CS & IT ENGINEERING

Algorithms

Introduction to Algorithms and Analysis

Lecture No.- 04

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Recap of Previous Lecture

- **Topic** Apriori Analysis ✓
- **Topic** Types of Analysis : W.C ; B.C ; Av.C
- **Topic** Worst-Case and Best-Case Behaviour
- **Topic**
- **Topic**

Topics to be Covered

Topics

Asymptotic Notations

Big – Oh, Big – Omega, Theta Notations

Small Notations

Topic: Asymptotic Notations

Time-Complexity ~ $T(n)$

- Step-count
- Order of Magnitude

$T(n) = \underline{1+n+n^2}$

$= \underline{\underline{4}}n^2 + \underline{\underline{8}}n + \underline{\underline{6}}$

A.S.N

repr. with a

Charact Suitable

Notation

Polynomial (n^x)

- Log
- Const
- Quad
- Cubic

Exponential

$a^n \; (a>1)$

Topic: Asymptotic Notations

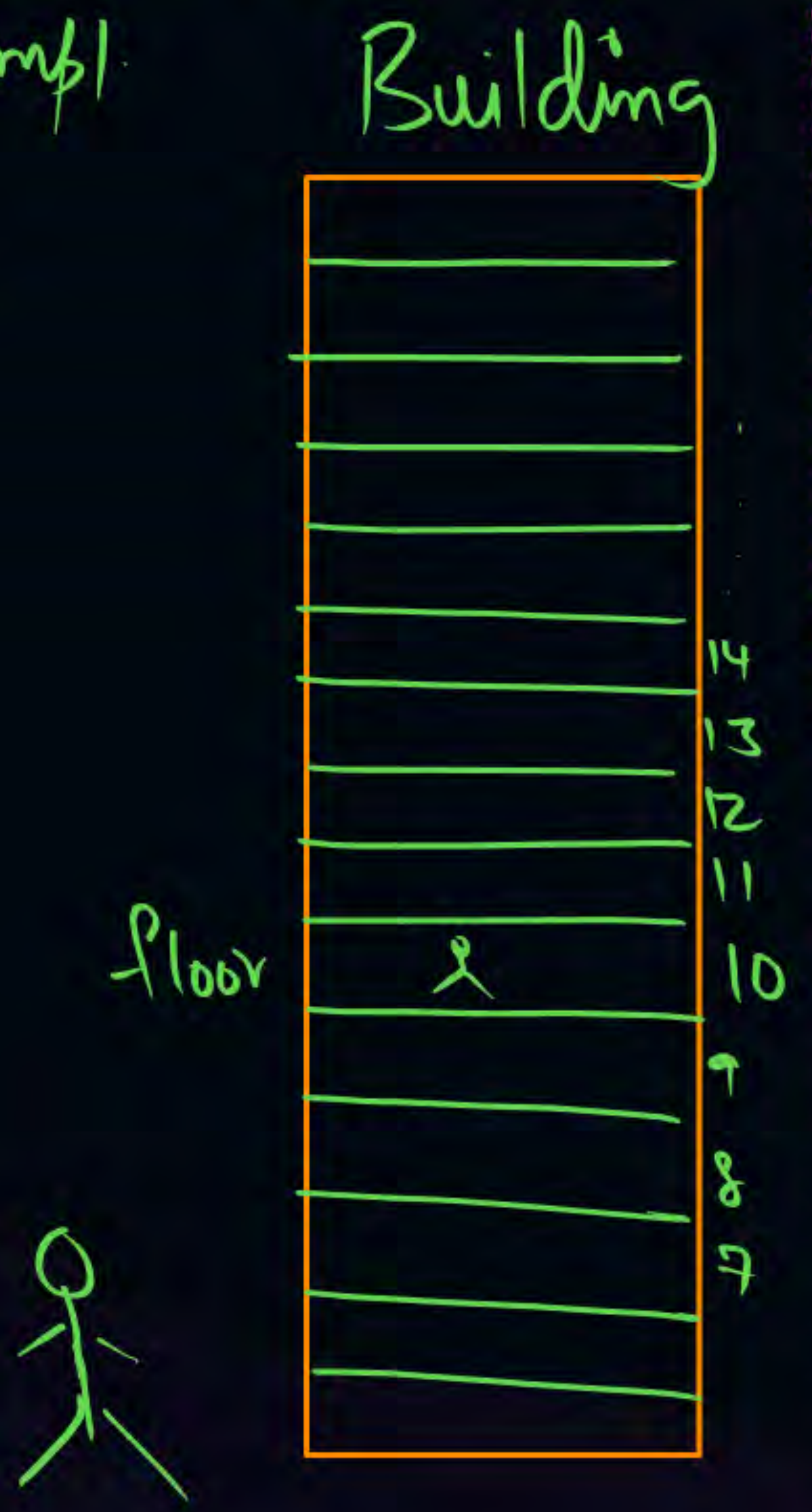
Asymplotatic Notations (ASN)

