Block on incline.

We write down and solve the ODE for a block on an (infinite) inclined plane.

We use distance along the incline as the coordinate s, oriented downward. This makes all tangential forces, accelerations, and velocities into scalars.

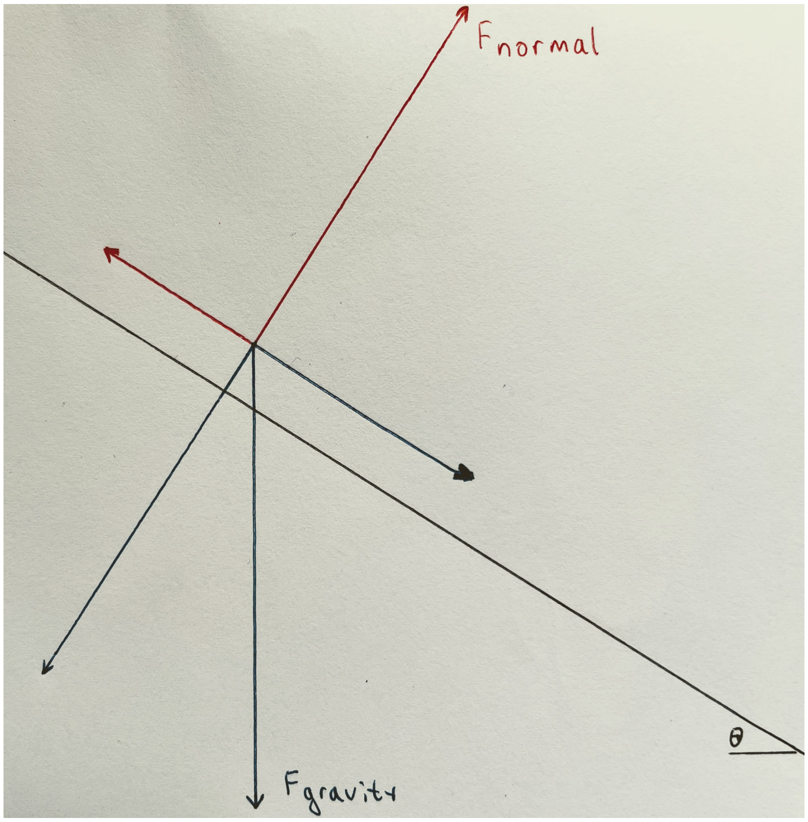

Figure 1: Force diagram.

$$|a_{\text{gravity}}| = g$$
$$|a_{\text{gravity-tangential}}| = g \sin \theta$$
$$|a_{\text{normal}}| = g \cos \theta$$
$$|a_{\text{friction}}| = \mu g \cos \theta$$

We take

$$c = g(\sin \theta - \mu \cos \theta).$$

No drag:

$$s'' = c$$
$$s(t) = s(0) + v(0)t + ct^2/2$$

Proportional drag:

$$a_{\text{drag}} = -rv$$

$$s'' = c - rs'$$

$$v' = c - rv$$

Then $v_{\text{terminal}} = \frac{c}{r}$ is the terminal velocity, and the velocity converges to it exponentially with rate r:

$$v(t) = v_{\text{terminal}} + (v(0) - v_{\text{terminal}})e^{-rt}$$

Set $v_{\text{terminal}} - v(0) = K$. Then

$$v(t) = v_{\text{terminal}} - Ke^{-rt}$$

And finally:

$$s(t) = s(0) + v_{\text{terminal}}t + \frac{K}{r}(e^{-rt} - 1).$$

Quadratic drag:

$$a_{\text{drag}} = -rv|v|$$

$$v' = c - r|v|v.$$

Special case. We shall start with $c = r = 1$,

$$v' = 1 - |v|v.$$

Then the equilibrium velocity is $v|v| = 1$ aka $v = 1$.

Apart from the trivial case $v(0) = 1$, there are 3 regimes depending on $v(0)$: a) when $v(0) > 1$, b) when $0 \le v(0) < 1$, c) when $v(0) < 0$.

When $v \ge 0$ the ODE is

$$v' = 1 - v^2.$$

When $v(0) = 0$ this is the characterizing equation of

$$v(t) = \tanh t.$$

(This is also a shifted version of the logistic ODE $u' = u(1-u)$ whose solution with $u'(0) = 1$ is the sigmoid; tanh is the shifted sigmoid.)

The ODE is time-invariant, which means that for those $v(0)$ which are in the range of tanh (i.e. between -1 and 1), the solution is $v(t) = \tanh(T + t)$ for (the unique) T such that $\tanh(T) = v(0)$ (in our case $v(0) \ge 0$ we have $T \ge 0$). That is

$$v(t) = \tanh(T + t) = \frac{\tanh T + \tanh t}{1 + \tanh T \tanh t} = \frac{v(0) + \tanh t}{1 + v(0) \tanh t}$$

Furthermore, the equation $v' = 1 - v^2$ is invariant under $v \leftrightarrow \frac{1}{v}$. Thus, for $v(0)$ with $v(0) > 1$, the solutions are of the form

$$v(t) = \coth(T + t) = \frac{1 + \coth T \coth t}{\coth T + \coth t} = \frac{1 + v(0) \coth t}{v(0) + \coth t}$$

This solves the regimes a and b.

