

## Problem Statement

Often in finite element methods, resolving a boundary layer requires the use of tightly packed elements throughout the mesh. In conforming elements, such an arrangement incurs the penalty of bad aspect ratio elements. The effect of pouring in additional conforming elements at the boundary layer further results in an undue number of elements, especially far away from the boundary layer itself. However, by relaxing the  $H^1$  constraint of the conforming spectral element method (SEM), one may achieve greater flexibility in the spectral decomposition, allowing for  $p$ -adaptivity of elements near boundary layers. This is known as the mortar spectral element method (MSEM).

## Approach

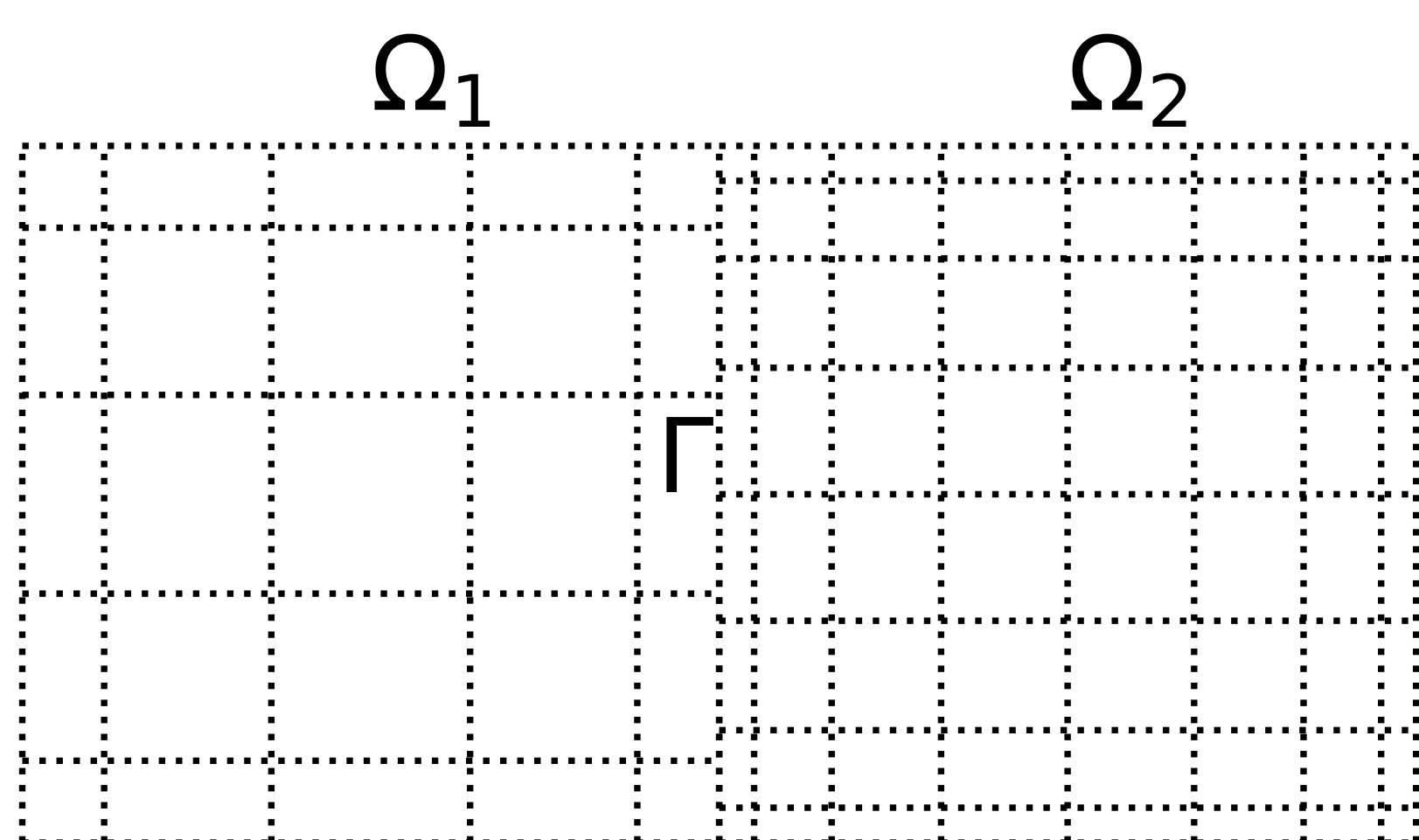


Figure: Example of a geometrically conforming domain decomposition,  $p_1 = 5$  and  $p_2 = 8$ .

