Solutions Excercise 4

Maksimov, Dmitrii
dmitrii.maksimov@fau.de
ko65beyp

Ilia, Dudnik
ilia.dudnik@fau.de
ex69ahum

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Exercise 1

Show: if $P = \{x \in \mathbb{R} | Ax \leq b\}$ is a non-empty polyhedron and $I = eq(P)$, then we have:

$$aff(P) = \{x \in \mathbb{R}^n | A_{I.}x = b_I\}$$

By definition:

$$aff(S) := \left\{ \sum_{i=1}^{t} \lambda_i a_i \; : \; t \geq 1 \text{ and finite}, a_1, \dots, a_t \in S, \lambda_1, \dots, \lambda_t \in \mathbb{R}, \sum_{i=1}^{t} \lambda_i = 1 \right\}$$

Let $\overline{x} \in aff(P)$, then:

$$\overline{x} = \lambda_1 x^1 + \cdots + \lambda_t x^t \text{ for some } x^1, \dots, x^t \in P, \lambda_1, \dots, \lambda_t \in \mathbb{R}, \sum_{i=1}^{t} \lambda_i = 1 \implies$$
$$A\overline{x} = \lambda_1 Ax^1 + \cdots + \lambda_t Ax^t = \left(\sum_{i=1}^{t} \lambda_i \right) b = b \implies$$
$$\overline{x} \in \{x \; : \; Ax = b\} \implies aff(P) = \{x \in \mathbb{R}^n | A_{I.}x = b_I\}$$

Exercise 2

Prove or disprove:

(a) Let K be a cone. Then $x + y \in K$ holds fo all $x, y \in K$ if and only if K is convex.

Consider two cones(convex and non-convex):

(a) A convex cone **(b)** A non-convex cone

Figure 1: Cones

From the illustration we can conclude, that:

- $\exists c = \lambda a + (1 - \lambda)b \ : \ z \notin K_{\text{non-convex}}, a, b \in K$
- $\nexists c = \lambda a + (1 - \lambda)b \ : \ z \notin K_{\text{convex}}, a, b \in K$

Since there exists $z = x + y$ and z goes through the point c, $z \in K$ if and only if K is convex.

(b) Each convex cone has at most one extreme point, namely the origin.

If there more than one extreme point in a cone (convex is not necessary), than $E\lambda \geq 0 \ : \ \lambda x \notin C, x \in C$, which contradicts the definition.

(c) A polyhedral cone of the form $K \ := \ x \in \mathbb{R}^n | Ax \leq 0($ with $A \in \mathbb{R}^{m \times x})$ has exactly one extreme point, namely the origin.

From my point of view it's the same as b.

Exercise 3

Let us represent each individual interval as a function:

$$
\begin{aligned}
f_{1,1} &= 3520.20 + 51.20x_{1,1} \\
f_{2,1} &= 82810.00 + 52.10x_{2,1} \\
f_{2,2} &= 82810.00 + 52.10 \cdot 20000 + 51.10x_{2,2} \\
f_{2,3} &= 82810.00 + 52.10 \cdot 20000 + 51.10 \cdot 40000 + 50.10x_{2,3} \\
f_{2,4} &= 82810.00 + 52.10 \cdot 20000 + 51.10 \cdot 40000 + 50.10 \cdot 20000 + 49.10x_{2,4} \\
f_{3,1} &= 60.50x_{3,1} \\
f_{3,2} &= 60.50 \cdot 50000 + 59.00x_{3,2},
\end{aligned}
$$

where $x_{i,j}$ - count of goods in i, j-interval.
Also, let us associate with each i, j - interval a binary variable $y_{i,j}$ such that, $y_{i,j} = 1$ if i, j - interval is chosen, else $y_{i,j} = 0$.

