Negotiated Binding Agreements*

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Abstract

I study binding agreements that can result from negotiation, where the agreement is over agents' behaviour in an underlying strategic environment. The strategic environment is represented by a game. To do so, I propose a negotiation protocol where, in each round of negotiation, agents make public proposals of the action they will take. The protocol terminates when these proposals are confirmed. Confirmation results in a binding agreement and payoffs are that of the agreed action profile. I provide easy-to-check necessary and sufficient conditions for the action profiles that can be agreed to using the solution concept of *Negotiated Binding Agreements*, a refinement of Subgame Perfect Equilibrium where agents only propose actions they could agree to. A full characterisation of these outcomes is provided for two-player games. I show that these general results are robust to variations in the negotiation procedure including in the timing of proposals, proposing actions for all agents, and the payoff of perpetual disagreement. I show that the necessary and sufficient conditions generalise when coalitions may jointly deviate in a cooperative way and are consistent with perturbed versions of the β-core.

Keywords: Agreements, Negotiation, Cooperation

JEL Codes: C70, C71, C72

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1 Introduction

Negotiations and their resulting agreements play an important role within the economy. For instance, firms and workers often negotiate in multiple rounds over the terms of the employment relation. When they reach an agreement and it is confirmed, then it is binding and enforced by, for instance, an outside party via the use of a contract. When prospective employees and employers negotiate, it is often the case that they do so over not only pay but the opportunity for flexible working, parental leave, vacation time, etc. In many cases, these non-monetary payments, or aspects that cannot be readily interpreted as a transfer, may be the only aspect of negotiation. For the example of an employment contract, this would be true if pay is fixed within a range, an especially common occurrence in public institutions. A similar process exists for buying a house, where the timing of the move and renovations needed may be the key object of negotiation. Cartels may also negotiate price fixing and committees may negotiate contributions to a public good, where, similarly, transfers may not be permitted. Given this, many important situations are better represented via an agreement over strategic choices between agents, rather than surplus division via transfers. However, due to the difficulty in modelling such negotiations over the agreement of strategic behaviour, there has been little progress in using tools that reflect this within theory, leading to little understanding of which agreements could occur within such applications.

Within this paper, I define a model of negotiation and corresponding solution concept that provides simple to use and well pinned down results, while maintaining consistent and fully rational beliefs. This allows for both ease of use in applications while having results be fully justified by preferences of all agents and their understanding of the negotiation procedure. This builds a bridge between a number of important strands of literature on the theory of binding agreements over strategic environments. Within the first, there are approaches that provide a negotiation procedure and companion solution concept that ensure agents always negotiate in an optimal way, regardless of previous behaviour within the negotiation (Kalai, 1981; Bhaskar, 1989; Chwe, 1994; Mariotti, 1997; Ray and Vohra, 2019). These concepts are reasonable but often intractable, due to the consistency of rationality they require and the richness of behaviour that these procedures allow for. Within the second, the approaches instead focus on what can be sustained using assumptions of behaviour that may not be consistent with rationality, particularly in the event of a deviation from the candidate agreement (Aumann, 1959, 1961; Currarini and Marini, 2003). These concepts take a cooperative view that abstracts from why such behaviour takes place. Doing so allows for a much larger degree of tractability. Nonetheless, they do not capture what can reasonably be used to prevent deviations and therefore may allow for predictions of unreasonable agreements in some applications.[1] Another strand of literature abstracts

[1] Scarf (1971) provides an early observation of this issue in reference to Aumann (1961)'s α-core, pointing to the potential unreasonable use of any punishment to prevent deviations, as such punishments need not ever be agreed upon.

from the negotiation process *within* a group and take a cooperative perspective, focusing on Pareto undominated actions that prevent new groups from breaking and forming (Ray and Vohra, 1997; Diamantoudi and Xue, 2007). These capture what can be sustained by such threats and are often tractable, but do not speak to what can be achieved within a group via a negotiation. Due to these polar difficulties, such concepts are often left as purely theoretical and do not allow for broad use in applications. This paper provides an approach that simultaneously allows for full rationality and well pinned down general results via a negotiation. This opens the door to understanding the behaviour that rational agents could agree upon when negotiating a binding agreement over their behaviour in an underlying game.

The negotiation protocol I consider, regarding the behaviour players should take in a game, takes the following form and is outlined in full in section 2. Suppose that, within a period, agents can make a public proposal of the action they will take. In doing so, they may consider all proposals, both their own and others, from all previous periods. Hence, agents' strategies in the negotiation map each history of past joint proposals to a new individual proposal. They continue making proposals in this form, with potentially infinitely many rounds, until the same proposal is made for two consecutive periods, used to ensure confirmation of their choice.[2] At this stage, a binding agreement to play this action profile, or outcomes, is made and is enforced by some outside institution such as a legal system. In this case, each agent receives the payoff of the resulting action profile. The payoff of perpetual disagreement is taken to be consistent with the limiting case of probabilistic termination in each period, where the current period's proposal is taken in the event of termination, taking the probability to 0. The assumed payoffs of perpetual disagreement are also consistent with a number of other interpretations that will be discussed later in the paper. This procedure will be referred to as the *baseline negotiation* procedure throughout and show that the results of the paper are consistent with a number of variations including in timing, proposing action profiles, and in the payoff of perpetual disagreement, which are studied in Appendix A.

As the aforementioned negotiation protocol defines a dynamic game with complete information, I explore a refinement of Subgame Perfect Equilibrium. I refer to this solution concept as *Negotiated Binding Agreements*. Firstly, this refinement only considers Subgame Perfect Equilibrium that result in agreement. Secondly, this refinement ensures players only propose actions that would be agreed upon on the continuation of *some* history, which I will refer to as a *no babbling* condition.[3] With this, Negotiated Binding Agreements captures two elements. As this paper is concerned with the agreements that can be made, this refinement ensures that uninteresting equilibria resulting in perpetual disagreement are not considered. Secondly, restricting attention to equilibria that are no babbling prevents

[2] Other methods of verification, if simultaneous and all respected would lead to the same results.

[3] This assumption can embed a form of no delay equilibrium used within bargaining games with a large number of players (Chatterjee et al., 1993).

unreasonable proposals from ever being made, in the sense that agents would not agree to play such an action. Action profiles of the baseline game that are selected by a Negotiated Binding Agreement are referred to as *supported* by a Negotiated Binding Agreement, and the set of such actions will be the key object of study within this paper.

This model allows for full rationality with respect to individual preferences, where agents perfectly understand the implications of a deviation, and these implications are justifiable via the use of a refinement of subgame perfection.[4] As this negotiation game has infinitely many histories, with different types of terminal histories, this is a complex object to consider. However, studying the Negotiated Binding Agreements allows for a tractable solution. I outline the key results here, which are presented in full in section 3.

Firstly, I provide necessary conditions for the proposals and outcomes of Negotiated Binding Agreements. I show that a necessary condition for any Negotiated Binding Agreement is that all proposed actions must survive iterated elimination of individually irrational actions, which I introduce in this paper. An action a is individually irrational if, given the most optimistic beliefs an agent can have when evaluating a, the payoff is still strictly worse than the minimum payoff they can receive from best responding to some action profile of others. Performing this process iteratively, deleting all individually irrational actions within a round before moving to the next, results in actions that survive iterated elimination of individually irrational actions. Secondly, I show it is necessary for agents to play in an *individually rational* way given they understand that others can only propose actions that survive iterated deletion of individually irrational actions. This leads to a minimal payoff each agent can receive from an outcome of a Negotiated Binding Agreement. The conditions of deletion and calculating the minimal payoffs are easy to implement and check for any finite game or game with smooth utility functions.

Secondly, I provide sufficient conditions for the action profiles that can be supported by a Negotiated Binding Agreement. I show that any profile which gives all players a payoff weakly higher than their "individual punishment" profile can be supported by a Negotiated Binding Agreement. The individual punishment profiles are such that a) the payoff for any other players' punishment is weakly better than the payoff of their own punishment and b) each player is prescribed to play their best response within the baseline game to the action profile of their punishment. This is similar, although more restrictive, to the approach of player-specific punishments used in the literature of infinitely repeated games, for example in Fudenberg and Maskin (1986) and Abreu et al. (1994), which will be discussed further within the paper. These sufficient conditions imply that any action profile in the baseline game that Pareto dominates a pure Nash equilibrium can always be supported in Negotiated Binding Agreement.

[4]This can be seen as in contrast with a strain of the cooperative literature that does not provide a justification for why such strategies take place in the event of deviations.

In two-player games, each player receiving a payoff higher than the individual punishment, as described above, is also necessary for an action profile to be supported by Negotiated Binding Agreements. Therefore, in this class of games, the described general sufficient conditions fully characterise the action profiles that can be supported by Negotiated Binding Agreements.

I explore three key applications. I use a simple three-firm Bertrand model as a leading example to display the key results of the paper and provide the intuition behind them.[5] In this example, in any action profile supported by a Negotiated Binding Agreement, the firm with the lowest marginal cost must at least partially serve the market. In section 4, I explore two other applications. Within the first, a public goods game, I show a contribution level can only be supported if it reaches a minimum threshold or if there is no contribution at all. With this, an agent would only agree to contribute if they are sufficiently "compensated" by the contributions of others. The second application considers a simple Cournot Duopoly with linear demand and heterogeneous marginal costs. I show that when marginal costs are the same, any profile of payoffs that gives both players positive profits is supportable. In contrast, when marginal costs are extremely different, only the efficient firm receiving their monopoly profit can be supported. The market scenarios of Bertrand and Cournot provide us with the simple intuition that a naturally more competitive firm cannot agree to be left out of the market.

Negotiated Binding Agreements as a solution concept allows for only individual deviations, but does not allow for the possibility that agents may be able to make an agreement to jointly deviate. To address this, I extend the solution concept to allow for the possibility of cooperative agreements within the very negotiation process. I do so by allowing coalitions of agents to jointly choose a new strategy, and will do so if it is profitable for all agents within the coalition. I allow for any possible set of coalitions to be permissible, including allowing for permissible coalitions to overlap. The novelty of permitting coalitions to overlap can offer new insights into environments when agents may be members of multiple groups simultaneously. For example, within international trade and politics, groups with international agreements regularly overlap. To provide one example, the Asia-Pacific Free Trade Area, which can be seen as a coalition in multinational negotiations, includes some members of the Association of Southeast Asian Nations, but other countries, such as the US, are present in Asia-Pacific Free Trade Area, and therefore can neither be seen as a partition nor merely a "copy" of the same coalition.

To capture the possibility of agents acting in such a way within the negotiation procedure, in section 5, I define the concept of $\mathcal{C}$-Negotiated Binding Agreement, where no coalition in a predefined set $\mathcal{C}$ can profitably deviate at any history. Further, as in Ne-

[5] In this environment, although it may be unreasonable for a binding agreement to enforced by a legal system, due to the collusive nature, it could be instead that it is enforced by a co-owner of all firms, amongst other interpretations.

gotiated Binding Agreement, a $\mathcal{C}$-Negotiated Binding Agreement requires a no babbling condition. In section 6, I show that the natural extension of the baseline necessary and sufficient conditions hold in this setting. Within $\mathcal{C}$-Negotiated Binding Agreement, players must only make proposals from the set of actions that survives iterated elimination of coalitionally irrational actions, defined in a similar way to individually irrational actions while taking the coalition-wide preferences into account. Further, all permissible coalition must play *coalitionally rationally*, where they cannot strictly improve the utility of all members, understanding that agents outside of the coalition will choose some constant action from the set that survives iterated elimination of coalitionally irrational actions. I argue that these conditions can be viewed as a perturbed version of the β-core of Aumann (1961). I provide sufficient conditions that use coalition-specific punishment and argue that these can be viewed as a further refined version of β-core, where further joint consistency is required. These conditions are generally demanding, but do provide sharp results in some interesting applications. For instance, I apply this concept to a simple Cournot model with fixed costs. I show that the Pareto efficient outcome where agents equally divide the monopoly quantity can be sustained in $\mathcal{C}$-Negotiated Binding Agreement.

Finally, I overview the related literature in section 8 and conclude the paper in section 9, pointing to a number of directions for future work.

2 Model

Let the game being negotiated over be $G = \langle N, (u_i, A_i)_{i \in N} \rangle$ where $N = \{1, 2, 3, ..., n\}$ is a finite set of players, A_i is a set of actions for each player with typical element $a_i \in A_i$, with a joint action $A = \times_{i \in N} A_i$ with typical element $a \in A$. u_i is utility function such that $u_i : A \to \mathbb{R}$ and u_i is bounded for all $i \in N$. Let $A_{-i} = \times_{j \neq i} A_j$. I make no further restrictions on the game. In particular, A need not be finite nor compact, nor u_i be continuous.

I will now define the *negotiation game* over G. There will be potentially infinitely many periods to reach an agreement, the process will take the following form. Within a period, each agent observes the history of all previous proposals of all agents and will make a proposal of the action they will take in G. Hence, agents' strategies in the negotiation game map each history of joint proposals to a new individual proposal. If all agents make the same proposal that they made in the previous period, that action profile is implemented in a binding way. Therefore the payoff of such a history is the payoff of the agreed upon action. If not, they continue to the next round and continue the same process until an agreement is made. When there is perpetual disagreement the payoffs are defined to be the lim sup and lim inf utility of the proposals made. This is consistent with the interpretation that an outside party decides the proposals, which must be taken from sufficiently far along

the path of proposals, that will be implemented. It is also consistent with there being some probability of the current period's proposal being accepted, taking this probability to 0. I define this formally below.

Let the set of partial histories consists of all $h = (a^1, a^2, ..., a^k)$ such that $a^t \neq a^{t-1}$ for any $t \leq k$ where $a^t = (a_i^t)_{i \in N}$ denotes the profile of proposals made in period t. I will denote the set of all partial histories by H. Proposals are assumed to be made simultaneously within a period, and therefore no history is such that only some agents have made proposals.

A history is terminal if, either:

a) the same action profile is proposed twice in consecutive periods, and no earlier occurrence of consecutive repetition is present. That is, $z = (a^1, ..., a^{k-1}, a^k)$ is terminal if $a^k = a^{k-1}$ and $a^m \neq a^{m-1}$ for all $m < k$. Let the set of such histories be denoted by Z' and refer to such histories as with *agreement*.

b) an infinite sequence where the same action profile is never proposed consecutively. Let the set of such histories be denoted by Z''. I will refer to these as histories with *perpetual disagreement*.

Let the set of all terminal histories be given by $Z = Z' \cup Z''$.

Let $U_i : Z \to \mathbb{R}$ denote the payoff for player $i \in N$ of the negotiation game.

Whenever $z = (a^1, ..., a^k) \in Z'$, that is a history that ends in agreement, let $U_i(z) = u_i(a^k)$ for all $i \in N$.

Whenever $z = (a^1, a^2, ..., a^k, ...) \in Z''$, that is a terminal history with perpetual disagreement, I assume that $U_i(z) \in [\liminf_{t \to \infty} u_i(a^t), \limsup_{t \to \infty} u_i(a^t)]$. This ensures the following properties hold:

1. For any $z' \in Z'$ and $z = (a^1, a^2, ..., a^k, ...) \in Z''$, if $\exists T \in \mathbb{N}$, such that $\inf_{a^t, t>T} u_i(a^t) > U_i(z')$ then $U_i(z) > U_i(z')$.

2. For any $z' \in Z'$ and $z = (a^1, a^2, ..., a^k, ...) \in Z''$, if $\exists T \in \mathbb{N}$ such that, $\sup_{a^t, t>T} u_i(a^t) \leq U_i(z')$, then $U_i(z) \leq U_i(z')$.

3. $U_i(z)$ is bounded for any history, and is bounded by the same bounds as u_i in G.

This assumes that the payoff of agents is based on proposals that are sufficiently far along the history. This is consistent with the interpretation that an outside party decides the proposals, which must be taken from sufficiently far along the path of proposals, that will be implemented, in a way that is known to the agents. Within finite games G it can also be interpreted as taking any weighted average of the proposals made infinitely often along

the entire path. Note that such a utility function always exists in this class of game and may meaningfully use *all* proposals sufficiently far along the path.[6]

This specification for the payoff of perpetual disagreement may embed, for example, the approach of infinitely repeated games with no discounting: i.e. using the limit of means criteria when well defined (Rubinstein, 1994; Aumann and Shapley, 1994). This can also be interpreted as a non-discounted version of the condition used within Kimya (2020). I reserve a discussion of the relation of this model to those for the literature review. Further to this, this restriction is consistent with the standard model, where the proposal today is implemented with probability $(1-\delta)$ for each period, while the process continues with probability δ, if the probability of continuation is taken to 1. This is formalised by the following lemma.

Lemma 1. *For* $z = (a^1, a^2, ..., a^t, ...) \in Z''$

$$\lim_{\delta \to 1}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(a^t) \in \left[\liminf_{k\to\infty} u_i(a^k), \liminf_{k\to\infty} u_i(a^k)\right]$$

I reserve this proof and all other proofs for appendix B.

Due to the structure of the game, where in each round agent's propose the action they can take, and the possibility to negotiate for many rounds. we can view this as having some similarity to the structure of repeated games. There are a few important changes. Firstly, repeated games only have one type of terminal history, where the game has been repeated the number of times it was specified to, be that some finite number or infinite. On the other hand, the structure of this negotiation game allows for two distinct types of terminal histories, those with agreement and those without. Further, repeated games use flow payoffs, receiving a payoff in each period of play. This negotiation game only allows for payoffs to be realised upon termination. Using a structure that has some of these repeated game type characteristics is a commonly used method within the literature (see Kalai 1981; Bhaskar 1989; Kimya 2020; Nishihara 2022, etc.).

[6] To see this note the following construction. For any history h let h^T denote the sequence that takes the first T elements of the sequence. For a history h denote the consequence $(v^1, v^2..., v^T, v^{T+1}, ...)$ to be such that $v_i \in \mathbb{R}^n$ such that $v_i^T = \sum_{a\in\{A|a\in h\}} \frac{|\{k\in\mathbb{N}|a^k=a, a^k\in h^T\}|}{T} u_i(a)$. Let $Conv(h) = \{v \in \mathbb{R}^n | \exists (v^{l_1}, v^{l_2}, ...) \subset (v^1, v^2..., v^T, v^{T+1}, ...) \text{ s.t. } v = \lim_{k\to\infty} v^{l_k}\}$. As $u_i(A)$ is bounded it follows that v_i^T is bounded for all T, and therefore so is v^T. Therefore there exists a convergent subsequence in $\mathbb{R}$. Therefore $Conv(h)$ is non-empty. Further, whenever the limit of the mean utility, $\lim_{T\to\infty}\sum_{a\in A}\frac{|\{k\in\mathbb{N}|a^k=a, a^k\in h^T\}|}{T}u_i(a)$ is well defined, $Conv(h)$ is a singleton containing only this value for each player. For $\sup[Conv(h)]_i$, the utility is only defined on infinite sequences, and therefore if there is a strict relation on all proposals after some finite T we ensure the relation translates to $u_i(h)$. Further, $\sup[Conv(h)]_i \geq \inf_{a^t, t>T} u_i(a^t)$ for any finite T. Therefore if $\inf_{a^t,t>T} u_i(a^t) > u_i(h')$ it follows that $\sup[Conv(h)]_i > u_i(h')$. Note that $\sup_{a^t,t>T} u_i(a^t) \geq \sup[Conv(h)]_i$ and therefore if $u_i(h') \geq \sup_{a^t,t>T} u_i(a^t)$ then $u_i(h') \geq \sup_{a^t,t>T} u_i(a^t)$. Further, as this value is bounded, we have the desired properties.

At each partial history $h \in H$ the strategy of $i \in N$ dictates the proposal i would make in the next round: $s_i : H \to A_i$. Let S_i be the space of all such mappings. Let $s : H \to A$ be the joint strategy, such that $s(h) = (s_i(h))_{i \in N}$.

For a partial history $h \in H$ and a joint strategy s let $(s|h)$ denote the continuation history of h given by s. That is, $(s|h) = z \in Z$ such that $z = (h, a^{'1}, a^{'2}, ..., a^{'k}, ...)$ where $a^{'1} = s(h)$, $a^{'2} = s((h, a^{'1}))$, $a^{'k} = s((h, a^{'1}, a^{'2}, ..., a^{'k-1}))$. With some abuse of notation, let $U_i(s|h) = U_i(z')$ where $z' \in Z'$ is defined as before and $U_i(s|h) = U_i(z'')$, where $(s|h) = (h, z'') \in Z''$. That is, only take the continuation of the history h for perpetual disagreement. When $z = (a^1, a^2, ..., a^k) \in Z'$, i.e. an agreement is made, let $a(z) = a^k$ and $a_i(z) = a_i^k$.

