2652227 Assignment 2

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1 Question 1

The characteristic equation for the given differential equation is:

$$r^2 - 4r + 8 = 0$$

The roots of this quadratic equation, we use the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 8}}{2 \cdot 1}$$

Simplifying further:

$$r = \frac{4 \pm \sqrt{16 - 32}}{2}$$

$$r = \frac{4 \pm \sqrt{-16}}{2}$$

$$r_1 = 2 + 2i \qquad ; \qquad r_2 = 2 - 2i$$

The complementary solution for the differential equation is then given by:

$$y_c(x) = e^{2x}(A\cos(2x) + B\sin(2x)) \qquad ; \qquad y_c(x) = Ae^{2x}\cos(2x) + Be^{2x}\sin(2x)$$

Ansatz of Particular equation :

$$y(x) = (Cx^2 + Dx + E)e^{2x}\cos(2x)x + (Fx^2 + Gx + I)e^{2x}\sin(2x)x$$

$$\begin{aligned} y'(x) = &\left((2F + 2C)x^3 + (2G + 2D + 3C)x^2 + (2I + 2E + 2D)x + E\right) e^{2x}\cos(2x) \\ &+ \left((2F - 2C)x^3 + (2G + 3F - 2D)x^2 + (2I + 2G - 2E)x + F\right) e^{2x}\sin(2x) \end{aligned}$$

$$
\begin{aligned}
y''(x) =& (8Fx^3 + (8G+12F+12C)x^2 + (8I+8G+8D+6C)x + 4I + 4E + 2D)e^{2x}\cos(2x) \\
&+ (-8Cx^3 + (12F-8D-12C)x^2 + (8G+6F-8E-8D)x + 4I + 2G - 4E)e^{2x}\sin(2x)
\end{aligned}
$$

substitute $y(x), y'(x), y''(X)$ into

$$y'' - 4y' + y = (2x^2 - 3x)e^{2x}\cos 2x + (10x^2 - x - 1)e^{2x}\sin 2x$$

$$
\begin{aligned}
y''(x) =& (8Fx^3 + (8G+12F+12C)x^2 + (8I+8G+8D+6C)x + 4I + 4E + 2D)e^{2x}\cos(2x) \\
&+ (-8Cx^3 + (12F-8D-12C)x^2 + (8G+6F-8E-8D)x + 4I + 2G - 4E)e^{2x}\sin(2x)
\end{aligned}
$$

$$
\begin{aligned}
-4y'(x) =& \big((-8F-8C)x^3 + (-8G-8D-12C)x^2 + (-8I-8E-8D)x - 4E\big)\, e^{2x}\cos(2x) \\
&+ \big((-8F+8C)x^3 + (-8G-12F+8D)x^2 + (-8I-8G+8E)x - 4I\big)\, e^{2x}\sin(2x)
\end{aligned}
$$

$$8y(x) = (8Cx^3 + 8Dx^2 + 8Ex)e^{2x}\cos(2x) + (8Fx^3 + 8Gx^2 + 8Ix)e^{2x}\sin(2x)$$

$$y'' - 4y' + y =$$

$$
\begin{aligned}
&e^{2x}\sin(2x)((-8C-8F+8C+8F)x^3 + (12F-8D-12C-8G-12F+8D+8G)x^2 \\
&\quad + (8G+6F-8E-8D-8I-8G+8E+8I)x + 4I + 2G - 4E - 4I) = (10x^2 - x - 1)e^{2x}\sin 2x \qquad (1) \\
&e^{2x}\cos(2x)((8F-8F-8C+8C)x^3 + (8G+12F+12C-8G-8D-12C+8D)x^2 \\
&\quad + (8I+8G+8D+6C-8I-8E-8D+8E)x + 4I + 4E + 2D - 4E) = (2x^2 - 3x)e^{2x}\cos 2x \qquad (2)
\end{aligned}
$$

$$
\begin{gathered}
e^{2x}\sin(2x) : -12Cx^2 + (6F-8D)x + 2G - 4E = 10x^2 - x - 1 \\
C = \frac{-5}{6}; 6F - 8D = -1; 2G - 4E = -1 \\
e^{2x}\cos(2x) : 12Fx^2 + (8G+6C)x + 4I + 2D = 2x^2 - 3x \\
12F = 2\ ;\ F = \frac{1}{6}\ ;\ 6F - 8D = -1\ ;\ D = \frac{2}{8} = \frac{1}{4}
\end{gathered}
$$

