0.5, 0$

	rho	w1_SR	w2_SR
0	0.5	-0.350552	1.350552
1	0.0	0.448996	0.551004

3.4 Diversification

The lower the correlation, the more of a diversification benefit there is. Both the GMV and the Sharpe weights shift toward holding more of a balance of assets instead of shorting one of them.

3.5 $\rho = 0.5$

3.5.1 Mean-Variance Efficient Frontier

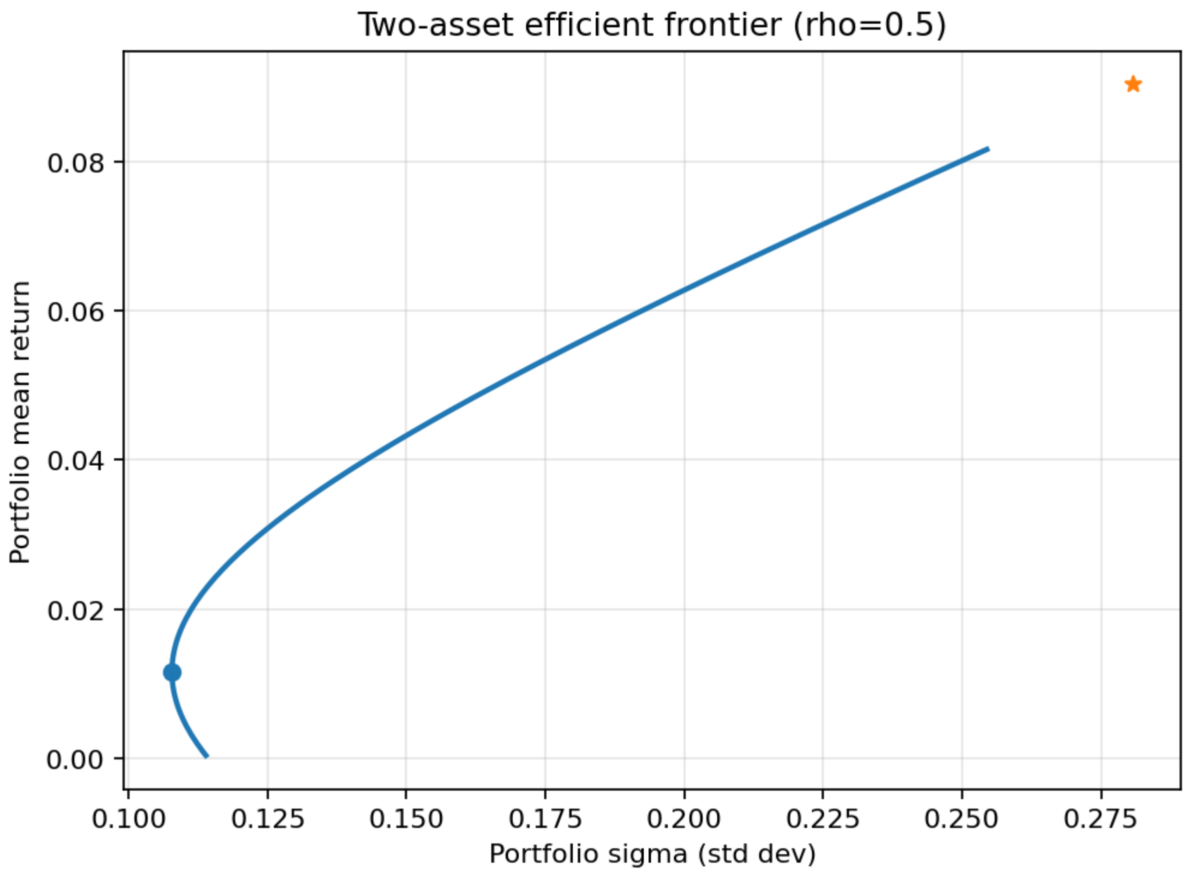

Figure 4: Two-asset efficient frontier (rho = 0.5); GMV (o) and Tangency (*).

