These notes were compiled for a reading group presentation, to offer additional material, practice, and possible motivations for research. Please excuse the lack of quality as these notes haven't been put through the usual scrutiny, and please suggest corrections if you see any errors.

References used are [NS06] (chps. 12, 14, 16), [MS17] (chp. 10), as well as talks by Nica and lectures by Speicher at his course. TODOs: Replace [sources] with citations.

1 Recap and motivation

In the past three weeks, we have covered several combinatorial aspects of free probability theory. As this will be the last week where the focus of our session will be new concepts in FPT, and as we enter the one-month mark of this reading group, let's take a moment to remind ourselves of the reason why we have been studying FPT in the first place.

1.1 Where we stand

A central concept in several recent results studying properties of random matrices has been free probability theory and the notion of convergence in distribution, and strong convergence. Most notably, and the reason we began this reading group, was the result in [BBvH23] which showed new bounds for the spectra of Gaussian random matrices, in terms of their convergence to *free elements* in a certain noncommutative probability space. Further work in [BC24] strengthened this notion of convergence in terms of arbitrary polynomials of Gaussian random matrices, and gave matching upper and lower bounds for this result, which was then alternatively formulated in [CGVTvH24].

1.2 Recap

As a standing assumption, throughout these notes, suppose that (A, φ) is a noncommutative $*$-probability space, and let $a, b \in A_{sa}$ be self-adjoint, with respective $*$-distributions μ and ν. Recall that μ and ν are both compactly supported, with their respective supports contained within $\mathbb{R}$.

In the previous week, Robert described the following concepts:

1. Free cumulants of random variables κ_n^a; (here, when it is clear according to context, we adopt the shorthand $k_n := \kappa_n^a$)
2. The moment-cumulant formula, which uniquely determined k_n when given $\varphi(a^1), \ldots, \varphi(a^n)$:
$$\varphi(a^n) = \sum_{\pi \in \mathrm{NC}(n)} \prod_{V \in \pi} k_{|V|} \tag{1}$$
3. The equivalent formulation of free independence in terms of free cumulants: i.e., $a, b \in A$ are free if and only if $\kappa_n^{a+b} = \kappa_n^a + \kappa_n^b$, for all $n \in \mathbb{N}_+$;
4. The vanishing of mixed cumulants of free elements;
5. The anti-isomorphic (bijective, cyclic, order-reversing) Kreweras complement $\mathrm{Kr} : \mathrm{NC}(n) \to \mathrm{NC}(n)$ such that, if $\pi \in \mathrm{NC}(n)$ with $|\pi| = k$, then $|\mathrm{Kr}(\pi)| = n - k$;
6. The canonical factorisation of the $\mathrm{NC}(n)$ lattice: if $\pi = \{V_1, \ldots, V_k\} \in \mathrm{NC}(n)$, then
$$[\mathbf{0}_n, \pi] = \{\sigma \in \mathrm{NC}(n) : \mathbf{0}_n \leq \sigma \leq \pi\} \cong \mathrm{NC}(|V_1|) \times \ldots \times \mathrm{NC}(|V_k|). \tag{2}$$

1.3 Motivation

At this point, we wish to move on from considering only the spectra and distributions of free elements to considering polynomials of noncommuting free elements. In other words, we wish to build a up a theory that will allow us to compute the $*$-distribution of some $p(a_1, \ldots, a_m)$, if it exists, when given the $*$-distributions μ_i of the free a_i(s).

To do so, we begin by considering the free analogue of additive and multiplicative convolutions, before reducing the problem of polynomial convolutions to these simpler cases. We begin, first, by introducing all the transforms that we shall be using.

2 Introducing the transforms

We can define the following transforms both for free elements and for their corresponding $*$-distributions. It makes sense to convert between these two notions without worrying, because the transforms are all defined in terms if either the moments or the free cumulants, which uniquely determine each other.

Additionally, note that we postpone proving the statements about the Cauchy transform for later sessions.

Definition 2.1 (Cauchy transform)**.** *Given a self-adjoint free element a (resp. a compactly supported probability distribution μ over $\mathbb{R}$), its Cauchy transform is a map $G_a \equiv G_\mu : \mathbb{C}_+ \to \mathbb{C}_-$ given by*

$$G_\mu : z \mapsto \int_{\mathbb{R}} \frac{1}{z-t}\, \mathrm{d}\mu(t). \tag{3}$$

Here, we denote $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ and $\mathbb{C}_- = \{z \in \mathbb{C} : \Im(z) < 0\}$.

