Quality and creativity
KATOOMBA HIGH SCHOOL

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2023

YEAR 12
YEARLY
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks

52%

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

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Year 12 Mathematics Extension 2 Section I - Answer Sheet

Student Name/Number Malachi Haskew

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

- If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct ↙

A ● B ● C ○ D ○

✓ 1. A ○ B ○ C ● D ○

2. (A) ○ B ○ C ● D ○

✓ 3. A ● B ○ C ○ D ○

4. A ● B ○ (C) ○ D ○

✓ 5. A ○ B ● C ○ D ○

✓ 6. A ○ B ○ C ○ D ●

✓ 7. A ● B ○ C ○ D ○

8. A ● B ○ C ○ (D) ○

9. A ○ (B) ○ C ● D ○

10. A ○ B ● C ○ (D) ○

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. Which of the following is an expression for $\int \frac{x}{\sqrt[3]{x^2+1}}dx$?

(A) $\frac{3}{2}\sqrt[3]{(x^2+1)^2}+C$

(B) $\frac{1}{2}\sqrt[3]{(x^2+1)^2}+C$

(C) $\frac{3}{4}\sqrt[3]{(x^2+1)^2}+C$

(D) $\frac{3}{2}\sqrt[3]{(x^2+1)^2}+C$ ←

2. What is the quadratic equation with solutions $2+3i$ and $2-3i$?

(A) $z^2-4z+13=0$

(B) $z^2-4z-13=0$

(C) $z^2+4z-5=0$

(D) $z^2+4z-13=0$

3. If two vectors are $\underset{\sim}{u}=\underset{\sim}{i}-\underset{\sim}{j}-\underset{\sim}{k}$ and $\underset{\sim}{v}=4\underset{\sim}{i}+12\underset{\sim}{j}-3\underset{\sim}{k}$, what is their scalar product ?

(A) -5

(B) 19

(C) $4\underset{\sim}{i}-12\underset{\sim}{j}+3\underset{\sim}{k}$

(D) $5\underset{\sim}{i}+11\underset{\sim}{j}-4\underset{\sim}{k}$

4. Given that $z=3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$, what is the value of $\bar{z}^3$?

(A) $9\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right)$

(B) $9\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

(C) $27\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right)$

(D) $27\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

5. What is the magnitude of the vector $\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$?

(A) $\sqrt{17}$

(B) $\sqrt{35}$

(C) 17

(D) 35

6. A particle moves in simple harmonic motion such that $v^2 + 9x^2 = k$.
What is the period of the particle's motion ?

(A) $\frac{2\pi}{k}$

(B) 3π

(C) $\frac{3k}{2\pi}$

(D) $\frac{2\pi}{3}$

$\frac{1}{2}v^2 = \frac{1}{2}(-9x^2 + k)$

$= -\frac{9}{2}x^2 + \frac{1}{2}k$

$a = \frac{d}{dx}(-\frac{9}{2}x^2 + \frac{1}{2}k)$

$= -\frac{9}{2}x$

7. The converse of $P \Rightarrow Q$ is:

(A) $Q \Rightarrow P$

(B) $Q \Leftrightarrow Q$

(C) (not P) $\Leftrightarrow$ (not Q)

(D) (not Q) $\Rightarrow$ (not P)

8. The velocity of a body moving in a straight line is given by $v = f(x)$ where x metres is the distance from origin and v is the velocity in metres per second. The acceleration of the body in ms^{-2} is given by:

