Solutions Excercise 2

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Exercise 1

(a) Consider the subset $M = \{v_1, v_2, v_3, v_4, v_5\} \in \mathbb{R}^5$ with:

$$\begin{aligned} v_1 &= (1,0,0,0,0)^T \\ v_2 &= (0,1,1,0,0)^T \\ v_3 &= (0,1,1,2,0)^T \\ v_4 &= (0,0,0,0,1)^T \\ v_5 &= (1,2,2,2,1)^T \end{aligned}$$

(i) the dimension of lin(M)

The dimension of lin(M) is the maximum number of linearly independent vectors in M:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 2 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow dim(lin(M)) = 4$$

(ii) the affine rank of M

affrang(M) is the cardinality of the largest affinely independent subset of M:

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow affrang(M) = 4$$

(iii) the dimension of aff(M)

dim(aff(M)) = affrang(M) = 4

(b) From linear independence follows affine independence. Show: The converse does not hold.

Let $v_1 = (2,0)^T$, $v_2 = (1,0)$, $v_3 = (1,1)$. They are affinely independent, since: $v_2 - v_1$ and $v_3 - v_1$ are linearly independent. However, $v_1 = 2v_2 \Rightarrow v_1, v_2$ and v_3 are affinely independent, but not linearly independent.

(c) Show: if $x_1, \dots, x_k, x_k + 1 \in \mathbb{R}^n$, $n \geq k$, are affinely independent, then $x_1 = x_{k+1}, \dots, x_k - x_{k+1}$ are lineraly independent

Consider some $\lambda_1, \dots, \lambda_k$, such that:

$$\sum_{i=1}^{k} \lambda_i (x_i - x_{k+1}) = 0.$$

In accordance with affinely independence: $\sum_{i=1}^{k+1} \lambda_i = 0$. Also,

$$\sum_{i=1}^{k+1} \lambda_i x_i = \sum_{i=1}^{k} \lambda_i (x_i - x_{k+1}) + \sum_{i=1}^{k+1} \lambda_i x_{k+1}.$$

Hence, $\sum_{i=1}^{k+1} \lambda_i x_i = 0 \Longrightarrow \lambda_i = 0 \forall i$

Exercise 2

Solve the following optimization problem using branch-and-bound and sketch the branch-andbound tree:

$$\begin{aligned} \max \quad & 4x_1 - x_2 \\ \text{s.t.} \quad & 7x_1 - 2x_2 \leq 14 \\ & x_2 \leq 3 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

For the LP relaxation

$$\begin{aligned} \max \quad & 4x_1 - x_2 \\ \text{s.t.} \quad & 7x_1 - 2x_2 \leq 14 \\ & x_2 \leq 3 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

has optimal solution at $(2.9, 3)$ with $Z = 8.4$. Then we consider two cases: $x_1 \geq 3$ and $x_1 \leq 2$.

1. $x_1 \geq 3$

   The linear programming relaxation

   $$\begin{aligned} \max \quad & 4x_1 - x_2 \\ \text{s.t.} \quad & 7x_1 - 2x_2 \leq 14 \\ & x_2 \leq 3 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_2 \geq 0 \\ & x_1 \geq 3 \end{aligned}$$

   has no feasible solution. Hence, the IP has no feasible solution either.

2. $x_1 \leq 2$

   The linear programming relaxation

   $$\begin{aligned} \max \quad & 4x_1 - x_2 \\ \text{s.t.} \quad & 7x_1 - 2x_2 \leq 14 \\ & x_2 \leq 3 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_1 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

   has optimal solution at $(2, 0.5)$ with $Z = 7.5$. Then we consider two cases: $x_2 \geq 1$ and $x_2 \leq 0$.

2.1. $x_2 \geq 1$

The linear programming relaxation

$$\begin{aligned} \max \quad & 4x_1 - x_2 \\ \text{s.t.} \quad & 7x_1 - 2x_2 \leq 14 \\ & x_2 \leq 3 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_1 \leq 2 \\ & x_2 \geq 1 \\ & x_1 \geq 0 \end{aligned}$$

has optimal solution at $(2, 1)$ with $Z = 7.0$. This is the optimal solution of the IP as well. Currently, the best value of Z for the original IP is Z = 7.

2.2. $x_2 \leq 0$

The linear programming relaxation

$$\begin{aligned} \max \quad & 4x_1 - x_2 \\ \text{s.t.} \quad & 7x_1 - 2x_2 \leq 14 \\ & x_2 \leq 0 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_1 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

has optimal solution at $(1.5, 0)$ with $Z = 6$. Then we consider two cases: $x_1 \geq 2$ and $x_1 \leq 1$.

