Solutions worksheet 01

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Exercise [Solving linear equation systems]

Solve the linear equation system $Ax = b$. A and b given as

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}$$

Gauss elimination method:

1. Augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 1 & 4 & 9 & | & 2 \\ 1 & 8 & 27 & | & 7 \end{bmatrix}$$

2. Add -1 times row (1) to both row (2) and row (3)

$$\begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 0 & 2 & 6 & | & -1 \\ 0 & 6 & 24 & | & 4 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

3. Divide all terms in row (2) by 2 and add -3 times row (2) to row (3)

$$\begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 0 & 1 & 3 & | & -\frac{1}{2} \\ 0 & 0 & 6 & | & 7 \end{bmatrix} \begin{matrix} \\ \frac{R_2}{2} \\ R_3 - 3R_2 \end{matrix}$$

4. Divide all terms in row (3) by 6

$$\begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 0 & 1 & 3 & | & -\frac{1}{2} \\ 0 & 0 & 1 & | & \frac{7}{6} \end{bmatrix} \begin{matrix} \\ \\ \frac{R_3}{6} \end{matrix}$$

5. Row echelon form

   Add -3 times row (3) to both row (2) and row (1)

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & \frac{7}{6} \end{array}\right] \begin{array}{l} R_1 - 3R_3 \\ R_2 - 3R_3 \\ \\ \end{array}$$

   Add -2 times row (2) to row (1)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{15}{2} \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & \frac{7}{6} \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ \\ \end{array}$$

Answer:

$$x = \begin{pmatrix} \frac{15}{2} \\ -4 \\ \frac{7}{6} \end{pmatrix}$$

Exercise [Norms]

Definition (Norm) — A mapping $\|\cdot\|$ from any (real) vector space V to the real numbers $\mathbb{R}$ is called a norm, whenever

1. $\|v + w\| \leq \|v\| + \|w\|$
2. $\|v\| = 0 \implies v = 0_V$
3. $\|\lambda v\| = |\lambda| \cdot \|v\|$

for all $\lambda \in \mathbb{R}$, $v, w \in V$

1. Let $V = \mathbb{R}^n$ for some $n \in \mathbb{N}$. The euclidean norm

   $$\|v\|_2 := \sqrt{\sum_{i=1}^{n} v_i^2}$$

   is a norm.

   Proof:

   (a) Proof of 1 statement

   $$\|v + w\|_2^2 := \sum_{i=1}^{n}(v_i+w_i)^2 = \sum_{i=1}^{n} v_i^2+2v_iw_i+w_i^2 = \sum_{i=1}^{n} v_i^2+\sum_{i=1}^{n} 2v_iw_i+\sum_{i=1}^{n} w_i^2 = \|v\|_2^2+2(v\cdot w)+\|w\|_2^2$$

   Taking into account the Cauchy-Schwarz Inequality

   $$|v \cdot w| \leq \|v\|_2 \cdot \|w\|_2$$

   which implies

   $$\|v\|_2^2 + 2(v \cdot w) + \|w\|_2^2 \leq \|v\|_2^2 + 2\|v\|\|w\| + \|w\|_2^2 = (\|v\| + \|w\|)^2$$

   Hence,

   $$\|v + w\|_2^2 \leq (\|v\|_2 + \|w\|_2)^2$$

   $$\|v + w\|_2 \leq \|v\|_2 + \|w\|_2$$

   as required

   (b) Proof of 2 statement

   $$\|v\|_2 := \sqrt{\sum_{i=1}^{n}(v_i)^2}$$

   $$\sqrt{\sum_{i=1}^{n} v_i^2} = 0 \iff v_i = 0,\ \forall i$$

   which implies

   $$v = 0_V$$

   as required

(c) Proof of 3 statement

$$\|\lambda v\|_2 := \sqrt{\sum_{i=1}^{n} (\lambda v_i)^2} = |\lambda| \cdot \sqrt{\sum_{i=1}^{n} v_i^2} = |\lambda| \cdot \|v\|$$

as required

Hence, the euclidean norm is a norm.