When $v \le 0$, we have

$$v' = 1 + v^2$$

which, for $v(0) = 0$ is solved by

$$v(t) = \tan(t)$$

and in general by

$$v(t) = \tan(T + t) = \frac{v(0) + \tan t}{1 - v(0) \tan t}.$$

In our case $v(0) = \tan T < 0$ we have $T \leq 0$.

Note that in this regime the trajectory will reach $v = 0$ at $t = -T \geq 0$ and will switch to $\tanh(t + T)$ for subsequent t. They have the same slope of 1, both have zero second derivative, and only differ in the third derivative.

The phase portrait for velocity looks like this:

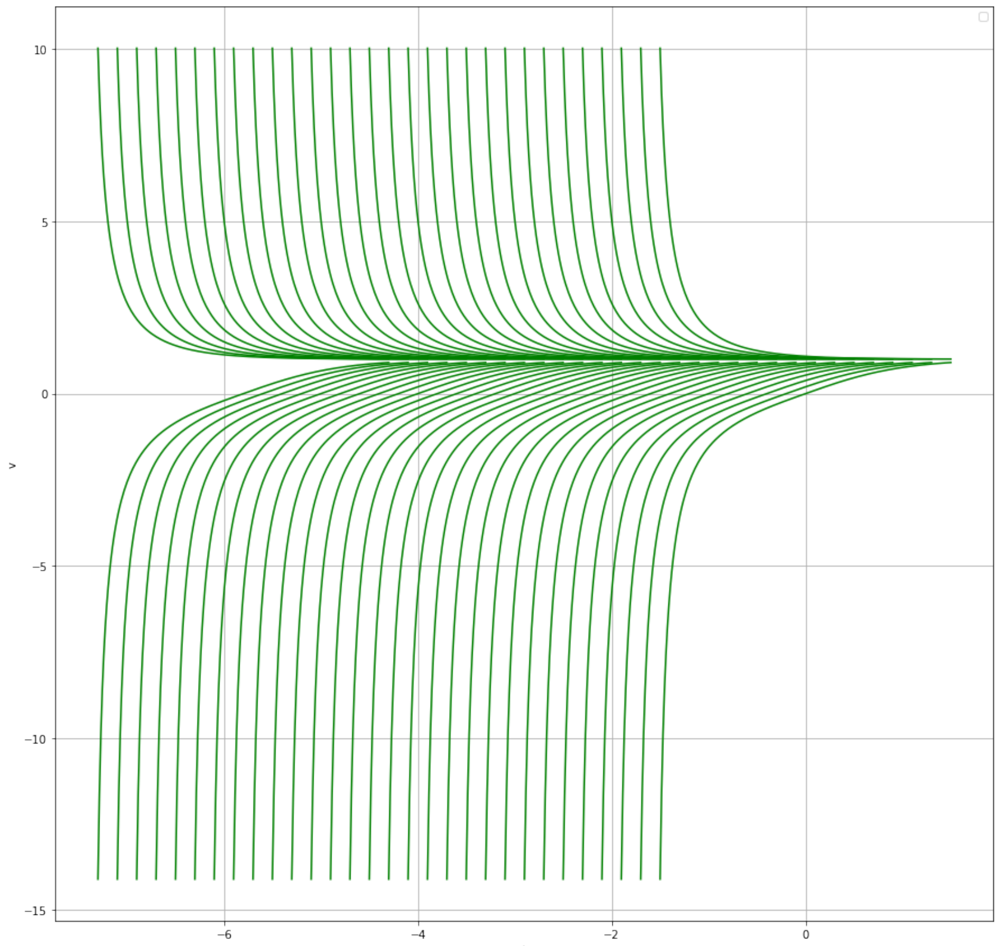

Figure 2: Phase Portrait

The colab which produced this image.

General case, velocity. To get the solutions to

$$v' = c - rv^2$$

we need to dilate time and stretch the function. That is, we are looking for a solution in the form $v(t) = Af(Bt)$ where $f' = 1 - f^2$ or $f^2 = 1 - f'$. Then $v' = c - rv^2$ translates to

$$ABf' = c - r(A^2 f^2) = c - r(A^2(1 - f'))$$

$$AB = rA^2, \quad c = rA^2$$

$$A = \sqrt{\frac{c}{r}}, \quad B = \sqrt{cr}.$$

The terminal velocity is now $V = \sqrt{\frac{c}{r}}$, so $A = V$ and $B = rV$.