- 1 Divide  $\Omega$  into  $E$  nonoverlapping subdomains, such that  $\Omega = \bigcup_{e=1}^E \Omega^e$ .
- 2 Assign (non-unique) mortars to the surfaces between two different subdomains, assigning dependent and independent sides.
- 3 Across the interface,  $\Gamma$ , between two elements  $i$  and  $j$ , impose an  $L_2$  continuity requirement  $\int_{\Gamma} (u_i - u_j) \psi d\tau = 0 \quad \forall \psi \in \mathbb{P}_{N_i-2}(\Gamma)$ .

For the two subdomain case with  $\Omega_1$  being the independent side,  $\Omega_2$  being the dependent side, and  $N_1 \leq N_2$ , the continuity requirement from (3) results in a simple interpolation scheme:

$$\tilde{u}^2(s) = \tilde{u}^1(s) + \alpha L_{N_2}(s) + \beta L_{N_2-1}(s),$$

where the coefficients,  $\alpha$ ,  $\beta$ , are expressed in terms of  $\tilde{u}_0^2$  and  $\tilde{u}_{N_2}^2$ . Imposing direct continuity on the vertices of  $\Omega_1$  and  $\Omega_2$ , this further simplifies into a (variable-resolution) conforming case, which is considered here.

## Test Cases

Solve in the rectangular domain (1)  $[-2, 2] \times [-1, 1]$  the following PDEs with homogeneous Dirichlet B.C. on  $\partial\Omega$ :

- 1  $-\Delta u = f$
- 2  $\frac{\partial u}{\partial t} + \mathbf{c} \cdot \nabla u = \nu \Delta u + f$ , for  $t \in [0, 3]$ .

Compare convergence rates against SEM ( $E = 1, 2$ ) and a MSEM ( $E = 2$ ) of varying polynomial degree.

