Department Mathematik
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3. Exercise Sheet of "Diskrete Optimierung I"

Required knowledge from the lecture (see lecture notes):

- Branch-and-bound
- (Integer) linear programs
- Basic definitions from linear algebra: convex hull, affine hull, affine independence, linear independence, dimension, ...
- Simplex algorithmus

Groupwork

Exercise G1 (Knapsack and Branch-and-Bound)

For $a, c \in \mathbb{R}^n_+$, the 0/1-knapsack problem is given by

$$\begin{array}{lrcl} \max & \sum_{i=1}^n c_i x_i & & \\ \text{s.t.} & \sum_{i=1}^n a_i x_i & \leq & b \\ & x & \in & \{0,1\}^n. \end{array}$$

(a) Show that the following greedy algorithm always finds an optimal solution to the LP-relaxation of the Knapsack-problem:
W.l.o.g., let the vectors a and b be sorted such that: $\frac{c_1}{a_1} \geq \cdots \geq \frac{c_n}{a_n}$. Further, let $k \in \{1, \dots, n\}$ be the index that satisfies $\sum_{i=1}^k a_i \leq b$ and $\sum_{i=1}^{k+1} a_i > b$. Now we define $x \in [0,1]^n$ by:

$$x_j = \begin{cases} 1 & \text{, if } j \in \{1, \dots, k\}, \\ \frac{b - \sum_{i=1}^k a_i}{a_{k+1}} & \text{, if } j = k+1, \\ 0 & \text{, else.} \end{cases}$$

(b) Use the branch-and-bound algorithm to compute an optimal solution to the 0/1 knapsack problem for $n = 7$, $b = 35$, $a = (3, 4, 4, 3, 15, 13, 16)^T$ and $c = (12, 12, 9, 15, 90, 26, 112)^T$.
To this end, select the LP relaxation value (obtained by using (a)) as an upper bound of each node. Further, generate a heuristic feasible solution in each node by flooring the non-integer component of the optimal solution to the LP relaxation. This yields a lower bound. The node selection strategy shall choose the node attaining the strongest lower bound.

Exercise G2 (Modelling feasible Sets)

Model each of the following subsets of $\mathbb{R}^2$ as the feasible region of a mixed-integer linear program:

(a) $M_1 = \{(1,1),\ (2,3),\ (3,1),\ (4,2)\}$

(b) $M_2 = ([0,1] \times \{1\}) \cup ([1,2] \times \{3\}) \cup ([2,3] \times \{2\}) \cup ([3,4] \times \{3\})$

(c) $M_3 = \big\{(x_1, x_2) \in \mathbb{R}^2 \mid -3 \le x_1 \le 3,\ |x_1| \le x_2 \le |x_1| + 1\big\}$.

Exercise G3 (Basic Solutions)

Consider the linear optimization problem

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \ge 0. \end{array}$$

Show: if the above LP posseses a non-degenerate optimal basic solution, then the corresponding dual LP possesses a unique optimal solution.

Homework

Exercise H1 („Warm-up"-Aufgabe: Affine & Linear Independence, Affine & Linear Hull, etc.) (3+2+2+2+4 points)

(a) Consider the subset $M = \{v_1, v_2, v_3, v_4, v_5\} \subset \mathbb{R}^5$ with:

$$\begin{aligned} v_1 &= (1,0,0,0,0)^T \\ v_2 &= (0,1,1,0,0)^T \\ v_3 &= (0,1,1,2,0)^T \\ v_4 &= (0,0,0,0,1)^T \\ v_5 &= (1,2,2,2,1)^T. \end{aligned}$$

Determine

i. the dimension of $\mathrm{lin}(M)$,

ii. the affine rank of M,

iii. the dimension of $\mathrm{aff}(M)$.

(b) From linear independence follows affine independence. Show: The converse does not hold.

(c) Show: If $x_1, \ldots, x_k, x_{k+1} \in \mathbb{R}^n$, $n \ge k$, are affinely independent, then $x_1 - x_{k+1}, \ldots, x_k - x_{k+1}$ are linearly independent.

(d) State (with proof) $n+1$ affinely independent vectors in $\mathbb{R}^n$.

(e) Let $M \subseteq \mathbb{R}^n$. Show:

i. $0 \in \mathrm{aff}(M) \Rightarrow \dim(M) = \mathrm{rang}(M)$.

ii. $0 \notin \mathrm{aff}(M) \Rightarrow \dim(M) = \mathrm{rang}(M) - 1$.

Exercise H2 (Branch-and-Bound) (6 points)

Solve the following optimization problem using branch-and-bound and sketch the branch-and-bound tree:

$$\begin{array}{rrcl} \max & 4x_1 - x_2 & & \\ \text{s.t.} & 7x_1 - 2x_2 & \leq & 14 \\ & x_2 & \leq & 3 \\ & 2x_1 - 2x_2 & \leq & 3 \\ & x_1,\, x_2 & \geq & 0 \\ & x_1,\, x_2 & \in & \mathbb{Z}. \end{array}$$

You may use an LP solver to solve the arising LP relaxations.

Exercise H3 (Simplex Algorithm) (6 points)

Solve the following optimization problem using the simplex algorithm. Note that the following LP is not given in standard form:

$$\begin{aligned} \max\ & 3x_1 + 2x_2 + 4x_3 \\ \text{s.t.}\ & x_1 + x_2 + 2x_3 \leq 4 \\ & 2x_1 + 3x_2 \leq 5 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Start in the point $x = (0, 0, 0)$.

For submitting your homework, you have two weeks time. You can submit them as a (fixed) team of 2 people. The submission is electronic via StudOn. For reaching 2/3 of all possible points from the homework, you obtain a bonus of 0,3 geade points in the oral exam (only applicable for grades between 1,3 and 4,0).

The submission of the 3rd homework is until **Friday, 26 November 2021**