3.5.2 Compare Arbitrary Weights for w_1 and w_2 to the Previous Frontier

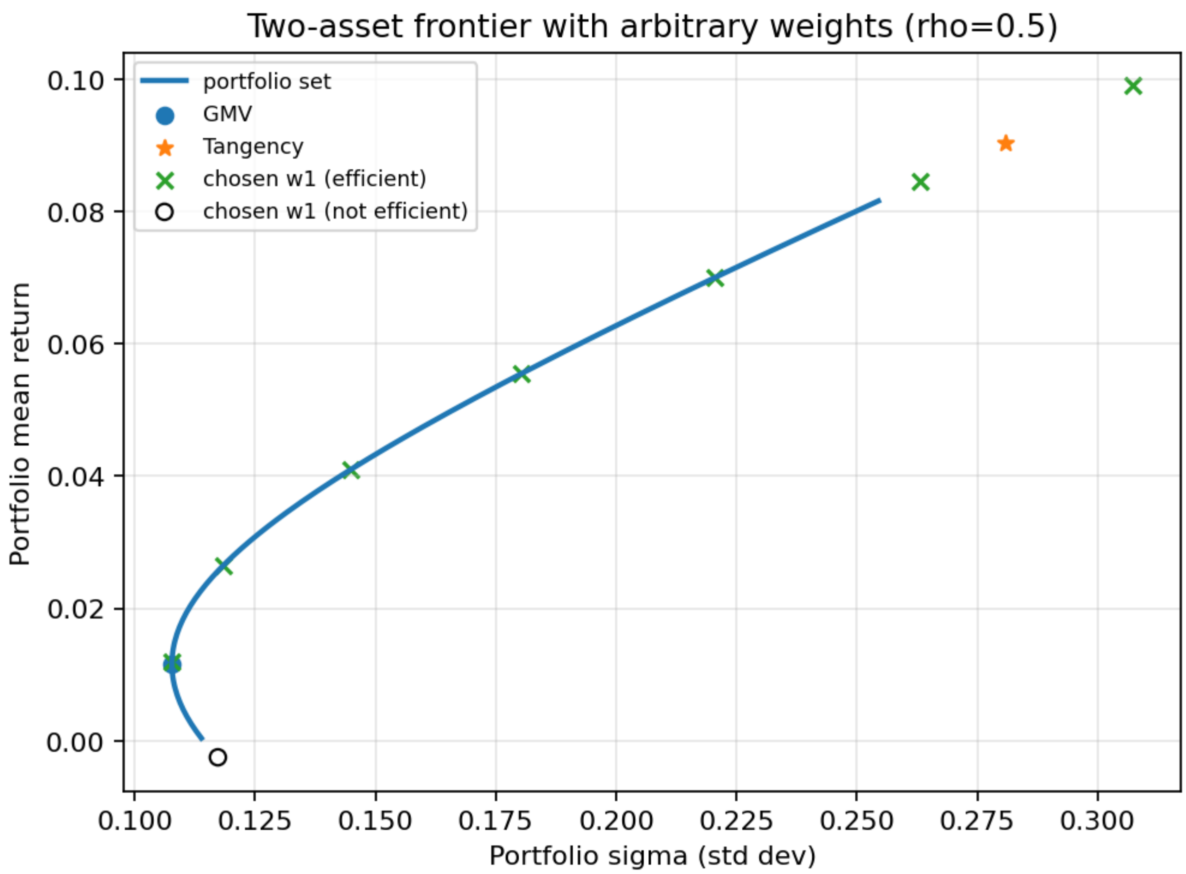

Figure 5: Frontier with arbitrary weights overlaid (rho = 0.5).

Table 13: Selected weights and their mean/sigma (rho = 0.5).

	w1	w2	mean	sigma
0	-0.50	1.50	0.0990	0.307301
1	-0.25	1.25	0.0845	0.263132
2	0.00	1.00	0.0700	0.220454
3	0.25	0.75	0.0555	0.180329
4	0.50	0.50	0.0410	0.144893
5	0.75	0.25	0.0265	0.118434
6	1.00	0.00	0.0120	0.107778
7	1.25	-0.25	-0.0025	0.117314

4 Forward Contracts

4.1 Prove $F_{(0,1)} = S_0(1 + r_f)$

No–arbitrage cash–and–carry. Assume a forward to *sell* one share at time 1 for price $F(0, 1) = F$.