Proposition 2.2. *G_a is analytic, and has a power series expansion centered at infinity on the disk $\{|z| > r\}$ where $r := \sup\{|t| : t \in \operatorname{supp}\, \mu\}$, given by*

$$G_a(z) = \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{z^{n+1}} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{\varphi(a^n)}{z^n} \tag{4}$$

Proof sketch. Recall that $\frac{1}{z-t} = \sum_{n=0}^{\infty} \frac{t^n}{z^{n+1}}$, and that $z - t \neq 0$ for all $z \in \mathbb{C}_+$ (so G_a has no poles to integrate over). ■

Without diving into the details, we briefly describe the Stieltjes inversion forumula which allows us to recover the distribution μ when given its Cauchy transform G_μ.

Theorem 2.3 (Stieltjes inversion)**.** *For any $\epsilon > 0$, let $h_{\mu,\epsilon}(t) := -\frac{1}{\pi}\Im[G_\mu(t+i\epsilon)]$. Then, $\frac{d\mu}{dt} = \lim_{\epsilon\to 0} h_{\mu,\epsilon}(t)$.*

Note that, using Proposition 2.2, it makes sense to interpret the Cauchy transform as a formal power series $G_a \in \mathbb{C}[[z]]$, which is exactly what we shall do for the rest of these notes. We now introduce the key transforms to study convolutions. As notation, denote by $\mathbb{C}_0[[z]]$ the set of all formal power series in z with zero constant term.

Definition 2.4 (Free transforms)**.** *Given $a \in A_{sa}$ (resp. μ over $\mathbb{R}$), we define three transforms:*

1. *The moment transform $M_a \in \mathbb{C}_0[[z]]$ given by*

$$M_a : z \mapsto \sum_{n=1}^{\infty} \varphi(a^n) z^n; \tag{5}$$

2. *The R-transform $R_a \in \mathbb{C}_0[[z]]$ given by*

$$R_a : z \mapsto \sum_{n=1}^{\infty} k_n z^n; \tag{6}$$

3. *The cumulant transform $C_a \in \mathbb{C}[[z]]$ given by*

$$C_a : z \mapsto 1 + R_a(z) \tag{7}$$

Immediately, from the definitions, we note that

Proposition 2.5. $G_a(z) = \frac{1}{z}(1 + M_a(1/z))$.

Proof. Note that $G_a(z) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{\varphi(a^n)}{z^n} = \frac{1}{z}\sum_{n=0}^{\infty}\varphi(a^n)(1/z)^n = \frac{1}{z}(1 + M_a(z))$. ■

Next, and most importantly, we obtain the functional equation for the R-transform:

Theorem 2.6 (Functional equation for R-transform)**.** $R_a[z(1 + M_a(z))] = M_a(z)$.

Before looking at the proof, let's look at a few useful applications of the R-transform and of this result.

Corollary 2.6.1 (Cauchy transform and R-transform)**.** *We note that*

1. $G_a[\frac{1}{z}(1 + R_a(z))] = z$;
2. $R_a[G_a(z)] + 1 = zG_a(z)$.

Proof. Define $K(z) = z^{-1}C_a(z)$ as the *"K-transform"*. Then, to prove 2. above, it suffices to show that $K_a[G_a(z)] = z$. In fact, we claim (without proof) that to prove 1., it also suffices to show that $G_a[K_a(z)] = z$. Now, all that remains is straightforward computation:

$$\begin{aligned}
K_a[G_a(z)] &= \frac{1}{G_a(z)}C_a[G_a(z)] \\
&= \frac{1}{G_a(z)}(1 + R_a[\tfrac{1}{z}(1 + M_a(1/z))]) && \text{(by Proposition 2.5)} \\
&= \frac{1}{G_a(z)}(1 + M_a(1/z)) && \text{(by Theorem 2.6)} \\
&= \frac{1}{z^{-1}(1 + M_a(1/z))}(1 + M_a(1/z)) = z && \text{(by Proposition 2.5)}
\end{aligned}$$

We omit the proof for 2., chalking it up to similar arithmetic with formal power series. ■

2.1 Sums of free random variables

Firstly, we remark (without proof) that the R-transform is commutative, associative, and satisfies the following property:

Proposition 2.7. *For any self-adjoint $a \in A_{sa}$ and any $\lambda \in \mathbb{C}$, we have $R_{\lambda a}(z) = \lambda R_a(\lambda z)$.*

Again, let $a, b \in A$ be free self-adjoint elements with corresponding $*$-distributions μ and ν. Then, $a + b \in A$ is also self-adjoint, and hence, also has a compactly supported $*$-distribution over $\mathbb{R}$. Denote this new distribution by $\mu \boxplus \nu$, and our goal now is to compute $\mu \boxplus \nu$ in terms of μ and ν.