2. Let $V = \mathbb{R}^n$ for some $n \in \mathbb{N}$. The mapping

$$\|v\|_{\frac{1}{2}} := (\sum_{i=1}^{n} \sqrt{|v_i|})^2$$

is a norm.

Proof: let $v = (0, 1)$, $w = (1, 0)$, so $v, w \in V$. Given that

$$\|v + w\|_{\frac{1}{2}} := (\sum_{i=1}^{2} \sqrt{|v_i + w_i|})^2 = 2^2 = 4$$

$$\|v\|_{\frac{1}{2}} + \|w\|_{\frac{1}{2}} := (\sum_{i=1}^{2} \sqrt{|v_i|})^2 + (\sum_{i=1}^{2} \sqrt{|w_i|})^2 = 1 + 1 = 2$$

$$\|v + w\|_{\frac{1}{2}} > \|v\|_{\frac{1}{2}} + \|w\|_{\frac{1}{2}}$$

Hence, the $\|\cdot\|_{\frac{1}{2}}$ is not a norm.

3. Let V be the space of convergent sequences. The mapping

$$\|v\|_{lim} := \lim_{n \to \infty} v_n$$

is a norm.

Proof:

$$\|\lambda \cdot v\|_{lim} := \lim_{n \to \infty} \lambda \cdot v_n = \lambda \cdot \lim_{n \to \infty} v_n = \lambda \cdot \|v\|_{lim} \neq |\lambda| \cdot \|v\|_{lim}$$

Hence, the $\|\cdot\|_{lim}$ is not a norm.

Exercise [Python, Pandas, K-Means]

(a) DS analysis

Let's start by describing DS

	eruptions	waiting
count	272.000000	272.000000
mean	3.487783	70.897059
std	1.141371	13.594974
min	1.600000	43.000000
25%	2.162750	58.000000
50%	4.000000	76.000000
75%	4.454250	82.000000
max	5.100000	96.000000

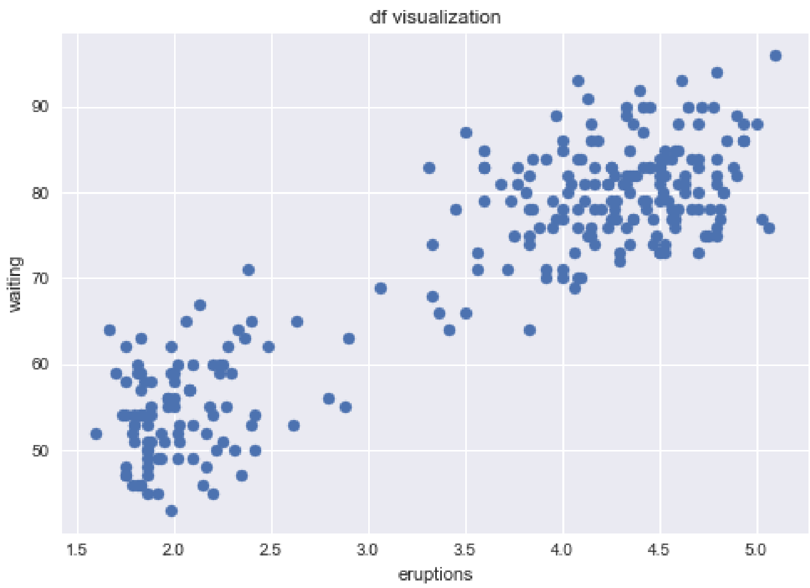
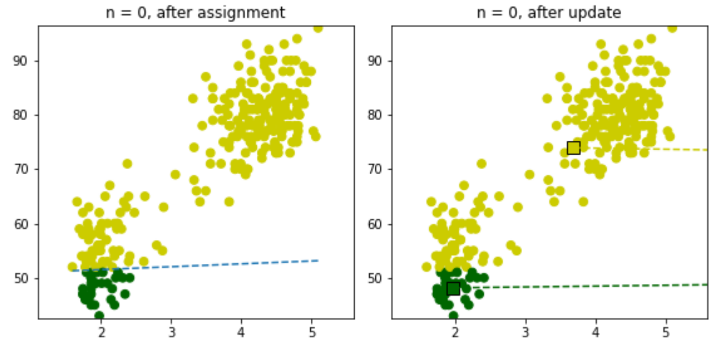
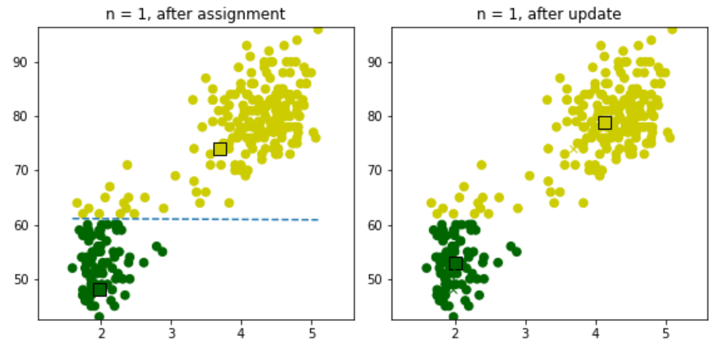


