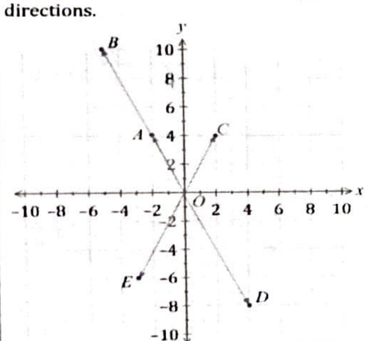
ACE Examination 2023
Year 12 Mathematics Extension 1 Yearly Examination
Worked solutions and marking guidelines

Section I

	Solution	Criteria
1.	$-1 \le 3x \le 1$ $-\frac{1}{3} \le x \le \frac{1}{3}$	1 Mark: A
2.	$\int \sin^2 4x dx = \frac{1}{2}\int (1-\cos 8x)dx$ $= \int \frac{(1-\cos 8x)}{2} dx$	1 Mark: B
3.	Check each option by differentiation Option (C) $N = 500 + 100e^{0.1t}$ also $100e^{0.1t} = N - 500$ $\frac{dN}{dt} = 100 \times 0.1e^{0.1t}$ $= 0.1 \times 100e^{0.1t}$ $= 0.1(N-500)$	1 Mark: C
4.	Graphically vector $\overrightarrow{OE}$ and $\overrightarrow{OC}$ are parallel but in opposite directions. 	1 Mark: C
5.	$\int_{\frac{3}{\sqrt{2}}}^{3} \frac{4}{\sqrt{9-x^2}} dx = 4 \times \int_{\frac{3}{\sqrt{2}}}^{3} \frac{1}{\sqrt{9-x^2}} dx$ $= 4\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_{\frac{3}{\sqrt{2}}}^{3}$ $= 4\left[\sin^{-1}\left(\frac{3}{3}\right) - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$ $= 4\left[\frac{\pi}{2} - \frac{\pi}{4}\right]$ $= \pi$	1 Mark: D

	Solution	Criteria
6.	Step 2: Assume true for $n = k$ $S_k = \frac{k(k+1)(2k+1)}{6}$ Step 3: To prove true for $n = k+1$ $S_{k+1} = \frac{(k+1)(k+2)(2k+3)}{6}$ $S_k + T_{k+1} = S_{k+1}$ LHS $= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ $= \frac{(k+1)}{6}[k(2k+1) + 6(k+1)]$ $= \frac{(k+1)}{6}(2k^2+7k+6)$ $= \frac{(k+1)(k+2)(2k+3)}{6}$ = RHS	1 Mark: D
7.	$\frac{dy}{dx} = \frac{2x}{y}$ $\int y dy = \int 2x dx$ $\frac{1}{2}y^2 = x^2 + C_1$ $y^2 = 2x^2 + C_2$	1 Mark: B
8.	Two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular if and only if $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ $\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1x_2 + y_1y_2$ $0 = 5 \times 4 + 2 \times (-x)$ $x = 10$	1 Mark: D
9.	Gravity is constant. Therefore the acceleration of the projectile remains constant during its entire flight.	1 Mark: A
10.	$A = \int_a^b y dx$ $= \int_1^4 x^3 - 2x^2 dx$ $= [\frac{1}{4}x^4 - \frac{2}{3}x^3]_1^4$ $= \left[\left(\frac{1}{4}4^4 - \frac{2}{3}4^3\right) - \left(\frac{1}{4}1^4 - \frac{2}{3}1^3\right)\right]$ $= 21\frac{3}{4}$ square units	1 Mark: B