## Results

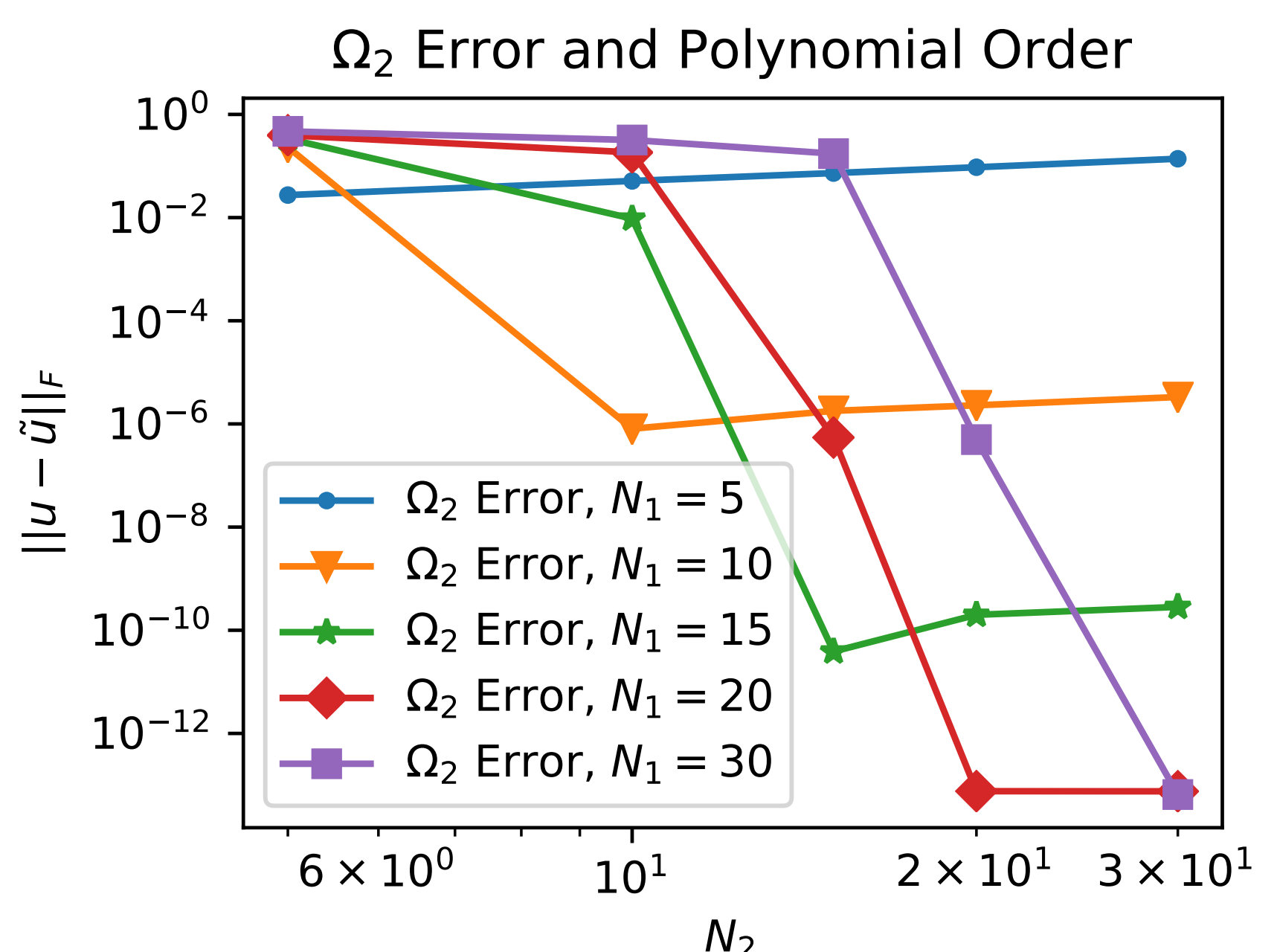


Figure: Error compared to true solution for Poisson's equation in  $\Omega_2$ .

As shown in the above figure, the MSEM converges for mixed polynomial orders. However, the error is minimized whenever the polynomial orders match.

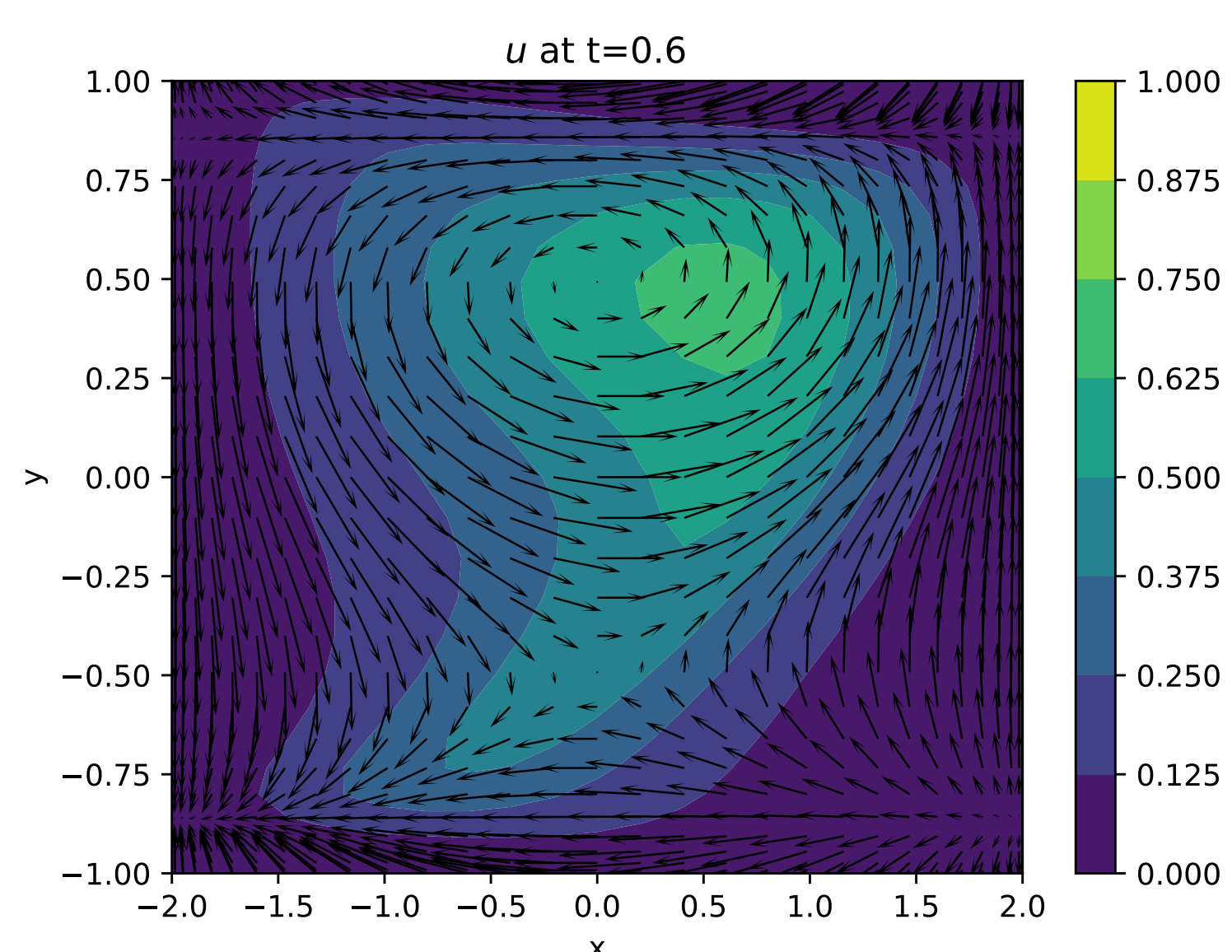


Figure: Convection-diffusion equation solution at  $t = 0.6s$  for  $E = 1$  SEM,  $p = 30$ .

