Poisson

- Poisson solve encompasses the majority of the solution time.
- Spectral element (SE): E elements with polynomial degree p, $n \approx Ep^3$ unknowns and $\mathcal{O}(Ep^6)$ nonzeros.
    - Matrix-free is a must: exploit tensor-product-sum factorization, $\mathcal{O}(Ep^4)$ cost to apply matrix-vector product[1].
    - Poor conditioning of system: $\kappa(A) \sim \mathcal{O}(h^{-p})$ requires preconditioning.

[1] Deville et al., *High-order methods for incompressible fluid flow*.

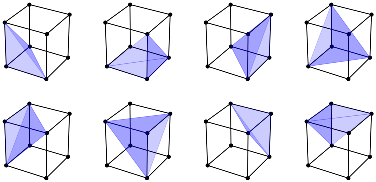
Solvers

- Solution projection for initial guess generation[2]
- Krylov subspace projection methods:
  - flexible PCG
  - PGMRES
- Preconditioners:
  - Low-order operator preconditioning (SEMFEM)
  - Geometric *p*-Multigrid (pMG), requiring a smoother:
    - Additive Schwarz (ASM) and restrictive additive Schwarz (RAS)
    - Chebyshev polynomial smoothing
      - Jacobi
      - ASM, RAS

[2] Fischer, "Projection techniques for iterative solution of Ax= b with successive right-hand sides".

SEMFEM

- Precondition high-order system with low-order discretizations with coinciding nodes
- Orszag[3] demonstrated $\kappa(\mathsf{M}^{-1}\mathsf{A}) \sim \pi^2/4$ scaling for second-order Dirichlet problems.
- Bello-Maldonado and Fischer[4] proposed using one-per-vertex scheme.
    - Same, but solve with AmgX[5] single pass V-cycle, damped Jacobi relaxation ($\omega = 0.9$).

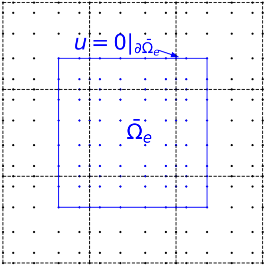


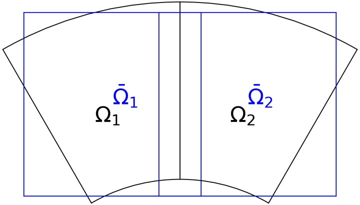
[3] Orszag, "Spectral Methods for Problems in Complex Geometrics".

[4] Bello-Maldonado and Fischer, "Scalable Low-Order Finite Element Preconditioners for High-Order Spectral Element Poisson Solvers".

[5] Naumov et al., "AmgX".

ASM, RAS Smoothers

(a) 

(b) 

- $M^{-1} := \sum_{e=1}^{E} W_e R_e^T \bar{A}_e^{-1} R_e$, subdomain (a)[6].
- How to form $\bar{A}_e^{-1}$?
  - Galerkin: $\bar{A}_e = R_e A R_e^T$, ruins $\mathcal{O}(p^3)$ storage, $\mathcal{O}(p^4)$ work per element.
  - Box-like approximation (b): recover $\mathcal{O}(p^3)$ storage, $\mathcal{O}(p^4)$ work per element using fast diagonalization method (FDM).

[6] Lottes and Fischer, "Hybrid Multigrid/Schwarz Algorithms for the Spectral Element Method"; Loisel et al., "On Hybrid Multigrid-Schwarz Algorithms".

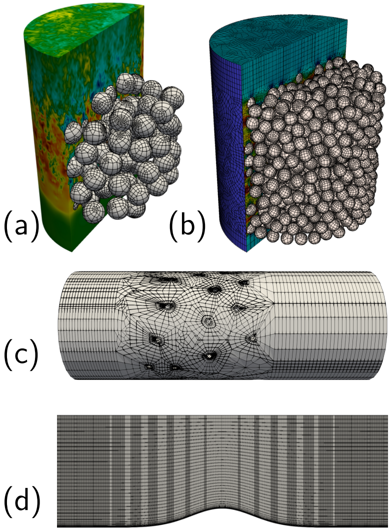
Chebyshev Smoothing

- Construct Chebyshev polynomial of SA minimum in interval $[\lambda_{min}, \lambda_{max}]$.
- Max eigenvalue estimate $\tilde{\lambda}$ obtained with 10 Arnoldi iterations, $(\lambda_{max}, \lambda_{min}) = (1.1, 0.1)\tilde{\lambda}$.
- S = invDiag(A)[7], or, more recently, a Schwarz smoother[8]
  - Chebyshev-acceleration *robustifies* point wise smoother (Jacobi)
  - Similarly, Chebyshev-acceleration applied to Schwarz smoothers improves multigrid convergence

