Poisson

- Poisson solve encompasses the majority of the solution time
- Spectral element (SE): E elements with polynomial degree p, $n \approx Ep^3$ unknowns and $\mathcal{O}(Ep^6)$ nonzeros
  - Matrix-free is a must: exploit tensor-product-sum factorization, $\mathcal{O}(Ep^4)$ cost to apply matrix-vector product[1]
  - Fast solvers require preconditioning

[1] Deville et al., *High-order methods for incompressible fluid flow.*

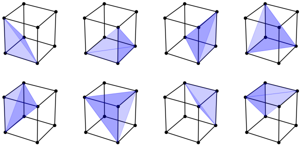
Solvers

- Solution projection for initial guess generation[2]
- Krylov subspace projection methods:
  - flexible PCG
  - PGMRES
- Preconditioners:
  - Low-order operator preconditioning (SEMFEM)
  - Geometric p-Multigrid (pMG), requiring a smoother:
    - Additive Schwarz (ASM) and restrictive additive Schwarz (RAS)
    - Chebyshev polynomial smoothing
      - Jacobi
      - ASM, RAS

[2] Fischer, "Projection techniques for iterative solution of Ax= b with successive right-hand sides".

SEMFEM

- Precondition high-order system using low-order discretizations with coinciding nodes
- Orszag[3] demonstrated $\kappa(\mathsf{M}^{-1}\mathsf{A}) \sim \pi^2/4$ scaling for second-order Dirichlet problems
- Bello-Maldonado and Fischer[4] proposed one-per-vertex scheme
    - Same, single V-cycle AmgX[5], damped Jacobi ($\omega = 0.9$)
- Other approaches exist, such as Pazner[6]



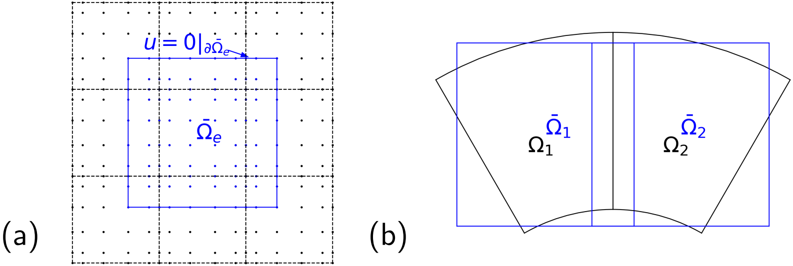
[3] Orszag, "Spectral Methods for Problems in Complex Geometrics".

[4] Bello-Maldonado and Fischer, "Scalable Low-Order Finite Element Preconditioners for High-Order Spectral Element Poisson Solvers".

[5] Naumov et al., "AmgX".

[6] Pazner, "Efficient low-order refined preconditioners for high-order matrix-free continuous and discontinuous Galerkin methods".

ASM, RAS Smoothers



- $M^{-1} := \sum_{e=1}^{E} W_e R_e^T \bar{A}_e^{-1} R_e$, subdomain (a)[7]
- How to form $\bar{A}_e^{-1}$?
  - Galerkin: $\bar{A}_e = R_e A R_e^T$, ruins $\mathcal{O}(p^3)$ storage, $\mathcal{O}(p^4)$ work per element
  - Box-like approximation (b): recover $\mathcal{O}(p^3)$ storage, $\mathcal{O}(p^4)$ work per element using fast diagonalization method (FDM)

[7] Lottes and Fischer, "Hybrid Multigrid/Schwarz Algorithms for the Spectral Element Method"; Loisel et al., "On Hybrid Multigrid-Schwarz Algorithms".

Chebyshev Smoothing

- Construct Chebyshev polynomial of SA minimum in interval $[\lambda_{min}, \lambda_{max}]$
- Max eigenvalue estimate $\tilde{\lambda}$ obtained with 10 Arnoldi iterations, $(\lambda_{max}, \lambda_{min}) = (1.1, 0.1)\tilde{\lambda}$
- S = invDiag(A)[8], or, more recently, a Schwarz smoother[9]
    - Chebyshev-acceleration *robustifies* point wise smoother (Jacobi)
    - Similarly, Chebyshev-acceleration applied to Schwarz smoothers improves multigrid convergence

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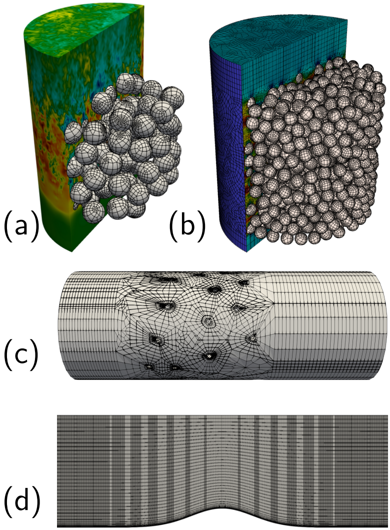
[8] Adams et al., "Parallel multigrid smoothing"; Kronbichler and Ljungkvist, "Multigrid for matrix-free high-order finite element computations on graphics processors".