Ah, the R-transform is perfect for this!

Theorem 2.8 (Additivity of the R-transform)**.** *With a, b as above, $R_{\mu\boxplus\nu} \equiv R_{a+b} = R_a + R_b \equiv R_\mu + R_\nu$.*

Proof. The result follows from the definition of the R-transform and the additivity of the free cumulants. ■

The task to recover the distribution $\mu \boxplus \nu$ from its R-transform remains. While this isn't always possible analytically, here is a workflow which applies to several non-trivial examples, when given μ, ν (or a, b) as above,

1. Compute either G_μ and G_ν, or M_μ and M_ν;
2. Compute R_μ and R_ν;
3. Compute $R_{\mu\boxplus\nu} = R_\mu + R_\nu$;
4. Compute $G_{\mu\boxplus\nu}$;
5. Apply Stieltjes inversion.

Consider the following examples.

Example 2.9. *Suppose $a_1, \dots, a_n \in A_{sa}$ are Bernoulli element, i.e., $\mu = \mu_j = \frac{1}{2}(\delta_{-1} + \delta_{+1})$ is the standard Bernoulli distribution, or equivalently $\varphi(a_j^m) = \frac{1+(-1)^m}{2}$. Then, $M_\mu(z) = z^2 + z^4 + \dots = \frac{z^2}{1-z^2}$, which gives us $z(1 + M_\mu(z)) = \frac{z}{1-z^2}$.*

Using Theorem 2.6, we get $R_a[\frac{z}{1-z^2}] = \frac{z^2}{1-z^2}$. Change variables to $w = \frac{z}{1-z^2}$, so $R_a(w) = zw$. Upon solving for z in terms of w, we have $z = \frac{-1+\sqrt{1+4w^2}}{2w}$. Thus, the R-transform is

$$R_\mu(w) = \frac{-1 + \sqrt{1 + 4w^2}}{2}.$$

Hence, we get $R_{\mu^{\boxplus n}}(z) = \frac{n}{2}(-1 + \sqrt{1 + 4z^2})$. From Corollary 2.6.1, we get

$$\frac{n}{2}(-1 + \sqrt{1 + 4G_{\mu^{\boxplus n}}(z)^2}) + 1 = zG_{\mu^{\boxplus n}}(z) - 1$$

Upon solving the resulting quadratic equation for $G_{\mu^{\boxplus n}}$, and applying Stieltjes inversion, we will recover the distribution $\mu^{\boxplus n}$. In particular, $G_{\mu^{\boxplus n}}$ is given by

$$G_{\mu^{\boxplus n}}(z) = \frac{(2 - n)z + \sqrt{(n - 2)^2 z^2 + 4(z^2 - n^2)(n - 1)}}{2(z^2 - n^2)}.$$

Remark 2.10. *We make two remarks for Example 2.9, which aren't related to our main goal at hand.*

1. *It's a simple computation to show that the special case of $n = 2$ yields $\mu \boxplus \mu$ as the arcsine distribution. Recall also that, in the classical case, the convolution of two Bernoulli distributions yields a Binomial distribution. This is why, sometimes, the arcsine distribution is called the "free Binomial distribution" in FPT literature.*

2. *This example of $\mu^{\boxplus n}$ is, perhaps, the origin of free probability. In [Kesten, need cite], Kesten considered these probability measures (for all $n \in \mathbb{R}_+$, not just $n \in \mathbb{N}$) to study random walks on groups. Voiculescu then used elements of this theory in his result distinguishing between the free groups with 2 and 3 generators. More generally, studying $\mu^{\boxplus n}$ in the context of modern FPT can lead to a definition of free convolution semigroups, similar to the classical Markov semigroups described in Lap Chi's notes from the Winter 2024 reading group [Lap Chi: need cite].*

2.2 Proof of the functional equation for the R-transform

Recall that the statement of Theorem 2.6 claims that $R_a[z(1+M_a)] = M_a$. The key idea in proving this claim is to show the coefficients of the formal power series on both sides agree, i.e., ${}_{[z^n]}R_a[z(1 + M_a)] = {}_{[z^n]}M_a$ for all $n \in \mathbb{N}_+$.