Repr. Time & Space Compl of Algo by functions

↓

(ASN)

U.B (Max) L.B (Min)



↓

Math Tool to obtain / repr. Bounds of functions

- Upper Bound <Max>
- Lower Bound <Min>
- Tight Bound

Topic: Asymptotic Notations

ASN

Big

Upper Bound

→ Big-oh : O

Lower Bound

→ Big-omega : Ω

→ Theta : Θ

Tight Bound

Small | Little

→ Little oh : o → Proper U·B

→ Little omega : ω → Proper L·B

⟹ Let 'f' & 'g' be functions from the set of integers/Reals to Real numbers;

① **Big-oh (O): Upper Bound**

$f(n)$ in $O(g(n))$ iff there exists some constants $c > 0$ and $n_0 > 0$

such that $\boxed{f(n) \leq c \cdot g(n), \text{ whenever } n \geq n_0}$

$f(n) = O(g(n))$

$f(n) \in \underbrace{O(g(n))}_{\text{set}}$

Topic: Asymptotic Notations

$$f(n) = 1 + n + n^2 \quad O(n^2)$$

$$1 + n + n^2 \le n^2 + n^2 + n^2$$

$$\underline{f(n)} \le \underset{c}{3} \cdot \underset{g}{n^2}, \quad n \ge \underset{n_0}{1}$$

$$\boxed{f(n) \text{ is } O(n^2)}$$

$$1 + n + n^2 \le 5 \cdot n^3, \quad n > 1$$

$$1 + 1 + 1 \le 5 \cdot 1$$

$$1 + 2 + 4 \le 5 \cdot 8$$

$f(n)$ is $O(n^2)$ ✓ : closest | Hightest U.B

$f(n)$ is $O(n^3)$ ✓

$f(n)$ is $O(n^4)$ ✓

Topic: Asymptotic Notations

Whenever we determine the U.B & L.B, we should find that function 'g' which is closest to the given function;

Construction:

(L.B) → 50L

B_2:

~~4~~ 45L

> 5000 rs

50L

B_1: U.B

< 100L

< 50L

200 Sq.yd.

(G+1) 3R 1H

House

A → B

5km

Abids

> 5s

(20m) ✓

1 Hr

1 Day

Ambani's

Business_Man :

- walk :
- Cycle : 15K
- Bike : 80K
- Car : 20L
- Jet : 100 cr

Investment

O(crs)

2) Big-omega (Ω): Lower Bound

$f(n)$ is $\Omega(g(n))$ iff there exists Const's 'c' & 'n_0' such that $\boxed{f(n) \geq c \cdot g(n), \text{ whenever } n \geq n_0;}$

1) $f(n) = 1+n+n^2$

- $\Omega(1)$ ✓
- $\Omega(n)$ ✓
- $\boxed{\Omega(n^2)}$ ✓

$O(n^2)$

$1+n+n^2 \leq 3 \cdot n^2$

$1+n+n^2 > 1 \cdot 1$, $n \geq 1$

$1+n+n^2 > 1 \cdot n$, $n > 1$

$\boxed{1+n+n^2 \geq 1 \cdot n^2, \; n > 1}$

c g

3) Theta (Θ) : Tight Bound :

$$f(n) \text{ is } \Theta(g(n)) \text{ iff } f(n) \text{ is } O(\underline{\underline{g(n)}})$$

$$\& \; f(n) \text{ is } \Omega(\underline{\underline{g(n)}})$$

$$\boxed{\underset{f}{\underline{1+n+n^2}} \begin{cases} O(\overset{g_2}{n^2}) \\ \Omega(\underset{g}{n^2}) \end{cases} \therefore \Theta(n^2)}$$

$$\boxed{c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)}$$

Topic : Analysis of Algorithms

Big O notation is a mathematical notation that describes the limiting behavior of a function when the argument tends towards a particular value or infinity.