[9] Phillips, Kerkemeier, and Fischer, "Tuning Spectral Element Preconditioners for Parallel Scalability on GPUs".

Navier-Stokes

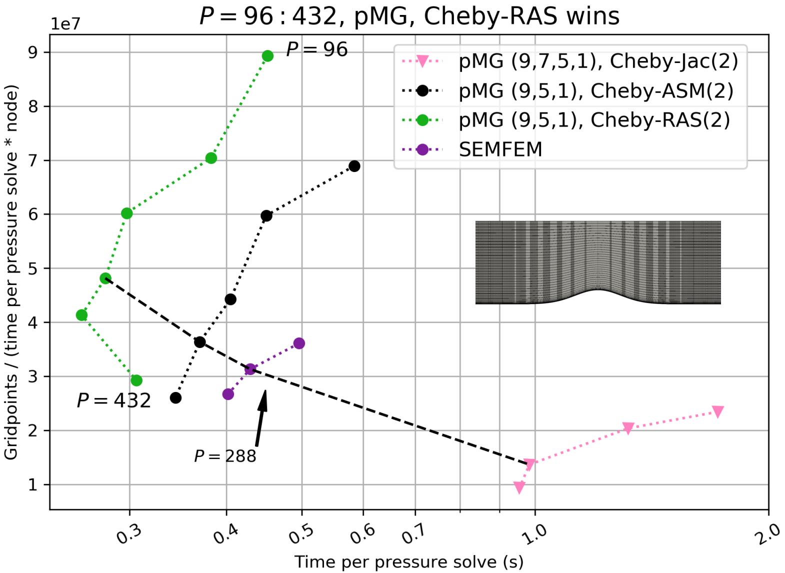


Case Name	E	p	n
146 pebble (a)	62K	7	21M
1568 pebble (b)	524K	7	180M
67 pebble (c)	122K	7	42M
Speed bump (d)	885K	9	645M
	CFL	Δt	$T_{restart}$
(a)	4	2×10^{-3}	10
(b)	4	5×10^{-4}	20
(c)	4	5×10^{-5}	10.6
(d)	0.8	2×10^{-3}	5.6
	Re	Tol	Steps
(a)	5000	10^{-4}	2000
(b)	5000	10^{-4}	2000
(c)	1460	10^{-4}	2000
(d)	10^6	10^{-5}	2000

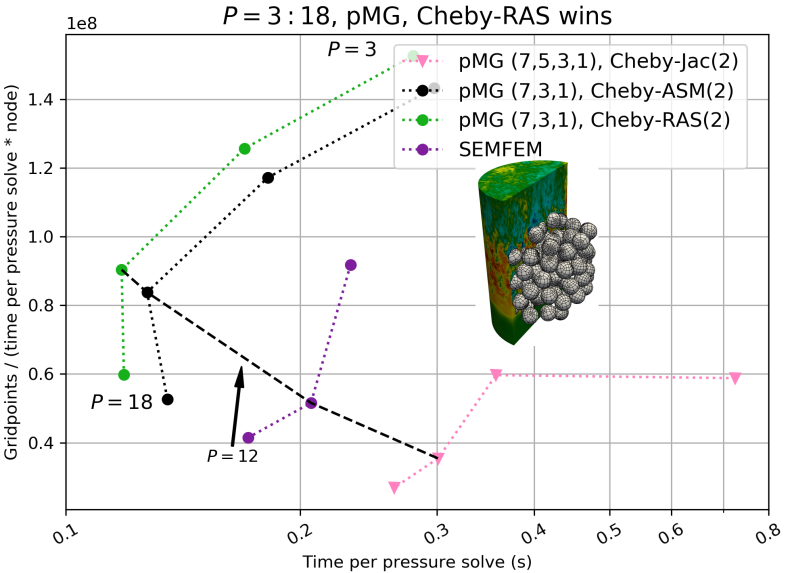
Results

- All results on Summit
    - 42 IBM Power9 CPUs per node
    - 6 NVIDIA V100 GPUs per node
- Each of the P ranks are assigned one GPU, 6 GPUs per node (unless $P < 6$)
- At coarsest level, solve using one BoomerAMG V-cycles[10] on the CPU
- pMG (7,3,1), Cheby-ASM(2) denotes
    - pMG preconditioning
    - 2nd-order Chebyshev-accelerated ASM smoother
    - $p = 7$, $p = 3$, and $p = 1$ as multigrid levels

[10] Henson and Yang, "BoomerAMG".