(A) $f'(x)$

(B) $f'(v)$

(C) $xf'(x)$

(D) $f(x)f'(x)$

$\frac{1}{2}v^2 = \frac{1}{2}f(x)f(x)$

$= \frac{d}{dx}(\frac{1}{2}f(x)^2$

$= f'(x)$

9. What are the values of real numbers p and q such that $(2 - i)$ is a root of the equation $z^3 + pz + q = 0$?

(A) $p = -11$ and $q = -20$

(B) $p = -11$ and $q = 20$

(C) $p = 11$ and $q = -20$

(D) $p = 11$ and $q = 20$

$z + i = \alpha$

$z - i = \beta$

$z + \dot{i} + z - \dot{i} = -$

10. What is the value of $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1 - \cos^2 x}} dx$?

(A) $-\pi$

(B) $-\frac{\pi}{2}$

(C) π

(D) $\frac{\pi}{2}$

$-\int_0^{\frac{\pi}{2}} \frac{-\sin x}{\sin x} dx$

$-[\sin^{-1}(\frac{\cos x}{1})]_0^{\pi/2}$

Section II

90 marks
Attempt questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) If $z = 4 - i$, express the following in the form $a + ib$ where a and b are real.

(i) $\overline{iz}$ **1**

(ii) $\dfrac{1}{z}$ **1**

(b) The point A has position vector $\overrightarrow{OA} = 2\underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k}$ relative to an origin O. **2**
Find a unit vector parallel to $\overrightarrow{OA}$.

(c) Find the value of $\displaystyle\int_0^{\sqrt{5}} \frac{x}{\sqrt{x^2+4}}\,dx$. **2**

no brackets?

(d) Let two complex numbers be $z_1 = 2\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$ and $z_2 = 2i$.

(i) On an Argand diagram sketch the vectors OA and OB to represent z_1 and z_2 respectively. **1**

(ii) Draw the vectors z_1+z_2 and z_1-z_2 on the same Argand diagram. **1**

(iii) What is the exact value of $\arg(z_1+z_2)$? **2**

(e) Find $\displaystyle\int \frac{1}{\sqrt{12+4x-x^2}}\,dx$. **2**

(f) Show that if $x \neq 1$ then $1 + x + x^2 + \ldots + x^n = \dfrac{x^{n+1}-1}{x-1}$ for $n \geq 1$. **3**

Question 12 (15 marks) **Marks**

(a) Use integration by parts to evaluate $\int x\, e^{-x} dx$ **3**

(b) (i) What is the expansion of $(1 + ia)^4$ in ascending powers of a ? **1**

(ii) Hence find the values of a such that. $(1 + ia)^4$ is real. **2**

(c) (i) Find real numbers A, B, C and D such that **2**

$$\frac{5x^3 - 3x^2 + 2x - 1}{x^2(x^2 + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}$$

(i) Hence find $\int \frac{5x^3 - 3x^2 + 2x - 1}{x^2(x^2 + 1)} dx$ **2**

double .

(d) A particle is moving in simple harmonic motion with its acceleration given by:

$$\ddot{x} = -12\sin 2t$$

Initially, the particle is at the origin and has a positive velocity of 6 m/s.

(i) Find the equation for the particle's velocity. **2**

(ii) Show that $\ddot{x} = -4x$. **3**

Question 13 (15 marks) **Marks**

(a) By completing the square, find $\int \frac{1}{9x^2+6x+5}dx$ **3**

(b) Lines l_1 and l_2 are given below, relative to a fixed origin O.

l_1: $\underset{\sim}{r} = (11\underset{\sim}{i} + 2\underset{\sim}{j} + 17\underset{\sim}{k}) + \lambda(-2\underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k})$

l_2: $\underset{\sim}{r} = (-5\underset{\sim}{i} + 11\underset{\sim}{j} + p\underset{\sim}{k}) + \mu(-3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$

where λ and μ are scalar parameters.

(i) Given that line l_1 and l_2 intersect, find the value of p. **3**

(ii) Hence find the point of intersection of line l_1 and l_2. **1**

(c) Show that $z = 2i$, $w = \sqrt{3} - i$ and $v = -\sqrt{3} - i$ are vertices of an equilateral triangle. **2**

