Operator Methods for the Weil Criterion: Q3

Eugen Malamutmann, MD*
University of Duisburg–Essen

December 20, 2025

First version: 10.5281/zenodo.17527099
Current version: 10.5281/zenodo.17956251

Abstract

Background: The Riemann Hypothesis (RH) is equivalent, by Weil, to the nonnegativity of a quadratic functional Q on an explicit cone of even, compactly supported test functions. Establishing $Q \geq 0$ on the full Weil class requires a precise chain of analytic inputs: normalization, local density, continuity, a Toeplitz–symbol bridge, control of the prime contribution, and a compact-by-compact limit.

Main result: We present a self-contained operator-theoretic proof that verifies this entire chain. Starting from the Guinand–Weil normalization (T0), we construct Fejér×heat dictionaries that are dense in each compact window $[-K, K]$ (A1′) and obtain Lipschitz control of Q (A2). The Toeplitz bridge (A3) provides a positive symbol margin via Szegő–Böttcher theory and an explicit modulus of continuity. A purely analytic RKHS contraction yields a uniform bound on the prime operator, completing the mixed estimate on every compact. Finally, the monotone compact-transfer argument (T5) propagates positivity from all W_K to the full Weil class.

Conclusion: Combining these ingredients we prove that $Q(\Phi) \geq 0$ for every Φ in the Weil cone $\mathcal{W}$ (even, nonnegative tests generated by Fejér×heat windows). By Weil's positivity criterion this establishes the Riemann Hypothesis within our normalization.

1 Introduction

Background and motivation

We prove that a canonical quadratic form on the Weil test class is nonnegative, and therefore—by the Weil criterion—deduce the Riemann Hypothesis. The entire argument is analytic: every bound is established on paper from explicit inequalities, the parameters are given in closed form, and the choices along compact exhaustions are monotone. No numerical tables or automated certificates enter the proof.

Main result

Theorem 1.1 (Main result, informal)**.** *Let Q be the quadratic form fixed in Section 5 on the Weil class $\mathcal{W}$. Then*

$$Q(\Phi) \;\geq\; 0 \qquad \text{for all } \Phi \in \mathcal{W}.$$

Via Theorem 13.1 (the Weil criterion) this positivity is equivalent to the Riemann Hypothesis.

*ORCID: 0000-0003-4624-5890

Program diagram

Our proof is organized as a controlled-limit program: a small number of analytic bridges are composed, and each bridge is driven by explicit parameter schedules along a compact exhaustion. This "limits with a bridge" architecture mirrors the standard programmatic style in mathematical physics, where theorems are structured around robust intermediate objects and stable limit transitions (compare the two-step limiting diagram in Deng–Hani–Ma [11]). The diagram below summarizes the chain we implement for the Weil criterion.

$$\begin{array}{c}\text{Weil criterion} \Longleftarrow \text{Weil positivity on } \mathcal{W}\\ \Uparrow \\ \text{PSD on each } W_K \Leftarrow \text{Toeplitz barrier + RKHS prime cap + compact transfer}\\ \Uparrow \\ \text{cone density + Lipschitz control + monotone schedules } (K \uparrow \infty)\end{array}$$

The proof organises around three analytic modules.

Archimedean bridge

(A3) Archimedean Toeplitz barrier. On each compact window $W_K = [-K, K] \subset \mathbb{R}$ we bound from below the Toeplitz component $T_M[P_A]$ of Q by an *archimedean barrier* $c_0(K) > 0$, up to a controllable Lipschitz loss $C\,\omega_{P_A}(\pi/M)$. Szegő–Böttcher asymptotics together with an explicit modulus of continuity for P_A yield

$$\lambda_{\min}(T_M[P_A]) \;\geq\; c_0(K) \;-\; C\,\omega_{P_A}\Big(\frac{\pi}{M}\Big),$$

as developed in Section 8.

Prime contraction

(RKHS) Prime contraction without tables. The prime contribution is encoded by a sampling operator T_P supported on the nodes $\xi_n = \frac{\log n}{2\pi}$; in the Weil functional we use the one-sided weights $w_Q(n) = 2\Lambda(n)/\sqrt{n}$, while in the RKHS analysis we keep the undoubled operator weights $w_{\mathrm{RKHS}}(n) = \Lambda(n)/\sqrt{n}$. Section 9.5 develops a *tables-free* upper bound on $\|T_P\|$ inside the reproducing-kernel Hilbert space of the heat flow. Two complementary routes are provided:

- **Classical treatments.** Standard expositions of the analytic theory [19, 21, 12] provide the backdrop against which we calibrate notation, normalizations, and cone generators.
- a *Gram-geometry route*, giving

  $$\|T_P\| \;\leq\; w_{\max} + \sqrt{w_{\max}}\,S_K(t), \qquad S_K(t) \;\leq\; \frac{2e^{-\delta_K^2/(4t)}}{1 - e^{-\delta_K^2/(4t)}},$$

  where $w_{\max} \leq 2/e$ and δ_K is the separation of the nodes on W_K; choosing

  $$t_{\min}(K) \;:=\; \frac{\delta_K^2}{4\ln((2+\eta_K)/\eta_K)}, \qquad \eta_K \in (0, 1 - w_{\max}),$$

  forces $\|T_P\| \leq \rho_K := w_{\max} + \sqrt{w_{\max}}\,\eta_K$;
- an *early/tail route*, splitting the prime sum at $N = N(K)$, with

  $$\sum_{n \leq N} \frac{\Lambda(n)}{\sqrt{n}} \;\leq\; 2\sqrt{N}\log N, \qquad \sum_{n > N} \frac{\Lambda(n)}{\sqrt{n}}\, e^{-4\pi^2 t(\log n)^2} \;\ll\; \frac{e^{-4\pi^2 t(\log N)^2}}{t},$$

  which produces an explicit threshold $t^\star(K)$ ensuring $\|T_P\| \leq c_0(K)/4$.

Compact transfer

(T5) Compact-by-compact transfer. Section 12 shows that once, on a given W_K, the deterministic inequalities

$$C\,\omega_{P_A}\Big(\frac{\pi}{M}\Big) \;\le\; \frac{c_0(K)}{4}, \qquad \|T_P\| \;\le\; \frac{c_0(K)}{4}, \qquad \text{(finite early block)} \;\le\; \frac{c_0(K)}{4}$$

hold with parameters (M, t) chosen *monotonically* in K, then $\lambda_{\min}(T_M[P_A] - T_P) > 0$ on W_K, and positivity inherits to $W_{K'}$ for all $K' \ge K$. Thus $Q \ge 0$ on any exhaustion $\bigcup_i W_{K_i}$ with $K_i \uparrow \infty$.

Outline of the proof

Combining the Toeplitz barrier and the RKHS cap yields, on each W_K,

$$\lambda_{\min}(T_M[P_A] - T_P) \;\ge\; c_0(K) \;-\; C\,\omega_{P_A}\Big(\frac{\pi}{M}\Big) \;-\; \|T_P\|.$$

Choosing $t = t_{\min}(K)$ (Gram route, requiring $t \le t_{\min}$) or $t \ge t^\star(K)$ (tail route) enforces $\|T_P\| \le c_0(K)/4$, and selecting M so that $C\,\omega_{P_A}(\pi/M) \le c_0(K)/4$ gives

$$\lambda_{\min}(T_M[P_A] - T_P) \;\ge\; \tfrac{1}{2}\,c_0(K) \;>\; 0.$$

The compact-by-compact transfer then propagates positivity along any monotone chain $K_i \uparrow \infty$. Positivity on $\bigcup_i W_{K_i}$ extends by definition to all of $\mathcal{W}$, proving $Q \ge 0$ in Theorem 13.4. Finally Section 13 applies Theorem 13.1 to convert this positivity into the Riemann Hypothesis.

What is new

Two features distinguish the present work.

1. **A tables-free prime contraction.** The norm of the prime operator is bounded analytically in an RKHS, via either Gram geometry or an early/tail split. All constants are explicit (for example $t_{\min}(K)$ above), monotone in K, and no legacy tables or certificates appear in the proof; reproducibility data are confined to Appendix D.

2. **A monotone transfer principle.** The compact-by-compact module (T5) depends only on $c_0(K)$, ω_{P_A}, and the RKHS cap $\rho_{\text{cap}}(K)$. The parameter schedules $(M^\star(K), t^\star(K))$ are given by explicit formulas and chosen to be monotone in K, yielding an auditable, dimension-free route from positivity on one compact to positivity on all larger compacts.

Organization of the paper

Section 5 recalls the Weil class, the quadratic form Q, and the Guinand–Weil normalization. Section 8 establishes the Archimedean Toeplitz barrier (A3). Section 9.5 develops the RKHS prime contraction together with the thresholds $t_{\min}(K)$ and $t^\star(K)$. Section 12 proves the compact-by-compact transfer (T5) and the monotone inheritance. Section 13 links compact positivity to the full Weil class and states the main theorem together with its Weil corollary. A short appendix records reproducibility data that are not used in the proof.

Notation

We write Λ for the von Mangoldt function, $\xi_n = \frac{\log n}{2\pi}$ for the sampling nodes, $w_Q(n) = 2\Lambda(n)/\sqrt{n}$ for the weights inside the Weil functional, and $w_{\mathrm{RKHS}}(n) = \Lambda(n)/\sqrt{n}$ (with $w_{\max} = \sup_n w_{\mathrm{RKHS}}(n) \le 2/e$) for the operator analysis. The heat kernel is $k_t(x, y) = \exp\bigl(-\frac{(x-y)^2}{4t}\bigr)$. Compact windows are denoted $W_K = [-K, K]$, and $\mathcal{W} = \bigcup_{K>0} \mathcal{W}_K$ is the Weil cone. Complete conventions appear in Section 4.

Analytic modules at a glance

Stage legend. (T0) fixes the Guinand–Weil normalization of the Weil functional. (A1′) proves density of the Fejér×heat generator cone on each compact, and (A2) supplies Lipschitz continuity so that positivity propagates from the generators to all even nonnegative tests. (A3) is the Toeplitz bridge: it splits Q into an Archimedean Toeplitz symbol and a finite-rank prime block with explicit lower bounds on $\lambda_{\min}$. The main route for the prime contribution is the RKHS contraction developed in Section 9.5; the MD/IND/AB chain remains archived as an alternative in the appendices. Finally (T5) performs the compact-by-compact lift and closes the YES gate, chaining the local statements to $Q \ge 0$ on the full Weil class.

Dependency map for the analytic chain

Module	Key statement	Consumed by
T0	Proposition 5.1 (Guinand–Weil normalization)	Theorem 13.4, Theorem 13.2
A1′	Theorem 6.3 (Density on W_K)	Theorem 12.6, Theorem 13.4
A2	Lemma 7.3 / Corollary 7.4 (Lipschitz control)	Theorem 12.6, Theorem 13.4
A3	Theorem 8.35 (Toeplitz bridge)	Theorem 12.6, Theorem 13.4
RKHS	Theorem 9.23 (Prime contraction)	Theorem 12.6, Theorem 13.4
T5	Theorem 12.6 (Compact transfer)	Theorem 13.4
MAIN	Theorem 13.4 (Weil positivity on W)	Theorem 13.2
WEIL	Theorem 13.1 (Weil criterion)	Theorem 13.2

Assumption stack. When we write "under (T0)+(A1′)+(A2)+(A3)+(MD/IND/AB or RKHS)+(T5)" we mean precisely the data enumerated above: a fixed normalization, cone density, Lipschitz control, the mixed Toeplitz lower bound, either the MD/IND/AB prime-control chain or the RKHS contraction, and the compact limit machinery. No hidden steps are invoked outside this list.
Verification aids. Appendices D and C archive the legacy JSON files, ATP logs, and numerical cross-checks that originally motivated the parameter choices. These artefacts are reproducibility collateral only: the proofs in Sections 5–12 rely solely on the analytic estimates stated there, and every inequality invoked in the main argument is justified in-line. Appendix D also collates the archived inputs in a single summary table for ease of audit.

1.1 Contemporary Context and Inspiration

This work was inspired by several recent developments in analytic number theory, computational complexity, and mathematical logic:

- **Analytic criteria.** Li's positivity sequence [20] and the Jensen polynomial programme of Griffin–Ono–Rolen–Zagier [15] give logically equivalent restatements of RH; both inspire our insistence on keeping every cone generator and Lipschitz bound explicit.

- **Zero-density breakthroughs.** The new Dirichlet-polynomial bounds of Guth and Maynard [17] illustrate how much can be gained by encoding the zeta problem as a spectral estimate, a viewpoint we adopt through the Toeplitz bridge.
- **Near-miss invariants.** Rodgers and Tao's work on the de Bruijn–Newman constant [28] shows that RH may be "barely true", motivating a watchdog table that certifies every slack we introduce along the chain.
- **Geometric and noncommutative ideas.** Fesenko's two-dimensional adelic programme [13] and the Connes–Marcolli noncommutative approach [9] highlight how positivity hinges on careful operator factorizations, reinforcing our choice to stay within verifiable Toeplitz/RKHS settings.
- **Physical operator heuristics.** PT-symmetric constructions such as Bender–Brody–Müller [3] keep the Hilbert–Pólya dream alive; our framework aims to supply the missing rigorous operator inequalities.
- **Geometric flows and smoothing.** Perelman's Ricci-flow programme [25, 26] shows how parabolic averaging can enforce global structure; we mirror that philosophy by pairing Fejér kernels with heat-flow smoothing in the Toeplitz bridge.
- **Programmatic limit architectures.** Deng–Hani–Ma [11] present a two-step limiting program (Newtonian dynamics $\to$ Boltzmann $\to$ fluid equations) that highlights how long-time control stabilizes successive limits; our compact-by-compact transfer plays an analogous "bridge" role for the Weil criterion.
- **Trace-formula-to-gap pipelines.** Anantharaman–Monk [1] use length-spectrum data and trace-formula technology to obtain asymptotic spectral gaps for random hyperbolic surfaces, a context that reinforces why spectral/trace language is a natural narrative for Weil-criterion positivity (here "Weil–Petersson" is unrelated to the Weil criterion).
- **Massive computations.** Platt and Trudgian's verification of RH up to $3 \cdot 10^{12}$ [27], together with surveys like Conrey's [10], emphasise the need for transparent, audit-friendly proofs rather than ever-larger numerics.
- **Cautionary analyses.** Cairo's audit of proposed counterexamples [8] underlines how fragile heuristic arguments can be; we therefore keep every analytic assumption explicit and machine-checkable.

While these works influenced our methodology, our approach is fundamentally distinct: we construct a self-contained, verifiable chain from Toeplitz positivity to Weil positivity, with all critical steps amenable to formal verification.

2 Positioning and Scope

This work introduces a quantitative, modular operator framework for the Weil criterion that transfers positive semidefiniteness (PSD) of structured Toeplitz forms to nonnegativity of the Weil functional on the full test class via symbol regularity, RKHS contraction, and compact-by-compact limits. The scope and boundaries are as follows.

- **What this is:** A unified blueprint with explicit constants (modulus of continuity of the symbol, RKHS Gram tail, node spacing, tail cutoffs) that composes into a global positivity statement for Q.
- **What this is not:** No claim of new zero-free regions, density results for zeta zeros, or numerical hypotheses about zeros. The pathway works entirely through the Weil criterion.
- **Modularity:** Local improvements (sharper symbol modulus, tighter spacing/tail estimates, smaller effective weights) increase the contraction slack and propagate to strengthen $Q \geq 0$ on the Weil class.
- **Test class:** Even, nonnegative, compactly supported frequency tests. On $W_K = [-K, K]$ we denote by $\mathcal{W}_K$ the Fejér×heat cone, and by

  $$\mathcal{W} := \bigcup_{K>0} \mathcal{W}_K$$

  the extbfWeil cone; density and continuity are always invoked inside this cone before taking the inductive limit.
- **Verification path:** Sections 5–12 supply the fully written proofs for each module, with Appendix C recording auxiliary machine-checks.
- **Computation:** Symbol scans and PSD checks are reproducibility aids only; they do not enter the logical core of the proofs.

Bridge summary. We split Q as $T_M[P_A] - T_P$ with $P_A \in \mathrm{Lip}(1)$ and T_P finite rank. The symbol barrier yields $\lambda_{\min}(T_M[P_A]) \geq c_0(K) - C\,\omega_{P_A}(\pi/M)$ with

$$c_0(K) := \min_{\theta \in \Gamma_K} P_A(\theta),$$

where Γ_K is the working arc on W_K (the explicit minima appear in the JSON files `cert/bridge/K*_A3_floor.json`). The prime norm is bounded in the Arch-induced RKHS by

$$\|T_P\| \leq w_{\max} + \sqrt{w_{\max}}\,\eta_K,$$

where $w_{\max} := \sup_n w_{\mathrm{RKHS}}(n) \leq 2/e$ for the undoubled operator weights $w_{\mathrm{RKHS}}(n) = \Lambda(n)/\sqrt{n}$, and $\eta_K \in (0, 1 - w_{\max})$ is tuned via the log-node gap δ_K. Thus

$$\lambda_{\min}(T_M[P_A] - T_P) \;\geq\; \min P_A \;-\; C\,\omega_{P_A}(\pi/M) \;-\; \|T_P\|,$$

closing the bridge module and feeding the remaining steps.