Section II

11(a)	Given $\frac{dh}{dt} = -0.01\ \text{ms}^{-1}$ and $r = 0.6$ m. Require $\frac{dV}{dt}$ $V = \pi r^2 h$ $\frac{dV}{dh} = \pi r^2$ $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $= \pi r^2 \times -0.01$ $= \pi \times 0.6^2 \times -0.01$ $= -0.011309\ldots$ ≈ -0.0113 ∴ The rate of change of volume is decreasing at 0.0113 m^3s^{-1}	2 Marks: Correct answer. 1 Mark: Makes significant progress using the chain rule.
11(b) (i)	$\underset{\sim}{u} + \underset{\sim}{v} = (\underset{\sim}{i} + 4\underset{\sim}{j}) + (4\underset{\sim}{i} - 2\underset{\sim}{j})$ $= 5\underset{\sim}{i} + 2\underset{\sim}{j}$	1 Mark: Correct answer.
11(b) (i)	$3\underset{\sim}{u} - \underset{\sim}{v} = 3(\underset{\sim}{i} + 4\underset{\sim}{j}) - (4\underset{\sim}{i} - 2\underset{\sim}{j})$ $= -\underset{\sim}{i} + 14\underset{\sim}{j}$	1 Mark: Correct answer.
11(c)	Step 1: To prove true for $n = 1$ LHS $= 1 \times 2^{1-1} = 1$ RHS $= 1 + (1-1)2^1 = 1$ Result is true for $n = 1$ Step 2: Assume true for $n = k$ $S_k = 1 + (k-1)\,2^k$ Step 3: To prove true for $n = k+1$ $S_{k+1} = 1 + k\,2^{k+1}$ $S_k + T_{k+1} = S_{k+1}$ LHS $= 1 + (k-1)\,2^k + (k+1)\,2^k$ $= 1 + 2^k(k - 1 + k + 1)$ $= 1 + 2k \times 2^k$ $= 1 + k\,2^{k+1}$ $=$ RHS Step 4: True by induction	3 marks: Correct answer. 2 marks: Proves the result true for $n = 1$ and attempts to use the result of $n = k$ to prove the result for $n = k + 1$. 1 mark: Proves the result true for $n = 1$.
11(d) (i)	$p = 0.4, n = 7$ $P(X = 4) = {}^7C_4\, 0.4^4\, 0.6^{7-4}$ $= 0.1935$	1 Mark: Correct answer.
11(d) (ii)	$E(X) = np$ $= 7 \times 0.4$ $= 2.8$	1 Mark: Correct answer.

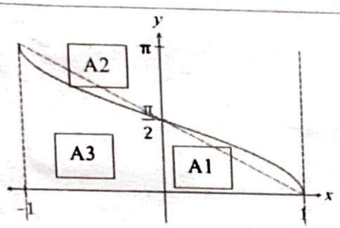
11(d) (iii)	$\text{Var}(X) = np(1-p)$ $= 7 \times 0.4(1 - 0.4)$ $= 1.68$	1 Mark: Correct answer.
11(e)	$\int \frac{1}{x^2 + 2x + 2}\,dx = \int \frac{1}{(x+1)^2 + 1}\,dx$ $= \tan^{-1}(x+1) + C$	2 Marks: Correct answer. 1 Mark: Completes the square.
11(f)	Let $\tan\frac{x}{2} = t$ then $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$ $\frac{7\sin x}{2} + 2\cos x = 4$ $\frac{7}{2} \times \frac{2t}{1+t^2} + 2 \times \frac{1-t^2}{1+t^2} = 4$ $\frac{7t + 2 - 2t^2}{1+t^2} = 4$ $7t + 2 - 2t^2 = 4 + 4t^2$ $6t^2 - 7t + 2 = 0$ $(3t-2)(2t-1) = 0$ $t = \frac{2}{3}$ or $t = \frac{1}{2}$ $\tan\frac{x}{2} = \frac{2}{3}$ or $\tan\frac{x}{2} = \frac{1}{2}$ $\therefore x = 53°8'$ or $67°23'$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Substitutes using the t-formula or equivalent merit.
12(a)	$u = 5 - x^2$ $\frac{du}{dx} = -2x$ $-\frac{1}{2}du = x\,dx$ $\int \frac{x}{(5-x^2)^3}\,dx = -\frac{1}{2}\int \frac{1}{u^3}\,du$ $= -\frac{1}{2} \times -\frac{1}{2}u^{-2} + C$ $= \frac{1}{4(5-x^2)^2} + C$	2 Marks: Correct answer. 1 Mark: Sets up the substitution correctly.
12(b)	Combination: selection from a group of items and the order is not important. ${}^9C_4 \times {}^7C_2 = 2646$	1 Mark: Correct answer.
12(c) (i)	Remainder theorem $P(x) = 2x^3 + 3x^2 - 29x - 60$ $P(-2) = 2(-2)^3 + 3(-2)^2 - 29(-2) - 60$ $= -6$ ∴ The remainder is −6.	1 mark: Correct answer.