Figure 1: visualization of *faithful.csv* data

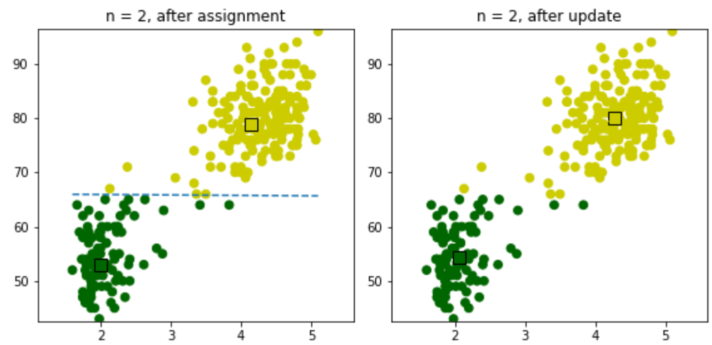
(b) KMeans clustering visualization



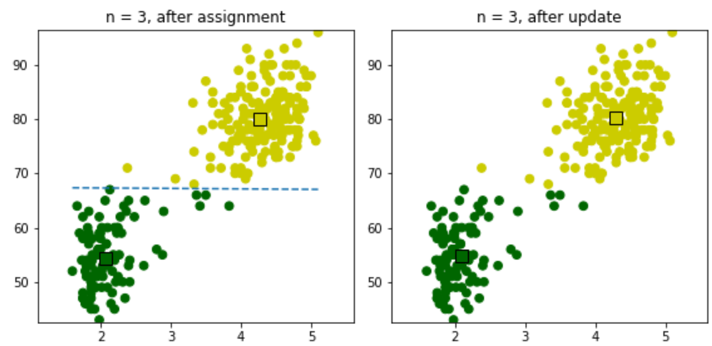
(a) iteration = 0



(b) iteration = 1



(c) iteration = 2



(d) iteration = 3

Figure 2: KMeans algorithm

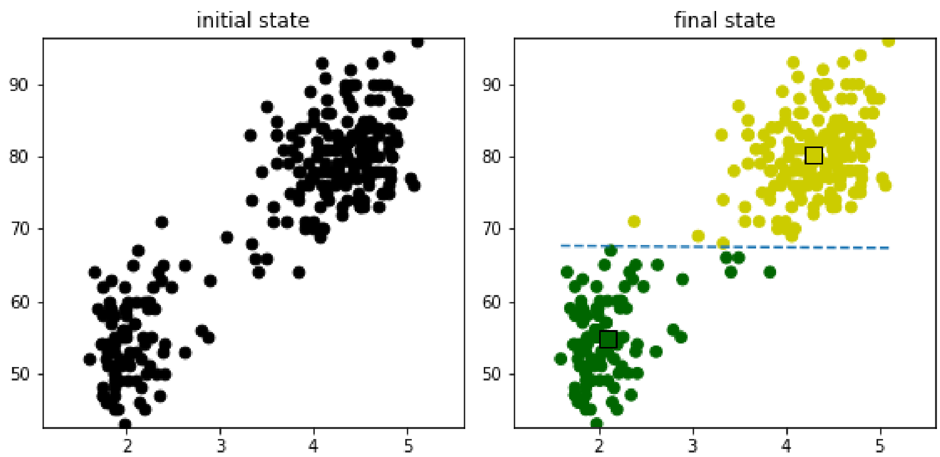


Figure 3: KMean clustering sumup

Exercise [Implementing EM for Clustering]

Exercise [Experiments with K-Means and EM]

Exercise [Theory of K-means]

- Show that the iterates of the algorithm satisfy

$$\frac{1}{2}\sum_{k=1}^{K}\sum_{x\in C_k^{(i)}}\left\|x-m_k^{(i)}\right\|^2 \leq \frac{1}{2}\sum_{k=1}^{K}\sum_{x\in C_k^{(i-1)}}\left\|x-m_k^{(i-1)}\right\|^2$$

To confirm the inequality above we will show that both of operations can never increase the clustering energy.

$$E(C^{(i)},m^{(i)}) := \frac{1}{2}\sum_{k=1}^{K}\sum_{x\in C_k^{(i)}}\left\|x-m_k^{(i)}\right\|^2 \triangleq \text{clustering energy}$$

1. $E(C^{(i)},m^{(i)}) < E(C^{(i-1)},m^{(i)})$

   From the logic of the algorithm: $C^{(i)}$ and $C^{(i-1)}$ are differently only if there is a point that finds a closer cluster center in $m^{(i)}$ than the one assigned to it by $C^{(i-1)}$.Hence,

