Researching of functions using the derivative

p.1. Basic theorems on differentiable functions

p.1.1. The increasing and decreasing of the function. Fermat's theorem

Definition. Function $y = f(x)$ *increases at a point* x_0, if $(\exists\delta > 0)$, such that

$$\left(\forall x \in u_\delta^0(x_0 - 0)\right)\{f(x) < f(x_0)\} \wedge \left(\forall x \in u_\delta^0(x_0 + 0)\right)\{f(x_0) < f(x)\}.$$

Definition. Function $y = f(x)$ *decreases at a point* x_0, if $(\exists\delta > 0)$, such that

$$\left(\forall x \in u_\delta^0(x_0 - 0)\right)\{f(x) > f(x_0)\} \wedge \left(\forall x \in u_\delta^0(x_0 + 0)\right)\{f(x_0) > f(x)\}.$$

Definition. The point x_0 is called *the maximum point of the function* $y = f(x)$, if

$$(\exists\delta > 0)\left(\forall x \in u_\delta^0(x_0)\right)\{f(x) \le f(x_0)\}.$$

The point x_0 is called *the minimum point of the function* $y = f(x)$, if

$$(\exists\delta > 0)\left(\forall x \in u_\delta^0(x_0)\right)\{f(x) \ge f(x_0)\}.$$

Theorem 1.

Let the function $y = f(x)$ be differentiable at the point x_0. Then, if $f'(x_0) > 0$, then f is a function $y = f(x)$ *increases at a point* x_0, and if $f'(x_0) < 0$, then f is a function $y = f(x)$ *decreases at a point* x_0.

p.1.2. Theorems about the average (Mean Value Theorems)

The average value theorems, also known as the Mean Value Theorems, is a fundamental concept in calculus.

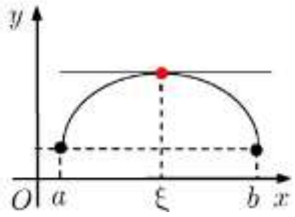
Theorem 2 (Fermat's theorem).

If the function $y = f(x)$ be differentiable at a point x_0 and has an extremum at the point x_0, then $f'(x_0) = 0$.

Theorem 3 (Roll's).

Let the $y = f(x)$ function:

1) continuous on the segment $[a, b]$;
2) differential on the interval (a, b);
3) $f(a) = f(b)$.



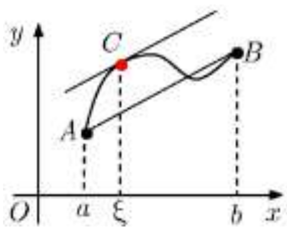
Then *there exists a point* $\xi \in (a, b)$ such that

$$f'(\xi) = 0.$$

Theorem 4 (Lagrange).

Let the function $y = f(x)$ be:

1) continuous on the segment $[a, b]$;
2) differential on the interval (a, b).



Then *there exists a point* $\xi \in (a, b)$ such that

$$f(b) - f(a) = f'(\xi)(b-a).$$

Theorem 5 (Cauchy).

Let the functions $f(x)$ and $g(x)$ are:

1) continuous on the segment $[a,b]$;
2) differentiable on the interval (a,b);
3) $\big(\forall x \in (a,b)\big)\,\{g'(x) \neq 0\}$.

Then *there exists a point* $\xi \in (a,b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Consequences of Lagrange's theorem.

Theorem 6.

Let the function $y = f(x)$ be:

1) differential on the interval (a,b);
2) $\big(\forall x \in (a,b)\big)\,\{f'(x) = 0\}$.

Then

$$f(x) = const \text{ on the interval } (a,b).$$

Theorem 7.

In order for a differentiable on an interval (a,b) function $y = f(x)$ to be non-decreasing (non-increasing) on this interval, it is necessary and sufficient that for any $x \in (a,b)$ is performed

$$f'(x) > 0 \quad (f'(x) < 0).$$

p.1.3. Taylor's theorem

Theorem 8 (Taylor's formula).