$8G + 6C = -1$; $8G + 6(\frac{-5}{6}) = -3$; $G = \frac{1}{4}$

so $2G - 4E = -1$; $2(\frac{1}{4}) - 4E = -1$; $E = \frac{3}{8}$

AND $4I + 2D = 0$; $4I + 2(\frac{1}{4}) = 0$; $I = -\frac{1}{8}$

so

$$y_p(x) = (-\frac{5}{6}x^2 + \frac{1}{4}x + \frac{3}{8})e^{2x}\cos 2x + (\frac{1}{6}x^2 + \frac{1}{4}x + \frac{1}{8})e^{2x}\sin 2x$$

`y(x) = Ae`$^{2x}\cos(2x) + Be^{2x}\sin(2x) + (-\frac{5}{6}x^2 + \frac{1}{4}x + \frac{3}{8})e^{2x}\cos 2x + (\frac{1}{6}x^2 + \frac{1}{4}x + \frac{1}{8})e^{2x}\sin 2x$

2 Question 2

1. a) The temperature of the food is increasing at a rate which is proportional to the current heat of the food at a give time t.

b) T(0) = 20 °C
Temperature of microwave = Ambient temperature = T_m = ?
$t_0 = 0$
k = rate of heat transfer

c) Assume that nothing else is affecting the temperature of the food.

d) Let T be the temperature of the food at some point in time t such that T = T(t)

e) $\frac{dT}{dt}$ = -k(T - T_m)

$if k > 0$; $(T - T_m) < 0$
Thus :
$\frac{dT}{dt}$ = k(T_m - T)
$k > 0$ so $T_m > T$

2. $\frac{dT}{dt}$ = k(T_m - T)

$$\int \frac{dT}{dt} = \int k(T_m - T)$$

-ln(T_m $- T$) = kt + c

$$T(t) = T_m \cdot A \cdot e^{-kt}$$

T(0) = 20 °C
20 = T_m - A ; T_m = 20 + A

$$T(t) = 20 + A - A \cdot e^{-kt}$$

3. T(30_s) = 42 °C ; T(60_s) = 62 °C

$$42 = 20 + A \cdot (1 - A \cdot e^{-30k}) \qquad (1)$$
$$62 = 20 + A \cdot (1 - A \cdot e^{-60k}) \qquad (2)$$

(2) : A = $\frac{42}{(1-e^{-30K})}$ (1) : 22 = $A \cdot (1 - e^{-60k})$ //
substitute (2) into (1)

$$22 = \frac{42 \cdot (1 - e^{-30k})}{(1 - e^{-60k})}$$

$42e^{-30k} - 22e^{-60k} = 20$ let u $= e^k$

$$42u^{-30} - 22u^{-60} = 20$$

$$u_1 = \sqrt{\sqrt[15]{\frac{11}{10}}}, u_2 = -\sqrt{\sqrt[15]{\frac{11}{10}}}, u_3 = 1, u_4 = -1$$

$e^k = \sqrt{\sqrt[15]{\frac{11}{10}}} : k = \frac{\ln\frac{11}{10}}{30}$
$e^k = -\sqrt{\sqrt[15]{\frac{11}{10}}}$: No solution for k $\in \mathbb{R}$
$e^k = 1 : k = 0$
$e^k = -1$: No solution for k $\in \mathbb{R}$

So :

$$k = \frac{\ln\frac{11}{10}}{30} \text{ or } k = 0$$

substitute K for A :

$$A = \frac{42}{(1 - e^{-60\cdot(\frac{\ln\frac{11}{10}}{30})})}; \qquad \text{A} = 242$$

$$T(t) = 20 + 242 - (242) \cdot e^{-kt}$$

so :

$$T(t) = 262 - 242e^{-kt}$$

$$T_m = \text{Temperature of oven} = 262\,^{\circ}\text{C}$$