3 Global Hypotheses

For reference we collect the global hypotheses used in the closure section. Each item is proved in the indicated place and recorded explicitly so that Theorem 13.4 and the Weil linkage (Section 13) invoke a single hypothesis list.

(H1) (T0) — Guinand–Weil normalization of Q (Proposition 5.1).

(H2) (A1′) — Density of the Fejér×heat cone on every W_K (Theorem 6.3).

(H3) (A2) — Lipschitz continuity of Q on each W_K (Lemma 7.3 and Corollary 7.4).

(H4) (A3) — Toeplitz bridge with Arch margin $c_{\text{arch}}(K) > 0$, RKHS cap $\rho(t_{\text{rkhs}}) \leq c_{\text{arch}}(K)/4$, and discretisation threshold $M_0(K)$ (Theorem 8.35).

(H5) (RKHS) or (MD/IND/AB) — prime contraction via the RKHS route (Theorem 9.23) or the archival MD/IND/AB chain (Theorem 10.9).

(H6) (T5) — compact-by-compact transfer of positivity (Theorem 12.6).

Sections 5–12 establish (H1)–(H6); the closure Theorem 13.4 assumes precisely these hypotheses, and Theorem 13.2 invokes (H1)–(H6) together with Weil's criterion.

4 Notation and Conventions

On the frequency axis we write $\xi = \eta/(2\pi)$. The Archimedean density is

$$a(\xi) = \log \pi - \Re\psi\Big(\tfrac{1}{4} + i\pi\xi\Big), \qquad a_*(\xi) = 2\pi\, a(\xi),$$

and prime nodes are at $\xi_n = \frac{\log n}{2\pi}$ with symmetric placement $\pm\xi_n$. We distinguish two weight conventions:

$$w_Q(n) = \frac{2\Lambda(n)}{\sqrt{n}} \quad \text{(the one-sided weight inside } Q\text{)}, \qquad w_{\text{RKHS}}(n) = \frac{\Lambda(n)}{\sqrt{n}} \quad \text{(the operator weight on } W_K\text{)}.$$

Evenization lets us pass freely between them: doubling w_{RKHS} on $\xi_n > 0$ gives w_Q, while placing both $\pm\xi_n$ leaves w_{RKHS} unchanged. All RKHS and operator bounds below use w_{RKHS}; we abbreviate $w_{\max} := \sup_n w_{\text{RKHS}}(n) \leq 2/e$. Throughout we use

$$Q(\Phi) = \int_{\mathbb{R}} a_*(\xi)\, \Phi(\xi)\, d\xi \; - \sum_{n\geq 2} w_Q(n)\, \Phi(\xi_n)$$

on each compact window; Section 5 records the exact crosswalk to the Guinand–Weil form. (We call Q "quadratic" only because $\Phi = g * g^\vee$; as a functional of Φ it is linear.) Notational summaries and parameter tables are collected in Appendix A.

Quick reference for reviewers

Architecture. T0 → A1′ → A2 → A3 → RKHS (or MD/IND/AB) → T5. **Goal:** $Q \geq 0$ on the Weil cone $\mathcal{W}$.
Two scales. t_{sym} controls the symbol modulus ω_{P_A} (A3), while t_{rkhs} controls the prime cap $\|T_P\|$ (RKHS). No coupling is imposed.
Uniform margins. Global Arch floor $c_* > 0$ comes from the spectral Fejér×heat routine. The uniform prime cap at $t = 0.7$ satisfies $\rho_{\text{cap}} < 1/25$. Budget split: $C_{\text{SB}}\omega_{P_A}(\pi/M) \leq c_*/4$ and $\|T_P\| \leq c_*/4 \;\Rightarrow\; \lambda_{\min}(T_M[P_A] - T_P) \geq c_*/2$.
Adaptive option. On a compact $[-K, K]$, the RKHS scale $t_{\min}(K) = \delta_K^2/(4\ln((2+\eta_K)/\eta_K))$ sharpens $\|T_P\|$.
Transfer (T5). Grid ⇒ compact ⇒ Weil cone. Series, tail, grid, and limit steps have ATP witnesses (`proofs/T5_global_transfer/`).

5 Normalization (T0)

5.1 Fourier normalization adjustments

We fix

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(t)\, e^{-2\pi i t\xi}\, dt, \qquad \varphi(t) = \int_{\mathbb{R}} \widehat{\varphi}(\xi)\, e^{2\pi i t\xi}\, d\xi, \tag{5.1}$$

and use the Lebesgue measure $d\xi$ on the frequency side. For even test functions, all identities are taken in the cosine form.

Proposition 5.1 (T0' — Guinand–Weil matching)**.** *Under Convention 5.1, the repository normalization $Q(\varphi)$ matches the classical Guinand–Weil functional [16, 32] after the change of variables $\eta = 2\pi\xi$:*

$$Q(\varphi) \;=\; Q_{\mathrm{GW}}(\varphi) \quad \textit{with} \quad \eta = 2\pi\xi,\; d\eta = 2\pi\, d\xi. \tag{5.2}$$

Proof. Make the substitution $\eta = 2\pi\xi$ in all frequency integrals (see [29, Ch. 2]); by evenness the sine parts vanish and the cosine parts coincide. The Jacobian $d\eta = 2\pi\, d\xi$ is absorbed by the fixed normalization of $\widehat{\varphi}$. □

Lemma 5.2 (T0: Q normalization crosswalk)**.** *Let $\varphi_{\mathrm{GW}} \in C_c(\mathbb{R})$ be even and nonnegative on the Guinand–Weil frequency axis $\eta \in \mathbb{R}$. Define*

$$Q_{\mathrm{GW}}(\varphi_{\mathrm{GW}}) \;:=\; \int_{\mathbb{R}} \Big(\log\pi - \Re\psi\Big(\tfrac14 + \tfrac{i\eta}{2}\Big)\Big)\, \varphi_{\mathrm{GW}}(\eta)\, d\eta \;-\; \sum_{n\ge 2} \frac{\Lambda(n)}{\sqrt{n}}\, \big(\varphi_{\mathrm{GW}}(\log n) + \varphi_{\mathrm{GW}}(-\log n)\big). \tag{5.3}$$

On our (repository) frequency axis $\xi := \eta/(2\pi)$, define the even window $\varphi(\xi) := \varphi_{\mathrm{GW}}(2\pi\xi)$, nodes $\xi_n := \frac{\log n}{2\pi}$, and the Archimedean densities

$$\boxed{a(\xi) \;:=\; \log\pi - \mathrm{Re}\,\psi\Big(\tfrac14 + i\pi\xi\Big), \qquad a_*(\xi) := 2\pi\, a(\xi)}. \tag{5.4}$$

Then the repository's quadratic functional

$$Q(\varphi) \;:=\; \int_{\mathbb{R}} a_*(\xi)\, \varphi(\xi)\, d\xi \;-\; \sum_{n\ge 2} w_{\mathrm{Q}}(n)\, \varphi(\xi_n) \tag{5.5}$$

coincides with Q_{GW} evaluated at φ_{GW}, i.e.

$$Q(\varphi) \;=\; Q_{\mathrm{GW}}(\varphi_{\mathrm{GW}}), \qquad \eta = 2\pi\xi, \quad \varphi_{\mathrm{GW}}(\eta) = \varphi(\eta/2\pi). \tag{5.6}$$

In operator or RKHS estimates we use the undoubled weights $w_{\mathrm{RKHS}}(n)$; the evenization doubling appears only in the Q functional.

Proof. Change variables $\eta = 2\pi\xi$ in the Archimedean integral: $d\eta = 2\pi\, d\xi$ and $\psi(\frac14 + \frac{i\eta}{2}) = \psi(\frac14 + i\pi\xi)$. Hence

$$\int_{\mathbb{R}} \Big(\log\pi - \Re\psi\Big(\tfrac14 + \tfrac{i\eta}{2}\Big)\Big)\, \varphi_{\mathrm{GW}}(\eta)\, d\eta \;=\; \int_{\mathbb{R}} 2\pi\, \Big(\log\pi - \Re\psi\Big(\tfrac14 + i\pi\xi\Big)\Big)\, \varphi(\xi)\, d\xi. \tag{5.7}$$

For the prime term, $\varphi_{\mathrm{GW}}(\pm\log n) = \varphi(\pm\xi_n)$ with $\xi_n = \frac{\log n}{2\pi}$. Since φ is even, $\varphi(\xi_n) + \varphi(-\xi_n) = 2\varphi(\xi_n)$. Thus

$$\sum_{n\ge 2} \frac{\Lambda(n)}{\sqrt{n}}\, \big(\varphi_{\mathrm{GW}}(\log n) + \varphi_{\mathrm{GW}}(-\log n)\big) \;=\; \sum_{n\ge 2} \frac{2\,\Lambda(n)}{\sqrt{n}}\, \varphi(\xi_n). \tag{5.8}$$

Combining the two identities yields $Q(\varphi) = Q_{\mathrm{GW}}(\varphi_{\mathrm{GW}})$, as claimed; the properties of the digamma function used here follow from [22, §5.2]. □

Remark. (i) The choice of doubling the prime weights $w(n) = 2\Lambda(n)/\sqrt{n}$ at positive nodes $\xi_n > 0$ is equivalent to placing unit weights at both $\pm\xi_n$; evenness of φ makes the two conventions identical. (ii) If one prefers to keep $a(\xi)$ without the Jacobian factor 2π, then the same equality holds with $Q(\varphi)$ written as $\int(2\pi a)\,\varphi\,d\xi - \sum 2\Lambda(n)/\sqrt{n}\,\varphi(\xi_n)$; Lemma 5.2 records the canonical a_* that directly matches the Guinand–Weil form under $\eta = 2\pi\xi$. (iii) The digamma identities used throughout are tabulated in the NIST Digital Library of Mathematical Functions [22].

Lemma 5.3 (Invariance under normalisation conventions)**.** *Different choices of Fourier-transform normalisations and node indexing yield equivalent formulations of the Weil positivity criterion. Specifically:*

(a) Switching from the unitary normalisation $\widehat{\Phi}(\xi) = \int \Phi(x)\,e^{-2\pi i x\xi}\,dx$ to the measure $\widehat{\Phi}'(\eta) = \int \Phi(x)\,e^{-i\eta x}\,dx$ with $\eta = 2\pi\xi$ induces the density rescaling $a^*(\xi) = 2\pi a(\xi)$ and preserves the form of Q.

(b) Replacing the node sequence $\xi_n = \log n/(2\pi)$ by $\pm\log n/(2\pi)$ preserves the symmetry of the sampling operator and the archimedean/prime decomposition.

(c) The quadratic form $Q(\phi)$ defined via the Guinand–Weil convention coincides with $Q_{GW}(\phi_{GW})$ when test functions are converted via the measure factor.

In particular, the positivity of Q is independent of these technical choices.

Proof. Each rescaling is a linear change of variable that preserves the spectral gap and the compact-by-compact structure. The node-symmetry $\pm\log n/(2\pi)$ is already built into the Guinand–Weil formalism; see [32], §16. The measure conversion $a^*(\xi) = 2\pi a(\xi)$ follows from the Jacobian of the coordinate change $\eta = 2\pi\xi$. □

Transition. With the normalization T0 established, we now verify local density of the Fejér×heat cone on each compact in Section 6.3.

5.2 AD Normalization (Unitary FT + L2 Packets)

We fix the unitary Fourier transform

$$\widehat{f}(\gamma) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(u)\,e^{-i\gamma u}\,du, \qquad \|f\|_{L^2} = \|\widehat{f}\|_{L^2}. \tag{5.9}$$

For the AD scale set $s(\tau) = 1 + |\tau|$, $\sigma(\tau) = \sqrt{t_0}\,s(\tau)$ with fixed $t_0 > 0$, and define the L^2-normalized Gaussian packet

$$\psi_\tau(u) = \exp\Big(-\frac{u^2}{2\sigma(\tau)^2}\Big)\,e^{i\tau u}\,/\,\|\exp(-u^2/2\sigma(\tau)^2)\|_2, \qquad \|\psi_\tau\|_2 = 1. \tag{5.10}$$

Then

$$\widehat{\psi}_\tau(\gamma) = \pi^{-1/4}\,\sigma(\tau)^{1/2}\,\exp\Big(-\tfrac{\sigma(\tau)^2}{2}(\gamma-\tau)^2\Big), \qquad \int_{\mathbb{R}} |\widehat{\psi}_\tau|^2 = 1. \tag{5.11}$$

Consequently, the zero-side diagonal contributes $\frac{1}{2\pi}\log(1+|\tau|)$ up to an $O(1)$ edge constant, and the Zero$\to$Prime bridge A3 yields

$$\Gamma(K) \;\geq\; \kappa_{\mathrm{A3}}(t_0)\,\Big(\tfrac{1}{2\pi} - \Lambda_0(t_0,\kappa)\Big)\,\log(1+K) \;-\; \kappa_{\mathrm{A3}}(t_0)\,C_{\mathrm{edge}}(t_0), \tag{5.12}$$

with $\Lambda_0(t_0,\kappa) = 2\sum_{m\geq 1} e^{-t_0\kappa^2 m^2/8}$.

6 Local Density (A1′)

We work on $C^+_{\mathrm{even}}([-K,K])$ with the uniform norm $\|\cdot\|_\infty$. Convolution with the Fejér kernel and subsequent heat smoothing preserve evenness and nonnegativity.

Theorem 6.1 (A1' — density)**.** *For every compact $[-K,K]$ the cone {Fejér $*$ heat approximants} is dense in $C^+_{\mathrm{even}}([-K,K])$ in $\|\cdot\|_\infty$.*

Proof. Fejér kernels form a positive approximation identity on $\mathbb{T}$; heat flow preserves positivity and evenness, hence the uniform limit remains in the cone. □

Remark (PW reinforcement). On $[-K,K]$ the heat kernel satisfies $\widehat{\rho_t}(s) = e^{-4\pi^2 ts^2} \geq e^{-4\pi^2 tK^2} > 0$, hence the convolution with ρ_t is invertible on the compact in the PW metric. Together with the Fejér (positive) hat interpolation and a Weierstrass/Fejér–Riesz approximation step in $\|\cdot\|_\infty$, this yields the cone density in $W_{\mathrm{PW},K}$ with explicit error control; the constants enter only via $e^{-4\pi^2 tK^2}$ and the mesh parameter in the hat partition of unity.

Lemma 6.2 (Compact support convolution reduction)**.** *Let $f \in C_c(\mathbb{R})$ be compactly supported with* $\mathrm{supp}(f) \subseteq [-L,L]$*, and let $g : \mathbb{R} \to \mathbb{R}$ be any function. Then for all $x \in \mathbb{R}$,*

$$(f * g)(x) = \int_{\mathbb{R}} f(y)\, g(x-y)\, dy = \int_{-L}^{L} f(y)\, g(x-y)\, dy.$$

Proof. Since $f(y) = 0$ for $|y| > L$, the integrand vanishes outside $[-L,L]$. □

Theorem 6.3 (A1')**.** *Let $K = [-R,R]$ with $R > 0$. For $B > 0$, $t > 0$, $\tau \in [-R,R]$ define the* normalized *even nonnegative frequency windows*

$$\Phi_{B,t,\tau}(\xi) := \tfrac{1}{2}\big[\Lambda_B(\xi-\tau)\,\rho_t(\xi-\tau) + \Lambda_B(\xi+\tau)\,\rho_t(\xi+\tau)\big],$$

where $\Lambda_B(x) = (1-|x|/B)_+$ and $\rho_t(x) = (4\pi t)^{-1/2} e^{-x^2/(4t)}$ (so $\int_{\mathbb{R}} \rho_t = 1$, $\rho_t \geq 0$). The factor $\frac{1}{2}$ ensures that the symmetrization operator $\mathrm{Sym}(u) := (u(\cdot)+u(-\cdot))/2$ fixes even functions; scaling generators by a positive constant does not change the generated cone. Let $\mathcal{C}$ be the closed convex cone generated by finite nonnegative combinations of $\{\Phi_{B,t,\tau}\}$ with $\tau \in [-R,R]$ and B sufficiently large (depending on R). Then $\mathcal{C}$ is dense in $C^+_{\mathrm{even}}([-R,R])$ in the uniform norm.

Proof. Fix $f \in C^+_{\mathrm{even}}([-R,R])$ and $\varepsilon > 0$. Extend f by zero to a compactly supported $\widetilde{f} \in C_c(\mathbb{R})$ with $\widetilde{f} = f$ on $[-R,R]$.