If the function $y = f(x)$ is n times continuously differentiable in the around $\mathcal{U}_\delta(x_0)$, then for $x \in \mathcal{U}_\delta(x_0)$ is performed

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{(n)!}(x - x_0)^n + R_n(x),$$

where

$R_n(x) = o((x - x_0)^n), x \to x_0$ is the residual term of the Taylor formula in Peano form,

or

$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}, \xi = x_0 + \theta(x - x_0), 0 < \theta < 1$ is the residual term of the Taylor formula in the Lagrange form.

In the case $x_0 = 0$ Taylor's formula is called *McLaurin's formula* .

Example. Decompose the function $y = e^x$ according to McLaurin's formula.

Decomposition of some functions according to McLaurin's formula.

1) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + o(x^n),\ x \to 0;$

2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + o(x^{2n+1}),\ x \to 0;$

3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n}),\ x \to 0;$

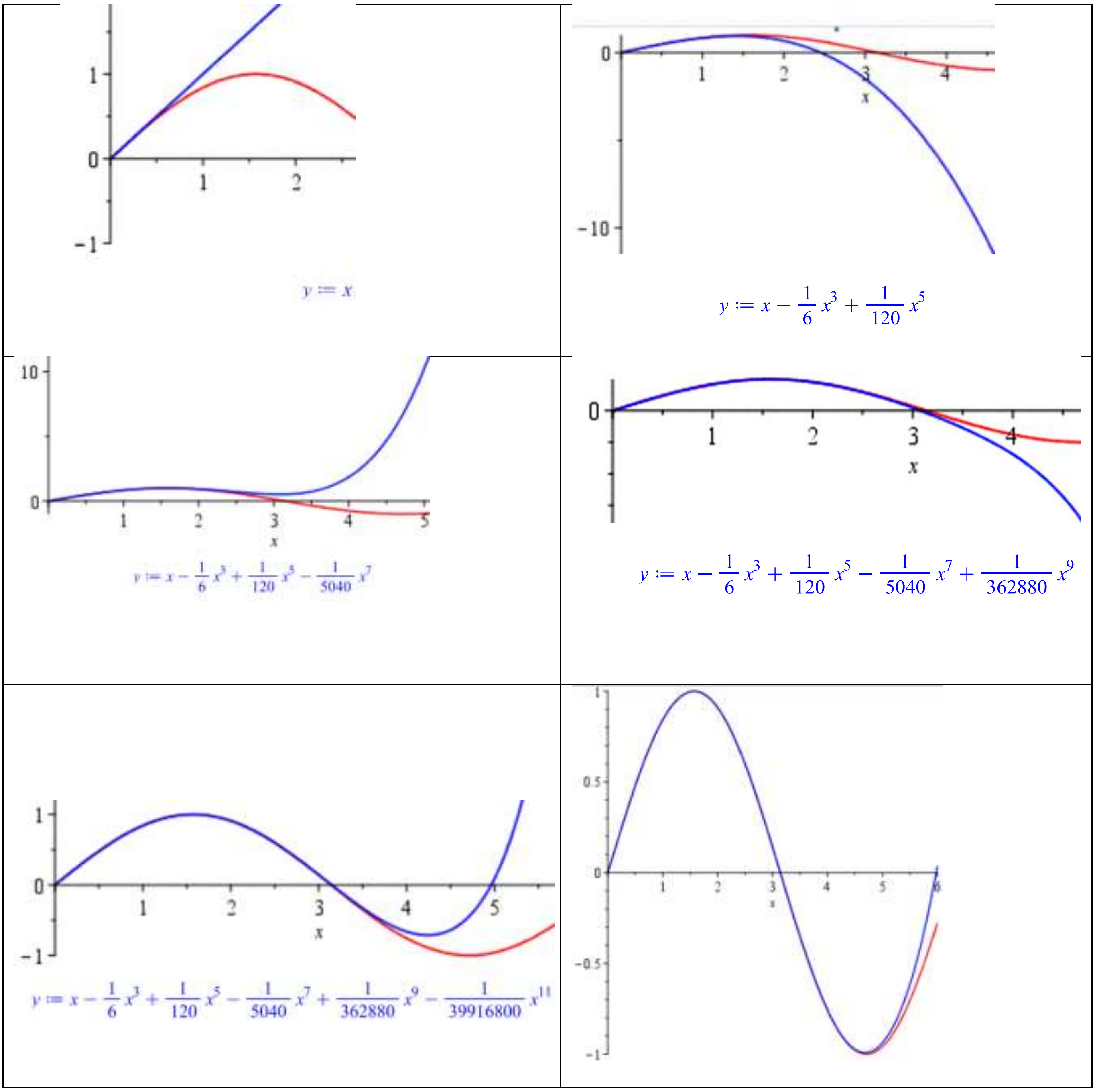
4) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{(-1)^{n+1} x^n}{n} + o(x^n),\ x \to 0.$

