Mortar Spectral Element Method

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Problem Statement

Often in finite element methods, resolving a boundary layer requires the use of tighly packed elements throughout the mesh. In conforming elements, such an arrangement incurs the penalty of bad aspect ratio elements. The effect of pouring in additional conforming elements at the boundary layer further results in an undue number of elements, especially far away from the boundary layer itself. However, by relaxing the H^1 constraint of the conforming spectral element method (SEM), one may achieve greater flexibility in the spectral decomposition, allowing for p-adaptivity of elements near boundary layers. This is known as the mortar spectral element method (MSEM).

Approach

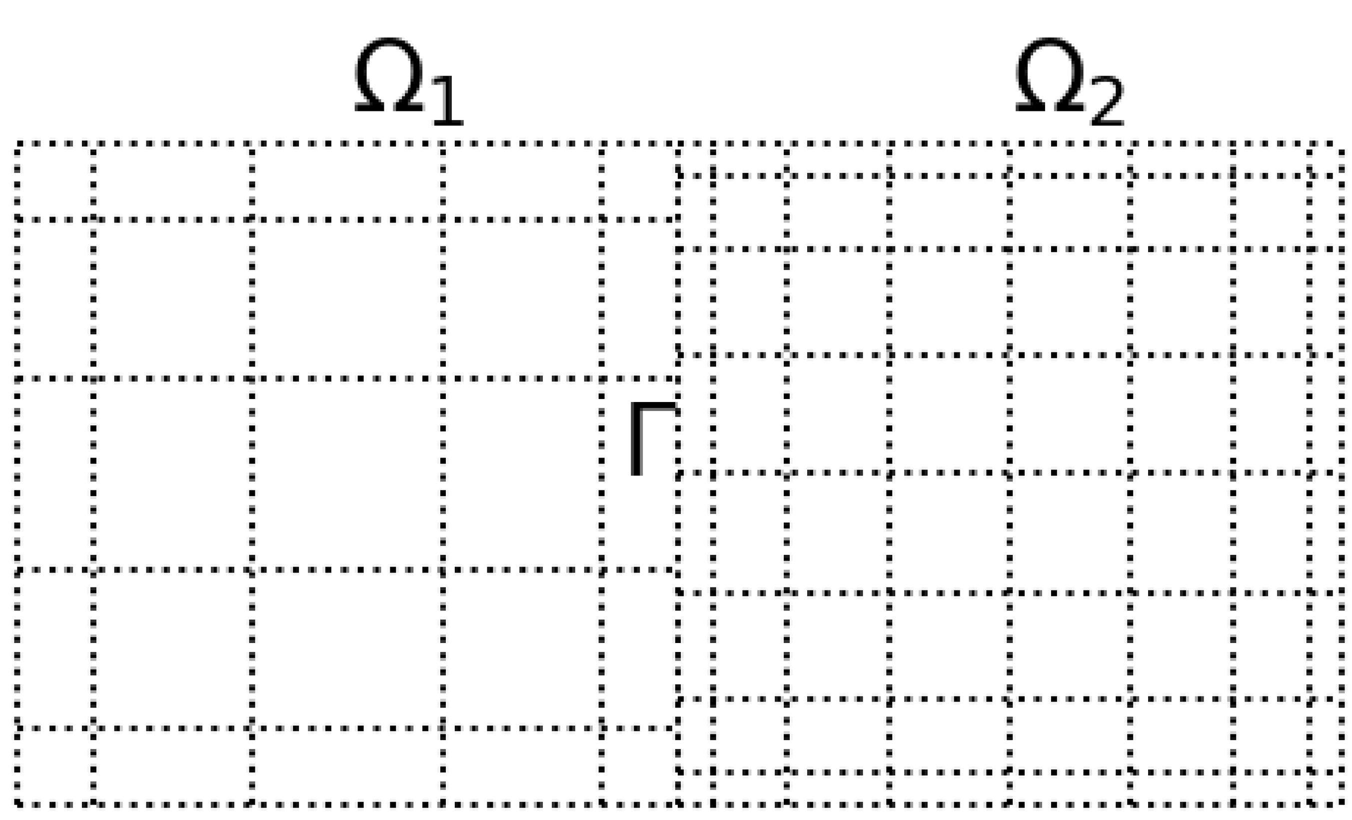

Figure: Example of a geometrically confroming domain decomposition, $p_1 = 5$ and $p_2 = 8$.

