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2. Exercise Sheet of "Diskrete Optimierung I"

Required knowledge (see lecture notes):

- Complexity classes $\mathcal{P}$ and $\mathcal{NP}$
- Karp-reduction and $\mathcal{NP}$-completeness

The SAT problem: Let $V = \{x_1, \ldots, x_n\}$ be a set of Boolean variables. A *literal* $x_{ij} \in X := \{x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n\}$ is a variable x_i or a negated variable $\bar{x}_i$. A Boolean formula is given in *conjunctive normal form (KNF)* if it is a conjunction (AND-statement) of disjunctions. Each such disjunction (OR-statement) connects only literals. A disjunction in a Boolean KNF formula is also called *clause*. Thus, the KNF formula is given by:

$$\mathcal{F} = \bigwedge_i \Big(\bigvee_j x_{ij} \Big).$$

An example is: $(x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_3 \vee x_4)$ for $V = \{x_1, x_2, x_3, x_4\}$ and $X = \{x_1, \bar{x}_1, x_2, \bar{x}_2, x_3, \bar{x}_3, x_4, \bar{x}_4\}$.

The *SAT* problem is defined as follows:

- Given: A Boolean KNF formula $\mathcal{F}$.
- Question: Is $\mathcal{F}$ satisfiable?

Groupwork

Exercise G1 (STABLE SET (SS))

Let $G = (V, E)$ be a graph, and let $k \in \mathbb{N}$ be a number. A subset $U \subseteq V$ is called a *stable set*, if no two nodes $u, v \in U$ are connected by an edge. We consider the problem STABLE SET: Does G contain a stable set consisting of k nodes?

Show: STABLE SET is $\mathcal{NP}$-complete. Use that SAT is $\mathcal{NP}$-complete.

Exercise G2 (CLIQUE)

Let $G = (V, E)$ be a graph, and let $k \in \mathbb{N}$ be a number. A *clique* is a subset $U \subseteq V$, such that every two nodes $u, v \in U$, $u \neq v$, are connected by an edge. We consider the problem CLIQUE: Does G contain a clique of cardinality k?

Show: CLIQUE is $\mathcal{NP}$-complete.

Exercise G3 (CLIQUE, VERTEX COVER (VC) und STABLE SET)

Let $G = (V, E)$ be a graph, and let $k \in \mathbb{N}$ be a number. A *vertex cover* in a graph $G = (V, E)$ is a set $U \subseteq V$, such that every edge $e \in E$ is incident to at least one node in U. We consider the problem VERTEX COVER: Does G contain a vertex cover of cardinality k?

(a) Show that VERTEX COVER is $\mathcal{NP}$-complete. Give a polynomial reduction from STABLE SET to VERTEX COVER.

(b) Alternatively, give the proof via a polynomial reduction from CLIQUE to VERTEX COVER.

Homework

Exercise H1 (The k-SAT Problem) (6 + 2 + 2 + 4 + 2 points)

The *k-SAT* problem is defined as follows:

- Given: Let $\mathcal{F}$ be a Boolean KNF formula where each clause contains exactly k literals.
- Question: Is $\mathcal{F}$ satisfiable?

(a) Show: 3-SAT is $\mathcal{NP}$-complete. *Hint:* Reduce SAT to 3-SAT.

(b) Next, show that 2-SAT is in $\mathcal{P}$. To do this, construct to a given 2-SAT instance consisting of n variables $x_1, \ldots, x_n$ and k clauses $C_1, \ldots, C_k$, a directed graph $\tilde{G} = (V, A)$. Let $V := \{x_1, \ldots, x_n, \bar{x}_1, \ldots \bar{x}_n\}$ and the edge set A chosen in such a way that for each clause $C = (l_1 \vee l_2)$, $l_1, l_2 \in V$, two directed edges $\bar{l}_1 \to l_2$ and $\bar{l}_2 \to l_1$. These edges model the relation that for satisfiability in any 2-SAT clause we have: If l_1 is false, then l_2 must be true and vice versa.

i. Give an example of a non-satisfiable 2-SAT formula and the corresponding auxiliary graph $\tilde{G}$.

ii. Show: If there exists a directed path from v to u in $\tilde{G}$ for $u, v \in V$, then there also exists a path from $\bar{u}$ to $\bar{v}$.

iii. Show: 2-SAT is not satisfiable if and only if there is a node $v \in V$ for which there exists both a directed path from v to $\bar{v}$ as well as a directed path from $\bar{v}$ to v.

iv. Show by i)–iii): 2-SAT $\in \mathcal{P}$

For submitting your homework, you have two weeks time. You can submit them as a (fixed) team of 2 people. The submission is electronic via StudOn. For reaching 2/3 of all possible points from the homework, you obtain a bonus of 0,3 geade points in the oral exam (only applicable for grades between 1,3 and 4,0).

The submission of the 2nd homework is until **Friday, 20 November 2021**