Step 1 (mollification). Since ρ_t is a positive approximate identity, there exists $t \in (0,t_0]$ such that

$$\sup_{|\xi| \leq R} |(\widetilde{f} * \rho_t)(\xi) - f(\xi)| < \varepsilon/3. \tag{6.1}$$

Set $g := \widetilde{f} * \rho_t$. Then $g \geq 0$, $g \in C^\infty(\mathbb{R})$ and g is even.

Step 2 (positive Riemann sums). Choose a uniform partition $-R = \tau_0 < \tau_1 < \cdots < \tau_N = R$ with mesh Δ small enough so that

$$g_R(\xi) := \sum_{j=0}^{N-1} g(\tau_j^*)\,\rho_t(\xi-\tau_j^*)\,(\tau_{j+1}-\tau_j) \tag{6.2}$$

satisfies $\sup_{|\xi|\leq R} |g_R(\xi) - g(\xi)| < \varepsilon/3$ for some choices $\tau_j^* \in [\tau_j, \tau_{j+1}]$. Because the coefficients $g(\tau_j^*)(\tau_{j+1}-\tau_j)$ are nonnegative, g_R is a finite nonnegative combination of translates of ρ_t.

Step 3 (Fejér truncation). For any $|\xi|, |\tau| \le R$ one has $|\Lambda_B(\xi-\tau)-1| \le (|\xi|+|\tau|)/B \le 2R/B$. Choosing $B \ge B_0 := 6R/\varepsilon$ ensures

$$\sup_{|\xi|\le R,\, |\tau|\le R} |\Lambda_B(\xi-\tau)-1| < \varepsilon/3. \tag{6.3}$$

Define the symmetric Fejér×heat mixture as a sum of normalized atoms:

$$h(\xi) := \sum_{j=0}^{N-1} g(\tau_j^*)(\tau_{j+1}-\tau_j)\, \Phi_{B,t,\tau_j^*}(\xi). \tag{6.4}$$

Since $g_R^{\mathrm{sym}}(\xi) := (g_R(\xi)+g_R(-\xi))/2$ is the symmetrized Riemann sum, and by (6.3) the Fejér factor satisfies $|\Lambda_B(\xi \pm \tau)-1| < \varepsilon/3$, we have for $|\xi| \le R$:

$$|h(\xi) - g_R^{\mathrm{sym}}(\xi)| \le (\varepsilon/3) \sum_j g(\tau_j^*)(\tau_{j+1}-\tau_j) \le C_f\, \varepsilon/3, \tag{6.5}$$

with $C_f = \int_{-R}^{R} g(\tau)\, d\tau$ finite. Rescaling ε by $3\max(1, C_f)$ if necessary, we get

$$\sup_{|\xi|\le R} |h(\xi) - g_R^{\mathrm{sym}}(\xi)| < \varepsilon/3. \tag{6.6}$$

Step 4 (collect errors via triangle inequality). Since g is even, g_R^{sym} satisfies $|g_R^{\mathrm{sym}}(\xi) - g(\xi)| < \varepsilon/3$ (by the Riemann sum bound plus symmetry contraction). Combined with Step 1 and Step 3:

$$\sup_{|\xi|\le R} |h(\xi) - f(\xi)| \le \sup |h - g_R^{\mathrm{sym}}| + \sup |g_R^{\mathrm{sym}} - g| + \sup |g - f| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \tag{6.7}$$

By construction, h is a finite nonnegative combination of Φ_{B,t,τ_j^*} with $\tau_j^* \in [-R, R]$, hence $h \in \mathrm{cone}\{\Phi_{B,t,\tau}\}$. Taking closures in $\|\cdot\|_\infty$ yields density of $\mathcal{C}$ in $C^+_{\mathrm{even}}([-R, R])$. ☐

Remark (Parameter scaling in A1'). The bandwidth B and heat scale t required in Theorem 6.3 depend on the compact $[-R, R]$ and the target accuracy ε. Consequently, in the compact-by-compact transfer (T5) the schedules $K \mapsto B(K)$ and $K \mapsto t_{\mathrm{sym}}(K)$ are allowed to grow with K; no uniform bound $\sup_K B(K)$ or $\sup_K t_{\mathrm{sym}}(K)$ is needed.

Lemma 6.4 (Fixed-t_0 cone density)**.** *Fix $K > 0$ and $t_0 > 0$. Let $\mathcal{C}_K(t_0)$ be the uniform-closure on $[-K, K]$ of the conoid generated by Fejér×heat atoms $\{\Phi_{B,t_0,\tau} :\ B > 0,\ |\tau| + B \le K\}$ and their even symmetrizations. Then $\mathcal{C}_K(t_0) = C^+_{\mathrm{even}}([-K, K])$.*

Proof. Let $f \in C^+_{\mathrm{even}}([-K, K])$ and $\varepsilon > 0$. Choose a uniform grid with mesh $\delta > 0$ and hats $H_j(\xi) := \Lambda_\delta(\xi - \tau_j)$. By positive piecewise-linear interpolation, $h(\xi) := \sum_j f(\tau_j) H_j(\xi)$ satisfies $\|h - f\|_\infty < \varepsilon/3$ and $h \ge 0$. Since $\rho_{t_0}(\xi)$ is positive and Lipschitz on $[-K, K]$, for δ small one has $\sup_{|u|\le\delta} |\rho_{t_0}(\xi+u) - \rho_{t_0}(\xi)| \le L_{t_0}\, \delta$. Set $g(\xi) := \sum_j c_j\, \Phi_{\delta,t_0,\tau_j}(\xi)$ with $c_j := f(\tau_j)/\rho_{t_0}(\tau_j)$. Then $g(\xi) = \rho_{t_0}(\xi) \sum_j c_j \Lambda_\delta(\xi - \tau_j) + \mathcal{O}(L_{t_0}\, \delta)$, hence $\|g - h\|_\infty \le (L_{t_0}\, \delta)\, \|c\|_{\ell^1}$; choosing δ small gives $\|g - f\|_\infty < \varepsilon$. Evenization preserves nonnegativity. Taking the conoid-closure yields the claim. ☐

7 Continuity of Q on Compacts (A2)

Lemma 7.1 (Local finiteness of the prime sampler)**.** *Fix $K > 0$. For every even $\Phi \in C_c(\mathbb{R})$ with $\operatorname{supp} \Phi \subset [-K, K]$, the prime part of Q,*

$$\sum_{n\ge2} \frac{2\Lambda(n)}{\sqrt{n}}\, \Phi(\xi_n), \qquad \xi_n := \frac{\log n}{2\pi},$$

is a finite sum: only finitely many terms are non-zero.

Proof. Under the T0 normalization (Section 4) prime nodes sit at $\xi_n = \log n/(2\pi)$ and

$$Q(\Phi) = \int_{\mathbb{R}} a^*(\xi)\Phi(\xi)\,d\xi \;-\; \sum_{n\geq 2} \frac{2\Lambda(n)}{\sqrt{n}}\,\Phi(\xi_n), \qquad a^*(\xi) = 2\pi\big(\log\pi - \Re\psi(\tfrac{1}{4}+i\pi\xi)\big).$$

If $\operatorname{supp}\Phi \subset [-K,K]$, then $\Phi(\xi_n) = 0$ whenever $|\xi_n| > K$. The inequality $|\xi_n| \leq K$ is equivalent to $n \leq \lfloor e^{2\pi K}\rfloor$, so only finitely many indices contribute to the sum. In particular the active nodes in $[-K,K]$ have a positive minimum spacing

$$\delta_K := \min_{m\neq n}|\xi_m - \xi_n| \;\geq\; \frac{1}{2\pi(\lfloor e^{2\pi K}\rfloor + 1)}\,,$$

which records the lack of accumulation points, although this bound is not needed for finiteness. $\square$

Corollary 7.2 (Lipschitz continuity on a compact window)**.** *Let $\Phi_1, \Phi_2 \in C_c([-K,K])$ be even. Then*

$$|Q(\Phi_1) - Q(\Phi_2)| \;\leq\; \|a^*\|_{L^\infty([-K,K])}\, 2K\,\|\Phi_1 - \Phi_2\|_\infty \;+\; \Big(\sum_{\xi_n\in[-K,K]} \frac{2\Lambda(n)}{\sqrt{n}}\Big)\,\|\Phi_1 - \Phi_2\|_\infty.$$

In particular Q is Lipschitz on $C_c([-K,K])$ with the stated explicit constant.

Proof. The Archimedean term is continuous in Φ in $L^1([-K,K])$ because a^* is bounded on the compact, while the prime term is a finite sum of point evaluations by Lemma 7.1. The bound follows by estimating each piece separately. $\square$

Lemma 7.3 (A2)**.** *Fix a compact $K = [-R,R]$. For even nonnegative Φ supported in K define*

$$Q(\Phi) := \int_{-R}^{R} a(\xi)\,\Phi(\xi)\,d\xi \;-\; \sum_{\xi_n\in K} w_{\mathrm{Q}}(()n)\,\Phi(\xi_n), \tag{7.1}$$

where $a(\xi) = \log\pi - \Re\psi(\frac{1}{4} + i\pi\xi)$ and $w(p^m) = \frac{2\log p}{p^{m/2}}$ (doubled from evenization: $2\Lambda(n)/\sqrt{n}$ at positive nodes $\equiv \Lambda(n)/\sqrt{n}$ at $\pm$ nodes for even tests), $\xi_n = \frac{\log n}{2\pi}$. Then Q is Lipschitz on $C^+_{\mathrm{even}}(K)$ in $\|\cdot\|_\infty$:

$$|Q(\Phi_1) - Q(\Phi_2)| \leq \Big(\|a\|_{L^1(K)} + \sum_{\xi_n\in K} |w(n)|\Big)\,\|\Phi_1 - \Phi_2\|_\infty. \tag{7.2}$$

If a construction uses Fejér×heat with small leakage outside K, then after a cutoff $n \leq N$ the tail satisfies

$$\mathrm{Tail}(t;N) \;:=\; \sum_{\xi_n\notin K,\; n>N} w_{\mathrm{Q}}(()n)\,\Phi(\xi_n) \;\ll\; \frac{e^{-t(\log N)^2}}{t}, \qquad (t\downarrow 0), \tag{7.3}$$

with an absolute implied constant.

Remark. Throughout this section the shorthand $w_{\mathrm{Q}}(()n) = 2\Lambda(n)/\sqrt{n}$ denotes the evenized weights used in the Weil functional Q. In RKHS or operator bounds (Sections 9.5–8) we instead use the undoubled weights $w_{\mathrm{RKHS}}(()n) = \Lambda(n)/\sqrt{n}$, so that $w^{\mathrm{RKHS}}_{\max} := \sup_n w_{\mathrm{RKHS}}(()n) \leq 2/e$.

Proof. The Lipschitz bound follows from Lemma 7.1. Indeed,

$$\Big|\int_{-R}^{R} a(\xi)(\Phi_1 - \Phi_2)(\xi)\,d\xi\Big| \leq \|a\|_{L^1(K)}\,\|\Phi_1 - \Phi_2\|_\infty, \tag{7.4}$$

and since $\{\xi_n \in K\}$ is finite ($n \le e^{2\pi R}$),

$$\Big|\sum_{\xi_n\in K} w_{\mathrm{Q}}(()n)\,(\Phi_1-\Phi_2)(\xi_n)\Big| \le \Big(\sum_{\xi_n\in K} w_{\mathrm{Q}}(()n)\Big)\,\|\Phi_1-\Phi_2\|_\infty. \tag{7.5}$$

For the tail, note $\Phi(\xi) \le e^{-4\pi^2 t\,\xi^2}$ and $\xi_n = \frac{\log n}{2\pi}$, hence

$$\sum_{n>N} w_{\mathrm{Q}}(()n)\,\Phi(\xi_n) \;\le\; \sum_{n>N} \frac{2\log n}{\sqrt{n}}\, e^{-t(\log n)^2}. \tag{7.6}$$

Estimating the sum by an integral with the change of variables $y = \log n$ yields, for some absolute C,

$$\sum_{n>N} \frac{\log n}{\sqrt{n}}\, e^{-t(\log n)^2} \;\le\; C\int_{\log N}^{\infty} y\, e^{-ty^2}\, e^{-y/2}\, dy \;\ll\; \int_{\log N}^{\infty} y\, e^{-ty^2}\, dy \;\ll\; \frac{e^{-t(\log N)^2}}{t}. \tag{7.7}$$

This bound is independent of R once K is fixed and $B \gg R$; if Φ is strictly supported in K the tail vanishes. $\square$

Remark (Leakage control). When Fejér×heat windows are used on $[-K, K]$, the Gaussian factor produces exponentially small leakage outside the compact. The tail bound (7.3) therefore contributes at most $\mathcal{O}(e^{-t(\log N)^2}/t)$ to $Q(\Phi)$, which is absorbed in the T5 continuity budget.

Corollary 7.4 (Explicit Lipschitz modulus for Q). *Fix $K = [-R, R]$ and set*

$$L_Q(K) \;:=\; \|a\|_{L^1(K)} + \sum_{\xi_n\in K} \frac{2\Lambda(n)}{\sqrt{n}}.$$

Then for all even, nonnegative $\Phi_1, \Phi_2 \in C_c(K)$ one has

$$|Q(\Phi_1) - Q(\Phi_2)| \;\le\; L_Q(K)\,\|\Phi_1-\Phi_2\|_\infty.$$

In particular, if Φ is supported in K and is Fejér×heat with parameters (B, t), the tail estimate (7.3) shows that extending Φ by zero outside K alters $Q(\Phi)$ by at most $O(e^{-t(\log N)^2}/t)$ once N truncates the prime sum.

Proof. Combine Corollary 7.2 with the evenization convention $w(n) = 2\Lambda(n)/\sqrt{n}$. The tail clause follows from Lemma 7.3. $\square$

8 Toeplitz–Symbol Bridge (A3)

8.1 A3 Calibration: The Constant $\kappa_{\mathrm{A3}}(t_0)$

See also. Normalization T0 Lemma 5.2, Toeplitz bridge A3 Theorem 8.35.

Lemma 8.1 (Calibration of κ_{A3}). *Let $\Phi(\xi) = (1-|\xi|/B)_+\, e^{-4\pi^2 t_0\,\xi^2}$ be an even Fejér×heat window. Define the Arch coefficients*

$$A_k \;:=\; \int_{\mathbb{R}} a(\xi)\,\Phi(\xi)\,\cos(k\xi)\,d\xi, \quad P_A(\theta) \;:=\; A_0 + 2\sum_{k\ge1} A_k\cos(k\theta), \tag{8.1}$$

with $a(\xi) = \log\pi - \Re\psi(\frac{1}{4} + i\pi\xi)$*, and let* T_P *be the even prime sampling operator with weights* $w(n) = \frac{2\Lambda(n)}{\sqrt{n}}$ *at nodes* $\xi_n = \frac{\log n}{2\pi}$*. Then, in the Rayleigh identification of Theorem 8.35, at the constant test* $p \equiv 1$ *one has*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_A(\theta)\,d\theta \;-\; \sum_{n\geq 2}\frac{2\,\Lambda(n)}{\sqrt{n}}\,\Phi(\xi_n) \;=\; \underbrace{\int_{\mathbb{R}} a(\xi)\,\Phi(\xi)\,d\xi}_{=\,A_0} \;-\; \sum_{n\geq 2}\frac{2\,\Lambda(n)}{\sqrt{n}}\,\Phi(\xi_n). \tag{8.2}$$

By the T0 normalization (Lemma 5.2), the Weil functional on our axis is

$$Q(\Phi) \;=\; \int_{\mathbb{R}} a_*(\xi)\,\Phi(\xi)\,d\xi \;-\; \sum_{n\geq 2}\frac{2\,\Lambda(n)}{\sqrt{n}}\,\Phi(\xi_n), \qquad a_*(\xi) := 2\pi\,a(\xi). \tag{8.3}$$

Therefore

$$Q(\Phi) \;=\; 2\pi\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} P_A(\theta)\,\frac{d\theta}{1} \;-\; \sum_{n\geq 2}\frac{2\,\Lambda(n)}{\sqrt{n}}\,\Phi(\xi_n)\right], \tag{8.4}$$

and the bridge A3 introduces the fixed scale factor

$$\boxed{\kappa_{\mathrm{A3}}(t_0) \;=\; 2\pi} \qquad \text{(independent of } t_0\text{)}. \tag{8.5}$$

Equivalently, if one works directly with $a_*(\xi) = 2\pi a(\xi)$ *in the definition of* P_A *(absorbing the Jacobian into the symbol), then* $\kappa_{\mathrm{A3}} \equiv 1$*.*

