

1 Sequences

A sequence is a list of numbers written in a defined order

$$a_1, a_2, a_3, \dots$$

Sequences can also be denoted

$$\{a_n\}$$
$$\{a_n\}_{n=1}^{\infty}$$

Converging sequences settle on a certain number while diverging sequences does not settle on a number.

Recursive sequence: A sequence that depends on previous values in the sequence.

2 Limit of a sequence

L'Hospital's rule: For differentiable functions f and g , if $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$ or $\pm\infty$, and $g'(x) \neq 0$ for all x , and $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

Limit laws:

Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and c be a constant.

$$\begin{aligned}\lim_{n \rightarrow \infty} (a_n + b_n) &= \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n \\ \lim_{n \rightarrow \infty} (a_n - b_n) &= \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n \\ \lim_{n \rightarrow \infty} (a_n b_n) &= \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n \\ \lim_{n \rightarrow \infty} c a_n &= c \lim_{n \rightarrow \infty} a_n \\ \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \text{ if } \lim_{n \rightarrow \infty} b_n \neq 0 \\ \lim_{n \rightarrow \infty} a_n^p &= \left(\lim_{n \rightarrow \infty} a_n \right)^p \text{ if } p > 0, a_n > 0\end{aligned}$$

If $\lim_{n \rightarrow \infty} |a_n| = 0$, $\lim_{n \rightarrow \infty} a_n = 0$.

If $\lim_{n \rightarrow \infty} a_n = L$ and function f is continuous at L , $\lim_{n \rightarrow \infty} f(a_n) = f(L)$.

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r .

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Squeeze theorem: If $a_n \leq b_n \leq c_n$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, $\lim_{n \rightarrow \infty} b_n = L$.

Monotonic sequences: A sequence a_n is increasing if $a_n < a_{n+1}$ for all $n \geq 1$ and is decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. Monotonic sequences are either increasing or decreasing.

Bounded sequences: A sequence a_n is bounded above if there is a number M such that $a_n \leq M$ for all $n \geq 1$. a_n is bounded below if there is a number m such that $m \leq a_n$ for all $n \geq 1$. If a_n is bounded above and below, then a_n is a bounded sequence.

If a sequence is bounded and monotonic, that sequence is convergent.

3 Ways to determine the limit

- Draw graph or construct table
- Break it up into smaller limits using the limit laws
- L'Hospital's Rule
- Determine how each term will change as n approaches infinity
- Use the squeeze theorem if it converges between two functions with the same limits
- Substitute a variable if a common pattern appears more than once. Remember to solve its limit to replace the limit variable.
- If it is raised to the n th power, calculate the natural log of both sides.

4 Series

Series: The sum of all terms of a sequence.

Infinite sum:

$$\sum_{n=1}^{\infty} a_n = S$$

Partial sum:

$$\sum_{K=1}^n a_K = S_n$$

Partial sums can be used to determine convergence of an infinite sum.

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{K=1}^n a_K$$

If $\{S_n\}$ converges, then $\lim_{n \rightarrow \infty} S_n = S$.

5 Geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + a^2 + a^3 + \cdots + ar^{n-1}$$

- If $r \leq -1$ or $r \geq 1$, the series will diverge
- If $-1 < r < 1$, the series will converge and the limits is

$$S = \frac{a}{1-r}$$

Theorems for convergent series

$$\begin{aligned}\sum_{n=1}^{\infty} ca_n &= c \sum_{n=1}^{\infty} a_n \\ \sum_{n=1}^{\infty} (a_n + b_n) &= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \\ \sum_{n=1}^{\infty} (a_n - b_n) &= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n\end{aligned}$$

Telescoping series: Each pair of consecutive terms have parts that cancel out, which makes it easy to find the closed form.

Test for divergence: If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

6 Integral test

Integral Test: Let f be a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

P-series test: The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

7 Comparison test

Comparison test: Let $\sum a_n$ and $\sum b_n$ be series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum a_n$ is convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum a_n$ is divergent.

8 Limit comparison test

Limit comparison test: Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

9 Alternating series

Alternating series: A series whose terms are alternately positive and negative.

$$\sum_{n=0}^{\infty} (-1)^{n-1} b_n = a_0 - a_1 + a_2 - a_3 + \cdots$$

Alternating series test: Given an alternating series $\sum (-1)^{n-1} b_n$, if b_n is decreasing and $\lim_{n \rightarrow \infty} b_n = 0$, then $\sum (-1)^{n-1} b_n$ is convergent.