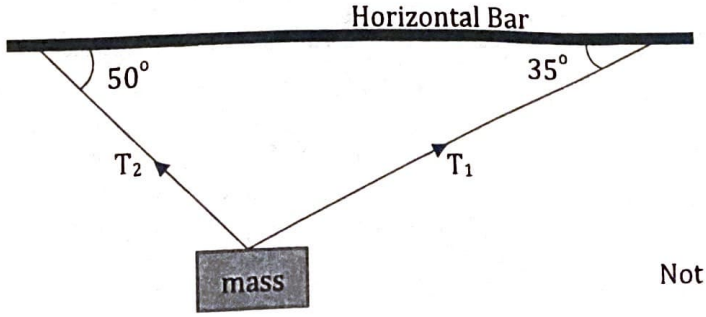
(d) A particle is moving in a straight line in simple harmonic motion. If the amplitude of the motion is 2 cm and the period of the motion is 4 seconds, calculate the:

(i) maximum velocity of the particle. **2**

(ii) speed of the particle when it is 1 cm from the centre of the motion. **2**

(e) Two light strings of different lengths connect a mass to a horizontal bar, as shown below. **2**

Given the tension in the longer string is T_1 and the tension in the shorter string is T_2, find the ratio of the tensions $\frac{T_1}{T_2}$ to 3 significant digits.



Not to scale

Question 14 (15 marks) **Marks**

(a) Express in modulus-argument form

(i) $-1+i$ **2**

(ii) $(-1+i)^n$, where n is a positive integer. **1**

(b) Given $a+b=m$, prove that, for $a>0, b>0$ and $m>0$ **3**

$$\frac{1}{a}+\frac{1}{b}>\frac{4}{m}$$

(c) A submarine of mass m is propelled through the water with a force of magnitude F.
It experiences resistance of magnitude kv for some constant $k>0$.
The submarine initially starts from rest. Velocity is measured in km/h and time is in hours.

(i) Show that the velocity of the submarine is given by **3**

$$v=\frac{F}{k}\left(1-e^{-\frac{kt}{m}}\right)$$

(ii) How long will it take the submarine to achieve half its terminal velocity, in terms of m and k? **2**

(iii) Find an equation for the displacement of the submarine at any given time, t, in terms of m, k and F **2**

(d) Prove by contradiction that if n is a positive integer then $\sqrt{4n-2}$ is always irrational. **2**

Question 15 (15 marks) **Marks**

(a) Solve the equation $z^2 = |z|^2 - 4$ **3**

(b) Use mathematical induction to show that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all positive integers $n \geq 1$. **4**

(c) Position vectors of the points A, B and C, relative to an origin O, are $-\underset{\sim}{i} - \underset{\sim}{j}$, $\underset{\sim}{j} + 2\underset{\sim}{k}$, and $4\underset{\sim}{i} + \underset{\sim}{k}$ respectively.

(i) Find $\overrightarrow{AB}$. **1**

(ii) Find $|\overrightarrow{AB}|$. **1**

(iii) Prove that $\angle ABC$ is a right angle. **3**

(d) A particle of mass m is moving in a straight line under the action of a force. **3**

$$F = \frac{m}{x^3}(6 - 10x)$$

What is the velocity in any position, if the particle starts from rest at $x = 1$?

Question 16 (15 marks) **Marks**

(a) (i) Let $I_n = \int_0^1 (1-x^r)^n dx$ where $r > 0$ for $n = 1, 2, 3, \ldots$ **3**

Show that $I_n = \dfrac{nr}{nr+1} I_{n-1}$

(ii) Hence or otherwise, find the exact value of $\int_0^1 (1-x^{\frac{3}{2}})^3 dx$ **2**

(b) Let a, b and c be real numbers such that $a > b > c > 1$.
Show that $a^{a-b}b^{b-c} > c^{a-c}$. **1**

(c) On an Argand diagram, sketch the inequalities $|z-1| \leq 3$ and $\text{Im}(z) \geq 3$ and indicate where they hold simultaneously. **3**

(d) The point A, with coordinates $(0, a, b)$ lies on the line l_1, which has the equation:

l_1: $\underset{\sim}{r} = 6\underset{\sim}{i} + 19\underset{\sim}{j} - \underset{\sim}{k} + \lambda(\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k})$

(i) Find the values of a and b. **2**

(ii) The point P lies on l_1 and is such that OP is perpendicular to l_1, where O is the origin. Find the position vector of point P. **4**

End of paper