$$\frac{1}{2}\sum_{k=1}^{K}\sum_{x\in C_k^{(i)}}\left\|x-m_k^{(i)}\right\|^2 < \frac{1}{2}\sum_{k=1}^{K}\sum_{x\in C_k^{(i-1)}}\left\|x-m_k^{(i)}\right\|^2$$

2. $E(C^{(i)},m^{(i)}) \leq E(C^{(i)},m^{(i-1)})$

   This statement is equivalent to the following

$$\frac{1}{2}\sum_{k=1}^{K}\sum_{x\in C_k}\|x-m_k\|^2 \leq \frac{1}{2}\sum_{k=1}^{K}\sum_{x\in C_k}\|x-a_k\|^2$$

, where $a = (a_1, \dots, a_k)$ with $a_k \in \mathbb{R}^M$ is an arbitrary point in the same space. Consider C_k for it:

$$\sum_{x \in C_k} \|x - m_k\|^2 \leq \sum_{x \in C_k} \|x - a_k\|^2$$

$$\begin{aligned}
\sum_{x \in C_k} \|x - a_k\|^2 &= \sum_{x \in C_k} \|(x - m_k) + (x - a_k)\|^2 \\
&= \sum_{x \in C_k} \|x - m_k\|^2 + \|m_k - a_k\|^2 + 2(x - m_k) \cdot (m_k - a_k) \\
&= \sum_{x \in C_k} \|x - m_k\|^2 + \sum_{x \in C_k} \|m_k - a_k\|^2 + 2 \sum_{x \in C_k} (x \cdot m_k - x \cdot a_k - m_k \cdot m_k + m_k \cdot a_k) \\
&\text{as} \sum_{x \in C_k} x = \sum_{x \in C_k} m_k \\
&= \sum_{x \in C_k} \|x - m_k\|^2 + |C_k| \|m_k - a_k\|^2 + 2 \cdot |C_k| (m_k \cdot m_k - m_k \cdot a_k - m_k \cdot m_k + m_k \cdot a_k) \\
&= \sum_{x \in C_k} \|x - m_k\|^2 + |C_k| \|m_k - a_k\|^2 \\
&\geq \sum_{x \in C_k} \|x - m_k\|^2
\end{aligned}$$