12(c) (ii)	Factor theorem $P(x) = 2x^3 + 3x^2 - 29x - 60$ $P(-3) = 2(-3)^3 + 3(-3)^2 - 29(-3) - 60$ $= 0$ $\therefore (x+3)$ is a factor of $P(x)$	1 mark: Correct answer.
12(c) (iii)	$$\begin{array}{r} 2x^2 - 3x - 20 \\ x+3\overline{)\,2x^3 + 3x^2 - 29x - 60} \\ \underline{2x^3 + 6x^2} \qquad\qquad\quad \\ -3x^2 \qquad\qquad\quad \\ \underline{-3x^2 - 9x} \qquad\quad \\ -20x \qquad \\ \underline{-20x - 60} \\ 0 \end{array}$$ $2x^3 + 3x^2 - 29x - 60 = (x+3)(2x^2 - 3x - 20)$ $= (x+3)(2x+5)(x-4)$ $\therefore P(x) = (x+3)(2x+5)(x-4)$	3 marks: Correct answer. 2 marks: Correctly completes the division. 1 mark: Shows some understanding of the process of division.
12(d) (i)	$\sqrt{3}\sin x + 3\cos x = R\sin(x+\alpha)$ $= R\sin x\cos\alpha + R\cos x\sin\alpha$ $R\cos\alpha = \sqrt{3}$ ① $R\sin\alpha = 3$ ② Equation ② divided by equation ① $\tan\alpha = \frac{3}{\sqrt{3}} = \sqrt{3}$ $\alpha = \frac{\pi}{3}$ Squaring and adding the equations $R^2(\sin^2\alpha + \cos^2\alpha) = 3^2 + (\sqrt{3})^2$ $R^2 = 12$ $R = 2\sqrt{3}\ (R > 0)$ $\therefore \sqrt{3}\sin x + 3\cos x = 2\sqrt{3}\sin\left(x + \frac{\pi}{3}\right)$	2 Marks: Correct answer. 1 Mark: Shows some understanding.
12(d) (ii)	$2\sqrt{3}\sin\left(x + \frac{\pi}{3}\right) = \sqrt{3}$ $\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$ $x + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \ldots$ $x = \frac{\pi}{2}$ or $\frac{11\pi}{6}$ for $0 \le x \le 2\pi$.	2 Marks: Correct answer. 1 Mark: Finds one solution or makes some progress using part (a).

12(e)	$V = \pi\int_a^b y^2\,dx$ $= \pi\int_0^\pi \left(3\cos\frac{x}{2}\right)^2 dx$ $= \pi\int_0^\pi 9\cos^2\frac{x}{2}\,dx$ $= \frac{9\pi}{2}\int_0^\pi (1 + \cos x)dx$ $= \frac{9\pi}{2}[x + \sin x]_0^\pi$ $= \frac{9\pi}{2}[(\pi + \sin\pi) - (0 + \sin 0)]$ $= \frac{9\pi^2}{2}$ cubic units	3 Marks: Correct answer. 2 Marks: Applies the double angle trig identity. 1 Mark: Sets up the integral for volume.
13(a) (i)	$T = A + Be^{-kt}$ or $Be^{-kt} = T - A$ $\frac{dT}{dt} = -kBe^{-kt}$ $= -k(T - A)$	1 Mark: Correct answer.
13(a) (ii)	Initially $t = 0$ $T = 60$ with $A = 12$ (surrounding temperature) $T = A + Be^{-kt}$ $60 = 12 + Be^{-k\times 0}$ $B = 48$ Also $t = 25$ $T = 30$ $30 = 12 + 48e^{-k\times 25}$ $e^{-25k} = \frac{18}{48} = \frac{3}{8}$ $-25k = \ln\frac{3}{8}$ $k = -\frac{1}{25}\ln\frac{3}{8}$ $= \frac{1}{25}\ln\frac{8}{3}$ We need to find t when $T = 15$ $15 = 12 + 48e^{-k\times t}$ $e^{-kt} = \frac{3}{48} = \frac{1}{16}$ $-kt = \ln\frac{1}{16}$ $t = \frac{1}{k}\ln 16$ $= \frac{25 \times \ln 16}{\ln\frac{8}{3}}$ $= 70.6695\ldots \approx 71$ *minutes.* $\therefore$ 71 minutes for the object's temperature to reach 15°C.	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Finds the value of B.