Alternating series estimation theorem: If $S = \sum (-1)^{n-1} b_n$ where $b_n > 0$ is the sum of a converging alternating series, then

$$|R_n| = |S - S_n| \leq b_{n+1}$$

10 Ratio test

Absolute convergence: Given a series $\sum a_n$, if $\sum |a_n|$ converges, then $\sum a_n$ is absolutely convergent.

Conditional convergence: A series $\sum a_n$ that is convergent but not absolutely convergent.

Ratio test: Given the series $\sum_{n=1}^{\infty} a_n$,

$$\text{if } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \begin{cases} < 1, \text{ then the series is absolutely convergent} \\ > 1 \text{ or } \infty, \text{ then the series is divergent} \\ = 1, \text{ the Ratio Test is inconclusive} \end{cases}$$

11 Root test

Root test: Given the series $\sum_{n=1}^{\infty} a_n$,

$$\text{if } \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \begin{cases} < 1, \text{ then the series is absolutely convergent} \\ > 1 \text{ or } \infty, \text{ then the series is divergent} \\ = 1, \text{ the Ratio Test is inconclusive} \end{cases}$$

12 Tests for convergence and divergence

- Geometric series
- Telescoping series
- Test for divergence
- Integral test
- P-series test
- Comparison test
- Limit comparison test
- Alternating series test
- Ratio test
- Root test

13 Power series

The power series:

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

This power series is centered around a .

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

- The series converges only when $x = a$
- The series converges for all x
- There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$

The interval of convergence is the interval for x such that the series converges.

The radius of convergence is half the distance of the interval of convergence.

Sometimes, such as for the ratio test and root test, the endpoints must be checked for convergence.

14 Representations of functions as power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

where $|x| < 1$

Term-by-term differentiation and integration:

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$
$$\int f(x)dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of both equations are R .

15 Taylor and Maclaurin Series

If the power series representation at a is

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

then its Taylor series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x - a)^n$$

Maclaurin series: The Taylor series but with $a = 0$

Important Maclaurin series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, R = \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, R = \infty$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, R = \infty$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, R = 1$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots = \sum_{n=0}^{\infty} \binom{k}{n} x^n, R = 1$$

16 Parametric equations

Unlike rectangular equations, parametric equations have a third variable t . It has an initial point and terminal point for the minimum and maximum t values respectively.

First derivative (Slope):

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

if $\frac{dx}{dt} \neq 0$

Second derivative (Concavity):

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$$

If a curve is defined by $x = f(t), y = g(t), \alpha \leq t \leq \beta$, the area under the curve is

$$\int_{\alpha}^{\beta} g(t) f'(t) dt$$

The arc length is

$$\int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

17 Polar coordinates

Converting from polar coordinates to Cartesian coordinates

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

Converting from Cartesian to polar coordinates

$$r^2 = x^2 + y^2$$

$$\tan(\theta) = \frac{y}{x}$$

Distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- If $f(\theta) = f(-\theta)$, the curve is symmetric about the polar axis.
- If $f(\theta) = f(\theta + \pi)$, the curve is symmetric about the pole.
- If $f(\theta) = f(\pi - \theta)$, the curve is symmetric about $\theta = \pi/2$.

Area of R :

$$\int_a^b \frac{1}{2} (f(\theta))^2 d\theta$$

Trig identities

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

18 Vectors

A vector:

- Has a magnitude and direction.
- Has an initial point and terminal point

- A vector from A to B is written \overrightarrow{AB}

Scalar multiplication of vectors: If c is a scalar and v is a vector, the scalar multiple cv is the vector whose length is $|c|$ times the length of v and whose direction is the same as v if $c > 0$ and is opposite to v if $c < 0$. If $c = 0$, then $cv = 0$.

Vector addition: The sum of vectors u and v , $u + v$, is a vector from the initial point of u to the terminal point of v . $u + v = v + u$.

Position vector: A vector that represents a coordinate relative to an origin.

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

Unit vector: Vectors that have a magnitude of 1.

\hat{i} = unit vector in x direction

\hat{j} = unit vector in y direction

\hat{k} = unit vector in z direction

Magnitude of a vector:

$$|a| = \sqrt{a_1^2 + a_2^2}$$

$$\hat{u} = \frac{\vec{a}}{|\vec{a}|}$$

Equation of a sphere:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$