We begin with a combinatorial argument for the right side:

$${}_{[z^n]}M_a(z) = \varphi(a^n) = \sum_{\pi \in \mathrm{NC}(n)} \prod_{V \in \pi} k_{|V|}.$$

Now, we parameterise each $\pi \in \mathrm{NC}(n)$ by its leftmost block, i.e. by $V_\pi = \{i_1, \dots, i_m : 1 = i_1 < \dots < i_m \le n\}$. Note that this leaves *pockets* of size $j_r = i_{r+1} - i_r$ between successive elements $i_r, i_{r+1} \in V$, and one additional *pocket* of size $n - i_m$ to the right of V. These pockets are all of non-negative size $j_r \ge 0$, and cover the remaining $n - m$ elements outside V, i.e. $\sum_{r \in [m]} j_r = n - m$. Furthermore, the restriction of π to these pockets must be well-defined, on account of being non-crossing, and we then obtain smaller non-crossing partitions $\pi_1, \dots, \pi_m$ such that $\pi_r \in \mathrm{NC}(j_r)$, for each $r \in [m]$.

As notation, we suppose $|\mathrm{NC}(0)| = 1$ to account for the case where V contains neighbouring elements. Effectively, our parameterisation is of the form:

$$\pi \longleftrightarrow (V; m; j_1, \dots, j_m; \pi_1, \dots, \pi_m). \tag{8}$$

Upon replacing the original summation with one suiting our parameterisation, we have

$$\begin{aligned}
\varphi(a^n) &= \sum_{m=1}^{n} \sum_{\substack{j_1,\dots,j_m \geq 0 \\ j_1+\dots+j_m=n-m}} \sum_{\substack{\pi_1 \in \mathrm{NC}(j_1) \\ \cdots \\ \pi_m \in \mathrm{NC}(j_m)}} k_m \cdot \left(\prod_{V_1 \in \pi_1} k_{|V_1|} \right) \dots \left(\prod_{V_m \in \pi_m} k_{|V_m|} \right) \\
&= \sum_{m=1}^{n} k_m \cdot \left[\sum_{\substack{j_1,\dots,j_m \geq 0 \\ j_1+\dots+j_m=n-m}} \left(\sum_{\pi_1 \in \mathrm{NC}(j_1)} \prod_{V_1 \in \pi_1} k_{|V_1|} \right) \dots \left(\sum_{\pi_m \in \mathrm{NC}(j_m)} \prod_{V_m \in \pi_m} k_{|V_m|} \right) \right] \\
&= \sum_{m=1}^{n} k_m \cdot \left[\sum_{\substack{j_1,\dots,j_m \geq 0 \\ j_1+\dots+j_m=n-m}} \varphi(a^{j_1}) \dots \varphi(a^{j_m}) \right] \\
&= \sum_{m=1}^{n} k_m \sum_{\substack{j_1,\dots,j_m \geq 0 \\ j_1+\dots+j_m=n-m}} \prod_{r=1}^{m} \varphi(a^{j_r})
\end{aligned}$$

We now move on towards considering the left side, where we argue using formal proof arithmetic:

$$\begin{aligned}
R_a[z(1+M_a(z))] &= \sum_{n=1}^{\infty} R_a[z(1+\textstyle\sum_{k\geq 1} \varphi(a^k) z^k)] \\
&= \sum_{n=1}^{\infty} k_n z^n \cdot (1+\textstyle\sum_{k\geq 1} \varphi(a^k) z^k)^n \\
&= \sum_{n=1}^{\infty} \xi_n z^n \qquad\qquad \text{(for some } \xi_n = {}_{[z^n]} R_a[z(1+M_a(z))])
\end{aligned}$$

Now, to compute the coefficient ξ_n, note that we can consider a contribution of $k_m z^m$ and obtain an additional z^{n-m} term from $(1+\sum_{k\geq 1} \varphi(a^k) z^k)^m$. In other words, we wish to find the coefficient of z^{n-m} in the expansion of:

$$\underbrace{\left(1+\sum_{k=1}^{\infty} \varphi(a^k) z^k\right) \dots \left(1+\sum_{k=1}^{\infty} \varphi(a^k) z^k\right)}_{n \text{ times}}$$

If we obtain a z^{j_1} term from the first bracket, and so on, until a z^{j_m} term from the last bracket, then we wish to impose the restriction $j_1+\dots+j_m = n-m$, where each $j_r \geq 0$. Thus, the desired expansion becomes

$$R_a[z(1+M_a(z))] = \sum_{n=1}^{\infty} \xi_n z^n = \sum_{n=1}^{\infty} z^n \cdot \left(\sum_{m=1}^{n} k_m \sum_{\substack{j_1,\dots,j_m \geq 0 \\ j_1+\dots+j_m=n-m}} \prod_{r=1}^{m} \varphi(a^{j_r}) \right)$$

Indeed, this agrees with the coefficient of z^n we found for the right side! ■

2.3 The free central limit theorem

As a demonstration of the practicality of the R-transform, let's derive an alternative proof for the free central limit theorem.