2.1. Solution Concept

This negotiation protocol defines a dynamic game with complete information therefore subgame perfect equilibria (SPE) and refinements of SPE can be seen as the appropriate solution concept.

Definition (Subgame Perfect Equilibria)**.** *s^* is subgame perfect equilibrium, if for all partial histories $h \in H$, for all $i \in N$, $U_i(s^*|h) \geq U_i(s_i, s^*_{-i}|h)$, for all $s_i \in S_i$.*

Due to the structure of the negotiation protocol, in any SPE agents must receive a payoff weakly higher than their inf-sup payoff. This is true for any history. This is formalised by the following lemma.

Lemma 2. *For any subgame perfect equilibrium s^*, for any partial history $h \in H$*

$$U_i(s^*|h) \geq \inf_{a_{-i} \in A_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$$

Note that the set of SPE trivially includes many perpetual disagreement outcomes. As this work is primarily focused on the agreements that can be supported by some equilibrium, I focus on SPE that reaches an agreement from the initial history. I will also restrict attention to the case where proposals of individuals can only be used if there is some continuation in which said proposal would make up part of an agreement. This rules out agents proposing actions that they would never agree to. I will refer to such property as agreements having *no babbling*. Note that similar concepts have been used within the literature on bargaining. For instance, the study of no delay equilibrium by Chatterjee et al. (1993), where the proposals can only be made if they could be accepted.[7] I will refer to this concept as *Negotiated Binding Agreement*.

[7] This is in the view of efficiency due to discounting in their model while in this model discounting is not assumed.

Definition 1 (Negotiated Binding Agreement)**.** *s^* is a Negotiated Binding Agreement supporting $a^* = a(s^*|\emptyset)$ if:*

a) s^* is a subgame perfect equilibrium.

b) No babbling: $\forall h \in H$, $\exists h' \in H$ such that $s_i^*(h) = a_i(s^*|h')$.

Further motivation can be found behind ensuring that the agreement is no babbling, as it rules out the possibility of making proposals that are always payoff irrelevant on the equilibrium path. These would be understood to be payoff irrelevant by agents due to complete information and correct beliefs of equilibrium. To see the use for this, it may be that an SPE may induce proposals that are not used for the purpose of agreement, even in the event an agreement is eventually reached. For instance, consider the following example.

Example 1. Take the following game

1\2	L	C	R
U	2, 3	0, 1	1, 1
D	3, 2	1, 1	1, 0

Notice that the individual min-max for player 1 is given by 1, while the individual min-max for player 2 is given by 2. By lemma 2, players cannot receive below their min-max payoff.

Now, consider the following SPE.

$$s_2^*(h) = \begin{cases} R & \text{if } h = (a^1, ..., a^k), a^k = (D, C) \text{ or } a^k = (D, L) \\ C & \text{if } h = (a^1, ..., a^k), a^k = (U, R) \\ L & \text{otherwise} \end{cases} \qquad s_1^*(h) = U, \quad \forall h \in H$$

On the path of play, starting from the initial history, this induces a path such that (U, L) is proposed within the first two periods and the game terminates. On the other hand, if the history is such that (D, C) has been proposed in the previous period, $t-1$, then (U, R) is proposed in period t, (U, C) in period $t+1$, (U, L) in period $t+2$ and the game terminates in period $t+3$ as (U, L) is proposed again. We can see that this strategy always leads to terminal histories that end in (U, L). Note that there is no profitable deviation for any history, and therefore this is a SPE. Clearly player 2 cannot improve their utility, as L is the best response to U in the baseline game, and player 1 only makes proposals of U. On the other hand, player 1 cannot profitably deviate as they cannot receive a payoff higher than 2 given the strategy of player 2. To see this, notice that due to the strategy of 2, it is only possible to terminate in an agreement with (U, L). Now consider any strategy of 1. In order to be profitable, it must be that player 2 plays L sufficiently often, as this (D, L)

is the only outcome that provides a higher payoff. However, by the strategy of player 2 the play cannot terminate in (D, L), therefore the only possibility of a profitable deviation is to induce perpetual disagreement where (D, L) is proposed frequently enough. In order for it to be perpetual disagreement, and given the strategy of 2 it must be C and R are played at least as frequently as (D, L). Therefore this leads to a payoff of at most 2. Therefore it cannot be that utility is improved.

However, within this game, from a history h such that $s^*(h) = (U, C)$, it is clear the strategy only delays the eventual agreement of (U, L), and therefore we do not have a Negotiated Binding Agreement. ▼

I now turn to a leading example.

2.2. Leading Example and Preview of Results

Here I provide a leading example to illustrate the key ideas in the paper. The environment will be a 3 player Bertrand Oligopoly with heterogeneous marginal costs and a unit demand. Specifically, there are three firms, $N = \{1, 2, 3\}$, who may each set an integer price $p_i \in \{0, 1, 2, 3, ..., 10\}$, where 10 is the highest price a firm may set.[8] Each firm faces a marginal cost c_i. It is assumed that: $c_1 = 1$, $c_2 = 3$ and $c_3 = 4$. The firms face a unit demand and the firms who set the lowest price share the demand equally. Therefore utility is given by their individual demand multiplied by the price they set minus their marginal costs. Therefore, when they do not set the lowest price they receive a utility of 0. Formally,

$$u_i(p) = \begin{cases} \frac{p_i - c_i}{|\operatorname{argmin}_{j \in \{1,2,3\}} p_j|} & \text{if } i \in \operatorname{argmin}_{j \in \{1,2,3\}} p_j \\ 0 & \text{if } i \notin \operatorname{argmin}_{j \in \{1,2,3\}} p_j \end{cases}$$

Note that in any Nash equilibrium firms 2 and 3 gain a utility of 0, while firm 1 gains a utility of at least 1. For instance, the profile of prices given by $(p_1^*, p_2^*, p_3^*) = (2, 3, 3)$ is a Nash equilibrium, yielding a payoff of 1 for player 1 and 0 for both other players.

First, we will understand what cannot be supported in a Negotiated Binding Agreement. Firstly, can any firm setting a price of 0 be supported in a Negotiated Binding Agreement s^*? If this were the case, then firm i would receive a strictly negative utility in equilibrium. However, such an agent could deviate to a new strategy that proposes a price of c_i at all partial histories; $s_i'(h) = c_i, \quad \forall h \in H$. If they do so, by the definition of U_i, we have that $U_i(s_i', s_{-i}^*|\emptyset) = 0 > U_i(s^*|\emptyset)$. This is *regardless* of s_{-i}^*, as either a) s_i', s_{-i}^* ends in agreement such that firm i agrees to c_i and improves their utility or b) s_i', s_{-i}^* results in

[8] It could equally be assumed that setting a price beyond 10 faces demand 0 without a change in the results.

perpetual disagreement, in which case, the payoff is defined with respect to the proposals along the path of s_i', s^*_{-i}, and therefore is defined by proposals involving only c_i for firm i. In either case, the utility is 0. In other words, setting a price of 0 is *individually irrational*, which will be formally introduced and discussed in the next section. This is as no matter the prices that others set, which may change depending on the price you set, the min-max payoff, which is weakly higher than that of setting a price of c_i, is higher than the payoff of setting a price of 0. Therefore it cannot be that any firm setting a price of 0 can be supported by a Negotiated Binding Agreement. By no babbling, it cannot be 0 is proposed at any history.

We have concluded that there is no Negotiated Binding Agreement in which 0 is *ever* proposed by any firm. With this, let us now consider whether any Negotiated Binding Agreement can support firm 2 or 3 agreeing to a price of 1. It must be that they would receive a strictly negative utility. This is as they certainly serve at least part of the market as 1 is the minimum price that can be set now we have eliminated the possibility of others setting a price of 0, and is below their marginal cost. Using the same logic as before, we can see that proposing a price of c_i is certainly better than agreeing to set a price of 1. In other words, for firm 2 or 3 a price of 1 does not survive *iterated elimination of individually irrational actions*, which will be formally defined and discussed in the next section. Therefore it cannot be that either firms 2 or 3 setting a price of 1 can be supported, and it cannot be that firm 2 or 3 proposes 1 at any history. Using the same induction, we may conclude that firm 1 will also never propose a price of 1 and firms 2 and 3 will never propose a price of 2.

In conclusion, it is necessarily the case that in any Negotiated Binding Agreement no proposal of firm 1 is less than 2, and no proposal of firms 2 and 3 is less than 3. Given that we have ruled out what cannot be supported by Negotiated Binding Agreement, we will go on to examine what can be supported. Firstly, notice that for any vector of prices that are supported by a Negotiated Binding Agreement it must be that firm 1 receives a payoff greater or equal to 1. As firms 2 and 3 never propose less than 3, if firm 1 were to deviate to a constant strategy that proposes a price of 2 for all histories firm 1 would guarantee themselves a payoff of at least 1. This is regardless of whether the history ends in perpetual disagreement or an agreement. Given this, we may conclude that any Negotiated Binding Agreement must provide firm 1 with a payoff of at least 1 to ensure such a deviation is not profitable. Similarly, firms 2 and 3 must receive a payoff of at least 0. We will now show that *any* such profile is supported, denoted by p^* such that $u_1(p^*) \geq 1$, $u_2(p^*) \geq 0$, and $u_3(p^*) \geq 0$. Consider the following strategy profile. In the initial history, $s_i^*(\emptyset) = p_i^*$. In the history where p^* has been proposed in the first round, players again propose p_i^* and the game terminates: $s_i^*(p^*) = p_i^*$. For any other history, let $s_1^*(h) = 2$, $s_2^*(h) = s_3^*(h) = 3$. Now let us consider whether a deviation is profitable. For firms 2 and 3 there is no incentive to deviate from the proposals at the initial history or the history p^*, as this will lead to firm 1 proposing 2 in all subsequent periods. Therefore

if the negotiation ends in an agreement for which firm 1 plays 2, which implies that the deviating firm receives a profit of at most 0, weakly worse than not deviating. Similarly, if the negotiation results in perpetual disagreement, it must be the payoff is with respect to firm 1 proposing a price of 2, which again cannot give a payoff greater than 0. Similarly, they cannot deviate from any other history. A similar logic holds for firm 1.

In conclusion, we have that any vector of prices p^* such that $u_1(p^*) \geq 1$, $u_2(p^*) \geq 0$, and $u_3(p^*) \geq 0$ can be supported by a Negotiated Binding Agreement.

With this, I move on to provide general necessary and sufficient conditions, which this example has already pointed to.

3 Negotiated Binding Agreement Outcomes

3.1. Necessary Conditions

Within this section, I characterise a number of necessary conditions for a Negotiated Binding Agreement outcomes and strategies. First, I will show that any action proposed at some history of Negotiated Binding Agreement must survive a procedure of iterated deletion of individually irrational actions. This procedure works inductively as follows. If an individual's action, regardless of the action profile of other agents chosen, always provides a payoff that is not individually rational, in the sense of inf-sup utility, then it is individually irrational. In the iterated elimination we can therefore remove said actions from consideration. Now, upon deleting such actions, we proceed inductively, where if an individual's action, regardless of the action profile of other agents chosen *within* the set that has survived iterated deletion of individually irrational actions, always provides a payoff that is not individually rational, in the sense of inf-sup utility, where the inf is taken *over the set of actions that survives iterated individual rationality*, then it does not survive iterated deletion of individually irrational actions. In this subsection, I show that any action that is proposed must survive iterated deletion of individually irrational actions. The formal definition of individual rationality and iterated individual rationality are formally defined below.

Definition 2 (Individually Irrational actions given $C_{-i} \subseteq A_{-i}$)**.** *$a_i \in A_i$ is individually irrational given $C_{-i} \subseteq A_{-i}$ if*

$$\inf_{a'_{-i} \in C_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) > \sup_{a_{-i} \in C_{-i}} u_i(a_i, a_{-i})$$

Denote the set of actions that are individually irrational given C_{-i} by $D_i(C_{-i})$.

As I do not require that the utility functions are continuous and defined over a compact set, the minimum or maximum need not exist. With this, I take the supremum and

infimum, which by the assumption that the utility function is bounded are always well defined. Whenever the game being considered has well defined maxima and minima I will refer to them as such, rather than using the infimum and supremum. This notion is similar to the notion of absolute dominance by Salcedo (2017), simultaneously developed in Halpern and Pass (2018), instead of comparing the best case of one action and the worst case of another we compare based on the best case of an action compared to the inf-sup. Therefore the set that survives elimination of individually irrational actions is smaller, as if an action is obviously dominated it is also individually irrational. Note that, if in a normal form game there is a single action that is not absolutely dominated given A_{-i}, then this action is an obviously dominant strategy as defined by Li (2017). Therefore if single action is not individually irrational it is also obviously dominant.

Definition 3 (Iterated Deletion of Individually Irrational Actions)**.** *Let $\tilde{A}_i^0 = A_i$ for all $i \in N$. Let $\tilde{A}_{-i}^0 = A_{-i}$. Then for all $m > 0$ let $\tilde{A}_i^m = \tilde{A}_i^{m-1} \backslash D_i(\tilde{A}_{-i}^{m-1})$ where $\tilde{A}_{-i}^{m-1} = \times_{j \neq i} \tilde{A}_j^{m-1}$.*

The set of actions that survive iterated deletion of individually irrational actions, or those that are iteratively individually rational, for i is given by $IIR_i = \bigcap_{m \geq 0} \tilde{A}_i^m$. Let $IIR = \times_{i \in N} IIR_i$.

Given these definitions, we can present the first necessary condition of Negotiated Binding Agreement, which states that any proposal must survive iterated elimination of individually irrational actions.

Theorem 1. *If s^* is a Negotiated Binding Agreement, then for all $h \in H$, $s_i^*(h) \in IIR_i$.*

Notice that this applies for all histories. Therefore any proposal being made must have survived iterated elimination of individually irrational actions. Notice that this is the exact process and result that was used in order to find the proposals that could occur within the leading example, resulting in no proposal including a price of 0 or 1 for any firm and firms 2 and 3 proposing a price of 2.

To better understand the set of actions that survives iterated elimination of individually irrational actions, note the following. In a large class of games, non-emptiness of the set of actions that are iteratively individually rational is implied by the fact that the set of actions that survive iterated elimination of never best responses to pure actions, a refinement of rationalizable strategies as defined by Bernheim (1984); Pearce (1984), also survive iterated elimination of individually irrational actions. This is formalised in the following definition and lemma.

Definition 4. *Let $a_i \in A_i$ be a be a never best response to a pure action in $C_{-i} \subseteq A_{-i}$ if, for all $a_{-i} \in C_{-i}$ there is some $a_i' \in A_i$ for which $u_i(a_i', a_{-i}) > u_i(a_i, a_{-i})$. Denote the set*

of actions that are never best responses to pure actions in C_{-i} by $NBR_i(C_{-i})$.

Let $B_i^0 = A_i$. Let $B_i^k = B_i^{k-1} \backslash NBR_i(A_{-i}^{k-1})$. Let $B^k = \times_{i \in N} B_i^k$ and $B_{-i}^k = \times_{j \neq i} B_j^k$. Let the set of actions that survive iterated elimination of never best responses to pure actions be given by $IENBR = \bigcap_{k \geq 1} B^k$.

Lemma 3. *The set of actions that survive iterated elimination of never best responses to pure actions also survives iterated elimination of iterated deletion of individually irrational actions: $IENBR \subseteq IIR$.*

Note that the set of actions that survives iterated elimination of never best replies is necessarily non-empty in finite games. Further, typically even more profiles may survive iterated elimination of individually irrationally actions than never best responses to pure actions. To see this, consider the following game.

Example 2. Consider the following prisoners' dilemma game.

1\2	C	D
C	3,3	0,4
D	4,0	1,1

D is strictly dominant for both players, hence the only profile that survives survive iterated elimination of never best responses to pure actions. Yet, in IIR, all action profiles survive. This is as the maximum payoff that C can receive is 3. The individually rational payoff is given by 1. Therefore, by definition, C is not individually irrational. ▼

Further, and importantly for the case of Negotiated Binding Agreements, the result of lemma 3 gives rise to the following corollary, that any pure Nash equilibrium is contained in IIR.

Corollary 1. *If a^{NE} is a pure Nash equilibrium of G then $a^{NE} \in IIR$.*

A further useful set of actions, and using a similar logic to example 2, we can find a chain of action profiles such that: for each player there is an action profile such that it prescribes their best response to the action prescribed to others in that profile, while the other profiles give them a weakly higher utility. Note, joint with lemma 1, this implies all profiles that Pareto dominate a pure Nash equilibrium remain in IIR. Having such a set of profiles within IIR will be further leveraged for sufficient conditions. This is given formally in the following lemma.

Lemma 4. *If $\{a^*, \underline{a}^1, .., \underline{a}^n\} \subseteq A$ satisfy:*

1. $\underline{a}_i^i \in \text{argmax}_{a_i \in A_i} u_i(a_i, \underline{a}_{-i}^i)$

2. $u_i(a^*) \geq u_i(\underline{a}^i)$

3. $u_i(\underline{a}^j) \geq u_i(\underline{a}^i)$ *for all* $i, j \in N$

Then $\{a^*, \underline{a}^1, .., \underline{a}^n\} \subseteq IIR$.

The next result provides further necessary conditions, that combine the necessary conditions in theorem 1 with individual rationality considerations over the set of actions that survive iterated elimination of individually irrational actions.

Theorem 2. *if* s^* *is a Negotiated Binding Agreement then*

$$U_i(s^*|h) \geq \inf_{a'_{-i} \in IIR_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a'_{-i})$$

for all $h \in H$ *and* $i \in N$.

I illustrate the use of this result with the same prisoner's dilemma game as in example 2.

Example 2. revisited Again consider the following prisoners' dilemma game

1\2	C	D
C	3,3	0,4
D	4,0	1,1

In this case, no actions are individually irrational for any player, as previously argued. However, notice that the min-max payoff for each player is 1. The min-max is given by 1, as the worst outcome is the other player selecting D. Therefore we conclude that no Negotiated Binding Agreement can support the action profile (D, C) or (C, D). However, the necessary conditions do not rule out the possibility of (C, C). ▼

Note that the inf-sup restricted to the set of actions that survives iterated elimination of individually irrational actions is always weakly higher than the inf-sup without this restriction.

Remark 1. *For any game* G *such that* $\underline{u}_i$ *is well defined the following inequality holds:*

$$\inf_{a'_{-i} \in IIR_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) \geq \inf_{a_{-i} \in A_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$$

Notice this inequality holds strictly within the leading example: the min-max payoff for firm 1 is 0, via other firms setting prices of 0, however the min-max payoff when we restrict ourselves to IIR is 1.

The results of this section bear resemblance to the analysis of infinitely repeated games, where individual rationality constraints must be satisfied. I reserve this discussion for the literature review in section 8.

Finally, before moving to the sufficient conditions for an action profile to be supported by a Negotiated Binding Agreement, note that in two-player games, where the action space is a compact subset of metric space and utility is continuous the conditions of lemma 4 are also necessary. That is, a^* can be supported by a Negotiated Binding Agreement only if there exists a punishment for each player, where each player is prescribed the best response to their punishment within their punishment profile, they prefer the other players' punishment to their own, and they prefer a^* to their punishment. This is formalised by the following theorem.

Theorem 3. *For any game G such that $N = \{1, 2\}$, A_i is compact subset of a metric space and u_i is continuous for all $i \in \{1, 2\}$, a^* is supported by a Negotiated Binding Agreement, s^*, only if $\exists\{\underline{a}^1, \underline{a}^2\} \subseteq A$ such that:*

1. $\underline{a}_i^i \in \text{argmax}_{a_i \in A_i} u_i(a_i, \underline{a}_{-i}^i)$
2. $u_i(a^*) \geq u_i(\underline{a}^i)$
3. $u_i(\underline{a}^j) \geq u_i(\underline{a}^i)$ *for all* $i, j \in N$

The proof is similar to that of theorem 5, and therefore is omitted.

3.2. Sufficient Conditions

The first sufficient condition I provide states that if the outcome of iterated elimination of individually irrational actions is unique for all players then that only profile can be supported by any Negotiated Binding Agreement.

Corollary 2 (Conditions for a Unique Outcome)**.** *For any game G such that A_i is compact subset of a metric space and u_i is continuous for all $i \in N$, if $IIR = \{a^*\}$ then there is a unique Negotiated Binding Agreement and it is such that $s_i^*(h) = a_i^*$ for all histories.*

Of course, this uniqueness result requires strong conditions. Nonetheless, examples of this result do exist. This would be true in example 1 where only (D, L) survives iterated elimination of individually irrational actions. This corollary is a joint implication of theorem 1 and theorem 4 that follow.

