Department Mathematik
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1. Exercise Sheet of "Diskrete Optimierung I"

Required knowledge:

- Basic knowledge in linear optimization
- Solution of linear optimization problems
- Strong and weak duality, optimality conditions

Groupwork

Exercise G1 (LP modelling, graphical solution)

Professor Milkman and his family have a business in which they sell dairy products, made from the milk of the 3 cows Daisy, Ermentrude and Florence. The 3 cows combined give 22 barrels of milk per week, from which Professor Milkman and his family produce ice cream and butter to sell them in their business. For producing 1 kilogram of butter, 2 barrels of milk are required, while 1 barrell of ice cream requires 3 barrels of milk. Of course, arbitrary smaller amounts of ice cream or milk can be produced as well. Professor Milkmans family can use at most 6 hours of labour for the production of ice cream or butter. In one hour, they are able to produce either 4 barrels of ice cream or 1 kilogram of butter. They make a profit of 5 Euros per barrel of ice cream and a profit of 4 Euros per kilogram of butter.

(a) Formulate a linear optimization problem (LP) whose solution gives the profit-maximizing production amounts of ice cream and milk.

(b) Determine an optimal solution in a graphical fashion. Draw the set of all feasible solutions and state the optimal objective function value.

(c) How can you generate lower and upper bounds for the optimal objective value from Part (b))?

(d) Verify the optimal value found in (b)) via the dual program (DP). Graphically solve the dual program as well.

(e) Briefly(!) state how the simplex method solves a linear program.

Exercise G2 (Linear optimization)

(a) Consider the linear program

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 - x_2 \leq 0, \\ & -x_1 + x_2 \leq -1. \end{aligned}$$

Determine the corresponding dual program and show that both primal and dual program are infeasible.

(b) Determine when, depending on the parameter $t \in \mathbb{R}$ when the linear program

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + tx_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

i. is infeasible,
ii. is unbounded,
iii. has an optimal solution.

(c) Show that the following optimization problem can be written as a linear program:

$$\begin{aligned} \min \quad & \max\{x_1, x_2\} \\ \text{s.t.} \quad & |x_1 + x_2 + x_3 + x_4| \leq 10, \\ & \max\{x_1, x_2\} \leq \min\{x_3, x_4\}, \\ & \frac{x_2 - x_4}{x_1 + x_3 + 1} \leq 4, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Exercise G3 (Transformation of linear optimization problems)

Consider the linear program

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0. \end{aligned}$$

Show that the following linear programs can be transformed to have the form of the above linear program:

(a)

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b, \\ & Cx \geq d, \\ & x \geq 0. \end{aligned}$$

(b)

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b. \end{aligned}$$

Homework

Exercise H1 (Linear Programs) (2 + 2 points)

(a) State the dual program corresponding to the following linear programs:

$$\begin{aligned} \max \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax \geq b, \\ & g^\top y = f, \\ & x, y \geq 0, \end{aligned}$$

Here, $f \in \mathbb{R}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $d, g \in \mathbb{R}^l$ and $A \in \mathbb{R}^{m \times n}$.

(b) Use Part (a) in order to show or disprove that $(x^*, y^*) = (\frac{13}{3}, 0, \frac{5}{3}, 1, 0)^\top$ is an optimal solution for

$$\begin{aligned} \max \quad & (6, -1, -3)x + (4, -6)y \\ \text{s.t.} \quad & \begin{pmatrix} -2 & -1 & \frac{17}{5} \\ -1 & 1 & -1 \end{pmatrix} x \geq \begin{pmatrix} -3 \\ -6 \end{pmatrix}, \\ & (2, -3)y = 2, \\ & x, y \geq 0. \end{aligned}$$

Remark: You may use software.

Exercise H2 (Duality, Complementary slackness theorem) (1 + 3 points)

Consider the following linear program:

$$\begin{aligned} \max \quad & 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5, \\ \text{s.t.} \quad & x_1 + 3x_2 + 5x_3 - 2x_4 + 2x_5 \leq 4, \\ & 4x_1 + 2x_2 - 2x_3 + x_4 + x_5 \leq 3, \\ & 2x_1 + 4x_2 + 4x_3 - 2x_4 + 5x_5 \leq 5, \\ & 3x_1 + x_2 + 2x_3 - x_4 - 2x_5 \leq 1, \\ & x_1, \ldots, x_5 \geq 0. \end{aligned}$$

(a) Formulate the corresponding dual program.

(b) Check if $x^* = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)^\top$ is an optimal primal solution via the complementary slackness theorem.

For submitting your homework, you have two weeks time. You can submit them as a (fixed) team of 2 people. The submission is electronic via StudOn. For reaching 2/3 of all possible points from the homework, you obtain a bonus of 0,3 geade points in the oral exam (only applicable for grades between 1,3 and 4,0).

The submission of the 1st homework is until **Friday, 5 November 2021**