2.2.1. $x_1 \geq 2$

The linear programming relaxation

$$\begin{aligned} \max \quad & 4x_1 - x_2 \\ \text{s.t.} \quad & 7x_1 - 2x_2 \leq 14 \\ & x_2 \leq 0 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_1 \leq 2 \\ & x_2 \geq 0 \\ & x_1 \geq 2 \end{aligned}$$

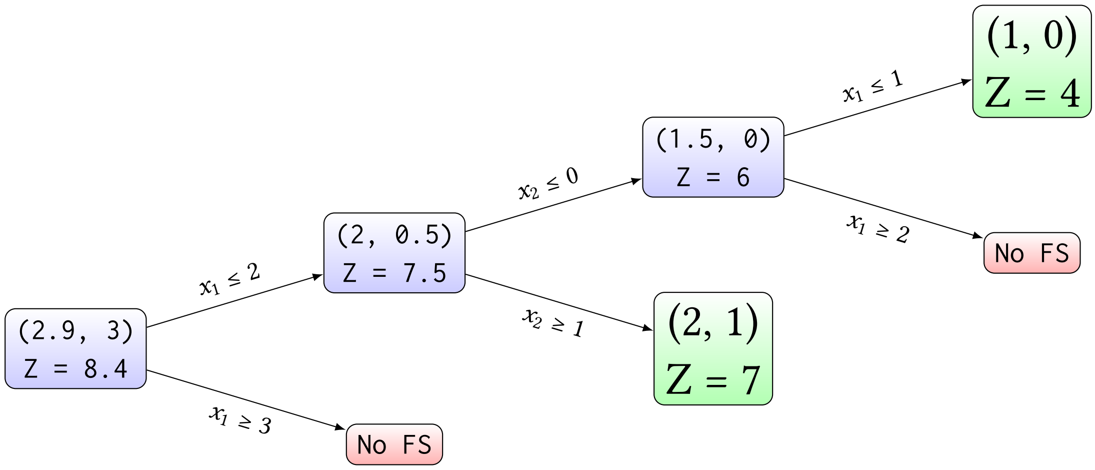
has no feasible solution. Hence, the IP has no feasible solution either.

2.2.2. $x_1 \leq 1$

The linear programming relaxation

$$\begin{aligned} \max \quad & 4x_1 - x_2 \\ \text{s.t.} \quad & 7x_1 - 2x_2 \le 14 \\ & x_2 \le 0 \\ & 2x_1 - 2x_2 \le 3 \\ & x_1 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

has optimal solution at $(1, 0)$ with $Z = 4.0$. This is the optimal solution of the IP as well. Currently, the best value of Z for the original IP is Z = 7.



Therefore, the solution (2, 1) with Z = 7 is the optimal solution.

Exercise 3

Solve the following optimization problem using the simplex algorithm.

$$
\begin{aligned}
\max \quad & 3x_1 + 2x_2 + 4x_3 \\
\text{s.t.} \quad & x_1 + x_2 + 2x_3 \leq 4 \\
& 2x_1 + 3x_2 \leq 5 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Let slack variables (s_1, s_2) be introduced in each of the restrictions of $\leq$ type, to convert them into equalities, resulting the system of linear equations:

$$
\begin{aligned}
x_1 + x_2 + 2x_3 + s_1 &= 4 \\
2x_1 + 3x_2 + s_2 &= 5.
\end{aligned}
$$

Matching the objective function to zero:

$$z - 3x_1 - 2x_2 - 4x_3 - 0s_1 - 0s_2 = 0$$

Simplex tableau method:

	x_1	x_2	x_3	s_1	s_2	
s_1	1	1	$2^{\star}$	1	0	4
s_2	2	3	0	0	1	5
z	-3	-2	-4	0	0	0
x_3	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	2
s_2	$2^{\star}$	3	0	0	1	5
z	-1	0	0	2	0	8
x_3	0	$\frac{1}{2}$	$-\frac{1}{4}$	1	$\frac{1}{2}$	$\frac{3}{4}$
x_1	1	$\frac{3}{2}$	0	0	$\frac{1}{2}$	$\frac{5}{2}$
z	0	$\frac{3}{2}$	0	2	0	$\frac{21}{2}$

Hence, $(2.5, 0, 0.75)$ with $z = 10.5$ is the optimal solution.