For the more general conditions, we require that each agent has a specific action profile, which I will denote $\underline{a}^i$. This can be thought of as the punishment of deviation used for i. For this action profile, i will best respond to $\underline{a}^i_{-i}$ in the baseline game G. The action that is sustained in Negotiated Binding Agreement, which I will denote a^*, must, for each player i, give a weakly higher payoff than $\underline{a}^i$. Further, I will require that the punishment of other agents gives a weakly higher payoff than the punishment for i. If such a collection of action profiles exist, then a^* can be supported by a Negotiated Binding Agreement. Notice by lemma 4, such a set of actions will be within IIR. In essence, this relies on player specific punishment strategies, that have been used for sufficiency for SPE in infinitely repeated games (Fudenberg and Maskin, 1986; Abreu et al., 1994). The requirements here are more stringent, as the profile used to punish i must use i's best response to the punishment in the baseline game. This is as there are no future payoff to compensate for abiding to the punishment, as the agreement to play such an action is binding and the game terminates. I discuss the reasoning for this disparity further within the literature review. I state this formally in the following theorem.

Theorem 4. *Take any game such that* $\exists\{a^*, \underline{a}^1, ..., \underline{a}^n\} \subseteq A$ *such that:*

1. $\underline{a}^i_i \in \text{argmax}_{a_i \in A_i} u_i(a_i, \underline{a}^i_{-i})$
2. $u_i(a^*) \geq u_i(\underline{a}^i)$
3. $u_i(\underline{a}^j) \geq u_i(\underline{a}^i)$ *for all* $i, j \in N$

Then a^* *can be supported in a Negotiated Binding Agreement.*

It is worth noting that any pure Nash equilibrium of the game G is indeed supported by a Negotiated Binding Agreement. This is immediately implied by the fact any pure Nash equilibrium, denoted by a^{NE}, are in IIR via lemma 1, and can be used as the punishment for all individuals. That is, $\underline{a}^i = a^{NE}$ for all players. Further, any action profile that Pareto dominates a pure Nash equilibrium can be sustained by this reasoning. However, in games where no pure Nash equilibrium exist there may exist a Negotiated Binding Agreement due to the above sufficient conditions.

Example 3. Take the following two-player game. For clarity, I have underlined the corresponding best responses in the baseline game.

1\2	L	C	R
T	7,7	<u>4</u>,4	0,<u>12</u>
M	4,<u>4</u>	0,0	<u>2</u>,3
D	<u>12</u>,0	3,<u>2</u>	1,1

Notice that there is no pure Nash equilibrium in this game. However, there exists a Negotiated Binding Agreement. Specifically, applying theorem 4 take $a^* = (T, L)$, while taking $\underline{a}^1 = (M, R)$ and $\underline{a}^2 = (D, C)$, which satisfies the assumptions. Therefore there exists a Negotiated Binding Agreement that supports (T, L), while there is no pure Nash equilibrium in the underlying game. ▼

In two-player games where the action space is a compact subset of a metric space and u_i is continuous for each player such, conditions are both necessary and sufficient.

Corollary 3. *For any game G such that $N = \{1, 2\}$, A_i is compact subset of a metric space and u_i is continuous for all $i \in \{1, 2\}$, a^* is supported by a Negotiated Binding Agreement, s^*, if and only if $\exists\{\underline{a}^1, \underline{a}^2\} \subseteq A$ such that:*

1. $\underline{a}^i_i \in \text{argmax}_{a_i \in A_i} u_i(a_i, \underline{a}^i_{-i})$
2. $u_i(a^*) \geq u_i(\underline{a}^i)$
3. $u_i(\underline{a}^j) \geq u_i(\underline{a}^i)$ *for all* $i, j \in N$

This is a direct implication of theorems 3 and 4.

Before moving forward, I point to the following corollary.

Corollary 4. *If a^{NE} is a pure Nash equilibrium of G such that:*

$$u_i(a^{NE}) = \min_{a_{-i} \in IIR_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i})$$

, i.e. the IIR minmax profiles are mutual, then a^* can be supported by a Negotiated Binding Agreement if and only if $u_i(a^*) \geq u_i(a^{NE})$.

This is a direct implication of theorems 2 and 4. This provides a class of games for which the Negotiated Binding Agreements are fully characterised by action profiles that Pareto Dominate a Nash equilibrium. Specifically, if that Nash equilibrium provides agents with their individually rational payoffs over the set of actions that survives iterated deletion of individually irrational actions, then an action profile can be supported by a Negotiated Binding Agreement if and only if said action profile Pareto Dominates this Nash equilibrium.

Further justification for the general sufficient conditions can be found. For a refinement of Negotiated Binding Agreements, where the focus is upon SPE that end in immediate agreement following from each history the sufficient conditions are also necessary in games where the action space is a compact subset of a metric space and utility is continuous. I refer to this solution as No Delay Negotiated Agreements and is similar to the no delay

equilibrium proposed by Chatterjee et al. (1993). Therefore, for the class of No Delay Negotiated Binding Agreements, I fully characterise the set of outcomes that can be supported. Here I formally define No Delay Negotiated Binding Agreement and state the formal result.

Definition 5 (No Delay Negotiated Binding Agreement)**.** *s^* is a No Delay Negotiated Binding Agreement supporting $a^* = a(s^*|\emptyset)$ if:*

a) s^* is a subgame perfect equilibrium.

b) No Delay: For all partial histories $h \in H$, $s^*(h) = s^*(h, s^*(h)) = a^*(s^*|h)$.

With this, I turn to formally stating the result.

Theorem 5. *For any game G such that A_i is compact subset of a metric space and u_i is continuous for all $i \in N$, a^* is supported by a No Delay Negotiated Binding Agreement, s^*, if and only if $\exists\{\underline{a}^1, ..., \underline{a}^n\} \subseteq A$ such that:*

1. $\underline{a}_i^i \in \text{argmax}_{a_i \in A_i} u_i(a_i, \underline{a}_{-i}^i)$
2. $u_i(a^*) \geq u_i(\underline{a}^i)$
3. $u_i(\underline{a}^j) \geq u_i(\underline{a}^i)$ *for all* $i, j \in N$

Finally, note that within the literature on agreements it is common to use the notion of perfect equilibrium of Selten (1988), for instance in Kalai (1981) and Bhaskar (1989). This is a subgame perfect equilibrium that does not make use of weakly dominated strategies at any history. Notice that this does not have a significant change in the results, and to ensure the sufficient conditions remain true for this refinement, as well as the no babbling and agreement for all histories condition, the only check is to ensure that the action $\underline{a}_i^i$ is not weakly dominated in the underlying game G. One would also need to rule out weakly dominated actions from IIR, keeping the necessary conditions the same for this refinement of IIR. This, for instance, would rule out the possibility of using the worst Nash equilibrium as a punishment in the Bertrand game in the leading example. However the other Nash equilibrium of the game, where $p_1^* = 3$, $\min\{p_2^*, p_3^*\} = 4$ would provide the same logical result, albeit changing the lower bar of utility that firm 1 must receive.

With these results, I now turn to some applications.

4 Applications

In this section, I explore two key applications of Negotiated Binding Agreement, within which the necessary and sufficient conditions allow for a full characterisation of Negotiated

Binding Agreement. These applications also provide the intuitions surrounding the proofs of the results presented in the paper.

In application 1, I explore a public good game and fully characterise the set action profiles that can be supported in Negotiated Binding Agreement. In this setting, contribution of an agent can only be supported if sufficiently many other agents also contribute. With this, full and no contribution can be supported. In application 2, I consider a simple Cournot Duopoly with potentially heterogeneous marginal costs. I fully characterise the set of actions that can be supported by Negotiated Binding Agreements and show that when marginal costs are the same, any profile of payoffs that gives both players positive profits is supportable. In contrast, when marginal costs are extremely different, only the firm with the lowest marginal cost receiving their monopoly profit can be supported.

Application 1. (Public Goods Game) $N = \{1, 2, ..., n\}$. Let $A_i = \{c, d\}$ for each i. Let $u_i(a) = 1 + k\left[\sum_{j \in N} \mathbf{1}_{a_j = c}\right] - \mathbf{1}_{a_i = c}$ with $k \in \left(\frac{1}{n}, 1\right)$.

Firstly notice that for any player it is strictly dominant to choose d and hence the only Nash equilibrium payoff is 1.

I will now construct a strategy that allows for any action profile that Pareto dominates the Nash equilibrium to be supported by Negotiated Binding Agreement. Specifically, let a^* denote an action profile such that $u_i(a^*) = 1 + k|\{i \in N : a_i^* = c\}| - \mathbf{1}_{a_i^* = c} \geq 1$. Now construct s^* as follows. Let $s_i^*(\emptyset) = s_i^*(a^*) = a_i^*$. For all other partial histories let $s_i^*(h) = d$. First, notice that for the partial histories $\emptyset$ and (a^*) we have that $s^*(h) = a^*$ while $a(s^*|h) = a^*$. Secondly, notice that for all other partial histories we have $s^*(h) = (d)_{i \in N}$ while $a(s^*|h) = (d)_{i \in N}$. Concluding the condition for a no babbling agreement is always satisfied. All that is left to show is that s^* is a subgame perfect equilibrium. Suppose not, there is some partial history $h \in H$ such that there is some other strategy $s_i \in S_i$ such that $U_i(s_i, s_{-i}^*|h) > U_i(s^*|h)$. There are two possible cases.

1. The first possibility is that $h = \emptyset$ or (a^*). Notice for a deviation to be profitable it must be such that $a(s_i, s_{-i}^*|h) \neq a^*$, as otherwise a strict inequality cannot hold. Given this, the strategy s_i, s_{-i}^* must induce a history $h' \neq (a^*, a^*)$. Therefore, it must be that all other players choose d for all periods other than the first. There are three possibilities.

   (a) Firstly, it may be that the strategy s_i, s_{-i}^* induces a terminal history, $z \in Z'$, with the agreement $(d)_{i \in N}$. This induces a payoff of 1 for player i, while $U_i(s^*|\emptyset) = u_i(a^*) \geq 1 = U_i(s_i, s_{-i}^*|\emptyset)$, therefore this cannot be profitable.

   (b) It may be that the strategy induces a terminal history, $z \in Z'$, with the agreement $((d)_{j \neq i}, c)$. However, this leads to a payoff of 0 for player i, while $U_i(s^*|\emptyset) = u_i(a^*) \geq 1 > 0 = U_i(s_i, s_{-i}^*|\emptyset)$, therefore this cannot be profitable.

(c) It may be that s_i, s^*_{-i} induces a terminal history, z such that d is played by all other players in all but the first period, and no agreement is made, i.e. $z \in Z''$. As no agreement is made, it must be that there are no two consecutive periods where the same action profile is played by all players it must be that s_i alternates between d and c. This implies that the lim sup of utilities induces by the proposals is 1. As $U_i(s^*|\emptyset) = u_i(a^*) \geq 1 = U_i(s_i, s^*_{-i}|\emptyset)$, this cannot be a profitable deviation.

2. Now suppose the history is partial and such that $h \neq \emptyset$ and $h \neq a^*$. No deviation leads to the agreement $(d)_{i\in N}$, with a payoff of 1. A deviation can only lead to the three cases examined above. Given this, the logic of the previous case remains true.

In conclusion, for any a^* such that $u_i(a^*) = 1 + k|\{i \in N : a^*_i = c\}| - \mathbf{1}_{a^*_i=c} \geq 1$ holds, we can provide a Negotiated Binding Agreement that supports such a profile. Further to this, it provides some intuition behind the sufficiency proof of theorem 4 and the result of said theorem would imply this result.

To explore this further, notice that this implies that $a^* = (d)_{i\in N}$ may be supported. Further to this, a number of action profiles that maintain contribution can be supported by a Negotiated Binding Agreement. Specifically, for some a^* such that there exists some i such that $a^*_i = c$, we have that $1 + k|\{i \in N : a^*_i = c\}| > k|\{i \in N : a^*_i = c\}|$, i.e. the number of players contributing have a strictly lower utility that those who are not. With this, we can see that any a^* such that $k|\{i \in N : a^*_i = c\}| \geq 1$. More succinctly, when the number of contributors is above a lower bound, $|\{i \in N : a^*_i = c\}| \geq \frac{1}{k}$, the action profile can be supported by a Negotiated Binding Agreement. As $\frac{1}{k} < n$ this implies that full cooperation can be sustained.

Finally, to show that this fully characterises the Negotiated Binding Agreement, suppose that there is some equilibrium s^* that supports some a^* such that $u_i(a^*) < 1$ for some $i \in N$. For this to be the case it must be that $a(s^*|\emptyset) = a^*$. Now consider a deviation of $i \in N$ such that $s_i(h') = d$ for all histories $h' \in H$ at $h = \emptyset$. Such a deviation ensures that in any terminal history the payoff is pinned down by $u_i(d, a_{-i})$, be that if the history ends in agreement or not. If it does not end in agreement, it is pinned down by between some $u_i(d, a_{-i})$ with $a_{-i} \in \{c, d\}^{n-1}$. However, $u_i(d, a_{-i}) \geq 1$ for all possible $a_{-i} \in \{c, d\}^{n-1}$. Therefore $U_i(s_i, s^*_{-i}|\emptyset) \geq 1 > U_i(s^*|\emptyset)$. Therefore it cannot be that s^* is a subgame perfect equilibrium and therefore cannot be a Negotiated Binding Agreement. ▼

Application 2. (Cournot Duopoly with Heterogeneous Marginal Costs) Consider a simple Cournot Duopoly model where $q_1, q_2 \in [0, b] = A_i$ where the inverse demand curve is given by $p(q_1, q_2) = \min\{b - q_1 - q_2, 0\}$. Let firms have heterogeneous costs, c_1 and c_2 where without loss of generality $c_1 \geq c_2 \geq 0$. Assume that $\frac{b+c_2}{2} \geq c_1$. Profits are given by

$$\pi_i(q_1, q_2) = q_i(p(q_1, q_2) - c_i)$$

Notice that the static best responses are given by

$$q_i^*(q_{-i}) = \begin{cases} 0 & \text{if } q_{-i} \geq b - c_i \\ \frac{b-c_i-q_{-i}}{2} & \text{if } q_{-i} < b - c_i \end{cases}$$

This leads to profits of

$$\pi_i(q_i^*(q_{-i}), q_{-i}) = \begin{cases} 0 & \text{if } q_{-i} \geq b - c_i \\ \left(\frac{b-c_i-q_{-i}}{2}\right)^2 & \text{if } q_{-i} < b - c_i \end{cases}$$

Consider the following strategy for proposals for supporting (q_1^*, q_2^*) such that $\pi_1(q_1^*, q_2^*) \geq 0$ and $\pi_2(q_1^*, q_2^*) \geq (c_1 - c_2)^2$ in a Negotiated Binding Agreement. Note given the assumption that $\frac{b+c_2}{2} \geq c_1$ it follows that $(\frac{b-c_2}{2})^2 \geq (c_1 - c_2)^2$ and therefore such a profile exists. $s^*(\emptyset) = s^*(h) = (q_1^*, q_2^*)$ such that $h = (q^1, q^2, ..., (q_1^*, q_2^*))$, $s^*(h') = (0, \underline{q}_1^1)$, where $\underline{q}_2^1 = b - c_1$ if $h' = (q^1, q^2, .., (q_1', q_2^*))$ for $q_1' \neq q_1^*$, $h' = (q^1, q^2, .., (q_1, \underline{q}_2^1))$, and $s^*(h'') = \underline{q}^2 = (b - 2c_1 + c_2, c_1 - c_2)$ for all other histories. Notice that such a strategy profile satisfies agreement for all histories and no babbling. Therefore all that is left is to check that s^* is a subgame perfect equilibrium. Suppose that firm 1 has a profitable deviation at any history. It cannot be that it is profitable to deviate from $h = \emptyset$ or $h = (q^1, q^2, ..., (q_1^*, q_2^*))$ as this leads to player 2 playing $b - c_1$ for all periods. Therefore firm 1 can receive a utility of at most 0 via any deviation, as the static best response to $b - c_1$ is to set a quantity of 0. The same logic holds for all other cases, as, by construction, the static utility at every period of any other history is weakly less than 0, no matter s_i'. Suppose that firm 2 has a profitable deviation. It cannot be that a profitable deviation exists from $h = \emptyset$ or $h = (q^1, q^2, ..., (q_1^*, q_2^*))$ as this will lead to firm 1 proposing $b - 2c_1 + c_2$ in all periods. Therefore the highest possible utility is given by the static best response utility to such a quantity, given by $(c_1 - c_2)^2$. By construction, $U_i(s^*|\emptyset) = U_i(s^*|(q^1, q^2, ..., q^*)) \geq (c_1 - c_2)^2$, therefore it cannot be profitable. Further, *any* deviation leads to $s_1(h) = b - 2c_1 + c_2$, for which the static best response in each period would be $c_1 - c_2$, and therefore leading to a payoff no higher than $(c_1 - c_2)^2$. Finally, note that $U_i(s^*|q^1, q^2, .., (q_1', q_2^*)) = U_i(s^*|q^1, q^2, .., (q_1, \underline{q}_2^1)) = (b - c_1)(c_1 - c_2) \geq (c_1 - c_2)^2$ and therefore it cannot be that it is profitable to deviate from such a history either. Note that by corollary 3, this fully characterises the set of payoffs and strategies that can be supported by a Negotiated Binding Agreement, as it gives both firms 1 and 2 the lowest possible payoffs they could best respond to, while maintaining that they prefer the punishment of the other firm to their own.

The payoff space is represented by figure 1.

Note that the construction provides us with some natural comparative statics. If $c_1 = c_2$, then both players may receive any payoff above 0. If $c_1 = \frac{b+c_2}{2}$ then we conclude that $\pi_2(\underline{q}^2) = \left(\frac{b-c_2}{2}\right)^2$, their monopoly profits.[9] ▼

[9]Note that if $c_1 > \frac{b+c_2}{2}$ then the only outcome that can be supported by a Negotiated Binding Agreement

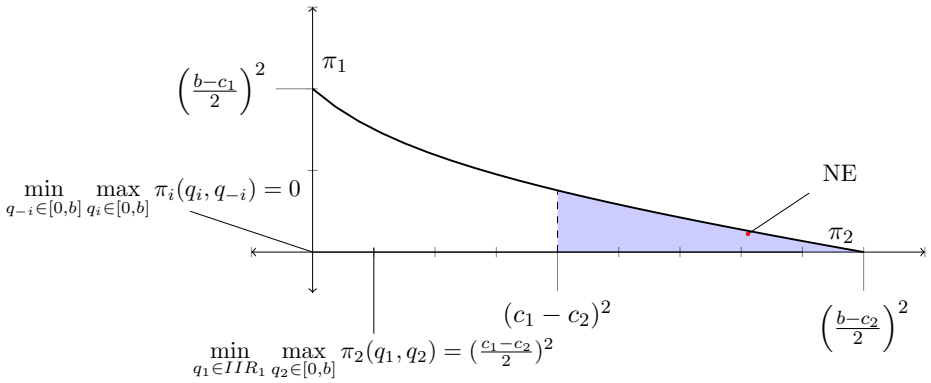

Figure 1: The payoff space of the Cournot Duopoly in example 2, where costs are such that $0 \leq c_2 < c_1 < \frac{c_2+b}{2}$. The black curve represents the payoff frontier. The shaded blue area represents that payoffs that can be sustained in Negotiated Binding Agreements.

5 Coalitional Deviations

In principle, a negotiation may be susceptible to a collection of agents making binding agreements over how they will play *within* the negotiation process itself. To address these concerns, I now extend the analysis to allow for this possibility. To do so, I include a collection of permissible coalitions, where a coalition may jointly deviate. The richest of all such possibilities is the power set of N, which allows *any* possible subset of players to jointly deviate.

In this analysis, I will look for the most robust form of equilibrium, that prevents any permissible coalition from deviating, where coalitions are permitted to agree to any deviation. This can be seen as stronger than necessary, as we may wish for the deviations to face the same criticism of stability, where these deviations must be the result of some agreement.[10] However, if it were possible to make a binding agreement to not make a new binding agreements, agents may take this option upon deviating. Therefore, in the context of binding agreements, if we do not wish to make assumptions surrounding the game that is induced to negotiate when a deviation occurs then this approach ensures no misspecification. That is, do we allow for agents within a coalition to have veto power? Do we allow for agents to make agreements over what can be within the agreement in the sense that they pre-commit to rule out some options? This can potentially allow for different conclusions in the outcome of the game. Nonetheless, if all deviations of a coalition are permitted, this includes the outcomes of processes, and therefore if we have an

is $q_1^* = 0$, $q_2^* = \frac{b-c_2}{2}$.

[10] This would be in line with a concept of renegotiation proofness a la Farrell and Maskin (1989) and Bernheim and Ray (1989).

equilibrium that allows for all possible deviations we certainly have an equilibrium when all such deviations are not allowed.

I first introduce the notation of a coalition and coalition configuration. A coalition configuration defines the set of coalitions that may make a binding agreement within the negotiation. I let a coalition configuration be denoted by $\mathcal{C}$, and only restrict $\mathcal{C}$ to be a cover of N. That is, for all $i \in N$, there is some coalition $C \in \mathcal{C}$ such that $i \in C$. For a coalition configuration $\mathcal{C}$, if $C \in \mathcal{C}$ I will refer to C as permissible.

Further to this, for a non-empty coalition $C \in \mathcal{C}$, let $a_C = (a_i)_{i \in C}$, $A_C = \times_{i \in C} A_i$, $s_C = (s_i)_{i \in C}$ and $S_C = \times_{i \in C} S_i$. Let $a_{-C} = (a_i)_{i \notin C}$, $A_{-C} = \times_{i \notin C} A_i$, $s_{-C} = (s_i)_{i \notin C}$ and $S_{-C} = \times_{i \notin C} S_i$. For a set $B \subset A$, which may or may not have a product structure, let $B_C = \{a_C \in A_C | \exists a'_{-C} \in A_{-C} \text{ s.t. } (a_C, a'_{-C}) \in B\}$ and $B_{-C} = \{a_{-C} \in A_{-C} | \exists a_C \in A_C \text{ s.t. } (a_C, a_{-C}) \in B\}$.

With this, I go on to define the natural extension of subgame perfect equilibrium when coalitions are permitted to jointly deviate. This will be referred to as $\mathcal{C}$-subgame perfect equilibrium and will require that strategies are such that, at no history, is there a way for *any* permissible coalition of players, $C \in \mathcal{C}$, to jointly deviate and improve the utility of all players within that coalition. In essence, this is assuming that, at any history, any permissible coalition may write a private binding agreement that dictates the behaviour they will take going forward. Note that the assumption that these agreements are private is important within this setting to ensure that the strategy of those outside are not dependent on the agreement itself. If the agreements were public, the concept would be closer to a coalitional version of Tennenholtz (2004)'s program equilibrium. I now define $\mathcal{C}$-Subgame Perfect equilibrium formally.

Definition ($\mathcal{C}$-Subgame Perfect Equilibrium)**.** *s^* is a $\mathcal{C}$-subgame perfect equilibrium if, for all partial histories $h \in H$, there does not exist a non-empty coalition $C \in \mathcal{C}$ and a joint strategy $s_C \in \times_{i \in C} S_i$, such that $u_i(s_C, s^*_{-C}|h) > U_i(s^*|h)$ for all $i \in C$.*

This concept includes a number of solution concepts, which I outline here:

1. Firstly, whenever $\mathcal{C} = \{\{i\}_{i \in N}\}$, $\mathcal{C}$-subgame perfect equilibrium and subgame perfect equilibrium of Selten (1965) coincide. Further to this, whenever $\{\{i\}_{i \in N}\} \subset \mathcal{C}$, $\mathcal{C}$-subgame perfect equilibrium is a refinement of subgame perfect equilibrium.

2. Whenever $\mathcal{C} = 2^N \backslash \{\emptyset\}$, $\mathcal{C}$-subgame perfect equilibrium coincides with the concept of strong perfect equilibrium of Rubinstein (1980). Whenever $\mathcal{C} = 2^N \backslash \{\emptyset\}$ I will refer to this concept as strong in its place. Note that any strong subgame perfect equilibrium would also be a $\mathcal{C}$-subgame perfect equilibrium for any $\mathcal{C}$.

3. Finally, when $\mathcal{C}$ is a partition of N, $\mathcal{C}$-subgame perfect equilibrium can be seen as the

extension of the coalitional equilibrium concept of Ray and Vohra (1997) to extensive form games.

I reserve a more in-depth discussion of the relation of this concept with these within the literature review.

Before defining the notion of Negotiated Binding Agreement with respect to this concept, it is worth noting that some coalition configurations can be seen as more reasonable than others in this case. Firstly, it seems reasonable to include all singletons within the coalition configuration, as allowing individuals to make unilateral deviations is in the essence of individual rationality. With this, I will concentrate the remainder of the analysis taking $\{i\}_{i\in N} \subseteq \mathcal{C}$ as implicit within the discussion, although it is not necessary for the formal results. I will also pay particular attention to the grand coalition being permitted $N \in \mathcal{C}$.

With this, I turn to defining $\mathcal{C}$-Negotiated Binding Agreement. This simply extends the notion of Negotiated Binding Agreement, instead of requiring a Negotiated Binding Agreement is a subgame perfect equilibrium, that has no babbling, I will require instead that it is a $\mathcal{C}$-subgame perfect equilibrium, that has a form of no babbling. Before formally defining this concept, note that the use of $\mathcal{C}$-subgame perfect equilibria when $N \in \mathcal{C}$, gives further justification for no babbling agreements, and indeed no delay agreements. To see this, suppose that there was some $\epsilon > 0$ cost for delay for all agents. If this were the case, then there would be no $\mathcal{C}$-subgame perfect equilibrium that concluded in more than two periods. To see this, suppose that the equilibrium concludes in a and did so in more than 2 periods from the current one. This is as if this were the case, the grand coalition containing all agents would be able to profitably deviate to a joint strategy that ends in two periods and concludes in a. With this, I turn to formally define $\mathcal{C}$-Negotiated Binding Agreement.

Definition 6 ($\mathcal{C}$-Negotiated Binding Agreement)**.** *s^* is a $\mathcal{C}$-Negotiated Binding Agreement supporting $a^* = a(s^*|\emptyset)$ if:*

1. *s^* is a $\mathcal{C}$-subgame perfect equilibrium*
2. *$\mathcal{C}$-no babbling: $\forall h \in H$, $\exists h' \in H$ such that $s^*_C(h) = a_C(s^*|h')$.*

Again, when $\mathcal{C} = 2^N \backslash \{\emptyset\}$ I refer to this as a strong Negotiated Binding Agreement.

Whenever $\{i\}_{i\in N} \subset \mathcal{C}$, $\mathcal{C}$-Negotiated Binding Agreement are a subset of Negotiated Binding Agreement and therefore necessary conditions still hold. However, we can strengthen these conditions, and provide conditions that hold for a general coalition configuration $\mathcal{C}$. I show that natural extensions of the necessary and sufficient conditions used for Negotiated Binding Agreement hold for $\mathcal{C}$-Negotiated Binding Agreement.

6 $\mathcal{C}$-Negotiated Binding Agreement Outcomes

6.1. Necessary Conditions

First, I will show that any action proposed at some history of $\mathcal{C}$-Negotiated Binding Agreement must survive a procedure of iterated deletion of coalitionally irrational actions. This procedure works inductively as follows. Consider some joint action of those within a coalition $C \in \mathcal{C}$, a_C. If, for a coalition $C \in \mathcal{C}$ there is some function, that maps the joint action of those outside of the coalition to a joint action of the coalition, which, even in the worst case said function can provide a higher payoff than the joint action a_C, then a_C is a coalitionally irrational joint action. This generalises the notion of individual rationality.[11] Notice this is essentially the notion of Aumann (1961)'s β-core. We may proceed inductively. Remove all coalitionally irrational actions for all coalitions $C \in \mathcal{C}$. Consider some joint action of those within a coalition $C \in \mathcal{C}$, a_C, which survives iterated elimination of coalitionally irrational actions up to some iteration k. If, for a coalition $C \in \mathcal{C}$ there is some function, that maps the joint actions that survive iterated coalitionally irrational actions of those outside of the coalition to a joint action of the coalition, which, even in the worst case of the joint actions outside outside of C that survives iterated elimination of coalitionally irrational actions is taken, said function provides a higher payoff than the joint action a_C, then a_C is a coalitionally irrational joint action at the iteration at hand. This provides a recursive version of Aumann (1961)'s β-core, where the "punishments" themselves must be justified. This, therefore, provides one answer to the question posed by Scarf (1971), providing a notion of the core for normal form games that is fully justified. The formal definition of coalitional rationality and iterated elimination of coalitionally irrational joint actions is formally defined below.

Definition 7. *For a coalition C, a joint action $a_C \in A_C$ is coalitionally irrational with respect to $B_{-C} \subseteq A_{-C}$ if, for some $a'_C : B_{-C} \to A_C$*

$$\inf_{a_{-C} \in B_{-C}} u_i(a'_C(a_{-C}), a_{-C}) > \sup_{a_{-C} \in B_{-C}} u_i(a_C, a_{-C}) \qquad \forall i \in C$$

Denote the set of joint actions that are coalitionally irrational with respect to B_{-C} by $D_C(B_{-C})$.

Definition 8 (Iterated Elimination of Coalitionally Irrationality actions with respect to $\mathcal{C}$)**.** *Let $\tilde{A}^0(\mathcal{C}) = A$. For $m > 0$ let $\tilde{A}^m(\mathcal{C}) = \tilde{A}^{m-1}(\mathcal{C}) \setminus \left[\bigcup_{C \in \mathcal{C}} \left[[D_C(\tilde{A}^{m-1}(\mathcal{C})_{-C})] \times A_{-C}\right]\right]$.*

Let the set of action profiles that survives iterated elimination of coalitionally irrational actions, or those that are iteratively coalitionally rational, with respect to $\mathcal{C}$ be denote $ICIR(\mathcal{C})$ where $ICIR(\mathcal{C}) = \bigcap_{m>0} \tilde{A}^m(\mathcal{C})$.

[11] For games with compact action spaces and continuous utility they are identical when $\mathcal{C} = \{\{i\}_{i \in N}\}$.

Note, unlike iterated elimination of individually irrational actions, iterated elimination of coalitionally irrational actions may be empty in finite games. To see this, consider the following example.

Example 4. Consider the following 2 player game where $\mathcal{C} = \{\{1,2\},\{1\},\{2\}\}$, i.e. both players may make unilateral deviations and may write a binding agreement at any history.

$1\backslash 2$	L	C	R
T	20,0	20,0	20,0
M	0,7.5	0,7.5	30,5
D	10,10	0,0	0,0

Notice that only (M,R) and (D,L) survive iterated elimination of coalitionally irrational actions for the coalition $C = \{1,2\}$. However, D cannot survive elimination of individually irrational actions for player 1, as the maximum payoff of D is 10 while the min-max utility for player 1 is 20. Therefore we conclude that within the first round of iterated elimination of coalitionally irrational actions only (M,R) survives. However, this implies that R is individually irrational with respect to M for player 2, as the profile (M,R) gives a payoff of 5 while the min-max utility, when restricting attention to player 1 playing R is 7.5. Therefore $ICIR(\mathcal{C}) = \emptyset$. ▼

However, it may be non-empty, even when a rich set of coalitions are permitted. Here I provide an example that shows how to find $ICIR(\mathcal{C})$. Before doing so, notice the following. If $\mathcal{C}' \subset \mathcal{C}$, then $ICIR(\mathcal{C}) \subseteq ICIR(\mathcal{C}')$. That is, if some action profile survives $ICIR(2^N \backslash \{\emptyset\})$ then it survives any other $\mathcal{C}$.

Example 5. Consider the following 2 player game where $\mathcal{C} = \{\{1,2\},\{1\},\{2\}\}$, i.e. both players may make unilateral deviations and may write a binding agreement at any history.

$1\backslash 2$	L	C	R
T	2,7	2,8	0,6
M	1,4	0,8	2,3
D	1,9	0,8	20,7.5

Notice that (D,R), and (D,L) and (T,C) are the set of Pareto efficient outcomes, therefore, as $\{1,2\} \in \mathcal{C}$, it must be all other action profiles are rules out in $\tilde{A}^1(\mathcal{C})$. Further, R is individually irrational for 2 as it provides a payoff of at most 7.5, while the min-max payoff is 8. We conclude that $\tilde{A}^1(\mathcal{C}) = \{(D,L),(T,C)\}$. Now notice that D is individually irrational for 1 with respect to $\tilde{A}^1_{-1}$, where $\tilde{A}^1_{-1} = \{L,C\}$, as the highest payoff that D can provide is 1 while the min-max payoff over this set is 2. We conclude that $\tilde{A}^2(\mathcal{C}) = \{(T,C)\}$. Finally, note that neither T or C are individually irrational given

$B_{-1} = \{C\}$ and $B_{-2} = \{T\}$ respectively. Therefore $ICIR(\mathcal{C}) = \{(T, C)\}$. ▼

One condition that ensures non-emptiness, regardless of the coalition configuration, is the existence of a strong Nash equilibrium.[12]

Lemma 5. *For any Strong Nash equilibrium a^{SNE} of G, $a^{SNE} \in ICIR(\mathcal{C})$ regardless of $\mathcal{C}$.*

With this definition, a similar necessary condition to theorem 1, linking $ICIR(\mathcal{C})$ and $\mathcal{C}$-Negotiated Binding Agreement, exists.

Theorem 6. *For any $\mathcal{C}$-Negotiated Binding Agreement, s^*, and any $h \in H$, $s^*(h) \in ICIR(\mathcal{C})$.*

Notice once again that this holds for all histories. Further to this, by the definition of $ICIR(\mathcal{C})$, whenever $N \in \mathcal{C}$, it follows that no proposal is coalitionally irrational for the coalition N. This implies that only proposals that are weakly Pareto optimal may be used.

The following corollary links the observation surrounding the potential emptiness of $ICIR(\mathcal{C})$ to emptiness of $\mathcal{C}$-Negotiated Binding Agreement.

Corollary 5. *If $ICIR(\mathcal{C}) = \emptyset$ then no $\mathcal{C}$-Negotiated Binding Agreement can exist.*

This is an immediate implication of theorem 6. Note that this is possible, i.e. in example 4, and may imply that there is no Negotiated Binding Agreement that is robust to the concerns of coalitions for a specific coalition structure $\mathcal{C}$.

Further to this, a result analogous to theorem 2 holds. This result will state that at any history h, s^* must give a payoff that is coalitionally rational for any coalition C, with respect to $[ICIR(\mathcal{C})]_{-C}$. A payoff is not coalitional rational, with respect to $[ICIR(\mathcal{C})]_{-C}$, if, for any punishment for deviation a coalition can find some joint action $a_C \in A_C$ such that the utility is higher for all agents. To understand the implications of this result more fully, I define a notion of the β-core Aumann (1961), which I refer to as the β-core with respect to $ICIR(\mathcal{C})$.

Definition 9. *$a^* \in A$ is in the β-core with respect to $ICIR(\mathcal{C})$ if, there is no $C \in \mathcal{C}$ and $a_C : [ICIR(\mathcal{C})]_{-C} \to A_C$ such that $\inf_{a_{-C} \in [ICIR(\mathcal{C})]_{-C}} u_i(a'_C(a_{-C}), a_{-C}) > u_i(a^*)$ for all $i \in C$.*

This includes a notion of making sure the action profile at hand is jointly coalitionally rational. Note the similarity to the β-core of Aumann (1961). Within the β-core the payoff

[12] Recall a strong Nash equilibrium is an action profile a^{SNE} such that for all $C \in 2^N \setminus \{\emptyset\}$ $\nexists a_C \in A_C$ such that $u_i(a_C, a^{SNE}_{-C}) > u_i(a^{SNE})$ for all $i \in C$.

of equilibria must be higher than the coalitional rational with respect to A_{-i}, in the sense that a coalition understands that they can only be punished for a deviation with a specific profile of actions. However, the actions used to prevent deviations are not necessarily justifiable. The β-core with respect to $ICIR(\mathcal{C})$ partially resolves this problem, as upon deviating the actions of others are restricted to a set of actions that is consistent with respect to itself and is defined in a similar way to the β-core restriction.

With this, I formalise the result connecting $\mathcal{C}$-Negotiated Binding Agreement to the β-core with respect to $ICIR(\mathcal{C})$ by the following theorem.

Theorem 7. *For any $\mathcal{C}$-Negotiated Binding Agreement s^* must be such that, for any history h, and for any coalition $C \in \mathcal{C}$, there is no $a'_C : [ICIR(\mathcal{C})]_{-C} \to A_C$ such that $\inf_{a_{-C} \in [ICIR(\mathcal{C})]_{-C}} u_i(a'_C(a_{-C}), a_{-C}) > U_i(s^*|h)$ for all $i \in C$.*

In other words, $a(s^*|h)$ must be in the β-core with respect to $ICIR(\mathcal{C})$ for all histories.

I reserve a discussion of the relation of this concept to other related concepts for the literature review.

This result can provide us with some insight into the types of agreements that may not be sustained. For instance, it may be that an outcome is both Pareto efficient and individually rational, yet it is not possible to sustain such an outcome via a $\mathcal{C}$-Negotiated Binding Agreement for $\{N, \{i\}_{i \in N}\} \subseteq \mathcal{C}$. This is illustrated by the following result.

Example 6. Consider the following two-player game and consider the richest set of coalitions $\mathcal{C} = \{\{1\}, \{2\}, \{1, 2\}\} = 2^N \backslash \{\emptyset\}$.

1\2	LL	L	R	RR
TT	**6,6**	0,4	**1,12**	0,0
T	4,0	0,0	**7,2**	1,1
D	**12,1**	**2,7**	4,4	0,8
DD	0,0	1,1	8,0	0,0

I have labelled the weakly Pareto efficient outcomes in bold blue font, and therefore must be the only actions in $\tilde{A}^1 = \{(TT, LL), (TT, R), (T, R), (D, LL), (D, L)\}$. No further deletion can take place, as the individually rational payoffs over this set are given by 2, the lowest payoff given by a profile in this set, therefore:

$$ICIR(2^N \backslash \{\emptyset\}) = \{(TT, LL), (TT, R), (T, R), (D, LL), (D, L)\}$$

Now notice that the outcome (TT, R) necessarily cannot be sustained in equilibrium, as it provides a payoff of 1, while the min-max payoff, given that player 2 must choose from $[ICIR(2^N \backslash \{\emptyset\})]_2 = \{LL, L, R\}$, is given by 2. Therefore we conclude that despite the fact that TT, R is Pareto efficient, and provides a higher payoff than the min-max over all possible profiles, it cannot be sustained in a $2^N \backslash \{\emptyset\}$-Negotiated Binding Agreement. ▼

With these results, I now turn to providing sufficient conditions for $\mathcal{C}$-Negotiated Binding Agreement.

6.2. Sufficient Conditions

Similarly to theorem 4, I provide sufficient conditions for $\mathcal{C}$-Negotiated Binding Agreement that prevent deviations from equilibrium based on the identity of the deviators, rather than the deviation they perform. Firstly, similarly to corollary 2, if there is only a single action profile consistent with $ICIR(\mathcal{C})$ then this must be supported in equilibrium, and further to this is the only profile that can be the outcome of $\mathcal{C}$-Negotiated Binding Agreement. I state this formally here.

Corollary 6. *If G is such that u_i is continuous and A_i is compact for all agents, if $ICIR(\mathcal{C}) = \{a^*\}$, then a^*, then s^* is a $\mathcal{C}$-Negotiated Binding Agreement if and only if $s_i^*(h) = a_i^*$ for all $h \in H$.*

Note that this condition may occur in more environments than corollary 2 when $\{i\}_{i\in N} \subset \mathcal{C}$, as $ICIR(\mathcal{C})$ may involve more deletion. However, as $ICIR(\mathcal{C})$ may be empty and leave us with no $\mathcal{C}$-Negotiated Binding Agreement.

Nonetheless, a more general set of sufficient conditions apply, as with theorem 4. To provide these conditions, I again rely on a structure that does not focus on the deviation that a coalition takes, but only on the deviating coalition. These are as before: a coalition must prefer the punishment of others to their own and a coalition must not be able to improve all members' utility by changing their action profile, holding the punishment used against them constant. Note, the inclusion of such profiles in $ICIR(\mathcal{C})$ is now required and not implied due to the rich deletion that can take place. This is formalised by the following theorem.

Theorem 8. *Take any game such that there is some $a^* = \underline{a}^N \in ICIR(\mathcal{C})$ and for all $C \in \mathcal{C}\backslash N$ $\exists \underline{a}^C \in ICIR(\mathcal{C})$ such that:*

1. *$\nexists a'_C \in A_C$ such that $u_i(a'_C, \underline{a}^C_{-C}) > u_i(\underline{a}^C)$ for all $i \in C$*
2. *for all $C \in \mathcal{C}$ there is some $i \in C$ such that $u_i(a^*) \geq u_i(\underline{a}^C)$*
3. *For all $C, C' \in \mathcal{C}$ there is some $i \in C$ such that $u_i(\underline{a}^{C'}) \geq u_i(\underline{a}^C)$*

Then a^* can be supported in a $\mathcal{C}$-Negotiated Binding Agreement.

Combining this result with the result of lemma 5, which states that if a strong Nash equilibrium exists, it must be within $ICIR(\mathcal{C})$, implies that any strong Nash equilibrium

can be supported in a $\mathcal{C}$-Negotiated Binding Agreement. However, these conditions can apply in games with no strong Nash equilibrium, and therefore are a more general set of conditions. To see this, consider the following example.

Example 6. revisited Consider again the following two-player game where all possible coalitions are permitted, $\mathcal{C} = 2^N \backslash \{\emptyset\}$.

1\2	LL	L	R	RR
TT	**6,6**	0,4	**1,12**	0,0
T	4,0	0,0	**7,2**	1,1
D	**12,1**	**2,7**	4,4	0,8
DD	0,0	1,1	8,0	0,0

Here there is no strong Nash equilibrium. In fact, as there is no pure Nash equilibrium, there is no pure coalition proof Nash equilibrium. However, the conditions of theorem 8 apply and from the previous analysis we know that:

$$ICIR(2^N \backslash \{\emptyset\}) = \{(TT, LL), (TT, R), (T, R), (D, LL), (D, L)\}$$

To see this, take, for example, $\underline{a}^N = a^* = (TT, LL)$, $\underline{a}^1 = (D, L)$ and $\underline{a}^2 = (T, R)$. Therefore TT, LL can be sustained in $2^N \backslash \{\emptyset\}$-Negotiated Binding Agreement. ▼

The sufficient conditions presented in theorem 8 can be seen as a further refinement of the β-core of Aumann (1961), where within the β-core any constant action profile of those outside of a coalition may be used in order to prevent deviations, whereas here we must satisfy additional conditions to ensure such a profile can be mutually justified by all coalitions. Note that this is not necessarily true in the notion of the β-core with respect to $ICIR(\mathcal{C})$, as some profiles within $ICIR(\mathcal{C})$ do not satisfy this notion of mutual individual rationality.[13]

I now turn to an application.

7 Application of $\mathcal{C}$-Negotiated Binding Agreement

As with strong Nash equilibrium, conditions for existence of a $\mathcal{C}$-Negotiated Binding Agreement are generically not satisfied. Nonetheless, there exist interesting applications for

[13] Shubik (2012) examines the 78 2x2 games which can be induced by strict ordinal preferences, of these 78, 67 allow for the sufficient conditions to be applied. Note that is only 2 less than the existence of Nash equilibrium in pure strategies. In this sense, the sufficient conditions apply to more scenarios than initial inspection may suggest.

which $\mathcal{C}$-Negotiated Binding Agreement exist. Consider the following Cournot game.

Application 3. (Symmetric Cournot with Fixed Cost) Consider a simple model of Cournot with fixed costs. These fixed costs depend on the total number of firms that enter the market. This captures a situation where the fixed cost is due to the purchase of equipment. The cost of equipment itself is dictated by the law of supply and demand and therefore this cost increases with the number of firms purchasing this.

I model this in the following way. Let there be $n = 4$ firms. Let each firm choose the quantity that they will sell, $q_i \geq 0$. Let total demand, as a function of the total quantity, be given by $\max\{b - \sum_{j=1}^{4} q_j, 0\}$, where $b > 0$. I assume that the marginal cost is constant and symmetric, therefore it is without loss to it to 0. Therefore gross profits for player $i \in \{1, 2, 3, 4\}$ are given by $\max\{(b - \sum_{j=1}^{n} q_j), 0\}q_i$. Let fixed costs take the following form: $\left(\frac{3}{32} b \sum_{j \neq i} \mathbf{1}_{q_j > 0}\right)^2 \mathbf{1}_{q_i > 0}$. Notice that this does indeed increase with the number of firms entering, and the first firm to enter the market may do so for free. Therefore utility takes the following form:

$$u_i(q) = \max\left\{\left(b - \sum_{j=1}^{4} q_j\right), 0\right\} q_i - \left(\frac{3}{32} b \sum_{j \neq i} \mathbf{1}_{q_j > 0}\right)^2 \mathbf{1}_{q_i > 0}$$

Notice that in this model the individual best response are given by the following expressions

$$q_i^*(q_{-i}) = \begin{cases} \left\{\frac{b - \sum_{j \neq i} q_j}{2}\right\} & \text{if } \sum_{j \neq i} q_j < b - \frac{3}{16} m \sum_{j \neq i} \mathbf{1}_{q_j > 0} \\ \left\{0, \frac{b - \sum_{j \neq i} q_j}{2}\right\} & \text{if } \sum_{j \neq i} q_j = b - \frac{3}{16} b \sum_{j \neq i} \mathbf{1}_{q_j > 0} \\ \{0\} & \text{if } \sum_{j \neq i} q_j > b - \frac{3}{16} b \sum_{j \neq i} \mathbf{1}_{q_j > 0} \end{cases}$$

Notice that this implies the following:

1. There is a Nash equilibrium when 2 firms enters the market, and both choose quantities of $q_i^* = \frac{b}{3}$. This leads to a payoff of $\frac{943}{9216} b^2$ for the firm who enters and 0 for those who do not. There are no other Nash equilibria of this game. Further, note that this is not a Strong Nash equilibrium, as the two producing firms could split the monopoly profits equally. Such quantities are coalition-proof Nash equilibrium.

2. There are many Pareto efficient outcomes. Any outcome such that the firms who sell, in aggregate sell the monopoly quantity, $\sum_{i=1}^{2} q_i = \frac{b}{2}$, while profits are strictly positive for all those who produce strictly positive quantities, are Pareto efficient. Note that there exists such a profile for any number of firms entering. For instance, all firms producing $\frac{b}{8}$ leads to profits of $u_i(q^{p,all}) = \frac{55}{1024} b^2 > 0$.

3. Any profile that is such that at least one player receives a payoff of at least $\frac{55}{1024}b^2$ is in the α-core and the β-core.

Consider $\mathcal{C} = 2^N \backslash \{\emptyset\}$. Let $q^* = (q_1^*, q_2^*, q_3^*, q_4^*)$ be the quantity that is trying to be sustained. I will argue that it is possible to sustain the efficient outcome where all agents produce in strong Negotiated Binding Agreement. That is, an agreement such that $q_i^* = \frac{b}{8}$ can be sustained. Consider the following strategies.

1. If $h = (q^1, q^2, ..., q^k)$ is such that $q_{-C}^{k-1} = s_{-C}^*((q^1, q^2, ..., q^{k-2}))$ and either
   (a) $q_l^k = s_l^*(q^1, q^2, ..., q^{k-1})$ for all $l \notin C$ and $q_j^k \neq s_j^*(q^1, q^2, ..., q^{k-1})$ for all $j \in C$
   (b) or $q_{-C}^k = \frac{16+\sqrt{137}}{64}b$ if $|C| = 3$, $q_{-C}^k = \left(\frac{16+\sqrt{137}}{64}b, \frac{16+\sqrt{137}}{64}b\right)$ if $|C| = 2$ and $q_{-C}^k = \left(\frac{16+\sqrt{137}}{64}b, \frac{16+\sqrt{137}}{64}b, 0\right)$ if $|C| = 1$

   Then:
   - $s_i^*(h) = \frac{16+\sqrt{137}}{64}b$ for $i = \min_{j \notin C} j$ if $|C| \leq 3$ or $i = 1$ if $|C| = 4$.
   - $s_i^*(h) = \frac{16+\sqrt{137}}{64}b$ for $i = \min_{j \notin C \backslash \{\min_{j \notin C} j\}} j$ if $|C| \leq 2$ or $i =$ mod $(j+2, 4)$, $j \in C$, $j \geq k$, $k \in C$ otherwise.
   - $s_i^*(h) = 0$ for all other $i \in N$.
2. For all other histories, let $s_i^*(h) = \frac{b}{8}$

The logic of this strategy is as follows. Suppose that we are at a history only one coalition has deviated in the penultimate period of the history, while in the previous period either all players have played the assigned strategy or only those within the deviating coalition have deviated. Note that this may involve a smaller coalition deviating in the penultimate period, while in the next a larger coalition deviates. If this is the case, assign two players to play a strategy that gives them exactly the payoff of all players entering and producing the efficient quantity, while all other players do not produce. At least one of these players is not within the deviating coalition if the cardinality of that coalition is 3 or less. At all other histories all agents propose their share of the equal division of the monopoly quantity.

Now I will show that this does indeed constitute a strong Negotiated Binding Agreement.

First consider a coalition deviating from a history that does not fall into case 2, where no deviation leads to the agreement that all firms enter and divide the monopoly quantity. It cannot be that the grand coalition deviates to improve the utility of all members. Therefore it must be that deviation does not involve one firm. By the structure of case 1, which any deviation must lead to, it is then the case that those outside the coalition are

proposing, in aggregate, at least $\frac{16+\sqrt{137}}{64}b$ in every period. As $\frac{16+\sqrt{137}}{64}b > \frac{1}{8}b$ it follows that it cannot be that all firms who deviate are producing and improving the utility of all members. Therefore it must be that the deviation only involves a coalition of at most two firms. It cannot be that they are both assigned to not produce in all periods, as this implies that the profits are bounded above by $\frac{249-32\sqrt{137}}{4096}b < 0$ if producing and 0 if not. This bound is the same if only one firm deviates. Therefore no profitable deviation can exist from case 2. Now suppose that a profitable deviation exists from case 1. By a similar logic, it cannot be that two firms deviate and improve their utility. This is as a "punishing" firm does not wish to deviate, as they would be punished for this. A "punished" firm also does not wish to deviate, as the punishment is sufficiently high to ensure that they do not wish to enter the market. Further, it cannot be that all agents jointly deviate, as there are two punishing firms, who, in aggregate, receive the utility that is the maximum that can be achieved for all firms entering. Therefore they have no incentive to do so. ▼

Before concluding the paper, I turn to some related literature.

8 Literature Review

Here I outline the key related literatures and the main related papers within them.

Mariotti (1997) has the closest model to this paper. Histories take a similar form, both terminal and partial, whereas in Mariotti (1997) the payoff of perpetual disagreement is normalised to $-\infty$. Mariotti (1997) similarly allows for a fully general specification of permissible coalitions. However, Mariotti (1997) looks at a different solution concept, where coalitions make new proposals only if it is strictly in their benefit to do so from the current proposal. In this sense, Mariotti (1997) takes a solution concept closer to those used in cooperative theory, whereas this paper takes a more non-cooperative approach. Mariotti (1997) provides some general necessary conditions and some sufficient conditions for two-player games. Both Kamada and Kandori (2020) and Caruana and Einav (2008) consider models of revision strategies, where agents may revise their strategies before some final time and the last proposed action is implemented in a binding way. This is a similar sense in which a twice repeated profile is binding within the model of this paper, however, in my model the time taken to have this binding agreement is not bounded. Kalai (1981) also looks at a model of negotiation, where agents may make proposals, where agents have finitely many periods to reach an agreement, and if agents change their proposal within a period then they are no longer permitted to change their proposal again. Kalai (1981) looks at the perfect equilibria of Selten (1988)[14] and show that only cooperation can be sustained in a 2 player prisoners' dilemma game. More recently, Nishihara (2022) has

[14] These are subgame perfect equilibria that do not permit the use of weakly dominated strategies at any history.

extended this result for Kalai's pre-play negotiation procedure to an n-player prisoners' dilemma. Bhaskar (1989) examines a model of pre-play agreement for a symmetric two-player Bertrand game. In a similar sense to this model, agents make proposals of the prices they will take, and have the opportunity to revise this proposal sequentially. If there is a sequence of 3 proposals, 2 for 1 player and 1 for the other, where the player who has proposed first and last does not revise on her last proposal, then the prices are implemented. Bhaskar (1989) looks at perfect equilibria of such a game and concludes that only the monopoly price can be sustained.

A number of papers have provided cooperative solutions for normal form games. Chwe (1994); Nakanishi (2009); Ray and Vohra (2019) all consider versions of the farsighted stable set, with Ray and Vohra (2019) being the closest to $\mathcal{C}$-Negotiated Binding Agreement as it looks for well defined strategies to pin down the stable set. The concept of Ray and Vohra (2019) and Chwe (1994) are similar to that of Mariotti (1997), while Mariotti (1997) demands full optimality with respect to the strategies of others in comparison to Chwe (1994) and allows for a general utility function in comparison to Ray and Vohra (2019). Both allow for a fully general specification of permissible coalitions. Ray and Vohra (2019) is similar to this paper, however assumes that there are no externalities across coalitions, in this paper I instead allow for externalities via the use of a game. Ray and Vohra (2019) provide general conditions for existence in games with a general coalition structure in games with transferable utility. Aumann (1959, 1961) defines strong Nash equilibrium, the α-core, and the β-core. In this paper, my solution of $\mathcal{C}$-Negotiated Binding Agreement lies somewhere between the β-core and strong Nash equilibrium, as agents are permitted to change their proposals when they observe a proposal of others change, but can only do so in a way consistent with rationality, and must be pinned down by an optimal strategy in the sense of equilibrium. In the work of Aumann (1961) the traditional cooperative approach of using any means to ensure the agreement is kept is used. The sufficient conditions provided in this paper are also close in nature to the β-core, where mutual coalitional rationality consistency of punishments is required. Bernheim et al. (1987) look at coalition proof Nash equilibria, which has a similar flavour of private agreements as $\mathcal{C}$-Negotiated Binding Agreements, however in that case they are non-binding, and as there is no observability as with proposals in the negotiation game, all other players outside of the agreement choose a constant strategy. Chander and Wooders (2020) define a notion of coalitional subgame perfect equilibrium for games with transferable utility, where a coalition's deviation payoff is with respect to the best subgame perfect equilibrium assuming all other players act without cooperation. Their solution relies on an assumption that agents know a coalition has deviated, but their solution is shown to be related to a perturbed version of the α-core. Herings et al. (2004); Ambrus (2006, 2009); Grandjean et al. (2017) all consider versions of coalitional rationalizability, all of which are different to the concept of iterated elimination of coalitionally irrational actions.

In contract theory, Jackson and Wilkie (2005); Yamada (2003); Ellingsen and Paltseva

(2016) all consider models of contracting where agents may all have an input into the contracts that are proposed and agreed upon. Negotiated Binding Agreement has a similar flavour in this respect, as all agents have a meaningful impact, i.e. rather than a take-it-or-leave-it offer, on the action they take. Kalai et al. (2010) and Peters and Szentes (2012) look at a notion similar to Tennenholtz (2004)'s program equilibrium, where agents may contract over contracts. This allows agents to specify reactions to deviations in full, and can allow for these to be fully specified at a higher level also. They do not consider this is a cooperative way, however conceptually this is similar to $\mathcal{C}$-Negotiated Binding Agreement, as this can be viewed as the agreements that result when agreements over how to negotiate can be made.

This paper can also be seen in the light of the Nash agenda pointed to in Nash (1953), as the sufficient conditions $\mathcal{C}$-Negotiated Binding Agreement can be seen as a perturbed version of the β-core, while it is the result of a fully specified game with fully consistent behaviour, and would be the equilibrium even when less general deviations are permitted. Notable contributions to this literature include Rubinstein (1982); Chatterjee et al. (1993) on bargaining. Recently, a working paper of Ismail (2021) looks at strategic cooperation in the view of this agenda, where an extensive form is constructed to allow cooperation to be a choice. In $\mathcal{C}$-Negotiated Binding Agreement, the underlying incentive cooperation is taken to be implicit rather than explicit.

A number of papers consider a form of communication for equilibrium selection. Notable examples are of Bernheim et al. (1987)'s coalition proof equilibrium, where coalitions are permitted to deviate, but can only do so in a non-binding way and therefore deviations must be self-enforcing a la Nash. Farrell and Maskin (1989) and developed simultaneously by Bernheim and Ray (1989) develop a similar concept of renegotiation proof equilibrium, where the grand coalition may deviate to a preferred SPE at a point in a repeated game. The closest work within this literature to this paper is that of Rabin (1994). Rabin (1994) explicitly models a negotiation over the choice of equilibrium, rather than the implicit process by the above papers.

The way payoffs are defined for perpetual disagreement can be seen as similar to the literature of infinitely repeated games with no discounting. Notably, when well defined, the limit of means criteria of Aumann and Shapley (1994); Rubinstein (1994) can be used. It may also embed a concept similar to that of the overtaking criteria of Rubinstein (1979). The sufficient conditions within the paper are also similar to the sufficient conditions of player specific punishment is used, for instance, in Fudenberg and Maskin (1986) and Abreu et al. (1994). It is more restrictive as player specific punishment only requires that their punishments' provide them an individually rational payoff and they prefer to punish rather than be punished. In contrast, I require that individuals are best responding to their punishment. This is used as there are no further reward from following their punishment, which are held in the continuation of an infinitely repeated game.

9 Conclusion

In this paper, I propose a tractable model of negotiation. This is represented by proposals within a period, where if agents all propose the same action in two consecutive periods a binding agreement is made to play said action profile. The payoff of perpetual disagreement is taken to be between the lim inf and lim sup of the payoffs induced by the proposals. I study a form of subgame perfect equilibrium where agents only propose the actions they expect to take upon agreeing at some history, therefore there is no babbling. I refer to this as a *Negotiated Binding Agreement*. I provide necessary and sufficient conditions for the outcomes and strategies of a Negotiated Binding Agreement. The necessary conditions provide an iterative procedure to delete actions that no agent should recommend, given by iterative deletion of individually irrational actions. The sufficient conditions rely on a form of player specific punishment, where agents must be prescribed the action that best responds to their punishment in the baseline game. I show that these results are robust to perturbations in appendix A.

I go on to explore two key applications. Firstly, I show that in a public goods game cooperation can be supported by a Negotiated Binding Agreement. In a Cournot Duopoly I show that when marginal costs are the same, any profile of payoffs such that each player receives positive profits is sustainable. In contrast, when marginal costs are very different only the firm with the lowest marginal cost receiving their monopoly profit is supported. In both examples, I fully characterise the Negotiated Binding Agreements.

I show how the necessary and sufficient conditions for the case where agents may only act unilaterally naturally generalise to the case where agents may act jointly, and show the link of these conditions to a perturbed version of the β-core of Aumann (1961). I apply this to a Cournot model, where the fixed costs of firms depends on the number of entrants. I show that within this setting, a fair Pareto optimal allocation can be sustained.

A number of questions remain open. Firstly, there is an opportunity for applied theory to use such a concept where appropriate. A number of applied theory papers have made use of cooperative solutions, for example in environmental agreements (Chander and Tulkens, 1997; Carraro, 1998; Carraro et al., 2006) and trade agreements (Aghion et al., 2007). Due to the easy-to-use conditions, the negotiation process, as well as the corresponding results of this paper, may also provide some interesting insights in some applied theoretical settings.

On a broader level, a further examination of reasonable deviations, and the expansion of the set of equilibria that this would allow calls for further attention. Additionally, extending the model to allow for asymmetric information, and therefore agreements occur without full knowledge of the outcome, provides an interesting, albeit challenging, question. I leave these for future work.

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A Appendix: Robustness

In this section, I outline how the results of this paper are robust to changes in how the negotiation game is defined. I do so as follows. In subsection A.1. I show that necessarily proposals can only be made from actions that survive iterated elimination of absolutely dominated actions, which are tightly related to those that survive iterated elimination of individually irrational actions, and the sufficient conditions hold if agents make proposals sequentially rather than simultaneously in each period. In subsection A.2. I show that, if the payoffs of the infinite histories are appropriately defined, both the necessary and sufficient conditions hold if agents may make proposals of the joint action, rather than just their own, in each period. In subsection A.3., I show that the sufficient conditions remain to be true in a model where the payoff of the infinitely terminal histories are taken to be worse than the payoff of any finite terminal history. An alternative specification, where, in the case of perpetual disagreement, agents believe that the worst agreeable action is played by all other players, while they may individually deviate, is considered in A.4.. Here both the necessary and sufficient conditions hold.

In essence, these robustness checks show how the drivers of the results. Specifically, that agents cannot use a non-agreement outcome as a threat of deviating, whereas timing and the proposals used are not so important for driving the results.

A.1. Robustness to Order of Proposals

As in section 2, let G be a game with bounded payoffs.

Define the negotiation game with order as follows.

Let $\mathcal{O} : N \to |N|$ be the order in which agents make proposals within a period. Note that this function may not be one-to-one, and therefore it may be that many agents make the proposals at the same time. Assume that if $\mathcal{O}(i) = k > 1$ then $\exists j \in N$ such that $\mathcal{O}(j) = k - 1$. That is, $\mathcal{O}$ naturally defines an order: if I am not first, then there must be someone who proposes before me. I also assume that $\mathcal{O}(i) = 1$ for some $i \in N$ to ensure the first proposer is labelled as such. Let $\mathcal{O}^{-1}(k) = \{i \in N | \mathcal{O}(i) = k\}$, that is, define $\mathcal{O}^{-1}(k)$ is the set of agents who make the k^{th} proposal.

A history will be the empty set, followed by a sequence of proposals for all agents, and then followed by the first k proposals within the last period. That is,

$$h = (a^1, a^2, ..., a^{k-1}, (a^k_{\mathcal{O}^{-1}(2)}, a^k_{\mathcal{O}^{-1}(1)}, ..., a^k_{\mathcal{O}^{-1}(l)}))$$

, with $l \leq n$, i.e. there may be agents who are yet to make a proposal within the current period.

A history is terminal if, either:

a) Where the same action profile is proposed twice in consecutive periods, and all agents have made a proposal within the last period, and no earlier occurrence of consecutive repetition is present. That is, $z = (a^1, ..., a^{k-1}, a^k)$ is terminal if $a^k = a^{k-1}$ and $a^m \neq a^{m-1}$ for all $m < k$. Let the set of such histories be denoted by $\tilde{Z}'$ and refer to this histories as ones where an *agreement* is made.

b) an infinite sequence where the same action profile is never proposed consecutively, and all agents have made a proposal within each period. Let the set of such histories be denoted by $\tilde{Z}''$. I will again refer to these as perpetual disagreement histories.

Let the set of terminal histories be given by $\tilde{Z} = \tilde{Z}' \cup \tilde{Z}''$. The set of all possible histories is all terminal histories, and all finite histories where there are no consecutive proposals that are the same action for all agents. Let the set of partial histories be denoted by $\tilde{H}$.

As before, whenever $z = (a^1, ..., a^k) \in \tilde{Z}'$ let $U_i(z) = u_i(a^k)$.

Whenever $z \in \tilde{Z}''$ let $U_i(z) \in [\liminf_{t\to\infty} u_i(a^t), \limsup_{t\to\infty} u_i(a^t)]$. Only take these definitions over well defined action profiles.

Let $\tilde{H}_i$ be the set of partial histories where $i \in N$ is active. That is $h \in \tilde{H}_i$ is such that $h = (a^1, a^2, ..., a^{k-1}, (a^k_{\mathcal{O}^{-1}(1)}, ..., a^k_{\mathcal{O}(i)-1}))$ when $\mathcal{O}(i) \neq 1$ and $h = (a^1, a^2, ..., a^{k-1}, a^k)$. the strategy of $i \in N$ dictates the proposal i would make at any history for which they are active: $s_i : \tilde{H}_i \to A_i$. Let S_i be the space off all such mappings.

For a partial history $h \in \tilde{H}$, let $U_i(s|h)$ denote the payoff that would be received from the terminal history that the strategy s would induce, starting from the history $h \in \tilde{H}$. I will refer to such a history as $(s|h)$. When $z \in \tilde{Z}'$, i.e. an agreement is made, let $a(h)$ as the action profile that terminates z.

I define subgame perfect equilibria for this model here:

Definition (Subgame Perfect Equilibria)**.** *s^* is subgame perfect equilibrium, if for all $i \in N$, for all partial histories where $i \in N$ is active $h \in \tilde{H}_i$, $U_i(s^*|h) \geq U_i(s_i, s^*_{-i}|h)$, for all $s_i \in S_i$.*

This leads to the natural definition of Negotiated Binding Agreement in this setting. To make the distinction clear, I refer to this as Negotiated Binding Agreement with order.

Definition 10 (Negotiated Binding Agreement with Order)**.** *s^* is a Negotiated Binding Agreement with order $\mathcal{O}$ supporting $a^* = a * (s^*|\emptyset)$ if:*

a) s^* is a subgame perfect equilibria.

b) For all $h \in \tilde{H}_i$ $\exists h' \in \tilde{H}_i$ *such that* $s_i(h) = a_i(s^*|h')$.

Now I show that some necessary conditions related in section 3 remains to be true for this specification of the model. First, I will show that any action proposed at some history of Negotiated Binding Agreement must survive a procedure of iterated deletion of absolutely dominated actions, also known as interdependent choice rationalizability (Salcedo, 2017) and minmax rationalizability (Halpern and Pass, 2018).

Definition 11 (Absolute Domination given $C_{-i} \subseteq A_{-i}$)**.** $a_i \in A_i$ *is absolutely dominated given* $C_{-i} \subseteq A_{-i}$ *if* $\exists a_i' \in A_i$ *such that*

$$\inf_{a_{-i} \in C_{-i}} u_i(a_i', a_{-i}) > \sup_{a_{-i} \in C_{-i}} u_i(a_i, a_{-i})$$

Denote the set of absolutely dominated actions given C_{-i} *by* $D_i(C_{-i})$.

As I do not require that the utility functions are continuous and defined over a compact set, the minimum or maximum need not exist. With this, I take the supremum and infimum, which by the assumption that the utility function is bounded are always well defined. Bar this change, the above definition is equivalent to that of Salcedo (2017). Note that, if in a normal form game there is a single action that is not absolutely dominated given A_{-i}, then this action is an obviously dominant strategy as defined by Li (2017).

Definition 12 (Iterated Elimination of Absolutely Dominated Actions)**.** *Let* $\tilde{A}_i^0 = A_i$ *for all* $i \in N$. *Let* $\tilde{A}_{-i}^0 = A_{-i}$. *Then for all* $m > 0$ *let* $\tilde{A}_i^m = \tilde{A}_i^{m-1} \backslash D_i(\tilde{A}_{-i}^{m-1})$ *where* $\tilde{A}_{-i}^{m-1} = \times_{j \neq i} \tilde{A}_j^{m-1}$.

The set of actions that survives Iterated Elimination of Absolutely Dominated Actions (IAD) for i *is given by* $IAD_i = \bigcap_{m \geq 0} \tilde{A}_i^m$. *Let* $IAD = \times_{i \in N} IAD_i$.

Note that if at each level of iteration, if the min-max and max min payoff are the same, then IAD coincides with IIR. Note that generically, the concept of iterated elimination of individually irrational actions and iterated elimination of absolutely dominated actions are different, for instance consider the following example.

Example 7. Consider the following two-player game.

$1\backslash 2$	L	R
U	$1, 2$	$-1, 0.5$
M	$-1, 1$	$1, 0.5$
D	$-0.7, 3$	$-0.7, 3$

Here, in iterated elimination of absolutely dominated actions, all profiles survive. However, if we consider iterated elimination of individually irrational actions, we may remove D, as the min-max payoff for player 1 is 1. Given this, we may also eliminate R for player 2, as her min-max payoff is 0.5. Finally, we remove M, therefore we conclude that iterated elimination of individually rational actions leads to the unique prediction of U, L, while iterated elimination of absolutely dominated actions allows for any action profile. ▼

These definitions lead to the following proposition.

Proposition 1. *For any order $\mathcal{O}$, if s^* is a Negotiated Binding Agreement with order then, for all histories for i is active $h \in \tilde{H}_i$, $s_i(h) \in IAD_i$.*

I reserve this proof, and all other proofs within this section, for the appendix C.

Further to this, the following proposition shows that the sufficient conditions are relevant within this specification of the model. Indeed, further to this, any outcome that can be sustained with a Negotiated Binding Agreement can be sustained within a model of negotiation with order, no matter the order. This is highlighted by the following proposition.

Proposition 2. *Take any order $\mathcal{O}$. a^* is supported in a Negotiated Binding Agreement then it is supported in Negotiated Binding Agreement with order $\mathcal{O}$.*

In essence, this shows that the order of proposals is not important, nor is important that the proposals are made simultaneously. Rather, the structure of the terminal histories, and the associated payoffs, as well as the ability for all agents to make some proposal, are the key features of the model. Within the next subsection, I go on to show that when the payoffs of infinite histories are correctly specified, the robustness of these results also holds when agents propose the action profile, rather than only their action. This further highlights this point.

A.2. Robustness to Joint Proposals

As in section 2, let G be a game with bounded payoffs.

Define the negotiation game with order as follows.

A history will be the empty set, followed by a sequence of proposals for all agents, where each agent may propose a joint action profile. That is,

$$h = ((a^{1,1}, a^{2,1}, ..., a^{n,1}), (a^{1,2}, a^{2,2}, ..., a^{n,2}), ..., (a^{1,k}, a^{2,k}, ..., a^{n,k}))$$

, where $a^{i,t} \in A$. With some abuse of notation, let $a^t = (a^{1,t}, a^{2,t}, ..., a^{n,t})$.

A history is terminal if, either:

a) Where the same action profile is proposed twice in consecutive periods by all agents and no earlier occurrence of consecutive repetition is present. That is, $h = (a^1, ..., a^{k-1}, a^k)$ is terminal if $a^k = a^{k-1}$, $a^{i,k} = a^{j,k}$ for all $i, j \in N$, and either $a^m \neq a^{m-1}$ for all $m < k$ or $a^{i,m} \neq a^{j,m}$ for some $i, j \in N$. Let the set of such histories be denoted by $\tilde{\tilde{Z}}'$ and refer to this histories as ones where an *agreement* is made.

b) an infinite sequence where the same action profile for all agents is never proposed consecutively. Let the set of such histories be denoted by $\tilde{\tilde{Z}}''$. Refer to these as perpetual disagreement histories.

Let the set of terminal histories be given by $\tilde{\tilde{Z}} = \tilde{\tilde{Z}}' \cup \tilde{\tilde{Z}}''$. The set of all possible histories is all terminal histories, and all finite histories where there are no consecutive proposals that are the same action for all agents. Let the set of all histories, terminal and partial, be given by $\tilde{\tilde{H}}$.

Whenever $z = (a^1, ..., a^k, ...) \in \tilde{\tilde{Z}}'$, let $\tilde{h} = ((a_i^{i,1})_{i \in N}, (a_i^{i,2})_{i \in N}, ..., (a_i^{i,k})_{i \in N}, ...)$, i.e., take the proposals that each agent makes for themselves. Let this sequence be denoted by $\tilde{z} = (\tilde{a}^1, \tilde{a}^2, ..., \tilde{a}^k, ...)$ Let bound of the $\liminf_{t \to \infty} u_i(\tilde{a}^t)$ and an upper bound of the $\limsup_{t \to \infty} u_i(\tilde{a}^t)$. This implies that if no agreement is made, then only your own proposals matter, you cannot impact what others do in this case.

the strategy of $i \in N$ dictates the proposal i would make when they are active: $s_i : \tilde{\tilde{H}} \to A$. Let S_i be the space off all such mappings.

With some abuse of notation, for a partial history $h \in \tilde{\tilde{H}}$, let $U_i(s|h)$ denote the payoff that would be received from the terminal history that the strategy s would induce, starting from the history $h \in \tilde{\tilde{H}}$. I will again refer to such a history as $(s|h)$. As before, when $z \in Z'$, i.e. an agreement is made, let $a(h)$ as the action profile that terminates z.

Definition (Subgame Perfect Equilibria)**.** *s^* is subgame perfect equilibrium, if for all $i \in N$, for all partial histories $h \in \tilde{\tilde{H}}$, for all $i \in N$, $U_i(s^*|h) \geq U_i(s_i, s^*_{-i}|h)$, for all $s_i \in S_i$.*

This leads to the definition of Negotiated Binding Agreement in this setting. To make the distinction clear, I refer to this as Negotiated Binding Agreement with all proposals.

Definition 13 (Negotiated Binding Agreement with all Proposals)**.** *s^* is a Negotiated Binding Agreement with all proposals supporting $a^* = a(s|\emptyset)$ if:*

a) s^* is a subgame perfect equilibria.

b) $\forall h \in \tilde{\tilde{H}}$ $\exists h' \in \tilde{\tilde{H}}$ such that $s_i(h) = a(s^*|h)$.

As before, the following proposition shows that the necessary conditions previously shown for Negotiated Binding Agreement hold for this specification of the model.

Proposition 3. *If s^* is a Negotiated Binding Agreement with all proposals, for all histories $h \in \tilde{\tilde{H}}$, $s_i(h) \in IIR_i$.*

Further, for any negotiated with order s^* be be such that, for any history $h \in \tilde{\tilde{H}}$, $U_i(s^*|h) \geq \underline{u}_i$ where

$$\underline{u}_i = \inf_{a_{-i} \in IIR_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$$

.

Further to this, the sufficient conditions also hold. This is captured by the following proposition, which shows us that any Negotiated Binding Agreement can be replicated by a Negotiated Binding Agreement with all proposals.

Proposition 4. *a^* is supported by a Negotiated Binding Agreement with all proposals if a^* is supported by a Negotiated Binding Agreement.*

This again highlights the important features and drivers of the results of the model. In essence, it is the ability for agents to make a meaningful impact on their payoff via their proposals, while ensuring they do not force other agents to take some action. This is highlighted by the idea that the payoffs of infinite terminal histories, i.e. when there is no agreement, take the actions for individuals that they propose for themselves. With this, I present one last robustness result for Negotiated Binding Agreement.

A.3. Robustness to Outside Options

Within this subsection, I take the model to be exactly as in section 2. That is, agents simultaneously propose the action that they will take. The only caveat is that whenever a terminal history is infinite they receive a payoff that is worse that the payoff of any agreement. That is, when $z \in Z''$ let $U_i(z) = \inf_{a \in A} u_i(a)$. Negotiated Binding Agreement can be defined as before. To distinguish between these cases I will refer to Negotiated Binding Agreement for the model in this subsection as *constant outside option Negotiated Binding Agreement*. In this setting, it is no longer true that the necessary conditions remain to be true. However, the sufficient conditions remain to be valid. This is highlighted by the following proposition.

Proposition 5. *If s^* is a Negotiated Binding Agreement then s^* is a constant outside option Negotiated Binding Agreement.*

As Negotiated Binding Agreement do need not to make use of the infinitely long terminal histories as part of equilibrium, this result shows us that they are important only for restricting deviations. That is, if we were to make such an option worse for each player, they have less incentive to deviate than before. Therefore Negotiated Binding Agreement captures a set of strategies and outcomes that work regardless of whether the outside option is specified as within this paper or normalised to be worse than any agreement as typically assumed in bargaining games.

A.4. Robustness to Worst Agreement of others for Perpetual Disagreement

In this section I discuss how the results are robust to an alternative specification of the payoffs of perpetual disagreement. Here, it will be assumed that agents believe that the actions of others will be pinned down by the worst outcome of agreement for all other agents, while they will be permitted to unilaterally deviate. Formally, this will be described as follows.

Let the game being negotiated over be $G = \langle N, (u_i, A_i)_{i \in N} \rangle$ where $N = \{1, 2, 3, ..., n\}$ is a finite set of players, A_i is a set of actions for each player, with a joint action $A = \times_{i \in N} A_i$. u_i is utility function such that $u_i : A \to \mathbb{R}$ and u_i is bounded for all $i \in N$. Let $A_{-i} = \times_{j \neq i} A_j$.

The set of partial histories consists of all $h = (a^1, a^2, ..., a^k)$ where $a^t = (a_i^t)_{i \in N}$ denotes the profile of proposals made in period t. I will denote the set of all partial histories by H. Proposals are assumed to be made simultaneously within a period, and therefore no history is such that only some agents have made proposals.

A history is terminal if, either:

a) Where the same action profile is proposed twice in consecutive periods, and no earlier occurrence of consecutive repetition is present. That is, $z = (a^1, ..., a^{k-1}, a^k)$ is terminal if $a^k = a^{k-1}$ and $a^m \neq a^{m-1}$ for all $m < k$. Let the set of such histories be denoted by Z' and refer to this histories as ones where an *agreement* is made.

b) an infinite sequence where the same action profile is never proposed consecutively. Let the set of such histories be denoted by Z''. I will refer to these as histories with *perpetual disagreement*.

Let the set of terminal histories be given by $Z = Z' \cup Z''$. The set of all possible histories is all terminal histories, and all finite histories where there are no consecutive proposals that are the same action for all agents.

Let U_i denote the payoff for player $i \in N$ of the negotiation game.

Whenever $z = (a^1, ..., a^k) \in Z'$, that is a history that ends in agreement let $U_i(z) = u_i(a^k)$ for all $i \in N$.

Let $s_i : H \to A_i$ be the strategy for each player. Notice that from any history $h \in H$ an agent can choose a strategy such that the continuation lies in Z'', regardless of the strategies of others.

Let A^{agree} be the set of agreements outcomes that can be supported in equilibrium. Let this set be constructed in the following way:

1. $\exists s^*$ be a strategy profile such that:
   (a) $s^*(\emptyset) = s^*(a) = a \in A^{agree}$.
   (b) For any $h \in H$, $\nexists s_i' \in S_i$ such that the continuation of $(s_i', s_{-i}^*|h) \in Z'$ and $U_i(s_i', s_{-i}^*|h) > U_i(s^*|h)$.
2. $\forall i \in N \; \forall a \in A^{agree} \; \exists a'_{-i} \in A_{-i}^{agree}$ such that $\sup_{a_i' \in A_i} u_i(a_i', a'_{-i}) \leq u_i(a)$.

1. ensures that agents can not reach a better agreement within the negotiation it self. 2. ensures that it is not the case that an agent can cause perpetual disagreement, and in doing so can induce others playing any of the actions that can be agreed upon, while ensuring that the deviating agent can increase their utility.

Here both the necessary and sufficient conditions presented in the main paper still hold, as demonstrated by the following two propositions.

Proposition 6. *For all $a \in A^{agree}$, $a \in IIR$.*

Further, if $a^* \in A^{agree}$, then $u_i(a^*) \geq \inf_{a_{-i} \in IIR_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$

Proposition 7. *If $\{a^*, \underline{a}^1, .., \underline{a}^n\} \subseteq A$ satisfy:*

1. $\underline{a}_i^i \in \text{argmax}_{a_i \in A_i} u_i(a_i, \underline{a}_{-i}^i)$
2. $u_i(a^*) \geq u_i(\underline{a}^i)$
3. $u_i(\underline{a}^j) \geq u_i(\underline{a}^i)$ *for all $i, j \in N$*

Then $\{a^*, \underline{a}^1, .., \underline{a}^n\} \subseteq A^{agree}$.

Notice that proposition 7 follows by definition of the agreement set and the fact that, by lemma 4, these actions are in IIR, and therefore I forgo the proof.

B Appendix: Proofs from Main Text

Lemma. *1* *For* $z = (a^1, a^2, ..., a^t, ...) \in Z''$

$$\lim_{\delta \to 1}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(a^t) \in \left[\liminf_{k\to\infty} u_i(a^k), \liminf_{k\to\infty} u_i(a^k)\right]$$

Proof. Notice that

$$\liminf_{k\to\infty} u_i(a^k) = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\liminf_{k\to\infty} u_i(a^k)$$

Therefore by continuity of subtraction we have that $\lim_{\delta\to 1}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(a^t) - \liminf_{k\to\infty} u_i(a^k) = \lim_{\delta\to 1}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\left(u_i(a^t) - \liminf_{k\to\infty} u_i(a^k)\right)$.

Note by definition of the lim inf, for all $\epsilon > 0$ $\exists T \in \mathbb{N}$ such that $\forall t > T$ we have that $u_i(a^t) - \liminf_{k\to\infty} u_i(a^k) > -\epsilon$. Therefore, for any such T, we may decompose the expression as follows.

$$\begin{aligned}
\lim_{\delta\to 1}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\left(u_i(a^t) - \liminf_{k\to\infty} u_i(a^k)\right) &= \lim_{\delta\to 1}(1-\delta)\sum_{t=1}^{T}\delta^{t-1}\left(u_i(a^t) - \liminf_{k\to\infty} u_i(a^k)\right) + \cdots \\
&\cdots + \lim_{\delta\to 1}(1-\delta)\sum_{t=T+1}^{\infty}\delta^{t-1}\left(u_i(a^t) - \liminf_{k\to\infty} u_i(a^k)\right) \\
&= \lim_{\delta\to 1}(1-\delta)\sum_{t=T+1}^{\infty}\delta^{t-1}\left(u_i(a^t) - \liminf_{k\to\infty} u_i(a^k)\right) \\
&> \lim_{\delta\to 1}(1-\delta)\sum_{t=T+1}^{\infty}\delta^{t-1}(-\epsilon) \\
&= \lim_{\delta\to 1} -\delta^{T+1}\epsilon \\
&= -\epsilon
\end{aligned}$$

Therefore we may conclude that $\lim_{\delta\to 1}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\left(u_i(a^t) - \liminf_{k\to\infty} u_i(a^k)\right) > -\epsilon$ $\forall\epsilon > 0$, concluding that $\lim_{\delta\to 1}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\left(u_i(a^t) - \liminf_{k\to\infty} u_i(a^k)\right) \geq 0$ and therefore $\lim_{\delta\to 1}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(a^t) \geq \liminf_{k\to\infty} u_i(a^k)$. The analogous proof works for showing $\lim_{\delta\to 1}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(a^t) \leq \limsup_{k\to\infty} u_i(a^k)$. □

Lemma. *2 For any subgame perfect equilibrium s^*, for any partial history $h \in H$*

$$U_i(s^*|h) \geq \inf_{a_{-i} \in A_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$$

Proof. Suppose not, $U_i(s^*|h) < \inf_{a_{-i} \in A_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$. For any $\epsilon > 0$, let $\tilde{a}_i : A_{-i} \to A_i$ be such that $u_i(\tilde{a}_i(a_{-i}), a_{-i}) > \sup_{a_i \in A_i} u_i(a_i, a_{-i}) - \epsilon$. Note such a function exists for any $\epsilon > 0$. Let $s'_i(h) = (\tilde{a}_i(s^*_{-i}(h')), s^*_{-i}(h'))$ for all $h' \in H$. It follows that $U_i(s'_i, s^*_{-i}|h)$ is either such that it ends in agreement, in which case $U_i(s'_i, s^*_{-i}|h) > \inf_{a_{-i} \in A_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i}) - \epsilon$ and therefore, as we can construct such a function for any $\epsilon > 0$, we conclude that $U_i(s'_i, s^*_{-i}|h) \geq \inf_{a_{-i} \in A_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$. On the other hand, it may be that $U_i(s'_i, s^*_{-i}|h)$ ends in perpetual disagreement. In which case $(s'_i, s^*_{-i}|h) = (a^1, a^2, ..., a^T, ...)$, where $a^t_i = \tilde{a}_i(a^t_{-i})$. Therefore

$$U_i(s'_i, s^*_{-i}|h) \geq \liminf_{t \to \infty} u_i(\tilde{a}_i(a^t_{-i}), a^t_{-i}) \geq \inf_{a_{-i} \in A_{-i}} u_i(\tilde{a}_i(a_{-i}), a_{-i}) > \inf_{a_{-i} \in A_{-i}} \sup_{a_i \in A_{-i}} u_i(a_i, a_{-i}) - \epsilon$$

and therefore $U_i(s'_i, s^*_{-i}|h) \geq \inf_{a_{-i} \in A_{-i}} \sup_{a_{i} \in A_{-i}} u_i(a_i, a_{-i})$. A contradiction as there exists a profitable deviation. □

Theorem. *1 If s^* is a Negotiated Binding Agreement, then for all $h \in H$, $s^*_i(h) \in IIR_i$.*

Proof. Suppose not, for some history $h' \in H$ we have that $s_i(h') = a_i$. By no babbling it follows that there exists some $h \in H$ such that $a_i(s|h) = a_i$. Therefore it must be that $U_i(s|h) = u_i(a(s|h)) \leq \sup_{a'_{-i} \in A_{-i}} u_i(a_i, a'_{-i})$. Take $\epsilon = \inf_{a'_{-i} \in A_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) - u_i(a(s|h)) > 0$. Take a function $\tilde{a}_i : A_{-i} \to A_i$ such that $u_i(\tilde{a}_i(a_{-i}), a_{-i}) > \sup_{a'_i \in A_i} u_i(a'_i, a_{-i}) - \epsilon$. Consider a deviation $s'_i(h'') = \tilde{a}_i(s_{-i}(h''))$ for all $h'' \in H$. It follows that

$$U_i(s'_i, s_{-i}|h) \geq \inf_{a_{-i} \in A_{-i}} u_i(\tilde{a}_i(a_{-i}), a_{-i}) > \inf_{a_{-i} \in A_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a_{-i}) - \epsilon$$

. Therefore it follows that $U_i(s'_i, s_{-i}|h) > u_i(a(s|h)) = U_i(s|h)$, concluding that a profitable deviation exists and therefore it cannot be that s is a subgame perfect equilibrium. By no babbling, we conclude that $s_i(h) \notin D_i(A_{-i})$ for any $h \in H$.

Now suppose by contradiction that, for all $j \in N$ $s_j(h') \in \tilde{A}^k_j \ \forall k < m$ and $h' \in H$ but for some $i \in N$ $s_j(h') = a_i \notin \tilde{A}^{m+1}_j$ for some $h' \in H$. By no babbling it must be that a) $s_{-i}(h') \in \tilde{A}^m_i$ for all h' and b) by no babbling there is some $h \in H$ for which $a_i(s|h) = a_i$. Therefore it must be that $U_i(s|h) = u_i(a(s|h)) \leq \sup_{a'_{-i} \in \tilde{A}^m_{-i}} u_i(a_i, a'_{-i})$. Take $\epsilon = \inf_{a'_{-i} \in \tilde{A}^m_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) - u_i(a(s|h)) > 0$. Take a function $\tilde{a}_i : \tilde{A}^m_{-i} \to A_i$ such

that $u_i(\tilde{a}_i(a_{-i}), a_{-i}) > \sup_{a'_i \in A_i} u_i(a'_i, a_{-i}) - \epsilon$ for all $a_{-i} \in \tilde{A}^m_{-i}$. Consider a deviation $s'_i(h'') = \tilde{a}_i(s_{-i}(h''))$ for all $h'' \in H$. It follows that

$$U_i(s'_i, s_{-i}|h) \geq \inf_{a_{-i} \in \tilde{A}^m_{-i}} u_i(\tilde{a}_i(a_{-i}), a_{-i}) > \inf_{a_{-i} \in \tilde{A}^m_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a_{-i}) - \epsilon$$

. Therefore it follows that $U_i(s'_i, s_{-i}|h) > u_i(a(s|h)) = U_i(s|h)$, concluding that a profitable deviation exists and therefore it cannot be that s is a subgame perfect equilibrium. By no babbling, we conclude that $s_i(h) \notin D_i(\tilde{A}^m_{-i})$ for any $h \in H$ and therefore $s_i(h) \in \tilde{A}^{k+1}_i$, a contradiction. □

Lemma. *3 The set of actions that survive iterated elimination of never best responses to pure actions it also survives iterated elimination of iterated deletion of individually irrational actions: $IENBR \subseteq IIR$.*

Proof. Note that $B^0 = \tilde{A}^0$. Now we will show that $B^k \subseteq \tilde{A}^k$ for all $k \geq 0$. By the inductive hypothesis suppose that $B^m \subseteq \tilde{A}^m$ for all $m < k$. Now notice that for any $a_i \in B^k_i$ we have that there is some $a_{-i} \in B^{k-1}_{-i} \subseteq \tilde{A}^{k-1}_{-i}$ such $u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i})$ for all $a'_i \in A_i$. It follows that $u_i(a_i, a_{-i}) \geq \inf_{a'_{-i} \in B^{k-1}_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) \geq \inf_{a'_{-i} \in \tilde{A}^{k-1}_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a'_{-i})$. Further, $u_i(a_i, a_{-i}) \leq \sup_{a''_{-i} \in B^k_{-i}} u_i(a_i, a''_{-i}) \leq \sup_{a''_{-i} \in \tilde{A}^k_{-i}} u_i(a_i, a''_{-i})$. Therefore we conclude that if $a_i \in B^k_i$ then $a_i \in \tilde{A}^k_i$. concluding the proof. □

Lemma. *4 If $\{a^*, \underline{a}^1, .., \underline{a}^n\} \subseteq A$ satisfy:*

1. $\underline{a}^i_i \in \text{argmax}_{a_i \in A_i} u_i(a_i, \underline{a}^i_{-i})$
2. $u_i(a^*) \geq u_i(\underline{a}^i)$
3. $u_i(\underline{a}^j) \geq u_i(\underline{a}^i)$ *for all* $i, j \in N$

Then $\{a^*, \underline{a}^1, .., \underline{a}^n\} \subseteq IIR$.

Proof. We proceed inductively. By definition, $\{a^*, \underline{a}^1, .., \underline{a}^n\} \subseteq A = \tilde{A}^0$.

Now suppose that $\{a^*, \underline{a}^1, .., \underline{a}^n\} \subseteq \tilde{A}^k$ for all $k \leq m$ for $m \geq 0$. Note that

$$\underline{a}^i_i \in \underset{a_i \in A_i}{\text{argmax}}\, u_i(a_i, \underline{a}^i_{-i}) = \arg \sup_{a_i \in A_i} u_i(a_i, \underline{a}^i_{-i})$$

and therefore

$$u_i(\underline{a}_i, \underline{a}_{-i}) \geq \inf_{a_{-i} \in \tilde{A}^m_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$$

. Therefore by definition

$$u_i(\underline{a}^j) \geq \inf_{a_{-i} \in \tilde{A}^m_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$$

and $u_i(a^*) \geq \inf_{a_{-i} \in \tilde{A}^m_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$. Further, we have that $\sup_{a_{-i} \in \tilde{A}^m_{-i}} u_i(\underline{a}^i_i, a_{-i}) \geq u_i(\underline{a}^i)$, $\sup_{a_{-i} \in \tilde{A}^m_{-i}} u_i(\underline{a}^i_j, a_{-i}) \geq u_i(\underline{a}^j)$ and $\sup_{a_{-i} \in \tilde{A}^m_{-i}} u_i(a_i*, a_{-i}) \geq u_i(a^*)$. Therefore we may conclude that $\underline{a}^i_i, \underline{a}^j_i, a^*_i \in \tilde{A}^{m+1}_i$. □

Theorem. *2 if s^* is a Negotiated Binding Agreement then $U_i(s^*|h) \geq \inf_{a'_{-i} \in IIR_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a'_{-i})$ for all $h \in H$ and $i \in N$.*

Proof. Suppose not, then there is some $i \in N$ and $h \in H$ such that that

$$\inf_{a'_{-i} \in IIR_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) > U_i(s^*|h)$$

. It must be that a) s^* is a subgame perfect equilibrium and b) by theorem 1 it must be that $s^*_{-i}(h) \in IIR_{-i}$ for all $h \in H$. Let $\epsilon = \inf_{a'_{-i} \in IIR_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) - U_i(s^*|h) > 0$. Construct $\tilde{a}_i : IIR_{-i} \to A_i$ such that $u_i(\tilde{a}_i(a_{-i}), a_{-i}) \geq \sup_{a_i \in A_i} u_i(a_i, a_{-i}) - \frac{\epsilon}{2}$ for all $a_{-i} \in IIR_{-i}$. Consider a deviation to $s'_i(h')$ such that $s'_i(h') = \tilde{a}(s^*_{-i}(h'))$ for all $h' \in H$ at the history h. It follows that $U_i(s'_i, s^*_{-i}|h) \geq \inf_{a_{-i} \in IIR_{-i}} u_i(\tilde{a}_i(a_{-i}), a_{-i}) = \inf_{a_{-i} \in IIR_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i}) - \frac{\epsilon}{2} = \frac{\inf_{a_{-i} \in IIR_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i}) + U_i(s^*|h)}{2} > U_i(s^*|h)$. A contradiction, as therefore s^* is not a subgame perfect equilibrium and therefore not a Negotiated Binding Agreement. □

Theorem. *4 Take any game such that $\exists \{a^*, \underline{a}^1, ..., \underline{a}^n\} \subseteq A$ such that:*

1. $\underline{a}^i_i \in \text{argmax}_{a_i \in A_i} u_i(a_i, \underline{a}^i_{-i}) = \text{argmax}_{a_i \in A_i} u_i(a_i, \underline{a}^i_{-i})$
2. $u_i(a^*) \geq u_i(\underline{a}^i)$
3. $u_i(\underline{a}^j) \geq u_i(\underline{a}^i)$ *for all* $i, j \in N$

Then a^* can be supported in a Negotiated Binding Agreement.

Proof. Note within this proof I maintain the notation a^k to refer to the k^{th} period proposal in a history h, while I use $\underline{a}^j$ to denote the action profile used in equilibrium as a punishment for j.

Let s^* be as follows:

1. if $h = (a^1, ..., a^k)$ is such that there is some $j \in N$, such that $a^{k-1}_{-j} = s^*_{-j}((a^1, ..., a^{k-2}))$ and either

   (a) $a^k_l = s^*_l(h \backslash a^{k-1}) \quad \forall l \neq j$ while $a^k_j \neq s^*_j(h \backslash a^{k-1})$

   (b) or $a^k_{-j} = \underline{a}^j_{-j}$

   then $s^*_i(h) = \underline{a}^j_i$.

2. $s^*_i(h) = a^*_i$ otherwise

First note that from any history the continuation is terminal within two periods and therefore no babbling is satisfied.

Now to show that s^* is a subgame perfect equilibrium. Suppose that a profitable deviation exists at a history $h \in H$ for $i \in N$. If the deviation does not include some different proposal within two periods of h it cannot be profitable, as the outcome remains the same. Therefore any deviation must occur within two periods. Any such deviation, denoted by s'_i, if it does not lead to the same terminal history and therefore cannot be profitable, of $i \in N$ must lead to $\underline{a}^i_{-i}$ for all periods following. Let the terminal history following the deviation be denoted by $(s^*_{-i}, s'_i|h) = (h, a^k, a^{k+1}, ..., a^t, ...)$. When $(s^*_{-i}, s'_i|h) \in Z'$ let

$$(s^*_{-i}, s'_i|h) = (h, a^{'1}, a^{'2}, ..., a((s^*_{-i}, s'_i|h)), a((s^*_{-i}, s'_i|h)), a((s^*_{-i}, s'_i|h)), ...)$$

, i.e let the agreement that $(s^*_{-i}, s'_i|h)$ concludes in be infinitely repeated at the end of the sequence, with some abuse of notation. However, by construction, it must be that $\limsup_{t\to\infty} u(a^t) \leq u_i(\underline{a}^i)$ and therefore it must be at least weakly worse than any terminal history of the strategy s^*. Therefore no profitable deviation exists. □

Theorem. *5 For any game G such that A_i is compact subset of a metric space and u_i is continuous for all $i \in N$, a^* is supported by a no delay Negotiated Binding Agreement, s^*, if and only if $\exists\{\underline{a}^1, ..., \underline{a}^n\} \subseteq A$ such that:*

1. $\underline{a}^i_i \in \text{argmax}_{a_i \in A_i} u_i(a_i, \underline{a}^i_{-i})$
2. $u_i(a^*) \geq u_i(\underline{a}^i)$
3. $u_i(\underline{a}^j) \geq u_i(\underline{a}^i)$ *for all* $i, j \in N$

Proof. By the construction of theorem 4 such a^* can be supported.

To see that only such a^* can be sustained, take any a^* such that it is supported by a no delay Negotiated Binding Agreement given by the SPE s^*. Denote $\tilde{A} = \{a \in A | \exists h \in H \text{ s.t. } s^*(h) = a\}$. Note by strict no babbling these completely define the set of actions that can be agreed upon. Further to this, not that $s^*_{-i}(h) \in \tilde{A}_{-i}$ for all $h \in H$ by strict no delay. As s^* is an SPE it must be that there is no profitable deviation. Notice that $U_i(s^*|h) \geq \max_{a_i \in A_i} \inf_{a_{-i} \in \tilde{A}_{-i}} u_i(a_i, a_{-i})$. Suppose not $U_i(s^*|h) < \max_{a_i \in A_i} \inf_{a_{-i} \in \tilde{A}_{-i}} u_i(a_i, a_{-i})$. It follows that $\max_{a_i \in A_i} \inf_{a_{-i} \in \tilde{A}_{-i}} u_i(a_i, a_{-i}) - U_i(s^*|h) > 0$. Consider a deviation to s'_i such that $s'_i(h') = s^*_i(h')$ for all h' such that $h = (h', h'')$ while $s'_i(h')$ is such that $u_i((s'_i, s^*_{-i})(h')) = \max_{a_i \in A_i} u_i(a_i, s^*_{-i}(h'))$ for all other histories. Suppose such a deviation leads to perpetual disagreement. Denote the sequence induced by such a strategy by $z' = (a^1, a^2, ..., a^t, ...)$. Notice that $u_i(a^t_i, a^t_{-i}) = \max_{a_i \in A_i} u_i(a_i, a^t_{-i})$. Note that therefore $u_i(a^t_i, a^t_{-i}) \geq \max_{a_i \in A_i} \inf_{a_{-i} \in \{a'_{-i} \in A_{-i} | a'_{-i} = a^k_{-i}\}} u_i(a_i, a_{-i})$ By definition,

$$\begin{aligned}
U_i(s_i, s^*_{-i}|h) &\geq \liminf_{t \to \infty} u_i(a^t) \\
&\geq \liminf_{t \to \infty} \max_{a_i \in A_i} \inf_{a_{-i} \in \{a'_{-i} \in A_{-i} | a'_{-i} = a^k_{-i}\}} u_i(a_i, a_{-i}) \\
&= \max_{a_i \in A_i} \inf_{a_{-i} \in \{a'_{-i} \in A_{-i} | a'_{-i} = a^k_{-i}\}} u_i(a_i, a_{-i}) \\
&\geq \max_{a_i \in A_i} \inf_{a_{-i} \in \tilde{A}_{-i}} u_i(a_i, a_{-i}) \\
\Rightarrow U_i(s_i, s^*_{-i}|h) &> U_i(s^*|h)
\end{aligned}$$

therefore it cannot be that s^* is an SPE if the deviation ends in perpetual disagreement. The argument for agreement is direct from the definition.

Therefore it must be that $U_i(s^*|h) \geq \max_{a_i \in A_i} \inf_{a_{-i} \in \tilde{A}_{-i}} u_i(a_i, a_{-i})$. As $\tilde{A}$ are agreed upon, it must therefore be that $\forall \tilde{a} \in \tilde{A}$ we have that $u_i(\tilde{a}) \geq \max_{a_i \in A_i} \inf_{a_{-i} \in \tilde{A}_{-i}} u_i(a_i, a_{-i})$. Therefore there must be some $a'_{-i} \in \bar{\tilde{A}}_{-i}$, where $\bar{\tilde{A}}_{-i}$ is the limit points of $\tilde{A}_{-i}$ such that $u_i(\tilde{a}) \geq \max_{a_i \in A_i} u_i(a_i, a'_{-i})$. As this holds for all $\tilde{a} \in \tilde{A}$ it follows that $u_i(a') \geq \max_{a_i \in A_i} u_i(a_i, a'_{-i})$ therefore $u_i(a') = \max_{a_i \in A_i} u_i(a_i, a'_{-i})$. therefore $\exists a^i \in \bar{\tilde{A}}$ such that $u_i(\tilde{a}) \geq u_i(a^i) = \max_{a_i \in A_i} u_i(a_i, a^i_{-i})$. Notice that: $u_i(\tilde{a}) \geq u_i(a^i)$ for all $\bar{\tilde{A}}$ and therefore $u_i(\underline{a}^j) \geq u_i(\underline{a}^i)$ and $u_i(a^*) \geq u_i(a^i)$. Therefore such a profile of action profiles must exist for a^* to be supported. □

Lemma. ***5*** *For any Strong Nash equilibrium a^{SNE} of G, $a^{SNE} \in ICIR(\mathcal{C})$ regardless of $\mathcal{C}$.*

Proof. As a^* is a strong Nash equilibrium, it follows that $\nexists C \in 2^N \backslash \{\emptyset\}, a'_C \in A_C$ such that $u_i(a'_C, a^*_{-C}) > u_i(a^*)$ for all $i \in C$. Therefore a^* is not coalitionally irrational. Now

suppose that $a^* \in \tilde{A}^m(\mathcal{C})$ for all $m < k$. Notice that by the same statement this implies that $a^* \in \tilde{A}^{m+1}(\mathcal{C})$. This implies that $a^* \in ICIR(\mathcal{C})$ for all $\mathcal{C}$. □

Theorem. *6 For any $\mathcal{C}$-Negotiated Binding Agreement, s^*, and any $h \in H$, $s^*(h) \in ICIR(\mathcal{C})$.*

Proof. Suppose not, for some history $h' \in H$ we have that $s_C(h') = a_C$. By C no babbling it follows that there exists some $h \in H$ such that $a_C(s|h) = a_C$. Therefore it must be that $U_i(s^*|h) = u_i(a(s^*|h)) \leq \sup_{a'_{-C} \in A_{-C}} u_i(a_C, a'_{-C})$ for all $i \in C$. By definition of a_C being not coalitionally rational, there exists a function $a'_C : A_{-C} \to A_C$ such that $\inf_{a_{-C} \in A_{-C}} u_i(a'_C(a_{-C}), a_{-C}) > \sup_{a'_{-C} \in A_{-C}} u_i(a_C, a'_{-C})$. Consider a deviation of C at history h such that $s_C(h') = a'_C(s_{-C}(h'))$ for all $h' \in H$. It follows that $U_i(s'_C, s^*_{-C}|h) \geq \inf_{a_{-C} \in A_{-C}} u_i(a'_C(a_{-C}), a_{-C}) > \sup_{a'_{-C} \in A_{-C}} u_i(a_C, a'_{-C}) \geq U_i(s^*|h)$ for all $i \in C$. Concluding that s^* is not a $\mathcal{C}$-subgame perfect equilibrium.

Now suppose by contradiction that $s(h') \in \tilde{A}^k(\mathcal{C})\ \forall k < m$ and $h' \in H$ but $s(h') = a \notin \tilde{A}^{m+1}(\mathcal{C})$ for some $h' \in H$. By definition, it must be that $a \in \bigcup_{C \in \mathcal{C}} [D_C(\tilde{A}^{m-1}(\mathcal{C})_{-C}) \times A_{-C}]$. Therefore it must be that $a_C \in D_C(\tilde{A}^{m-1}(\mathcal{C})_{-C})$ for some $C \in \mathcal{C}$. By $\mathcal{C}$-no babbling we have that $\exists h \in H$ such that $a_C = a^*_C(s^*|h)$. By definition of coalition rationality given $\tilde{A}^{m-1}(\mathcal{C})_{-C}$, as $a_C \in D_C(\tilde{A}^{m-1}(\mathcal{C})_{-C})$ there must be some that there is some $a'_C : \tilde{A}^{m-1}(\mathcal{C})_{-C}$ such that $\inf_{a_{-C} \in \tilde{A}^{m-1}(\mathcal{C})_{-C}} u_i(a'_C(a_{-C}), a_{-C}) > \sup_{a'_{-C} \in \tilde{A}^{m-1}(\mathcal{C})_{-C}} u_i(a_C, a'_{-C})$. Consider a deviation of C at history h such that $s_C(h') = a'_C(s_{-C}(h'))$ for all $h' \in H$. It follows that

$$U_i(s'_C, s^*_{-C}|h) \geq \inf_{a_{-C} \in \tilde{A}^{m-1}(\mathcal{C})_{-C}} u_i(a'_C(a_{-C}), a_{-C}) > \sup_{a'_{-C} \in \tilde{A}^{m-1}(\mathcal{C})_{-C}} u_i(a_C, a'_{-C})$$

. Therefore $U_i(s'_C, s^*_{-C}|h) > U_i(s^*|h)$ for all $i \in C$. Concluding that s^* is not a $\mathcal{C}$-subgame perfect equilibrium. A contradiction.

□

Theorem. *7 For any $\mathcal{C}$-Negotiated Binding Agreement s^* must be such that, for any history h, and for any coalition $C \in \mathcal{C}$, there is no $a'_C : [ICIR(\mathcal{C})]_{-C} \to A_C$ such that $\inf_{a_{-C} \in [ICIR(\mathcal{C})]_{-C}} u_i(a'_C(a_{-C}), a_{-C}) > U_i(s^*|h)$ for all $i \in C$.*

In other words, $a(s^*|h)$ must be in the β-core with respect to $ICIR(\mathcal{C})$ for all histories.

Proof. Suppose this is not the case. There is some $C \in \mathcal{C}$ $a'_C : [ICIR(\mathcal{C})]_{-C} \to A_C$ such that

$$\inf_{a_{-C} \in [ICIR(\mathcal{C})]_{-C}} u_i(a'_C(a_{-C}), a_{-C}) > U_i(s^*|h)$$

for all $i \in C$. It must be that s^* is a $\mathcal{C}$-subgame perfect equilibrium, and therefore there cannot exist a profitable deviation for C. Notice that $s_i^*(h) \in [ICIR(\mathcal{C})]_i$ for all $i \in N$. Consider a joint deviation from coalition C to a strategy s'_C such that $s'_C(h) = a'_C(s^*_{-C}(h))$ for all $h \in H$. By the definition of the utilities that this can induce, it is clear that

$$U_i(s'_C, s^*_{-C}|h) \geq \inf_{a_{-C} \in [ICIR(\mathcal{C})]_{-C}} u_i(a'_C(a_{-C}), a_{-C})$$

for all $i \in C$, and therefore $u_i(s'_C, s^*_{-C}|h) > U_i(s^*|h)$ for all $i \in C$. In conclusion, s^* cannot be a $\mathcal{C}$-subgame perfect equilibrium, and therefore cannot be a $\mathcal{C}$-Negotiated Binding Agreement. □

Theorem. *8 Take any game G such that there is some $a^* = \underline{a}^N \in ICIR(\mathcal{C})$ and for all $C \in \mathcal{C} \backslash N$ $\exists \underline{a}^C \in ICIR(\mathcal{C})$ such that:*

1. *$\nexists a'_C \in A_C$ such that $u_i(a'_C, \underline{a}^C_{-C}) > u_i(\underline{a}^C)$ for all $i \in C$*
2. *for all $C \in \mathcal{C}$ there is some $i \in C$ such that $u_i(a^*) \geq u_i(\underline{a}^C)$*
3. *For all $C, C' \in \mathcal{C}$ there is some $i \in C$ such that $u_i(\underline{a}^{C'}) \geq u_i(\underline{a}^C)$*

Then a^* can be supported in a $\mathcal{C}$-Negotiated Binding Agreement.

Proof. 1. if $h = (a^1, ..., a^k)$ is such that there is some $C \in \mathcal{C}$, such that $a^{k-1}_{-C} = s^*_{-C}((a^1, ..., a^{k-2}))$ and either

   (a) $a^k_l = s^*_l(h \backslash a^{k-1}) \quad \forall l \notin C$ while $a^k_j \neq s^*_j(h \backslash a^{k-1})$ for all $j \in C$

   (b) or $a^k_{-C} = \underline{a}^C_{-C}$

   then $s^*_i(h) = \underline{a}^C_i$.

2. $s^*_i(h) = a^*_i$ otherwise

Now I will show that s^* is a $\mathcal{C}$-Negotiated Binding Agreement.

First, I will show that s^* is a $\mathcal{C}$-subgame perfect equilibrium. First, by assumption, at no history can N deviate as a coalition to improve all their utilities if $N \in \mathcal{C}$, as all $\underline{a}^C$ are weakly Pareto optimal in this case by the definition of $ICIR(\mathcal{C})$. Now assume that some other coalition $C \in \mathcal{C}$ has a profitable deviation. Now,suppose that $a_j \neq s^*_j(h)$ for all $j \in C$, then it cannot be profitable as it leads to a history that induces the $\underline{a}^C_{-C}$ for all periods. Now suppose that $a_j \neq s^*_j(h)$ for all $j \in B$, where $B \subset C$, while $a^*_j = s^*_j(h)$. Then it must induce a path such that either a member of B is worse off, or further deviations

within C take place. Either way, it cannot be that this is a profitable deviation.

As all histories end within 2 periods we satisfy the condition of no babbling agreements and therefore we have a $\mathcal{C}$-Negotiated Binding Agreement. □

C Proofs for Appendix A

Proposition. *1 If s^* is a Negotiated Binding Agreement with order then, for all histories for i is active $h \in \tilde{H}_i$, $s_i(h) \in IAD_i$.*

Proof. By induction. Firstly, note that $s_i(h) = a_i \notin D_i(A_{-i})$ for all $h \in \tilde{H}_i$. To see this suppose by contradiction it is not the case. Then $s_i^*(h) = a_i \in D_i(A_{-i})$ for some $i \in N$ and some history $h \in \tilde{H}_i$. It must be that $a_i(s^*|h) = a_i$ for some $h' \in H$. Given this, $U_i(s^*|h) \leq \sup_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})$. Further, s^* is a subgame perfect equilibrium, and therefore there is no profitable deviation for s_i^* at any history for which i is active, including h'. Notice that as $a_i \in D_i(A_{-i})$ then $\exists a_i' \in A_i$ such that $\inf_{a'_{-i} \in A_{-i}} u_i(a_i', a'_{-i}) > \sup_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})$. Now consider a strategy s_i' such that $s_i'(h'') = a_i'$ for all h'' for which i is active. Notice that, by construction of s_i', the history $(s_i', s_{-i}^*|h')$ must either terminate in a_i' or be such that only action profiles with a_i' appear after h. In either case, we can conclude that $U_i(s_i', s_{-i}^*|h') \geq \inf_{a'_{-i} \in A_{-i}} u_i(a_i', a'_{-i}) > \sup_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \geq U_i(s^*|h')$. A contradiction to s^* be a subgame perfect equilibrium.

By the inductive hypothesis, suppose that $s_i^*(h) \in \tilde{A}_i^m$ for all $h \in \tilde{H}_i$ and $i \in N$. Now suppose by contradiction that $s_i^*(h) = a_i \in D_i(\tilde{A}_{-i}^m)$. It must be that $a_i(s^*|h') = a_i$. Given this, $U_i(s^*|h') \leq \sup_{a_{-i} \in A_{-i}^m} u_i(a_i, a_{-i})$, as $s_{-i}^*(h'') \in \tilde{A}_{-i}^m$ for all $h'' \in \tilde{H}_i$. Further, s^* is a subgame perfect equilibrium, and therefore there is no profitable deviation for s_i^* at any history, including h'. Notice that as $a_i \in D_i(\tilde{A}_{-i}^m)$ then $\exists a_i' \in A_i$ such that $\inf_{a'_{-i} \in \tilde{A}_{-i}^m} u_i(a_i', a'_{-i}) > \sup_{a_{-i} \in \tilde{A}_{-i}^m} u_i(a_i, a_{-i})$. Now consider a strategy s_i' such that $s_i'(h'') = a_i'$ for all $h'' \in \tilde{H}_i$. Notice that, by definition and construction of s_i' $U_i(s_i', s_{-i}|h')$ must only be constructed using the utility of $u_i(a_i', \cdot)$, as either $(s_i', s_{-i}|h') \in Z'$, in which case it must terminate in a_i' by definition, or $(s_i', s_{-i}|h') \in Z''$, in which case all histories following h' use only a_i'. Further, as $s_{-i}^*(h'') \in \tilde{A}_{-i}^m$ that from this history on the only action profiles proposed are a_i', a'_{-i} such that $a'_{-i} \in \tilde{A}_{-i}^m$. Given this, we can conclude that $U_i(s_i', s_{-i}|h') \geq \inf_{a'_{-i} \in \tilde{A}_{-i}^m} u_i(a_i', a'_{-i}) > \sup_{a_{-i} \in \tilde{A}_{-i}^m} u_i(a_i, a_{-i}) \geq U_i(s^*|h')$. A contradiction to s^* be a subgame perfect equilibrium. □

Proposition. *2 Take any order $\mathcal{O}$. a^* is supported in a Negotiated Binding Agreement then it is supported in Negotiated Binding Agreement with order $\mathcal{O}$.*

Proof. We will show that if a^* is sustained in a Negotiated Binding Agreement then it can be sustained in a Negotiated Binding Agreement with order $\mathcal{O}$ for any order. Take any order $\mathcal{O}$. Take s^* that sustains a^* in a Negotiated Binding Agreement. Let $s'_i : \tilde{H}_i \to A_i$ such that, for all $h \in \tilde{H}_i$ such that $h = (h', (a_{\mathcal{O}^{-1}(1)}, ..., a_{\mathcal{O}^{-1}(i)-1}))$ we have that $s'_i(h) = s^*_i(h')$. First note that $a(s'|\emptyset) = a^*$ and $a(s'|h') = a(s^*|h)$ whenever $h' = h$ while $h' \in \tilde{H}$ and $h \in H$. Next we will show that s' is subgame perfect. Suppose not, there is some $i \in N$ for which there exists some $h \in H'_i$ and some $s''_i \in S_i$ such that $U_i(s''_i, s'_{-i}|h) > U_i(s'|h)$. However, given agents are rational and the structure of s', they can replicate any deviation from s'_i with a deviation from s^*_i. With this, we must conclude that s^*_i is not subgame perfect. A contradiction. Concluding that s' is a Negotiated Binding Agreement with order $\mathcal{O}$, leading to the outcome a^*. □

Proposition. *3 If s^* is a Negotiated Binding Agreement with all proposals, for all histories $h \in \tilde{\tilde{H}}$, $s_i(h) \in IIR_i$.*

Further, for any negotiated with order s^* be be such that, for any history $h \in \tilde{\tilde{H}}$, $U_i(s^*|h) \geq \underline{u}_i$ where

$$\underline{u}_i = \inf_{a_{-i} \in IIR_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$$

.

Proof. By induction. Firstly, note that $s_i(h) = a \notin D_i(A)$ for all $h \in \tilde{\tilde{H}}$. Suppose by contradiction it is the case. Then $s^*_i(h) = a \in D(A)$ for some $i \in N$ and some history $h \in \tilde{\tilde{H}}$. It must be that $a(s^*|h') = a$ for some history $h' \in H$. This implies that $[s^*_i(h')]_j = a_j \in D_j(A_{-j})$ for some j. Given this, $U_j(s^*|h) \leq \sup_{a_{-j} \in A_{-j}} u_j(a_j, a_{-j})$. Further, s^* is a subgame perfect equilibrium, and therefore there is no profitable deviation for s^*_j at any history, including h. Notice that as $a_j \in D_j(A_{-j})$ then, for all $\epsilon > 0$ $\exists a'_j : A_{-i} \to A_j$ such that $u_i(a'_j(a_{-i}), a'_{-j}) > \sup_{a_{-j} \in A_{-j}} u_i(a_j, a_{-j}) - \epsilon$. Now consider a strategy s'_j such that $s'_j(h'') = (a'_j(s_{-j}(h'')), a''_{-j})$, for some $a''_{-j} \in A_{-j}$ for all h''. Notice that, by construction of s'_i, the history $(s'_j, s^*_{-j}|h')$ must either terminate in a'_j or be such that only action profiles with a'_j appear after h'. In either case, we can conclude that $U_j(s'_j, s^*_{-j}|h') \geq \inf_{a'_{-j} \in A_{-j}} u_i(a'_j(a'_{-j}), a'_{-j}) > \sup_{a_{-j} \in A_{-j}} u_i(a_j, a_{-j}) \geq U_j(s^*|h')$. A contradiction to s^* be a subgame perfect equilibrium.

By the inductive hypothesis, suppose that $s^*_i(h) \in \tilde{A}^m$ for all $h \in \tilde{H}_i$ and $i \in N$. Now suppose by contradiction that $s^*_i(h) = a \in D(\tilde{A}^m)$. It must be that $a(s^*|h') = a$ for some history $h' \in H$. Further, for some $j \in N$ $a_j \in D(\tilde{A}^m)$. Without loss of generality let $j = i$. Given this, $U_i(s^*|h') \leq \sup_{a_{-i} \in A^m_{-i}} u_i(a_i, a_{-i})$, as $s^*_{-i}(h'') \in \times_{j \neq i} \tilde{A}^m$ for all $h'' \in \tilde{\tilde{H}}$. Further, s^* is a subgame perfect equilibrium, and therefore there is no profitable deviation for s^*_i at any history, including h. Notice that as $a_j \in D_j(A_{-j})$ then, for all $\epsilon > 0$

$\exists a'_j : \tilde{A}^m_{-i} \to A_j$ such that $u_i(a'_j(a_{-i}), a'_{-j}) > \sup_{a_{-j} \in \tilde{A}^m_{-j}} u_i(a_j, a_{-j}) - \epsilon$. Now consider a strategy s'_i such that $s'_i(h'') = (a'_i(s^*_{-i}(h'')), a''_{-i})$, with $a''_{-i} \notin A_{-i}$ for all $h'' \in \tilde{H}_i$. Notice that, by definition and construction of s'_i $U_i(s'_i, s^*_{-i}|h')$ must only be constructed using the utility of $u_i(a'_i, \cdot)$, as with the before logic, we can only terminate in histories that have a'_i infinitely repeated or an agreement is reached with a'_i. Given this, we can conclude that $U_i(s'_i, s^*_{-i}|h') \geq \inf_{a'_{-i} \in \tilde{A}^m_{-i}} u_i(a'_i(a'_{-i}), a'_{-i}) > \sup_{a_{-i} \in \tilde{A}^m_{-i}} u_i(a_i, a_{-i}) \geq U_i(s^*|h')$. A contradiction to s^* be a subgame perfect equilibrium.

As proposals are simultaneous, the logic of showing that for any negotiated with order s^* be be such that, for any history $h \in \tilde{H}$, $U_i(s^*|h) \geq \underline{u}_i$ where

$$\underline{u}_i = \inf_{a_{-i} \in IIR_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$$

is identical to theorem 2, where s'_i is selected to intentionally cause perpetual disagreement. □

Proposition. *4 a^* is supported by a Negotiated Binding Agreement with all proposals if a^* is supported by a Negotiated Binding Agreement.*

Proof. If a^* is supported by a Negotiated Binding Agreement then a^* is supported by a all proposal Negotiated Binding Agreement. Take s^* that supports a^* in a Negotiated Binding Agreement. Construct $s'_i : \tilde{\tilde{H}} \to A$ as follows. Let $s'_i(h'') = s^*(\tilde{h''})$, where $\tilde{h''}$ is as defined to define payoffs of infinite histories. Clearly if s^*_i is optimal so is s'_i as a deviation to a partial infinite history leads to the same payoff that could be achieved under s^*_{-i}. A deviation to another terminal history must be such that it could not be achieved under a deviation from s^*_i. However, by definition of s'_i, this cannot be the case. □

Proposition. *5 If s^* is a Negotiated Binding Agreement then s^* is a constant outside option Negotiated Binding Agreement.*

Proof. As s^* is a Negotiated Binding Agreement it must be that s^* is a subgame perfect equilibrium with the terminal infinite histories giving a payment as defined in section 2. As s^* never dictates that a history should be infinite and terminal, it follows that there is no profitable deviation where the outcome leads to a deterministic outcome. It follows that the payoff on the path remains the same when the model of a constant outside option is taken. Finally, as there is no profitable deviation when the deviation would induce a terminal infinite history when the payoff is defined as in section 2, there cannot be a profitable deviation when the constant outside option is taken. Therefore s^* is a constant outside option Negotiated Binding Agreement. □

Proposition. *6 For all $a \in A^{agree}$, $a \in IIR$.*

Further, if $a^* \in A^{agree}$, then $u_i(a^*) \geq \inf_{a_{-i} \in IIR_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i})$.

Proof. We proceed inductively. First by contradiction suppose that $a_i \in D_i(A_{-i})$ while $a_i \in A^{agree}$. As $a_i \in D_i(A_{-i})$ it follows that $\inf_{a'_{-i} \in A_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) > \sup_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i})$. Therefore, it follows that $\forall a'_{-i} \in A^{agree}$ for which $\sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) > \sup_{a_{-i}} u_i(a_i, a_{-i})$. Therefore we conclude that for any $a_{-i} \in A^{agree}_{-i}$ we have that $\sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) > u_i(a_i, a_{-i})$, violating 2.. Therefore we conclude that $A^{agree} \subseteq \tilde{A}^1$.

Inductively, assume that $A^{agree} \subseteq \tilde{A}^k$ for all $k > 0$, we will show that $A^{agree} \subseteq \tilde{A}^{k+1}$. Suppose not, there is some $a \in \tilde{A}^{k+1}$ such that $a \notin A^{agree}$. It follows that for some $a_i \in A^{agree}_i$, while $a_i \in D_i(\tilde{A}^k_{-i})$. As $a_i \in D_i(\tilde{A}^k_{-i})$ it follows that t $\inf_{a'_{-i} \in \tilde{A}^k_{-i}} \sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) > \sup_{a_{-i} \in \tilde{A}^k_{-i}} u_i(a_i, a_{-i})$. Therefore, it follows that $\forall a'_{-i} \in A^{agree}$ for which $\sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) > \sup_{a_{-i}} u_i(a_i, a_{-i})$. Therefore we conclude that for any $a_{-i} \in A^{agree}_{-i}$ we have that $\sup_{a'_i \in A_i} u_i(a'_i, a'_{-i}) > u_i(a_i, a_{-i})$, violating 2.. Therefore we conclude that $A^{agree} \subseteq \tilde{A}^{k+1}$. Therefore we can conclude that $A^{agree} \subseteq IIR$.

Finally, $u_i(a^*) \geq \inf_{a_{-i} \in IIR_{-i}} \sup_{a_i \in A_i} u_i(a_i, a_{-i}) = \underline{u}_i, \forall a^* \in A^{agree}$ is immediately implied by 2.. □