13(b)	$x = \sin^2\theta$ $dx = 2\sin\theta\cos\theta d\theta$ $\int_0^{\frac{1}{2}}\sqrt{\frac{x}{1-x}}\,dx = \int_0^{\frac{\pi}{4}}\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}} \times 2\sin\theta\cos\theta d\theta$ $= \int_0^{\frac{\pi}{4}}\sqrt{\frac{\sin\theta}{\cos\theta}} \times 2\sin\theta\cos\theta d\theta$ $= 2\int_0^{\frac{\pi}{4}}\sin^2\theta d\theta$ $= \int_0^{\frac{\pi}{4}}(1-\cos2\theta)d\theta$ $= \left[\theta - \frac{1}{2}\sin2\theta\right]_0^{\frac{\pi}{4}}$ $= \left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(0 - \frac{1}{2}\sin0\right)$ $= \frac{\pi}{4} - \frac{1}{2}$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Obtains the correct integrand in terms of θ or equivalent merit.
13(c) (i)	Let p be the probability of hitting the target ($p = 0.87$) The shooter takes fifty shots ($n = 50$) $P(X = x) = {}^nC_x\, p^x(1-p)^{n-x}$ $P(X = 40) = {}^{50}C_{40}(0.87)^{40}(0.13)^{50-40}$ $\approx 0.053934 \ldots.$ $= 0.0539$	1 Mark: Correct answer.
13(c) (ii)	Misses at most 2 targets then $x = 48, 49$ *and* 50. $P(X \geq 48) = P(X = 48) + P(X = 49) + P(X = 50)$ $= {}^{50}C_{48}(0.87)^{48}(0.13)^{50-48} + {}^{50}C_{49}(0.87)^{49}(0.13)^{50-49}$ $+{}^{50}C_{50}(0.87)^{50}(0.13)^{50-50}$ $= 0.0338941 \ldots$ ≈ 0.0339	2 Marks: Correct answer. 1 Mark: Shows some understanding.
13(d)	$\frac{d}{dx}(x\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$	2 Marks: Correct answer. 1 Mark: States the derivative of $\sin^{-1}x$.
13(e) (i)	$\lvert \underset{\sim}{a} \rvert = \sqrt{x^2+y^2}$ $= \sqrt{6^2+(-3)^2}$ $= \sqrt{45} = 3\sqrt{5}$	1 mark: Correct answer.
13(e) (ii)	$\lvert \underset{\sim}{b} \rvert = \sqrt{x^2+y^2}$ $= \sqrt{(-1)^2+5^2}$ $= \sqrt{26}$	1 Mark: Correct answer.
13(e) (iii)	$\underset{\sim}{a} \cdot \underset{\sim}{b} = x_1\, x_2 + y_1\, y_2$ $= 6 \times (-1) + (-3) \times 5$ $= -21$	1 Mark: Correct answer.

14(a) (i)	To find the horizontal range. At P we know $y = 0$ $0 = -\frac{1}{2}gt^2 + Vt\sin\theta$ $0 = t\left(V\sin\theta - \frac{1}{2}gt\right)$ $t = 0$ or $t = \frac{2V\sin\theta}{g}$ To find the OP or horizontal range $x = Vt\cos\theta$ $OP = V\left(\frac{2V\sin\theta}{g}\right)\cos\theta$ $= \frac{2V^2\sin\theta\cos\theta}{g}$ $= \frac{V^2\sin2\theta}{g}$ $(\sin2\theta = 2\sin\theta\cos\theta)$	2 Marks: Correct answer. 1 Mark: Finds $t = \frac{2V\sin\theta}{g}$ *or* equivalent merit.
14(a) (ii)	$\frac{dx}{dt} = V\cos\theta$ $\frac{dy}{dt} = V\sin\theta - gt$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ $= \frac{V\sin\theta - gt}{V\cos\theta}$ When $\theta = \frac{\pi}{3}$ and $t = \frac{2V}{\sqrt{3}g}$ $\frac{dy}{dx} = \frac{V\sin\frac{\pi}{3} - g\left(\frac{2V}{\sqrt{3}g}\right)}{V\cos\frac{\pi}{3}}$ $= \frac{V \times \frac{\sqrt{3}}{2} - \frac{2V}{\sqrt{3}}}{V \times \frac{1}{2}}$ $= -\frac{1}{\sqrt{3}}$ Let α be the angle between the curve and the positive direction of the x-axis at that time. $\tan\alpha = -\frac{1}{\sqrt{3}}$ $\alpha = \frac{5\pi}{6}$ from positive x axis	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Finds $\frac{dy}{dx}$ or equivalent *merit*.
14(b)	$y = e^{mx}, \frac{dy}{dx} = me^{mx}, \frac{d^2y}{dx^2} = m^2e^{mx}$ $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$ $m^2e^{mx} + me^{mx} - 6e^{mx} = 0$ $(m^2 + m - 6)e^{mx} = 0$ $(m-2)(m+3)e^{mx} = 0$ $\therefore m = 2$ or $m = -3$	2 Marks: Correct answer. 1 Mark: Shows some understanding.