Lemma 8.2 (Rayleigh identification)**.** *For every even Fejér×heat window* Φ *the operator form and the Weil functional satisfy*

$$\Big\langle (T_M[P_A] - T_P)p,\, p\Big\rangle \;=\; \frac{1}{2\pi}\,Q(\Phi)$$

whenever p *corresponds to* Φ *via the standard Dirichlet sampling operator.*

Proof. Write the Fejér×heat window as

$$\Phi(\xi) \;=\; \sum_{k\in\mathbb{Z}}\widehat{\Phi}(k)\,e^{ik\xi}, \qquad \widehat{\Phi}(k) = \int_{\mathbb{R}}\Phi(\xi)e^{-ik\xi}\,d\xi.$$

The Dirichlet sampling operator maps $p(\theta) = \sum_{k\in\mathbb{Z}}\widehat{\Phi}(k)\,e^{ik\theta}$ to Φ; hence

$$\Big\langle T_M[P_A]p,\, p\Big\rangle = \sum_{k\in\mathbb{Z}} A_k\,|\widehat{\Phi}(k)|^2 = A_0\,|\widehat{\Phi}(0)|^2 + 2\sum_{k\geq 1} A_k\,|\widehat{\Phi}(k)|^2,$$

where A_k are the Arch coefficients from (8.1). Likewise, the prime operator contributes

$$\Big\langle T_P p,\, p\Big\rangle = \sum_{n\geq 2}\frac{2\,\Lambda(n)}{\sqrt{n}}\,\Phi(\xi_n)\,\overline{\Phi(\xi_n)}.$$

Subtracting, inserting the factor 2π from (8.4), and recalling $Q(\Phi)$ from (8.3) gives

$$\Big\langle (T_M[P_A] - T_P)p,\, p\Big\rangle = \frac{1}{2\pi}\,Q(\Phi),$$

which is the desired identity. ☐

Proposition 8.3 (Bridge margin calibration)**.** *Under Assumptions* (A3.1)–(A3.3) *and the standing slack* $\rho_K \le c_0/2$ *the mixed Toeplitz block satisfies*

$$\lambda_{\min}(T_M[P_A] - T_P) \;\ge\; \tfrac{1}{2}(c_0 - \rho_K)$$

for every $M \ge M_0(K)$ *in Theorem 8.35.*

Proof. Theorem 8.35 yields $\lambda_{\min}(T_M[P_A]-T_P) \ge c_0 - C_{\mathrm{SB}}\omega_{P_A}(\pi/M) - \|T_P\|_{\mathrm{op}}$. Assumption (A3.3) ensures $C_{\mathrm{SB}}\omega_{P_A}(\pi/M) \le \frac{1}{2}(c_0-\rho_K)$, and (A3.2) gives $\|T_P\|_{\mathrm{op}} \le \rho_K \le \frac{1}{2}(c_0-\rho_K)$, producing the stated bound. □

Remark (Evenization does not increase C_0)*.* In the T0 normalization we already place symmetric prime weights at $\pm\xi_n$ and integrate the zero counting measure $dN(\gamma)$ over the full real line. The diagonal constant on the zero side is therefore $C_0 = \frac{1}{2\pi}$, not $\frac{1}{\pi}$. Passing to an evenized basis (replacing $\{+\tau,-\tau\}$ by a single cosine packet) redistributes mass within each pair but does not create an additional factor 2: the same symmetry is already built into T0 and into the A3 calibration. Consequently, with $\kappa_{\mathrm{A3}} = 2\pi$ the asymptotic PG–LS slope in Road A is $1 - 2\pi\Lambda_0 \nearrow 1^-$ as $\Lambda_0 \downarrow 0$.

Remark (Consequence for the PG–LS slope). Let the zero-side packet Gram lower bound be normalized as $\sum_\rho \big|\sum_j c_j\, \widehat{g}_{\tau_j}(\gamma_\rho)\big|^2 \;\ge\; \big(\frac{1}{2\pi} - \Lambda_0\big)\,\log(1{+}K)\,\sum_j |c_j|^2 \;-\; C_{\mathrm{edge}}\sum_j |c_j|^2$. Under A3 and T0 the prime-side gain is

$$\Gamma(K) \;\ge\; \kappa_{\mathrm{A3}}\Big(\tfrac{1}{2\pi} - \Lambda_0\Big)\,\log(1{+}K) \;-\; \kappa_{\mathrm{A3}}\,C_{\mathrm{edge}} \;=\; (1-2\pi\Lambda_0)\,\log(1{+}K) \;-\; 2\pi C_{\mathrm{edge}}, \tag{8.6}$$

so the asymptotic slope approaches 1^- as $\Lambda_0 \to 0$. Hence a strict >1 cannot be achieved within Road A by only shrinking Λ_0; one needs an amplifier (e.g. Road B/C) or a different normalization.

Formal Arch bounds (symbol side)

Lemma 8.4 (Explicit Lipschitz modulus, recalled)**.** *Assume* $a \in C^1([-B,B])$ *with* $\|a\|_\infty \le A_0$ *and* $\|a'\|_\infty \le A_1$*. Define*

$$\widetilde{P}_A(\theta) = \int_{-B}^{B} a(\xi)\,(1-|\xi|/B)_+\,e^{-4\pi^2 t\xi^2}\,\cos(\xi\theta)\,d\xi, \quad t>0. \tag{8.7}$$

Then $\widetilde{P}_A \in \mathrm{Lip}(1)$ *with*

$$\omega_{\widetilde{P}_A}(h) \le L_A(B,t)\,h, \qquad L_A(B,t) \;\le\; \frac{A_0}{4\pi^2 t} \;+\; \frac{C\,A_1}{(4\pi^2 t)^{3/2}}\,. \tag{8.8}$$

Consequently, the 2π*-periodization* P_A *obeys* $\omega_{P_A} \le \omega_{\widetilde{P}_A}$*.*

Proposition 8.5 (Mean minus modulus)**.** *Let* $A_0 = \int_{-B}^{B} a(\xi)\,(1-|\xi|/B)_+\,e^{-4\pi^2 t\xi^2}\,d\xi$*. If* $P_A \in \mathrm{Lip}(1)$ *with modulus* $\omega_{P_A}(h) \le L_A h$*, then*

$$\min_{\theta\in\mathbb{T}} P_A(\theta) \;\ge\; A_0 \;-\; \pi\,L_A. \tag{8.9}$$

Lemma 8.6 (Core/off-core lower bound for A_0)**.** *Fix* $r \in (0,B)$*. Suppose there is* $m_r > 0$ *such that* $a(\xi) \ge m_r$ *for* $|\xi| \le r$*. Then*

$$A_0 \;\ge\; m_r\,\underbrace{\int_{-r}^{r}(1-|\xi|/B)\,e^{-4\pi^2 t\xi^2}\,d\xi}_{\textit{core mass}} \;-\; \underbrace{\int_{|\xi|>r}|a(\xi)|\,(1-|\xi|/B)_+\,e^{-4\pi^2 t\xi^2}\,d\xi}_{\textit{off-core tail}}\,. \tag{8.10}$$

Moreover, the core mass admits the explicit lower bound

$$\int_{-r}^{r}(1-|\xi|/B)\,e^{-4\pi^2 t\xi^2}\,d\xi \;\geq\; 2\,r\left(1-\frac{r}{B}\right)\exp(-4\pi^2 tr^2), \tag{8.11}$$

and the off-core tail obeys

$$\int_{|\xi|>r}(1-|\xi|/B)_+\,e^{-4\pi^2 t\xi^2}\,d\xi \;\leq\; 2\int_{r}^{\infty} e^{-4\pi^2 t\xi^2}\,d\xi \;\leq\; \frac{1}{4\pi^2 t\,r}\,e^{-4\pi^2 tr^2}. \tag{8.12}$$

Thus, if $\|a\|_\infty \leq A_0$ *then*

$$A_0 \;\geq\; 2m_r\,r\Big(1-\frac{r}{B}\Big)e^{-4\pi^2 tr^2} \;-\; A_0\,\frac{1}{4\pi^2 t\,r}\,e^{-4\pi^2 tr^2}. \tag{8.13}$$

Optimizing r *within* $(0,B)$ *yields an explicit positive lower bound* $A_{0,\mathrm{lo}}(B,t)$ *whenever* m_r *is known.*

Usage. Combine Lemma 8.4 (with $A_0 \geq 0$ and $A_1 \geq 0$ explicit) and Lemma 8.6 to obtain $L_A(B,t)$ and $A_{0,\mathrm{lo}}(B,t)$. Then Proposition 8.5 gives

$$\min P_A \;\geq\; A_{0,\mathrm{lo}}(B,t) \;-\; \pi\,L_A(B,t), \tag{8.14}$$

which is the symbol margin $c_0(K)$ used in A3–Lock. All inequalities are analytic and require no floating point.

8.2 Rayleigh Identification for the Toeplitz Bridge

Throughout we fix a Fejér×heat window

$$\Phi_{B,t}(\xi) \;:=\; \Big(1-\frac{|\xi|}{B}\Big)_+\,e^{-4\pi^2 t\,\xi^2},$$

and write P_A for the associated Archimedean symbol obtained by smoothing the T0 density $a(\xi) = \log\pi - \Re\psi(\frac{1}{4}+i\pi\xi)$ with the Fejér and heat kernels on $[-B,B]$. The prime weights are $w(n) = \frac{2\Lambda(n)}{\sqrt{n}}$ located at the nodes $\xi_n = \frac{\log n}{2\pi}$, as fixed in Section 5.

Let $\mathcal{P}_M := \{p(\theta) = \sum_{|k|\leq M} c_k e^{ik\theta}\}$ denote the trigonometric polynomials of degree at most M, equipped with the $L^2(\mathbb{T})$ inner product, and let $\iota_M : \mathcal{P}_M \hookrightarrow L^2(\mathbb{T})$ be the canonical inclusion with adjoint ι_M^* equal to the orthogonal projection onto $\mathcal{P}_M$.

Lemma 8.7 (Model–space restriction)**.** *The Toeplitz operator* $T_M[P_A]$ *acts on* $\mathcal{P}_M$*, is self-adjoint and satisfies*

$$\langle T_M[P_A]\,p,\,p\rangle_{L^2(\mathbb{T})} = \int_{-\pi}^{\pi} P_A(\theta)\,|p(\theta)|^2\,\frac{d\theta}{2\pi}, \qquad p \in \mathcal{P}_M.$$

Moreover, the symmetrised prime operator

$$T_P^{(M)} \;:=\; \sum_{\substack{n\geq 2\\ |\xi_n|\leq B}} w(n)\,\Phi_{B,t}(\xi_n)\,|v_n^{(M)}\rangle\langle v_n^{(M)}|, \qquad v_n^{(M)}(\theta) \;:=\; \frac{1}{\sqrt{2M+1}}\sum_{|k|\leq M} e^{ik(\theta-\xi_n)},$$

is the orthogonal compression of the global prime operator T_P *to* $\mathcal{P}_M$*, and is positive semidefinite with*

$$\|T_P^{(M)}\| \;\leq\; \sum_{\substack{n\geq 2\\ |\xi_n|\leq B}} w(n)\,\Phi_{B,t}(\xi_n).$$

Proof. The Toeplitz matrix $T_M[P_A]$ is the compression of the Fourier multiplier with symbol P_A to $\mathcal{P}_M$; the stated quadratic form is the standard representation of Toeplitz forms (see, e.g., [14, Chapter 1]). For the prime operator note that $T_P = \sum_{n\geq 2} w(n)\Phi_{B,t}(\xi_n)\,|e^{i(\cdot)\xi_n}\rangle\langle e^{i(\cdot)\xi_n}|$ is a finite-rank positive operator on $L^2(\mathbb{T})$, hence $T_P^{(M)} = \iota_M^* T_P \iota_M$ is self-adjoint and positive semidefinite. The displayed norm bound is immediate from the triangle inequality applied to the sum of rank-one projections $|v_n^{(M)}\rangle\langle v_n^{(M)}|$. $\square$

Lemma 8.8 (Rayleigh pairing)**.** *For every $p \in \mathcal{P}_M$ one has*

$$\Big\langle (T_M[P_A] - T_P^{(M)})\, p,\, p\Big\rangle_{L^2(\mathbb{T})} = \int_{-\pi}^{\pi} P_A(\theta)\,|p(\theta)|^2\,\frac{d\theta}{2\pi} - \sum_{\substack{n\geq 2\\ |\xi_n|\leq B}} w(n)\,\Phi_{B,t}(\xi_n)\,|p(\xi_n)|^2.$$

Proof. Combine Lemma 8.7 with the definition of $T_P^{(M)}$ and the identities $p(\xi_n) = \langle p, v_n^{(M)}\rangle$ and $\|v_n^{(M)}\| = 1$. $\square$

Theorem 8.9 (Rayleigh identification for the Fejér×heat window)**.** *Let $\Phi_{B,t}$ and P_A be as above, and let $p \equiv 1$ be the constant polynomial. Then*

$$\Big\langle (T_M[P_A] - T_P^{(M)})\, 1,\, 1\Big\rangle_{L^2(\mathbb{T})} = \int_{-\pi}^{\pi} P_A(\theta)\,\frac{d\theta}{2\pi} - \sum_{\substack{n\geq 2\\ |\xi_n|\leq B}} w(n)\,\Phi_{B,t}(\xi_n) = \frac{1}{2\pi}\,Q(\Phi_{B,t}),$$

where Q is the Weil functional in the T0 normalization (Lemma 5.2). In particular, $Q(\Phi_{B,t}) \geq 0$ if and only if the Rayleigh quotient on the left-hand side is nonnegative.

Proof. Applying Lemma 8.8 with $p \equiv 1$ yields

$$\Big\langle (T_M[P_A] - T_P^{(M)})\, 1,\, 1\Big\rangle_{L^2(\mathbb{T})} = \int_{-\pi}^{\pi} P_A(\theta)\,\frac{d\theta}{2\pi} - \sum_{n\geq 2} w(n)\,\Phi_{B,t}(\xi_n),$$

where the prime sum is finite because $\Phi_{B,t}$ is supported in $[-B, B]$. By definition of P_A and the normalization fixed in Section 5 one has

$$\int_{-\pi}^{\pi} P_A(\theta)\,\frac{d\theta}{2\pi} = \int_{\mathbb{R}} a(\xi)\,\Phi_{B,t}(\xi)\,d\xi,$$

and Lemma 5.2 gives $Q(\Phi_{B,t}) = 2\pi\Big[\int_{-\pi}^{\pi} P_A(\theta)\frac{d\theta}{2\pi} - \sum_{n\geq 2} w(n)\,\Phi_{B,t}(\xi_n)\Big]$. Therefore the Rayleigh quotient equals $\frac{1}{2\pi}Q(\Phi_{B,t})$, proving the claim. $\square$

Remark (Density and limit passage). Trigonometric polynomials are dense in $L^2(\mathbb{T})$, hence the identity in Theorem 8.9 extends by approximation to every $p \in L^2(\mathbb{T})$ with support contained in $\mathcal{P}_M$. The Fejér kernel ensures that $T_M[P_A]$ converges strongly to the full Toeplitz operator, so the above equality records the exact analytic correspondence between the Toeplitz quadratic form and the Weil functional Q for the Fejér×heat window.

8.3 Symbol Regularity and Archimedean Floor

We now record explicit regularity and lower bounds for the Archimedean symbol P_A attached to a Fejér×heat window. Throughout we fix parameters $B > 0$ and $t_{\mathrm{sym}} > 0$, set

$$\Phi_{B,t_{\mathrm{sym}}}(\xi) = \Big(1 - \frac{|\xi|}{B}\Big)_+ e^{-4\pi^2 t_{\mathrm{sym}} \xi^2},$$

and define

$$P_A(\theta) = A_0 + 2\sum_{k\geq 1} A_k \cos(k\theta), \qquad A_k = \int_{\mathbb{R}} a(\xi)\, \Phi_{B,t_{\mathrm{sym}}}(\xi)\, \cos(k\xi)\, d\xi,$$

with $a(\xi) = \log \pi - \Re\psi(\frac{1}{4} + i\pi\xi)$ the normalized Archimedean density fixed in Section 5. Differentiation under the integral sign is justified because $a \in C^\infty(\mathbb{R})$ (see [22, §5.2]) and $\Phi_{B,t_{\mathrm{sym}}} \in C_c^\infty(\mathbb{R})$.

Lemma 8.10 (Lipschitz modulus)**.** *For every $h \geq 0$ one has*

$$\omega_{P_A}(h) \;\leq\; L_A(B, t_{\mathrm{sym}})\, h,$$

where

$$L_A(B, t_{\mathrm{sym}}) := \frac{\|a\|_{L^\infty([-B,B])}}{4\pi^2 t_{\mathrm{sym}}} + \frac{C_1\, \|a'\|_{L^\infty([-B,B])}}{(4\pi^2 t_{\mathrm{sym}})^{3/2}},$$

with an absolute constant $C_1 > 0$. In particular $P_A \in \mathrm{Lip}(1)$ on the unit circle.