[7] Adams et al., "Parallel multigrid smoothing"; Kronbichler and Ljungkvist, "Multigrid for matrix-free high-order finite element computations on graphics processors".

[8] Phillips, Kerkemeier, and Fischer, "Tuning Spectral Element Preconditioners for Parallel Scalability on GPUs".

Navier-Stokes

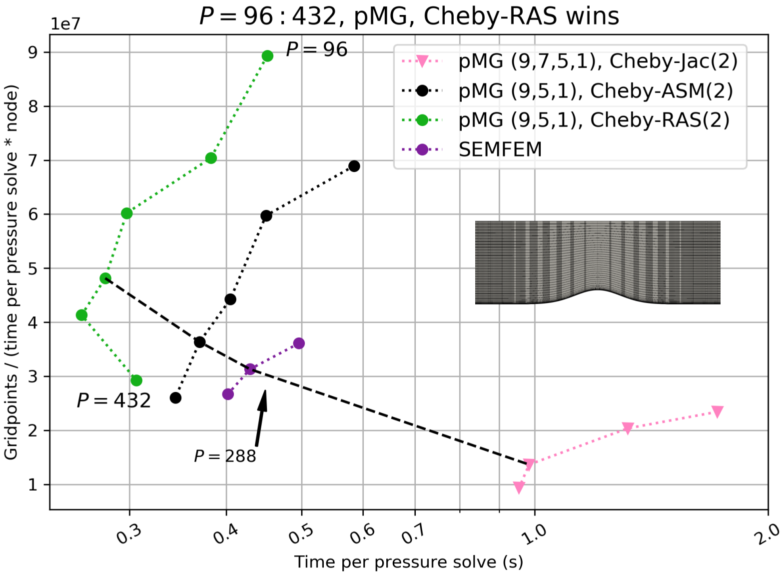


Case Name	E	p	n
146 pebble (a)	62K	7	21M
1568 pebble (b)	524K	7	180M
67 pebble (c)	122K	7	42M
Speed bump (d)	885K	9	645M
	CFL	Δt	$T_{restart}$
(a)	4	2×10^{-3}	10
(b)	4	5×10^{-4}	20
(c)	4	5×10^{-5}	10.6
(d)	0.8	2×10^{-3}	5.6
	Re	Tol	Steps
(a)	5000	10^{-4}	2000
(b)	5000	10^{-4}	2000
(c)	1460	10^{-4}	2000
(d)	10^6	10^{-5}	2000