Theorem 2.11 (Free central limits)**.** *For $n \in \mathbb{N}_+$, let $a_1, \dots, a_n, \dots$ be free elements that are all identically distributed and centered, with unit variance. Then, as $N \to \infty$, we have $a := (a_1 + \dots + a_N)/\sqrt{N}$ converging to s (the standard semicircular element) in distribution.*

Proof. It suffices to prove that $\mathcal{R}_a(z) := z^{-1}R_a(z)$ converges to $\mathcal{R}_s(z) := z^{-1}R_s(z)$, which would in turn prove equality for the R-transforms, and hence, the respective moments via the moment-cumulant formula. Recall that $R_s(z) = z^2$, and note that $\mathcal{R}$ also satisfies Proposition 2.7. Further recall that $k_1 = \varphi(a)$ and also $k_2 = \varphi(a^2) - \varphi(a)^2$ (hence, k_2 is also called the *free variance*).

We now have

$$\begin{aligned}
\mathcal{R}_a(z) &= \frac{1}{\sqrt{N}}\mathcal{R}_{a_1+\ldots+a_N}(z/\sqrt{N}) \\
&= \frac{N}{\sqrt{N}}\mathcal{R}_{a_1}(z/\sqrt{N}) \\
&= \sqrt{N}\cdot\sum_{n=0}^{\infty} k_{n+1}\frac{z^n}{N^{n/2}} \\
&= \sqrt{N}\cdot\left(k_1 + \frac{k_2}{\sqrt{N}}z + \sum_{n=2}^{\infty} k_{n+1}\frac{z^n}{N^{n/2}}\right) \\
&= 0 + z + \mathcal{O}(1/\sqrt{N}) && (k_1 = 0 \text{ and } k_2 = 1)
\end{aligned}$$

Since $z + \mathcal{O}(1/\sqrt{N}) \to z = \mathcal{R}_s(z)$ as $N \to \infty$, we obtain the desired result. ∎

3 The free multiplicative convolution

This section will use the concepts of canonically factoring the $\mathrm{NC}(n)$ lattice and properties of the Kreweras complement Kr (or just K) which were introduced in the previous week.

We begin by introducing the free multiplicative convolution as a binary operation on formal power series in $\mathbb{C}_0[[z]]$, i.e. a map $\mathbb{C}_0[[z]] \times \mathbb{C}_0[[z]] \to \mathbb{C}_0[[z]]$. We then pull it back towards a binary operation on probability distributions via the R-transform.

Definition 3.1 (Free multiplicative convolution of formal power series)**.** *Given $f, g \in \mathbb{C}_0[[z]]$ with $f(z) = \sum_{n=1}^{\infty}\alpha_n z^n$ and $g(z) = \sum_{n=1}^{\infty}\beta_n z^n$, we define $f \boxtimes g \in \mathbb{C}_0[[z]]$ by $(f \boxtimes g)(z) := \sum_{n=1}^{\infty}\gamma_n z^n$, where*

$$\gamma_n := \sum_{\pi\in\mathrm{NC}(n)}\left(\prod_{V\in\pi}\alpha_{|V|}\right)\left(\prod_{W\in\mathrm{Kr}(\pi)}\beta_{|W|}\right). \tag{9}$$

Consider the following examples, for some small n:

Example 3.2 (Small n)**.** *With f and g as in the preceding definition, we now compute γ_n for $n = 1, 2, 3$.*

1. $\mathrm{NC}(1)$ *consists of a single partition, which is its own Kreweras complement. Thus, $\gamma_1 = \alpha_1\beta_1$.*
2. $\mathrm{NC}(2)$ *consists of two partitions, and the Kreweras complement is cyclic between the two. Thus, $\gamma_1 = \alpha_1^2\beta_2 + \alpha_2\beta_1^2$.*
3. *The structure of* $\mathrm{NC}(3)$ *is the first non-trivial lattice. Upon repeating the computation, we get*

$$\gamma_3 = \alpha_1^3\beta_3 + 3\alpha_1\alpha_2\beta_1\beta_2 + \alpha_3\beta_1^3.$$