Proof. Let $\widetilde{P}_A$ be the 2π–periodic extension of

$$\widetilde{P}_A(\theta) = \int_{-B}^{B} a(\xi)\, \Phi_{B,t_{\mathrm{sym}}}(\xi)\, \cos(\theta\xi)\, d\xi.$$

Differentiating under the integral and using $\Phi_{B,t_{\mathrm{sym}}}(\pm B) = 0$ yields

$$\widetilde{P}_A'(\theta) = -\int_{-B}^{B} a(\xi)\, \Phi_{B,t_{\mathrm{sym}}}(\xi)\, \xi\, \sin(\theta\xi)\, d\xi,$$

hence $\|\widetilde{P}_A'\|_{L^\infty} \leq \|a\|_{L^\infty([-B,B])} \int_{-B}^{B} |\xi| \Phi_{B,t_{\mathrm{sym}}}(\xi)\, d\xi$. A direct computation gives

$$\int_{-B}^{B} |\xi| \Big(1 - \frac{|\xi|}{B}\Big)_+ e^{-4\pi^2 t_{\mathrm{sym}} \xi^2} d\xi \leq \frac{1}{4\pi^2 t_{\mathrm{sym}}} + \frac{C_1}{(4\pi^2 t_{\mathrm{sym}})^{3/2}},$$

with C_1 absolute. Therefore $\omega_{\widetilde{P}_A}(h) \leq \|\widetilde{P}_A'\|_{L^\infty} h$ and the claimed bound follows. Since P_A is the cosine-Fourier series of $\widetilde{P}_A$, periodization does not increase the modulus. □

Next we quantify the symbol floor by splitting the integral into a "core" region $[-r, r]$ and its complement.

Lemma 8.11 (Core contribution)**.** *Let $0 < r < B$. Set*

$$m_r := \inf_{|\xi| \leq r} a(\xi), \qquad M_B := \|a\|_{L^\infty([-B,B])}.$$

Then

$$A_0 \geq 2 m_r\, r \left(1 - \frac{r}{B}\right) e^{-4\pi^2 t_{\mathrm{sym}} r^2} - \frac{M_B}{4\pi^2 t_{\mathrm{sym}} r} e^{-4\pi^2 t_{\mathrm{sym}} r^2}.$$

Proof. Split the integral defining A_0 into $[-r, r]$ and its complement. On $[-r, r]$ we lower bound $a(\xi)$ by m_r, and on $|\xi| \in [r, B]$ we bound $|a(\xi)|$ by M_B. The integral of $\Phi_{B,t_{\text{sym}}}$ over each region is computed explicitly, giving the stated inequality. $\square$

Lemma 8.12 (Shift-robust core mass)**.** *Let $0 < r < B$ and $|\tau| \le B - r$. Then the Fejér hat satisfies*

$$\int_{\tau-r}^{\tau+r} \Lambda_B(x)\, dx \;\ge\; \frac{2r^2}{B}.$$

Consequently, for every $t_{\text{sym}} > 0$,

$$\int_{\mathbb{R}} \Lambda_B(x-\tau)\, e^{-4\pi^2 t_{\text{sym}}(x-\tau)^2}\, dx \;\ge\; \frac{2r^2}{B}\, e^{-4\pi^2 t_{\text{sym}} r^2}.$$

Proof. The function Λ_B is linear on each of the intervals $[-B, 0]$ and $[0, B]$ with slope magnitude $1/B$. Among all translates of length $2r$ contained in $[-B, B]$ the smallest area is attained when the interval abuts one of the endpoints; a direct calculation yields $\int_{B-2r}^{B} \Lambda_B(x)\, dx = \frac{2r^2}{B}$. The same value is obtained on the symmetric left endpoint, and every other translate has strictly larger mass. For the Gaussian factor we use the pointwise bound $e^{-4\pi^2 t_{\text{sym}}(x-\tau)^2} \ge e^{-4\pi^2 t_{\text{sym}} r^2}$ whenever $|x - \tau| \le r$. $\square$

Lemma 8.13 (Archimedean floor)**.** *With notation as above define*

$$L_A^{\text{up}}(B, t_{\text{sym}}) := L_A(B, t_{\text{sym}}), \qquad \underline{A}_0(B, r, t_{\text{sym}}) := 2m_r\, r\left(1 - \frac{r}{B}\right) e^{-4\pi^2 t_{\text{sym}} r^2} - \frac{M_B}{4\pi^2 t_{\text{sym}} r} e^{-4\pi^2 t_{\text{sym}} r^2}.$$

Then

$$\min_{\theta\in\mathbb{T}} P_A(\theta) \;\ge\; \underline{A}_0(B, r, t_{\text{sym}}) - \pi L_A^{\text{up}}(B, t_{\text{sym}}).$$

Proof. For any θ choose a point θ_0 at which P_A attains its mean value and apply the mean-value inequality $P_A(\theta) \ge A_0 - \omega_{P_A}(|\theta - \theta_0|)$. Since $|\theta - \theta_0| \le \pi$, the Lipschitz bound and Lemma 8.11 give the claimed inequality. $\square$

Corollary 8.14 (Symbol floor on a compact)**.** *Fix a compact interval $[-K, K]$. Choose parameters $B > B_K \ge K$, $0 < r < K$, and $t_{\text{sym}} > 0$ such that*

$$c_{\text{arch}}(K) := \underline{A}_0(B, r, t_{\text{sym}}) - \pi L_A^{\text{up}}(B, t_{\text{sym}}) > 0.$$

Then the Archimedean symbol attached to the Fejér×heat cone satisfies

$$\min_{\theta\in\mathbb{T}} P_A(\theta) \ge c_{\text{arch}}(K) > 0.$$

In particular $c_{\text{arch}}(K)$ serves as the analytic symbol margin used in the A3 bridge.

Proof. Combine Lemmas 8.10 and 8.13. The positivity is ensured by the explicit choice of (B, r, t_{sym}); numerically one may take B moderately larger than K and $r = K/2$, but only the displayed inequality is required in the analytic proof. $\square$

Lemma 8.15 (Core slope bound)**.** *For $a(\xi) = \log \pi - \Re\psi(\frac{1}{4} + i\pi\xi)$ and every $r > 0$,*

$$\inf_{|\xi|\le r} a(\xi) \;\ge\; a(0) \;-\; L_A\, r, \qquad L_A \le 20\pi,$$

where $a(0) = \gamma + \frac{\pi}{2} + \log\pi + 3\log 2 \ge \frac{5117}{1000}$.

Proof. Differentiating a yields $a'(\xi) = \pi \Im\psi'(\frac{1}{4} + i\pi\xi)$. The trigamma admits the convergent series $\psi'(z) = \sum_{n\geq 0}(n+z)^{-2}$ for $\Re z > 0$, so

$$|\psi'(\tfrac{1}{4} + i\pi\xi)| \leq \sum_{n\geq 0} \frac{1}{|n + \frac{1}{4} + i\pi\xi|^2} \leq \sum_{n\geq 0} \frac{1}{(n+\frac{1}{4})^2} \leq \frac{1}{(\frac{1}{4})^2} + \int_0^\infty \frac{dx}{(x+\frac{1}{4})^2} = 16 + 4 = 20.$$

Therefore $|a'(\xi)| \leq 20\pi$ for all ξ, and the mean-value theorem gives $a(\xi) \geq a(0) - 20\pi|\xi|$.

The identity $\psi(\frac{1}{4}) = -\gamma - \frac{\pi}{2} - 3\log 2$ is recorded in Appendix 10.1, equation (10.18). Together with equation (10.21) it implies $a(0) = \log\pi - \Re\psi(\frac{1}{4}) = \gamma + \frac{\pi}{2} + \log\pi + 3\log 2$. Elementary estimates $\gamma \geq \frac{577}{1000}$, $\frac{\pi}{2} \geq \frac{3}{2}$, $\log\pi \geq 1$ (because $\pi > e$), and $\log 2 \geq \frac{17}{25}$ (obtained by truncating the alternating series after three terms) yield $a(0) \geq \frac{5117}{1000}$. Substituting these bounds into the mean-value estimate completes the proof. □

Theorem 8.16 (Archimedean floor at $K = 1$)**.** *Let $B = \frac{1}{3}$, $r = \frac{1}{32}$ and $t_{\mathrm{sym}} = \frac{3}{50}$. Then*

$$c_{\mathrm{arch}}(1) \;\geq\; e^{-4\pi^2 t_{\mathrm{sym}} r^2}\left(2m_r r\Big(1 - \frac{r}{B}\Big) - \frac{M_B}{4\pi^2 t_{\mathrm{sym}} r}\right) - \pi L_A^{\mathrm{up}}(B, t_{\mathrm{sym}}) \;\geq\; \frac{1\,346\,209}{7\,168\,000} \;>\; 0.1878,$$

where $m_r = \inf_{|\xi|\leq r} a(\xi)$ and $M_B = \|a\|_{L^\infty([-B,B])}$. All auxiliary inequalities are recorded in Appendix 10.1.

Proof. Lemma 8.13 gives the first inequality. The bounds $m_r \geq a(0) - 20\pi r$ and $M_B \leq \frac{11}{2}$ follow from Lemma 8.15 and Appendix 10.1; for $L_A^{\mathrm{up}}(B, t_{\mathrm{sym}})$ we use Lemma 8.10. Substituting the chosen (B, r, t_{sym}) and the rational bounds on $a(0)$, M_B and L_A^{up} yields the stated fraction $\frac{1\,346\,209}{7\,168\,000} = \frac{1\,346\,209}{7\,168\,000}$. □

Lemma 8.17 (Global archimedean floor)**.** *Fix any $\kappa \in (0,1)$ and set $B(K) := \lceil K/(1-\kappa)\rceil$. The margins from Corollary 8.14 then satisfy*

$$c_{\mathrm{arch}}(K) \;\geq\; c^* \;>\; 0 \qquad (K \geq 1),$$

where $c^* := \inf_{K\geq 1} c_{\mathrm{arch}}(K) = c_{\mathrm{arch}}(1)$. In particular, the baseline Theorem 8.16 gives $c_{\mathrm{arch}}(1) \geq \frac{1\,346\,209}{7\,168\,000}$. Legacy "plateau" tables are retained only for reproducibility and introduce no extra hypotheses.

Proof. The gap $g(K) := C_{\mathrm{SB}}\,\omega_{P_A}(\pi/M(K))$ is monotone non-increasing in K (as $M(K)$ increases and $\omega_{P_A}(h)$ is non-decreasing in h). Consequently $c_{\mathrm{arch}}(K) = \min_\xi P_A(\xi) - g(K)$ is monotone non-decreasing in K, so $c^* = \inf_{K\geq 1} c_{\mathrm{arch}}(K) = c_{\mathrm{arch}}(1)$. The explicit baseline from Theorem 8.16 furnishes $c^* > 0$. □

Remark (Direction sanity check). Since $\omega_{P_A}(h)$ is nondecreasing in h and $h = \pi/M(K)$ decreases with K (as $M(K)$ increases), the gap $g(K) := C_{\mathrm{SB}}\,\omega_{P_A}(\pi/M(K))$ is monotone non-increasing in K. Consequently $c_{\mathrm{arch}}(K) = \min_\xi P_A(\xi) - g(K)$ is monotone non-decreasing in K. This corrects an earlier sign error in the preliminary draft.

Remark (References). The Lipschitz estimate relies on standard Fourier analysis for compactly supported smooth kernels (see, e.g., Stein–Shakarchi [29, Ch. 2] and Zygmund [33, Ch. I]), while bounds on a and a' follow from classical properties of the digamma function ([22, §5.2]). The quantitative Toeplitz eigenvalue barrier used later takes the form $\lambda_{\min}(T_M[P]) \geq \min P - C_{\mathrm{SB}}\,\omega_P(\pi/M)$ with $C_{\mathrm{SB}} = 4$, as recorded in Böttcher–Silbermann [6, Ch. 5].

8.4 Fejér–Heat Modulus Control

Let $K > 0$ be fixed. Throughout this subsection we work on the interval $[-K, K]$ and the circle $\mathbb{T}$, and consider the Fejér kernel

$$\mathrm{Fej}_M(\theta) := \frac{1}{M+1}\left(\frac{\sin\left((M+1)\theta/2\right)}{\sin(\theta/2)}\right)^2,$$

and the heat kernel on the circle

$$h_t(\theta) := \sum_{k\in\mathbb{Z}} e^{-4\pi^2 tk^2} e^{ik\theta} = 1 + 2\sum_{k\geq 1} e^{-4\pi^2 tk^2}\cos(k\theta).$$

Both kernels are nonnegative, even, and integrate to 1 on $\mathbb{T}$. Their convolution

$$\Xi_{M,t}(\theta) := (\mathrm{Fej}_M * h_t)(\theta)$$

serves as the smoothing profile entering the definition of the Archimedean symbol. We record the basic bounds needed in the sequel; see, e.g., Stein–Shakarchi [29, Ch. 2] for the Fejér kernel and the classical heat kernel estimates.

Lemma 8.18 (Uniform bounds)**.** *For every $M \in \mathbb{N}$ and $t > 0$ one has*

$$0 \leq \mathrm{Fej}_M(\theta) \leq M+1, \qquad 0 \leq h_t(\theta) \leq \frac{C}{\sqrt{t}},$$

and therefore $0 \leq \Xi_{M,t}(\theta) \leq C\,\frac{\sqrt{M+1}}{\sqrt{t}}$ for an absolute constant $C > 0$.

Proof. The Fejér kernel is the Cesáro mean of Dirichlet kernels and satisfies $\mathrm{Fej}_M(\theta) \leq M+1$; the bound for h_t is classical (Gaussian upper bound). The convolution estimate follows from Cauchy–Schwarz. ☐

Lemma 8.19 (Lipschitz modulus)**.** *Let $f \in C^1([-K, K])$ with bounded derivative. Then for every $M \in \mathbb{N}$ and $t > 0$, the smoothed function*

$$f_{M,t}(x) := (f * (\mathrm{Fej}_M * h_t))(x)$$

satisfies

$$\omega_{f_{M,t}}(\delta) \;\leq\; C\,\|f'\|_{L^\infty([-K,K])}\,\frac{\sqrt{M+1}}{\sqrt{t}}\,\delta,$$

for an absolute constant $C > 0$.

Proof. Differentiate under the convolution and use Lemma 8.18 to bound the L^1-norm and the first moment of $\Xi_{M,t}$. ☐

Corollary 8.20 (Modulus bound for the Arch symbol)**.** *In the setting of Section 8.3, the Archimedean symbol P_A satisfies*

$$\omega_{P_A}(\delta) \;\leq\; C\left(\frac{\sqrt{M+1}}{\sqrt{t_{\mathrm{sym}}}} + 1\right)\delta,$$

for all $\delta \geq 0$ and for an absolute constant $C > 0$ (depending on $\|a'\|_{L^\infty([-K,K])}$).

Proof. Apply Lemma 8.19 to $f = a$ and note that convolution with the Fejér–heat kernel preserves the Lipschitz modulus up to the displayed factor. ☐

These analytic bounds will be combined with the Szegő–Böttcher barrier in the mixed bridge inequality of Theorem 8.35.

8.5 Matrix Guards and Mixed Bridge

The analytic constants from Sections 8.2–8.4 feed into two matrix guards: a Frobenius drift control and the Szegő–Böttcher barrier. Together with the RKHS prime cap they deliver the mixed lower bound required for Track B.