14(c) (i)	$y = \cos^{-1}x$ is symmetrical about the red dotted line shown opposite. $A = \frac{1}{2}bh$ $= \frac{1}{2} \times 2 \times \pi$ $= \pi$ square units 	2 Marks: Correct answer. 1 Mark: Recognises the graph is symmetrical or equivalent merit.
	Alternate method - integrate wrt y. x = cos y $A_1 = \int_0^{\frac{\pi}{2}} \cos y\ dy$ $= [\sin y]_0^{\frac{\pi}{2}}$ $= \sin\frac{\pi}{2} - \sin 0$ $= 1$ $A_2 = A_1 = 1$ by symmetry $A_3 = 1 \times \pi - A_2 = \pi - 1 =$ Hence area under curve = $A_1 + A_3$ $= 1 + \pi - 1$ $= \pi$ square units	
14(c) (ii)	$y = \cos^{-1}x$ $x = \cos y$ $x^2 = \cos^2 y$ $= \frac{1}{2}[\cos 2y + 1]$ $V = 2 \times \int_0^{\frac{\pi}{2}} \pi x^2\, dy$ $= \int_0^{\frac{\pi}{2}} \pi[\cos 2y + 1]\, dy$ $= \pi\left[\frac{1}{2}\sin 2y + y\right]_0^{\frac{\pi}{2}}$ $= \pi \times \left[\frac{1}{2} \times 0 + \frac{\pi}{2}\right]$ $= \frac{\pi^2}{2}$ cubic units	3 marks: Correct answer. 2 marks: Makes significant progress. 1 mark: Sets up the integral using the double angle.

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14(d)	Step 1: To prove true for $n = 1$ $8^{2\times1+1} + 6^{2\times1-1} = 8^3 + 6^1$ $= 518 = 7 \times 74$ Result is true for $n = 1$ Step 2: Assume true for $n = k$ $8^{2k+1} + 6^{2k-1} = 7m$ (1) Step 3: To prove true for $n = k + 1$ $8^{2(k+1)+1} + 6^{2(k+1)-1} = 7p$ LHS $= 8^{2(k+1)+1} + 6^{2(k+1)-1}$ $= 8^{2k+3} + 6^{2k+1}$ $= 8^2 \times 8^{2k+1} + 6^2 \times 6^{2k-1}$ $= 8^2 \times 8^{2k+1} + 8^2 \times 6^{2k-1} - 8^2 \times 6^{2k-1} + 6^2 \times 6^{2k-1}$ $= 8^2(8^{2k+1} + 6^{2k-1}) + (6^2 - 8^2)\, 6^{2k-1}$ $= 64 \times 7m - 28 \times 6^{2k-1}$ from (1) $= 7(64m - 4 \times 6^{2k-1})$ $= 7p$ $=$ RHS Step 4: True by induction Alternately, can substitute: $6^{2k-1} = 7m - 8^{2k+1}$ LHS $= 8^{2(k+1)+1} + 6^{2(k+1)-1}$ $= 8^{2k+3} + 6^{2k+1}$ $= 8^2 \times 8^{2k+1} + 6^2 \times 6^{2k-1}$ $= 8^2(8^{2k+1}) + 6^2(7m - 8^{2k+1})$ from (1) $= 64(8^{2k+1}) + 6^2 \times 7m - 36(8^{2k+1})$ $= 28(8^{2k+1}) + 7 \times 6^2 m$ $= 7 \times 4(8^{2k+1}) + 7 \times 36m$ $= 7(4 \times 8^{2k+1} + 36m)$ $= 7p$ $=$ RHS	3 marks: Correct answer. 2 marks: Proves the result true for $n = 1$ and attempts to use the result of $n = k$ to prove the result for $n = k + 1$. 1 mark: Proves the result true for $n = 1$.