The examples above ideally suggest that $\boxtimes$ is commutative, as an operation on formal power series. Indeed, this is true, and we have:

Theorem 3.3 (Properties of free multiplicative convolution)**.** *As an operation on formal power series, $\boxtimes$ is associative and commutative. Further, the formal power series* $\mathrm{id}(z) = z$ *is the multiplicative unit with respect to $\boxtimes$.*

Proof. We omit the proof of associativity, as it is more involved than the other properties. To prove commutativity, consider γ_n to be the coefficient of z^n in $(g \boxtimes f)$, where f, g are as above. Then:

$$\begin{aligned}
\gamma_n &= \sum_{\pi \in \mathrm{NC}(n)} \left(\prod_{V \in \pi} \beta_{|V|} \right) \left(\prod_{W \in \mathrm{Kr}(\pi)} \alpha_{|W|} \right) \\
&= \sum_{\substack{\rho \in \mathrm{NC}(n) \\ \pi = \mathrm{Kr}(\rho)}} \left(\prod_{V \in \mathrm{Kr}(\rho)} \beta_{|V|} \right) \left(\prod_{W \in \mathrm{Kr}^2(\rho)} \alpha_{|W|} \right) && \text{(since Kr is bijective)} \\
&= \sum_{\substack{\rho \in \mathrm{NC}(n) \\ \pi = \mathrm{Kr}(\rho)}} \left(\prod_{W \in \mathrm{Kr}^2(\rho)} \alpha_{|W|} \right) \left(\prod_{V \in \mathrm{Kr}(\rho)} \beta_{|V|} \right)
\end{aligned}$$

Since Kr^2 simply applies the right-shift permutation on all elements, it preserved the sizes of the sets in the original partition. Thus, this implies

$$\gamma_n = \sum_{\rho \in \mathrm{NC}(n)} \left(\prod_{W \in \rho)} \alpha_{|W|} \right) \left(\prod_{V \in \mathrm{Kr}(\rho)} \beta_{|V|} \right).$$

This matches the coefficient of z^n in $(f \boxtimes g)(z)$ as given in the definition, hence proving commutativity. Next, we compute the coefficients in $(f \boxtimes \mathrm{id})(z)$:

$$\gamma_n = \sum_{\pi \in \mathrm{NC}(n)} \left(\prod_{V \in \pi)} \alpha_{|V|} \right) \left(\prod_{W \in \mathrm{Kr}(\pi)} \mathbf{1}_{|W|,1} \right) = \alpha_n \cdot 1,$$

The second equality follows from the observation that the second product is 1 if and only if $\pi = \mathbf{1}_n \in \mathrm{NC}(n)$ (i.e., $\mathrm{Kr}(\pi)$ consists of singletons), and is 0 otherwise. Using commutativity of $\boxtimes$, id is the multiplicative unit. ■

Corollary 3.3.1. $(\mathbb{C}_0[[z]], \boxtimes)$ *is a monoid.*

We explicitly define two additional formal power series, which will be related to the ζ and Mőb functions defined in the previous week. Due to this relation, we borrow the same names.

Definition 3.4 (ζ and Mőb)**.** *Define* $\zeta, \textit{Mőb} \in \mathbb{C}_0[[z]]$ *by* $\zeta(z) := \sum_{n=1}^{\infty} z^n$ *and* $\textit{Mőb}(z) = \sum_{n=1}^{\infty} (-1)^{n-1} C_n z^n$.

In fact, Mőb is precisely the Mőbius inversion function on poset intervals, when expressed as a formal power series! Thus, we claim without proof:

Proposition 3.5. $\zeta \boxtimes \textit{Mőb} = \mathrm{id}$.

Additionally, recall that we previously proved $m_n = k_n * \zeta$, as functions on posets (where m_n denotes the moment of order n). We can readily lift this to a statement for the respective formal power series as:

Proposition 3.6. *For all self-adjoint* $a \in A_{sa}$*, we have* $M_a = R_a \boxtimes \zeta$*.*

Corollary 3.6.1. *For all self-adjoint* $a \in A_{sa}$*, we have* $R_a = M_a \boxtimes \textit{Mőb}$*.*

Proof. This follows immediately from Proposition 3.5. ■

We are now ready to pull this operation back to yield an operation on probability distribution. However, before doing so, we must make an important interjection.