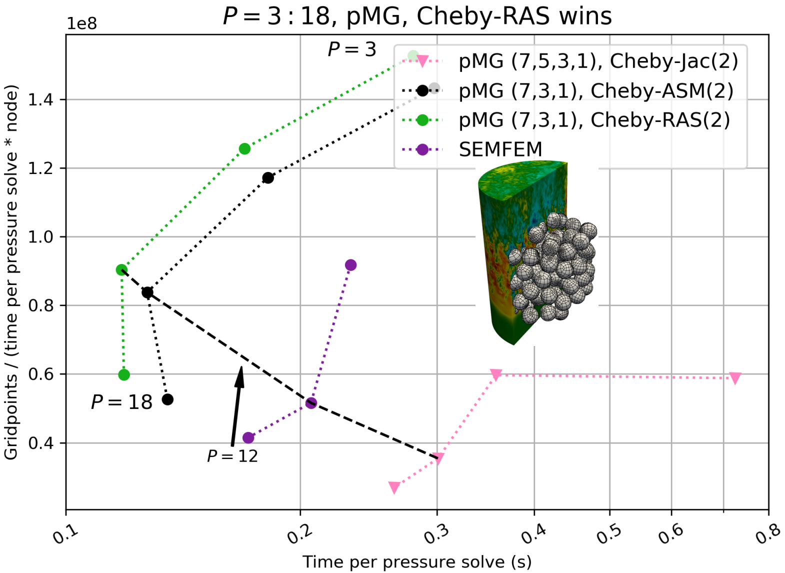
Results

- All results on Summit (42 IBM Power9 CPUs, 6 NVIDIA V100 GPUs per node).
- Each of the P ranks are assigned one GPU, 6 GPUs per node (unless $P < 6$).
- At coarsest level, solve using one BoomerAMG V-cycles[9]
- pMG (7,3,1), Cheby-ASM(2) denotes pMG preconditioning with a 2nd-order Chebyshev-accelerated ASM smoother with $p = 7$, $p = 3$, and $p = 1$ as multigrid levels.

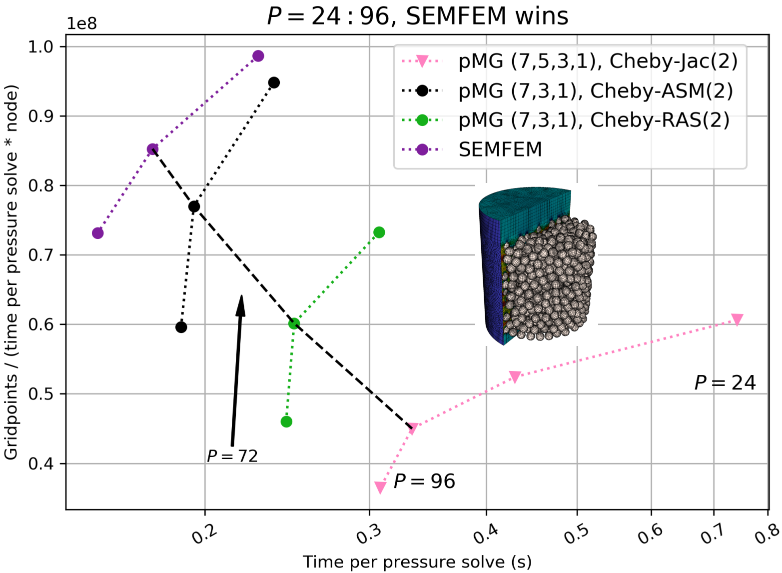
[9] Henson and Yang, "BoomerAMG".



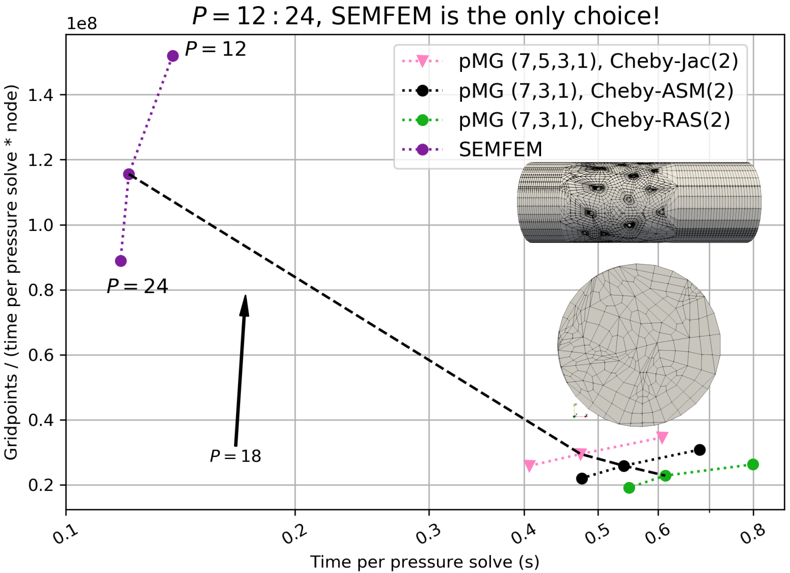
Strong scaling results on Summit for the Boeing speed bump problem.



Strong scaling results on Summit for the 146 pebble case.



Strong scaling results on Summit for the 1568 pebble case.



Strong scaling results on Summit for the 67 pebble case.