1. Divide Ω into E nonoverlapping subdomains, such that $\Omega = \bigcup_{e=1}^{E} \Omega^e$.
2. Assign (non-unique) mortars to the surfaces between two different subdomains, assigning dependent and independent sides.
3. Across the interface, Γ, between two elements i and j, impose an L_2 continuity requirement $\int_\Gamma (u_i - u_j)\psi d\tau = 0 \ \forall \psi \in \mathbb{P}_{N_i-2}(\Gamma)$.

For the two subdomain case with Ω_1 being the independent side and Ω_2 being the dependent side and $N_1 \leq N_2$, the continuity requirement from (3) results in a simple interpolation scheme:

$$\tilde{u}^2(s) = \tilde{u}^1(s) + \alpha L_{N_2}(s) + \beta L_{N_2-1}(s),$$

where the coefficients, α, β, are expressed in terms of $\tilde{u}_0{}^2$ and $\tilde{u}^2_{N_2}$. Imposing direct continuity on the vertices of Ω_1 and Ω_2, this further simplifiers into a (variable-resolution) conforming case, which is considered here.

Test Cases

Solve in the rectangular domain (1) $[-2,2] \times [-1,1]$ the following PDEs with homogeneous Dirichlet B.C. on $\partial\Omega$:

1. $-\Delta u = f$
2. $\dfrac{\partial u}{\partial t} + \boldsymbol{c} \cdot \nabla u = \nu \Delta u + f$, for $t \in [0,3]$.

Compare convergence rates against SEM ($E = 1,2$) and a MSEM ($E = 2$) of varying polynomial degree.