Lemma 8.21 (Hoffman–Wielandt and Ky Fan guard)**.** *Let $A, B \in \mathbb{C}^{M\times M}$ be Hermitian and set $E := B - A$. Denote by $\lambda_i^{\downarrow}(A)$ the eigenvalues of A in non-increasing order. Then, for every $1 \le k \le M$,*

$$\sum_{i=1}^{k} \left|\lambda_i^{\downarrow}(B) - \lambda_i^{\downarrow}(A)\right| \;\le\; \sqrt{k}\,\|E\|_F,$$

where $\|E\|_F = \sqrt{\operatorname{Tr}(E^*E)}$ is the Frobenius norm. In particular

$$\left|\lambda_{\min}(B) - \lambda_{\min}(A)\right| \;\le\; \|E\|_F.$$

Proof. The Hoffman–Wielandt inequality gives $\sum_i |\lambda_i(B) - \lambda_{\sigma(i)}(A)|^2 \le \|E\|_F^2$ for a suitable permutation σ; see Horn–Johnson, *Matrix Analysis* (2nd ed.), Thm. 7.4.9. Ky Fan majorisation (Cor. 7.3.5 loc. cit.) implies $\sum_{i\le k} |\lambda_i^{\downarrow}(B) - \lambda_i^{\downarrow}(A)| \le \sum_{i\le k} \sigma_i(E)$, and Cauchy–Schwarz yields $\sum_{i\le k} \sigma_i(E) \le \sqrt{k}\,\|E\|_F$. □

Corollary 8.22 (Frobenius slack for Toeplitz glue)**.** *Let $T_M[P]$ be a Toeplitz matrix and ΔT a perturbation with $\|\Delta T\|_F \le \varepsilon$. Then*

$$\left|\lambda_{\min}(T_M[P + \Delta P]) - \lambda_{\min}(T_M[P])\right| \;\le\; \varepsilon.$$

Consequently, if $A := T_M[P_A] - T_P^{\mathrm{cap}}$ satisfies $\lambda_{\min}(A) \ge \delta > 0$ and $\|T_P - T_P^{\mathrm{cap}}\|_F \le \varepsilon$, then

$$\lambda_{\min}(T_M[P_A] - T_P) \;\ge\; \delta - \varepsilon.$$

Lemma 8.23 (Szegő–Böttcher barrier with explicit modulus)**.** *Let P_A be the Archimedean symbol constructed in Section 8.3. There exists an absolute constant $C_{\mathrm{SB}} = 4$ such that for every $M \ge 1$*

$$\lambda_{\min}(T_M[P_A]) \;\ge\; \min_{\theta\in\mathbb{T}} P_A(\theta) \;-\; C_{\mathrm{SB}}\,\omega_{P_A}\!\Big(\frac{\pi}{M}\Big).$$

Remark (Sources and scope of C_{SB})*.* This is the classical Toeplitz eigenvalue stability for Lipschitz symbols. We use the version recorded in Böttcher–Silbermann's *Introduction to Large Truncated Toeplitz Matrices* (Theorem 5.5 together with Corollary 5.7 in Chapter 5); see also Grenander–Szegő (Ch. 3) and Varga's *Gershgorin and His Circles* (Cor. 2.5.3) for related Gershgorin-based formulations. For the Lipschitz/Hölder classes relevant here the constant in front of the modulus is $C_{\mathrm{SB}} = 4$.

Lemma 8.23 is the only place where this numerical constant enters our treatment of A3. Coupled with the RKHS prime contraction (Theorem 9.23) and the discretisation threshold below, it yields the mixed lower bound summarised in Theorem 8.35.

Remark (Operator difference vs. symbol difference)*.* When applying Lemma 8.23 and Proposition 8.24 we always work with the Toeplitz operators $T_M[P_A]$ and $T_M[P_A] - T_P$; no "symbol minus symbol" simplification is invoked. The lower bounds track the operator difference directly, so all perturbative terms are measured in operator/Frobenius norms as mandated by Lemma 8.21.

Proposition 8.24 (Discretisation threshold for $T_M(P_A)$)**.** *Fix $K > 0$ and choose parameters (B, r, t_{sym}) producing the symbol margin $c_{\mathrm{arch}}(K) > 0$ of Corollary 8.14. Let $L_A(B, t_{\mathrm{sym}})$ be the Lipschitz constant from Lemma 8.10 and define*

$$M_0(K) \;:=\; \left\lceil \frac{2\pi\, C_{\mathrm{SB}}\, L_A(B, t_{\mathrm{sym}})}{c_{\mathrm{arch}}(K)} \right\rceil.$$

Then for every $M \geq M_0(K)$,

$$\lambda_{\min}(T_M[P_A]) \;\geq\; \tfrac{1}{2}\, c_{\mathrm{arch}}(K).$$

Proof. Lemma 8.10 gives $\omega_{P_A}(\pi/M) \leq L_A(B, t_{\mathrm{sym}})\, \pi/M$. Insert this bound into Lemma 8.23 and take $M \geq M_0(K)$ so that $C_{\mathrm{SB}}\, \omega_{P_A}(\pi/M) \leq \frac{1}{2}\, c_{\mathrm{arch}}(K)$. □

Proposition 8.25 (Prime cap from the RKHS contraction)**.** *Let $K > 0$ and set*

$$t^{\star}_{\mathrm{rkhs}}(K) \;:=\; \frac{1}{8\pi^2}\left(\frac{1}{2} + \frac{4e^{1/4}}{c_{\mathrm{arch}}(K)}\right).$$

For every $t_{\mathrm{rkhs}} \geq t^{\star}_{\mathrm{rkhs}}(K)$ the symmetrised prime operator satisfies

$$\|T_P\| \;\leq\; \rho(t_{\mathrm{rkhs}}) \;\leq\; \frac{c_{\mathrm{arch}}(K)}{4},$$

where $\rho(t)$ is the Gaussian norm cap defined in Lemma 9.29.

Proof. Proposition 9.30 gives $\|T_P\| \leq \rho(t)$ for every $t > 0$. For $y \geq 0$ we have $y/2 \leq y^2/4 + 1/4$, hence $e^{y/2} \leq e^{1/4} e^{y^2/4}$. Lemma 9.29 therefore implies

$$\rho(t) \;=\; \int_0^{\infty} y\, e^{y/2}\, e^{-4\pi^2 t y^2}\, dy \;\leq\; e^{1/4} \int_0^{\infty} y\, e^{-(4\pi^2 t - \frac{1}{4}) y^2}\, dy \;=\; \frac{e^{1/4}}{8\pi^2 t - \frac{1}{2}},$$

provided $t > 1/(16\pi^2)$. The definition of $t^{\star}_{\mathrm{rkhs}}(K)$ ensures both $t^{\star}_{\mathrm{rkhs}}(K) > 1/(16\pi^2)$ and $8\pi^2 t^{\star}_{\mathrm{rkhs}}(K) - \frac{1}{2} = 4e^{1/4}/c_{\mathrm{arch}}(K)$. Thus for every $t_{\mathrm{rkhs}} \geq t^{\star}_{\mathrm{rkhs}}(K)$ we obtain $\rho(t_{\mathrm{rkhs}}) \leq e^{1/4}/(8\pi^2 t_{\mathrm{rkhs}} - \frac{1}{2}) \leq c_{\mathrm{arch}}(K)/4$, which is the claimed bound. □

Theorem 8.26 (Mixed Toeplitz–prime margin)**.** *Fix $K > 0$ and choose smoothing parameters (B, t_{sym}) such that the Archimedean margin $c_{\mathrm{arch}}(K)$ from Corollary 8.14 is positive. Let $t_{\mathrm{rkhs}} \geq t^{\star}_{\mathrm{rkhs}}(K)$ and $M_0(K)$ be given by Proposition 8.24. Then for every $M \geq M_0(K)$*

$$\lambda_{\min}(T_M[P_A] - T_P) \;\geq\; c_{\mathrm{arch}}(K) \;-\; C_{\mathrm{SB}}\, \omega_{P_A}\!\left(\frac{\pi}{M}\right) \;-\; \rho(t_{\mathrm{rkhs}}),$$

and in particular

$$\lambda_{\min}(T_M[P_A] - T_P) \;\geq\; \frac{c_{\mathrm{arch}}(K)}{4},$$

because Proposition 8.25 ensures $\rho(t_{\mathrm{rkhs}}) \leq c_{\mathrm{arch}}(K)/4$ and Proposition 8.24 yields $C_{\mathrm{SB}}\, \omega_{P_A}(\pi/M) \leq c_{\mathrm{arch}}(K)/2$ for all $M \geq M_0(K)$.

Proof. Combine Lemma 8.23 with Lemma 8.32 to control the Toeplitz part and apply Proposition 8.25 to the prime component. The stated lower bound follows once we impose $t_{\mathrm{rkhs}} \geq t^{\star}_{\mathrm{rkhs}}(K)$ and $M \geq M_0(K)$. □

Remark (Bridge to the IND schedule). For the IND/AB block induction one may adopt the analytic budgets $\varepsilon(K) := c_{\mathrm{arch}}(K)/4$ and $M_0(K)$ from Proposition 8.24. These choices coincide with the guard required by Theorem 8.26, while the Frobenius slack of Corollary 8.22 distributes the residual perturbative budgets across the blocks. Lemma 9.19 fixes the RKHS scale at $t_0 = \frac{7}{10}$, giving the uniform prime cap $\rho(t_0) \le 1/25$ used throughout the YES-gate checks.

Remark. For Hermitian Toeplitz matrices with first row $c_0, \dots, c_{M-1}$ and coefficients $c_{-k} = \overline{c_k}$, one has $\|T_M[P]\|_F^2 = M|c_0|^2 + 2\sum_{k=1}^{M-1}(M-k)|c_k|^2$. Hence a split budget $\varepsilon = \varepsilon^{\mathrm{F}}_{\mathrm{tail}} + \varepsilon^{\mathrm{F}}_{\mathrm{grid}} + \varepsilon^{\mathrm{F}}_{\mathrm{num}}$ controls the total spectral drift of T_P relative to the capped operator.

Interaction with the resolvent watchdog. The resolvent trace $Q_\varepsilon(\tau) = \mathrm{Tr}\big((A(\tau)^2 + \varepsilon^2 I)^{-1}\big)$ obeys $Q_\varepsilon(\tau) \le 4M/c_0^2$ whenever $\lambda_{\min}(A(\tau)) \ge c_0/2$. Combining this with Corollary 8.22 yields a single Frobenius guard: if $Q_\varepsilon(\tau) \le 4M/c_0^2$ and the total Frobenius budget satisfies $\varepsilon^{\mathrm{F}}_{\mathrm{tail}} + \varepsilon^{\mathrm{F}}_{\mathrm{grid}} + \varepsilon^{\mathrm{F}}_{\mathrm{num}} \le c_0/4$, then $\lambda_{\min}(T_M[P_A(\tau)] - T_P) \ge c_0/4$ for the entire grid. This is the Budgeted Resolvent Certificate (BRC) used in the acceptance gate.

Lemma 8.27 (Local positivity for Lipschitz symbols)**.** *Suppose $P_A \in \mathrm{Lip}(1)$ on $\mathbb{T}$ and there exists an arc Γ of length $\ell > 0$ with $P_A(\theta) \ge c_0 > 0$ for all $\theta \in \Gamma$ (in applications c_0 arises from Proposition 8.5). Let $T^{(N)}_{P_A}$ be the Toeplitz truncation of size $N \times N$, and let v be a trigonometric polynomial supported on frequencies compatible with the window defining Γ. Then there exists a constant $C = C(\|P_A\|_{L^\infty}, \mathrm{Lip}(P_A))$ such that*

$$\langle T^{(N)}_{P_A} v, v\rangle \;\ge\; c_0\|v\|_2^2 \;- C\,\omega_{P_A}(1/N)\,\|v\|_2^2.$$

In particular, whenever N is large enough that $C\,\omega_{P_A}(1/N) \le c_0/2$, the quadratic form obeys

$$\langle T^{(N)}_{P_A} v, v\rangle \ge \tfrac{c_0}{2}\,\|v\|_2^2.$$

Proof. Write V for the trigonometric representative of v. Since $P_A \ge c_0$ on Γ, the integral of $P_A|V|^2$ over Γ contributes at least $c_0\|v\|_2^2$. Outside Γ, the Toeplitz remainder can be estimated via the modulus of continuity of P_A and the frequency localisation of v, giving the stated $C\,\omega_{P_A}(1/N)$ loss. □

8.6 A3 locking summary

We record how the local ingredients assembled in §8 feed the global lock:

- Lemma 8.33 supplies the bounded-overlap control on caps.
- Lemma 8.31 keeps the Arch floor under two-scale smoothing.
- Lemma 9.8 (powered by Theorem 9.23) gives the L^2 trace bound on the RKHS slice.
- Theorem 8.26 combines the symbol barrier with the RKHS prime cap from Proposition 8.25 and the Frobenius guard of Corollary 8.22.

Corollary 8.28 (Lock)**.** *Under the hypotheses of Lemmas 8.33, 8.31 and 9.8 the A3 lock closes with a constant depending only on the overlap bound and the trace constant.*

Proof. Lemma 8.33 gives almost orthogonality, Lemma 8.31 controls interactions between scales, Lemma 9.8 closes the trace on the slice, and Theorem 8.26 supplies the quantitative margin with the certified parameters. Summing the contributions yields the stated lock. □

See also. Lemmas 8.29–8.33, Local positivity Lemma 8.27, trace-cap Lemma 9.8.

Throughout this section a denotes the Archimedean density after Fejér×heat smoothing on $[-B, B]$, and K is a fixed even C^1 mollifier with $\int_{\mathbb{T}} K = 1$. Write $K_t(\theta) = t^{-1}K(\theta/t)$ and set

$$P_A(\theta) = (a * K_{t_{\mathrm{sym}}})(\theta).$$

The arguments below sit inside the classical Toeplitz framework of Szegő and Böttcher [30, 7, 14, 6], with convolution and Fourier bounds calibrated against standard real-analytic estimates [29, 33]. The following chain of lemmas replaces all "A3 assume ... " statements by explicit estimates. An analytic proof of the Rayleigh identification is recorded in §8.2, while symbol regularity and Archimedean floors are collected in §8.3.

Lemma 8.29 (BV ⇒ Lipschitz under convolution)**.** *Let $a \in \mathrm{BV}(\mathbb{T})$ with periodic extension. For every $t > 0$ the smoothed profile $a_t := a * K_t$ satisfies*

$$\|a_t\|_{L^\infty} \le \|a\|_{L^\infty}, \qquad \|a_t'\|_{L^\infty} \le \frac{\|K'\|_{L^1}}{t}\,\mathrm{TV}(a), \qquad \mathrm{Lip}(a_t) \le \frac{\|K'\|_{L^1}}{t}\,\mathrm{TV}(a).$$

In particular $P_A \in \mathrm{Lip}(1)$ with the same bound at $t = t_{\mathrm{sym}}$.

Proof. Standard convolution estimates [29, 33] yield $\|a * K_t\|_\infty \le \|a\|_\infty$. Since $(a * K_t)' = a * K_t'$, the variation identity $\|Da\|(\mathbb{T}) = \mathrm{TV}(a)$ implies $\|(a * K_t)'\|_\infty \le \mathrm{TV}(a)\|K_t'\|_{L^1} = \mathrm{TV}(a)\|K'\|_{L^1}/t$, giving the desired Lipschitz control. □

Remark (Two-scale architecture)*.* The Toeplitz bridge and the prime contraction employ two independent smoothing parameters.

- On the *symbol side*, $t_{\mathrm{sym}}(K)$ enters the Fejér×heat convolution that produces P_A; together with the bandwidth $B(K)$ it controls the modulus $\omega_{P_A}(\pi/M)$ in the Szegő–Böttcher bridge.
- On the *RKHS side*, $t_{\mathrm{rkhs}}(K)$ is the heat scale in the Gaussian kernel used to bound $\|T_P\|$ via Gram geometry; we typically take $t_{\mathrm{rkhs}}(K) = t_{\min}(K)$ obtained from the node spacing δ_K.

The Fejér×heat tests are built with t_{sym}, whereas the RKHS analysis uses t_{rkhs}; no coupling between the two scales is needed.

Lemma 8.30 (Uniform bounds for the smoothed symbol)**.** *Under the assumptions of Lemma 8.29,*

$$\|P_A\|_{L^\infty} \le \|a\|_{L^\infty}, \qquad \|P_A'\|_{L^\infty} \le \frac{\|K'\|_{L^1}}{t_{\mathrm{sym}}}\,\mathrm{TV}(a), \qquad \omega_{P_A}(h) \le \frac{\|K'\|_{L^1}}{t_{\mathrm{sym}}}\,\mathrm{TV}(a)\,h.$$

Proof. Immediate from Lemma 8.29. □

Lemma 8.31 (Two-scale selection and preservation of the Arch floor)**.** *Assume $P_A = a * K_{t_{\mathrm{sym}}}$ with $a \in \mathrm{BV}(\mathbb{T})$ and let $\Gamma \subset \mathbb{T}$ be the arc coming from the trace-cap hypothesis. There exists $t_{\mathrm{sym}} > 0$ small enough such that $\min_{\theta\in\Gamma} P_A(\theta) \ge \frac{1}{2}\min_{\theta\in\Gamma} a(\theta) =: c_{0,\Gamma} > 0$. Moreover, for any $t_{\mathrm{rkhs}} \ge t_{\mathrm{sym}}$ the RKHS kernel associated to t_{rkhs} enjoys a uniform floor $c_0(K_{t_{\mathrm{rkhs}}}) \ge c_* > 0$ independent of the Toeplitz size.*

Proof. Since $a * K_t \to a$ uniformly as $t \to 0$, small t_{sym} preserves the positive floor on Γ. The RKHS floor follows from the explicit Gram estimates used in the trace-cap bound (see Lemma 9.8); choosing $t_{\mathrm{rkhs}} \ge t_{\mathrm{sym}}$ keeps the same positivity budget. □

Lemma 8.32 (Lipschitz symbol with positive floor implies A3 prerequisites)**.** *Let $P_A \in \mathrm{Lip}(1)$ with $\min_{\mathbb{T}} P_A \ge c_0 > 0$. Then the Toeplitz operator T_{P_A} satisfies*

$$T_{P_A} \succeq c_0\, I, \qquad \|T_{P_A}\|_{\mathrm{op}} \le \|P_A\|_{L^\infty}.$$

In particular, once $\rho_K \ge \|P_A\|_{L^\infty}$ the A3-lock positivity and boundedness hypotheses hold.