Notice that, even if a and b are free and self-adjoint in A, it need not be true that their product ab is self-adjoint. Thus, we cannot guarantee that ab has a corresponding $*$-distribution if we solely assume that $a, b \in A_{sa}$. We thus make the stronger assumption that a and b are positive elements, i.e., $a, b \in A_+$ (where positive elements are those with positive spectra, or equivalently, $A_+ = \{a^* a : a \in A\}$) with corresponding

$*$-distributions μ and ν. This then guarantees that ab is itself positive, and thus has a $*$-distribution, denoted by $\mu \boxtimes \nu$.

Remark 3.7. *It would actually suffice to let exactly one of a or b to be positive, and the other self-adjoint. Indeed, if $a \in A_+$ and $b \in A_{sa}$, then a has a positive square-root $\sqrt{a} \in A_+$. Thus, we get:*

$$\begin{aligned} \varphi((\sqrt{a}b\sqrt{a})^n) &= \varphi(\sqrt{a}(ba)^{n-1}b\sqrt{a}) \\ &= \varphi(\sqrt{a}\sqrt{a}(ba)^{n-1}b) = \varphi((ab)^n). \qquad \text{(tracial property of } \varphi) \end{aligned}$$

Since $(\sqrt{a}b\sqrt{a})^* = \sqrt{a}b\sqrt{a}$ is self-adjoint, it has a corresponding $*$-distribution.

We now introduce the following theorom.

Theorem 3.8. *Given free elements a, b as above, we remark*

1. *$M_{ab} = R_a \boxtimes M_b = M_a \boxtimes R_b$;*
2. *$R_{ab} = R_a \boxtimes$.*

Before proving this theorem, note that the two statements above are equivalent! Indeed, suppose that $M_{ab} = R_a \boxtimes M_b$. Then, upon right-multiplying by Mőb, we obtain $R_{ab} = R_a \boxtimes R_b$. Conversely, if we assume $R_{ab} = R_a \boxtimes R_b$, then left- or right-multiplying by ζ yields $M_{ab} = R_a \boxtimes M_b = M_a \boxtimes R_b$.

Proof. It suffices to prove only the first equality in the first statment. Once again, consider coefficients of the power series. We have:

$$\varphi((ab)^n) = \sum_{\sigma \in \mathrm{NC}(2n)} \prod_{U \in \sigma} \underbrace{\kappa_U((a, b, \ldots, a, b)|_U))}_{=0 \text{ unless } \substack{U \subseteq \{2,4,\ldots,2n\} \text{ or} \\ U \subseteq \{1,3,\ldots,2n-1\}}}$$

Introduce the terminology $\mathrm{NC}_{o/e}(m)$ for the set of all noncrossing partitions σ where each set in the partition $U \in \sigma$ is contained entirely within either the even numbers or the odd numbers.

Given $\pi \in \mathrm{NC}(n)$, let $\pi^o = \{2V - 1 : V \in \pi\}$ denote a non-crossing partition of $2[n] - 1$. Similarly, let $\pi^e = \{2V : V \in \pi\}$ be the corresponding non-crossing partition of $2[n]$. Note that, for any $\sigma \in \mathrm{NC}_{o/e}(2n)$, there exist some $\pi, \rho \in \mathrm{NC}(n)$ such that $\sigma = \pi^o \sqcup \rho^e$. However, not just any π^o and ρ^e will yield such a σ!

Indeed, upon fixing some π^o, note that $\mathrm{Kr}(\pi)^e$ is the coarsest (in the sense of the reverse refinement order) partition of $\{2, 4, \ldots, 2n\}$ that does not cross with π^o. Thus, the set of all possible ρ such that $\rho^e \sqcup \pi^o$ is non-crossing is precisely the interval $[\mathbf{0}_n, \mathrm{Kr}(\pi)] \cong \mathrm{NC}(|Y_1|) \times \ldots \times \mathrm{NC}(|Y_q|)$, where we let $\mathrm{Kr}(\pi) = \{Y_1, \ldots, Y_q\}$. Now, we have:

$$.\varphi((ab)^n) = \sum_{\pi \in \mathrm{NC}(n)} \sum_{\rho \in [\mathbf{0}_n, \mathrm{Kr}(\pi)]} \left(\prod_{V \in \pi} \kappa^a_{|V|} \right) \left(\prod_{W \in \rho} \kappa^b_{|W|} \right)$$