Hence, the corresponding optimization problem:

$$
\begin{aligned}
\min \quad & y_{1,1}f_{1,1} + y_{2,1}f_{2,1} + y_{2,2}f_{2,2} + y_{2,3}f_{2,3} + y_{2,4}f_{2,4} + y_{3,1}f_{3,1} + y_{3,2}f_{3,2} + x_{1,1} + x_{2,2} + x_{2,3} + x_{2,4} + x_{3,1} + x_{3,2} \\
\text{s.t.} \quad & y_{1,1}x_{1,1} + \\
& y_{2,1}x_{2,1} + y_{2,2}(20000 + x_{2,2}) + y_{2,3}(60000 + x_{2,3}) + y_{2,4}(80000 + x_{2,4}) + \\
& y_{3,1}x_{3,1} + y_{3,2}(50000 + x_{3,2}) = 150000 \\
& y_{1,1} \le 1 \\
& y_{2,1} + y_{2,2} + y_{2,3} + y_{2,4} \le 1 \\
& y_{3,1} + y_{3,2} \le 1 \\
& x_{1,1} \le 50000 \\
& x_{2,1} \le 20000 \\
& x_{2,2} \le 40000 \\
& x_{2,3} \le 20000 \\
& x_{2,4} \le 20000 \\
& x_{3,1} \le 50000 \\
& x_{3,2} \le 30000 \\
& x_{1,1}, x_{2,2}, x_{2,3}, x_{2,4}, x_{3,1}, x_{3,2} \ge 0 \\
& y_{1,1}, y_{2,2}, y_{2,3}, y_{2,4}, y_{3,1}, y_{3,2} \in \{0, 1\} \\
& x_{1,1}, x_{2,2}, x_{2,3}, x_{2,4}, x_{3,1}, x_{3,2} \in \mathbb{Z}
\end{aligned}
$$

We added $x_{1,1} + x_{2,2} + x_{2,3} + x_{2,4} + x_{3,1} + x_{3,2}$ in an objective function in order to fix such problem: if $y_{2,1} = 1$ and $y_{2,2} = 0$ then $x_{2,2}$ can be any value ≥ 0, but for this objective function $x_{2,2}$ will be equal to zero.

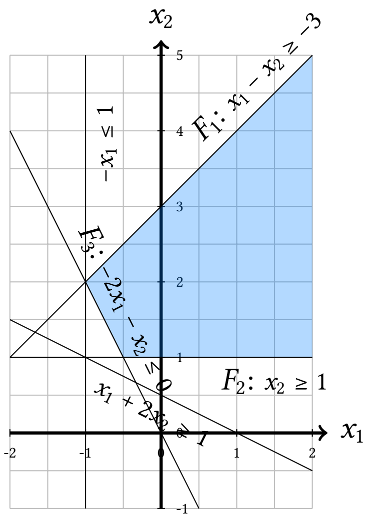
The objective function is twice differentiable. Hence, we can examine Hessian. It's obvious that Hessian matrix is a positive semidefinite $\Rightarrow$ the objective function is convex.Since, each constraint is a convex function, covex set is also convex.

Exercise 4

Consider the polyhedron P, which is given by the following five inequalities:

$$x_1 + 2x_2 \geq 1, -x_1 \leq 1, x_1 - x_2 \geq -3, x_2 \geq 1, -2x_1 - x_2 \leq 0$$

(a) Make a drawing of the polyhedron P.



(b) Determine at the hand of your drawing all faces of the polyhedron and state a defining inequaliti for each face.

$$F_1 = P \cap \{x \in \mathbb{R}^2 \; : \; x_1 - x_2 \geq -3\}$$
$$F_2 = P \cap \{x \in \mathbb{R}^2 \; : \; x_2 \geq 1\}$$
$$F_3 = P \cap \{x \in \mathbb{R}^2 \; : \; -2x_1 - x_2 \leq 0\}$$

(c) State at the hand of your drawing a matrix A and a vector b such that both $P = P(A, b)$ holds and the system $Ax \leq b$ is irredundant.

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \\ -2 & -1 \end{pmatrix}, b = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

(d) Find a matrix B and a vector c such that P is eqivalent to $P^{=}(B, c)$. Is $P = P^{=}(B, c)$?

By definition: $P^{=}(B, c) := \{x \; : \; Bx = c, x \geq 0\}$. Let $x_i = x_i^+ - x_i^-$ where $x_i^+, x_i^- \geq 0$ and y_1, y_2, y_3 are slack variables, , then the system of inequalities can be written as an equivalent system of equalities:

$$B = \begin{pmatrix} -1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ -2 & 2 & -1 & 1 & 0 & 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

, where $x = (x_1^+, x_1^-, x_2^+, x_2^-, y_1, y_2, y_3)^T$. Hence, $P^{=}(B, c) = \{x \in R^7 \; : \; Bx = c, x \geq 0\}$.

P.S. It would be great if you provide a solution where it is wrong.