Results

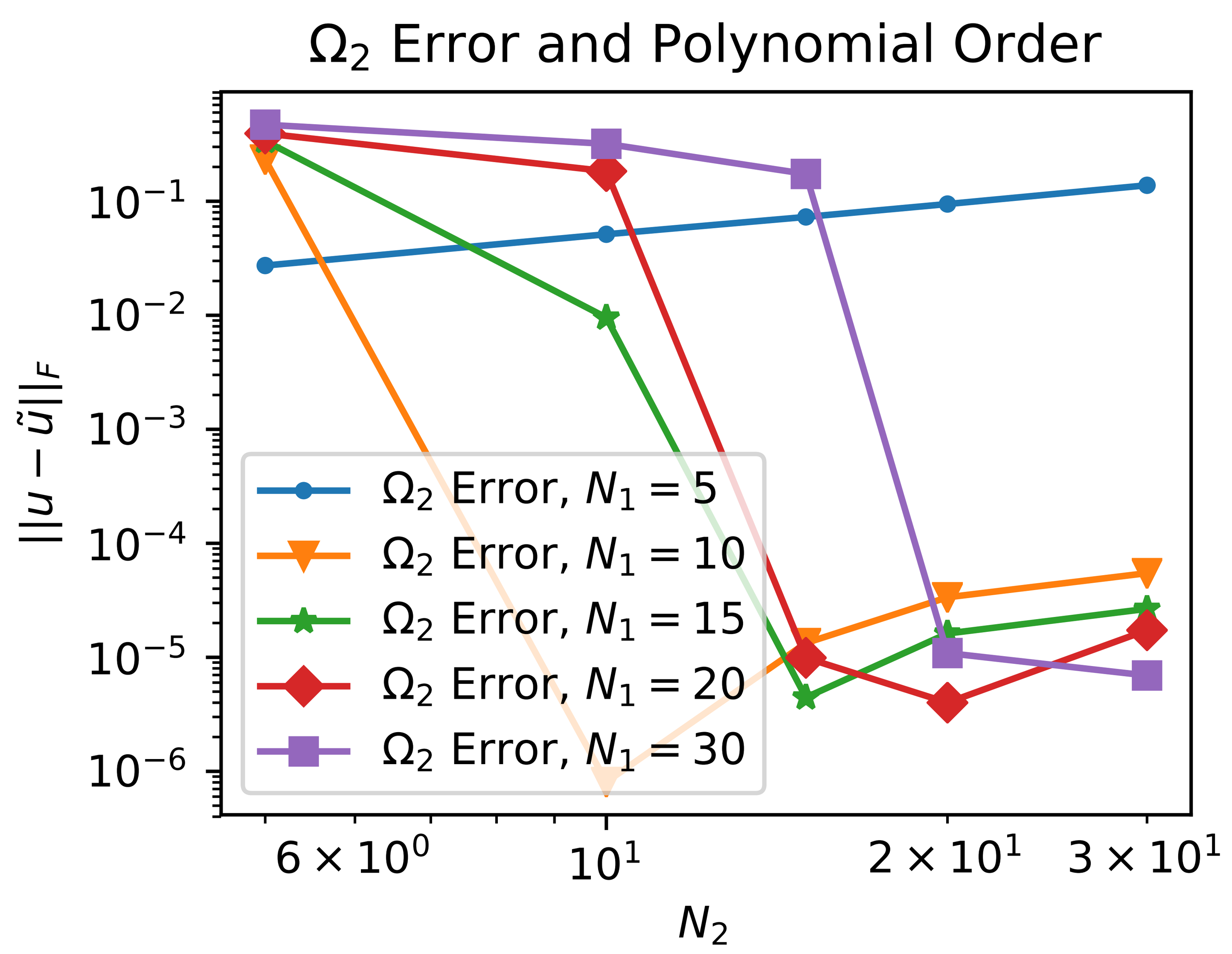

Figure: Error compared to true solution for Poisson's equation in Ω_2.

As shown in the above figure, the MSEM converges for mixed polynomial orders. However, the error is minimized whenever the polynomial orders match.

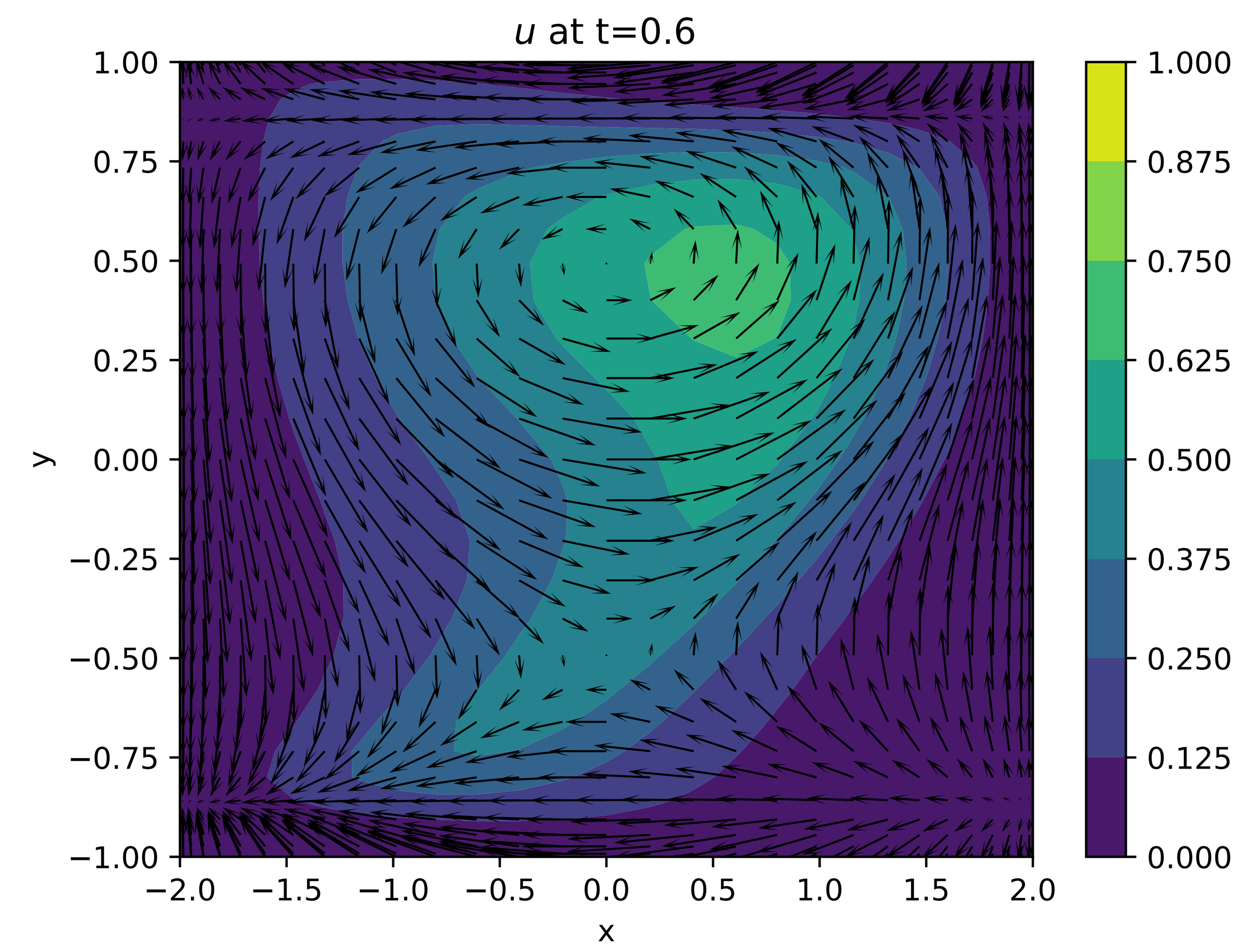

Figure: Convection-diffusion equation solution at $t = 0.6s$ for $E = 1$ SEM, $p = 30$.