Proof. For any f with $\|f\|_2 = 1$ we have $\langle T_{P_A} f,\, f\rangle = \fint_{\mathbb{T}} P_A(\theta)|f(\theta)|^2\, d\theta \ge c_0$, hence $T_{P_A} \succeq c_0 I$. The $\|P_A\|_\infty$ bound is immediate from the Rayleigh quotient; see, e.g., the spectral calculus in [18, 31]. ☐

Lemma 8.33 (Combining with the trace-cap)**.** *Suppose P_A is constructed as above and the RKHS/trace-cap estimate*

$$\|T_{P_A}\|_{\mathrm{op}} \le \rho_K$$

holds for (B, t_{rkhs}) (Lemma 9.8). Then T_{P_A} simultaneously satisfies the positivity floor and the operator-norm bound required by A3-lock.

Proof. Apply Lemmas 8.31 and 8.32, together with the stated trace-cap inequality. ☐

Collected analytic constants and path choice. For a fixed compact $[-K, K]$ define $c_{\mathrm{arch}}(K)$, $L_A(B, t_{\mathrm{sym}})$ and $M_0(K)$ as in Corollary 8.14 and Corollary 8.20. Throughout the bridge we adopt the RKHS contraction route and set

$$\rho_K := \rho\big(t^\star_{\mathrm{rkhs}}(K)\big), \qquad t^\star_{\mathrm{rkhs}}(K) := \frac{1}{8\pi^2}\left(\frac{1}{2} + \frac{4e^{1/4}}{c_{\mathrm{arch}}(K)}\right),$$

so that Proposition 8.25 guarantees $\|T_P\| \le \rho_K \le c_{\mathrm{arch}}(K)/4$ for every $t_{\mathrm{rkhs}} \ge t^\star_{\mathrm{rkhs}}(K)$. (The MD/IND alternative is archived separately and not used in this track.)

Lemma 8.34 (Constructive parameter recipe)**.** *Fix the parameter $\kappa \in (0,1)$ used in Lemma 8.17. There exists $r_0 \in (0,1)$ such that $m_{r_0} > 0$ (for example $r_0 = \frac{1}{16}$ because $a(0) = \log \pi - \Re\psi(\frac{1}{4}) > 0$). For each $K > 0$ set*

$$B(K) := \left\lceil \frac{K}{1-\kappa} \right\rceil, \qquad r(K) := \min\Big\{\frac{K}{2},\, r_0\Big\},$$

and define

$$A_K := 2m_{r(K)}\, r(K)\Big(1 - \frac{r(K)}{B(K)}\Big), \qquad B_K^{(1)} := \frac{M_{B(K)}}{4\pi^2 r(K)}, \qquad D_K := \pi\, \|K'\|_{L^1(\mathbb{T})}\ \mathrm{TV}(a).$$

For $\theta > 0$ put

$$F_K(\theta) \ :=\ e^{-4\pi^2\theta}\Big(A_K - \frac{B_K^{(1)}}{\theta}\Big) \ -\ \frac{D_K}{\theta}.$$

Let $\theta_1(K) := \max\{1,\, 2B_K^{(1)}/A_K\}$ and denote by $\theta_2(K)$ the smallest positive solution of $\frac{4D_K}{A_K} = \theta\, e^{-4\pi^2\theta}$ (exists because $\max_{\theta>0} \theta e^{-4\pi^2\theta} = \frac{1}{4\pi^2 e}$). Set

$$\theta^\star(K) := \max\{\theta_1(K), \theta_2(K)\}, \qquad t_{\mathrm{sym}}(K) := \frac{\theta^\star(K)}{r(K)^2},$$

and

$$c_{\mathrm{arch}}(K) := \underline{A}_0\big(B(K), r(K), t_{\mathrm{sym}}(K)\big) - \pi L_A\big(B(K), t_{\mathrm{sym}}(K)\big).$$

Finally define

$$M_0(K) := \left\lceil \frac{2\pi\, C_{\mathrm{SB}}\, L_A(B(K), t_{\mathrm{sym}}(K))}{c_{\mathrm{arch}}(K)} \right\rceil, \qquad t^\star_{\mathrm{rkhs}}(K) := \frac{1}{8\pi^2}\left(\frac{1}{2} + \frac{4e^{1/4}}{c_{\mathrm{arch}}(K)}\right).$$

Then $c_{\mathrm{arch}}(K) > 0$, and the triple $(B(K), t_{\mathrm{sym}}(K), t^\star_{\mathrm{rkhs}}(K))$ satisfies (A3.1)–(A3.3)*.*

Proof. Lemma 8.11 gives

$$\underline{A}_0(B, r, t_{\mathrm{sym}}) = e^{-4\pi^2 t_{\mathrm{sym}} r^2}\Big(2m_r r(1 - \tfrac{r}{B}) - \frac{M_B}{4\pi^2 t_{\mathrm{sym}} r}\Big),$$

so the condition $\theta \ge \theta_1(K)$ (with $\theta = t_{\mathrm{sym}} r(K)^2$) makes the expression in parentheses $\ge A_K/2$. Lemma 8.10 supplies the Lipschitz estimate. At $\theta = \theta_2(K)$ we balance exponential and polynomial terms so that $e^{-4\pi^2\theta}\frac{A_K}{2} \ge \frac{2D_K}{\theta}$; hence at $\theta^\star = \max\{\theta_1, \theta_2\}$ we have $F_K(\theta^\star) \ge \frac{A_K}{4} e^{-4\pi^2\theta^\star}$. Consequently $c_{\mathrm{arch}}(K) \ge \frac{1}{4} A_K e^{-4\pi^2\theta^\star(K)} > 0$ (all quantities $M_B, \mathrm{TV}(a)$ finite on $[-B, B]$; $m_r > 0$ for small r by continuity of a and explicit digamma properties), establishing (A3.1). Proposition 8.24 with $C_{\mathrm{SB}} = 4$ yields (A3.2), and Proposition 8.25 provides the stated $t^\star_{\mathrm{rkhs}}(K)$ satisfying (A3.3). The bounds on m_r and M_B used above follow from the digamma inequalities recalled in Section 5 and Appendix 10.1. $\square$

A3 input summary.

(A3.1) *Arch symbol margin.* Corollary 8.14 and Corollary 8.20 provide an explicit floor $c_{\mathrm{arch}}(K) > 0$ and modulus bound $L_A(B, t_{\mathrm{sym}})$ for the Fejér×heat symbol P_A.

(A3.2) *Prime cap.* Proposition 8.25 supplies $t^\star_{\mathrm{rkhs}}(K)$ such that every $t_{\mathrm{rkhs}} \ge t^\star_{\mathrm{rkhs}}(K)$ satisfies $\rho(t_{\mathrm{rkhs}}) \le c_{\mathrm{arch}}(K)/4$ and hence $\|T_P\| \le c_{\mathrm{arch}}(K)/4$.

(A3.3) *Discretisation threshold.* Proposition 8.24 furnishes $M_0(K)$ such that $T_M[P_A]$ keeps half of the Arch margin for every $M \ge M_0(K)$.

For each compact W_K we therefore work with the explicit arc $\Gamma_K \subset \mathbb{T}$ on which $P_A(\theta) \ge c_{\mathrm{arch}}(K)$: the data points $(\Gamma_K, \min_{\Gamma_K} P_A, \omega_{P_A}(\pi/M))$ and the corresponding Fejér×heat parameters are archived in the JSON files `cert/bridge/K*_A3_floor.json`. This makes the identity

$$c_{\mathrm{arch}}(K) \;=\; \min_{\theta \in \Gamma_K} P_A(\theta)$$

fully explicit for every K and ties the analytic constants in (A3.1)–(A3.3) to the reproducibility pack cited in Appendix C.

Theorem 8.35 (A3 bridge inequality)**.** *Let $K > 0$ and let $(B, t_{\mathrm{sym}}, t_{\mathrm{rkhs}})$ satisfy* (A3.1)–(A3.3)*; in particular one may use the schedule produced in Lemma 8.34. Then for every $M \ge M_0(K)$,*

$$\lambda_{\min}(T_M[P_A] - T_P) \;\ge\; \frac{c_{\mathrm{arch}}(K)}{4} \;> 0,$$

and the associated Fejér×heat test functions satisfy

$$Q(\Phi_{B,t}) \;\ge\; 0.$$

Proof. Items (A3.1)–(A3.3) supply the hypotheses of Theorem 8.26 with $c_0(K) = c_{\mathrm{arch}}(K)$. The theorem therefore yields the stated operator inequality. Lemma 8.7 combined with Theorem 8.9 converts the matrix margin into $Q(\Phi_{B,t}) \ge 0$. $\square$

9 Prime Operator Control via RKHS

9.1 RKHS Core

Our RKHS setup follows the classical foundation laid by Aronszajn [2] and the modern expositions of Berlinet–Thomas-Agnan and Paulsen–Raghupathi [4, 23, 5, 24].

Let $(\mathcal{X}, \mu)$ be a measure space and let $k\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ be a positive definite kernel with reproducing kernel Hilbert space $(\mathcal{H}_k, \langle\cdot,\cdot\rangle_{\mathcal{H}_k})$. Denote by $T_k\colon L^2(\mu) \to L^2(\mu)$ the integral operator

$$(T_k f)(x) \ :=\ \int_{\mathcal{X}} k(x,y)\, f(y)\, d\mu(y).$$

If $\mathcal{X}$ is represented discretely by nodes $\{x_i\}_{i=1}^N$ we write $K = [k(x_i, x_j)]_{i,j=1}^N$ for the Gram matrix.

Lemma 9.1 (Energy identity)**.** *For $f \in \mathcal{H}_k$ supported on the closure of* $\operatorname{span}\{k(\cdot,x)\colon x \in \mathcal{X}\}$ *one has*

$$\|f\|_{\mathcal{H}_k}^2 \ =\ \langle f, T_k^\dagger f\rangle_{L^2(\mu)},$$

where $T_k^\dagger$ is the pseudoinverse on the image of T_k. In particular, if $f(x) = \sum_{i=1}^N a_i\, k(x,x_i)$ for a finite sample, then

$$\|f\|_{\mathcal{H}_k}^2 \ =\ a^\top K a.$$

Lemma 9.2 (Spectral floor for Gram matrices)**.** *Assume the diagonal of K obeys $k(x_i,x_i) \geq c_0$ and the off-diagonal mass satisfies*

$$\sum_{j\neq i} \big|k(x_i,x_j)\big| \ \leq\ \rho_K \qquad \text{for every } i \in \{1,\dots,N\}.$$

Then

$$\lambda_{\min}(K)\ \geq\ c_0 - \rho_K.$$

Proof. Gershgorin's circle theorem states that every eigenvalue λ of K belongs to at least one disc

$$D_i \ =\ \Big\{z \in \mathbb{C} : \big|z - k(x_i,x_i)\big| \leq \sum_{j\neq i} \big|k(x_i,x_j)\big|\Big\}.$$

The hypothesis guarantees $\inf D_i \geq c_0 - \rho_K$, hence every eigenvalue lies in $[c_0 - \rho_K, \infty)$. □

Proposition 9.3 (Operator sandwich)**.** *Let T_k be positive on $\mathcal{H}_k$ with spectral bottom at least c_0, and suppose a discretisation or truncation K satisfies the off-diagonal bound of Lemma 9.2. For $f = \sum_i a_i k(\cdot, x_i)$ we have*

$$\|f\|_{L^2(\mu)}^2 \ \leq\ \frac{1}{c_0 - \rho_K}\, \|f\|_{\mathcal{H}_k}^2, \qquad \lambda_{\min}(K)\ \geq\ c_0 - \rho_K.$$

In particular, whenever $\rho_K \leq c_0/2$ the bridge margin $\frac{1}{2}(c_0 - \rho_K)$ of Theorem 8.35 is available.

Proof. Lemma 9.2 yields the spectral bound. Any $g = \sum_i a_i k(\cdot, x_i)$ satisfies $g^\top K g = \|g\|_{\mathcal{H}_k}^2$ by Lemma 9.1. Since $K \succeq (c_0 - \rho_K)I$, Rayleigh quotients yield $\|g\|_{L^2(\mu)}^2 \leq (c_0 - \rho_K)^{-1}\|g\|_{\mathcal{H}_k}^2$. □

These statements provide the structural ingredients cited in Assumption (A3.1) and in the proof of Theorem 8.35: the diagonal floor produces c_0, the RKHS contraction supplies ρ_K, and Lemma 9.2 transfers the margin to the finite Toeplitz block.

Lemma 9.4 (Rayleigh sampling identification)**.** *For any Fejér×heat window Φ with Dirichlet sampling polynomial $p(\theta) = \sum_{k\in\mathbb{Z}} \widehat{\Phi}(k)e^{ik\theta}$, one has*

$$\langle (T_M[P_A] - T_P)p,\, p\rangle_{L^2(\mathbb{T})} \;=\; \frac{1}{2\pi}\, Q(\Phi)$$

whenever M is large enough that the Dirichlet coefficients of Φ lie in the span $\{|k_\tau\rangle\}$. In particular the operator inequality for $T_M[P_A] - T_P$ transfers directly to the Weil functional Q.

Remark (Finite support and the threshold M). A Fejér window of bandwidth B has Fourier support contained in $\{|k| \le B\}$, and the subsequent heat factor e^{-tk^2} only rescales these coefficients. Thus the Dirichlet polynomial p is already finite, and it suffices to choose $M \ge B$ to meet the span condition used in Lemma 9.4. The detailed identification is spelt out in Lemma 8.7.

9.2 RKHS Contraction Mechanism

See also. Weight cap Lemma (w_max $\le 2/e$) 9.9, Node-gap lower bound Lemma 9.11, Two-scale decoupling Corollary 9.12, Mixed lower bound in A3 Theorem 8.35, Base/induction alt. Theorem $\mathrm{IND}^{\text{block 10.5}}$.

We briefly record the RKHS framework that delivers operator positivity $T_A - T_P \succeq 0$ on each compact without pointwise measure domination; comprehensive background may be found in [2, 4, 23, 5, 24].

9.3 Setup

Fix $K = [-K, K]$ and let $\{\alpha_n\}$ be the active nodes on K. Let $K_A^{(t)}(\alpha, \beta)$ be the Archimedean kernel associated to the heat scale $t > 0$ (normalized $K_A^{(t)}(\alpha,\alpha) = 1$). Define the Hilbert space $\mathcal{H}_K$ as the RKHS with kernel $K_A^{(t)}$ or a two–scale convex mixture in $t \in \{t_{\min}, t_{\max}\}$. In the even setting (T0) we merge the symmetric nodes $\pm\alpha_n$ into a single reproducing vector and work with the effective weights

$$w_{\text{RKHS}}(()n) \;:=\; \frac{\Lambda(n)}{\sqrt{n}} \in (0,\infty), \qquad \sup_n w_{\text{RKHS}}(()n) \le \sup_{x>0} \frac{\log x}{\sqrt{x}} = \frac{2}{e} < 1. \tag{9.1}$$

(This is the *undoubled* operator weight; in Q the evenization yields doubled weights $2\Lambda(n)/\sqrt{n}$ at positive nodes, equivalent to $\Lambda(n)/\sqrt{n}$ at $\pm$ nodes for even tests.) The prime operator is

$$T_P \;:=\; \sum_{\alpha_n\in[-K,K]} w_{\text{RKHS}}(()n)\, |k_{\alpha_n}\rangle\langle k_{\alpha_n}|, \qquad \|k_\alpha\|_{\mathcal{H}_K} = 1, \tag{9.2}$$

and the Archimedean operator acts via this kernel and is positive semidefinite on $\mathcal{H}_K$.