Big O is a member of a family of notations invented by Paul Bachmann, Edmund Landau, and others, collectively called Bachmann-Landau notation or asymptotic notation. The letter O was chosen by Bachmann to stand for Ordnung, meaning the order of approximation.

In computer science, big O notation is used to classify algorithms according to how their run time or space requirements grow as the input size grows.

In analytic number theory, big O notation is used to express a bound on the difference between an arithmetical function and a better understood approximation; a famous example of such a difference is the remainder term in the prime number theorem.

Topic : Analysis of Algorithms

Big O notation is also used in many other fields to provides similar estimates.

Big O notation characterizes functions according to their growth rates: different functions with the same asymptotic growth rate may be represented using the same O notation. The letter O is used because the growth rate of a function is also referred to as the order of the function. A description of a function in terms of big O notation usually only provides an upper bound on the growth rate of the function.

Associated with big O notation are several related notations, using the symbols O, Ω, ω and Θ, to describe other kinds of bounds on asymptotic growth rates.

Topic : Analysis of Algorithms

Definition: A theoretical measure of the execution of an algorithm, usually the time or memory needed, given the problem size n, which is usually the number of items. Informally, saying some equation $f(n) = O(g(n))$ means it is less than some constant multiple of $g(n)$. The notation is read, "f of n is big oh of g of n".

Formal Definition: $f(n) = O(g(n))$ means there are positive constants c and k, such that $0 < f(n) \leq cg(n)$ for all $n \geq K$. The values of c and k must be fixed for the function f and must not depend on n.

Big-Omega Notation (Ω):

Similar to big O notation, big Omega (Ω) function is used in computer science to describe the performance or complexity of an algorithm. If a running time is $\Omega(f(n))$, then for large enough n, the running time is at least k.f(n) for some constant k.

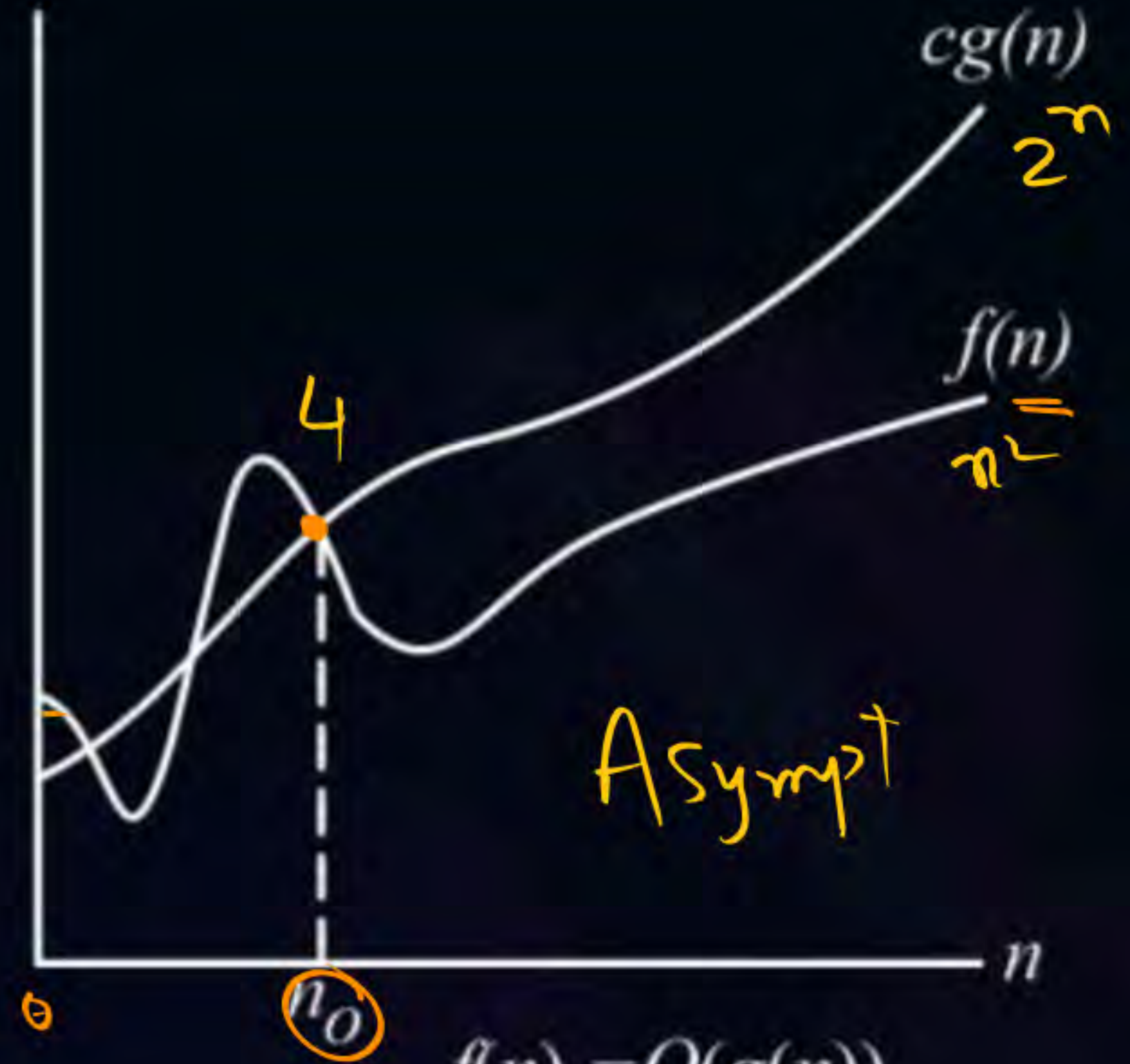
The formal definitions associated with the Big Notation are as follows:

