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4. Exercise Sheet of "Diskrete Optimierung I"

Required knowledge from the lectures (see lecture notes):

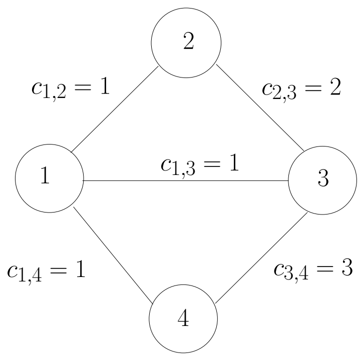
- Basic terms from graph theory, (integer) linear programs
- Polyhedra, Cones, Faces
- Dimension polyhedron/polytope

Groupwork

Exercise G1 (Motivation and entry point to polyhedral theory)
In this exercise we consider minimal degree-constrained spanning-trees:
Let an undirected graph $G = (V, E)$ and weights c_e, $e \in E$ be given, $n = |V|$ vertices and $m = |E|$ edges. A minimal spanning tree T is a subgraph of G that is a tree. It contains all vertices of V and has minimal edge weight. A tree is a connected, circle free, graph. There are algorithms (Kruskal, Prim) that solve the minimal spanning-tree problem in polynomial time.
From now on it shall additionally hold: Each vertex in a minimal spanning tree must have a degree of at most $k \in N$. In general we may not expect to find an algorithm solving this problem in polynomial time (it is $\mathcal{NP}$-hard). What can we do now?

a **Homework:** Show that a tree of n vertices consists of exactly $n-1$ edges. One may use: *Auxiliary Lemma*: Each tree with $n \geq 2$ vertices has at most two *leaves* (vertices of degree one)

b Let the following undirected graph G be given:



Write down all incidence vectors $v_1, \ldots, v_l$, $l \in \mathbb{N}$, of degree constrained spanning tress with $k \leq 2$ in G.

c Let $P = \text{conv}(\{v_1, \dots v_l\})$ be the inner description of the polytope induced by the convex hull of all incidence vectors of part b). A full outer description of P is given by:

$$x_{24} = 0 \tag{1}$$
$$x_{12} + x_{13} + x_{14} + x_{23} + x_{34} = 3 \tag{2}$$
$$x_{13} + x_{14} + x_{23} + x_{34} \geq 2 \tag{3}$$
$$x_{14} + x_{34} \geq 1 \tag{4}$$
$$x_{23} + x_{34} \geq 1 \tag{5}$$
$$x_{13} \geq 0 \tag{6}$$
$$x_{34} \leq 1,\ x_{23} \leq 1,\ x_{14} \leq 1 \tag{7}$$
$$x_{13} + x_{23} + x_{34} \leq 2 \tag{8}$$
$$x_{13} + x_{14} + x_{34} \leq 2 \tag{9}$$

i Explain briefly why the constraints are feasible for P

ii Reason why the outer description of P through (1)–(9) can be useful to solve the minimal-degree-constrained-spanning-tree problem. How can one obtain an optimal solution?

d Formulate the minimal-spanning-tree problem as an integer linear program. Reason why your set of constraints characterizes spanning trees.

e Formulate the minimal-degree-constrained-spanning-tree problem as an integer linear program.

f Determine the dimension of P, defined as the dimension of $\text{aff}(P)$ (i.e. the maximal number of affine independent vectors in P -1).

Exercise G2 (Polyhedron?)

Which of the following sets are polyhedra? Prove or disprove:

(a) $M_1 := \{X \in \mathbb{R}^{n \times n} \mid a_1^T X a_1 \leq a_2^T X a_2\}$, mit $a_1, a_2 \in \mathbb{R}^n$.

(b) $M_2 := \{x \in \mathbb{R}^n \mid x \geq 0,\ e^T x = 1, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_i a_i^2 = b_2\}$, with $a_1, \dots, a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$, and the vector $e \in \mathbb{R}^n$, whose components are all equal to 1.

(c) $M_3 := \{x \in \mathbb{R}^n \mid x \geq 0, x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1\}$.

Exercise G3 (Integrality)

Consider the integer program

$$\begin{array}{lrcrcl}
\max & x_2 & & & & \\
\text{s.t.} & -3x_1 & + & 4x_2 & \leq & 4 \\
 & 3x_1 & + & 2x_2 & \leq & 11 \\
 & 2x_1 & - & x_2 & \leq & 5 \\
 & & & x_1, x_2 & \geq & 0 \\
 & & & x_1, x_2 & \in & \mathbb{Z}.
\end{array}$$

Draw a picture in order to answer the following questions: what is the optimal value of the linear relaxation? What is the optimal value of the integer problem? What is the convex hull of all feasible solutions of the integer program (state the defining inequalities)?

Homework

Exercise H1 (Affine hull, equality set) (6 points)

Show: If $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is a non-empty polyhedron and $I = \text{eq}(P)$, then we have:

$$\text{aff}(P) = \{x \in \mathbb{R}^n \mid A_{I\cdot} x = b_I\}$$

Exercise H2 (Cone) (2 + 2 + 2 points)

Prove or disprove:

(a) Let K be a cone. Then $x + y \in K$ holds for all $x, y \in K$ if and only if K is convex.

(b) Each convex cone has at most obe extreme point, namely the origin.

(c) A polyhedral cone of the form $K := \{x \in \mathbb{R}^n \mid Ax \leq 0\}$ (with $A \in \mathbb{R}^{m \times n}$) has exactly one extreme point, namely the origin.

Exercise H3 (Modelling) (6 points)

A Buyer wants to buy 150,000 pieces of a certain good. Three sellers make offers, which are described in the following table. For each offer, it states the fix costs (they arise independently of how much is bought) and the price per piece in monetary units. The latter can vary depending on the amount of good bought. Furthermore, the delivery capacity of the sellers is bounded.

Let x_1, x_2 resp. x_3 the numer of pieces bought from seller 1, 2 resp. 3. The goal is to buy in such a way that the total costs are minimal.

Verkäufer	Fixkosten	Stückpreis	Menge
1	3,520.20	51.20	$0 < x_1 \leq 50,000$
2	82,810.00	52.10	$0 < x_2 \leq 20,000$
		51.10	$20,000 < x_2 \leq 60,000$
		50.10	$60,000 < x_2 \leq 80,000$
		49.10	$80,000 < x_2 \leq 100,000$
3	0	60.50	$0 < x_3 \leq 50,000$
		59.00	$50,000 < x_3 \leq 80,000$

The table is to be understand that e.g. when buying from seller 2, the 20,001-st piece is offered at a cheaper price than the first 20,000 pieces.

Formulate the above problem as an optimization problem and investigate whether the objective function is convex or concave and whether the feasible set is convex.

Exercise H4 (Polyhedron) (2 + 3 + 1 + 2 points)

Consider the polyhedron P, which is given by the following five inequalities:

$$x_1 + 2x_2 \geq 1, \qquad -x_1 \leq 1, \qquad x_1 - x_2 \geq -3, \qquad x_2 \geq 1, \qquad -2x_1 - x_2 \leq 0.$$

(a) Make a drawing of the polyhedron P.

(b) Determine at the hand of your drawing all faces of the polyhedron and state a defining inequality for each face.

(c) State at the hand of your drawing a matrix A and a vector b such that both $P = P(A, b)$ holds and the system $Ax \leq b$ is irredundant.

(d) Find a matrix B and a vector c such that P is equivalent to $P^{=}(B, c)$. Is $P = P^{=}(B, c)$?

The submission of the 4th homework is until **Friday, 10 December**