Non-negative initial velocity. If $v(0) \geq V$ the solution is

$$v(t) = V \coth(rV(t+T)) = V \frac{V + v(0)\coth(rVt)}{v(0) + V\coth(rVt)}$$
$$= V \frac{\tanh(rVt) + \frac{v(0)}{V}}{\frac{v(0)}{V}\tanh(rVt) + 1}$$

Here $T = \frac{1}{rV}\text{arcoth}(\frac{v(0)}{V}) > 0$. (There is a singularity in the past, at $t = -T$).

If $V > v(0) \geq 0$ the solution is

$$v(t) = V \tanh(rV(t+T)) = V \frac{v(0) + V\tanh(rVt)}{V + v(0)\tanh(rVt)}$$
$$= V \frac{\frac{v(0)}{V} + \tanh(rVt)}{1 + \frac{v(0)}{V}\tanh(rVt)}.$$

Here $T = \frac{1}{rV}\text{artanh}(\frac{v(0)}{V}) > 0$. (The solution has crossed zero in the past, at $t = -T$).

The formulas coincide, and, in fact give $v(t) = V$ when $v(0) = V$.

Negative initial velocity. If $v(0) < 0$ then initially the solution is

$$v(t) = V \tan(rV(t+T)) = V \frac{v(0) + V\tan(rVt)}{V - v(0)\tan(rVt)}$$
$$= V \frac{\frac{v(0)}{V} + \tan(rVt)}{1 - \frac{v(0)}{V}\tan(rVt)}.$$

Here $T = \frac{1}{rV}\arctan(\frac{v(0)}{V}) < 0$. Then, after $t = -T$:

$$v(t) = V\tanh(rV(t+T)) = V\frac{\tanh(\arctan\frac{v(0)}{V}) + \tanh(rVt)}{1+\tanh(\arctan\frac{v(0)}{V})\tanh(rVt)}.$$

General case, position. The distance travelled is the integral of velocity.

Non-negative initial velocity. For the case $v(0) \geq V$

$$s(t) - s(0) = \int_0^t v(\tau)\,d\tau = \int_0^t V\coth(rV(\tau+T))\,d\tau$$

Recalling $rV = B$, $u = B(\tau+T)$ so $du = Bd\tau$.

$$s(t) - s(0) = \frac{1}{r}\int_{BT}^{B(t+T)} \coth u\,du =$$
$$\frac{1}{r}\ln\frac{\sinh B(t+T)}{\sinh BT} = \frac{1}{r}\ln(\cosh Bt + \sinh Bt\coth BT) =$$
$$= \frac{1}{r}\ln(\cosh rVt + \frac{v(0)}{V}\sinh rVt)$$

Similarly, for the case $V > v(0) \geq 0$

$$s(t) - s(0) = \int_0^t v(\tau)\,d\tau = \int_0^t V\tanh(rV(\tau+T))\,d\tau$$

Again, $rV = B$, $u = B(\tau+T)$ so $du = Bd\tau$.

$$s(t) - s(0) = \frac{1}{r}\int_{BT}^{B(t+T)} \tanh u\,du =$$
$$\frac{1}{r}\ln\frac{\cosh B(t+T)}{\cosh BT} = \frac{1}{r}\ln(\cosh Bt + \sinh Bt\tanh BT)$$
$$= \frac{1}{r}\ln(\cosh rVt + \frac{v(0)}{V}\sinh rVt)$$

Note that these formulas coincide. In fact this holds also when $V = v(0)$ where it becomes $\frac{1}{r}rVt = Vt$, as expected.

Negative initial velocity. For $v(0) < 0$ there are two regimes, before switching direction and after. Initially, when $t < -T$ (and hence $Bt < \frac{\pi}{2}$)

$$s(t) - s(0) = \int_0^t v(\tau)\, d\tau = \int_0^t V \tan(rV(\tau + T))\, d\tau$$

$$s(t) - s(0) = \frac{1}{r}\int_{BT}^{B(t+T)} \tan u\, du =$$

$$= -\frac{1}{r}\ln\frac{\cos B(t+T)}{\cos BT} = -\frac{1}{r}\ln(\cos Bt - \sin Bt \tan BT)$$

$$= -\frac{1}{r}\ln(\cos rVt - \frac{v(0)}{V}\sin rVt)$$

Note that at $t = -T$

$$s(-T) - s(0) = \frac{1}{r}\ln(\cos BT) = \frac{1}{r}\ln(\cos(\arctan\frac{v(0)}{V})) =$$

$$= -\frac{1}{2r}\ln(1 + \left(\frac{v(0)}{V}\right)^2).$$

Then, when $t > -T$,

$$s(t) - s(0) = -\frac{1}{2r}\ln(1 + \left(\frac{v(0)}{V}\right)^2) + \frac{1}{r}\ln(\cosh rV(t+T))$$

$$-\frac{1}{2r}\ln(1 + \left(\frac{v(0)}{V}\right)^2) + \frac{1}{r}\ln(\cosh(rVt + \arctan\frac{v(0)}{V})).$$