- $f(n) = O(g(n))$ means $c \cdot g(n)$ is an upper bound on $f(n)$. Thus there exists some constant c such that $f(n)$ is always $\leq c.g(n)$, for large enough n (i.e., $n \geq no$ for some constant n_0).
- $f(n) = \Omega(g(n))$ means $c.g(n)$ is a lower bound on $f(n)$. Thus there exists some constant c such that $f(n)$ is always $\geq c.\ g(n)$, for all $n \geq no$.

Topic: Analysis of Algorithms

$f(n) = n^2$; $g(n) = 2^n$

n	f	g
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32
6	36	64
7	49	128



$f(n) = O(g(n))$

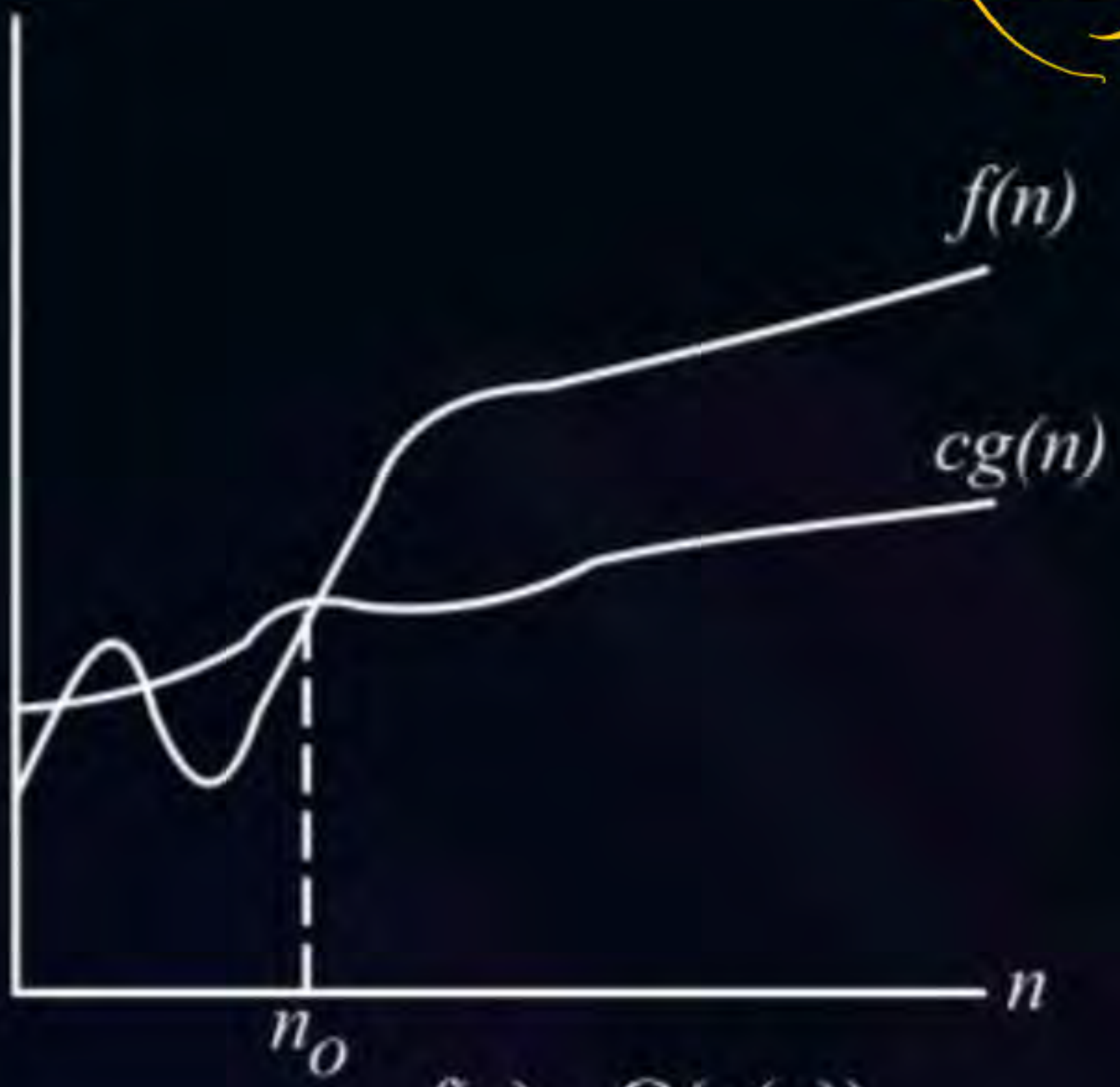
Set

$O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0\}$.

We write $f(n) = O(g(n))$ to indicate that a function $f(n)$ is a member of the set $O(g(n))$. Note that $f(n) = \Theta(g(n))$ implies $f(n) = O(g(n))$, since Θ-notation is a stronger notion than O-notation. Written set-theoretically, we have

Topic: Analysis of Algorithms

(Big-omega)