Hence,

$$\frac{1}{2} \sum_{k=1}^{K} \sum_{x \in C_k^{(i)}} \left\| x - m_k^{(i)} \right\|^2 \leq \frac{1}{2} \sum_{k=1}^{K} \sum_{x \in C_k^{(i-1)}} \left\| x - m_k^{(i-1)} \right\|^2$$

- Every data point x_n must be assigned to precisely one class in oder to the algorithm be converging. It was shown above.
- It is not a big deal to extend first step to an arbitrary norm. The problem is how to calculate centers of clusters.
- In that case result of clustering depends on realization in programm. Final result significantly depends on the initial statements of centers.

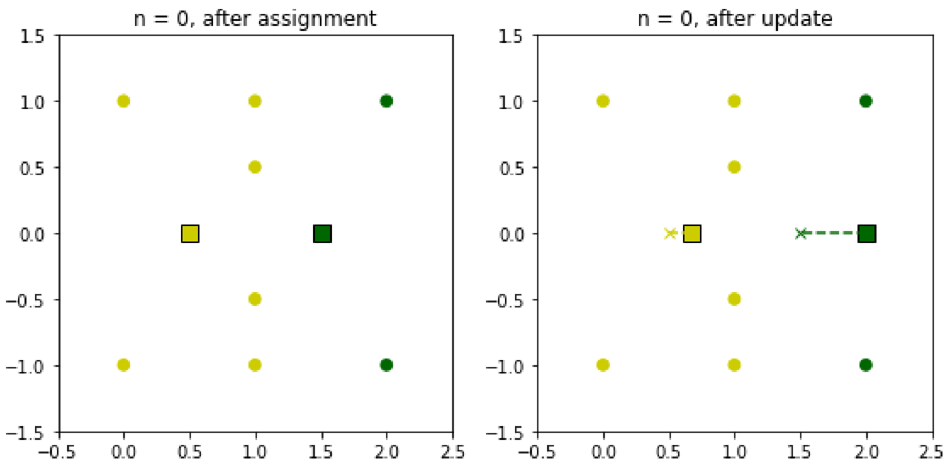


Figure 4: KMeans algorithm

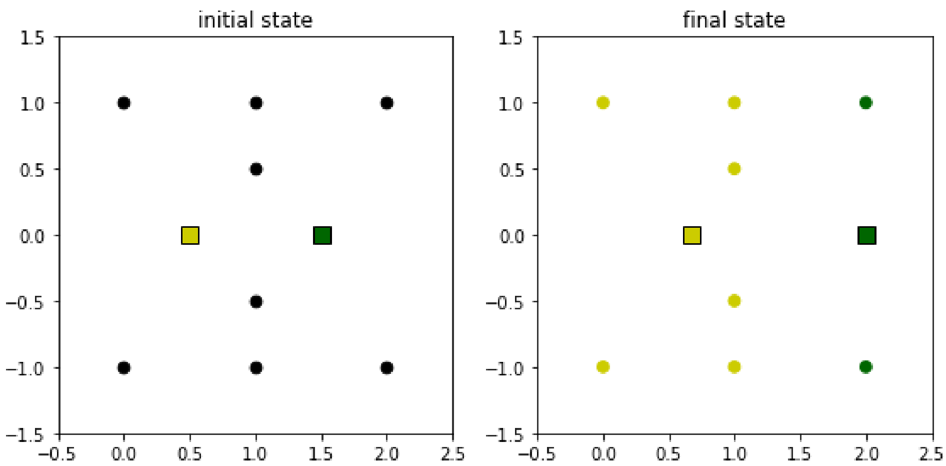


Figure 5: KMean clustering sumup