(i) Suppose $F > S(0)(1 + r_f)$ (forward overpriced). At $t = 0$: borrow $S(0)$ at the risk–free rate and buy one share; short one forward. At $t = 1$: deliver the share into the short forward, receive F, and repay the loan $S(0)(1 + r_f)$. Profit $= F - S(0)(1 + r_f) > 0$ regardless of the stock path.

(ii) Suppose $F < S(0)(1 + r_f)$ (forward underpriced). At $t = 0$: short–sell one share to receive $S(0)$ and invest at the risk–free rate; go long one forward. At $t = 1$: use the long forward to buy the share for F and return it to the lender of the short sale; your invested cash is $S(0)(1 + r_f)$. Profit $= S(0)(1 + r_f) - F > 0$ path–independent.

No–arbitrage eliminates both cases, so $F(0, 1) = \boxed{S(0)(1 + r_f)}$.

4.2 Find $\mathbb{E}^{\mathbb{Q}}[S_1]$

The risk-neutral probability is defined as

$$q = \frac{r_f - d}{u - d}.$$

Then

$$\mathbb{E}^{\mathbb{Q}}[S_1] = S_0\big(q(1 + u) + (1 - q)(1 + d)\big) = S_0\,(1 + r_f).$$

Equivalently, for simple returns R_S, we have $\mathbb{E}^{\mathbb{Q}}[R_S] = r_f$.

4.3 Derive Futures Payoff Formula

A long forward entered at $t = 0$ with delivery price $F(0,1) = F$ pays at $t = 1$

$$V(1) = S_1 - F = \begin{cases} S_0U - F, & \text{with prob } \pi, \\ S_0D - F, & \text{with prob } 1 - \pi. \end{cases}$$

The long breaks even when $V(1) = 0$, i.e., when $S_1 = F$ at maturity. With a fairly priced forward, $\mathbb{E}^{\mathbb{Q}}[V(1)] = 0$.

Table 14: Long-forward payoff by state when the forward is fairly priced.

	state	prob	S1	F0	payoff_long_forward
0	up	0.70229	103.155272	104.0	-0.844728
1	down	0.29771	96.135692	104.0	-7.864308

4.4 Demonstrate Arbitrage Trading Strategy

Here $F^* = S_0(1 + r_f) = 100 \times 1.04 = 104$.

(a) $F(0,1) = 104$: $F = F^*$, so no arbitrage; the forward is fairly priced.

(b) $F(0,1) = 105$ (overpriced: $F > F^*$): *Cash–and–carry.* At $t = 0$ borrow \$100 and buy one share; short one forward. At $t = 1$ deliver the share, receive \$105, repay \$104; profit \$1 riskless.

(c) $F(0,1) = 103$ (underpriced: $F < F^*$): *Reverse cash–and–carry.* At $t = 0$ short–sell one share for \$100 and invest the proceeds; go long one forward. At $t = 1$ use the forward to buy the share for \$103, return it to the lender, and withdraw \$104 from the risk–free account; profit \$1 riskless.

Table 15: Fair forward price from cash-and-carry.

	S0	RF	F_fair
0	100.0	0.04	104.0

Table 16: Risk-neutral probability and expectation check.

	S0	U	D	RF	q	valid_q	E_Q_S1	S0*(1+RF)
0	100.0	1.031553	0.961357	0.04	1.120339	False	104.0	104.0

Table 17: Mispricing classification and strategy for F = 104, 105, 103.

	F	F_fair	mispricing	strategy
0	104.0	104.0	0.0	fair (no arbitrage)
1	105.0	104.0	1.0	short forward (overpriced); cash-and-carry
2	103.0	104.0	-1.0	long forward (underpriced); reverse cash-and-carry