5) $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n + o(x^n),\ x \to 0.$

6) $\mathrm{tg}\, x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots + \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} + o(x^{2n-1}).$

7) $\mathrm{arctg}\, x = x - \frac{x^3}{3} + \ldots + \frac{(-1)^n x^{2n+1}}{2n+1} + o(x^{2n+1}).$

An example of the approximation of a function $y = \sin x$ with increasing number of terms in McLaurin's formula.



	$y := x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11} + \frac{1}{6227020800}x^{13} - \frac{1}{1307674368000}x^{15}$

p.1.4. L'Hôpital's rule

Theorem (L'Hôpital's rule).

If:

1) functions $f(x)$ and $g(x)$ defined and differentiated in some punctured around of points x_0;
2) $g'(x) \neq 0$ in all points of this circle;
3) $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$, or $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \infty$;
4) there exists $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)} = L$,

then exists $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)}$, and

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L.$$

Consequence.

If also $\lim\limits_{x\to x_0} f'(x) = \lim\limits_{x\to x_0} g'(x) = 0$ (or $\lim\limits_{x\to x_0} f'(x) = \lim\limits_{x\to x_0} g'(x) = \infty$), and the conditions of the previous theorem apply to the first and second derivatives, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \lim_{x\to x_0} \frac{f''(x)}{g''(x)},$$

etc.

Example.

$$\lim_{x\to 0} \frac{\sin x}{x} = \left[\frac{0}{0}\right] = \lim_{x\to 0} \frac{\sin' x}{x'} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Example.

$$\lim_{x\to 0} \frac{\sin^2 x}{x^2} = \left[\frac{0}{0}\right] = \lim_{x\to 0} \frac{(\sin^2 x)'}{x^{2\prime}} = \lim_{x\to 0} \frac{2\sin x \cos x}{2x} = \lim_{x\to 0} \frac{2\sin x}{2x} = \left[\frac{0}{0}\right] \underset{x\to 0}{=} \lim \frac{(2\sin x)'}{(2x)'} =$$

$$= \lim_{x\to 0} \frac{2\cos x}{2} = 1.$$

p.2. Study of functions and construction of function graphs

p.2.1. Signs of monotonicity of the function

The relationship between the character of monotonicity of a function differentiable on a certain interval and the sign of its derivative was previously established:

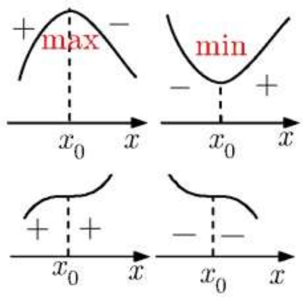
- $f'(x) \geq 0 \Leftrightarrow f(x)$ be non-decreasing;
- $f'(x) \leq 0 \Leftrightarrow f(x)$ be non-increasing;
- $f'(x) > 0 \Rightarrow f(x)$ be increasing;
- $f'(x) < 0 \Rightarrow f(x)$ be decreasing;
- $f'(x) \equiv 0 \Leftrightarrow f(x) \equiv \mathrm{const}$;

p.2.2. Sufficient conditions of extremum

Fermat's theorem was previously formulated, which is a necessary condition for the extremum.

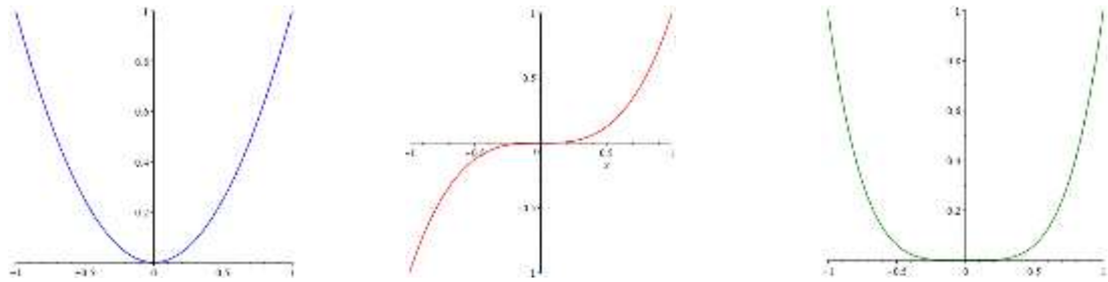
Theorem 10 (Fermat's).
If the function $y = f(x)$ is differentiable at a point x_0 and has an extremum at the point x_0, then $f'(x_0) = 0$.