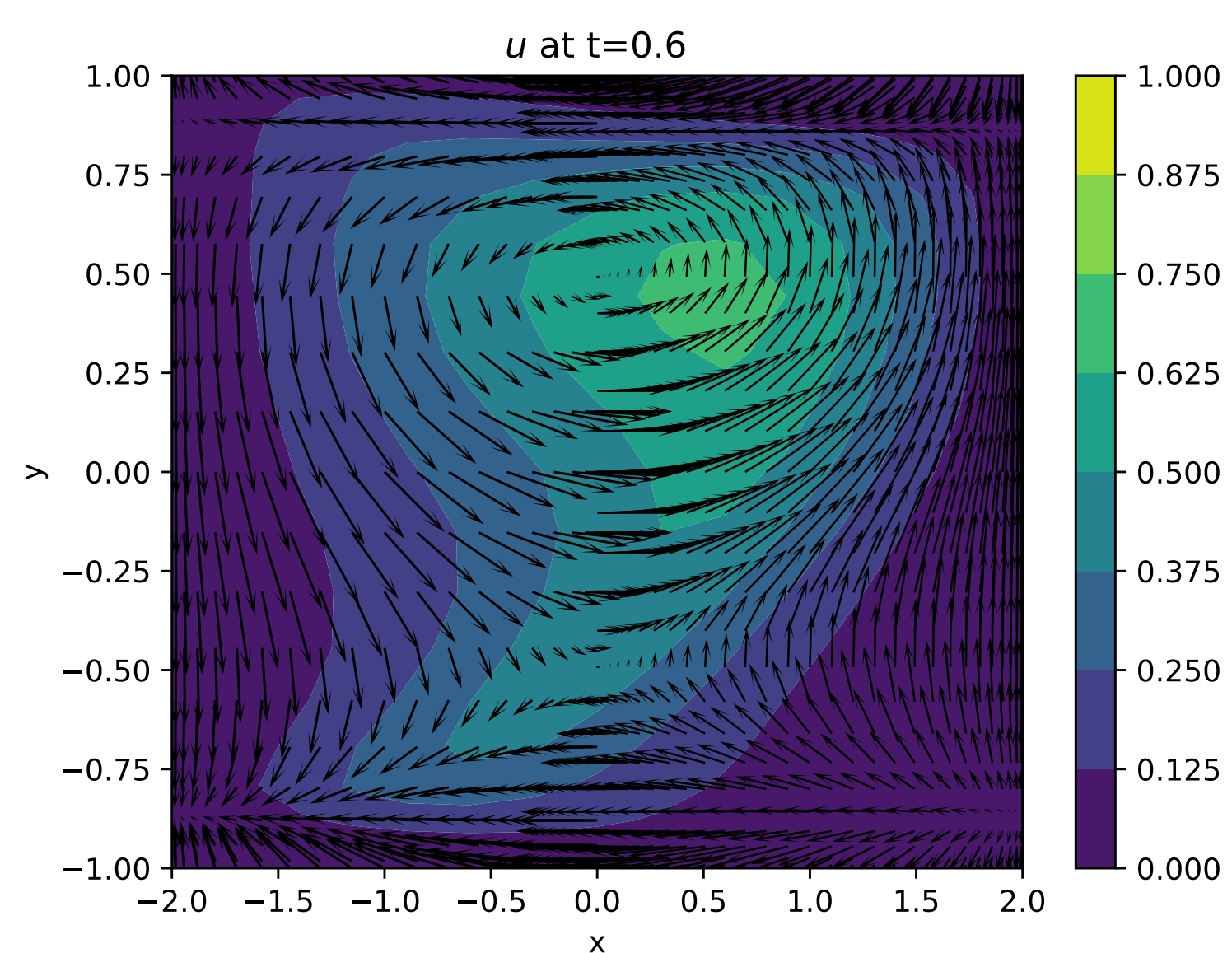


Figure: Convection-diffusion equation solution at  $t = 0.6s$  for  $E = 2$  MSEM,  $p_1 = 20$ ,  $p_2 = 30$ .

## References

- M. O. Deville, P. F. Fischer, and E. H. Mund, High-Order Methods for Incompressible Fluid Flow. Cambridge: Cambridge University Press, 2002.  
 \* <https://github.com/MalachiTimothyPhillips/MSE>