By identifying each such ρ with some tuple $(\rho_1, \ldots, \rho_q)$ (where $\rho_j \in \mathrm{NC}(|Y_j|)$ for all j), we have:

$$\begin{aligned} \varphi((ab)^n) &= \sum_{\pi \in \mathrm{NC}(n)} \left[\left(\prod_{V \in \pi} \kappa^a_{|V|} \right) \cdot \sum_{\substack{\rho_1 \in \mathrm{NC}(|Y_1|) \\ \rho_q \in \mathrm{NC}(|Y_q|)}} \left(\prod_{W_1 \in \rho_1} \kappa^b_{|W_1|} \right) \cdots \left(\prod_{W_q \in \rho_q} \kappa^b_{|W_q|} \right) \right] \\ &= \sum_{\pi \in \mathrm{NC}(n)} \left[\left(\prod_{V \in \pi} \kappa^a_{|V|} \right) \cdot \left(\sum_{\rho_1 \in \mathrm{NC}(|Y_1|)} \prod_{W_1 \in \rho_1} \kappa^b_{|W_1|} \right) \cdots \left(\sum_{\rho_q \in \mathrm{NC}(|Y_q|)} \prod_{W_q \in \rho_q} \kappa^b_{|W_q|} \right) \right] \\ &= \sum_{\pi \in \mathrm{NC}(n)} \left[\left(\prod_{V \in \pi} \kappa^a_{|V|} \right) \cdot \left(\prod_{Y \in \mathrm{Kr}(\pi} \varphi(b^{|Y|}) \right) \right] \end{aligned}$$

Note that this is precisely the coefficient of z^n in the formal power series $R_a \boxtimes M_b$. ■

Remark 3.9. *As an operation on compactly supported probability distributions on $\mathbb{R}_+$, the associativity of $\boxtimes$ is no surprise at all. Indeed, let $\mu \boxtimes \nu$ be the $*$-distribution of $ab \in A_+$, and take ξ to be the $*$-distribution of $c \in A_+$. Then, we wish for $(\mu \boxtimes \nu) \boxtimes \xi$ to be the $*$-distribution for $abc \in A_+$, i.e., have its n-th moments as $\varphi((abc)^n) = \varphi((\sqrt{ab}c\sqrt{ab})^n) = \varphi((\sqrt{bc}a\sqrt{bc})^n)$.*

4 Onwards?

4.1 Polynomials??

Consider the polynomial $p = a_1b_1a_1 + a_2b_2a_2$ for positive free elements a_1, a_2, b_1, b_2 when $\{a_1, a_2\}$ is free from $\{b_1, b_2\}$. Clearly, p is self-adjoint, and thus, has a $*$-distribution. However, while we can obtain the distribution of $a_1b_1a_1$ and $a_2b_2a_2$ using the concepts mentioned above, we still can't say anything about their sum, i.e., p. This is because the two summands need not be free themselves.

Unfortunately, this shows we still don't have all the tools to establish a theory for the spectra (resp. distributions) of polynomials of noncommuting free elements. But we're nearly there! Before reading the relevant papers for the reading group, all that remains is to lift the free probability theory we have developed to one over matrices over such $*$-probability spaces, and consider an operator-valued version of FPT.

4.2 What maths remains?

At this point, to go through all relevant (main) general results in [Lehner: need cite] and [BBvH23], we need to introduce the following, in addition to the previous point about distributions of polynomials.

1. We need to introduce resolvents of matrices, for which BBvH give new asymptotic bounds.
2. We need to introduce Gaussian random matrices and their asymptotic freeness. for the setup of BBvH and its applications to discrepancy theory.
3. We must describe cumulants of operators on Fock space, for the proof of Lehner Theorem 1.3 and Corollary 1.5.

I recommend browsing through [Spe98] if interested in looking at applications of the combinatorics of operator-valued FPT to TCS.

References

[BBvH23] Afonso S. Bandeira, March T. Boedihardjo, and Ramon van Handel. Matrix concentration inequalities and free probability. *Inventiones mathematicae*, 234(1):419–487, June 2023.

[BC24] Charles Bordenave and Benoit Collins. Norm of matrix-valued polynomials in random unitaries and permutations, 2024.

[CGVTvH24] Chi-Fang Chen, Jorge Garza-Vargas, Joel A. Tropp, and Ramon van Handel. A new approach to strong convergence, 2024.

[MS17] James Mingo and Roland Speicher. *Free Probability and Random Matrices*, volume 35 of *Fields Institute Monographs*. Springer New York, NY, 2017.

[NS06] Alexandru Nica and Roland Speicher. *Lectures on the Combinatorics of Free Probability*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2006.

[Spe98] Roland Speicher. *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, volume 132 of *Memoirs of the American Mathematical Society*. Amer. Math. Soc., 1998.