$f(n) = \Omega(g(n))$

$O(n^3)$

c	$n \cdot \log n$
n	$n^2 \cdot \log n$
n^2	$\sqrt{n}$
$\log n$	

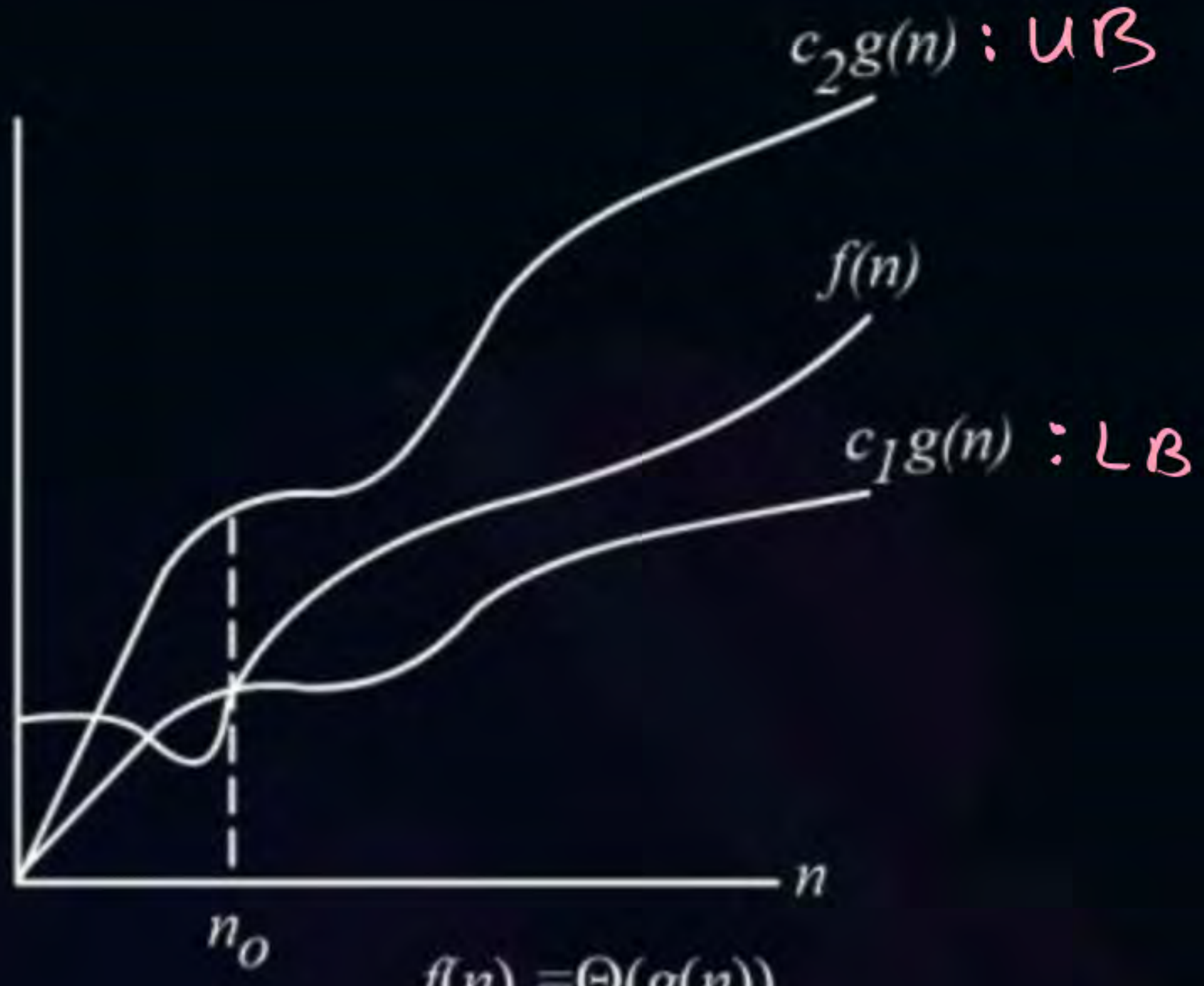
$c \leq a \cdot n^3$

$n \leq a \cdot n^3$

f $n^2 \leq a \cdot n^3$

- $f(n) = \Theta(g(n))$ means $c_1.\, g(n)$ is an upper bound on $f(n)$ and $c_2.g(n)$ is a lower bound on $f(n)$, for all $n \geq$ no. Thus there exist constants c_1 and c_2 such that $f(n) \leq c_1.g(n)$ and $f(n) \geq c_2.g(n)$. This means that $g(n)$ provides a nice, tight bound on $f(n)$.

Topic: Analysis of Algorithms



$f(n) = \Theta(g(n))$

For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. ■

1) $f(n) = 1 + n + n^2 \begin{cases} O(n^2) \\ \Omega(n^2) \end{cases} \therefore \Theta(n^2)$

2) $f(n) = n \begin{cases} O(n) \\ \Omega(n) \end{cases}$

$\boxed{n \le c \cdot n} \quad n > 1$

