Solutions Excercise 1

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Exercise 1

(a) State the dual program corresponding to the following linear programs:

$$\begin{aligned} \max \quad & c^T x + d^T y \\ \text{s.t.} \quad & Ax \geq b \\ & g^T y = f \\ & x, y \geq 0 \end{aligned}$$

Here, $f \in \mathbb{R}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $d, g \in \mathbb{R}^l$ and $A \in \mathbb{R}^{m \times n}$.

Following to the general rules for finding the dual of a given linear program, the corresponding dual program:

$$\begin{aligned} \min \quad & b^T u + f \cdot v \\ \text{s.t.} \quad & A^T u \geq c \\ & g \cdot v \geq d \\ & u \leq 0 \end{aligned}$$

(b) Use Part (a) in order to show or disprove that $(x^\star, y^\star) = (\frac{13}{3}, 0, \frac{5}{3}, 1, 0)^T$ is an optimal solution for

$$\begin{aligned} \max \quad & (6, -1, -3)x + (4, -6)y \\ \text{s.t.} \quad & \begin{pmatrix} -2 & -1 & \frac{17}{5} \\ -1 & 1 & -1 \end{pmatrix} x \geq \begin{pmatrix} -3 \\ -6 \end{pmatrix} \\ & (2, -3)y = 2 \\ & x, y, \geq 0 \end{aligned}$$

Using python and SciPy check whether $(x^\star, y^\star) = (\frac{13}{3}, 0, \frac{5}{3}, 1, 0)^T$ is an optimal solution and $W^\star = Z^\star$. The code implementation can be founded here Code. Due to SciPy restrictions linear function to be minimized: $-Z = -c^T x - d^T y$

```
     con: array([0.])
     fun: -25.0
 message: 'Optimization terminated successfully.'
     nit: 3
   slack: array([0., 0.])
  status: 0
 success: True
       x: array([4.33333333, 0.        , 1.66666667, 1.        , 0.        ])
```

Figure 1: Solution of primal problem

As we can see $x = (4.3, 0, 1.6, 1., 0.)^T = (x^\star, y^\star) = (\frac{13}{3}, 0, \frac{5}{3}, 1, 0)^T$ and $Z^\star = 25$.

Now, solve the dual problem

```
     con: array([], dtype=float64)
     fun: 25.0
 message: 'Optimization terminated successfully.'
     nit: 3
   slack: array([0.0000000e+00, 0.0000000e+00, 4.4408921e-16, 0.0000000e+00,
       0.0000000e+00])
  status: 0
 success: True
       x: array([-1.66666667, -2.66666667,  2.        ])
```

Figure 2: Solution of dual problem

Since $W^\star = Z^\star$ and all above mentioned $(x^\star, y^\star) = (\frac{13}{3}, 0, \frac{5}{3}, 1, 0)^T$ is an optimal solution.

Exercise 2

Consider the following program:

$$
\begin{aligned}
\max \quad & 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5 \\
\text{s.t.} \quad & x_1 + 3x_2 + 5x_3 - 2x_4 + 2x_5 \le 4 \\
& 4x_1 + 2x_2 - 2x_3 + x_4 + x_5 \le 3 \\
& 2x_1 + 4x_2 + 4x_3 - 2x_4 + 5x_5 \le 5 \\
& 3x_1 + x_2 + 2x_3 - x_4 - 2x_5 \le 1 \\
& x_1, \dots, x_5 \ge 0
\end{aligned}
$$

(a) Formulate the corresponding dual program

Following to the general rules for finding the dual of a given linear program, the corresponding dual program:

$$
\begin{aligned}
\min \quad & 4y_1 + 3y_2 + 5y_3 + y_4 \\
\text{s.t.} \quad & y_1 + 4y_2 + 2y_3 + 3y_4 \ge 7 \\
& 3y_1 + 2y_2 + 4y_3 + y_4 \ge 6 \\
& 5y_1 - 2y_2 + 4y_3 + 2y_4 \ge 5 \\
& -2y_1 + y_2 - 2y_3 - y_4 \ge -2 \\
& 2y_1 + y_2 + 5y_3 - 2y_4 \ge 3 \\
& y_1, \dots, y_4 \ge 0
\end{aligned}
$$

(b) Check if $x^\star = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)^T$ is an optimal solution via the complementary slackness theorem.

- 1 constraint: $0 + 4 + \frac{10}{3} - \frac{10}{3} + 0 \le 4 \Longrightarrow 4 \le 4 \longrightarrow$ tight
- 2 constraint: $0 + \frac{8}{3} - \frac{4}{3} + \frac{5}{3} + 0 \le 3 \Longrightarrow 3 \le 3 \longrightarrow$ tight
- 3 constraint: $0 + \frac{16}{3} + \frac{8}{3} - \frac{10}{3} + 0 \le 5 \Longrightarrow \frac{14}{3} \le 5 \longrightarrow$ slack
- 4 constraint: $0 + \frac{4}{3} + \frac{4}{3} - \frac{5}{3} + 0 \le 1 \Longrightarrow 1 \le 1 \longrightarrow$ tight
- $x_i^\star \ge 0$ for $i \in [1, \dots, 5]$

Hence, $y_1, y_2, y_4 \ge 0$, $y_3 = 0$.

Since $x_2, x_3, x_4 > 0$, complementary slackness demand that the corresponding constraints should be tight. Putting this together:

$$
\begin{cases}
3y_1^\star + 2y_2^\star + y_4^\star = 6 \\
5y_1^\star - 2y_2^\star + 2y_4^\star = 5 \\
-2y_1^\star + y_2^\star - y_4^\star = -2
\end{cases}
$$

$y^\star = (1, 1, 0, 1)$. Check whether $y^\star$ satisfies the dual feasibility conditions:

- $y_i^\star \ge 0$ for $i \in [1, \dots, 4]$
- 1 constraint: $1 + 4 + 3 \ge 7$ - yes
- 5 constraint: $2 + 1 - 2 \ge 3$ - no

It is, so (1, 1, 0, 1) is infeasible $\Longrightarrow$ $x^\star$ is not an optimal primal solution.