Theorem 11 (the first sufficient condition for the extremum).
Let the function $y = f(x)$ be continuous on the interval (a, b) and be differentiable in some around $u_\delta^0(x_0)$, then:
1) if, passing through the point x_0, the derivative $f'(x)$ changes sign, then the point x_0 is *the extremum point* of the function $y = f(x)$, and if from "+" to "-", then the point x_0 is *the maximum point*, if from "-" to "+", then the point x_0 is *the minimum point*;
2) if, passing through the point x_0, the derivative $f'(x)$ does not change its sign, then the point x_0 is not an extremum point of the function $y = f(x)$.



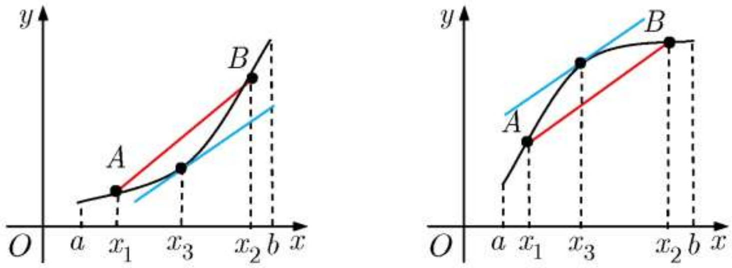
Theorem 12 (the second sufficient condition for the extremum).
Let the function $y = f(x)$ be defined on the interval (a, b), $x_0 \in (a, b)$, $f'(x_0) = 0$, and $f''(x_0) \neq 0$. Then x_0 is the extremum point of the function $y = f(x)$, and:
if $f''(x_0) > 0$, then x_0 is *the minimum point*;
if $f''(x_0) < 0$, then x_0 is *the maximum point*.

Example. $y = x^2$; $y = x^3$; $y = x^4$.



p.2.3. Convexity of the function

Definition. A function $y = f(x)$ defined and differentiable on the interval (a, b) is called *convex downward* (*convex upward*), if the graph of the function lies not lower (not higher) than its tangent drawn at any point of the interval (a, b).



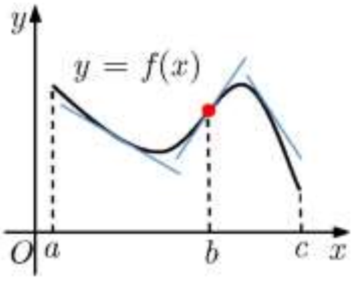
Theorem 13.

Let the function $y = f(x)$ be twice differentiable on the interval (a, b).

If $\big(\forall x \in (a, b)\big)\{f''(x) \geq 0\}$, then the function $y = f(x)$ is convex downward, and

if $\big(\forall x \in (a, b)\big)\{f''(x) \leq 0\}$, then the function $y = f(x)$ is convex upward on the interval (a, b).

Definition. A point x_0 from the domain of the function $y = f(x)$ is called *an inflection point* of this function, if there exists such $\delta > 0$ that in the one-sided δ-arounds of the point x_0, the function $y = f(x)$ has a different character of convexity.



Theorem 14 (necessary condition for inflection).

Let the function $y = f(x)$ be twice continuously differentiable on the interval (a, b) and let the point $x_0 \in (a, b)$ be the inflection point of the function $y = f(x)$. Then $f''(x_0) = 0$.

Example. $y = x^3$.

Theorem 15 (the first sufficient condition for inflection).

Let the function $y = f(x)$ be defined on the interval (a, b) and twice differentiable in some around of the point x_0. If the second derivative $f''(x)$ changes sign when passing through a point x_0, then the point x_0 is an inflection point of the function $y = f(x)$.