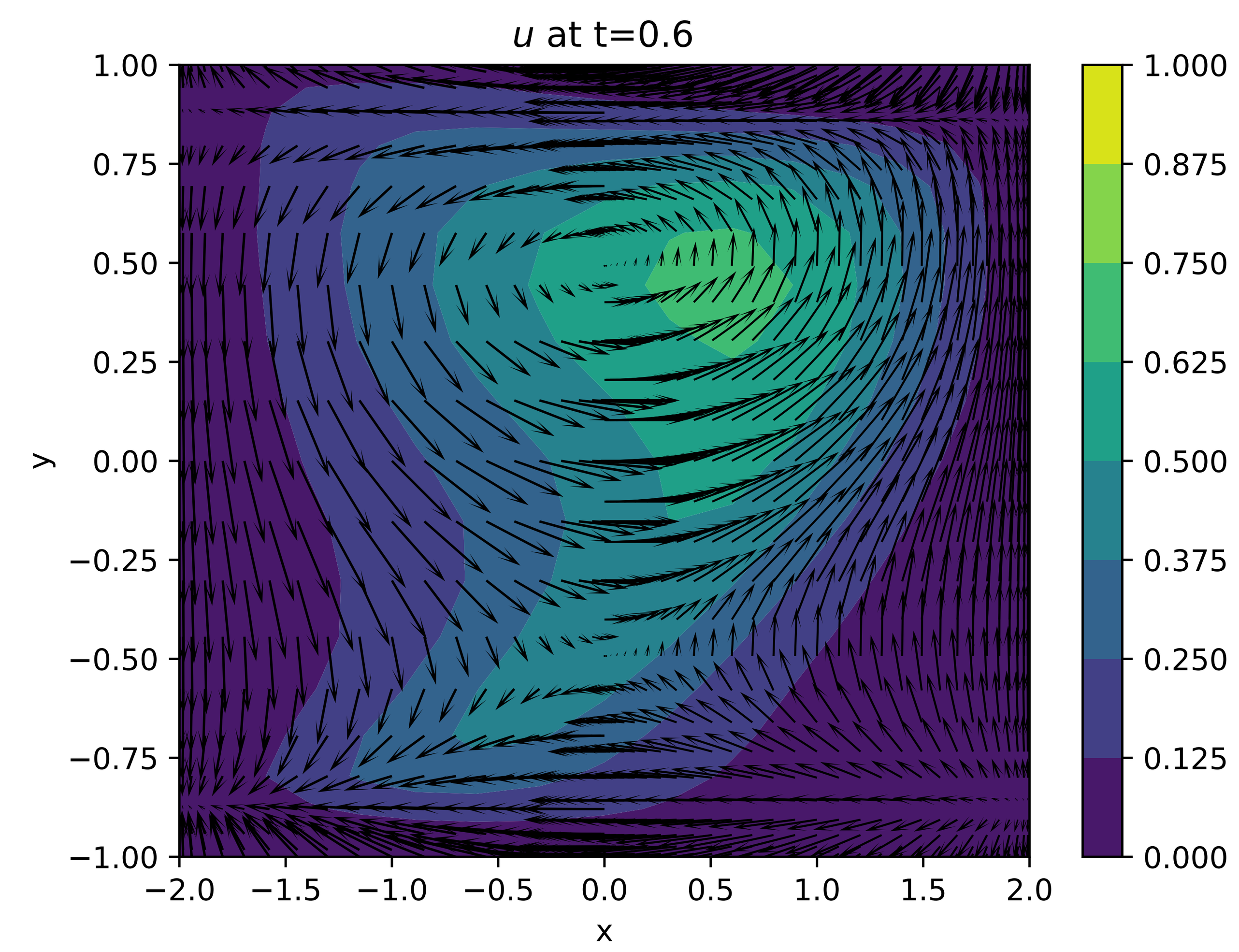

Figure: Convection-diffusion equation solution at $t = 0.6s$ for $E = 2$ MSEM, $p_1 = 20$, $p_2 = 30$.

References

M. O. Deville, P. F. Fischer, and E. H. Mund, High-Order Methods for Incompressible Fluid Flow. Cambridge: Cambridge University Press, 2002.
* https://github.com/MalachiTimothyPhillips/MSE