$n \le 2 \cdot n, \quad n > 1$

$n \ge d \cdot n, \quad n > 1$

$\therefore \Theta(n)$

$\ge \frac{1}{2} \cdot n$

3) $f(n) = 2^{100} \begin{cases} O(1) \\ \Omega(1) \end{cases} \therefore \Theta(1)$

$\boxed{2^{100} \le 3^{100} \cdot 1}$

$f(n) = c \begin{cases} O(1) \\ \Omega(1) \end{cases} \therefore \Theta(1)$

4) $f(n) = n + \log n$ $\begin{cases} O(n) \\ \Omega(n) \end{cases}$ $\therefore \Theta(n)$

5) $f(n) = \sqrt{n} + \log n$ $\begin{cases} O(\sqrt{n}) \\ \Omega(\sqrt{n}) \end{cases}$

$\therefore \Theta(\sqrt{n})$ ✓

$n + \log n \le n + n$

$\le 2 \cdot n$, $n > 1$

c

$n + \log n \ge 1 \cdot n$, $n > 1$

c

$\log \sqrt{n}$

$\log (n)^{1/2}$

$\frac{1}{2} \cdot \log n$ $>$

$(\sqrt{n})$

$\sqrt{64}$

8 $>$

$\log_2 n$

$\log_2 64$

6

$\log \log n$

$$f(n) = n^{1/\log n} = O(\quad)$$

THANK - YOU