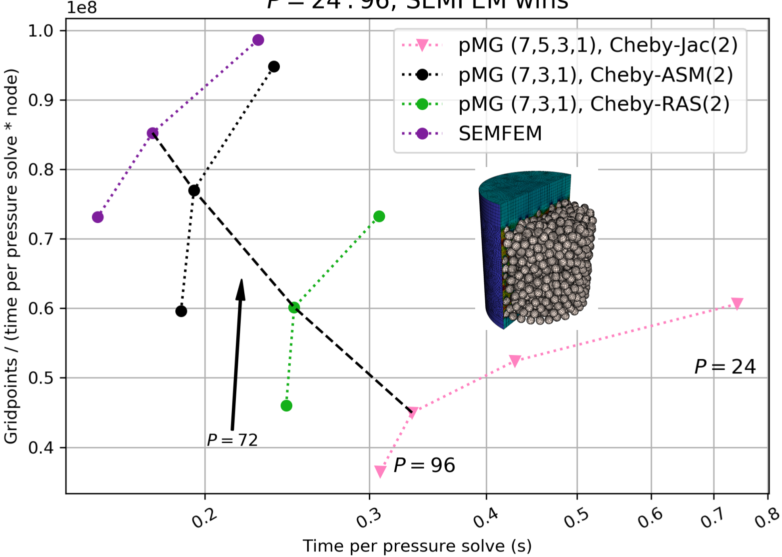


Strong scaling results on Summit for the Boeing speed bump problem.

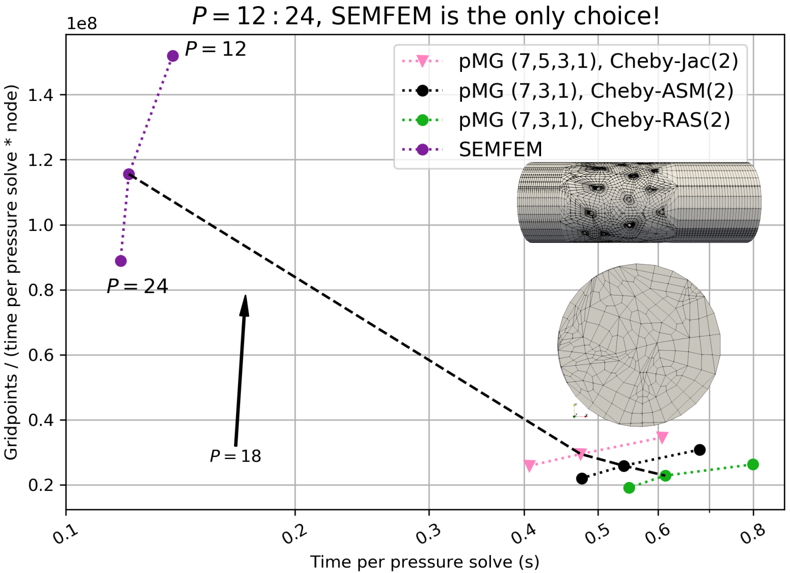


Strong scaling results on Summit for the 146 pebble case.

$P = 24:96$, SEMFEM wins



Strong scaling results on Summit for the 1568 pebble case.



Strong scaling results on Summit for the 67 pebble case.

Summary

- Extensive exploration[11] of
  - pMG: Chebyshev-accelerated Schwarz
  - pMG: Chebyshev-accelerated Jacobi
  - SEMFEM

	t_{SEMFEM}	t_{pMG}	Ratio	P
Speed bump	0.43	0.28	1.54	288
146 pebble	0.21	0.12	1.75	12
1568 pebble	0.18	0.19	0.95	72
67 pebble[12]	0.12	0.47	0.26	18

Table: Ratio of SEMFEM time to best pMG (fixed P)

- Polyalgorithmic solution strategies (with autoselection) are important for general-purpose production-level codes

[11] Phillips, Kerkemeier, and Fischer, "Tuning Spectral Element Preconditioners for Parallel Scalability on GPUs".

[12] tet-to-hex mesh