Theorem 16 (the second sufficient condition for inflection).

Let the function $y = f(x)$ be defined on the interval (a, b), $x_0 \in (a, b)$, $f''(x_0) = 0$, and $f'''(x_0) \neq 0$. Then the point x_0 is the inflection point of the function $y = f(x)$.

Example. $y = x^3$; $y = x^4$; $y = x^5$.

p.2.4. Asymptotes of the function

Definition. If at least one of the one-sided limits $\lim\limits_{x \to x_0 - 0} f(x)$, $\lim\limits_{x \to x_0 + 0} f(x)$ is equal to ∞, then the straight line $x = x_0$ is called *a vertical asymptote* of function $y = f(x)$.

Definition. Line $y = kx + b$ is called *a slant (or oblique) asymptote* of function $y = f(x)$ if

$$\lim_{x\to\infty}\big(f(x)-(kx+b\ )\big)=0.$$

Theorem 17.

In order for a function $y=f(x)$ to have a slant asymptote, it is necessary and sufficient that the limits exist

$$\lim_{x\to\infty}\frac{f(x)}{x}=k,\quad \lim_{x\to\infty}(f(x)-kx)=b.$$

p.2.5. A complete scheme for constructing a graph of a function

Researching and construction of the function graph are carried out according to the following scheme.

1) Find the domain (if possible also the domain of values) of the function.
2) Establish possible symmetries of the graph (study the function for parity, oddity, periodicity).
3) Find possible points of intersection of the graph with the coordinate axes.
4) Determine the possible breakpoints of the function and its asymptotes.
5) Using the first derivative, monotonicity intervals and extremum points are found.
6) Using the second derivative, convexity intervals and inflection points are found.
7) Build a graph of the function.

Example. Plot the graph of the function $y=\frac{2x^3}{x^2-4}$.

1) $D(x)=\{-\infty;-2\}\cup\{-2;2\}\cup\{2;+\infty\}$.
2) $f(-x)=-f(x)\ \Rightarrow$ the function is odd, so its graph is symmetric with respect to the origin.
3) $y(0)=0$.

We analyze: at $x\in(0,2)\ f(x)<0$, at $x\in(2,+\infty)\ f(x)>0$.

4) A straight line $x=2$ is a vertical asymptote, because

$$\lim_{x\to2-0}\frac{2x^3}{x^2-4}=-\infty;\quad \lim_{x\to2+0}\frac{2x^3}{x^2-4}=+\infty.$$

Next, we look for an slant asymptote $y=kx+b$:

$$k=\lim_{x\to\infty}\frac{f(x)}{x}=k=\lim_{x\to\infty}\frac{2x^2}{x^2-4}=2,$$

$$b=\lim_{x\to\infty}(f(x)-kx)=\lim_{x\to\infty}\left(\frac{2x^3}{x^2-4}-2x\right)=\lim_{x\to\infty}\frac{8x}{x^2-4}=0,$$

therefore, the graph of the function has a slant asymptote $y=2x$.

Let's find the first derivative:

$$y'=\frac{6x^2(x^2-4)-4x^4}{(x^2-4)^2}=\frac{2x^2(x^2-12)}{(x^2-4)^2}.$$

$y'=0$ at $x=0$, $x=\pm2\sqrt{3}$, and at $x\to2$ $y\to\infty$. Consider the behavior of the derivative of $x>0$. Since $y'\big(2\sqrt{3}-0\big)<0$, and $y'\big(2\sqrt{3}-0\big)>0$, then the point with coordinates $\big(2\sqrt{3};6\sqrt{3}\big)$ is the minimum point of the function and on the intervals $(0;2)$ and $\big(2;\ 2\sqrt{3}\big)$ the function decreases, and on the interval $\big(2\sqrt{3};+\infty\big)$ it increases.

The second derivative:

$$y''=\frac{16x(x^2+12)}{(x^2-4)^3}.$$

Converts to zero at $x=0$, and to infinity at $x=2$. In between $x\in(0;2)$ $y''<0$, therefore the graph of the function is convex up, and on the interval $(2;+\infty)$ it is convex down. Origin $O(0;0)$ is the inflection point of the function.

Using the research conducted above, we construct a graph of the function: