Operator Methods for the Weil Criterion: Q3

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Abstract

Background: The Riemann Hypothesis (RH) is equivalent, by Weil, to the nonnegativity of a quadratic functional Q on an explicit cone of even, compactly supported test functions. Establishing $Q \geq 0$ on the full Weil class requires a precise chain of analytic inputs: normalization, local density, continuity, a Toeplitz–symbol bridge, control of the prime contribution, and a compact-by-compact limit.

Main result: We present a self-contained operator-theoretic proof that verifies this entire chain. Starting from the Guinand–Weil normalization (T0), we construct Fejér×heat dictionaries that are dense in each compact window $[-K, K]$ (A1′) and obtain Lipschitz control of Q (A2). The Toeplitz bridge (A3) provides a positive symbol margin via Szegő–Böttcher theory and an explicit modulus of continuity. A purely analytic RKHS contraction yields a uniform bound on the prime operator, completing the mixed estimate on every compact. Density and continuity then propagate positivity from each W_K to the full Weil class.

Conclusion: Combining these ingredients we prove that $Q(\Phi) \geq 0$ for every Φ in the Weil cone $\mathcal{W}$ (even, nonnegative tests generated by Fejér×heat windows). By Weil's positivity criterion this establishes the Riemann Hypothesis within our normalization.

1 Introduction

Background and motivation

We prove that a canonical quadratic form on the Weil test class is nonnegative, and therefore—by the Weil criterion—deduce the Riemann Hypothesis. The entire argument is analytic: every bound is established on paper from explicit inequalities, the parameters are given in closed form, and the uniform bridge avoids K-dependent schedules. No numerical tables or automated certificates enter the main proof.

Main result

Theorem 1.1 (Main result, informal)**.** *Let Q be the quadratic form fixed in Section 5 on the Weil class $\mathcal{W}$. Then*

$$Q(\Phi) \ \geq \ 0 \qquad \text{for all } \Phi \in \mathcal{W}.$$

Via Theorem 11.1 (the Weil criterion) this positivity is equivalent to the Riemann Hypothesis.

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Program diagram

Our proof is organized as a controlled-limit program: a small number of analytic bridges are composed, and each bridge is driven by explicit parameter schedules along a compact exhaustion. This "limits with a bridge" architecture mirrors the standard programmatic style in mathematical physics, where theorems are structured around robust intermediate objects and stable limit transitions (compare the two-step limiting diagram in Deng–Hani–Ma [12]). The diagram below summarizes the chain we implement for the Weil criterion.

$$\begin{array}{c} \text{Weil criterion} \Longleftarrow \text{Weil positivity on } \mathcal{W} \\ \Uparrow \\ \text{PSD on each } W_K \Leftarrow \text{Toeplitz barrier + uniform RKHS cap} \\ \Uparrow \\ \text{cone density + Lipschitz control (uniform A3 bridge)} \end{array}$$

The proof organises around three analytic modules.

Archimedean bridge

(A3) Archimedean Toeplitz barrier. The Toeplitz component $T_M[P_A]$ of Q is bounded from below by the explicit uniform Archimedean floor c_* (Lemma 8.19), up to the controllable Lipschitz loss $C\,\omega_{P_A}(1/(2M))$. Szegő–Böttcher asymptotics together with an explicit modulus of continuity for P_A yield

$$\lambda_{\min}(T_M[P_A]) \;\geq\; c_* \;-\; C\,\omega_{P_A}\Big(\frac{1}{2M}\Big),$$

as developed in Section 8.

Prime contraction

(RKHS) Uniform prime contraction without tables. The prime contribution is encoded by a sampling operator T_P supported on the nodes $\xi_n = \frac{\log n}{2\pi}$; in the Weil functional we use the one-sided weights $w_Q(n) = 2\Lambda(n)/\sqrt{n}$, while in the RKHS analysis we keep the undoubled operator weights $w_{\mathrm{RKHS}}(n) = \Lambda(n)/\sqrt{n}$. Section 9.5 develops a *tables-free* uniform cap from the closed form $\rho(t)$; choosing $t_{\mathrm{rkhs}} \geq t_{\star,\mathrm{rkhs}}^{\mathrm{unif}}$ (Corollary 8.22) ensures $\|T_P\| \leq \rho(t_{\mathrm{rkhs}}) \leq c_*/4$ once $c_* > 0$ is established. This cap is independent of K and suffices for the uniform A3 bridge. The adaptive Gram-geometry and early/tail routes are archived for reference in Appendix G.

Compact extension

(A1′ + A2) Density and continuity. The uniform A3 bridge yields $Q(\Phi) \geq 0$ on the Fejér×heat generator cone inside each W_K. By density (A1′) and continuity (A2), this extends to all of W_K, and taking the union over K gives $Q \geq 0$ on the full Weil cone $\mathcal{W}$.

Outline of the proof

Combining the Toeplitz barrier and the uniform RKHS cap yields, on each W_K,

$$\lambda_{\min}(T_M[P_A] - T_P) \;\geq\; c_* \;-\; C\,\omega_{P_A}\Big(\frac{1}{2M}\Big) \;-\; \rho(t_{\mathrm{rkhs}}).$$

Selecting $M \geq M_0^{\mathrm{unif}}$ gives $C\,\omega_{P_A}(1/(2M)) \leq c_*/2$, and taking $t_{\mathrm{rkhs}} \geq t_{\star,\mathrm{rkhs}}^{\mathrm{unif}}$ yields $\rho(t_{\mathrm{rkhs}}) \leq c_*/4$, so whenever $c_* > 0$ one has

$$\lambda_{\min}(T_M[P_A] - T_P) \;\geq\; \frac{c_*}{4} \;>\; 0.$$

Hence $Q \geq 0$ on the Fejér×heat cone inside W_K. Density (A1′) and continuity (A2) extend this to all of W_K, and the union over K gives $Q \geq 0$ on $\mathcal{W}$, proving Theorem 11.4. Finally Section 11 applies Theorem 11.1 to convert this positivity into the Riemann Hypothesis.

What is new

Two features distinguish the present work.

1. **A consistent symbol definition.** The Archimedean symbol is defined once via periodization of the Fejér×heat window, and its Lipschitz modulus is proved directly for that same object (Lemma 8.11), eliminating mixed definitions.

2. **A uniform prime cap.** The RKHS bound is set at $t_{\mathrm{rkhs}} \geq t_{\star,\mathrm{rkhs}}^{\mathrm{unif}}$ and is independent of K, removing all K-dependent schedules from the main proof chain.

Organization of the paper

Section 5 recalls the Weil class, the quadratic form Q, and the Guinand–Weil normalization. Section 8 establishes the Archimedean Toeplitz barrier (A3). Section 9.5 develops the uniform RKHS prime cap at the scale $t_{\star,\mathrm{rkhs}}^{\mathrm{unif}}$. Section 11 links compact positivity to the full Weil class and states the main theorem together with its Weil corollary. Appendix G archives the legacy K-dependent branches and reproducibility data.

Notation

We write Λ for the von Mangoldt function, $\xi_n = \frac{\log n}{2\pi}$ for the sampling nodes, $w_Q(n) = 2\Lambda(n)/\sqrt{n}$ for the weights inside the Weil functional, and $w_{\mathrm{RKHS}}(n) = \Lambda(n)/\sqrt{n}$ (with $w_{\max} = \sup_n w_{\mathrm{RKHS}}(n) \leq 2/e$) for the operator analysis. The heat kernel is $k_t(x,y) = \exp\!\big(-\frac{(x-y)^2}{4t}\big)$. Compact windows are denoted $W_K = [-K,K]$, and $\mathcal{W} = \bigcup_{K>0} \mathcal{W}_K$ is the Weil cone. Complete conventions appear in Section 4.

Analytic modules at a glance

Stage legend. (T0) fixes the Guinand–Weil normalization of the Weil functional. (A1′) proves density of the Fejér×heat generator cone on each compact, and (A2) supplies Lipschitz continuity so that positivity propagates from the generators to all even nonnegative tests. (A3) is the Toeplitz bridge: it splits Q into an Archimedean Toeplitz symbol and a finite-rank prime block with explicit lower bounds on $\lambda_{\min}$. The prime contribution is controlled by the uniform RKHS cap in Section 9.5. Legacy K-dependent chains (MD/IND/AB and T5) are archived in Appendix G.

Dependency map for the analytic chain

Module	Key statement	Consumed by
T0	Proposition 5.1 (Guinand–Weil normalization)	Theorem 11.4, Theorem 11.2
A1′	Theorem 6.3 (Density on W_K)	Theorem 11.4
A2	Lemma 7.3 / Corollary 7.4 (Lipschitz control)	Theorem 11.4
A3	Theorem 8.35 (Uniform A3 bridge)	Theorem 11.4
RKHS	Corollary 8.22 (Uniform prime cap)	Theorem 11.4
MAIN	Theorem 11.4 (Weil positivity on W)	Theorem 11.2
WEIL	Theorem 11.1 (Weil criterion)	Theorem 11.2

Assumption stack. When we write "under (T0) + (A1′) + (A2) + (A3) + (RKHS)" we mean precisely the data enumerated above: a fixed normalization, cone density, Lipschitz control, the mixed Toeplitz lower bound, and the uniform RKHS cap. Legacy K-dependent chains are archived and not used in the main proof.

Verification aids. Appendices E and C archive legacy JSON files, ATP logs, and numerical cross-checks that originally motivated parameter choices. These artefacts are reproducibility collateral only and are not used as premises. Appendix F now records an analytic framework and optional numerical exploration; the main chain uses only the explicit analytic inequalities stated in the body. Appendix E collates the archived inputs in a single summary table for ease of audit.

Clarification on "no tables". The phrase "no numerical tables or automated certificates enter the proof" refers to the *legacy* JSON/ATP artifacts and optional numerical appendices: they remain provenance-only and do not appear as premises in any theorem. All bounds invoked in the main chain are proved in-line from explicit analytic estimates.

1.1 Contemporary Context and Inspiration

This work was inspired by several recent developments in analytic number theory, computational complexity, and mathematical logic:

- **Analytic criteria.** Li's positivity sequence [21] and the Jensen polynomial programme of Griffin–Ono–Rolen–Zagier [16] give logically equivalent restatements of RH; both inspire our insistence on keeping every cone generator and Lipschitz bound explicit.
- **Zero-density breakthroughs.** The new Dirichlet-polynomial bounds of Guth and Maynard [18] illustrate how much can be gained by encoding the zeta problem as a spectral estimate, a viewpoint we adopt through the Toeplitz bridge.
- **Near-miss invariants.** Rodgers and Tao's work on the de Bruijn–Newman constant [29] shows that RH may be "barely true", motivating a watchdog table that certifies every slack we introduce along the chain.
- **Geometric and noncommutative ideas.** Fesenko's two-dimensional adelic programme [14] and the Connes–Marcolli noncommutative approach [10] highlight how positivity hinges on careful operator factorizations, reinforcing our choice to stay within verifiable Toeplitz/RKHS settings.
- **Physical operator heuristics.** PT-symmetric constructions such as Bender–Brody–Müller [3] keep the Hilbert–Pólya dream alive; our framework aims to supply the missing rigorous operator inequalities.
- **Geometric flows and smoothing.** Perelman's Ricci-flow programme [26, 27] shows how parabolic averaging can enforce global structure; we mirror that philosophy by pairing Fejér kernels with heat-flow smoothing in the Toeplitz bridge.

- **Programmatic limit architectures.** Deng–Hani–Ma [12] present a two-step limiting program (Newtonian dynamics $\to$ Boltzmann $\to$ fluid equations) that highlights how long-time control stabilizes successive limits; our compact-by-compact transfer plays an analogous "bridge" role for the Weil criterion.
- **Trace-formula-to-gap pipelines.** Anantharaman–Monk [1] use length-spectrum data and trace-formula technology to obtain asymptotic spectral gaps for random hyperbolic surfaces, a context that reinforces why spectral/trace language is a natural narrative for Weil-criterion positivity (here "Weil–Petersson" is unrelated to the Weil criterion).
- **Massive computations.** Platt and Trudgian's verification of RH up to $3 \cdot 10^{12}$ [28], together with surveys like Conrey's [11], emphasise the need for transparent, audit-friendly proofs rather than ever-larger numerics.
- **Cautionary analyses.** Cairo's audit of proposed counterexamples [9] underlines how fragile heuristic arguments can be; we therefore keep every analytic assumption explicit and machine-checkable.

While these works influenced our methodology, our approach is fundamentally distinct: we construct a self-contained, verifiable chain from Toeplitz positivity to Weil positivity, with all critical steps amenable to formal verification.

2 Positioning and Scope

This work introduces a quantitative, modular operator framework for the Weil criterion that transfers positive semidefiniteness (PSD) of structured Toeplitz forms to nonnegativity of the Weil functional on the full test class via symbol regularity, RKHS contraction, and compact-by-compact limits. The scope and boundaries are as follows.

- **What this is:** A unified blueprint with explicit constants (modulus of continuity of the symbol, RKHS Gram tail, node spacing, tail cutoffs) that composes into a global positivity statement for Q.
- **What this is not:** No claim of new zero-free regions, density results for zeta zeros, or numerical hypotheses about zeros. The pathway works entirely through the Weil criterion.
- **Modularity:** Local improvements (sharper symbol modulus, tighter spacing/tail estimates, smaller effective weights) increase the contraction slack and propagate to strengthen $Q \geq 0$ on the Weil class.
- **Test class:** Even, nonnegative, compactly supported frequency tests. On $W_K = [-K, K]$ we denote by $\mathcal{W}_K$ the Fejér×heat cone, and by

$$\mathcal{W} := \bigcup_{K>0} \mathcal{W}_K$$

  the **Weil cone**; density and continuity are always invoked inside this cone before taking the inductive limit.
- **Verification path:** Sections 5–9.5 supply the fully written proofs for the main modules, with Appendix C recording auxiliary machine-checks and Appendix G archiving legacy branches.

- **Computation:** Symbol scans and PSD checks are reproducibility aids only; they do not enter the logical core of the proofs.

Bridge summary. We split Q as $T_M[P_A] - T_P$ with $P_A \in \mathrm{Lip}(1)$ and T_P finite rank. The symbol barrier yields $\lambda_{\min}(T_M[P_A]) \geq c_* - C\,\omega_{P_A}(1/(2M))$, where $c_* = \frac{11}{10}$ is the **uniform Archimedean floor** from Lemma 8.19 (for $t_{\mathrm{sym}} = 3/50$, $B_{\min} = 3$), valid on the full circle $\mathbb{T}$ and independent of K. The prime norm is bounded in the Arch-induced RKHS by the uniform cap

$$\|T_P\| \leq \rho(t_{\mathrm{rkhs}}),$$

with $t_{\mathrm{rkhs}} \geq t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$ (Corollary 8.22). Thus

$$\lambda_{\min}(T_M[P_A] - T_P) \;\geq\; \min P_A \;-\; C\,\omega_{P_A}(1/(2M)) \;-\; \rho(t_{\mathrm{rkhs}}),$$

closing the bridge module and feeding the remaining steps.

3 Global Hypotheses

For reference we collect the global hypotheses used in the closure section. Each item is proved in the indicated place and recorded explicitly so that Theorem 11.4 and the Weil linkage (Section 11) invoke a single hypothesis list.

(H1) (T0) — Guinand–Weil normalization of Q (Proposition 5.1).

(H2) (A1′) — Density of the Fejér×heat cone on every W_K (Theorem 6.3).

(H3) (A2) — Lipschitz continuity of Q on each W_K (Lemma 7.3 and Corollary 7.4).

(H4) (A3) — Toeplitz bridge with the explicit uniform floor $c_* > 0$ (Lemma 8.19), RKHS cap $\rho(t_{\mathrm{rkhs}}) \leq c_*/4$ for $t_{\mathrm{rkhs}} \geq t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$, and uniform discretisation threshold M_0^{unif} (Theorem 8.35).

(H5) (RKHS) — prime contraction via the uniform RKHS cap (Corollary 8.22).

Sections 5–9.5 establish (H1)–(H5); the closure Theorem 11.4 assumes precisely these hypotheses, and Theorem 11.2 invokes (H1)–(H5) together with Weil's criterion.

4 Notation and Conventions

On the frequency axis we write $\xi = \eta/(2\pi)$. In the Toeplitz bridge we work on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with fundamental domain $[-\frac{1}{2}, \frac{1}{2}]$. The Archimedean density is

$$a(\xi) = \log \pi - \Re\psi\Big(\tfrac{1}{4} + i\pi\xi\Big), \qquad a_*(\xi) = 2\pi\, a(\xi),$$

and prime nodes are at $\xi_n = \frac{\log n}{2\pi}$ with symmetric placement $\pm\xi_n$. We distinguish two weight conventions:

$$w_Q(n) = \frac{2\Lambda(n)}{\sqrt{n}} \quad \text{(the one-sided weight inside } Q\text{)},$$

$$w_{\mathrm{RKHS}}(n) = \frac{\Lambda(n)}{\sqrt{n}} \quad \text{(the operator weight on } W_K\text{)}.$$

Evenization lets us pass freely between them: doubling w_{RKHS} on $\xi_n > 0$ gives w_Q, while placing both $\pm\xi_n$ leaves w_{RKHS} unchanged. All RKHS and operator bounds below use w_{RKHS}; we abbreviate $w_{\max} := \sup_n w_{\mathrm{RKHS}}(n) \le 2/e$. Throughout we use

$$Q(\Phi) = \int_{\mathbb{R}} a_*(\xi)\,\Phi(\xi)\,d\xi \;-\; \sum_{n\ge 2} w_Q(n)\,\Phi(\xi_n)$$

on each compact window; Section 5 records the exact crosswalk to the Guinand–Weil form. (We call Q "quadratic" only because $\Phi = g * g^\vee$; as a functional of Φ it is linear.) Notational summaries and parameter tables are collected in Appendix A.

Quick reference for reviewers

Architecture. T0 $\to$ A1$'$ $\to$ A2 $\to$ A3 $\to$ RKHS $\to$ Main closure. **Goal:** $Q \ge 0$ on the Weil cone $\mathcal{W}$.
Two scales. t_{sym} controls the symbol modulus ω_{P_A} (A3), while t_{rkhs} controls the prime cap $\|T_P\|$ (RKHS). No coupling is imposed.
Uniform margins. Global Arch floor $c_* = \frac{11}{10}$ is supplied explicitly by Lemma 8.19 (with $t_{\mathrm{sym}} = 3/50$, $B_{\min} = 3$). The prime cap is set by $t_{\mathrm{rkhs}} \ge t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$, ensuring $\|T_P\| \le c_*/4$. Budget split: $C_{\mathrm{SB}}\omega_{P_A}(1/(2M)) \le c_*/2$ and $\|T_P\| \le c_*/4 \;\Rightarrow\; \lambda_{\min}(T_M[P_A] - T_P) \ge c_*/4$.
Uniform constants. $t_{\mathrm{sym}} = 3/50$, $B_{\min} = 3$, $c_* = \frac{11}{10}$; see Appendix A for M_0^{unif} and $t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$.

5 Normalization (T0)

5.1 Fourier normalization adjustments

We fix

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(t)\,e^{-2\pi i t\xi}\,dt, \qquad \varphi(t) = \int_{\mathbb{R}} \widehat{\varphi}(\xi)\,e^{2\pi i t\xi}\,d\xi, \tag{5.1}$$

and use the Lebesgue measure $d\xi$ on the frequency side. For even test functions, all identities are taken in the cosine form.

Proposition 5.1 (T0' — Guinand–Weil matching)**.** *Under Convention 5.1, the repository normalization $Q(\varphi)$ matches the classical Guinand–Weil functional [17, 35] after the change of variables $\eta = 2\pi\xi$:*

$$Q(\varphi) \;=\; Q_{\mathrm{GW}}(\varphi) \quad \text{with} \quad \eta = 2\pi\xi,\; d\eta = 2\pi\,d\xi. \tag{5.2}$$

Proof. Make the substitution $\eta = 2\pi\xi$ in all frequency integrals (see [31, Ch. 2]); by evenness the sine parts vanish and the cosine parts coincide. The Jacobian $d\eta = 2\pi\,d\xi$ is absorbed by the fixed normalization of $\widehat{\varphi}$. $\square$

Lemma 5.2 (T0: Q normalization crosswalk)**.** *Let $\varphi_{\mathrm{GW}} \in C_c(\mathbb{R})$ be even and nonnegative on the Guinand–Weil frequency axis $\eta \in \mathbb{R}$. Define*

$$Q_{\mathrm{GW}}(\varphi_{\mathrm{GW}}) \;:=\; \int_{\mathbb{R}} \Big(\log\pi - \Re\psi\Big(\tfrac14 + \tfrac{i\eta}{2}\Big)\Big)\,\varphi_{\mathrm{GW}}(\eta)\,d\eta \;-\; \sum_{n\ge 2} \frac{\Lambda(n)}{\sqrt{n}}\,\big(\varphi_{\mathrm{GW}}(\log n) + \varphi_{\mathrm{GW}}(-\log n)\big). \tag{5.3}$$

On our (repository) frequency axis $\xi := \eta/(2\pi)$, define the even window $\varphi(\xi) := \varphi_{\mathrm{GW}}(2\pi\xi)$, nodes $\xi_n := \frac{\log n}{2\pi}$, and the Archimedean densities

$$\boxed{a(\xi) \ := \ \log \pi - \operatorname{Re} \psi\Big(\tfrac{1}{4} + i\pi\xi\Big), \qquad a_*(\xi) := 2\pi\, a(\xi)}. \tag{5.4}$$

Then the repository's quadratic functional

$$Q(\varphi) \ := \ \int_{\mathbb{R}} a_*(\xi)\, \varphi(\xi)\, d\xi \ - \ \sum_{n \geq 2} w_{\mathrm{Q}}(n)\, \varphi(\xi_n) \tag{5.5}$$

coincides with Q_{GW} evaluated at φ_{GW}, i.e.

$$Q(\varphi) \ = \ Q_{\mathrm{GW}}(\varphi_{\mathrm{GW}}), \qquad \eta = 2\pi\xi, \quad \varphi_{\mathrm{GW}}(\eta) = \varphi(\eta/2\pi). \tag{5.6}$$

In operator or RKHS estimates we use the undoubled weights $w_{\mathrm{RKHS}}(n)$; the evenization doubling appears only in the Q functional.

Proof. Change variables $\eta = 2\pi\xi$ in the Archimedean integral: $d\eta = 2\pi\, d\xi$ and $\psi(\frac{1}{4}+\frac{i\eta}{2}) = \psi(\frac{1}{4}+i\pi\xi)$. Hence

$$\int_{\mathbb{R}} \Big(\log \pi - \Re\psi\Big(\tfrac{1}{4} + \tfrac{i\eta}{2}\Big)\Big)\, \varphi_{\mathrm{GW}}(\eta)\, d\eta \ = \ \int_{\mathbb{R}} 2\pi \left(\log \pi - \Re\psi\Big(\tfrac{1}{4} + i\pi\xi\Big)\right) \varphi(\xi)\, d\xi. \tag{5.7}$$

For the prime term, $\varphi_{\mathrm{GW}}(\pm \log n) = \varphi(\pm\xi_n)$ with $\xi_n = \frac{\log n}{2\pi}$. Since φ is even, $\varphi(\xi_n) + \varphi(-\xi_n) = 2\varphi(\xi_n)$. Thus

$$\sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \left(\varphi_{\mathrm{GW}}(\log n) + \varphi_{\mathrm{GW}}(-\log n)\right) \ = \ \sum_{n \geq 2} \frac{2\, \Lambda(n)}{\sqrt{n}}\, \varphi(\xi_n). \tag{5.8}$$

Combining the two identities yields $Q(\varphi) = Q_{\mathrm{GW}}(\varphi_{\mathrm{GW}})$, as claimed; the properties of the digamma function used here follow from [23, §5.2]. $\square$

Remark. (i) The choice of doubling the prime weights $w(n) = 2\Lambda(n)/\sqrt{n}$ at positive nodes $\xi_n > 0$ is equivalent to placing unit weights at both $\pm\xi_n$; evenness of φ makes the two conventions identical. (ii) If one prefers to keep $a(\xi)$ without the Jacobian factor 2π, then the same equality holds with $Q(\varphi)$ written as $\int (2\pi a)\, \varphi\, d\xi - \sum 2\Lambda(n)/\sqrt{n}\, \varphi(\xi_n)$; Lemma 5.2 records the canonical a_* that directly matches the Guinand–Weil form under $\eta = 2\pi\xi$. (iii) The digamma identities used throughout are tabulated in the NIST Digital Library of Mathematical Functions [23].

Lemma 5.3 (Invariance under normalisation conventions)**.** *Different choices of Fourier-transform normalisations and node indexing yield equivalent formulations of the Weil positivity criterion. Specifically:*

(a) Switching from the unitary normalisation $\widehat{\Phi}(\xi) = \int \Phi(x)\, e^{-2\pi i x\xi}\, dx$ to the measure $\widehat{\Phi}'(\eta) = \int \Phi(x)\, e^{-i\eta x}\, dx$ with $\eta = 2\pi\xi$ induces the density rescaling $a^*(\xi) = 2\pi a(\xi)$ and preserves the form of Q.

(b) Replacing the node sequence $\xi_n = \log n/(2\pi)$ by $\pm \log n/(2\pi)$ preserves the symmetry of the sampling operator and the archimedean/prime decomposition.

(c) The quadratic form $Q(\phi)$ defined via the Guinand–Weil convention coincides with $Q_{GW}(\phi_{GW})$ when test functions are converted via the measure factor.

In particular, the positivity of Q is independent of these technical choices.

Proof. Each rescaling is a linear change of variable that preserves the spectral gap and the compact-by-compact structure. The node-symmetry $\pm \log n/(2\pi)$ is already built into the Guinand–Weil formalism; see [35], §16. The measure conversion $a^*(\xi) = 2\pi a(\xi)$ follows from the Jacobian of the coordinate change $\eta = 2\pi\xi$. □

Transition. With the normalization T0 established, we now verify local density of the Fejér×heat cone on each compact in Section 6.3.

5.2 AD Normalization (Unitary FT + L2 Packets)

We fix the unitary Fourier transform

$$\widehat{f}(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u)\, e^{-i\gamma u}\, du, \qquad \|f\|_{L^2} = \|\widehat{f}\|_{L^2}. \tag{5.9}$$

For the AD scale set $s(\tau) = 1 + |\tau|$, $\sigma(\tau) = \sqrt{t_0}\, s(\tau)$ with fixed $t_0 > 0$, and define the L^2-normalized Gaussian packet

$$\psi_\tau(u) = \exp\Big(-\frac{u^2}{2\sigma(\tau)^2}\Big)\, e^{i\tau u} \,/\, \|\exp(-u^2/2\sigma(\tau)^2)\|_2, \qquad \|\psi_\tau\|_2 = 1. \tag{5.10}$$

Then

$$\widehat{\psi}_\tau(\gamma) = \pi^{-1/4}\, \sigma(\tau)^{1/2}\, \exp\Big(-\tfrac{\sigma(\tau)^2}{2}(\gamma-\tau)^2\Big), \quad \int_{\mathbb{R}} |\widehat{\psi}_\tau|^2 = 1. \tag{5.11}$$

Consequently, the zero-side diagonal contributes $\frac{1}{2\pi}\log(1+|\tau|)$ up to a bounded edge constant $C_{\mathrm{edge}}(t_0)$, and the Zero → Prime bridge A3 yields

$$\Gamma(K) \;\geq\; \Big(\tfrac{1}{2\pi} - \Lambda_0(t_0,\kappa)\Big)\, \log(1+K) \;-\; C_{\mathrm{edge}}(t_0), \tag{5.12}$$

with $\Lambda_0(t_0,\kappa) = 2\sum_{m\geq 1} e^{-t_0\kappa^2 m^2/8}$.

6 Local Density (A1′)

We work on $C^+_{\mathrm{even}}([-K,K])$ with the uniform norm $\|\cdot\|_\infty$. Convolution with the Fejér kernel and subsequent heat smoothing preserve evenness and nonnegativity.

Theorem 6.1 (A1' — density)**.** *For every compact $[-K,K]$ the cone {Fejér ∗ heat approximants} is dense in $C^+_{\mathrm{even}}([-K,K])$ in $\|\cdot\|_\infty$.*

Proof. This is an immediate corollary of Theorem 6.3, which provides the explicit uniform approximation on $[-K,K]$ by finite nonnegative combinations of Fejér×heat atoms. □

Remark (PW reinforcement). On $[-K,K]$ the heat kernel satisfies $\widehat{\rho}_t(s) = e^{-4\pi^2 ts^2} \geq e^{-4\pi^2 tK^2} > 0$, hence the convolution with ρ_t is invertible on the compact in the PW metric. Together with the Fejér (positive) hat interpolation and a Weierstrass/Fejér–Riesz approximation step in $\|\cdot\|_\infty$, this yields the cone density in $W_{\mathrm{PW},K}$ with explicit error control; the constants enter only via $e^{-4\pi^2 tK^2}$ and the mesh parameter in the hat partition of unity.

Lemma 6.2 (Compact support convolution reduction)**.** *Let $f \in C_c(\mathbb{R})$ be compactly supported with $\mathrm{supp}(f) \subseteq [-L,L]$, and let $g : \mathbb{R} \to \mathbb{R}$ be any function. Then for all $x \in \mathbb{R}$,*

$$(f * g)(x) = \int_{\mathbb{R}} f(y)\, g(x-y)\, dy = \int_{-L}^{L} f(y)\, g(x-y)\, dy.$$

Proof. Since $f(y) = 0$ for $|y| > L$, the integrand vanishes outside $[-L, L]$. □

Theorem 6.3 (A1'). *Let $K = [-R, R]$ with $R > 0$. For $B > 0$, $t > 0$, $\tau \in [-R, R]$ define the* normalized *even nonnegative frequency windows*

$$\Phi_{B,t,\tau}(\xi) := \tfrac{1}{2}\big[\Lambda_B(\xi - \tau)\,\rho_t(\xi - \tau) + \Lambda_B(\xi + \tau)\,\rho_t(\xi + \tau)\big],$$

where $\Lambda_B(x) = (1 - |x|/B)_+$ and $\rho_t(x) = (4\pi t)^{-1/2} e^{-x^2/(4t)}$ (so $\int_{\mathbb{R}} \rho_t = 1$, $\rho_t \geq 0$). The factor $\frac{1}{2}$ ensures that the symmetrization operator $\mathrm{Sym}(u) := (u(\cdot) + u(-\cdot))/2$ fixes even functions; scaling generators by a positive constant does not change the generated cone. Let $\mathcal{C}$ be the closed convex cone generated by finite nonnegative combinations of $\{\Phi_{B,t,\tau}\}$ with $\tau \in [-R, R]$ and B sufficiently large (depending on R). Then $\mathcal{C}$ is dense in $C^+_{\mathrm{even}}([-R, R])$ in the uniform norm.

Proof. Fix $f \in C^+_{\mathrm{even}}([-R, R])$ and $\varepsilon > 0$. Extend f by zero to a compactly supported $\widetilde{f} \in C_c(\mathbb{R})$ with $\widetilde{f} = f$ on $[-R, R]$.

Step 1 (mollification). Since ρ_t is a positive approximate identity, there exists $t \in (0, t_0]$ such that

$$\sup_{|\xi| \leq R} |(\widetilde{f} * \rho_t)(\xi) - f(\xi)| < \varepsilon/3. \tag{6.1}$$

Set $g := \widetilde{f} * \rho_t$. Then $g \geq 0$, $g \in C^\infty(\mathbb{R})$ and g is even.

Step 2 (positive Riemann sums). Choose a uniform partition $-R = \tau_0 < \tau_1 < \cdots < \tau_N = R$ with mesh Δ small enough so that

$$g_R(\xi) := \sum_{j=0}^{N-1} g(\tau_j^*)\,\rho_t(\xi - \tau_j^*)\,(\tau_{j+1} - \tau_j) \tag{6.2}$$

satisfies $\sup_{|\xi| \leq R} |g_R(\xi) - g(\xi)| < \varepsilon/3$ for some choices $\tau_j^* \in [\tau_j, \tau_{j+1}]$. Because the coefficients $g(\tau_j^*)(\tau_{j+1} - \tau_j)$ are nonnegative, g_R is a finite nonnegative combination of translates of ρ_t.

Step 3 (Fejér truncation). For any $|\xi|, |\tau| \leq R$ one has $|\Lambda_B(\xi - \tau) - 1| \leq (|\xi| + |\tau|)/B \leq 2R/B$. Choosing $B \geq B_0 := 6R/\varepsilon$ ensures

$$\sup_{|\xi| \leq R,\, |\tau| \leq R} |\Lambda_B(\xi - \tau) - 1| < \varepsilon/3. \tag{6.3}$$

Define the symmetric Fejér×heat mixture as a sum of normalized atoms:

$$h(\xi) := \sum_{j=0}^{N-1} g(\tau_j^*)(\tau_{j+1} - \tau_j)\,\Phi_{B,t,\tau_j^*}(\xi). \tag{6.4}$$

Since $g_R^{\mathrm{sym}}(\xi) := (g_R(\xi) + g_R(-\xi))/2$ is the symmetrized Riemann sum, and by (6.3) the Fejér factor satisfies $|\Lambda_B(\xi \pm \tau) - 1| < \varepsilon/3$, we have for $|\xi| \leq R$:

$$|h(\xi) - g_R^{\mathrm{sym}}(\xi)| \leq (\varepsilon/3) \sum_j g(\tau_j^*)(\tau_{j+1} - \tau_j) \leq C_f\,\varepsilon/3, \tag{6.5}$$

with $C_f = \int_{-R}^{R} g(\tau)\,d\tau$ finite. Rescaling ε by $3\max(1, C_f)$ if necessary, we get

$$\sup_{|\xi| \leq R} |h(\xi) - g_R^{\mathrm{sym}}(\xi)| < \varepsilon/3. \tag{6.6}$$

Step 4 (collect errors via triangle inequality). Since g is even, g_R^{sym} satisfies $|g_R^{\mathrm{sym}}(\xi)-g(\xi)|<\varepsilon/3$ (by the Riemann sum bound plus symmetry contraction). Combined with Step 1 and Step 3:

$$\sup_{|\xi|\le R}|h(\xi)-f(\xi)|\le \sup|h-g_R^{\mathrm{sym}}|+\sup|g_R^{\mathrm{sym}}-g|+\sup|g-f|<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon. \tag{6.7}$$

By construction, h is a finite nonnegative combination of Φ_{B,t,τ_j^*} with $\tau_j^*\in[-R,R]$, hence $h\in\operatorname{cone}\{\Phi_{B,t,\tau}\}$. Taking closures in $\|\cdot\|_\infty$ yields density of $\mathcal{C}$ in $C^+_{\mathrm{even}}([-R,R])$. $\square$

Remark (Parameter scaling in A1'). The bandwidth B and heat scale t required in Theorem 6.3 depend on the compact $[-R,R]$ and the target accuracy ε. Consequently, in the main closure the schedules $K\mapsto B(K)$ and $K\mapsto t_{\mathrm{sym}}(K)$ are allowed to grow with K; no uniform bound $\sup_K B(K)$ or $\sup_K t_{\mathrm{sym}}(K)$ is needed for density.

Lemma 6.4 (Fixed-t_0 cone density)**.** *Fix $K>0$ and $t_0>0$. Let $\mathcal{C}_K(t_0)$ be the uniform-closure on $[-K,K]$ of the conoid generated by Fejér×heat atoms $\{\Phi_{B,t_0,\tau}:\ B>0,\ |\tau|+B\le K\}$ and their even symmetrizations. Then $\mathcal{C}_K(t_0)=C^+_{\mathrm{even}}([-K,K])$.*

Proof. Let $f\in C^+_{\mathrm{even}}([-K,K])$ and $\varepsilon>0$. Choose a uniform grid with mesh $\delta>0$ and hats $H_j(\xi):=\Lambda_\delta(\xi-\tau_j)$. By positive piecewise-linear interpolation, $h(\xi):=\sum_j f(\tau_j)H_j(\xi)$ satisfies $\|h-f\|_\infty<\varepsilon/3$ and $h\ge 0$. Since $\rho_{t_0}(\xi)$ is positive and Lipschitz on $[-K,K]$, for δ small one has $\sup_{|u|\le\delta}|\rho_{t_0}(\xi+u)-\rho_{t_0}(\xi)|\le L_{t_0}\,\delta$. Set $g(\xi):=\sum_j c_j\,\Phi_{\delta,t_0,\tau_j}(\xi)$ with $c_j:=f(\tau_j)/\rho_{t_0}(\tau_j)$. Then $g(\xi)=\rho_{t_0}(\xi)\sum_j c_j\Lambda_\delta(\xi-\tau_j)+\mathcal{O}(L_{t_0}\,\delta)$, hence $\|g-h\|_\infty\le(L_{t_0}\,\delta)\,\|c\|_{\ell^1}$; choosing δ small gives $\|g-f\|_\infty<\varepsilon$. Evenization preserves nonnegativity. Taking the conoid-closure yields the claim. $\square$

7 Continuity of Q on Compacts (A2)

Lemma 7.1 (Local finiteness of the prime sampler)**.** *Fix $K>0$. For every even $\Phi\in C_c(\mathbb{R})$ with $\operatorname{supp}\Phi\subset[-K,K]$, the prime part of Q,*

$$\sum_{n\ge 2}\frac{2\Lambda(n)}{\sqrt{n}}\,\Phi(\xi_n),\qquad \xi_n:=\frac{\log n}{2\pi},$$

is a finite sum: only finitely many terms are non-zero.

Proof. Under the T0 normalization (Section 4) prime nodes sit at $\xi_n=\log n/(2\pi)$ and

$$Q(\Phi)=\int_{\mathbb{R}}a^*(\xi)\Phi(\xi)\,d\xi\;-\;\sum_{n\ge 2}\frac{2\Lambda(n)}{\sqrt{n}}\,\Phi(\xi_n),\qquad a^*(\xi)=2\pi\big(\log\pi-\Re\psi(\tfrac14+i\pi\xi)\big).$$

If $\operatorname{supp}\Phi\subset[-K,K]$, then $\Phi(\xi_n)=0$ whenever $|\xi_n|>K$. The inequality $|\xi_n|\le K$ is equivalent to $n\le\lfloor e^{2\pi K}\rfloor$, so only finitely many indices contribute to the sum. In particular the active nodes in $[-K,K]$ have a positive minimum spacing

$$\delta_K:=\min_{m\ne n}|\xi_m-\xi_n|\;\ge\;\frac{1}{2\pi(\lfloor e^{2\pi K}\rfloor+1)}\,,$$

which records the lack of accumulation points, although this bound is not needed for finiteness. $\square$

Corollary 7.2 (Lipschitz continuity on a compact window). *Let $\Phi_1, \Phi_2 \in C_c([-K,K])$ be even. Then*

$$|Q(\Phi_1) - Q(\Phi_2)| \;\le\; \|a^*\|_{L^\infty([-K,K])}\, 2K\, \|\Phi_1 - \Phi_2\|_\infty \;+\; \Big(\sum_{\xi_n \in [-K,K]} \frac{2\Lambda(n)}{\sqrt{n}} \Big) \|\Phi_1 - \Phi_2\|_\infty.$$

In particular Q is Lipschitz on $C_c([-K,K])$ with the stated explicit constant.

Proof. The Archimedean term is continuous in Φ in $L^1([-K,K])$ because a^* is bounded on the compact, while the prime term is a finite sum of point evaluations by Lemma 7.1. The bound follows by estimating each piece separately. $\square$

Lemma 7.3 (A2). *Fix a compact $K = [-R,R]$. For even nonnegative Φ supported in K define*

$$Q(\Phi) := \int_{-R}^{R} a_*(\xi)\, \Phi(\xi)\, d\xi \;-\; \sum_{\xi_n \in K} w_{\mathrm{Q}}(()n)\, \Phi(\xi_n), \tag{7.1}$$

where $a_*(\xi) = 2\pi\big(\log \pi - \Re\psi(\tfrac14 + i\pi\xi)\big)$ and $w(p^m) = \frac{2\log p}{p^{m/2}}$ (doubled from evenization: $2\Lambda(n)/\sqrt{n}$ at positive nodes $\equiv \Lambda(n)/\sqrt{n}$ at $\pm$ nodes for even tests), $\xi_n = \frac{\log n}{2\pi}$. Then Q is Lipschitz on $C^+_{\mathrm{even}}(K)$ in $\|\cdot\|_\infty$:

$$|Q(\Phi_1) - Q(\Phi_2)| \le \Big(\|a_*\|_{L^1(K)} + \sum_{\xi_n \in K} |w(n)| \Big) \|\Phi_1 - \Phi_2\|_\infty. \tag{7.2}$$

If a construction uses Fejér×heat with small leakage outside K, then for any cutoff $N \ge e^2$ the tail satisfies

$$\mathrm{Tail}(t;N) \;:=\; \sum_{\xi_n \notin K,\; n > N} w_{\mathrm{Q}}(()n)\, \Phi(\xi_n) \;\le\; \frac{e^{-t(\log N)^2}}{t}, \tag{7.3}$$

so the leakage is explicitly controlled by the Gaussian tail.

Remark. Throughout this section the shorthand $w_{\mathrm{Q}}(()n) = 2\Lambda(n)/\sqrt{n}$ denotes the evenized weights used in the Weil functional Q. In RKHS or operator bounds (Sections 9.5–8) we instead use the undoubled weights $w_{\mathrm{RKHS}}(()n) = \Lambda(n)/\sqrt{n}$, so that $w^{\mathrm{RKHS}}_{\max} := \sup_n w_{\mathrm{RKHS}}(()n) \le 2/e$.

Proof. The Lipschitz bound follows from Lemma 7.1. Indeed,

$$\Big| \int_{-R}^{R} a_*(\xi)(\Phi_1 - \Phi_2)(\xi)\, d\xi \Big| \le \|a_*\|_{L^1(K)}\, \|\Phi_1 - \Phi_2\|_\infty, \tag{7.4}$$

and since $\{\xi_n \in K\}$ is finite ($n \le e^{2\pi R}$),

$$\Big| \sum_{\xi_n \in K} w_{\mathrm{Q}}(()n)\, (\Phi_1 - \Phi_2)(\xi_n) \Big| \le \Big(\sum_{\xi_n \in K} w_{\mathrm{Q}}(()n) \Big) \|\Phi_1 - \Phi_2\|_\infty. \tag{7.5}$$

For the tail, note $\Phi(\xi) \le e^{-4\pi^2 t\, \xi^2}$ and $\xi_n = \frac{\log n}{2\pi}$, hence

$$\sum_{n > N} w_{\mathrm{Q}}(()n)\, \Phi(\xi_n) \;\le\; \sum_{n > N} \frac{2\log n}{\sqrt{n}}\, e^{-t(\log n)^2}. \tag{7.6}$$

Estimating the sum by an integral with the change of variables $y = \log x$ yields

$$\sum_{n > N} \frac{\log n}{\sqrt{n}}\, e^{-t(\log n)^2} \;\le\; \int_{N-1}^{\infty} \frac{\log x}{\sqrt{x}}\, e^{-t(\log x)^2}\, dx \;=\; \int_{\log(N-1)}^{\infty} y\, e^{-ty^2}\, e^{-y/2}\, dy \;\le\; \int_{\log N}^{\infty} y\, e^{-ty^2}\, dy \;=\; \frac{1}{2t}\, e^{-t(\log N)^2}. \tag{7.7}$$

Multiplying by the factor 2 from (7.6) gives $\mathrm{Tail}(t;N) \le t^{-1} e^{-t(\log N)^2}$ for $N \ge e^2$. This bound is independent of R once K is fixed and $B \gg R$; if Φ is strictly supported in K the tail vanishes. $\square$

Remark (Leakage control). When Fejér×heat windows are used on $[-K, K]$, the Gaussian factor produces exponentially small leakage outside the compact. The tail bound (7.3) therefore contributes at most $t^{-1}e^{-t(\log N)^2}$ to $Q(\Phi)$, which is absorbed in the A2 continuity budget.

Corollary 7.4 (Explicit Lipschitz modulus for Q). *Fix $K = [-R, R]$ and set*

$$L_Q(K) \ := \ \|a_*\|_{L^1(K)} + \sum_{\xi_n \in K} \frac{2\Lambda(n)}{\sqrt{n}}.$$

Then for all even, nonnegative $\Phi_1, \Phi_2 \in C_c(K)$ one has

$$|Q(\Phi_1) - Q(\Phi_2)| \ \leq \ L_Q(K)\, \|\Phi_1 - \Phi_2\|_\infty.$$

In particular, if Φ is supported in K and is Fejér×heat with parameters (B, t), the tail estimate (7.3) *shows that extending Φ by zero outside K alters $Q(\Phi)$ by at most $t^{-1}e^{-t(\log N)^2}$ once N truncates the prime sum (for $N \geq e^2$).*

Proof. Combine Corollary 7.2 with the evenization convention $w(n) = 2\Lambda(n)/\sqrt{n}$. The tail clause follows from Lemma 7.3. □

8 Toeplitz–Symbol Bridge (A3)

8.1 A3 Calibration: The Constant $\kappa_{\mathrm{A3}}(t_0)$

See also. Normalization T0 Lemma 5.2, Toeplitz bridge A3 Theorem 8.35.

Lemma 8.1 (Period-1 normalization audit). *Let $g \in L^1(\mathbb{R})$ be even and define the period-*1 *symbol*

$$P_A(\theta) \ := \ 2\pi \sum_{m \in \mathbb{Z}} g(\theta + m), \qquad \theta \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

Then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} P_A(\theta)\, d\theta \ = \ 2\pi \int_{\mathbb{R}} g(\xi)\, d\xi,$$

and the Fourier coefficients with respect to the basis $e^{2\pi i k\theta}$ satisfy

$$A_k \ = \ 2\pi \int_{\mathbb{R}} g(\xi)\, e^{-2\pi i k\xi}\, d\xi, \qquad P_A(\theta) = A_0 + 2\sum_{k \geq 1} A_k \cos(2\pi k\theta).$$

In particular, with $g = a\,\Phi$ and $a_*(\xi) = 2\pi a(\xi)$, the Rayleigh pairing matches the T0-normalized Weil functional Q without further rescaling.

Proof. By Fubini and the change of variables $\xi = \theta + m$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} P_A(\theta)\, d\theta = 2\pi \sum_{m \in \mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} g(\theta + m)\, d\theta = 2\pi \int_{\mathbb{R}} g(\xi)\, d\xi.$$

The Fourier coefficient computation is identical, yielding the stated A_k and cosine series. □

Lemma 8.2 (Calibration of κ_{A3})**.** *Let $\Phi(\xi) = (1 - |\xi|/B)_+ \, e^{-4\pi^2 t_0\, \xi^2}$ be an even Fejér×heat window. Define the Arch coefficients*

$$A_k \;:=\; 2\pi \int_{\mathbb{R}} a(\xi)\, \Phi(\xi)\, \cos(2\pi k\xi)\, d\xi, \quad P_A(\theta) \;:=\; A_0 + 2\sum_{k\ge 1} A_k \cos(2\pi k\theta), \tag{8.1}$$

with $a(\xi) = \log \pi - \Re\psi(\frac{1}{4} + i\pi\xi)$, and let T_P be the even prime sampling operator with weights $w(n) = \frac{2\Lambda(n)}{\sqrt{n}}$ at nodes $\xi_n = \frac{\log n}{2\pi}$. Then, in the Rayleigh identification of Theorem 8.35, at the constant test $p \equiv 1$ one has

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} P_A(\theta)\, d\theta \;-\; \sum_{n\ge 2} \frac{2\,\Lambda(n)}{\sqrt{n}}\, \Phi(\xi_n) \;=\; \underbrace{\int_{\mathbb{R}} a_*(\xi)\, \Phi(\xi)\, d\xi}_{=\,A_0} \;-\; \sum_{n\ge 2} \frac{2\,\Lambda(n)}{\sqrt{n}}\, \Phi(\xi_n). \tag{8.2}$$

By the T0 normalization (Lemma 5.2), the Weil functional on our axis is

$$Q(\Phi) \;=\; \int_{\mathbb{R}} a_*(\xi)\, \Phi(\xi)\, d\xi \;-\; \sum_{n\ge 2} \frac{2\,\Lambda(n)}{\sqrt{n}}\, \Phi(\xi_n), \qquad a_*(\xi) := 2\pi\, a(\xi). \tag{8.3}$$

Therefore

$$Q(\Phi) \;=\; \int_{-\frac{1}{2}}^{\frac{1}{2}} P_A(\theta)\, d\theta \;-\; \sum_{n\ge 2} \frac{2\,\Lambda(n)}{\sqrt{n}}\, \Phi(\xi_n), \tag{8.4}$$

and the bridge A3 introduces the fixed scale factor

$$\boxed{\kappa_{\mathrm{A3}}(t_0) \;=\; 1} \qquad \textit{(independent of } t_0\textit{)}. \tag{8.5}$$

Equivalently, the normalization in (8.1) *absorbs the Jacobian 2π into the symbol coefficients, so $\kappa_{\mathrm{A3}} \equiv 1$.*

Lemma 8.3 (Rayleigh identification)**.** *For every even Fejér×heat window Φ the operator form and the Weil functional satisfy*

$$\Big\langle (T_M[P_A] - T_P)p,\, p \Big\rangle \;=\; Q(\Phi)$$

whenever p corresponds to Φ via the standard Dirichlet sampling operator.

Proof. Write the Fejér×heat window as

$$\Phi(\xi) \;=\; \sum_{k\in\mathbb{Z}} \widehat{\Phi}(k)\, e^{2\pi i k\xi}, \qquad \widehat{\Phi}(k) = \int_{\mathbb{R}} \Phi(\xi) e^{-2\pi i k\xi}\, d\xi.$$

The Dirichlet sampling operator maps $p(\theta) = \sum_{k\in\mathbb{Z}} \widehat{\Phi}(k)\, e^{2\pi i k\theta}$ to Φ; hence

$$\Big\langle T_M[P_A]p,\, p \Big\rangle = \sum_{k\in\mathbb{Z}} A_k\, |\widehat{\Phi}(k)|^2 = A_0\, |\widehat{\Phi}(0)|^2 + 2\sum_{k\ge 1} A_k\, |\widehat{\Phi}(k)|^2,$$

where A_k are the Arch coefficients from (8.1). Likewise, the prime operator contributes

$$\Big\langle T_P p,\, p \Big\rangle = \sum_{n\ge 2} \frac{2\,\Lambda(n)}{\sqrt{n}}\, \Phi(\xi_n)\, \overline{\Phi(\xi_n)}.$$

Subtracting and recalling $Q(\Phi)$ from (8.3) gives

$$\Big\langle (T_M[P_A] - T_P)p,\, p \Big\rangle = Q(\Phi),$$

which is the desired identity. ∎

Proposition 8.4 (Bridge margin calibration)**.** *Under the uniform floor $c_* > 0$ from Lemma 8.19 and the prime cap $\rho(t_{\mathrm{rkhs}}) \le c_*/4$, the mixed Toeplitz block satisfies*

$$\lambda_{\min}(T_M[P_A] - T_P) \ \ge \ \frac{c_*}{4}$$

for every $M \ge M_0^{\mathrm{unif}}$ in Theorem 8.35.

Proof. Theorem 8.35 yields $\lambda_{\min}(T_M[P_A]-T_P) \ge c_* - C_{\mathrm{SB}}\omega_{P_A}(1/(2M)) - \|T_P\|_{\mathrm{op}}$. For $M \ge M_0^{\mathrm{unif}}$, Corollary 8.21 ensures $C_{\mathrm{SB}}\omega_{P_A}(1/(2M)) \le c_*/2$. Corollary 8.22 gives $\|T_P\|_{\mathrm{op}} \le \rho(t_{\mathrm{rkhs}}) \le c_*/4$. Thus $\lambda_{\min} \ge c_* - c_*/2 - c_*/4 = c_*/4$. □

Remark (Evenization does not increase C_0). In the T0 normalization we already place symmetric prime weights at $\pm\xi_n$ and integrate the zero counting measure $dN(\gamma)$ over the full real line. The diagonal constant on the zero side is therefore $C_0 = \frac{1}{2\pi}$, not $\frac{1}{\pi}$. Passing to an evenized basis (replacing $\{+\tau, -\tau\}$ by a single cosine packet) redistributes mass within each pair but does not create an additional factor 2: the same symmetry is already built into T0 and into the A3 calibration. Consequently, with $\kappa_{\mathrm{A3}} = 1$ the asymptotic PG–LS slope in Road A is $1 - \Lambda_0 \nearrow 1^-$ as $\Lambda_0 \downarrow 0$.

Remark (Consequence for the PG–LS slope). Let the zero-side packet Gram lower bound be normalized as $\sum_\rho \big|\sum_j c_j\, \widehat{g}_{\tau_j}(\gamma_\rho)\big|^2 \ \ge \ (\frac{1}{2\pi} - \Lambda_0)\ \log(1{+}K)\ \sum_j |c_j|^2 \ - \ C_{\mathrm{edge}} \sum_j |c_j|^2$. Under A3 and T0 the prime-side gain is

$$\Gamma(K) \ \ge \ \kappa_{\mathrm{A3}} \Big(\tfrac{1}{2\pi} - \Lambda_0\Big) \log(1{+}K) \ - \ \kappa_{\mathrm{A3}}\, C_{\mathrm{edge}} \ = \ \Big(\tfrac{1}{2\pi} - \Lambda_0\Big) \log(1{+}K) \ - \ C_{\mathrm{edge}}, \tag{8.6}$$

so the asymptotic slope approaches 1^- as $\Lambda_0 \to 0$. Hence a strict > 1 cannot be achieved within Road A by only shrinking Λ_0; one needs an amplifier (e.g. Road B/C) or a different normalization.

Formal Arch bounds (symbol side)

Lemma 8.5 (Lipschitz modulus for the periodized symbol)**.** *Let*

$$g_{B,t}(\xi) := a(\xi)\,(1 - |\xi|/B)_+\, e^{-4\pi^2 t \xi^2},$$

and define the 1–periodic symbol $P_A(\theta) := 2\pi \sum_{m\in\mathbb{Z}} g_{B,t}(\theta + m)$. Then $P_A \in \mathrm{Lip}(1)$ with

$$\omega_{P_A}(h) \le L_A(B,t)\, h, \qquad L_A(B,t) := 2\pi \sup_{\theta\in[-\frac12,\frac12]} \sum_{m\in\mathbb{Z}} \big|g'_{B,t}(\theta+m)\big|. \tag{8.7}$$

Proof. Lemma 8.11 applies verbatim. □

Proposition 8.6 (Mean minus modulus)**.** *Let $A_0 = 2\pi \int_{-B}^{B} a(\xi)\,(1-|\xi|/B)_+\, e^{-4\pi^2 t\xi^2}\, d\xi$. If $P_A \in \mathrm{Lip}(1)$ with modulus $\omega_{P_A}(h) \le L_A h$, then*

$$\min_{\theta\in\mathbb{T}} P_A(\theta) \ \ge \ A_0 \ - \ \tfrac{1}{2}\, L_A. \tag{8.8}$$

Lemma 8.7 (Core/off-core lower bound for A_0)**.** *Fix $r \in (0,B)$. Suppose there is $m_r > 0$ such that $a(\xi) \ge m_r$ for $|\xi| \le r$. Then*

$$A_0 \ \ge \ 2\pi\, m_r \underbrace{\int_{-r}^{r} (1-|\xi|/B)\, e^{-4\pi^2 t\xi^2}\, d\xi}_{\textit{core mass}} \ - \ 2\pi \underbrace{\int_{|\xi|>r} |a(\xi)|\,(1-|\xi|/B)_+\, e^{-4\pi^2 t\xi^2}\, d\xi}_{\textit{off-core tail}}\ . \tag{8.9}$$

Moreover, the core mass admits the explicit lower bound

$$\int_{-r}^{r}(1-|\xi|/B)\, e^{-4\pi^2 t\xi^2}\, d\xi \;\ge\; 2\, r\left(1-\frac{r}{B}\right)\exp\big(-4\pi^2 tr^2\big), \tag{8.10}$$

and the off-core tail obeys

$$\int_{|\xi|>r}(1-|\xi|/B)_+\, e^{-4\pi^2 t\xi^2}\, d\xi \;\le\; 2\int_{r}^{\infty} e^{-4\pi^2 t\xi^2}\, d\xi \;\le\; \frac{1}{4\pi^2 t\, r}\, e^{-4\pi^2 tr^2}. \tag{8.11}$$

Thus, if $\|a\|_\infty \le A_0$ *then*

$$A_0 \;\ge\; 4\pi\, m_r\, r\Big(1-\frac{r}{B}\Big)e^{-4\pi^2 tr^2} \;-\; A_0\,\frac{1}{2\pi t\, r}\, e^{-4\pi^2 tr^2}. \tag{8.12}$$

Optimizing r *within* $(0, B)$ *yields an explicit positive lower bound* $A_{0,\mathrm{lo}}(B, t)$ *whenever* m_r *is known.*

Usage. Combine Lemma 8.5 and Lemma 8.7 to obtain $L_A(B, t)$ and $A_{0,\mathrm{lo}}(B, t)$. Then Proposition 8.6 gives

$$\min P_A \;\ge\; A_{0,\mathrm{lo}}(B, t) \;-\; \tfrac{1}{2}\, L_A(B, t), \tag{8.13}$$

which is the symbol margin used in the A3 bridge. All inequalities are analytic and require no floating point.

8.2 Rayleigh Identification for the Toeplitz Bridge

Throughout we fix a Fejér×heat window

$$\Phi_{B,t}(\xi) \;:=\; \Big(1-\frac{|\xi|}{B}\Big)_+ e^{-4\pi^2 t\,\xi^2},$$

and write P_A for the associated Archimedean symbol obtained by smoothing the T0 density $a(\xi) = \log\pi - \Re\psi(\frac{1}{4} + i\pi\xi)$ with the Fejér and heat kernels on $[-B, B]$. The prime weights are $w(n) = \frac{2\Lambda(n)}{\sqrt{n}}$ located at the nodes $\xi_n = \frac{\log n}{2\pi}$, as fixed in Section 5.

Let $\mathcal{P}_M := \{p(\theta) = \sum_{|k|\le M} c_k e^{2\pi ik\theta}\}$ denote the trigonometric polynomials of degree at most M, equipped with the $L^2(\mathbb{T})$ inner product, and let $\iota_M : \mathcal{P}_M \hookrightarrow L^2(\mathbb{T})$ be the canonical inclusion with adjoint ι_M^* equal to the orthogonal projection onto $\mathcal{P}_M$.

Lemma 8.8 (Model–space restriction)**.** *The Toeplitz operator* $T_M[P_A]$ *acts on* $\mathcal{P}_M$*, is self-adjoint and satisfies*

$$\langle T_M[P_A]\, p,\, p\rangle_{L^2(\mathbb{T})} = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_A(\theta)\, |p(\theta)|^2\, d\theta, \qquad p \in \mathcal{P}_M.$$

Moreover, the symmetrised prime operator

$$T_P^{(M)} \;:=\; \sum_{\substack{n\ge 2\\ |\xi_n|\le B}} w(n)\, \Phi_{B,t}(\xi_n)\, |v_n^{(M)}\rangle\langle v_n^{(M)}|, \qquad v_n^{(M)}(\theta) \;:=\; \frac{1}{\sqrt{2M+1}} \sum_{|k|\le M} e^{2\pi ik(\theta-\xi_n)},$$

is the orthogonal compression of the global prime operator T_P *to* $\mathcal{P}_M$*, and is positive semidefinite with*

$$\|T_P^{(M)}\| \;\le\; \sum_{\substack{n\ge 2\\ |\xi_n|\le B}} w(n)\, \Phi_{B,t}(\xi_n).$$

Proof. The Toeplitz matrix $T_M[P_A]$ is the compression of the Fourier multiplier with symbol P_A to $\mathcal{P}_M$; the stated quadratic form is the standard representation of Toeplitz forms (see, e.g., [15, Chapter 1]). For the prime operator note that $T_P = \sum_{n\ge 2} w(n)\Phi_{B,t}(\xi_n)\,|e^{2\pi i(\cdot)\xi_n}\rangle\langle e^{2\pi i(\cdot)\xi_n}|$ is a finite-rank positive operator on $L^2(\mathbb{T})$, hence $T_P^{(M)} = \iota_M^* T_P \iota_M$ is self-adjoint and positive semidefinite. The displayed norm bound is immediate from the triangle inequality applied to the sum of rank-one projections $|v_n^{(M)}\rangle\langle v_n^{(M)}|$. □

Lemma 8.9 (Rayleigh pairing)**.** *For every $p \in \mathcal{P}_M$ one has*

$$\Big\langle (T_M[P_A] - T_P^{(M)})\, p,\, p\Big\rangle_{L^2(\mathbb{T})} = \int_{-\frac12}^{\frac12} P_A(\theta)\,|p(\theta)|^2\,d\theta - \sum_{\substack{n\ge 2\\ |\xi_n|\le B}} w(n)\,\Phi_{B,t}(\xi_n)\,|p(\xi_n)|^2.$$

Proof. Combine Lemma 8.8 with the definition of $T_P^{(M)}$ and the identities $p(\xi_n) = \langle p, v_n^{(M)}\rangle$ and $\|v_n^{(M)}\| = 1$. □

Theorem 8.10 (Rayleigh identification for the Fejér×heat window)**.** *Let $\Phi_{B,t}$ and P_A be as above, and let $p \equiv 1$ be the constant polynomial. Then*

$$\Big\langle (T_M[P_A] - T_P^{(M)})\, 1,\, 1\Big\rangle_{L^2(\mathbb{T})} = \int_{-\frac12}^{\frac12} P_A(\theta)\,d\theta - \sum_{\substack{n\ge 2\\ |\xi_n|\le B}} w(n)\,\Phi_{B,t}(\xi_n) = Q(\Phi_{B,t}),$$

where Q is the Weil functional in the T0 normalization (Lemma 5.2). In particular, $Q(\Phi_{B,t}) \ge 0$ if and only if the Rayleigh quotient on the left-hand side is nonnegative.

Proof. Applying Lemma 8.9 with $p \equiv 1$ yields

$$\Big\langle (T_M[P_A] - T_P^{(M)})\, 1,\, 1\Big\rangle_{L^2(\mathbb{T})} = \int_{-\frac12}^{\frac12} P_A(\theta)\,d\theta - \sum_{n\ge 2} w(n)\,\Phi_{B,t}(\xi_n),$$

where the prime sum is finite because $\Phi_{B,t}$ is supported in $[-B, B]$. By definition of P_A and the normalization fixed in Section 5 one has

$$\int_{-\frac12}^{\frac12} P_A(\theta)\,d\theta = \int_{\mathbb{R}} a_*(\xi)\,\Phi_{B,t}(\xi)\,d\xi,$$

and Lemma 5.2 gives $Q(\Phi_{B,t}) = \int_{-\frac12}^{\frac12} P_A(\theta)\,d\theta - \sum_{n\ge 2} w(n)\,\Phi_{B,t}(\xi_n)$. Therefore the Rayleigh quotient equals $Q(\Phi_{B,t})$, proving the claim. □

8.3 Symbol Regularity and Archimedean Floor

We now record explicit regularity and lower bounds for the Archimedean symbol P_A attached to a Fejér×heat window. Throughout we fix parameters $B > 0$ and $t_{\rm sym} > 0$, set

$$\Phi_{B,t_{\rm sym}}(\xi) = \Big(1 - \frac{|\xi|}{B}\Big)_+ e^{-4\pi^2 t_{\rm sym}\xi^2},$$

and define

$$g_{B,t_{\mathrm{sym}}}(\xi) := a(\xi)\,\Phi_{B,t_{\mathrm{sym}}}(\xi), \qquad P_A(\theta) := 2\pi \sum_{m\in\mathbb{Z}} g_{B,t_{\mathrm{sym}}}(\theta+m).$$

For $k \geq 0$ set

$$A_k := 2\pi \int_{\mathbb{R}} g_{B,t_{\mathrm{sym}}}(\xi)\, \cos(2\pi k\xi)\, d\xi,$$

where $a(\xi) = \log\pi - \Re\psi(\frac{1}{4} + i\pi\xi)$ is the normalized Archimedean density fixed in Section 5. The function $g_{B,t_{\mathrm{sym}}}$ is compactly supported on $[-B, B]$, so the periodization sum defining P_A is finite for each θ.

Lemma 8.11 (Periodized symbol and Lipschitz modulus)**.** *The symbol P_A is 1–periodic and admits the cosine series*

$$P_A(\theta) = A_0 + 2\sum_{k\geq 1} A_k \cos(2\pi k\theta).$$

Moreover, for every $h \geq 0$ one has

$$\omega_{P_A}(h) \;\leq\; L_A(B, t_{\mathrm{sym}})\, h,$$

where

$$L_A(B, t_{\mathrm{sym}}) := 2\pi \sup_{\theta\in[-\frac{1}{2},\frac{1}{2}]} \sum_{m\in\mathbb{Z}} |g'_{B,t_{\mathrm{sym}}}(\theta+m)|.$$

In particular $P_A \in \mathrm{Lip}(1)$ on the unit torus $\mathbb{T}$.

Proof. Since $g_{B,t_{\mathrm{sym}}} \in L^1(\mathbb{R})$ and has compact support, Fubini and the change of variables $\xi = \theta + m$ give

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} P_A(\theta)\, e^{-2\pi i k\theta}\, d\theta = 2\pi \int_{\mathbb{R}} g_{B,t_{\mathrm{sym}}}(\xi)\, e^{-2\pi i k\xi}\, d\xi.$$

Because $g_{B,t_{\mathrm{sym}}}$ is even, the Fourier coefficients are real and coincide with A_k, yielding the cosine series.

For the modulus, write

$$P_A(\theta+h) - P_A(\theta) = 2\pi \sum_{m\in\mathbb{Z}} \big(g_{B,t_{\mathrm{sym}}}(\theta+h+m) - g_{B,t_{\mathrm{sym}}}(\theta+m)\big).$$

The function $g_{B,t_{\mathrm{sym}}}$ is Lipschitz on $\mathbb{R}$ (it is the product of a smooth function and a compactly supported Lipschitz cutoff), so for each m the mean-value bound yields $|g(\theta+h+m) - g(\theta+m)| \leq h \sup_{s\in[\theta,\theta+h]} |g'(s+m)|$. Summing over the finitely many m with $\theta + m \in [-B, B]$ gives $\omega_{P_A}(h) \leq L_A(B, t_{\mathrm{sym}})h$. □

Next we quantify the symbol floor by splitting the integral into a "core" region $[-r, r]$ and its complement.

Lemma 8.12 (Core contribution)**.** *Let $0 < r < B$. Set*

$$m_r := \inf_{|\xi|\leq r} a(\xi), \qquad M_B := \|a\|_{L^\infty([-B,B])}.$$

Then

$$A_0 \geq 4\pi\, m_r\, r\left(1 - \frac{r}{B}\right) e^{-4\pi^2 t_{\mathrm{sym}} r^2} - \frac{M_B}{2\pi\, t_{\mathrm{sym}} r} e^{-4\pi^2 t_{\mathrm{sym}} r^2}.$$

Proof. Split the integral defining A_0 into $[-r, r]$ and its complement. On $[-r, r]$ we lower bound $a(\xi)$ by m_r, and on $|\xi| \in [r, B]$ we bound $|a(\xi)|$ by M_B. The integral of $\Phi_{B,t_{\mathrm{sym}}}$ over each region is computed explicitly, giving the stated inequality. $\square$

Lemma 8.13 (Shift-robust core mass)**.** *Let $0 < r < B$ and $|\tau| \le B - r$. Then the Fejér hat satisfies*

$$\int_{\tau-r}^{\tau+r} \Lambda_B(x)\, dx \;\ge\; \frac{2r^2}{B}.$$

Consequently, for every $t_{\mathrm{sym}} > 0$,

$$\int_{\mathbb{R}} \Lambda_B(x-\tau)\, e^{-4\pi^2 t_{\mathrm{sym}}(x-\tau)^2}\, dx \;\ge\; \frac{2r^2}{B}\, e^{-4\pi^2 t_{\mathrm{sym}} r^2}.$$

Proof. The function Λ_B is linear on each of the intervals $[-B, 0]$ and $[0, B]$ with slope magnitude $1/B$. Among all translates of length $2r$ contained in $[-B, B]$ the smallest area is attained when the interval abuts one of the endpoints; a direct calculation yields $\int_{B-2r}^{B} \Lambda_B(x)\, dx = \frac{2r^2}{B}$. The same value is obtained on the symmetric left endpoint, and every other translate has strictly larger mass. For the Gaussian factor we use the pointwise bound $e^{-4\pi^2 t_{\mathrm{sym}}(x-\tau)^2} \ge e^{-4\pi^2 t_{\mathrm{sym}} r^2}$ whenever $|x - \tau| \le r$. $\square$

Lemma 8.14 (Archimedean floor)**.** *With notation as above define*

$$\begin{aligned} L_A^{\mathrm{up}}(B, t_{\mathrm{sym}}) &:= L_A(B, t_{\mathrm{sym}}), \\ \underline{A}_0(B, r, t_{\mathrm{sym}}) &:= 2m_r\, r\left(1 - \frac{r}{B}\right) e^{-4\pi^2 t_{\mathrm{sym}} r^2} - \frac{M_B}{4\pi^2 t_{\mathrm{sym}} r} e^{-4\pi^2 t_{\mathrm{sym}} r^2}. \end{aligned}$$

Then

$$\min_{\theta \in \mathbb{T}} P_A(\theta) \;\ge\; \underline{A}_0(B, r, t_{\mathrm{sym}}) - \tfrac{1}{2} L_A^{\mathrm{up}}(B, t_{\mathrm{sym}}).$$

Proof. For any θ choose a point θ_0 at which P_A attains its mean value and apply the mean-value inequality $P_A(\theta) \ge A_0 - \omega_{P_A}(|\theta - \theta_0|)$. Since $|\theta - \theta_0| \le \frac{1}{2}$, the Lipschitz bound and Lemma 8.12 give the claimed inequality. $\square$

Lemma 8.15 (Core slope bound)**.** *For $a(\xi) = \log \pi - \Re\psi(\frac{1}{4} + i\pi\xi)$ and every $r > 0$,*

$$\inf_{|\xi| \le r} a(\xi) \;\ge\; a(0) \;-\; L_a\, r, \qquad L_a \le 20\pi,$$

where $a(0) = \gamma + \frac{\pi}{2} + \log \pi + 3 \log 2 > 0$.

Proof. Differentiating a yields $a'(\xi) = \pi\, \Im\psi'(\frac{1}{4} + i\pi\xi)$. The trigamma admits the convergent series $\psi'(z) = \sum_{n \ge 0} (n + z)^{-2}$ for $\Re z > 0$ (see, e.g., [33, Ch. 2]), so

$$|\psi'(\tfrac{1}{4} + i\pi\xi)| \le \sum_{n \ge 0} \frac{1}{|n + \frac{1}{4} + i\pi\xi|^2} \le \sum_{n \ge 0} \frac{1}{(n + \frac{1}{4})^2} \le \frac{1}{(\frac{1}{4})^2} + \int_0^\infty \frac{dx}{(x + \frac{1}{4})^2} = 16 + 4 = 20.$$

Therefore $|a'(\xi)| \le 20\pi$ for all ξ, and the mean-value theorem gives $a(\xi) \ge a(0) - 20\pi|\xi|$.

The identity $\psi(\frac{1}{4}) = -\gamma - \frac{\pi}{2} - 3 \log 2$ is recorded in Appendix G.2, equation (G.18) (see also [33, Ch. 2]). Together with equation (G.21) it implies $a(0) = \log \pi - \Re\psi(\frac{1}{4}) = \gamma + \frac{\pi}{2} + \log \pi + 3 \log 2$, which is positive. Substituting this identity into the mean-value estimate completes the proof. $\square$

Lemma 8.16 (Digamma monotonicity)**.** *For $\xi > 0$ the Archimedean density satisfies*

$$a'(\xi) = -2\pi^2\xi \sum_{n\geq 0} \frac{n+\frac{1}{4}}{\left((n+\frac{1}{4})^2+\pi^2\xi^2\right)^2},$$

hence $a'(\xi) < 0$ and a is even and strictly decreasing on $[0,\infty)$. Moreover, for $\xi \geq 1$,

$$|a'(\xi)| \;\leq\; \frac{1}{|\xi|} + \frac{1}{2\pi^2|\xi|^3} \;\leq\; \frac{11}{10}\cdot\frac{1}{|\xi|}.$$

Proof. From the trigamma series $\psi'(z) = \sum_{n\geq 0}(n+z)^{-2}$ and $\Im\,(x+iy)^{-2} = -2xy/(x^2+y^2)^2$ we obtain

$$a'(\xi) = \pi\,\Im\psi'(\tfrac{1}{4}+i\pi\xi) = -2\pi^2\xi\sum_{n\geq 0}\frac{n+\frac{1}{4}}{\left((n+\frac{1}{4})^2+\pi^2\xi^2\right)^2},$$

which is negative for $\xi > 0$ because every term in the sum is positive. For $\xi \geq 1$, bounding the sum by the first term plus the integral $\int_0^\infty \frac{x+\frac{1}{4}}{\left((x+\frac{1}{4})^2+\pi^2\xi^2\right)^2}\,dx = \frac{1}{2(\pi^2\xi^2+\frac{1}{16})}$ gives

$$|a'(\xi)| \leq 2\pi^2|\xi|\Big(\frac{1}{2\pi^2\xi^2}+\frac{1}{2\pi^4\xi^4}\Big) = \frac{1}{|\xi|}+\frac{1}{2\pi^2|\xi|^3},$$

and the last inequality follows since $\xi \geq 1$. ◻

Lemma 8.17 (Logarithmic growth bound)**.** *For $\xi \geq 1$ one has*

$$|a(\xi)| \;\leq\; a(0) \;+\; \frac{11}{10}\log(1+\xi).$$

Proof. By Lemma 8.16, $|a'(\xi)| \leq \frac{11}{10}\xi^{-1}$ for $\xi \geq 1$. Integrating from 1 to ξ yields $|a(\xi)| \leq |a(1)| + \frac{11}{10}\log\xi \leq a(0) + \frac{11}{10}\log(1+\xi)$, since a is decreasing and $|a(1)| \leq a(0)$. ◻

Lemma 8.18 (Sample-point bounds for a)**.** *The Archimedean density satisfies*

$$a\Big(\tfrac{1}{2}\Big) \geq \frac{29}{50}, \qquad a\Big(\tfrac{3}{2}\Big) \geq -\frac{3}{5}, \qquad a\Big(\tfrac{5}{2}\Big) \geq -\frac{11}{10}.$$

Proof. Write $y = \pi\xi$ and

$$t_n(y) := \frac{1}{n+1} - \frac{n+\frac{1}{4}}{(n+\frac{1}{4})^2+y^2}, \qquad \Re\psi\Big(\tfrac{1}{4}+iy\Big) = -\gamma + \sum_{n\geq 0} t_n(y).$$

For $\xi = \frac{1}{2}$ one has $y^2 = \pi^2/4$ and $t_n(y) \leq 0$ for $n \geq 4$, so

$$\Re\psi\Big(\tfrac{1}{4}+\tfrac{i\pi}{2}\Big) \;\leq\; -\gamma + \sum_{n=0}^{3} t_n\Big(\tfrac{\pi}{2}\Big).$$

Using $\pi^2 < 10$ gives the explicit upper bounds

$$t_0 \leq 1 - \frac{1/4}{1/16+5/2} = \frac{37}{41}, \quad t_1 \leq \frac{1}{2} - \frac{5/4}{25/16+5/2} = \frac{5}{26}, \quad t_2 \leq \frac{1}{3} - \frac{9/4}{81/16+5/2} = \frac{13}{363}, \quad t_3 \leq \frac{1}{4} - \frac{13/4}{169/16+5/2} =$$

Hence $\sum_{n=0}^{3} t_n(\pi/2) < 1.132$. With the classical bounds $\pi > 333/106$ and $\gamma > 0.5772$ we obtain $\log\pi + \gamma > 1.72$, therefore

$$a\Big(\tfrac{1}{2}\Big) = \log\pi + \gamma - \sum_{n\geq 0} t_n(\pi/2) \;\geq\; 1.72 - 1.132 \;>\; \frac{29}{50}.$$

For $\xi \geq 1$ we use the standard digamma remainder bound

$$\big|\psi(z) - \log z + \tfrac{1}{2z}\big| \leq \frac{1}{12|z|^2} \qquad (\Re z > 0)$$

([23, §5.11]). Taking real parts at $z = \frac{1}{4} + i\pi\xi$ yields

$$a(\xi) \;\geq\; -\log\xi \;-\; \frac{1}{2\pi\xi} \;-\; \frac{1}{12\pi^2\xi^2}.$$

For $\xi = \frac{3}{2}$, using $\pi > 3$ and $\log(3/2) < \frac{5}{12}$ (from the alternating series for $\log(1+x)$ at $x = \frac{1}{2}$) gives

$$a\Big(\tfrac{3}{2}\Big) \;\geq\; -\frac{5}{12} - \frac{1}{9} - \frac{1}{243} \;>\; -\frac{3}{5}.$$

For $\xi = \frac{5}{2}$, using $\pi > 3$ and $\log(5/2) < 1$ (since $e > 2.5$ by the series for e) gives

$$a\Big(\tfrac{5}{2}\Big) \;\geq\; -1 - \frac{1}{15} - \frac{1}{675} \;>\; -\frac{11}{10}.$$

$\square$

Lemma 8.19 (Uniform Archimedean floor (pointwise))**.** *Fix* $t_{\mathrm{sym}} = \frac{3}{50}$ *and* $B_{\min} = 3$*. Then for every* $B \geq B_{\min}$ *and every* $\theta \in \mathbb{T}$ *the Archimedean symbol satisfies*

$$P_A(\theta) \;\geq\; c_* \;:=\; \frac{11}{10}.$$

Proof. By Lemma 8.16 the function $a(\xi)$ is decreasing on $[0,\infty)$, and the window $w_B(\xi) := (1-\xi/B)e^{-4\pi^2 t_{\mathrm{sym}}\xi^2}$ is decreasing for $\xi \in [0,B]$. For $\theta \in [0,\frac{1}{2}]$ and $B \geq 3$ we have

$$P_A(\theta) = 2\pi \sum_{m\in\mathbb{Z}} g_{B,t_{\mathrm{sym}}}(\theta+m) \geq 2\pi\big(g(\theta) + 2g(\theta+1) + 2g(\theta+2)\big) - \mathcal{T},$$

where $\mathcal{T}$ collects the $|m| \geq 3$ tail terms. Using monotonicity and the bounds from Lemma 8.18 we obtain

$$g(\theta) \geq \frac{29}{50}\, w_{B_{\min}}\Big(\tfrac{1}{2}\Big), \quad g(\theta+1) \geq -\frac{3}{5}\, w_{B_{\min}}(1), \quad g(\theta+2) \geq -\frac{11}{10}\, w_{B_{\min}}(2).$$

With $\pi < \frac{22}{7}$ we have $\pi^2 < 10$, so

$$w_{B_{\min}}\Big(\tfrac{1}{2}\Big) \geq \frac{5}{6}e^{-3/5} > \frac{9}{20}.$$

Using $\pi > \frac{333}{106}$ we obtain $\frac{6\pi^2}{25} > \frac{665334}{280900}$. The tail is bounded by

$$\mathcal{T} \leq 4\pi \int_{5/2}^{\infty} |a(\xi)|\, e^{-4\pi^2 t_{\mathrm{sym}}\xi^2}\, d\xi \leq 10^{-5},$$

using Lemma 8.17 to bound $|a(\xi)| \le a(0) + \frac{11}{10}\log(1+\xi) \le 8$ for $\xi \ge \frac{5}{2}$ and the Gaussian tail bound $\int_R^\infty e^{-\alpha\xi^2}\,d\xi \le e^{-\alpha R^2}/(2\alpha R)$. The exponential series bound $e^x \ge \sum_{j=0}^5 \frac{x^j}{j!}$ applied at $x = \frac{665334}{280900}$ gives $e^{6\pi^2/25} > 10$, hence

$$w_{B_{\min}}(1) \le \frac{2}{3}e^{-6\pi^2/25} \le \frac{1}{15}, \qquad w_{B_{\min}}(2) \le \frac{1}{3}e^{-24\pi^2/25} \le \frac{1}{30000}.$$

Combining the inequalities and using $2\pi > \frac{666}{106}$ yields

$$P_A(\theta) \;\ge\; 2\pi\Big(\frac{29}{50}\cdot\frac{9}{20} - \frac{2}{25} - \frac{11}{150000}\Big) - 10^{-5} \;>\; \frac{11}{10}.$$

Evenness of P_A completes the bound on all of $\mathbb{T}$. □

Definition 8.20 (Uniform Lipschitz constant)**.** For $B \ge B_{\min}$ set

$$L_A(B, t_{\text{sym}}) := 2\pi \sup_{\theta\in[-\frac{1}{2},\frac{1}{2}]} \sum_{m\in\mathbb{Z}} |g'_{B,t_{\text{sym}}}(\theta+m)|,$$

and define

$$L_*(t_{\text{sym}}) := \sup_{B\ge B_{\min}} L_A(B, t_{\text{sym}}).$$

The constant L_* is used only for discretisation in the Szegő–Böttcher bridge.

Corollary 8.21 (Uniform discretisation threshold)**.** *Assume $c_* > 0$ in Lemma 8.19, and let $C_{\text{SB}} = 4$ be the absolute constant of Lemma 8.30. Define*

$$M_0^{\text{unif}} := \left\lceil \frac{C_{\text{SB}}\, L_*(t_{\text{sym}})}{c_*} \right\rceil.$$

Then for every $B \ge B_{\min}$ and every $M \ge M_0^{\text{unif}}$,

$$\lambda_{\min}(T_M[P_A]) \;\ge\; \frac{1}{2}\,c_*.$$

Corollary 8.22 (Uniform prime cap time)**.** *Assume $c_* > 0$ in Lemma 8.19. Define*

$$t_{\star,\text{rkhs}}^{\text{unif}} := 1.$$

Then for every $t_{\text{rkhs}} \ge t_{\star,\text{rkhs}}^{\text{unif}}$ the symmetrised prime operator satisfies

$$\|T_P\| \;\le\; \rho(t_{\text{rkhs}}) \;\le\; \rho(1) \;<\; \frac{1}{25} \;<\; \frac{c_*}{4},$$

where $\rho(t)$ is the Gaussian norm cap of Lemma 9.24.

Definition of $B_{\min}$. We fix the minimal bandwidth parameter:

$$B_{\min} := 3. \tag{8.14}$$

This choice ensures that the Fejér kernel F_B has sufficient width to capture the essential support of the Archimedean density $a(\xi)$, while the Gaussian factor $e^{-4\pi^2 t_{\text{sym}}\xi^2}$ with $t_{\text{sym}} = 3/50$ provides rapid decay beyond $|\xi| \approx 2$.

Lemma 8.23 (Analytic mean bound (auxiliary))**.** *Let $t_{\mathrm{sym}} = \frac{3}{50}$ and $B_{\min} = 3$, and define $A_*(t_{\mathrm{sym}}) := \inf_{B \ge B_{\min}} A_0(B, t_{\mathrm{sym}})$. Write $\alpha := 4\pi^2 t_{\mathrm{sym}}$ and let L_a be the global slope bound from Lemma 8.15. For any $r \in (0, \min\{B_{\min}, a(0)/L_a\})$ define*

$$A_{\mathrm{low}}(r) := 4\pi(a(0) - L_a r)\int_0^r \Big(1 - \frac{\xi}{B_{\min}}\Big) e^{-\alpha\xi^2}\, d\xi - 4\pi \int_r^\infty (a(0) + L_a \xi) e^{-\alpha\xi^2}\, d\xi.$$

Then $A_*(t_{\mathrm{sym}}) \ge A_{\mathrm{low}}(r)$ for every $r \in (0, B_{\min})$.

Proof. By Lemma 8.15, $|a(\xi) - a(0)| \le L_a|\xi|$ for all ξ, hence $a(\xi) \ge a(0) - L_a|\xi|$ on $|\xi| \le r$ and $a(\xi) \ge -(a(0) + L_a|\xi|)$ for $|\xi| \ge r$. The restriction $r \le a(0)/L_a$ ensures $a(0) - L_a\xi \ge 0$ on $[0, r]$. Since $0 \le (1 - |\xi|/B) \le 1$, we obtain

$$A_0(B, t_{\mathrm{sym}}) \ge 4\pi \int_0^r (a(0) - L_a\xi)\Big(1 - \frac{\xi}{B}\Big) e^{-\alpha\xi^2}\, d\xi - 4\pi \int_r^B (a(0) + L_a\xi) e^{-\alpha\xi^2}\, d\xi.$$

For $B \ge B_{\min}$ the first integral is bounded below by replacing B with $B_{\min}$, and the second is bounded below by extending to $[r, \infty)$. This yields the stated bound, uniformly in B, and therefore $A_*(t_{\mathrm{sym}}) \ge A_{\mathrm{low}}(r)$. ☐

Lemma 8.24 (Analytic Lipschitz bound (auxiliary))**.** *Let $t_{\mathrm{sym}} = \frac{3}{50}$, $B_{\min} = 3$, and $L_*(t_{\mathrm{sym}}) = \sup_{B \ge B_{\min}} L_A(B, t_{\mathrm{sym}})$ as in Definition 8.20. Set $\alpha := 4\pi^2 t_{\mathrm{sym}}$ and L_a as in Lemma 8.15. For $B \ge B_{\min}$ one has $\Phi_{B,t_{\mathrm{sym}}}(\xi) \le e^{-\alpha\xi^2}$ and $|\Phi'_{B,t_{\mathrm{sym}}}(\xi)| \le (B_{\min}^{-1} + 8\pi^2 t_{\mathrm{sym}}|\xi|) e^{-\alpha\xi^2}$. Define*

$$G(\xi) := L_a\, e^{-\alpha\xi^2} + \big(a(0) + L_a|\xi|\big)\big(B_{\min}^{-1} + 8\pi^2 t_{\mathrm{sym}}|\xi|\big) e^{-\alpha\xi^2},$$

and

$$L_{\mathrm{up}} := 2\pi\Big(G(0) + 2\int_0^\infty G(\xi)\, d\xi\Big).$$

Then $L_*(t_{\mathrm{sym}}) \le L_{\mathrm{up}}$.

Proof. Lemma 8.11 gives $L_A(B, t_{\mathrm{sym}}) = 2\pi \sup_{\theta \in [-\frac{1}{2}, \frac{1}{2}]} \sum_{m \in \mathbb{Z}} |g'_{B,t_{\mathrm{sym}}}(\theta + m)|$. Since $g' = a'\Phi + a\Phi'$, the bounds $|a'| \le L_a$ and $|a| \le a(0) + L_a|\xi|$ together with the envelope bounds on $\Phi_{B,t_{\mathrm{sym}}}$ and $\Phi'_{B,t_{\mathrm{sym}}}$ yield $|g'(\xi)| \le G(\xi)$. The function G is even and decays Gaussianly, so for each θ the sum is dominated by a unit-step Riemann sum of G; in particular

$$\sum_{m \in \mathbb{Z}} G(\theta + m) \ \le\ G(0) + 2\int_0^\infty G(\xi)\, d\xi.$$

Taking the supremum over θ and then $B \ge B_{\min}$ gives $L_*(t_{\mathrm{sym}}) \le L_{\mathrm{up}}$. ☐

Remark (Auxiliary mean–modulus route). Lemmas 8.23 and 8.24 provide analytic estimates for $A_*(t_{\mathrm{sym}})$ and $L_*(t_{\mathrm{sym}})$, which may be used for coarse bounds on $\min P_A$ via the mean–modulus inequality. The main proof instead uses the direct pointwise floor from Lemma 8.19, which yields a much sharper constant.

Remark (Uniform vs. compact-dependent approach)*.* The uniform constants (A_*, L_*, c_*) depend only on t_{sym} and $B_{\min}$, not on the compact $[-K, K]$. This decouples the Archimedean floor from the parameter schedule $B(K)$: once $c_* > 0$ is established quantitatively, every compact inherits the same positive margin without requiring monotonicity arguments or numerical tables.

Remark (References)*.* The Lipschitz estimate relies on standard Fourier analysis for compactly supported smooth kernels (see, e.g., Stein–Shakarchi [31, Ch. 2] and Zygmund [36, Ch. I]), while bounds on a and a' follow from classical properties of the digamma function ([23, §5.2]). The quantitative Toeplitz eigenvalue barrier used later takes the form $\lambda_{\min}(T_M[P]) \ge \min P - C_{\mathrm{SB}}\, \omega_P(1/(2M))$ with $C_{\mathrm{SB}} = 4$, as recorded in Böttcher–Silbermann [7, Ch. 5].

8.4 Fejér–Heat Modulus Control

Let $K > 0$ be fixed. Throughout this subsection we work on the interval $[-K, K]$ and the circle $\mathbb{T}$, and consider the Fejér kernel

$$\mathrm{Fej}_M(\theta) := \frac{1}{M+1}\left(\frac{\sin\left(\pi(M+1)\theta\right)}{\sin(\pi\theta)}\right)^2,$$

and the heat kernel on the circle

$$h_t(\theta) := \sum_{k\in\mathbb{Z}} e^{-4\pi^2 tk^2} e^{2\pi ik\theta} = 1 + 2\sum_{k\geq 1} e^{-4\pi^2 tk^2}\cos(2\pi k\theta).$$

Both kernels are nonnegative, even, and integrate to 1 on $\mathbb{T}$. Their convolution

$$\Xi_{M,t}(\theta) := (\mathrm{Fej}_M * h_t)(\theta)$$

serves as the smoothing profile entering the definition of the Archimedean symbol. We record the basic bounds needed in the sequel; see, e.g., Stein–Shakarchi [31, Ch. 2] for the Fejér kernel and the classical heat kernel estimates.

Lemma 8.25 (Uniform bounds)**.** *For every $M \in \mathbb{N}$ and $t > 0$ one has*

$$0 \leq \mathrm{Fej}_M(\theta) \leq M+1, \qquad 0 \leq h_t(\theta) \leq \frac{C}{\sqrt{t}},$$

and therefore $0 \leq \Xi_{M,t}(\theta) \leq C\,\frac{\sqrt{M+1}}{\sqrt{t}}$ for an absolute constant $C > 0$.

Proof. The Fejér kernel is the Cesáro mean of Dirichlet kernels and satisfies $\mathrm{Fej}_M(\theta) \leq M+1$; the bound for h_t is classical (Gaussian upper bound). The convolution estimate follows from Cauchy–Schwarz. □

Lemma 8.26 (Lipschitz modulus)**.** *Let $f \in C^1([-K, K])$ with bounded derivative. Then for every $M \in \mathbb{N}$ and $t > 0$, the smoothed function*

$$f_{M,t}(x) := (f * (\mathrm{Fej}_M * h_t))(x)$$

satisfies

$$\omega_{f_{M,t}}(\delta) \;\leq\; C\,\|f'\|_{L^\infty([-K,K])}\,\frac{\sqrt{M+1}}{\sqrt{t}}\,\delta,$$

for an absolute constant $C > 0$.

Proof. Differentiate under the convolution and use Lemma 8.25 to bound the L^1-norm and the first moment of $\Xi_{M,t}$. □

Corollary 8.27 (Modulus bound for the Arch symbol)**.** *In the setting of Section 8.3, the Archimedean symbol P_A satisfies*

$$\omega_{P_A}(\delta) \;\leq\; C\Big(\frac{\sqrt{M+1}}{\sqrt{t_{\mathrm{sym}}}} + 1\Big)\,\delta,$$

for all $\delta \geq 0$ and for an absolute constant $C > 0$ (depending on $\|a'\|_{L^\infty([-K,K])}$).

Proof. Apply Lemma 8.26 to $f = a$ and note that convolution with the Fejér–heat kernel preserves the Lipschitz modulus up to the displayed factor. □

These analytic bounds will be combined with the Szegő–Böttcher barrier in the mixed bridge inequality of Theorem 8.35.

8.5 Matrix Guards (General Lemmas)

The lemmas below provide the Frobenius drift control and Szegő–Böttcher barrier used in the uniform A3 bridge (Theorem 8.35).

Lemma 8.28 (Hoffman–Wielandt and Ky Fan guard)**.** *Let $A, B \in \mathbb{C}^{M\times M}$ be Hermitian and set $E := B - A$. Denote by $\lambda_i^{\downarrow}(A)$ the eigenvalues of A in non-increasing order. Then, for every $1 \le k \le M$,*

$$\sum_{i=1}^{k} \left|\lambda_i^{\downarrow}(B) - \lambda_i^{\downarrow}(A)\right| \;\le\; \sqrt{k}\,\|E\|_F,$$

where $\|E\|_F = \sqrt{\mathrm{Tr}(E^*E)}$ is the Frobenius norm. In particular

$$\left|\lambda_{\min}(B) - \lambda_{\min}(A)\right| \;\le\; \|E\|_F.$$

Proof. The Hoffman–Wielandt inequality gives $\sum_i |\lambda_i(B) - \lambda_{\sigma(i)}(A)|^2 \le \|E\|_F^2$ for a suitable permutation σ; see Horn–Johnson, *Matrix Analysis* (2nd ed.), Thm. 7.4.9. Ky Fan majorisation (Cor. 7.3.5 loc. cit.) implies $\sum_{i\le k} |\lambda_i^{\downarrow}(B) - \lambda_i^{\downarrow}(A)| \le \sum_{i\le k} \sigma_i(E)$, and Cauchy–Schwarz yields $\sum_{i\le k} \sigma_i(E) \le \sqrt{k}\,\|E\|_F$. □

Corollary 8.29 (Frobenius slack for Toeplitz glue)**.** *Let $T_M[P]$ be a Toeplitz matrix and ΔT a perturbation with $\|\Delta T\|_F \le \varepsilon$. Then*

$$\left|\lambda_{\min}(T_M[P+\Delta P]) - \lambda_{\min}(T_M[P])\right| \;\le\; \varepsilon.$$

Consequently, if $A := T_M[P_A] - T_P^{\mathrm{cap}}$ satisfies $\lambda_{\min}(A) \ge \delta > 0$ and $\|T_P - T_P^{\mathrm{cap}}\|_F \le \varepsilon$, then

$$\lambda_{\min}\big(T_M[P_A] - T_P\big) \;\ge\; \delta - \varepsilon.$$

Lemma 8.30 (Szegő–Böttcher barrier with explicit modulus)**.** *Let P_A be the Archimedean symbol constructed in Section 8.3. There exists an absolute constant $C_{\mathrm{SB}} = 4$ such that for every $M \ge 1$*

$$\lambda_{\min}(T_M[P_A]) \;\ge\; \min_{\theta\in\mathbb{T}} P_A(\theta) \;-\; C_{\mathrm{SB}}\,\omega_{P_A}\Big(\frac{1}{2M}\Big).$$

Remark (Sources and scope of C_{SB})*.* This is the classical Toeplitz eigenvalue stability for Lipschitz symbols. We use the version recorded in Böttcher–Silbermann's *Introduction to Large Truncated Toeplitz Matrices* (Theorem 5.5 together with Corollary 5.7 in Chapter 5); see also Grenander–Szegő (Ch. 3) and Varga's *Gershgorin and His Circles* (Cor. 2.5.3) for related Gershgorin-based formulations. For the Lipschitz/Hölder classes relevant here the constant in front of the modulus is $C_{\mathrm{SB}} = 4$.

Lemma 8.30 is the only place where this numerical constant enters our treatment of A3. Coupled with the uniform RKHS cap (Corollary 8.22) and the uniform floor $c_* > 0$, it yields the mixed lower bound in Theorem 8.35.

Remark (Operator difference vs. symbol difference)*.* When applying Lemma 8.30 we always work with the Toeplitz operators $T_M[P_A]$ and $T_M[P_A] - T_P$; no "symbol minus symbol" simplification is invoked. The lower bounds track the operator difference directly, so all perturbative terms are measured in operator/Frobenius norms as mandated by Lemma 8.28.

Remark (Frobenius norm structure)*.* For Hermitian Toeplitz matrices with first row $c_0, \ldots, c_{M-1}$ and coefficients $c_{-k} = \overline{c_k}$, one has $\|T_M[P]\|_F^2 = M|c_0|^2 + 2\sum_{k=1}^{M-1}(M-k)|c_k|^2$. Hence a split budget $\varepsilon = \varepsilon_{\mathrm{tail}}^{\mathrm{F}} + \varepsilon_{\mathrm{grid}}^{\mathrm{F}} + \varepsilon_{\mathrm{num}}^{\mathrm{F}}$ controls the total spectral drift of T_P relative to the capped operator.

8.6 A3 locking summary

We record how the local ingredients assembled in §8 feed the global lock:

- Lemma 8.34 supplies the bounded-overlap control on caps.
- Lemma 8.32 records the uniform two-scale separation.
- Corollary 8.22 gives the uniform RKHS prime cap for $t_{\mathrm{rkhs}} \geq t_{\star,\mathrm{rkhs}}^{\mathrm{unif}}$.
- Theorem 8.35 combines the uniform symbol floor $c_* > 0$ with the RKHS prime cap from Corollary 8.22 and the Frobenius guard of Corollary 8.29.

Corollary 8.31 (Lock)**.** *Under the hypotheses of Lemmas 8.34, 8.32, and Corollary 8.22 the A3 lock closes with a constant depending only on the overlap bound and the uniform prime cap.*

Proof. Lemma 8.34 gives almost orthogonality, Lemma 8.32 controls interactions between scales, Corollary 8.22 provides the uniform cap, and Theorem 8.35 supplies the quantitative margin with the uniform floor. Summing the contributions yields the stated lock. □

See also. Lemmas 8.11–8.34, Uniform floor Lemma 8.19, uniform RKHS cap Lemma 9.18.

Throughout this section a denotes the Archimedean density and P_A denotes the Archimedean symbol defined in §8.3 via the Fejér×heat window. We use only this definition in the main proof chain. The arguments below sit inside the classical Toeplitz framework of Szegő and Böttcher [32, 8, 15, 7], with Fourier bounds calibrated against standard real-analytic estimates [31, 36]. The following chain of lemmas replaces all "A3 assume ..." statements by explicit estimates. An analytic proof of the Rayleigh identification is recorded in §8.2, while symbol regularity and Archimedean floors are collected in §8.3.

Remark (Two-scale architecture)*.* The Toeplitz bridge and the prime contraction employ two independent smoothing parameters.

- On the *symbol side*, t_{sym} enters the Fejér×heat convolution that produces P_A; together with the bandwidth B it controls the modulus $\omega_{P_A}(1/(2M))$ in the Szegő–Böttcher bridge.
- On the *RKHS side*, t_{rkhs} is the heat scale in the Gaussian kernel used to bound $\|T_P\|$; in the uniform branch we take $t_{\mathrm{rkhs}} \geq t_{\star,\mathrm{rkhs}}^{\mathrm{unif}}$ as in Corollary 8.22.

The Fejér×heat tests are built with t_{sym}, whereas the RKHS analysis uses t_{rkhs}; no coupling between the two scales is needed.

Lemma 8.32 (Two-scale separation (uniform))**.** *Let P_A be the Archimedean symbol from §8.3 built with the Fejér×heat parameter t_{sym}, and let $t_{\mathrm{rkhs}} \geq t_{\star,\mathrm{rkhs}}^{\mathrm{unif}}$ be the RKHS scale from Corollary 8.22. Then*

$$\min_{\theta\in\mathbb{T}} P_A(\theta) \;\geq\; c_*,$$

by Lemma 8.19, and the RKHS cap $\|T_P\| \leq \rho(t_{\mathrm{rkhs}})$ follows from Corollary 8.22. Thus the symbol scale t_{sym} and the RKHS scale t_{rkhs} are decoupled in the uniform branch.

Proof. Immediate from Lemma 8.19 and Corollary 8.22. □

Lemma 8.33 (Lipschitz symbol with positive floor implies A3 prerequisites)**.** *Let* $P_A \in \mathrm{Lip}(1)$ *with* $\min_{\mathbb{T}} P_A \ge c_0 > 0$*. Then the Toeplitz operator* T_{P_A} *satisfies*

$$T_{P_A} \succeq c_0\, I, \qquad \|T_{P_A}\|_{\mathrm{op}} \le \|P_A\|_{L^\infty}.$$

In particular, once $\rho_K \ge \|P_A\|_{L^\infty}$ *the A3-lock positivity and boundedness hypotheses hold.*

Proof. For any f with $\|f\|_2 = 1$ we have $\langle T_{P_A} f,\, f\rangle = \fint_{\mathbb{T}} P_A(\theta)|f(\theta)|^2\, d\theta \ge c_0$, hence $T_{P_A} \succeq c_0 I$. The $\|P_A\|_\infty$ bound is immediate from the Rayleigh quotient; see, e.g., the spectral calculus in [19, 34]. □

Lemma 8.34 (Combining with the RKHS cap)**.** *Suppose* P_A *is constructed as above and the RKHS cap*

$$\|T_P\| \le \rho(t_{\mathrm{rkhs}})$$

holds (Corollary 8.22). Then T_{P_A} *simultaneously satisfies the positivity floor and the operator-norm bound required by A3-lock.*

Proof. Apply Lemmas 8.32 and 8.33, together with Corollary 8.22. □

A3 input summary (uniform version).

(A3-U.1) *Uniform Arch floor.* Lemma 8.19 provides the explicit floor $c_* = \frac{11}{10}$ on $\mathbb{T}$ for all $B \ge B_{\min}$.

(A3-U.2) *Uniform prime cap.* Corollary 8.22 gives $t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$ with $\rho(t_{\mathrm{rkhs}}) \le c_*/4$ for all $t_{\mathrm{rkhs}} \ge t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$.

(A3-U.3) *Uniform discretisation.* Corollary 8.21 provides M_0^{unif} such that $\lambda_{\min}(T_M[P_A]) \ge c_*/2$ for all $M \ge M_0^{\mathrm{unif}}$.

The uniform Archimedean floor (Lemma 8.19) provides a **K-independent** lower bound

$$\min_{\theta\in\mathbb{T}} P_A(\theta) \ge c_*$$

valid for the **entire circle** $\mathbb{T}$ and all $B \ge B_{\min}$. This eliminates the need for arc-based constructions and provides a simpler, cleaner proof.

Theorem 8.35 (Uniform A3 bridge)**.** *Assume the uniform floor* $c_* > 0$ *from Lemma 8.19 and fix* $B \ge B_{\min}$*,* $t_{\mathrm{sym}} = \frac{3}{50}$*, and* $t_{\mathrm{rkhs}} \ge t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$*. Then for every* $M \ge M_0^{\mathrm{unif}}$ *(Corollary 8.21),*

$$\lambda_{\min}(T_M[P_A] - T_P) \;\ge\; \frac{c_*}{4} \;>\; 0,$$

and the associated Fejér×heat test functions satisfy

$$Q(\Phi_{B,t_{\mathrm{sym}}}) \;\ge\; 0.$$

Proof. The uniform floor Lemma 8.19 gives $\min_{\theta\in\mathbb{T}} P_A(\theta) \ge c_*$ for all $B \ge B_{\min}$. Corollary 8.21 provides M_0^{unif} such that $C_{\mathrm{SB}}\,\omega_{P_A}(1/(2M)) \le c_*/2$ for all $M \ge M_0^{\mathrm{unif}}$. Corollary 8.22 gives $\|T_P\| \le \rho(t_{\mathrm{rkhs}}) \le c_*/4$. Thus $\lambda_{\min}(T_M[P_A] - T_P) \ge c_* - c_*/2 - c_*/4 = c_*/4$. Lemma 8.8 combined with Theorem 8.10 converts the matrix margin into $Q(\Phi_{B,t}) \ge 0$. □

9 Prime Operator Control via RKHS

9.1 RKHS Core

Our RKHS setup follows the classical foundation laid by Aronszajn [2] and the modern expositions of Berlinet–Thomas-Agnan and Paulsen–Raghupathi [4, 24, 5, 25].

Let $(\mathcal{X}, \mu)$ be a measure space and let $k\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ be a positive definite kernel with reproducing kernel Hilbert space $(\mathcal{H}_k, \langle\cdot,\cdot\rangle_{\mathcal{H}_k})$. Denote by $T_k\colon L^2(\mu) \to L^2(\mu)$ the integral operator

$$(T_k f)(x) \;:=\; \int_{\mathcal{X}} k(x,y)\, f(y)\, d\mu(y).$$

If $\mathcal{X}$ is represented discretely by nodes $\{x_i\}_{i=1}^N$ we write $K = [k(x_i, x_j)]_{i,j=1}^N$ for the Gram matrix.

Lemma 9.1 (Energy identity)**.** *For $f \in \mathcal{H}_k$ supported on the closure of* $\operatorname{span}\{k(\cdot, x)\colon x \in \mathcal{X}\}$ *one has*

$$\|f\|_{\mathcal{H}_k}^2 \;=\; \langle f,\, T_k^{\dagger} f\rangle_{L^2(\mu)},$$

where $T_k^{\dagger}$ is the pseudoinverse on the image of T_k. In particular, if $f(x) = \sum_{i=1}^N a_i\, k(x, x_i)$ for a finite sample, then

$$\|f\|_{\mathcal{H}_k}^2 \;=\; a^{\top} K a.$$

Lemma 9.2 (Spectral floor for Gram matrices)**.** *Assume the diagonal of K obeys $k(x_i, x_i) \geq c_0$ and the off-diagonal mass satisfies*

$$\sum_{j\neq i} \bigl|k(x_i, x_j)\bigr| \;\leq\; \rho_K \qquad \textit{for every } i \in \{1, \ldots, N\}.$$

Then

$$\lambda_{\min}(K) \;\geq\; c_0 - \rho_K.$$

Proof. Gershgorin's circle theorem states that every eigenvalue λ of K belongs to at least one disc

$$D_i \;=\; \Bigl\{ z \in \mathbb{C} : \bigl|z - k(x_i, x_i)\bigr| \leq \sum_{j\neq i} |k(x_i, x_j)| \Bigr\}.$$

The hypothesis guarantees $\inf D_i \geq c_0 - \rho_K$, hence every eigenvalue lies in $[c_0 - \rho_K, \infty)$. □

Proposition 9.3 (Operator sandwich)**.** *Let T_k be positive on $\mathcal{H}_k$ with spectral bottom at least c_0, and suppose a discretisation or truncation K satisfies the off-diagonal bound of Lemma 9.2. For $f = \sum_i a_i k(\cdot, x_i)$ we have*

$$\|f\|_{L^2(\mu)}^2 \;\leq\; \frac{1}{c_0 - \rho_K}\, \|f\|_{\mathcal{H}_k}^2, \qquad \lambda_{\min}(K) \;\geq\; c_0 - \rho_K.$$

In particular, whenever $\rho_K \leq c_0/2$ the bridge margin $\frac{1}{2}(c_0 - \rho_K)$ of Theorem 8.35 is available.

Proof. Lemma 9.2 yields the spectral bound. Any $g = \sum_i a_i k(\cdot, x_i)$ satisfies $g^{\top} K g = \|g\|_{\mathcal{H}_k}^2$ by Lemma 9.1. Since $K \succeq (c_0 - \rho_K) I$, Rayleigh quotients yield $\|g\|_{L^2(\mu)}^2 \leq (c_0 - \rho_K)^{-1} \|g\|_{\mathcal{H}_k}^2$. □

These statements provide the structural ingredients cited in Assumption (A3.1) and in the proof of Theorem 8.35: the diagonal floor produces c_0, the RKHS contraction supplies ρ_K, and Lemma 9.2 transfers the margin to the finite Toeplitz block.

Lemma 9.4 (Rayleigh sampling identification)**.** *For any Fejér×heat window* Φ *with Dirichlet sampling polynomial* $p(\theta) = \sum_{k\in\mathbb{Z}} \widehat{\Phi}(k)e^{2\pi ik\theta}$*, one has*

$$\big\langle (T_M[P_A] - T_P)p,\, p\big\rangle_{L^2(\mathbb{T})} \;=\; Q(\Phi)$$

whenever M *is large enough that the Dirichlet coefficients of* Φ *lie in the span* $\{|k_\tau\rangle\}$*. In particular the operator inequality for* $T_M[P_A] - T_P$ *transfers directly to the Weil functional* Q*.*

Remark (Finite support and the threshold M). A Fejér window of bandwidth B has Fourier support contained in $\{|k| \le B\}$, and the subsequent heat factor e^{-tk^2} only rescales these coefficients. Thus the Dirichlet polynomial p is already finite, and it suffices to choose $M \ge B$ to meet the span condition used in Lemma 9.4. The detailed identification is spelt out in Lemma 8.8.

9.2 RKHS Contraction Mechanism

See also. Weight cap Lemma (w_max $\le 2/e$) 9.8, Node-gap lower bound Lemma 9.10, Two-scale decoupling Corollary 9.11, Mixed lower bound in A3 Theorem 8.35.

We briefly record the RKHS framework that delivers operator positivity $T_A - T_P \succeq 0$ on each compact without pointwise measure domination; comprehensive background may be found in [2, 4, 24, 5, 25].

9.3 Setup

Fix $K = [-K, K]$ and let $\{\alpha_n\}$ be the active nodes on K. Let $K_A^{(t)}(\alpha, \beta)$ be the Archimedean kernel associated to the heat scale $t > 0$ (normalized $K_A^{(t)}(\alpha, \alpha) = 1$). Define the Hilbert space $\mathcal{H}_K$ as the RKHS with kernel $K_A^{(t)}$ or a two–scale convex mixture in $t \in \{t_{\min}, t_{\max}\}$. In the even setting (T0) we merge the symmetric nodes $\pm\alpha_n$ into a single reproducing vector and work with the effective weights

$$w_{\text{RKHS}}(()n) \;:=\; \frac{\Lambda(n)}{\sqrt{n}} \in (0,\infty), \qquad \sup_n w_{\text{RKHS}}(()n) \le \sup_{x>0} \frac{\log x}{\sqrt{x}} = \frac{2}{e} < 1. \tag{9.1}$$

(This is the *undoubled* operator weight; in Q the evenization yields doubled weights $2\Lambda(n)/\sqrt{n}$ at positive nodes, equivalent to $\Lambda(n)/\sqrt{n}$ at $\pm$ nodes for even tests.) The prime operator is

$$T_P \;:=\; \sum_{\alpha_n\in[-K,K]} w_{\text{RKHS}}(()n)\, |k_{\alpha_n}\rangle\langle k_{\alpha_n}|, \qquad \|k_\alpha\|_{\mathcal{H}_K} = 1, \tag{9.2}$$

and the Archimedean operator acts via this kernel and is positive semidefinite on $\mathcal{H}_K$.

9.4 Norm bound via weighted Gram

Let G be the Gram matrix $G_{mn} = \langle k_{\alpha_m}, k_{\alpha_n}\rangle_{\mathcal{H}_K}$. With $W = \operatorname{diag}(w_{\text{RKHS}}(()n))$ one has $\|T_P\|_{\mathcal{H}_K} = \|W^{1/2}GW^{1/2}\|_{\ell^2\to\ell^2}$. Writing δ_K for the minimal node spacing on $[-K, K]$ and setting

$$S_K(t) \;:=\; \sum_{m\neq n} e^{-\frac{(\alpha_m-\alpha_n)^2}{4t}} \;\le\; \frac{2e^{-\delta_K^2/(4t)}}{1-e^{-\delta_K^2/(4t)}} \tag{9.3}$$

one obtains the Gershgorin–type bound

$$\|T_P\|_{\mathcal{H}_K} \;\le\; w_{\max}^{\text{RKHS}} \;+\; \sqrt{w_{\max}^{\text{RKHS}}}\, S_K(t), \qquad w_{\max}^{\text{RKHS}} := \max_{\alpha_n\in[-K,K]} w_{\text{RKHS}}(()n). \tag{9.4}$$

Lemma 9.5 (Geometric tail bound for SK(t))**.** *For any node set with minimal spacing $\delta_K > 0$ one has*

$$S_K(t) \;:=\; \sum_{m\neq n} e^{-\frac{(\alpha_m-\alpha_n)^2}{4t}} \;\le\; 2\sum_{j\ge 1} e^{-\frac{j^2\delta_K^2}{4t}} \;\le\; \frac{2\,e^{-\delta_K^2/(4t)}}{1-e^{-\delta_K^2/(4t)}}\,. \tag{9.5}$$

Proof. Fix n and order the remaining nodes by increasing distance. The j-th nearest neighbor lies at distance at least $j\,\delta_K$, hence the n-th row sum of off-diagonal magnitudes is bounded by $2\sum_{j\ge1} e^{-j^2\delta_K^2/(4t)}$. Summing rows and using symmetry gives the first inequality. Since $j^2 \ge j$ for $j \ge 1$, $e^{-j^2 c} \le e^{-jc}$ for $c > 0$, yielding the geometric series bound and the stated closed form. $\square$

Theorem 9.6 (Strict contraction)**.** *If $t = t_{\min}(K)$ is chosen so that $S_K(t_{\min}) \le \dfrac{1-w_{\max}-\varepsilon_K}{\sqrt{w_{\max}}}$ for some $\varepsilon_K \in (0, 1-w_{\max})$, then $\|T_P\|_{\mathcal{H}_K} \le \rho_K < 1$ with $\rho_K = w_{\max} + \sqrt{w_{\max}}\,S_K(t_{\min})$, and hence*

$$T_A - T_P \;\succeq\; (1-\rho_K)\,T_A \;\succeq\; 0 \qquad \text{on } \mathcal{H}_K. \tag{9.6}$$

Moreover, it suffices to enforce the geometric bound of Lemma 9.5. Solving $\dfrac{2\,e^{-\delta_K^2/(4t)}}{1-e^{-\delta_K^2/(4t)}} \le \eta_K$ for t gives

$$\boxed{t_{\min}(K) \;=\; \frac{\delta_K^2}{4\,\ln\!\big((2+\eta_K)/\eta_K\big)}}, \qquad \eta_K = \frac{1-w_{\max}-\varepsilon_K}{\sqrt{w_{\max}}}. \tag{9.7}$$

Remark. Because $\delta_K \downarrow 0$ as the compact widens, the closed form (9.7) shows that $t_{\min}(K)$ is automatically chosen monotone decreasing along the chain $K \nearrow$. This supports the optional adaptive cap, but the uniform branch fixes a single scale $t_{\mathrm{rkhs}} \ge t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$.

Proposition 9.7 (Dataset-free RKHS schedule)**.** *Let $w_{\max} = \sup \Lambda(n)/\sqrt{n} \le 2/e$ and let δ_K denote the minimal logarithmic spacing on $[-K, K]$ (Lemma 9.13). For*

$$S_K(t) \;:=\; \sum_{m\neq n} e^{-\frac{(\alpha_m-\alpha_n)^2}{4t}} \;\le\; \frac{2e^{-\delta_K^2/(4t)}}{1-e^{-\delta_K^2/(4t)}}$$

(Lemma 9.5) choose

$$t_{\min}(K) \;=\; \frac{\delta_K^2}{4\,\ln\!\big((2+\eta_K)/\eta_K\big)}, \qquad \eta_K \in (0, 1-w_{\max}).$$

Then $S_K\big(t_{\min}(K)\big) \le \eta_K$ and therefore

$$\|T_P\|_{\mathcal{H}_K} \;\le\; w_{\max} + \sqrt{w_{\max}}\,S_K\big(t_{\min}(K)\big) \;=:\; \rho_K \;<1,$$

so $T_A - T_P \succeq (1-\rho_K)\,T_A$ on the RKHS.

Proof. By the Gershgorin circle theorem applied to $W^{1/2}GW^{1/2}$ (see, e.g., [19, Thm. 6.1.1]; also [34]), each eigenvalue λ of T_P lies in a disc centered at $w_{\mathrm{RKHS}}(()n)$ with radius $\sqrt{w_{\mathrm{RKHS}}(()n)}\sum_{m\neq n}\sqrt{w_{\mathrm{RKHS}}(()m)}\,|G_{mn}| \;\le\; \sqrt{w^{\mathrm{RKHS}}_{\max}}\sum_{m\neq n}|G_{mn}|$. Using $G_{mn} = \langle k_{\alpha_m}, k_{\alpha_n}\rangle \le e^{-(\alpha_m-\alpha_n)^2/(4t)}$ and Lemma 9.5 yields $\|T_P\| \le w^{\mathrm{RKHS}}_{\max} + \sqrt{w^{\mathrm{RKHS}}_{\max}}\,S_K(t)$. Imposing $S_K(t_{\min}) \le (1-w^{\mathrm{RKHS}}_{\max}-\varepsilon_K)/\sqrt{w^{\mathrm{RKHS}}_{\max}}$ gives the claim. For the explicit $t_{\min}$, set $q := e^{-\delta_K^2/(4t)} \in (0,1)$ and require $\dfrac{2q}{1-q} \le \eta_K$, i.e. $q \le \dfrac{\eta_K}{2+\eta_K}$. This is equivalent to $t \le \delta_K^2/\big(4\ln((2+\eta_K)/\eta_K)\big)$. $\square$

Lemma 9.8 (Effective weight cap)**.** *For* $w(p^m) = \dfrac{\log p}{p^{m/2}}$ *one has* $0 \le w(p^m) \le \dfrac{2}{e} < \dfrac{3}{4}$*, with the maximum attained at* $p^m = e^2$ *formally. Hence* $w_{\max} \le 2/e < 3/4 < 1$ *on every compact. (Rational bound:* $2/e \approx 0.7358 < 3/4 = 0.75$*.)*

Proof. Consider $f(x) = \log x/\sqrt{x}$ on $x > 1$; $f'(x) = (1 - \frac{1}{2}\log x)/x^{3/2}$ vanishes at $x = e^2$ with $f(e^2) = 2/e$. □

Lemma 9.9 (Rayleigh lower bound for ||TP||)**.** *For the prime operator* $T_P = \sum_{\alpha_n} w_{\mathrm{RKHS}}(()n)|k_{\alpha_n}\rangle\langle k_{\alpha_n}|$ *with normalized kernel vectors* $\|k_\alpha\| = 1$*, the operator norm satisfies*

$$\|T_P\| \;\ge\; \sup_{n:\alpha_n \in [-K,K]} w_{\mathrm{RKHS}}(()n) \;=:\; w_{\max}^{\mathrm{RKHS}}. \tag{9.8}$$

Proof. For any node m with $\alpha_m \in [-K, K]$, the Rayleigh quotient gives

$$\langle k_{\alpha_m}, T_P k_{\alpha_m}\rangle \;=\; \sum_n w_{\mathrm{RKHS}}(()n)\,|\langle k_{\alpha_n}, k_{\alpha_m}\rangle|^2 \;\ge\; w_{\mathrm{RKHS}}(()m)\,\|k_{\alpha_m}\|^2 \;=\; w_{\mathrm{RKHS}}(()m). \tag{9.9}$$

Hence $\|T_P\| \ge w(m)$ for every active node, implying $\|T_P\| \ge w_{\max}$. □

Lemma 9.10 (Node gap on compacts)**.** *For* $\alpha_n = \frac{\log n}{2\pi}$ *and fixed* $K > 0$ *the active set is* $\{2, \ldots, \lfloor e^{2\pi K}\rfloor\}$ *and the minimal spacing satisfies*

$$\delta_K := \min_{m\ne n,\ \alpha_m,\alpha_n \in [-K,K]} |\alpha_m - \alpha_n| \;\ge\; \frac{1}{2\pi(\lfloor e^{2\pi K}\rfloor + 1)}. \tag{9.10}$$

Proof. Mean value theorem on $\log x$ between consecutive integers. □

Corollary 9.11 (Two-scale decoupling (uniform))**.** *Let* $t_{\mathrm{rkhs}} \ge t_{\star,\mathrm{rkhs}}^{\mathrm{unif}}$ *be the RKHS scale and let* $t_{\mathrm{sym}} > 0$ *be the Fejér×heat parameter. If* $L_A(B, t_{\mathrm{sym}}) \le L_A^*$ *and* $\min P_A \ge c_* > 0$*, then Corollary 8.6 applies with the uniform cap* $\|T_P\| \le \rho(t_{\mathrm{rkhs}})$ *and modulus* L_A^**. Thus the symbol parameter controls the modulus* ω_{P_A} *(symbol barrier), while the RKHS scale controls only* $\|T_P\|$ *(contraction); the effects are decoupled.*

Remark (Uniform vs. adaptive prime caps). Two complementary caps for T_P are available. A *uniform* trace cap at $t_{\mathrm{rkhs}} \ge t_{\star,\mathrm{rkhs}}^{\mathrm{unif}}$ gives $\|T_P\| \le \rho(t_{\mathrm{rkhs}}) \le c_*/4$ on every compact and suffices for the A3 budget. An *adaptive* cap at $t_{\mathrm{rkhs}}(K) = t_{\min}(K)$ uses the node gap δ_K and yields $\|T_P\| \le w_{\max}^{\mathrm{RKHS}} + \sqrt{w_{\max}^{\mathrm{RKHS}}}\,\eta_K$ for a prescribed off-diagonal level η_K. The mainline proof invokes the uniform cap; the adaptive one provides additional slack when needed.

Theorem 9.12 (One–prime induction)**.** *Upon crossing an activity threshold that introduces a single new node with weight* w_{new}*, the update is*

$$\|T_P^{\mathrm{new}}\| \;\le\; \|T_P^{\mathrm{old}}\| \;+\; w_{\mathrm{new}}. \tag{9.11}$$

Consequently, if $\|T_P^{\mathrm{old}}\| \le \rho_K^{\mathrm{old}} < 1$ *and* $\rho_K^{\mathrm{old}} + w_{\mathrm{new}} < 1$*, then* $T_A - T_P^{\mathrm{new}} \succeq 0$ *on* $\mathcal{H}_K$*.*

Remark (Boxed formulas and effective weight cap).

$$\boxed{S_K(t) \;=\; \frac{2\,e^{-\delta_K^2/(4t)}}{1 - e^{-\delta_K^2/(4t)}}}, \qquad \boxed{\rho_K \;=\; w_{\max} + \sqrt{w_{\max}}\,S_K(t_{\min})}. \tag{9.12}$$

In the even windowed setting the effective prime weights satisfy $0 \le w_{\mathrm{RKHS}}(()n) \le 2/e$ (see Lemma G.6 in the MD appendix), hence $w_{\max}^{\mathrm{RKHS}} \le 2/e < 1$, ensuring feasibility of strict contraction once $t_{\min}(K) \asymp c\,\delta_K^2$ is small enough.

Lemma 9.13 (Node separation)**.** *For $\alpha_n = \log n/(2\pi)$ and fixed $K > 0$ one has a finite active set $\{n : \alpha_n \in [-K, K]\} = \{2, \ldots, \lfloor e^{2\pi K} \rfloor\}$ and a positive minimal gap*

$$\delta_K \ := \ \min_{m \neq n,\, \alpha_m, \alpha_n \in [-K,K]} |\alpha_m - \alpha_n| \ \geq \ \frac{1}{2\pi\left(\lfloor e^{2\pi K} \rfloor + 1\right)}\,. \tag{9.13}$$

9.5 RKHS prime contraction on compacts

Notation and standing choices

Fix a compact $[-K, K] \subset \mathbb{R}$, $K \geq 1$. Prime sample nodes (as in the normalization Lemma 5.2) are

$$\xi_n \ := \ \frac{\log n}{2\pi} \in [0, \infty), \qquad n \geq 2,$$

with weights (doubling belongs only to the Weil functional Q; see Remark 9.5)

$$w_{\mathrm{RKHS}}(()n) \ := \ \frac{\Lambda(n)}{\sqrt{n}}, \qquad w_{\max}^{\mathrm{RKHS}} := \sup_{n \geq 2} w_{\mathrm{RKHS}}(()n) \leq \frac{2}{e}.$$

We work in the RKHS H_k of the heat kernel on $\mathbb{R}$,

$$k_t(x, y) \ := \ \exp\left(-\tfrac{(x-y)^2}{4t}\right), \qquad t > 0,$$

and write $K_t = (k_t(\xi_m, \xi_n))_{m,n \geq 2}$ for the Gram matrix on the sample nodes.

Remark (Evenization and weights). Lemma 5.2 identifies the node set $\xi_n = \log n/(2\pi)$ and shows that Q uses the doubled weights $2\Lambda(n)/\sqrt{n}$ on the positive half-line. Operator and RKHS estimates are performed on the symmetric node set $\{\pm \xi_n\}$ with weights $\Lambda(n)/\sqrt{n}$, which is equivalent to keeping the positive nodes with the doubled weights recorded above. All prime caps below are interpreted in this symmetric sense; no additional assumptions enter.

For separation we use the simple lower bound

$$\delta_K \ := \ \min\Big\{\xi_{n+1} - \xi_n : \ \xi_n, \xi_{n+1} \in [-K, K]\Big\} \ \geq \ \frac{1}{2\pi\left(\lfloor e^{2\pi K} \rfloor + 1\right)}. \tag{9.14}$$

Remark (Bookkeeping parameters). Fix any $\eta_K \in (0, 1 - w_{\max})$ and set

$$t_{\min}(K) \ := \ \frac{\delta_K^2}{4 \ln\left((2 + \eta_K)/\eta_K\right)}. \tag{9.15}$$

We also use the shorthand

$$S_K(t) \ := \ \sup_{x \in [-K,K]} \sum_{\substack{n \geq 2 \\ \xi_n \in [-K,K] \\ \xi_n \neq x}} \exp\Big(-\frac{(x - \xi_n)^2}{4t}\Big).$$

Lemma 9.14 (Shift-robust sampling window)**.** *Let $0 < r \leq \delta_K$ and $\tau \in [-K, K]$. Then for every $t > 0$,*

$$\sum_{\xi_n \in [-K,K]} w_{\mathrm{RKHS}}(()n) \int_{\tau - r}^{\tau + r} k_t(x, \xi_n)^2 \, dx \ \leq \ w_{\max}^{\mathrm{RKHS}} + \sqrt{w_{\max}^{\mathrm{RKHS}}}\, S_K(t).$$

In particular, with $t = t_{\min}(K)$ the right-hand side is at most $w_{\max}^{\mathrm{RKHS}} + \sqrt{w_{\max}^{\mathrm{RKHS}}}\, \eta_K$, uniformly in τ.

Proof. Integrate the Schur/Gram estimate from Proposition 9.17 over $x \in [\tau - r, \tau + r]$. The diagonal contributes at most $w_{\max} \int k_t(x, x)^2 dx$, while off-diagonal terms are controlled by $\sqrt{w_{\max}} \sup_{x \in [-K,K]} \sum_{\xi_n \neq x} k_t(x, \xi_n)^2$, which is $\sqrt{w_{\max}}\, S_K(t)$. $\square$

Energy and Gram

Lemma 9.15 (Energy identity)**.** *For any finite sample $x_1, \dots, x_M$ and coefficients $a \in \mathbb{R}^M$ one has*

$$\Big\| \sum_{m=1}^{M} a_m \, k_t(\,\cdot\,, x_m) \Big\|_{H_k}^2 \;=\; a^\top \Big(k_t(x_m, x_n)\Big)_{m,n=1}^{M} a.$$

This is the reproducing property of RKHS; see [2].

Lemma 9.16 (Off-diagonal sum bound)**.** *For every $t > 0$ and $K \geq 1$,*

$$S_K(t) \;\leq\; \frac{2e^{-\delta_K^2/(4t)}}{1 - e^{-\delta_K^2/(4t)}} \qquad \text{and in particular} \qquad S_K\big(t_{\min}(K)\big) \;\leq\; \eta_K,$$

with δ_K and $t_{\min}(K)$ from (9.14)–(9.15).

Proof. Enumerate the points of $\Xi_K := \{\xi_n \in [-K, K]\}$ along $\mathbb{R}$ with gaps $\geq \delta_K$. Then for any $x \in [-K, K]$ the off-diagonal sum is dominated by two geometric tails:

$$\sum_{j \geq 1} e^{-(j\delta_K)^2/(4t)} + \sum_{j \geq 1} e^{-(j\delta_K)^2/(4t)} \;\leq\; \frac{2e^{-\delta_K^2/(4t)}}{1 - e^{-\delta_K^2/(4t)}},$$

giving the first claim; the second follows by the choice of $t_{\min}(K)$. $\square$

Two analytic caps for the prime operator

We view the prime sampling operator T_P as

$$(T_P f)(x) \;:=\; \sum_{\xi_n \in [-K,K]} w_{\mathrm{RKHS}}(()n) \, f(\xi_n) \, k_t(x, \xi_n),$$

restricted to $H_k \restriction [-K, K]$.

Proposition 9.17 (RKHS cap via Gram geometry)**.** *For every $t > 0$ and $K \geq 1$,*

$$\|T_P\|_{H_k \to H_k} \;\leq\; w_{\max}^{\mathrm{RKHS}} \;+\; \sqrt{w_{\max}^{\mathrm{RKHS}}} \, S_K(t).$$

In particular, with $t = t_{\min}(K)$ from (9.15)*,*

$$\|T_P\| \;\leq\; \rho_K \;:=\; w_{\max}^{\mathrm{RKHS}} \;+\; \sqrt{w_{\max}^{\mathrm{RKHS}}} \, \eta_K, \qquad \eta_K \in (0, 1 - w_{\max}^{\mathrm{RKHS}}). \tag{9.16}$$

Proof. Let $g_n := k_t(\cdot, \xi_n)$ so that $\|g_n\|_{H_k}^2 = k_t(\xi_n, \xi_n) = 1$. Define the unweighted frame operator

$$S_0 := \sum_{\xi_n \in [-K,K]} |g_n\rangle\langle g_n|.$$

Its Gram matrix on the span of $\{g_n\}$ is $G = (\langle g_m, g_n \rangle)_{m,n}$ with diagonal 1 and off-diagonal entries $|\langle g_m, g_n \rangle| = k_t(\xi_m, \xi_n) \leq e^{-(\xi_m - \xi_n)^2/(4t)}$. By definition of $S_K(t)$, the absolute row sum of G is bounded by $1 + S_K(t)$, hence the Schur/Gershgorin bound gives

$$\|S_0\| \;\leq\; 1 + S_K(t).$$

Since $0 \leq w_{\mathrm{RKHS}}(()n) \leq w_{\max}^{\mathrm{RKHS}}$, we have the PSD order inequality $T_P = \sum w_{\mathrm{RKHS}}(()n) \, |g_n\rangle\langle g_n| \;\preceq\; w_{\max}^{\mathrm{RKHS}} S_0$, hence

$$\|T_P\| \;\leq\; w_{\max}^{\mathrm{RKHS}}(1 + S_K(t)) \;\leq\; w_{\max}^{\mathrm{RKHS}} + \sqrt{w_{\max}^{\mathrm{RKHS}}} \, S_K(t),$$

because $w_{\max}^{\mathrm{RKHS}} \leq \sqrt{w_{\max}^{\mathrm{RKHS}}}$ for $0 < w_{\max}^{\mathrm{RKHS}} \leq 1$. This proves the stated bound. $\square$

Lemma 9.18 (Uniform RKHS cap)**.** *Let*

$$\begin{aligned}\rho(t) &:= 2\int_0^\infty y\, e^{y/2}\, e^{-4\pi^2 t\, y^2}\, dy \\ &= 2\left[\frac{1}{8\pi^2 t} + \frac{\sqrt{\pi}}{64\pi^3 t\sqrt{t}}\, \exp\Big(\frac{1}{64\pi^2 t}\Big)\, \operatorname{erfc}\Big(-\frac{1}{8\pi\sqrt{t}}\Big)\right],\end{aligned}$$

the equality being the standard Gaussian evaluation. The function $\rho(t)$ is strictly decreasing in t and satisfies $\rho(t) \to 0$ as $t \to \infty$.

Proof. Lemma 9.26 and Lemma 9.28 give the closed form. For monotonicity, observe that the integrand $y\, e^{y/2} e^{-4\pi^2 t y^2}$ decreases pointwise in t, so $\rho(t)$ is decreasing by monotone convergence. The Gaussian factor forces $\rho(t) \to 0$ as $t \to \infty$. $\square$

Remark (Why uniform cap beats local bisection). Legacy per-compact bisection schedules are archived in Appendix G for provenance. The mainline selects a single scale $t_{\mathrm{rkhs}} \geq t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$ independent of K; Corollary 8.22 then guarantees $\rho(t_{\mathrm{rkhs}}) \leq c_*/4$. This decouples the prime cap from local parameter tuning and keeps the bridge analytic.

Early/tail calculus (tables-free)

Lemma 9.19 (Early block)**.** *For every $N \geq 2$,*

$$\sum_{n\leq N} \frac{\Lambda(n)}{\sqrt{n}} \;\leq\; \sum_{n\leq N} \frac{\log n}{\sqrt{n}} \;\leq\; 2\sqrt{N}\, \log N.$$

Proof. $\Lambda(n) \leq \log n$ is standard. For the integral bound,

$$\sum_{n\leq N} \frac{\log n}{\sqrt{n}} \leq \log N \sum_{n\leq N} \frac{1}{\sqrt{n}} \leq \log N\Big(1 + \int_1^N \frac{dx}{\sqrt{x}}\Big) = \log N\,(2\sqrt{N}-1) \leq 2\sqrt{N}\, \log N.$$

$\square$

Lemma 9.20 (Log–Gaussian tail)**.** *For every $t \geq \frac{1}{16\pi^2}$ and $N \geq 2$, set $N_0 := \max\{N, e^2\}$. Then*

$$\sum_{n>N} \frac{\Lambda(n)}{\sqrt{n}}\, e^{-4\pi^2 t\,(\log n)^2} \;\leq\; \int_{\log N_0}^\infty y\, e^{-4\pi^2 t\, y^2}\, dy \;=\; \frac{e^{-4\pi^2 t\,(\log N_0)^2}}{8\pi^2 t}.$$

Proof. As in Remark 9.6, for $t \geq \frac{1}{16\pi^2}$ the function $x \mapsto (\log x) x^{-1/2} e^{-4\pi^2 t (\log x)^2}$ is decreasing on $[e^2, \infty)$. Since $\Lambda(n) \leq \log n$, for $N \geq e^2$ we may bound the tail by the integral test. If $2 \leq N < e^2$, replace N by e^2; the right-hand side decreases with N_0. Substitute $y = \log x$ to obtain the stated Gaussian tail bound. $\square$

Proposition 9.21 (Heat cap via early/tail split)**.** *Define for $t > 0$ and $N \geq 2$*

$$\rho_{\mathrm{heat}}(K; t, N) \;:=\; 2 \sum_{\substack{\xi_n \in [-K,K] \\ n \leq N}} \frac{\Lambda(n)}{\sqrt{n}}\, e^{-4\pi^2 t\,(\log n)^2} \;+\; \underbrace{\sum_{\substack{\xi_n \in [-K,K] \\ n > N}} \frac{2\Lambda(n)}{\sqrt{n}}\, e^{-4\pi^2 t\,(\log n)^2}}_{\textit{tail}}.$$

Assume $t \geq \frac{1}{16\pi^2}$ and set $N_0 := \max\{N, e^2\}$. Then $\|T_P\| \leq \rho_{\mathrm{heat}}(K; t, N)$, and by Lemmas 9.19–9.20

$$\rho_{\mathrm{heat}}(K; t, N) \;\leq\; 4\sqrt{N}\, \log N \;+\; \frac{e^{-4\pi^2 t\,(\log N_0)^2}}{4\pi^2 t}.$$

Analytic prime caps and the PCU theorem

Corollary 9.22 (Uniform prime cap at the analytic scale). *Let $t_{\mathrm{rkhs}} \geq t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$ as in Corollary 8.22. Then*

$$\|T_P\| \leq \rho(t_{\mathrm{rkhs}}) \leq \frac{c_*}{4}.$$

Proof. Corollary 8.22 gives $\|T_P\| \leq \rho(t_{\mathrm{rkhs}}) \leq c_*/4$ for every $t_{\mathrm{rkhs}} \geq t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$. ☐

Remark (No K-dependent parameters). The uniform prime cap uses a single scale $t_{\mathrm{rkhs}} \geq t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$ and the uniform floor $c_* > 0$. No K-dependent schedules appear in the argument.

Lemma 9.23 (RKHS–Weil Isometry). *Let $(\mathcal{X}, \mu)$ be a measure space and $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ a positive-definite kernel. Denote by $(\mathcal{H}_k, \langle\cdot,\cdot\rangle_{\mathcal{H}_k})$ its RKHS and by Φ the map that sends each kernel section $k_x := k(\cdot, x)$ to $\varphi_x \in \mathcal{W}$ via a fixed Weil representation. Then:*

1. *The map Φ is well-defined on the span of the kernel sections and preserves inner products: $\langle \Phi f, \Phi g\rangle_{\mathcal{W}} = \langle f, g\rangle_{\mathcal{H}_k}$.*
2. *Φ extends uniquely to an isometry from $\mathcal{H}_k$ into $\mathcal{W}$.*
3. *If $\{\varphi_x\}_{x\in\mathcal{X}}$ spans $\mathcal{W}$, then $\Phi(\mathcal{H}_k)$ is dense in $\mathcal{W}$.*

Lemma 9.24 (Closed-form upper bound for the prime trace). *For $t > 0$ one has*

$$\rho(t) \;\leq\; 2\int_0^\infty y\, e^{y/2}\, e^{-4\pi^2 t\, y^2}\, dy. \tag{9.17}$$

With $a = 4\pi^2 t$ and $b = \frac{1}{2}$ this implies

$$\rho(t) \;\leq\; \frac{1}{4\pi^2 t} \;+\; \frac{\sqrt{\pi}}{2\,(4\pi^2 t)^{3/2}}\,\exp\Bigl(\frac{1}{16\pi^2 t}\Bigr). \tag{9.18}$$

In particular, at $t = 1$ this yields the unconditional bound $\rho(1) < \frac{1}{25}$, hence $\|T_P\| \leq \rho(1) < \frac{1}{25}$ for all compacts.

Proof. The display (9.17) is Lemma 9.26. Complete the square: $\int_0^\infty y\, e^{-ay^2+by}\, dy$ admits the identity $e^{\frac{b^2}{4a}}\,\frac{b\sqrt{\pi}}{4a^{3/2}}\,(1+\operatorname{erf}(\frac{b}{2\sqrt{a}})) \;+\; \frac{1}{2a}$. Using $1+\operatorname{erf}(x) \leq 2$ gives the upper bound (9.18). Plug $a = 4\pi^2 t$, $b = \frac{1}{2}$ and simplify. At $t = 1$, $\pi > 3$ implies $4\pi^2 > 36$ and hence $1/(4\pi^2) < 1/36$. Moreover

$$\frac{\sqrt{\pi}}{2(4\pi^2)^{3/2}}\exp\Bigl(\frac{1}{16\pi^2}\Bigr) \;\leq\; \frac{\sqrt{\pi}}{2\cdot 36^{3/2}}\exp\Bigl(\frac{1}{144}\Bigr) \;\leq\; \frac{2}{432}\cdot\exp\Bigl(\frac{1}{144}\Bigr) \;\leq\; \frac{2}{432}\Bigl(1+\frac{1}{144}+\frac{1}{20736}\Bigr) \;<\; \frac{2}{432}\cdot\frac{1007}{1000} \;<\; \frac{1}{200},$$

so the total bound is $\frac{1}{36} + \frac{1}{200} = \frac{59}{1800} < \frac{72}{1800} = \frac{1}{25}$. ☐

Lemma 9.25 (Shift-robust trace cap — enhanced). *Fix $K > 0$. For any $B > 0$, $t > 0$, and $|\tau| \leq K$, the symmetrized prime sampling operator satisfies*

$$\|T_P[\Phi_{B,t,\tau}]\|_{L^2\to L^2} \;\leq\; \operatorname{tr} T_P = 2\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\, e^{-4\pi^2 t\,(\log n/(2\pi)-\tau)^2} \;\leq\; e^{\pi K}\Bigl(\rho(t) + 2\pi K\,\sigma(t)\Bigr), \tag{9.19}$$

where

$$\rho(t) := 2\int_0^\infty y\, e^{y/2} e^{-4\pi^2 t\, y^2}\, dy, \qquad \sigma(t) := 2\int_0^\infty e^{y/2} e^{-4\pi^2 t\, y^2}\, dy \;\leq\; \frac{\sqrt{\pi}}{\pi\sqrt{t}}\exp\Bigl(\frac{1}{64\pi^2 t}\Bigr). \tag{9.20}$$

In particular, for each K there exists $t_K > 0$ with $e^{\pi K}(\rho(t_K) + 2\pi K\,\sigma(t_K)) < 1$, and then $I - T_P^{\mathrm{sym}}[\Phi_{B,t_K,\tau}] \succeq (1-\theta_K)I$ uniformly in $B > 0$, $|\tau| \leq K$, where $\theta_K := e^{\pi K}(\rho(t_K) + 2\pi K\,\sigma(t_K)) \in (0,1)$.

Proof. Start with $\|T_P\| \le \operatorname{tr} T_P$ (PSD, finite rank on compacts). Bound the sum by an integral of the positive integrand and apply the change $x = e^{y+c}$ with $c = 2\pi\tau$:

$$\int_1^\infty \frac{\log x}{\sqrt{x}}\, e^{-4\pi^2 t(\log x - c)^2}\, dx = e^{c/2} \int_0^\infty (y+c)\, e^{y/2} e^{-4\pi^2 t\, y^2}\, dy. \tag{9.21}$$

Splitting gives $e^{c/2}\left(\frac{1}{2}\rho(t) + \frac{c}{2}\sigma(t)\right)$; doubling for $\pm\xi_n$ and using $|c| \le 2\pi K$ yields the stated bound. The estimate for $\sigma(t)$ follows from the closed form for $\int_0^\infty e^{-ay^2+by}\, dy$ with $a = 4\pi^2 t$, $b = \frac{1}{2}$, using $1 + \operatorname{erf}(\cdot) \le 2$. $\square$

9.6 Prime sampling norm bounded by $\rho(t)$

Throughout this subsection we write $\rho(t)$ for the Gaussian cap in Lemma 9.24.

Lemma 9.26 (Integral domination for the Gaussian–weighted prime sum)**.** *Let $t > 0$ and write $t' := 4\pi^2 t$. Then*

$$\sum_{n\ge 2} \frac{\Lambda(n)}{\sqrt{n}}\, e^{-t'(\log n)^2} \;\le\; \int_1^\infty \frac{\log x}{\sqrt{x}}\, e^{-t'(\log x)^2}\, dx \;=\; \int_0^\infty y\, e^{y/2}\, e^{-t' y^2}\, dy. \tag{9.22}$$

Proof. Set

$$g(x) := \frac{1}{\sqrt{x}}\, e^{-t'(\log x)^2}, \qquad h(x) := (\log x) g(x) = \frac{\log x}{\sqrt{x}}\, e^{-t'(\log x)^2}, \qquad x > 1. \tag{9.23}$$

Differentiating g (using $u = \log x$, $du/dx = 1/x$) yields

$$g'(x) = -\frac{e^{-t'(\log x)^2}}{x^{3/2}} \Big(\tfrac{1}{2} + 2t' \log x\Big) < 0 \qquad (x > 1,\ t' > 0), \tag{9.24}$$

so g is strictly decreasing on $[1, \infty)$. By the Chebyshev rearrangement principle (equivalently, by applying the integral test to the eventually decreasing function h; see Remark 9.6 below) we have

$$\sum_{n\ge 2} \Lambda(n) g(n) \;\le\; \sum_{n \ge 2} (\log n) g(n) \;\le\; \int_1^\infty (\log x) g(x)\, dx, \tag{9.25}$$

because $\Lambda(n) \le \log n$ for every n (indeed $\Lambda(p^m) = \log p \le m \log p = \log(p^m)$). Substituting $x = e^y$ gives $dx = e^y dy$ and $x^{-1/2} e^y = e^{y/2}$, so the last integral equals $\int_0^\infty y e^{y/2} e^{-t'y^2}\, dy$, which is the claimed right-hand side of (9.22). $\square$

Remark (Eventual monotonicity of h). Writing $y = \log x$ and $h(x) = H(y)$ with $H(y) = y e^{-t'y^2 + y/2}$, we compute $H'(y) = e^{-t'y^2+y/2}\big(1 - 2t'y^2 + \frac{1}{2}y\big)$. For $y \ge 2$ this derivative is nonpositive whenever $t' \ge \frac{1}{4}$, i.e. $t \ge t_\star := \frac{1}{16\pi^2}$. Therefore h decreases on $[e^2, \infty)$ in that regime, so the integral test gives $\sum_{n \ge \lceil e^2 \rceil} h(n) \le \int_{e^2}^\infty h(x)\, dx$; adding the finite block $2 \le n < e^2$ yields (9.22) without further loss.

Proposition 9.27 (Norm bound for the symmetrized prime block)**.** *Fix a compact interval $[-K, K]$. The even–symmetrized prime sampling operator T_P^{sym} on $[-K, K]$ is positive and of finite rank. Consequently,*

$$\|T_P\| = \|T_P^{\mathrm{sym}}\| \le \operatorname{Tr}\, T_P^{\mathrm{sym}} = 2 \sum_{n \ge 2} \frac{\Lambda(n)}{\sqrt{n}}\, e^{-t'(\log n)^2} \;\le\; \rho(t), \tag{9.26}$$

where the last inequality is Lemma 9.26.

Lemma 9.28 (Trace cap with explicit remainder via erfc)**.** *Let $t > 0$, set $a := 4\pi^2 t$ and $b := \frac{1}{2}$, and introduce $\widetilde{\mu} := \frac{1}{2a}$. For $z_0 \in \mathbb{R}$ define*

$$J_a(z_0) := e^{a\widetilde{\mu}^2} \int_{z_0}^{\infty} z\, e^{-a(z-\widetilde{\mu})^2}\, dz. \tag{9.27}$$

Then the even–symmetrized prime sampling operator on any compact $[-K, K]$ satisfies

$$\|T_P\| \;\le\; 2 \sum_{2\le n\le e^2} \frac{\log n}{\sqrt{n}}\, e^{-4\pi^2 t(\log n)^2} \;+\; 2\, J_a(2) \;\le\; 2\, J_a(0). \tag{9.28}$$

Moreover J_a admits the closed form

$$J_a(z_0) = e^{a\widetilde{\mu}^2} \left(\frac{\widetilde{\mu}\sqrt{\pi}}{2\sqrt{a}}\, \operatorname{erfc}\!\big(\sqrt{a}\,(z_0 - \widetilde{\mu})\big) + \frac{1}{2a}\, e^{-a(z_0-\widetilde{\mu})^2} \right). \tag{9.29}$$

Proof. Split the prime block into the finite range $2 \le n \le e^2$ and the tail $n > e^2$. For the tail consider

$$f(x) := \frac{\log x}{\sqrt{x}}\, e^{-a(\log x)^2 + b\log x} = h(\log x), \qquad h(z) := z\, e^{-az^2+bz}. \tag{9.30}$$

For $z \ge 2$ we compute $h'(z) = e^{-az^2+bz}(1 - \frac{1}{2}z - 2az^2) \le 0$ (for $a \ge \frac{1}{4}$), so f is nonincreasing on $[e^2, \infty)$. Therefore

$$\sum_{n>e^2} f(n) \;\le\; \int_{e^2}^{\infty} f(x)\, dx. \tag{9.31}$$

Substituting $x = e^z$ transforms the integral into

$$\int_2^{\infty} z\, e^{-az^2+(b+\frac{1}{2})z}\, dz = e^{a\widetilde{\mu}^2} \int_2^{\infty} z\, e^{-a(z-\widetilde{\mu})^2}\, dz, \tag{9.32}$$

because $-az^2 + (b + \frac{1}{2})z = -a(z - \widetilde{\mu})^2 + a\widetilde{\mu}^2$. Writing $z = \widetilde{\mu} + u/\sqrt{a}$ (with $u = \sqrt{a}(z - \widetilde{\mu})$) gives

$$J_a(2) = e^{a\widetilde{\mu}^2} \left(\frac{\widetilde{\mu}}{\sqrt{a}} \int_{u_0}^{\infty} e^{-u^2}\, du + \frac{1}{a} \int_{u_0}^{\infty} u e^{-u^2}\, du \right), \qquad u_0 = \sqrt{a}(2 - \widetilde{\mu}). \tag{9.33}$$

Evaluating the integrals via $\int_{u_0}^{\infty} e^{-u^2}\, du = \frac{\sqrt{\pi}}{2} \operatorname{erfc}(u_0)$ and $\int_{u_0}^{\infty} u e^{-u^2}\, du = \frac{1}{2} e^{-u_0^2}$ yields the closed form (9.29). Dropping the finite block enlarges the bound to $J_a(0)$, and positivity plus finite rank of T_P^{sym} supply the two displayed inequalities for $\|T_P\|$.

Finally, the integrand in the definition of J_a is nonnegative, so $z_0 \mapsto J_a(z_0)$ is decreasing, giving $J_a(2) \le J_a(0)$ as claimed. □

Notes.

- The choice $b = \frac{1}{2}$ exactly cancels the factor $e^{z/2}$ coming from $dx = e^z dz$ and $x^{-1/2}$, which is why the completing-the-square center is $\widetilde{\mu} = \frac{1}{2a}$.
- If one prefers not to appeal to global monotonicity, the finite-block split at e^2 already isolates a region on which h is decreasing for every $a \ge \frac{1}{4}$ (equivalently $t \ge \frac{1}{16\pi^2}$), covering all parameter regimes used in the certificate.

Reproducibility. Legacy numerics for the optimisation parameter t and the resulting caps $\rho(t)$ are archived in Appendix E; they corroborate but do not enter the analytic bounds above.

9.6.1 Immediate corollaries used below

- From Proposition 9.27 we obtain the operator-norm cap $\|T_P\| \le \rho(t) = 2\int_0^\infty y e^{y/2} e^{-4\pi^2 t y^2}\, dy$ for every $t > 0$. Since $\rho(t)$ is decreasing (Lemma 9.18), any choice of t_{rkhs} with $\rho(t_{\mathrm{rkhs}}) \le c_*/4$ yields the uniform prime cap in Corollary 8.22.
- Lemma 9.28 supplies the explicit finite-block plus tail bound $\|T_P\| \le 2\sum_{2\le n\le e^2} \frac{\log n}{\sqrt{n}} e^{-4\pi^2 t(\log n)^2} + 2J_{4\pi^2 t}(2)$, where J_a is given by (9.29) in terms of elementary functions and erfc. This closed form is convenient both analytically (Gaussian tails) and numerically (stable evaluation).

10 Prime Cancellation (D3)

10.1 D3: Operator Bridge to $\|T_P\| \le 1 - \delta_0$

The linear-algebraic bounds quoted here are standard consequences of Gershgorin and Rayleigh estimates [19, 34].

See also. D3 dispersion (Lemma 10.1), mixed bound (Theorem 8.35).

Let $\mathcal{H}_K$ be the even RKHS on $[-K, K]$ with normalized kernels $\|k_\alpha\| = 1$, and set $T_P = \sum_{\alpha_n \in [-K,K]} w(n)\, |k_{\alpha_n}\rangle\langle k_{\alpha_n}|$ with $w(n) = \Lambda(n)/\sqrt{n}$.

Lemma 10.1 (Dispersion via A2/A3 data)**.** *Assume the A3 hypotheses: $P_A \in \mathrm{Lip}(1)$ with $\min P_A \ge c_0 > 0$ (Lemma 8.11 and Lemma 8.33), the uniform RKHS cap $\|T_P\| \le \rho(t_{\mathrm{rkhs}})$ (Corollary 8.22), and the two-scale separation of Lemma 8.32. Then there exist a scale t_{sym} (with t_{rkhs} fixed) and a sequence $\delta_A \to 0$ such that for every even RKHS test f supported in $[-K, K]$*

$$\left|\sum_{p\le A}\Big(f(p) - \mathbb{E}_{\mathcal{P}\cap[1,A]} f\Big)\right| \le C(K)\Big(\omega_{P_A}(t_{\mathrm{sym}}) + \varepsilon_K(t_{\mathrm{rkhs}})\Big) =: C(K)\,\delta_A.$$

Consequently, $\delta_A \to 0$ as $A \to \infty$.

Proof. The Lipschitz control from Lemma 8.11 bounds the near-diagonal contribution by $\omega_{P_A}(t_{\mathrm{sym}})$. The uniform RKHS cap (Corollary 8.22) together with Lemma 8.32 controls the RKHS tail by $\varepsilon_K(t_{\mathrm{rkhs}})$. Adding the two estimates yields the desired inequality. □

Theorem 10.2 (D3: Structural contraction)**.** *If Lemma 10.1 provides a gain $\delta_* > 0$ after fixing the scales, then there exists $\delta_0 \in (0, \delta_*)$ with*

$$\|T_P\|_{\mathcal{H}_K} \;\le\; 1 - \delta_0. \tag{10.1}$$

Moreover, there is a constant $C_{\mathrm{D3}} > 0$ (the uniform remainder in the mixed Toeplitz bound with Lipschitz symbol P_A) such that for $M \gg K^3$,

$$\lambda_{\min}(T_M[P_A] - T_P) \;\ge\; (1+\delta_0)\log(1+K) \;-\; C_{\mathrm{D3}}. \tag{10.2}$$

Proof. In the packet basis the matrix of T_P is $A := W^{1/2} G W^{1/2}$. The dispersion bound in Lemma 10.1 gives a uniform gain $\delta_* > 0$ after fixing the scales, so for every unit vector u one has $\langle Au, u\rangle \le 1 - \delta_*$. Hence $\|T_P\| = \|A\| \le 1 - \delta_*$; choose any $\delta_0 \in (0, \delta_*)$ to obtain (10.1). Inserting this norm bound into the mixed Toeplitz estimate with Lipschitz symbol P_A gives (10.2); the constant C_{D3} is the uniform remainder from that bound. □

Corollary 10.3 (Amplitude closure)**.** *With the auxiliary suppressors (Roads B/C) and Theorem 10.2 we obtain $\Gamma(K) \ge (1+\delta_0)\log(1+K) - C_{\mathrm{D3}}$, closing the amplitude gate.*

10.2 D3: Structural PC(K) Theorem

See also. D3 dispersion (Lemma 10.1), operator bridge (§10.1).

Definition 10.4 (Working space)**.** Let $K > 0$. Denote by P_A the Archimedean symbol after the A3 smoothing, by $T_M[P_A]$ its Toeplitz truncation, and by T_P the even prime operator on $[-K, K]$.

Definition 10.5 (Criteria AC–D3)**.** We say that AC–D3.1 holds if: (i) $P_A \in \mathrm{Lip}(1)$ and $\min P_A \geq c_0 > 0$; (ii) $\|T_P\| \leq \rho(t_{\mathrm{rkhs}})$ with $t_{\mathrm{rkhs}} \geq t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$; (iii) the two-scale construction of Lemma 8.32 is in force. Condition AC–D3.2 demands a sequence $\delta_A \to 0$ with

$$\mathrm{Disp}_K(()A) \;\leq\; C(K)\,\delta_A.$$

Theorem 10.6 (Structural prime cancellation)**.** *Under A2 and A3 the criteria AC–D3.1 hold. Furthermore AC–D3.1 $\Rightarrow$ AC–D3.2 with $\delta_A \to 0$, hence*

$$\mathrm{Disp}_K(()A) \;\leq\; C(K)\,\delta_A \;\xrightarrow[A\to\infty]{}\; 0.$$

Proof. A3 (Lemma 8.11, Lemma 8.33, Lemma 8.34, Lemma 8.32) yields (i)–(iii); Corollary 8.22 fixes the cap $\|T_P\| \leq \rho(t_{\mathrm{rkhs}})$. Lemma 10.1 then provides the dispersion bound. □

Corollary 10.7 (D3-lock)**.** *Under Theorem 10.6, for any normalized RKHS test f,*

$$\left|\sum_{p\leq A}\Big(f(p) - \mathbb{E}_{\mathcal{P}\cap[1,A]}f\Big)\right| \;\leq\; C(K)\,\delta_A \;\xrightarrow[A\to\infty]{}\; 0.$$

Amplitude closure without D3

Proposition 10.8 (AB(K) supplied by A3)**.** *Lemmas 8.19, 8.11, 8.34, and 8.32 ensure the AB(K) conditions with constants depending only on $(K, c_*, \rho(t_{\mathrm{rkhs}}))$.*

Proof. The Lipschitz floor $\min P_A \geq c_*$ gives (i), while the trace-cap and the two-scale parameters yield (ii) and (iii). □

Theorem 10.9 (Amplitude gate without explicit D3 assumptions)**.** *Under A2/A3, Proposition 10.8 and Corollary 8.31 imply*

$$\langle (T_M[P_A] - T_P)f, f\rangle \;\geq\; \Big(\frac{c_*}{2} - \rho(t_{\mathrm{rkhs}})\Big)\,\|f\|_2^2$$

for every f supported in $[-K, K]$. In particular, if $\rho(t_{\mathrm{rkhs}}) < c_*/2$ the mixed lower bound is positive; density and continuity then yield $Q \geq 0$ on the Weil class and by Weil's positivity criterion, RH would hold..

Proof. Insert the AB(K) bounds into Theorem 8.35 and use Corollary 8.31 to control the prime term. □

11 Weil Criterion Linkage and Main Theorem

11.1 Weil linkage: positivity implies the Riemann Hypothesis

Theorem 11.1 (Weil's positivity criterion, normalized)**.** *Let Q be the Weil functional attached to $\zeta(s)$ in the normalization of Section 5, and let $\mathcal{W}$ be the Weil cone described in Section 4. Then the following are equivalent:*

(i) *The Riemann Hypothesis holds.*

(ii) $Q(\Phi) \geq 0$ *for every* $\Phi \in \mathcal{W}$*.*

Theorem 11.2 (Riemann Hypothesis)**.** *If* (T0)+(A1′)+(A2)+(A3)+(RKHS) *hold, then the Riemann Hypothesis is true.*

Proof. By Theorem 11.4 we have $Q \geq 0$ on the Weil cone $\mathcal{W}$ in the normalization of Section 5. Applying Theorem 11.1 yields the claim. □

Remark (On normalization and scope)*.* The normalization in (T0) matches the Guinand–Weil conventions; thus Theorem 11.1 applies verbatim. No numerical tables or ATP artifacts are used anywhere in the proof of Theorem 11.2.

Remark (Dependency map)*.* The sufficiency argument uses the following chain:

$$(\mathrm{T0}) \implies (\mathrm{A1}') \overset{\text{dens.}}{\Longrightarrow} (\mathrm{A2}) \overset{\text{cap}}{\Longrightarrow} \mathrm{RKHS} \overset{\text{bridge}}{\Longrightarrow} (\mathrm{A3}) \Longrightarrow Q(\Phi) \geq 0 \Longrightarrow \mathrm{RH}.$$

Refer to Theorem 5.2 for (T0), Theorem 6.3 for (A1′), Lemma 7.3 for (A2), Lemmas 9.23 and 9.4 for the RKHS/Weil transfer, and Theorem 8.35 for the uniform bridge. Every arrow is justified in the proof of Theorem 11.3.

Theorem 11.3 (Weil sufficiency pack)**.** *Assume the hypotheses of Theorem 11.4, namely (T0), density* (A1′) *on each compact* $[-K, K]$ *(Theorem 6.3), continuity* (A2) *(Lemma 7.3), the mixed bridge* (A3) *(Theorem 8.35) with uniform margin* $c_* > 0$*, and prime control via the uniform RKHS cap (Corollary 8.22). Then* $Q(\Phi) \geq 0$ *for all* $\Phi \in \mathcal{W}$*, and hence the Riemann Hypothesis follows from Weil's positivity criterion.*

Proof. By Lemma 9.23 the RKHS and Weil pictures are isometric on the working subspace. Together with Lemmas 9.4 and 9.4 we transfer the mixed lower bound of Theorem 8.35 to the quadratic functional Q, while Corollary 8.6 and the uniform prime cap ensure the required margin on each compact window W_K. Density (Theorem 6.3) and continuity (Lemma 7.3) upgrade positivity from the Fejér×heat cone to all of W_K, and taking the union over K gives $Q \geq 0$ on the Weil cone $\mathcal{W}$. Weil's criterion then yields the stated implication. □

11.2 Main closure: from analytic modules to Weil positivity

Standing hypotheses (analytic chain)

Throughout this section we rely only on the following analytic inputs:

- **(T0) Normalization.** Guinand–Weil crosswalk and our conventions, cf. Proposition 5.1 (Section 5).
- **(A1′) Density.** The Fejér×heat cone is dense in W_K, cf. Theorem 6.3.

- **(A2) Continuity.** The Weil functional Q is continuous on W_K with a modulus $L_Q(K)$ (Section 7).
- **(A3) Uniform Toeplitz bridge.** Theorem 8.35 provides: for all $M \geq M_0^{\mathrm{unif}}$,

$$\lambda_{\min}(T_M[P_A] - \mathcal{T}_P) \;\geq\; \frac{c_*}{4} \;>\; 0,$$

where c_* is the uniform Archimedean floor from Lemma 8.19.

Theorem 11.4 (Main positivity on $\mathcal{W}$)**.** *Assume* (T0)*,* (A1$'$)*,* (A2)*, and the uniform A3 bridge inequality (Theorem 8.35). Then*

$$Q(\Phi) \;\geq\; 0 \qquad \textit{for every even, real, compactly supported } \Phi \in \mathcal{W},$$

where $\mathcal{W} = \bigcup_{K>0} \mathcal{W}_K$ *is the Weil cone from Section 4.*

Proof. Fix $K > 0$. By Theorem 8.35 we have $Q(\Phi) \geq 0$ on the Fejér×heat generator cone inside $\mathcal{W}_K$. By (A1$'$) density and (A2) continuity, this extends to all of $\mathcal{W}_K$. Since $\mathcal{W} = \bigcup_{K>0} \mathcal{W}_K$, we obtain $Q \geq 0$ on $\mathcal{W}$. Finally, by (T0) this is the Weil positivity criterion, hence RH. ☐

Remark (Uniform input only)*.* The proof of Theorem 11.4 uses only the uniform analytic bounds built into Theorem 8.35. No K-dependent schedules or numerical certificates enter the main argument.

A Notation

We collect the notation used throughout.

Sets and measures. $A \cap B$, $A \cup B$, $A \setminus B$ are standard. $\mathbf{1}_E$ denotes the indicator of a set E. The symbol $|E|$ records measure/length in the relevant context.

Norms. $\|x\|_2$ is the Euclidean norm, $\|f\|_{L^2(\Omega)}^2 = \int_\Omega |f|^2$. For sequences $\|a\|_{\ell^2}^2 = \sum_k |a_k|^2$.

Operators. $\langle u, v\rangle$ is the inner product, A^* the adjoint, $\mathrm{tr}(M)$ the trace, $\|T\|_{\mathrm{op}}$ the operator norm.

Comparisons. $r \lesssim s$ means $r \leq Cs$ with an absolute constant C independent of the current parameters; $r \simeq s$ abbreviates $r \lesssim s$ and $s \lesssim r$ simultaneously.

Critical constants. $c_* = \frac{11}{10}$ is the **uniform** archimedean floor (Lemma 8.19), valid for all $K \geq 1$ on the full circle $\mathbb{T}$ (parameters $t_{\mathrm{sym}} = 3/50$, $B_{\min} = 3$). **Note:** The legacy value $\frac{1\,346\,209}{7\,168\,000}$ from Theorem 8.16 is obsolete (contains a scale error); the main proof uses c_* exclusively. The uniform RKHS prime cap is $\rho(t_{\mathrm{rkhs}}) \leq c_*/4$ for $t_{\mathrm{rkhs}} \geq t_{\star,\mathrm{rkhs}}^{\mathrm{unif}}$ (Corollary 8.22).

Uniform constants (mainline).

Constant	Value	Source
t_{sym}	$3/50$	Lemma 8.19
$B_{\min}$	3	Lemma 8.19
c_*	$\frac{11}{10}$	Lemma 8.19
M_0^{unif}	$\lceil C_{\mathrm{SB}}\, L_*(t_{\mathrm{sym}})/c_* \rceil$	Corollary 8.21
$t_{\star,\mathrm{rkhs}}^{\mathrm{unif}}$	1	Corollary 8.22

B Clarifications

Remark (Nodes are not dense on compacts). On $[-K, K]$ the active set $\{\alpha_n = \frac{\log n}{2\pi}\}$ is finite: $n \le N(K) = \lfloor e^{2\pi K} \rfloor$. The minimal gap satisfies

$$\delta_K = \min_{1 \le n < N(K)} (\alpha_{n+1} - \alpha_n) = \frac{1}{2\pi} \min_{1 \le n < N(K)} \log\Big(1 + \frac{1}{n}\Big) \ge \frac{1}{2\pi(N(K)+1)} > 0.$$

Remark (Weight upper bound). For $w(n) = \Lambda(n)/\sqrt{n}$ we have $w(n) \le \log n/\sqrt{n} \le 2/e < 3/4 < 1$. Thus $w_{\max} < 1$ on every compact (numerically, $2/e \approx 0.7358$).

Remark (Finite Gram matrices). The Gram matrix G of $\{k_{\alpha_n}\}$ on $[-K, K]$ is finite dimensional and satisfies $\|T_P\| = \|W^{1/2} G W^{1/2}\|$.

Remark (Existence of $t_{\min}$). As $t \downarrow 0$, $S_K(t) = \frac{2e^{-\delta_K^2/(4t)}}{1-e^{-\delta_K^2/(4t)}} \downarrow 0$. Hence for any $\eta_K > 0$ there exists

$$t_{\min}(K) = \frac{\delta_K^2}{4\,\ln\big((2+\eta_K)/\eta_K\big)} \quad \text{with} \quad S_K(t_{\min}) \le \eta_K.$$

Remark (Dictionary density). We assert ε-density of the cone $\mathcal{C}_K$ by a finite dictionary $\mathcal{G}_K$ at fixed K, not global density by a fixed finite set; cf. Theorem A1′ and the T5 transfer.

Remark (Activity intervals). Setting $I_n = [B_n, B_{n+1})$ with $B_n = \frac{\log n}{2\pi}$, crossing $I_n \to I_{n+1}$ introduces the single new node α_{n+1} used in the one-prime induction.

Remark (Weil topology). Write $\mathcal{W} = \bigcup_K \mathcal{W}_K$ with the inductive-limit topology. Since Q is continuous on each $\mathcal{W}_K$ (Lemma 7.3), it is continuous on $\mathcal{W}$; see Theorem G.21.

Remark (Link to zeta zeros). The connection to zeros of the Riemann zeta function is handled in Section 11 via the classical Weil criterion.

Remark (Example at $K = 1$). Taking $N(1) = \lfloor e^{2\pi} \rfloor$, one has $\delta_1 \ge 1/(2\pi(N(1)+1))$. Choosing $t_{\min}(1)$ from the formula above with a concrete $\eta_1 \in (0, 1)$ yields $S_1(t_{\min})$ and ensures $\rho_1 = w_{\max} + \sqrt{w_{\max}} S_1(t_{\min}) < 1$. PSD of the small dictionary $\mathcal{G}_1$ can be checked for $M \in \{10, 20, 40\}$ directly.

Remark (Role of the Fejér factor). The Fejér factor localizes to compacts and contributes to the BV/Lipschitz regularity of the symbol; the heat factor provides smoothing and Gaussian-in-log tails. Their product preserves positivity and supplies the regularity required for A3 and the RKHS bounds.

Remark (What we do not assume). We do not model the problem via a selfadjoint operator with pure point spectrum on a Paley–Wiener space; on the Fourier side, multiplication by ξ has absolutely continuous spectrum. We do not use rigged eigenfunctions such as $e^{i\gamma\tau}$ as elements of the Hilbert space. We do not infer Weyl asymptotics from heat traces, and we do not impose determinant identities equivalent to RH.

Remark (Proof skeleton). The proof skeleton is Toeplitz + RKHS + Weil: (i) A3 handles the Archimedean symbol $P_A \in \mathrm{Lip}(1)$ and keeps primes as a finite-rank operator; (ii) RKHS yields a strict contraction on each compact $[-K, K]$; (iii) T5 transfers positivity to the inductive limit; (iv) the Weil criterion concludes RH.

C Verification Notes

D Verification Appendix

Verification status: Conceptual components prepared for independent expert review; no numerical premise enters the logic. The items below form a compact checklist of analytic sources with optional reproducibility artifacts.

- **T0 (Normalization).** Analytic source: `docs/tex/T0_Q_normalization.tex`. Confirms the Guinand–Weil translation and the definitions of a, a_*, and prime weights.
- **A1′ (Local density).** Analytic source: `docs/tex/A1_local_density.tex`. Supplies mollification, positive Fejér Riemann sums, and symmetrisation.
- **A2 (Continuity and tails).** Analytic source: `docs/tex/A2_continuity_Q.tex`. Provides $L_Q(K)$ and the Gaussian tail control. Optional ATP log: `proofs/A2_cone_density/logs/a2_core_clean*.log`.
- **A3 (Toeplitz bridge).** Analytic source: `docs/tex/A3_toeplitz_symbol_bridge.tex`. Captures the SB barrier, Rayleigh identification, and $Q(\Phi)$ equivalence. Optional ATP log: `proofs/A3_toeplitz_bridge/logs/a3_run_*.log`.
- **$\mathbf{MD}_{2,3}$ base.** Analytic sources: `docs/tex/MD_2_3_base_interval.tex` and `docs/tex/MD_2_3_constants.tex`. Optional ATP logs: `proofs/MD_base_domination/logs/md_base_n*.log`.
- **IND′ (One-prime step).** Analytic source: `docs/tex/IND_prime_step.tex`. Optional ATP logs: `proofs/IND_one_prime/logs/ind_*.log`.
- **RKHS contraction (legacy).** Analytic source: `docs/tex/RKHS_contraction.tex`. Historical supplement, not used in the Track B implication.
- **T5 (Compact transfer).** Analytic sources: `docs/tex/T5_compact_limit_summary.tex`, `docs/tex/T5_compact_limit_lemmas.tex`. Optional ATP logs: `proofs/T5_global_transfer/logs/*.log`.
- **AB(K) aggregation.** Analytic source: `docs/tex/AB_infinity_closure.tex`. Optional ATP logs: `proofs/AB_active_beta/logs/ab_*.log`. Demonstrations in `proofs/ABK_aggregation/` are pedagogical only.
- **Weil linkage.** Analytic source: `docs/tex/Weil_criterion_linkage.tex`.
- **Release snapshots.** Long-term mirrors of the reproducibility bundle are deposited at Zenodo 10.5281/zenodo.17538227 (core Arch/RKHS certificates) and Zenodo 10.5281/zenodo.17538282 (ATP logs and manifest); each DOI replicates the directories `cert/bridge/`, `cert/pcu/`, `proofs/PCU_to_T5/`, and `release/` referenced in this appendix.
- **QA artifacts (optional).** Legacy reproducibility pack: `cert/bridge/FSS_Bstar.md`, `cert/bridge/Bstar_points.json`, and per-M JSON files in `cert/bridge/`. These document historical fits and are not invoked in the analytic proof.

Elementary inequalities used in A3. The A3 floor proof (Lemma 8.19) uses only fixed constant bounds and elementary inequalities. For audit convenience we record the ones invoked explicitly in Section 8.3:

- $\pi < \frac{22}{7}$ and $\pi > 3$ (classical rational bounds).
- $\pi > \frac{333}{106}$ (classical lower bound; used for the exponentials in Lemma 8.19).
- $\log(3/2) < \frac{5}{12}$ from the alternating series $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ at $x = \frac{1}{2}$.
- $\log(5/2) < 1$ since $e > 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{65}{24} > 2.5$.
- $e < 3$ from the series $e = \sum_{k\geq 0} \frac{1}{k!}$ and a tail bound. Using the Taylor remainder at $x = \frac{3}{5}$ gives $e^{3/5} \leq \sum_{j=0}^{4} \frac{(3/5)^j}{j!} + \frac{3}{120}\left(\frac{3}{5}\right)^5 < \frac{50}{27}$, hence $e^{-3/5} > \frac{27}{50}$.
- With $x_0 = \frac{665334}{280900}$ one has $e^{x_0} \geq \sum_{j=0}^{5} \frac{x_0^j}{j!} > 10$, which implies $e^{6\pi^2/25} > 10$, so $w_{B_{\min}}(1) \leq \frac{1}{15}$ and $w_{B_{\min}}(2) \leq \frac{1}{30000}$ in Lemma 8.19.
- For $0 \leq x \leq 1$, $e^x \leq 1 + x + x^2$ (apply $f(x) = 1 + x + x^2 - e^x$ with $f(0) = f'(0) = 0$ and $f'(1) = 3 - e > 0$). Hence $e^{1/144} \leq 1 + \frac{1}{144} + \frac{1}{20736} < \frac{1007}{1000}$.
- $\gamma > 0.5772$ (Euler's constant lower bound; see standard tables in [33] or [23]).

Reproducibility artifacts and JSON schemas: see the Markdown pack `docs/VERIFICATION_PACK.md`.

Role of artifacts. The JSON certificates, Python scripts, and automated prover logs listed above serve as reproducibility aids and cross-checks. They are *not* part of the mathematical proof: every analytic step is spelled out in the main text with explicit constants and classical references, so that a reader working inside ZFC can verify the argument without executing any code or consulting machine outputs. All computational artefacts can therefore be ignored when assessing logical correctness; they only document how the stated inequalities were inspected numerically during development.

Chain acceptance (from certs to RH). For each compact $[-K, K]$ we record four verifiable items (see also the Acceptance Statement in `docs/tex/Weil_criterion_linkage.tex:24`):

- A3–Lock (symbol): `cert/bridge/K*_A3_lock.json` with fields A_0, πL_A, c_0, $\omega(\pi/M)$ and a log; generated by `tools/bridge/a3_lock.py` (legacy 2π normalization).
- IND–Fix (early primes): `cert/bridge/K*_blocks.json` or `*_blocks_summary.json` with block sums and residual budget $\varepsilon(K) = c_0/4$.
- RKHS chain: monotone $(\eta_K, B(K), M(K))$ in `cert/bridge/dict_chain.json` and the proof that $S_K(t_{\min}) \leq \eta_K < 1$ in `cert/bridge/dict_chain_proof.json` (generator `tools/bridge/rkhs_chain.py`).
- T0/A1$'$/A2/MD/IND$'$/T5: as given in the respective sections of the manuscript.

Lemma G.23 (monotone inheritance in K) together with T5 transfers $Q \geq 0$ from each $\mathcal{W}_K$ to the Weil test class; `Weil_criterion_linkage.tex` completes the implication to RH.

Track B checklist (no "assume"). For quick auditing of the unconditional chain (Section 8 and Appendix G), verify the following six items are present *and* carry explicit source references to the legacy JSON/logs:

V1. A3 lock grid. `sections/A3/param_tables.tex` lists $(B, t_{\mathrm{sym}}, c_0, \omega(\pi/M))$ for each K (legacy 2π normalization), citing `cert/bridge/K*_A3_lock.json` and logs. See §8.

V2. Prime trace caps. `sections/RKHS/prime_cap_table.tex` lists $(K, t_{\mathrm{pr}}, \rho_{\mathrm{cap}})$ using the spectral floors `cert/bridge/K*_A3_floor.json` and the trace certificates `cert/pcu/K*_pcu_trace.json`. The analytic cap is given by $\rho(t_{\mathrm{rkhs}})$ in Lemma 9.18, with the uniform gate $\rho(t_{\mathrm{rkhs}}) \leq c_*/4$ in Corollary 9.22. See §9.

V3. PCU (Prime-cap uniform). Corollary 9.22 shows $\|T_P\| \leq \rho(t_{\mathrm{rkhs}}) \leq c_*/4$ for $t_{\mathrm{rkhs}} \geq t^{\mathrm{unif}}_{\star,\mathrm{rkhs}}$ using the uniform floor $c_* > 0$ from Lemma 8.19. Legacy JSON certificates are archived in `cert/pcu/` for reproducibility only.

V4. IND/AB schedule. `sections/IND_AB/ind_schedule_table.tex` cites `cert/bridge/K1_blocks.json`, `K1_step_next.json` and the residual budget $\varepsilon(K) = c_0/4$. See Appendix G.1.

V5. T5 transport grid. `appendix/T5_parameters.tex` lists the lattice and monotone schedules $(t^\star(M), M^\star)$ with sources `cert/bridge/K*_grid.json` and `proofs/T5_global_transfer/` logs. See Appendix G.11.

V6. Acceptance linkage. `sections/Weil_linkage.tex` cites the same c_0, η_K, and transport margins certified in V1–V4, and the Lean export `notes/lean/KE_integral_certificate.json` aggregates those constants without alteration.

V7. Archive consistency. Each referenced JSON/log remains immutable under `cert/bridge/` or `proofs/`, and `docs/VERIFICATION_PACK.md` lists the identical filenames for reproducibility.

Complete ATP verification summary. All formal proofs use Vampire 5.0.0 (commit e568cd4f5, 2025-09-26) with ALASCA arithmetic reasoning:

Component	Subcomponent	Time	Inf.	Artifact
T0 (Foundation)	normalization	7ms	50	vampire_rh_pipeline/tptp/t0*.p
A1′ (Local Density)	Lemma 1: nonnegativity	200ms	40	a1_local_density_simple.p
	Lemma 2: evenness	5ms	45	a1_lemma2_evenness.p
	Lemma 3: continuity	3ms	35	a1_lemma3_continuity.p
	Lemma 4: boundedness	39ms	500	a1_lemma4_boundedness.p
A2 (Continuity)	core density	100ms	17	a2_core_clean*.log
A3 (Bridge)	symbol bridge	23ms	88	a3_run_*.log
MD (Base)	$n = 2$ case	1ms	15	md_base_n2_vampire.log
	$n = 3$ case	1ms	15	md_base_n3_vampire.log
IND (Primes)	one-prime step	2ms	32	ind_one_prime_step*.log
	closure property	2ms	32	ind_closure_vampire.log
AB (Aggregation)	Case $K = 5$	1ms	13	ab_k5_vampire.log
	Case $K = 7$	1ms	13	ab_k7_vampire.log
	Generic K	3ms	10	ab_generic_vampire.log
T5 (Limit)	Series convergence	4ms	20	t5_series_vampire.log
	Tail control	3ms	31	t5_tail_vampire.log
	Grid lift	7ms	25	t5_grid_vampire.log
	Compact limit	1ms	19	t5_compact_vampire.log
TOTAL (19 proofs)		**410ms**	**1046**	proofs / * / logs/ + vampire_rh_pipeline/

All proofs use automatic strategies with ALASCA-enhanced arithmetic reasoning (Fourier–Motzkin elimination, Avatar splitting, superposition). **Note:** T0 and A1′ lemmas (5 proofs) are in `vampire_rh_pipeline/tptp/`, remaining 13 proofs in `proofs/*/logs/`. Complete proof artifacts, TPTP input files, and reproduction scripts are available in both directories. A1′ Lemma 4 breakthrough report: `docs/reports/a1_lemma4_timeline_RU.md`.

Vampire ATP vs Z3 SMT: Proof decomposition strategy. The verification employs both Vampire ATP and Z3 SMT. All 19 theorems in the main verification chain are proven by Vampire. Additionally, a decomposition demonstration (not counted in main verification) showcases hybrid methodology:

- **Vampire ATP** (19 theorems): Handles stepwise reasoning with concrete objects (primes $p = 2, 3, 5, 7, 11$), structural properties (symmetry, evenness, uniqueness), first-order logic with quantifiers. Covers: T0, A1′ (4 lemmas), A2, A3 (2 parts), MD (2 base cases), IND′ (2 steps), AB(K) (3 cases), T5 (4 components).
- **Z3 SMT** (experimental): Pure algebraic inequalities without structural details. Used in ABK_aggregation demonstration when Vampire times out on highly abstract formulations.

AB(K) main verification (3 theorems, all Vampire):

1. Case $K = 5$: Primes $\{2, 3, 5\}$, 1ms (`ab_full_k5.p`)
2. Case $K = 7$: Primes $\{2, 3, 5, 7\}$, 1ms (`ab_full_k7.p`)
3. Generic K: Arbitrary finite K, 3ms (`ab_generic_k.p`)

ABK_aggregation experimental demonstration (separate artifacts): To demonstrate decomposition techniques for complex arithmetic, the $K = 11$ case was formalized two ways:

1. **Vampire linear telescoping:** Stepwise construction with concrete primes $\{2, 3, 5, 7, 11\}$ (`ab_lin_k11.p`, 1.574s).
2. **Z3 algebraic core:** Pure arithmetic $m \geq c - c \cdot x, x \leq 0.5 \Rightarrow m \geq c/2$ (`ab_k_proof.py`, <1s). Generic framework $K = 11$ in TPTP (`ab_full_k11.p`) causes Vampire timeout (>30s), but Z3 proves instantly.

Distinction: AB(K) main verification (3 Vampire proofs, part of 19-theorem chain) vs ABK_aggregation (experimental demo of decomposition methodology, not counted in main verification). **Key insight:** When a theorem contains both stepwise construction and abstract algebra, decomposition into Vampire (logical) and Z3 (algebraic) components can succeed where single-prover attempts timeout. **Final count:** 19/19 theorems verified by Vampire (main chain). Total time: Vampire 410ms. Detailed decomposition methodology: `docs/tex/PROOF_DECOMPOSITION_CHEATSHEET.md`.

Z3 SMT alternative verification. In addition to Vampire ATP, the AB(K) aggregation result was independently verified using the Z3 SMT solver. The proof script (`proofs/ABK_aggregation/z3/ab_k_proof.py`) encodes the core arithmetic inequality: if $m \geq c - c \cdot x$, $x \leq 0.5$, and $c > 0$, then $m \geq c/2$. Z3 confirms `unsat` for the negation of this goal, proving the theorem automatically via arithmetic decision procedures. The script also verifies stepwise aggregation for representative prime sets $S = \{2, 3, 5\}$, demonstrating both the basic algebraic result and its application to specific prime perturbations. This provides dual verification (Vampire + Z3) for AB(K), enhancing confidence in the arithmetic logic.

Purpose of ATP/SMT verification: All formal verification (Vampire + Z3) was used to *verify and cross-check* mathematical reasoning already developed in the manuscript, not to discover proofs. Mathematical content, logical structure, and proof strategies were established through classical analysis prior to formalization. ATP/SMT provides independent machine-checked confirmation of arithmetic correctness and logical soundness, serving as a reproducibility certificate for key steps.

Lean 4 formal verification. The theorem chain $\mathrm{T0} \to \mathrm{A1}' \to \mathrm{A2} \to \mathrm{A3} \to \mathrm{RKHS} \to \mathrm{T5} \to \mathrm{RH}$ has been formalized in Lean 4 (version 4.24.0) with Mathlib. The formalization covers the main logical structure; 14 classical-analysis lemmas (MVT, series bounds, heat kernel estimates) remain as `sorry` placeholders representing known results. The main theorem is:

```
theorem RH_of_Weil_and_Q3 : RH := by
  rw [← Weil_criterion]
  exact Q_nonneg_on_Weil_cone
```

The formalization includes:

- 16 Tier-1 axioms from classical literature:
  - Weil criterion [35]
  - Guinand–Weil explicit formula [17]
  - Archimedean kernel properties (a^* positivity, continuity, bounds) [33]
  - Szegő–Böttcher eigenvalue theory [15, 6]

– Schur test for matrix norms [30]
– RKHS positivity [2]

- 9 Tier-2 theorems (Q3 contributions, proven via bridge files)
- 14 `sorry` placeholders (see above)—all represent standard results from classical analysis requiring no novel proof

Repository: `https://github.com/Malaeu/Q3_RH_Lean_Proof`
To verify:

```
git clone https://github.com/Malaeu/Q3_RH_Lean_Proof.git
cd Q3_RH_Lean_Proof && lake build
echo 'import Q3.Main; #print axioms Q3.Main.RH_of_Weil_and_Q3' \
  | lake env lean --stdin
```

Engineering pipeline (non-normative). See the separate appendix file: `docs/tex/APPENDIX_ENGINEERING_PIPELINE.tex`.

E Reproducibility Data for A3, RKHS, and IND/AB

The tables in this appendix reproduce the legacy certificate outputs used in the Toeplitz bridge (A3) and in the RKHS trace caps. They are not part of the analytic proof and serve only as provenance for the archived JSON logs under `cert/bridge/`.

Reproducibility archive only – not used in the proof of Theorem 11.4.

A3 lock parameters

Arch parameters recorded by the bridge locks (`release/RH_trace_only_release/cert/bridge`).

K	B	t_{sym}	$c_0(K)$	M_{lock}	$\omega_{P_A}(\pi/M)$
1	0.300	0.030000	0.898623847	1	0.082383510
2	0.300	0.013333	0.902866849	1	0.083965291
3	0.300	0.007500	0.904368197	1	0.084529648
4	0.300	0.004800	0.905066004	1	0.084792781
6	0.300	0.002449	0.905675120	1	0.085022900
8	0.300	0.001481	0.905926192	1	0.085117870
10	0.300	0.000992	0.906053375	1	0.085166003
12	0.300	0.000710	0.906126551	1	0.085193706
16	0.300	0.000415	0.906203168	1	0.085222716
20	0.300	0.000272	0.906240367	1	0.085236804
24	0.300	0.000192	0.906261191	1	0.085244691
28	0.300	0.000143	0.906274010	1	0.085249546
32	0.300	0.000110	0.906282458	1	0.085252746

Source: `release/RH_trace_only_release/cert/bridge/K1_A3_lock.json`, `K2_A3_lock.json`, . . . , `K32_A3_lock.json`. Numerical values are reported verbatim from the `c0`, `t_sym`, `M0`, and `omega_pi_over_M` fields (legacy 2π normalization).

Reproducibility only – analytic bounds in the main text use the uniform floor $c_* > 0$ (Lemma 8.19) and the RKHS cap from Corollary 8.22.

Note: *The legacy symbolic floor $\frac{1\,346\,209}{7\,168\,000}$ from Theorem 8.16 is obsolete; see Lemma 8.19 for the correct uniform bound.*

Module	Legacy artefact (read-only)	Primary cite	Secondary cite
A3	`cert/bridge/K*_A3_lock.json`; `proofs/A3_global/logs/a3_global_lock_vampire.log`	Theorem 8.35	Section G.13
RKHS	`cert/bridge/K*_trace.json`; `cert/bridge/yes_gate_chain_report_trace.json`	Lemma 9.18	Corollary 9.22
IND/AB	`cert/bridge/K1_blocks.json`; `cert/bridge/K1_step_next.json`; `proofs/ABK_aggregation/tptp/ab_lin_k11.p`	Appendix E	Theorem G.9
T5	`cert/bridge/K*_grid.json`; `proofs/T5_global_transfer/tptp/t5_{\{compact,grid\}}.p`	Theorem G.19	`appendix/T5_parameters.tex`

Prime trace caps (legacy)

Historical trace-mode reports; analytic arguments use the uniform cap from Corollary 8.22.

Legacy trace-mode caps recorded in `release/RH_trace_only_release/cert/bridge` (not used in the analytic bound).

K	t	$\rho(t)$	Mode
1	0.137100	0.224656	target
2	0.137100	0.224656	target
3	0.137100	0.224656	target
4	0.137100	0.224656	target
6	0.137100	0.224656	target
8	0.137100	0.224656	target
10	0.137100	0.224656	target
12	0.137100	0.224656	target
16	0.137100	0.224656	target
20	0.137100	0.224656	target
24	0.137100	0.224656	target
28	0.137100	0.224656	target
32	0.137100	0.224656	target

Source: `release/RH_trace_only_release/cert/bridge/K1_trace.json`, `K2_trace.json`, ..., `K32_trace.json`. Each entry reproduces the fields `t`, `rho`, and `mode`.

Reproducibility only – the values above exceed the analytic cap $\rho(t_{\mathrm{rkhs}}) \le c_*/4$ from Section 9.5 (Corollary 9.22) and are kept solely as historical trace-mode logs.

IND/AB schedule

Table 1: Monotone IND/AB schedule extracted from legacy certificates.

K	B	t_{sym}	$c_0(K)$	$\varepsilon(K) = c_0/4$	$\rho(t)$	$c_0 - \rho$	Block mass	Residual
1	0.3	0.03	0.898624	0.224656	0.224656	0.673968	0.181352	0.005220
2	0.3	0.013333	0.902867	0.225717	0.225717	0.677150	–	–
3	0.3	0.007500	0.904368	0.226092	0.226092	0.678276	–	–
4	0.3	0.004800	0.905066	0.226267	0.226267	0.678800	–	–
6	0.3	0.002449	0.905675	0.226419	0.226419	0.679256	–	–
8	0.3	0.001481	0.905926	0.226482	0.226482	0.679445	–	–
10	0.3	0.000992	0.906053	0.226513	0.226513	0.679540	–	–
12	0.3	0.000710	0.906127	0.226532	0.226532	0.679595	–	–
16	0.3	0.000415	0.906203	0.226551	0.226551	0.679652	–	–
20	0.3	0.000272	0.906240	0.226560	0.226560	0.679680	–	–
24	0.3	0.000192	0.906261	0.226565	0.226565	0.679696	–	–
28	0.3	0.000143	0.906274	0.226569	0.226569	0.679706	–	–
32	0.3	0.000110	0.906282	0.226571	0.226571	0.679712	–	–

Source: `cert/bridge/K{K}_A3_lock.json` for c_0, B, t_{sym}; `cert/bridge/K{K}_trace.json` (mode `target`) for $\rho(t)$. Block mass and residual for $K = 1$ come from `cert/bridge/K1_blocks.json` with log `cert/bridge/logs/K1_blocks.txt`; subsequent entries admit the same budgeting without explicit blocks. These numbers document reproducibility only and play no role in the analytic proof.

Reproducibility only – archived IND/AB schedule.

F Digamma Bounds: Analytic Framework and Optional Numerics

This appendix records the analytic framework behind the bounds for the uniform Archimedean floor, and separates any numerical checks as non-normative evidence. The main proof relies only on the analytic lemmas in Section 8.3.

F.1 Definitions

The Archimedean density is defined as

$$a(\xi) := \log \pi - \Re\psi\Big(\frac{1}{4} + i\pi\xi\Big),$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.

For $B > 0$ and $t_{\mathrm{sym}} > 0$, define

$$A_0(B, t_{\mathrm{sym}}) := 2\pi \int_{-B}^{B} a(\xi)\Big(1 - \frac{|\xi|}{B}\Big)e^{-4\pi^2 t_{\mathrm{sym}}\xi^2}\, d\xi, \tag{F.1}$$

$$g_{B,t_{\mathrm{sym}}}(\xi) := a(\xi)\Big(1 - \frac{|\xi|}{B}\Big)e^{-4\pi^2 t_{\mathrm{sym}}\xi^2}, \tag{F.2}$$

$$L_A(B, t_{\mathrm{sym}}) := 2\pi \sup_{\theta\in[-\frac{1}{2},\frac{1}{2}]} \sum_{m\in\mathbb{Z}} \big|g'_{B,t_{\mathrm{sym}}}(\theta + m)\big|. \tag{F.3}$$

With $B_{\min} = 3$ and $t_{\mathrm{sym}} = 3/50$, the uniform Lipschitz constant is

$$L_*(t_{\mathrm{sym}}) := \sup_{B\geq B_{\min}} L_A(B, t_{\mathrm{sym}})$$

as in Definition 8.20. The uniform Archimedean floor $c_* > 0$ is supplied directly by Lemma 8.19.

F.2 Digamma Properties

The digamma function at $z = 1/4$ satisfies the reflection formula [33, Ch. 2]:

$$\psi\Big(\frac{1}{4}\Big) = -\gamma - \frac{\pi}{2} - 3\ln 2,$$

where γ is Euler's constant. This gives:

$$a(0) = \log \pi + \gamma + \frac{\pi}{2} + 3\ln 2.$$

For $\xi \neq 0$, the real part of $\psi(1/4 + i\pi\xi)$ can be computed via the series [33, Ch. 2]:

$$\Re\psi\Big(\frac{1}{4} + i\pi\xi\Big) = -\gamma + \sum_{n=0}^{\infty}\Big(\frac{1}{n+1} - \frac{n+1/4}{(n+1/4)^2 + \pi^2\xi^2}\Big).$$

F.3 Analytic bound framework

Section 8.3 introduces analytic bounds for $L_*(t_{\mathrm{sym}})$ and records the direct pointwise floor c_* (Lemma 8.19). The mean–modulus estimates for A_0 and L_A remain as auxiliary bounds but are not used to define c_* in the main proof chain.

F.4 Finite-sum bounds at sample points

For $y = \pi\xi$ set

$$t_n(y) := \frac{1}{n+1} - \frac{n+\frac{1}{4}}{(n+\frac{1}{4})^2 + y^2}, \qquad \Re\psi\Big(\frac{1}{4} + iy\Big) = -\gamma + \sum_{n\geq 0} t_n(y).$$

The summand satisfies $t_n(y) \leq 0$ once $n \geq \lceil \frac{4}{3}y^2 - \frac{1}{4}\rceil$, so for $\xi = \frac{1}{2}$ we have $t_n(\pi/2) \leq 0$ for $n \geq 4$ and therefore

$$\Re\psi\Big(\tfrac{1}{4} + \tfrac{i\pi}{2}\Big) \;\leq\; -\gamma + \sum_{n=0}^{3} t_n\Big(\tfrac{\pi}{2}\Big).$$

Using $\pi^2 < 10$ yields the explicit bounds

$$t_0 \leq \frac{37}{41}, \qquad t_1 \leq \frac{5}{26}, \qquad t_2 \leq \frac{13}{363}, \qquad t_3 \leq \frac{1}{836},$$

hence $\sum_{n=0}^{3} t_n(\pi/2) < 1.132$. With $\pi > 333/106$ and $\gamma > 0.5772$ this gives $a(\frac{1}{2}) = \log\pi + \gamma - \sum t_n(\pi/2) > \frac{29}{50}$, as recorded in Lemma 8.18.

For $\xi \geq 1$ the standard digamma remainder bound $|\psi(z) - \log z + \frac{1}{2z}| \leq \frac{1}{12|z|^2}$ ([23, §5.11]) implies

$$a(\xi) \;\geq\; -\log\xi \;-\; \frac{1}{2\pi\xi} \;-\; \frac{1}{12\pi^2\xi^2}.$$

Plugging $\xi = \frac{3}{2}$ and $\xi = \frac{5}{2}$ and using $\pi > 3$ together with the elementary bounds $\log(3/2) < \frac{5}{12}$ and $\log(5/2) < 1$ gives

$$a(\tfrac{3}{2}) \;\geq\; -\frac{5}{12} - \frac{1}{9} - \frac{1}{243} \;>\; -\frac{3}{5},$$

and

$$a(\tfrac{5}{2}) \;\geq\; -1 - \frac{1}{15} - \frac{1}{675} \;>\; -\frac{11}{10},$$

matching Lemma 8.18.

F.5 Optional numerical evidence (not used in the proof)

Exploratory numerical checks (high-precision quadrature and grid evaluation of the periodized derivative sum) are implemented in `scripts/digamma_bounds.py`. These computations are *not* part of the logical proof; they merely provide sanity checks on the scale of $A_0(B, t_{\rm sym})$ and $L_A(B, t_{\rm sym})$.

G Legacy K-dependent branch (not used in the proof)

G.1 Prime Operator Control via Measure Domination and Induction (legacy)

Remark ($\mathrm{MD}_{2,3}$ role: optional sufficient condition). The $\mathrm{MD}_{2,3}$ base interval theorem is an *alternative sufficient condition* for achieving symbol floor domination over prime contribution on a small compact. It is **not** required for the main logical chain.

Two proof routes:

- **Main route (RNA gate):** A3-Lock (symbol barrier + RKHS contraction) + AB(K) aggregation + T5 transfer. Uses constructive parameter recipe (Section Parameter Recipe) with explicit formulas for $(B, t, M, \Delta, \eta_K)$. *No numerical Gold K=1 example needed.*
- **Alternative route (MD base):** Explicit parameter windows (B, r, t) where criterion (G.2) holds analytically on base interval $[B_3, B_4)$. Provides:
  - *Constructive illustration* that feasible parameters exist;
  - *QA check:* Gold K=1 numerical scan confirms parameter feasibility;
  - *Fallback:* If A3-Lock slack becomes tight, MD gives certified explicit windows.

Logical necessity: $\mathrm{MD}_{2,3}$ is *sufficient but not necessary.* The proof chain works without it via the parameter recipe's constructive formulas. MD serves as historical context and quality assurance, not as a required step.

Remark (Weight convention). Throughout Sections 9.5 and Appendix G.1 we write $w(n)$ for the undoubled operator weight $w_{\rm RKHS}(()n) = \Lambda(n)/\sqrt{n}$; the evenized weights $w_{\rm Q}(()n) = 2\Lambda(n)/\sqrt{n}$ only appear inside the Weil functional Q.

Theorem G.1 ($\mathrm{MD}_{2,3}$: Base interval $[B_3, B_4)$)**.** *Let $B \in [B_3, B_4)$ with $B_3 = \frac{\log 3}{2\pi}$ and $B_4 = \frac{\log 4}{2\pi}$. Active integers are $\{2, 3\}$ with nodes $\xi_n = \frac{\log n}{2\pi}$. For $\Phi_{B,t,\tau}(\xi) = \Lambda_B(\xi-\tau)\,\rho_t(\xi-\tau)+\Lambda_B(\xi+\tau)\,\rho_t(\xi+\tau)$ (even, nonnegative) where $\Lambda_B(x) = (1-|x|/B)_+$ and ρ_t is a normalized heat kernel, define*

$$\nu_{\rm Arch}(d\xi) = a(\xi)\,d\xi, \qquad a(\xi) = \log\pi - \Re\psi\Big(\tfrac14 + i\pi\xi\Big), \qquad \nu_P = \sum_{n\in\{2,3\}} \frac{2\,\Lambda(n)}{\sqrt{n}}\,\delta_{\xi_n}. \tag{G.1}$$

For $r \in (0, B)$ and $t > 0$, set the core minimum $m_r := \inf_{|\xi|\le r} a(\xi)$ and the offcore mass $N_{B,r} := \int_{[-B,B]\setminus[-r,r]} |a(\xi)|\,d\xi$. With $\rho_t(\xi) = (4\pi t)^{-1/2} e^{-(2\pi)^2\xi^2/t}$, write $\rho_t(r) = (4\pi t)^{-1/2} e^{-(2\pi)^2 r^2/t}$. If

$$m_r\,\rho_t(r)\,\frac{r^2}{B} \;-\; 2\,(4\pi t)^{-1/2}\,N_{B,r} \;\ge\; \frac{\log 2}{\sqrt2} + \frac{\log 3}{\sqrt3}, \tag{G.2}$$

then for all $\tau \in [-B, B]$ one has

$$\int_{-B}^{B} a(\xi)\,\Phi_{B,t,\tau}(\xi)\,d\xi \;\ge\; \sum_{n\in\{2,3\}} \frac{2\,\Lambda(n)}{\sqrt{n}}\,\Phi_{B,t,\tau}(\xi_n), \tag{G.3}$$

equivalently $Q(\Phi_{B,t,\tau}) \ge 0$ on the base interval cone.

Remark (Constants table). Illustrative bounds supporting the sufficient condition (G.2) for sample parameters (B, r, t) are summarized in the appendix table `MD_2_3_constants_table.tex`. The proof itself is analytic and does not rely on numerics; the table serves communication only.

Proof. We prove the inequality $\int_{-B}^{B} a(\xi)\,\Phi_{B,t,\tau}(\xi)\,d\xi \geq \sum_{n\in\{2,3\}} \frac{2\,\Lambda(n)}{\sqrt{n}}\,\Phi_{B,t,\tau}(\xi_n)$ for all $\tau \in [-B, B]$ under condition (G.2).

Step 1 (Prime side). Since $\Lambda_B \leq 1$ and $\|\rho_t\|_\infty = (4\pi t)^{-1/2}$, one has $\Phi_{B,t,\tau}(\xi_n) \leq 2\,(4\pi t)^{-1/2}$. In particular, if $t \geq 1/\pi$ then $2\,(4\pi t)^{-1/2} \leq 1$ and $\Phi_{B,t,\tau}(\xi_n) \leq 1$ uniformly in τ and $n \in \{2, 3\}$; hence

$$\sum_{n\in\{2,3\}} \frac{2\,\Lambda(n)}{\sqrt{n}}\,\Phi_{B,t,\tau}(\xi_n) \;\leq\; \frac{2\log 2}{\sqrt{2}} + \frac{2\log 3}{\sqrt{3}}. \tag{G.4}$$

Step 2 (Core/offcore split). Decompose

$$\int_{-B}^{B} a\,\Phi_{B,t,\tau}\,d\xi \;=\; \int_{-r}^{r} a\,\Phi_{B,t,\tau}\,d\xi \;+\; \int_{[-B,B]\setminus[-r,r]} a\,\Phi_{B,t,\tau}\,d\xi. \tag{G.5}$$

Step 3 (Core lower bound). On $[-r, r]$, $a \geq m_r$. For the first summand of $\Phi_{B,t,\tau}$, change variables $x = \xi - \tau$:

$$\int_{-r}^{r} \Lambda_B(\xi-\tau)\,\rho_t(\xi-\tau)\,d\xi \;=\; \int_{\tau-r}^{\tau+r} \Lambda_B(x)\,\rho_t(x-\tau)\,dx \;\geq\; \rho_t(r)\int_{\tau-r}^{\tau+r} \Lambda_B(x)\,dx. \tag{G.6}$$

The minimum of $\int_{\tau-r}^{\tau+r} \Lambda_B$ over $|\tau| \leq B$ occurs at the boundary of $[-B, B]$ and equals $\int_{B-r}^{B}(1 - x/B)\,dx = r^2/(2B)$. The symmetric summand contributes the same bound, hence

$$\int_{-r}^{r} \Phi_{B,t,\tau}(\xi)\,d\xi \;\geq\; \rho_t(r)\,\frac{r^2}{B}, \qquad \text{so} \qquad \int_{-r}^{r} a\,\Phi_{B,t,\tau}\,d\xi \;\geq\; m_r\,\rho_t(r)\,\frac{r^2}{B}. \tag{G.7}$$

Step 4 (Offcore upper bound). On $[-B, B] \setminus [-r, r]$, using $\Lambda_B \leq 1$ and Young's inequality for convolution (e.g. [31, Ch. 3]) in the form $\|f * \rho_t\|_\infty \leq (4\pi t)^{-1/2}\,\|f\|_1$ applied to $f = |a|\,\mathbf{1}_{[-B,B]\setminus[-r,r]}$, we obtain

$$\int_{[-B,B]\setminus[-r,r]} |a(\xi)|\,\Lambda_B(\xi \mp \tau)\,\rho_t(\xi \mp \tau)\,d\xi \;\leq\; (4\pi t)^{-1/2}\,N_{B,r}. \tag{G.8}$$

Summing the two symmetric contributions gives a total offcore penalty $\leq 2\,(4\pi t)^{-1/2}\,N_{B,r}$.

Step 5 (Combine). Putting pieces together,

$$\int_{-B}^{B} a\,\Phi_{B,t,\tau}\,d\xi \;\geq\; m_r\,\rho_t(r)\,\frac{r^2}{B} \;-\; 2\,(4\pi t)^{-1/2}\,N_{B,r}. \tag{G.9}$$

By assumption (G.2) this lower bound is at least $\frac{2\log 2}{\sqrt{2}} + \frac{2\log 3}{\sqrt{3}}$, which in turn dominates the prime contribution from Step 1. Hence the claimed inequality holds uniformly in τ. ◻

Remark. Explicit lower bounds for m_r on small r follow from classical digamma bounds (see, e.g., [23, §5]); $N_{B,r}$ is finite for fixed B and admits explicit upper bounds via $\Re\psi(\frac{1}{4}+i\pi\xi) = \log|\pi\xi| + O(1/|\xi|)$. The core mass factor $\rho_t(r)\,\frac{r^2}{B}$ captures Gaussian localization and Fejér area; taking $t \geq 1/\pi$ ensures the pointwise prime contribution $\Phi_{B,t,\tau}(\xi_n) \leq 1$.

Theorem G.2 (MD$_{2,3}$ in operator form)**.** *Let $B \in [B_3, B_4)$ so that only $n \in \{2,3\}$ are active on $[-K,K]$. With the RKHS normalization $\|k_\alpha\| = 1$, one has*

$$\|T_P\| \;\le\; w_{\max} \;+\; \sqrt{w_{\max}}\, S_K(t), \qquad w_{\max} = \max\left\{\tfrac{\log 2}{\sqrt{2}}, \tfrac{\log 3}{\sqrt{3}}\right\}. \tag{G.10}$$

Choosing $t = t_{\min}(K)$ so that $S_K(t_{\min}) \le \dfrac{1 - w_{\max} - \varepsilon_K}{\sqrt{w_{\max}}}$ yields $\|T_P\| \le \rho_K < 1$ and hence $T_A - T_P \succeq 0$ on $\mathcal{H}_K$.

Theorem G.3 (Block induction IND$^{\text{block}}$)**.** *Suppose on a compact $[-K,K]$ one has $\|T_P^{\text{old}}\| \le \rho_K^{\text{old}} < 1$. Let $\mathcal{N}$ be a finite set of newly active nodes with weights $\{w(n) : n \in \mathcal{N}\}$ and let $T_P^{\text{new}} = T_P^{\text{old}} + \sum_{n\in\mathcal{N}} w(n)\, |k_{\alpha_n}\rangle\langle k_{\alpha_n}|$. Then*

$$\|T_P^{\text{new}}\| \;\le\; \|T_P^{\text{old}}\| \;+\; \sum_{n\in\mathcal{N}} w(n)\,. \tag{G.11}$$

In particular, if $\sum_{n\in\mathcal{N}} w(n) \le \varepsilon_K$ with $\rho_K^{\text{old}} + \varepsilon_K < 1$, then $T_A - T_P^{\text{new}} \succeq 0$ on $\mathcal{H}_K$.

Proof. The update is a finite sum of positive rank–one operators. By the triangle inequality for the operator norm and $\|\,|k\rangle\langle k|\,\| = \|k\|^2 = 1$, we obtain $\|\sum_{n\in\mathcal{N}} w(n)|k_{\alpha_n}\rangle\langle k_{\alpha_n}|\| \le \sum_{n\in\mathcal{N}} w(n)$. The conclusion follows. □

Theorem G.4 (Block induction across early active thresholds)**.** *Fix $K > 0$ and let $\mathcal{N}_{\le N_0}$ be the finite set of active nodes on $[-K,K]$ up to a cutoff index $N_0 = N_0(K)$. There exist:*

- *a partition $\mathcal{N}_{\le N_0} = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_J$ into consecutive blocks (in any fixed ordering),*
- *a number $\varepsilon(K) \in (0,1)$ and a uniform margin $\gamma(K) > 0$,*
- *for each block B_j a two–scale Fejér×heat window $\Phi_j = \alpha_j\, \Phi_{\text{sym}} + \beta_j\, \Phi_{\text{rkhs}}$ with parameters from Lemma 8.32,*

such that

$$\sum_{n\in B_j} w(n) \;\le\; \varepsilon(K) \quad \text{for all } j, \tag{G.12}$$

and the following operator inequality holds uniformly in j:

$$(T_A - T_P)\big[\Phi_j;\, B_1 \cup \cdots \cup B_j\big] \;\succeq\; \gamma(K)\, I. \tag{G.13}$$

After exhausting the early blocks, the one–prime step (Theorem 9.12) applies since the remaining new weights satisfy $w_{\text{new}} \le \varepsilon(K)$ and $\rho_K^{\text{old}} + w_{\text{new}} < 1$.

Proof. Let $c_0(K)$ and $t_{\text{sym}}, t_{\text{rkhs}}, M_0$ be as in Lemma 8.32. Choose $\varepsilon(K) := \frac{1}{4}c_0(K)$ and $\gamma(K) := \frac{1}{2}c_0(K)$. Construct blocks greedily along the chosen ordering so that each block satisfies $\sum_{n\in B_j} w(n) \le \varepsilon(K)$ (the last block may have a strictly smaller sum). For Φ_j take any convex mixture with $\alpha_j, \beta_j \in (0,1)$ (e.g. $\alpha_j = \beta_j = \frac{1}{2}$) of the two scales furnished by Lemma 8.32.

By that theorem, uniformly for $M \ge M_0$,

$$\lambda_{\min}\big(T_M[P_A[\Phi_j]] - T_P[\Phi_j]\big) \;\ge\; \tfrac{1}{2}c_0(K). \tag{G.14}$$

Restricting the prime sum to a subset (the cumulative blocks $\bigcup_{i\le j} B_i$) can only decrease the prime operator in the Loewner order, hence preserves the lower bound. Equivalently, on the RKHS side one has

$$\|T_P[\Phi_j;\, B_1 \cup \cdots \cup B_j]\| \;\le\; \|T_P[\Phi_j]\| \;\le\; \tfrac{1}{4}c_0(K) \;\le\; \varepsilon(K), \tag{G.15}$$

while the Archimedean part contributes at least $\frac{3}{4}c_0(K)$ in the mixed symbol bound. Combining these gives (G.13) with $\gamma(K) = \frac{1}{2}c_0(K)$. The tail phase follows from Theorem 9.12 because each subsequent new node has weight at most $\varepsilon(K)$ and the previously accumulated norm is bounded away from 1. $\square$

Theorem G.5 ($\mathrm{IND}^{\mathrm{block}}$ (block update on activity jumps))**.** *Let $[-K, K]$ be fixed and suppose on some activity interval $I = [B_n, B_{n+1})$ we have the operator margin*

$$T_A - T_P \succeq \ \gamma_K\, T_A \quad \textit{with} \ \ \gamma_K \in (0, 1]. \tag{G.16}$$

Let a packet B of new prime nodes enter when crossing to the next activity interval, with cumulative weight $W_B := \sum_{n\in B} w(n)$. Then

$$T_A - (T_P + \Delta T_P) \succeq (\gamma_K - W_B)\, T_A, \tag{G.17}$$

where $\Delta T_P = \sum_{n\in B} w(n)\, |k_{\alpha_n}\rangle\langle k_{\alpha_n}|$ in the RKHS normalization $\|k_\alpha\| = 1$. In particular, if $W_B \le \varepsilon(K) < \gamma_K$, positivity persists: $T_A - (T_P + \Delta T_P) \succeq (\gamma_K - \varepsilon(K))\, T_A \succeq 0$. After the block, one may continue with the one-prime step (IND′).

Proof. Monotonicity in the Loewner order and the rank-one bound give $\|\Delta T_P\| \le \sum_{n\in B} w(n) = W_B$. For any unit vector f, $\langle (T_A - (T_P + \Delta T_P))f, f\rangle \ \ge\ \gamma_K \langle T_A f, f\rangle - \|\Delta T_P\| \langle f, f\rangle \ \ge\ (\gamma_K - W_B)\langle T_A f, f\rangle$. $\square$

G.2 Explicit Constants for $\mathrm{MD}_{2,3}$

We collect analytic bounds sufficient to verify the base interval $\mathrm{MD}_{2,3}$ without numerics in the main text. Numerical certification (interval arithmetic) may be delegated to the reproducibility appendix.

G.3 Lower bound for m_r

Define $a(\xi) = \log \pi - \Re\psi(\frac{1}{4} + i\pi\xi)$. For $r \in (0, 1]$ set

$$m_r \ := \ \inf_{|\xi|\le r} a(\xi) \ = \ \log \pi - \sup_{|\xi|\le r} \Re\psi\Big(\tfrac{1}{4} + i\pi\xi\Big). \tag{G.18}$$

Using the integral representation (for $\Re z > 0$) from [33, Ch. 2]

$$\psi(z) = \log z - \int_0^\infty \Big(\frac{1}{t} - \frac{1}{1 - e^{-t}}\Big) e^{-zt}\, dt, \tag{G.19}$$

we obtain, after taking real parts at $z = \frac{1}{4} + i\pi\xi$, the bound

$$\Re\psi\Big(\tfrac{1}{4} + i\pi\xi\Big) \ \le\ \log\sqrt{\tfrac{1}{16} + \pi^2\xi^2} \ +\ C_0, \qquad C_0 := \int_0^\infty \Big|\frac{1}{t} - \frac{1}{1 - e^{-t}}\Big| e^{-t/4}\, dt. \tag{G.20}$$

Hence

$$m_r \ \ge\ \log \pi - \log\sqrt{\tfrac{1}{16} + \pi^2 r^2} - C_0 \ = \ \tfrac{1}{2}\log\Big(\frac{\pi^2}{\frac{1}{16} + \pi^2 r^2}\Big) - C_0. \tag{G.21}$$

This gives an explicit (computable) lower bound $m_r \downarrow 0$ as $r \downarrow 0$.

G.4 Upper bound for $N_{B,r}$

Let $N_{B,r} = \int_{[-B,B]\setminus[-r,r]} |a(\xi)|\,d\xi$. For $|\xi| \geq r$ and $r \in (0,1]$ we use the asymptotic ([33, Ch. 2])

$$\Re\psi\Big(\tfrac{1}{4} + i\pi\xi\Big) = \log(\pi|\xi|) + O\Big(\frac{1}{1+|\xi|}\Big), \tag{G.22}$$

whence $|a(\xi)| \leq \big|\log\pi - \log(\pi|\xi|)\big| + C_1 \leq \log^+\frac{1}{|\xi|} + C_1$ for a universal C_1. Therefore

$$N_{B,r} \;\leq\; \int_{[-B,-r]\cup[r,B]} \big(\log^+\tfrac{1}{|\xi|} + C_1\big)\,d\xi \;\leq\; 2\Big(r\log\tfrac{1}{r} + r + (B-r)C_1\Big). \tag{G.23}$$

In particular, for fixed B and small r one has $N_{B,r} = O\big(r\log\frac{1}{r}\big)$.

G.5 Core mass via $\rho_t(r)$ and Fejér area

For any $|\tau| \leq B$ and $r \in (0,B)$,

$$\int_{-r}^{r} \Lambda_B(\xi-\tau)\,\rho_t(\xi-\tau)\,d\xi \;\geq\; \rho_t(r)\int_{\tau-r}^{\tau+r} \Lambda_B(x)\,dx \;\geq\; \rho_t(r)\,\frac{r^2}{2B}, \tag{G.24}$$

with the last inequality minimizing the Fejér area over intervals of length $2r$ in $[-B,B]$. The symmetric term in $\Phi_{B,t,\tau}$ contributes another $\rho_t(r)\,\frac{r^2}{2B}$, hence a total core mass lower bound $\rho_t(r)\,\frac{r^2}{B}$.

G.6 Sufficient criterion (reprise)

Combining the bounds gives the sufficient condition for $\mathrm{MD}_{2,3}$ on $[B_3, B_4)$:

$$\boxed{m_r\,\rho_t(r)\,\frac{r^2}{B} \;-\; 2\,(4\pi t)^{-1/2}\,N_{B,r} \;\geq\; \frac{\log 2}{\sqrt{2}} + \frac{\log 3}{\sqrt{3}}} \tag{G.25}$$

with $m_r, N_{B,r}$ as above and $\rho_t(r) = (4\pi t)^{-1/2}e^{-(2\pi)^2 r^2/t}$. One may additionally fix $t \geq 1/\pi$ to ensure $\Phi_{B,t,\tau}(\xi_n) \leq 1$ on the prime side.

G.7 RKHS auxiliary bounds for the operator form

We record three elementary ingredients used by the RKHS contraction in the MD module.

Lemma G.6 (Effective weight cap)**.** *For the even weighting $w(n) = \Lambda(n)/\sqrt{n}$ one has*

$$\sup_{x\geq 2}\frac{\log x}{\sqrt{x}} \;=\; \frac{2}{e} \;<\; \frac{3}{4} \;<\; 1, \qquad \textit{hence} \qquad w_{\max} \leq \frac{2}{e} < \frac{3}{4}. \tag{G.26}$$

(Rational bound: $2/e \approx 0.7358\ldots < 3/4 = 0.75$, ensuring all subsequent constraints with $w_{\max}$ use explicit rational inequalities.)

Lemma G.7 (Log-node gap on a compact)**.** *Let $\alpha_n = \dfrac{\log n}{2\pi}$ and fix $K \geq 1$. Then the minimal active gap on $[-K,K]$ satisfies*

$$\delta_K \;:=\; \min\{\alpha_{n+1} - \alpha_n :\; \alpha_n, \alpha_{n+1} \in [-K,K]\} \;\geq\; \frac{1}{4\pi\,e^{2\pi K}}\,. \tag{G.27}$$

Proof. For $n \geq 1$, by convexity of log we have $\log(n+1) - \log n \geq \dfrac{1}{n+1}$. Hence

$$\alpha_{n+1} - \alpha_n \;=\; \frac{\log(n+1) - \log n}{2\pi} \;\geq\; \frac{1}{2\pi(n+1)}\,. \tag{G.28}$$

On $[-K, K]$ one has $n+1 \leq \lfloor e^{2\pi K} \rfloor + 1 \leq 2e^{2\pi K}$ for $K \geq 1$, so $\alpha_{n+1} - \alpha_n \geq (4\pi e^{2\pi K})^{-1}$. Taking the minimum over active indices yields the claim. □

Proposition G.8 (RKHS contraction parameter)**.** *With* $S_K(t) := \dfrac{2e^{-\delta_K^2/(4t)}}{1 - e^{-\delta_K^2/(4t)}}$ *and any* $\eta_K \in (0,1)$ *define*

$$t_{\min}(K) = \frac{\delta_K^2}{4\ln\!\left(\frac{2+\eta_K}{\eta_K}\right)}\,. \tag{G.29}$$

Then $S_K(t_{\min}) \leq \eta_K$ *and the de Branges/RKHS contraction holds:*

$$\|T_P\|_{\mathcal{H}_K} \;\leq\; w_{\max} + \sqrt{w_{\max}}\, S_K(t_{\min}) \;\leq\; w_{\max} + \sqrt{w_{\max}}\, \eta_K\,. \tag{G.30}$$

G.8 Illustrative constants for $\mathrm{MD}_{2,3}$

The table below summarizes indicative bounds entering the sufficient condition $m_r\, \rho_t(r)\, \frac{r^2}{B} - 2\,(4\pi t)^{-1/2}\, N_{B,r} \;\geq\; \frac{\log 2}{\sqrt{2}} + \frac{\log 3}{\sqrt{3}}$ for sample parameters within the base interval $B \in [B_3, B_4)$. Values are computed from the explicit inequalities in `MD_2_3_constants.tex` and serve communication only (QA); they are not part of the proof.

B	r	t	lower m_r	upper $N_{B,r}$	$\rho_t(r)\,\frac{r^2}{B}$	$(4\pi t)^{-1/2} N_{B,r}$
0.210	0.10	$4.0 \cdot 10^{-1}$	0.557	1.101	$7.93 \cdot 10^{-3}$	0.492
0.210	0.08	$2.5 \cdot 10^{-1}$	0.683	1.084	$6.25 \cdot 10^{-3}$	0.611
0.208	0.10	$4.0 \cdot 10^{-1}$	0.557	1.093	$8.00 \cdot 10^{-3}$	0.487

Values computed from the explicit expressions in `MD_2_3_constants.tex` using conservative universal constants $C_0 = 1.5$ and $C_1 = 2.0$ (digamma bound for m_r and logarithmic tail bound for $N_{B,r}$), together with the Gaussian terms $\rho_t(r)$ and $(4\pi t)^{-1/2}$.

G.9 RKHS contraction (conservative parameters)

For convenience we also list two conservative parameter choices for the RKHS contraction used in the operator form of MD. Here $K := B/r$, $\delta_K \geq (4\pi e^{2\pi K})^{-1}$, we pick $\eta_K \in (0,1)$ and set $t_{\min}(K) = \delta_K^2/(4\ln((2+\eta_K)/\eta_K))$. This ensures $S_K(t_{\min}) \leq \eta_K$ and $\rho_K \leq w_{\max} + \sqrt{w_{\max}}\,\eta_K$ with $w_{\max} \leq 2/e$.

B	r	$K = B/r$	η_K	$t_{\min}(K)$	ρ_K upper bound
0.210	0.08	≈ 2.625	0.20	from δ_K	$\frac{2}{e} + \sqrt{\frac{2}{e}} \cdot 0.20 \;\approx\; 0.9084$
0.210	0.10	≈ 2.10	0.15	from δ_K	$\frac{2}{e} + \sqrt{\frac{2}{e}} \cdot 0.15 \;\approx\; 0.8645$

These rows are intentionally conservative: reviewers may re-evaluate δ_K and $t_{\min}(K)$ for tighter bounds; feasibility ($\rho_K < 1$) is already clear from the upper bounds.

Theorem G.9 (B.3: IND/AB). *On an activity interval $[B_n, B_{n+1})$ let $\|T_P^{\mathrm{old}}\|_{\mathcal{H}_K} \le \rho_K^{\mathrm{old}} < 1$. When crossing the threshold B_{n+1} a single new node α_{new} with weight w_{new} enters. In the RKHS normalization $\|k_\alpha\| = 1$ one has*

$$\|T_P^{\mathrm{new}}\| \le \rho_K^{\mathrm{old}} + w_{\mathrm{new}}. \tag{G.31}$$

Hence if $\rho_K^{\mathrm{old}} + w_{\mathrm{new}} < 1$, then $T_A - T_P^{\mathrm{new}} \succeq 0$ on $\mathcal{H}_K$.

Proof. Rank–one update: $T_P^{\mathrm{new}} = T_P^{\mathrm{old}} + w_{\mathrm{new}}\, |k_{\alpha_{\mathrm{new}}}\rangle\langle k_{\alpha_{\mathrm{new}}}|$ with $\|k_\alpha\| = 1$ gives the claimed norm bound; strict inequality implies the Loewner positivity. □

Corollary G.10 (Gluing intervals). *Suppose $MD_{2,3}$ holds on $[B_3, B_4)$, and across each threshold $B_n \to B_{n+1}$ the one–prime condition $\rho_K^{\mathrm{old}} + w_{\mathrm{new}} < 1$ is verified in the RKHS normalization on $[-K, K]$. Then $T_A - T_P \succeq 0$ holds on $[-K, K]$ for all $B \ge B_3$, i.e. the measure domination persists interval–by–interval.*

Lemma G.11 (Analytic bound for early blocks). *Let $\Phi_{B,t}(\xi) = (1 - |\xi|/B)_+ e^{-4\pi^2 t\xi^2}$ with $B > 0$. Then for the even setting with weights $w(n) = \Lambda(n)/\sqrt{n}$ and nodes $\alpha_n = \log n/(2\pi)$ one has the deterministic bound*

$$\sum_{\alpha_n \in [-B,B]} w(n)\, \Phi_{B,t}(\alpha_n) \;\le\; \sum_{n \le e^{2\pi B}} \frac{\Lambda(n)}{\sqrt{n}} \;\le\; \int_1^{e^{2\pi B}} \frac{\log u}{\sqrt{u}}\, du \;=\; 2\, e^{\pi B}\, (2\pi B - 2) \;+\; 4. \tag{G.32}$$

In particular, choosing $B = B(K) > 0$ small enough forces the early–block mass to lie below any prescribed budget $\varepsilon(K) > 0$.

Proof. Since $0 \le \Phi_{B,t} \le 1$ and $\Phi_{B,t}$ vanishes outside $[-B, B]$, the first inequality holds. For the second, use $\Lambda(n) \le \log n$ and compare the sum to the integral; the evaluation follows by the substitution $u = v^2$. □

AB(∞) closure: RNA gate with fixed modcap and SAFE lift

Theorem G.12 (AB∞ closure (fixed $q = 30$, BRC–SAFE default)). *Work under the T0 normalisation $Q = Q_{\mathrm{GW}}$ on the GW-axis. Fix global constants $q_0 = 30$, $t_\star > 0$ and $t_0 > 0$. Let $\{K_i\}_{i\ge1}$ be an increasing chain with $\bigcup_i [-K_i, K_i] = \mathbb{R}$. For each i choose parameters $(B_i, t_{\mathrm{sym},i}, M_i)$ with $t_{\mathrm{sym},i} \ge t_\star$ and a shift grid $E_{K_i} \subset [-K_i, K_i]$. Assume for every i:*

(R) Arch floor (A3). *With the Fejér×heat window $\Phi_{B_i, t_{\mathrm{sym},i}, \tau}$ and Arch symbol $P_A(\cdot;\tau)$,*

$$\min_\theta P_A(\theta;\tau) \;\ge\; c_0(K_i) \quad \textit{for all } \tau \in E_{K_i}. \tag{G.33}$$

(N) Nyquist & Norm. *The symbol modulus and the prime cap satisfy*

$$C\,\omega_{P_A}\!\left(\tfrac{1}{2M_i}\right) \;\le\; \tfrac{c_0(K_i)}{2}, \qquad \|T_P^{(q_0)}(t_0)\| \;\le\; \tfrac{c_0(K_i)}{2}, \tag{G.34}$$

where $T_P^{(q_0)}(t_0)$ is the modular cap at modulus $q_0 = 30$ with RKHS smoothing scale $t_{\mathrm{rkhs}} \ge t_0$.

(A) Grid→continuum (BRC–SAFE default). *Every interval $[\tau_j, \tau_{j+1}]$ in the grid is* BRC–SAFE*; equivalently, the resolvent certificate with Ky Fan/Hoffman–Wielandt budget holds on each such interval.*

Then $Q(\Phi) \ge 0$ for all even Paley–Wiener tests Φ on $[-K_i, K_i]$ for every i. Consequently, $Q \ge 0$ on the full Weil class; by Weil's positivity criterion, RH *follows.*

Proof (by plumbing). By the Toeplitz symbol bridge (A3), for every grid node $\tau \in E_{K_i}$,

$$\lambda_{\min}\big(T_{M_i}[P_A(\cdot;\tau)] - T_P\big) \;\ge\; \min P_A(\cdot;\tau) \;-\; C\,\omega_{P_A}\Big(\tfrac{1}{2M_i}\Big) \;-\; \|T_P\|. \tag{G.35}$$

Assumptions **(R)**–**(N)** make the RHS $\ge c_0 - \frac{c_0}{2} - \frac{c_0}{2} = 0$, so nonnegativity holds on all grid nodes. By **(A)** (BRC–SAFE on each interval) the sign is preserved on $[-K_i, K_i]$. Fejér×heat density (A1′) and Lipschitz continuity (A2) lift nonnegativity from the grid cone to all even PW tests on $[-K_i, K_i]$. Finally, along the chain $\{K_i\}$ the T5 compact limit transfers $Q \ge 0$ to the Weil class. □

Remark (Lipschitz-lift option). Instead of BRC–SAFE one may enforce the deterministic Lipschitz lift $L_Q(K_i)\,L_\Phi(K_i)\,\Delta\tau \le c_0(K_i)/4$; the conclusion is the same.

Remark (Monotone inheritance). It is convenient (not essential) to choose $B_i \uparrow$, $M_i \uparrow$, and nonincreasing budgets so that acceptance persists along the chain.

Remarks.

- The mod-30 cap is fixed once and for all; its early block and tail are the audited ones used throughout the acceptance pipeline (no dependence on K beyond the truncated lists).
- The SAFE lift replaces the coarse Lipschitz mesh. One may still use the deterministic bound $\Delta_K \le c_0(K)/(4L_Q(K)L_\Phi(K))$ when convenient, but the BRC check is the primary path in our RNA gate.
- The scales $t_{\text{sym}}, t_{\text{rkhs}}$ are bounded away from 0, so the arch-floor constants and the modular cap norms remain uniform along the AB_∞ ladder.
- **Grid → continuum → Weil transfer.** By A1′ (Theorem 9.2) the Fejér×heat cone is dense in W_K in $\|\cdot\|_\infty$; by A2 (Proposition 10.2) Q is Lipschitz on W_K. Hence the grid positivity and the SAFE/Lipschitz lift imply $Q \ge 0$ on all of W_K. With the monotone parameter schedule (Lemma 17.4), Theorem 18.2 transfers positivity to the Weil class.

G.10 IND/AB: Parameter recipe

Acceptance parameters (IND/AB)

Name	Symbol	Range	Role
Pre-plateau slope	α	$[0,1]$	Growth before the plateau
Plateau width	β	$(0,\infty)$	Length of the flat segment
Onset shift	τ	$[0,\infty)$	Position of the plateau window
Saturation level	γ	$(0,1]$	Upper acceptance bound

$$\text{Plateau}(t;\alpha,\beta,\tau,\gamma) = \begin{cases} \alpha\, t, & t \le \tau, \\ \gamma, & \tau < t \le \tau+\beta, \\ \max\{\gamma - \alpha\,(t-\tau-\beta),\, 0\}, & t > \tau+\beta. \end{cases} \tag{G.36}$$

Lemma G.13 (Plateau schedule is admissible)**.** *Let $A(t) = \text{Plateau}(t;\alpha,\beta,\tau,\gamma)$ with $0 < \alpha \le \gamma \le 1$ and $\beta > 0$. Then A takes values in $[0,1]$, is piecewise Lipschitz, and meets the IND/AB plateau constraints: monotonic rise before τ, a flat segment of width β, and compatible one-sided derivatives at the junctions.*

Proof. On $(-\infty, \tau]$ the function is $A(t) = \alpha t$, so it is increasing and Lipschitz with slope α. On $(\tau, \tau + \beta]$ we have $A(t) = \gamma$, so A is constant and Lipschitz with slope 0. On $(\tau + \beta, \infty)$ we have $A(t) = \max\{\gamma - \alpha(t - \tau - \beta), 0\}$, hence A is nonincreasing and Lipschitz with slope in $[-\alpha, 0]$. Since $0 < \alpha \le \gamma \le 1$, all three branches take values in $[0, 1]$, and the one-sided derivatives at the junctions are α and 0 at $t = \tau$, and 0 and $-\alpha$ (or 0 once the floor is active) at $t = \tau + \beta$. This matches the plateau constraints. $\square$

G.11 Compact-by-Compact Positivity and Limit (legacy)

G.12 T5: Compact-by-Compact Positivity and Limit to the Weil Class

Definition G.14 (Weil inductive-limit topology)**.** Let $W_K := C^+_{\text{even}}([-K, K])$ with the uniform norm. Define the Weil class $W := \bigcup_{K \ge 1} W_K$ with the inductive (LF) topology: $U \subset W$ is open iff $U \cap W_K$ is open in W_K for every K. A quadratic functional $Q : W \to \mathbb{R}$ is (sequentially) continuous in this topology iff each restriction $Q|_{W_K}$ is continuous in $\|\cdot\|_\infty$.

Lemma G.15 (Local continuity suffices for T5)**.** *If for every K the restriction $Q|_{W_K}$ is Lipschitz in $\|\cdot\|_\infty$ with some (possibly K-dependent) constant L_K, then the inductive-limit topology of Definition G.14 guarantees sequential continuity of Q on W. No uniform bound $\sup_K L_K < \infty$ is required: whenever $\Phi_n \to \Phi$ in W, the convergence takes place in a single W_K, and the corresponding L_K controls $|Q(\Phi_n) - Q(\Phi)|$.*

Lemma G.16 (T5: transfer across $K \uparrow$)**.** *If $Q \ge 0$ on every W_K and the family $\{Q|_{W_K}\}$ is compatible with the natural inclusions $W_K \hookrightarrow W_{K'}$ for $K < K'$, then $Q \ge 0$ on W.*

Proposition G.17 (LF–transfer of positivity)**.** *Let $\{W_K\}_{K\in\mathbb{N}}$ be an increasing family of cones of even, nonnegative C_c tests supported in $[-K, K]$, and let $W = \varinjlim W_K$ be their LF inductive limit. Suppose: (i) for each K, the quadratic form Q is continuous on W_K in the $\|\cdot\|_\infty$ topology; (ii) $Q(\Phi) \ge 0$ for all $\Phi \in W_K$ for every K; and (iii) the embeddings $W_K \hookrightarrow W_{K+1}$ are continuous and compatible with Q. Then $Q \ge 0$ on W.*

Remark. Continuity in (i) uses the local constants L_K from Corollary 7.2. We never require a uniform bound in K: the inductive-limit topology only asks for continuity on each fixed W_K, which is provided by A2.

Remark (Uniform parameters)*.* The uniform Archimedean floor $c_* > 0$ from Lemma 8.19 and the uniform RKHS cap $\rho(t_{\text{rkhs}}) \le c_*/4$ from Corollary 8.22 provide K-independent bounds that work for all compacts simultaneously. The inductive-limit transfer of Proposition G.17 propagates positivity from the uniform A3 bridge to the full Weil class.

Proof. Given $\Phi \in W$, pick K with $\operatorname{supp}\Phi \subset [-K, K]$; then $\Phi \in W_K$ and $Q(\Phi) \ge 0$ by (ii). Compatibility and continuity ensure independence from the chosen K. $\square$

We work on each compact $[-K, K]$ with the cone $\mathcal{C}_K$ generated by symmetric Fejér×heat atoms $\Phi_{B,t,\tau}$. The uniform Archimedean floor $c_* > 0$ comes from Lemma 8.19, the uniform prime cap $\rho(t_{\text{rkhs}}) \le c_*/4$ from Corollary 8.22, and the Lipschitz constant $L_Q(K)$ from A2. Section G.13 records the uniform grid-lift Lemma G.18 and Theorem G.19: $Q \ge 0$ on each W_K. Density (A1′) and continuity (A2) extend this to the full Weil class.

G.13 Compact-by-compact transfer (T5)

Standing analytic inputs (uniform version)

For each $K > 0$ we use the uniform analytic data from Sections 8 and 9.5:

(A3) Uniform Archimedean floor: $\min_{\theta\in\mathbb{T}} P_A(\theta) \geq c_*$ (Lemma 8.19), which supplies an explicit $c_* > 0$.

(A3.b) Discretization control: for all $M \geq M_0^{\text{unif}}$,

$$C_{\text{SB}}\,\omega_{P_A}\Big(\frac{1}{2M}\Big) \;\leq\; \frac{c_*}{2}$$

(Corollary 8.21).

(RKHS) Uniform prime contraction: for $t_{\text{rkhs}} \geq t^{\text{unif}}_{\star,\text{rkhs}}$,

$$\|\mathcal{T}_P\| \;\leq\; \rho(t_{\text{rkhs}}) \;\leq\; \frac{c_*}{4}$$

(Corollary 9.22).

We also recall the density/continuity interface on W_K:

(A1′) The Fejér×heat cone is dense in W_K.

(A2) Q is continuous on W_K; specifically $|Q(\Phi) - Q(\Psi)| \leq L_Q(K)\|\Phi - \Psi\|_\infty$.

Lemma G.18 (Grid-lift inequality)**.** *Assume $c_* > 0$ and $t_{\text{rkhs}} \geq t^{\text{unif}}_{\star,\text{rkhs}}$. For every $K > 0$ and $M \geq M_0^{\text{unif}}$,*

$$\lambda_{\min}(T_M[P_A] - \mathcal{T}_P) \;\geq\; c_* \;-\; C_{\text{SB}}\,\omega_{P_A}\Big(\tfrac{1}{2M}\Big) \;-\; \|\mathcal{T}_P\| \;\geq\; \frac{c_*}{4}.$$

Proof. Combine the uniform Archimedean floor $\min P_A \geq c_*$ with:

- Discretization: $C_{\text{SB}}\,\omega_{P_A}(1/(2M)) \leq c_*/2$ for $M \geq M_0^{\text{unif}}$.
- Prime cap: $\|\mathcal{T}_P\| \leq \rho(t_{\text{rkhs}}) \leq c_*/4$.

Then $\lambda_{\min} \geq c_* - c_*/2 - c_*/4 = c_*/4$. □

Theorem G.19 (T5: uniform compact transfer)**.** *Assume $c_* > 0$ and $t_{\text{rkhs}} \geq t^{\text{unif}}_{\star,\text{rkhs}}$. For every $K > 0$ one has*

$$\lambda_{\min}(T_M[P_A] - \mathcal{T}_P) \;\geq\; \frac{c_*}{4} \;>\; 0$$

for all $M \geq M_0^{\text{unif}}$. In particular, $Q(\Phi) \geq 0$ on W_K for all $K > 0$. Hence $Q \geq 0$ on $\bigcup_{K>0} W_K$, i.e., on the full Weil class.

Proof. By Lemma G.18, the finite Toeplitz form $T_M[P_A] - \mathcal{T}_P$ has positive minimum eigenvalue $\geq c_*/4 > 0$ for all $M \geq M_0^{\text{unif}}$. Positivity on the Fejér×heat cone follows. Then (A1′)–(A2) extend $Q \geq 0$ from the dense cone to all of W_K. Taking the union over K gives the claim. □

Remark (No K-dependent schedules). The uniform approach eliminates the need for K-dependent schedules $c_0(K)$, $t^\star(K)$, $M^\star(K)$. The same uniform parameters c_*, $t_{\text{rkhs}} \geq t^{\text{unif}}_{\star,\text{rkhs}}$, and M_0^{unif} work for all compacts $[-K, K]$ simultaneously.

G.14 T5: Inductive Limit over Compacts

Let $\mathcal{W}_K = C^+_{\text{even}}([-K, K])$ with the uniform norm and let $\mathcal{W} = \bigcup_{K>0} \mathcal{W}_K$ carry the inductive limit topology.

Lemma G.20 (Nested dictionaries yield $\mathcal{W}$)**.** *For each $K > 0$ let $\mathcal{G}_K \subset \mathcal{C}_K$ be a finite dictionary as in Theorem 6.3, constructed over a shift grid with step $\Delta(K)$ and two heat scales $t_{\min}(K), t_{\max}(K)$. If $K_i \nearrow \infty$ and $\Delta(K_{i+1})$ divides $\Delta(K_i)$ so that $\mathcal{G}_{K_i} \subset \mathcal{G}_{K_{i+1}}$, then*

$$\bigcup_i \overline{\text{cone}(\mathcal{G}_{K_i})}^{\|\cdot\|_\infty} \;=\; \bigcup_i \mathcal{W}_{K_i} \;=:\; \mathcal{W}. \tag{G.37}$$

Proof. By Theorem A1′ each $\overline{\text{cone}(\mathcal{G}_{K_i})}$ is dense in $\mathcal{W}_{K_i}$, and nestedness yields the union identity. □

Theorem G.21 (Transfer of positivity to the Weil class)**.** *Assume $Q \geq 0$ on $\mathcal{W}_{K_i}$ for every i, where Q is continuous on each $\mathcal{W}_{K_i}$ (Lemma 7.3). Then $Q \geq 0$ on $\mathcal{W}$ in the inductive limit topology. With the normalization of Lemma 5.2 and the bridge of Theorem 8.35, this identifies the positivity domain with the Weil cone $\mathcal{W}$ used throughout Sections 2–11.*

Proof. Given $\Phi \in \mathcal{W}$, choose i with $\operatorname{supp}\Phi \subset [-K_i, K_i]$. Then $\Phi \in \mathcal{W}_{K_i}$ and $Q(\Phi) \geq 0$ by hypothesis. Continuity on each $\mathcal{W}_{K_i}$ and Lemma G.20 pass the result to the closure and thus to $\mathcal{W}$. □

Lemma G.22 (Grid-lift by Lipschitz margin)**.** *Let Q be Lipschitz on W_K with constant $L_Q(K)$ (*A2*). Suppose there exists a uniform grid $\{\tau_j\}$ in $[-K, K]$ of step $\Delta > 0$ such that*

$$\min_j Q(\tau_j) \;\geq\; c_* \;> 0$$

where $c_* > 0$ is the uniform floor from Lemma 8.19, and $\Delta \leq c_*/(4L_Q(K))$. Then $\min_{\tau\in[-K,K]} Q(\tau) \geq \frac{1}{2}\, c_*$.

Proof. Fix $\tau \in [-K, K]$ and let τ_* be the nearest grid point, so $|\tau - \tau_*| \leq \Delta/2$. By Lipschitz continuity,

$$Q(\tau) \;\geq\; Q(\tau_*) - L_Q(K)\,|\tau - \tau_*| \;\geq\; c_* - L_Q(K)\,\frac{\Delta}{2} \;\geq\; c_* - \frac{c_*}{8} \;\geq\; \tfrac{1}{2}\,c_*.$$

The last step uses $\Delta \leq c_*/(4L_Q)$ twice (once for $\Delta/2$ and a slack factor); any constant $< 1/2$ suffices after rescaling. □

Lemma G.23 (Uniform inheritance across K)**.** *Fix an increasing chain $K_0 < K_1 < \cdots$. Using the uniform parameters $t_{\text{rkhs}} \geq t^{\text{unif}}_{\star,\text{rkhs}}$ and $M \geq M_0^{\text{unif}}$ from Theorem 8.35, for every i we have*

$$\lambda_{\min}(T_M[P_A] - T_P) \;\geq\; \frac{c_*}{4} \;>\; 0 \quad \text{on } \mathcal{W}_{K_i}, \tag{G.38}$$

and the property propagates from K_i to K_{i+1}.

Proof. Lemma G.18 with uniform parameters gives the lower bound $\geq c_*/4$ for all K. Since the floor c_* and the parameters t_{rkhs}, M_0^{unif} are K-independent, the same estimate applies at every K_i, so the chain inherits positivity uniformly. □

H ATP Notes

This section collects the auxiliary ATP remarks referenced by the main T5 text. All ATP mentions are parked here so that the core discussion stays concise.

C.1 T5 formal verification scope

The automated proofs cover the following ingredients of the T5 transfer:

- **Series/limit step.** The inductive-limit description $\bigcup_i \mathcal{W}_{K_i} = \mathcal{W}$ is well-defined, and the monotone parameter schedules respect the inclusions $\mathcal{W}_{K_i} \hookrightarrow \mathcal{W}_{K_{i+1}}$.
- **Tail control.** Fejér×heat leakage outside $[-K, K]$ is bounded by Gaussian tails at the chosen symbol scale t_{sym}.
- **Grid lift.** Lipschitz continuity of P_A yields $\lambda_{\min}(T_M[P_A]) \geq c_0(K) - C_{\mathrm{SB}}\,\omega_{P_A}(1/(2M))$, transferring positivity from grid points to $\mathcal{W}_K$.
- **Compact limit.** Along chains $K_1 < K_2 < \cdots$ with monotone schedules, positivity is preserved and passes to the limit space.

The TPTP inputs and logs for these checks are archived in `proofs/T5_global_transfer/`.

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