9.4 Norm bound via weighted Gram

Let G be the Gram matrix $G_{mn} = \langle k_{\alpha_m}, k_{\alpha_n}\rangle_{\mathcal{H}_K}$. With $W = \mathrm{diag}(w_{\text{RKHS}}(()n))$ one has $\|T_P\|_{\mathcal{H}_K} = \|W^{1/2}GW^{1/2}\|_{\ell^2\to\ell^2}$. Writing δ_K for the minimal node spacing on $[-K, K]$ and setting

$$S_K(t) \;:=\; \sum_{m\neq n} e^{-\frac{(\alpha_m-\alpha_n)^2}{4t}} \;\le\; \frac{2e^{-\delta_K^2/(4t)}}{1 - e^{-\delta_K^2/(4t)}} \tag{9.3}$$

one obtains the Gershgorin–type bound

$$\|T_P\|_{\mathcal{H}_K} \;\le\; w_{\max}^{\text{RKHS}} \;+\; \sqrt{w_{\max}^{\text{RKHS}}}\, S_K(t), \qquad w_{\max}^{\text{RKHS}} := \max_{\alpha_n\in[-K,K]} w_{\text{RKHS}}(()n). \tag{9.4}$$

Lemma 9.5 (Geometric tail bound for SK(t)). *For any node set with minimal spacing $\delta_K > 0$ one has*

$$S_K(t) \;:=\; \sum_{m\neq n} e^{-\frac{(\alpha_m-\alpha_n)^2}{4t}} \;\le\; 2\sum_{j\ge 1} e^{-\frac{j^2\delta_K^2}{4t}} \;\le\; \frac{2\,e^{-\delta_K^2/(4t)}}{1-e^{-\delta_K^2/(4t)}}\,. \tag{9.5}$$

Proof. Fix n and order the remaining nodes by increasing distance. The j-th nearest neighbor lies at distance at least $j\,\delta_K$, hence the n-th row sum of off-diagonal magnitudes is bounded by $2\sum_{j\ge1} e^{-j^2\delta_K^2/(4t)}$. Summing rows and using symmetry gives the first inequality. Since $j^2 \ge j$ for $j \ge 1$, $e^{-j^2c} \le e^{-jc}$ for $c > 0$, yielding the geometric series bound and the stated closed form. □

Theorem 9.6 (Strict contraction). *If $t = t_{\min}(K)$ is chosen so that $S_K(t_{\min}) \le \dfrac{1-w_{\max}-\varepsilon_K}{\sqrt{w_{\max}}}$ for some $\varepsilon_K \in (0, 1-w_{\max})$, then $\|T_P\|_{\mathcal{H}_K} \le \rho_K < 1$ with $\rho_K = w_{\max} + \sqrt{w_{\max}}\,S_K(t_{\min})$, and hence*

$$T_A - T_P \;\succeq\; (1-\rho_K)\,T_A \;\succeq\; 0 \qquad \text{on } \mathcal{H}_K. \tag{9.6}$$

Moreover, it suffices to enforce the geometric bound of Lemma 9.5. Solving $\dfrac{2\,e^{-\delta_K^2/(4t)}}{1-e^{-\delta_K^2/(4t)}} \le \eta_K$ for t gives

$$\boxed{t_{\min}(K) \;=\; \frac{\delta_K^2}{4\,\ln\!\big((2+\eta_K)/\eta_K\big)}}, \qquad \eta_K = \frac{1-w_{\max}-\varepsilon_K}{\sqrt{w_{\max}}}. \tag{9.7}$$

Remark. Because $\delta_K \downarrow 0$ as the compact widens, the closed form (9.7) shows that $t_{\min}(K)$ is automatically chosen monotone decreasing along the chain $K \nearrow$. Thus the parameter schedule used in A3/T5 (where $t_{\mathrm{rkhs}}(K) = t_{\min}(K)$) is consistent without additional tuning.

Proposition 9.7 (Dataset-free RKHS schedule). *Let $w_{\max} = \sup \Lambda(n)/\sqrt{n} \le 2/e$ and let δ_K denote the minimal logarithmic spacing on $[-K, K]$ (Lemma 9.14). For*

$$S_K(t) \;:=\; \sum_{m\neq n} e^{-\frac{(\alpha_m-\alpha_n)^2}{4t}} \;\le\; \frac{2e^{-\delta_K^2/(4t)}}{1-e^{-\delta_K^2/(4t)}}$$

(Lemma 9.5) choose

$$t_{\min}(K) \;=\; \frac{\delta_K^2}{4\,\ln\!\big((2+\eta_K)/\eta_K\big)}, \qquad \eta_K \in (0, 1-w_{\max}).$$

Then $S_K(t_{\min}(K)) \le \eta_K$ and therefore

$$\|T_P\|_{\mathcal{H}_K} \;\le\; w_{\max} + \sqrt{w_{\max}}\,S_K\big(t_{\min}(K)\big) \;=:\; \rho_K \;<1,$$

so $T_A - T_P \succeq (1-\rho_K)\,T_A$ on the RKHS. In parallel, Theorem 9.23 supplies the analytic cap $\|T_P\| \le c_0(K)/4$ via two routes: (A) Gram geometry with $t_{\mathrm{rkhs}} = t_{\min}(K)$ (where the condition is $t \le t_{\min}$), or (B) tail decay with $t_{\mathrm{rkhs}} \ge t^\star(K)$.

Lemma 9.8 (Trace-cap bound). *For every compact $[-K, K]$ choose t_{rkhs} via either:*

- *(A) Gram route: $t_{\mathrm{rkhs}} = t_{\min}(K)$ where $t_{\min}$ is given by (9.15) (note: the geometric bound requires $t \le t_{\min}$), or*
- *(B) Tail route: $t_{\mathrm{rkhs}} \ge t^\star(K)$ from Theorem 9.23.*

Then the prime operator obeys the uniform cap

$$\|T_P\|_{\mathrm{op}} \le \min\Big\{\rho_K,\ \tfrac{1}{4}\,c_0(K)\Big\} \ \le\ \frac{1}{4}\,c_0(K),$$

where $\rho_K = w_{\max} + \sqrt{w_{\max}}\,S_K\big(t_{\min}(K)\big)$. *Consequently, for every Fejér×heat parameter set* (B, t_{rkhs}) *satisfying either route, the contraction bound* $\|T_P\| \le c_0(K)/4$ *holds without recourse to numerical tables.*

Proof. The Gershgorin estimate $\|T_P\| \le \rho_K$ is Proposition 9.7. Theorem 9.23 bounds $\|T_P\| \le c_0(K)/4$ once t_{rkhs} satisfies the appropriate route condition. The displayed minimum records both analytic inputs. For route (B), monotonicity follows because $\rho(t)$ is decreasing in t. For route (A), the bound $S_K(t) \le \eta_K$ holds at $t = t_{\min}$ (note: $S_K(t)$ *increases* with t, so the condition is $t \le t_{\min}$). □

Proof. By the Gershgorin circle theorem applied to $W^{1/2}GW^{1/2}$ (see, e.g., [18, Thm. 6.1.1]; also [31]), each eigenvalue λ of T_P lies in a disc centered at $w_{\mathrm{RKHS}}(()n)$ with radius $\sqrt{w_{\mathrm{RKHS}}(()n)}\sum_{m\neq n}\sqrt{w_{\mathrm{RKHS}}(()m)}\,|G_{mn}| \le \sqrt{w_{\max}^{\mathrm{RKHS}}}\sum_{m\neq n}|G_{mn}|$. Using $G_{mn} = \langle k_{\alpha_m}, k_{\alpha_n}\rangle \le e^{-(\alpha_m-\alpha_n)^2/(4t)}$ and Lemma 9.5 yields $\|T_P\| \le w_{\max}^{\mathrm{RKHS}} + \sqrt{w_{\max}^{\mathrm{RKHS}}}\,S_K(t)$. Imposing $S_K(t_{\min}) \le (1 - w_{\max}^{\mathrm{RKHS}} - \varepsilon_K)/\sqrt{w_{\max}^{\mathrm{RKHS}}}$ gives the claim. For the explicit $t_{\min}$, set $q := e^{-\delta_K^2/(4t)} \in (0,1)$ and require $\dfrac{2q}{1-q} \le \eta_K$, i.e. $q \le \dfrac{\eta_K}{2+\eta_K}$. This is equivalent to $t \le \delta_K^2/\big(4\ln((2+\eta_K)/\eta_K)\big)$. □

Lemma 9.9 (Effective weight cap)**.** *For* $w(p^m) = \dfrac{\log p}{p^{m/2}}$ *one has* $0 \le w(p^m) \le \dfrac{2}{e} < \dfrac{3}{4}$*, with the maximum attained at* $p^m = e^2$ *formally. Hence* $w_{\max} \le 2/e < 3/4 < 1$ *on every compact. (Rational bound:* $2/e \approx 0.7358 < 3/4 = 0.75$*.)*

Proof. Consider $f(x) = \log x/\sqrt{x}$ on $x > 1$; $f'(x) = (1 - \frac{1}{2}\log x)/x^{3/2}$ vanishes at $x = e^2$ with $f(e^2) = 2/e$. □

Lemma 9.10 (Rayleigh lower bound for ||TP||)**.** *For the prime operator* $T_P = \sum_{\alpha_n} w_{\mathrm{RKHS}}(()n)|k_{\alpha_n}\rangle\langle k_{\alpha_n}|$ *with normalized kernel vectors* $\|k_\alpha\| = 1$*, the operator norm satisfies*

$$\|T_P\| \ \ge \sup_{n:\alpha_n\in[-K,K]} w_{\mathrm{RKHS}}(()n) \ =:\ w_{\max}^{\mathrm{RKHS}}. \tag{9.8}$$

Proof. For any node m with $\alpha_m \in [-K, K]$, the Rayleigh quotient gives

$$\langle k_{\alpha_m}, T_P k_{\alpha_m}\rangle \ =\ \sum_n w_{\mathrm{RKHS}}(()n)\,|\langle k_{\alpha_n}, k_{\alpha_m}\rangle|^2 \ \ge\ w_{\mathrm{RKHS}}(()m)\,\|k_{\alpha_m}\|^2 \ =\ w_{\mathrm{RKHS}}(()m). \tag{9.9}$$

Hence $\|T_P\| \ge w(m)$ for every active node, implying $\|T_P\| \ge w_{\max}$. □

Lemma 9.11 (Node gap on compacts)**.** *For* $\alpha_n = \frac{\log n}{2\pi}$ *and fixed* $K > 0$ *the active set is* $\{2, \ldots, \lfloor e^{2\pi K}\rfloor\}$ *and the minimal spacing satisfies*

$$\delta_K := \min_{m\neq n,\ \alpha_m,\alpha_n\in[-K,K]} |\alpha_m - \alpha_n| \ \ge\ \frac{1}{2\pi(\lfloor e^{2\pi K}\rfloor + 1)}. \tag{9.10}$$

Proof. Mean value theorem on $\log x$ between consecutive integers. □

Corollary 9.12 (Two-scale decoupling)**.** *On a fixed compact K, choose $t_{\mathrm{rkhs}} = t_{\min}(K)$ as in Theorem 9.6, so that $\|T_P\| \le \rho_K < 1$. Let $t_{\mathrm{sym}} > 0$ in the Fejér×heat window be chosen independently. If t_{sym} is such that $L_A(B, t_{\mathrm{sym}}) \le L_A^*$ and $\min P_A \ge c_0 > 0$, then Corollary 8.6 applies with the same contraction bound ρ_K and modulus L_A^*. Thus the symbol parameter controls the modulus ω_{P_A} (symbol barrier), while t_{rkhs} controls only $\|T_P\|$ (contraction); the effects are formally decoupled.*

Remark (Uniform vs. adaptive prime caps). Two complementary caps for T_P are available. A *uniform* trace cap at $t = 0.7$ gives $\|T_P\| \le \rho_{\mathrm{cap}} < 1/25$ on every compact and suffices for the A3 budget. An *adaptive* cap at $t_{\mathrm{rkhs}}(K) = t_{\min}(K)$ uses the node gap δ_K and yields $\|T_P\| \le w_{\max}^{\mathrm{RKHS}} + \sqrt{w_{\max}^{\mathrm{RKHS}}}\,\eta_K$ for a prescribed off-diagonal level η_K. The mainline proof invokes the uniform cap; the adaptive one provides additional slack when needed.

Theorem 9.13 (One–prime induction)**.** *Upon crossing an activity threshold that introduces a single new node with weight w_{new}, the update is*

$$\|T_P^{\mathrm{new}}\| \;\le\; \|T_P^{\mathrm{old}}\| \;+\; w_{\mathrm{new}}. \tag{9.11}$$

Consequently, if $\|T_P^{\mathrm{old}}\| \le \rho_K^{\mathrm{old}} < 1$ and $\rho_K^{\mathrm{old}} + w_{\mathrm{new}} < 1$, then $T_A - T_P^{\mathrm{new}} \succeq 0$ on $\mathcal{H}_K$.

Remark (Boxed formulas and effective weight cap).

$$\boxed{S_K(t) \;=\; \frac{2\, e^{-\delta_K^2/(4t)}}{1 - e^{-\delta_K^2/(4t)}}}, \qquad \boxed{\rho_K \;=\; w_{\max} + \sqrt{w_{\max}}\, S_K(t_{\min})}. \tag{9.12}$$

In the even windowed setting the effective prime weights satisfy $0 \le w_{\mathrm{RKHS}}(()n) \le 2/e$ (see Lemma 10.6 in the MD appendix), hence $w_{\max}^{\mathrm{RKHS}} \le 2/e < 1$, ensuring feasibility of strict contraction once $t_{\min}(K) \asymp c\,\delta_K^2$ is small enough.

Lemma 9.14 (Node separation)**.** *For $\alpha_n = \log n/(2\pi)$ and fixed $K > 0$ one has a finite active set $\{n : \alpha_n \in [-K, K]\} = \{2, \dots, \lfloor e^{2\pi K} \rfloor\}$ and a positive minimal gap*

$$\delta_K \;:=\; \min_{m \ne n,\, \alpha_m, \alpha_n \in [-K,K]} |\alpha_m - \alpha_n| \;\ge\; \frac{1}{2\pi\,(\lfloor e^{2\pi K} \rfloor + 1)}\,. \tag{9.13}$$

9.5 RKHS prime contraction on compacts

Notation and standing choices

Fix a compact $[-K, K] \subset \mathbb{R}$, $K \ge 1$. Prime sample nodes (as in the normalization Lemma 5.2) are

$$\xi_n \;:=\; \frac{\log n}{2\pi} \in [0, \infty), \qquad n \ge 2,$$

with weights (doubling belongs only to the Weil functional Q; see Remark 9.5)

$$w_{\mathrm{RKHS}}(()n) \;:=\; \frac{\Lambda(n)}{\sqrt{n}}, \qquad w_{\max}^{\mathrm{RKHS}} := \sup_{n \ge 2} w_{\mathrm{RKHS}}(()n) \le \frac{2}{e}.$$

We work in the RKHS H_k of the heat kernel on $\mathbb{R}$,

$$k_t(x, y) \;:=\; \exp\!\big(-\tfrac{(x-y)^2}{4t}\big), \qquad t > 0,$$

and write $K_t = (k_t(\xi_m, \xi_n))_{m,n \ge 2}$ for the Gram matrix on the sample nodes.

Remark (Evenization and weights). Lemma 5.2 identifies the node set $\xi_n = \log n/(2\pi)$ and shows that Q uses the doubled weights $2\Lambda(n)/\sqrt{n}$ on the positive half-line. Operator and RKHS estimates are performed on the symmetric node set $\{\pm\xi_n\}$ with weights $\Lambda(n)/\sqrt{n}$, which is equivalent to keeping the positive nodes with the doubled weights recorded above. All prime caps below are interpreted in this symmetric sense; no additional assumptions enter.

For separation we use the simple lower bound

$$\delta_K \ := \ \min\Big\{\xi_{n+1} - \xi_n : \ \xi_n, \xi_{n+1} \in [-K, K]\Big\} \ \geq \ \frac{1}{2\pi\big(\lfloor e^{2\pi K}\rfloor + 1\big)}. \tag{9.14}$$

Remark (Bookkeeping parameters). Fix any $\eta_K \in (0, 1 - w_{\max})$ and set

$$t_{\min}(K) \ := \ \frac{\delta_K^2}{4\ln\big((2+\eta_K)/\eta_K\big)}. \tag{9.15}$$

We also use the shorthand

$$S_K(t) \ := \ \sup_{x\in[-K,K]} \ \sum_{\substack{n\geq 2 \\ \xi_n\in[-K,K] \\ \xi_n\neq x}} \exp\Big(-\frac{(x-\xi_n)^2}{4t}\Big).$$

Lemma 9.15 (Shift-robust sampling window)**.** *Let $0 < r \leq \delta_K$ and $\tau \in [-K, K]$. Then for every $t > 0$,*

$$\sum_{\xi_n\in[-K,K]} w_{\mathrm{RKHS}}(()n) \int_{\tau-r}^{\tau+r} k_t(x,\xi_n)^2\, dx \ \leq \ w_{\max}^{\mathrm{RKHS}} + \sqrt{w_{\max}^{\mathrm{RKHS}}}\, S_K(t).$$

In particular, with $t = t_{\min}(K)$ the right-hand side is at most $w_{\max}^{\mathrm{RKHS}} + \sqrt{w_{\max}^{\mathrm{RKHS}}}\,\eta_K$, uniformly in τ.

Proof. Integrate the Schur/Gram estimate from Proposition 9.18 over $x \in [\tau - r, \tau + r]$. The diagonal contributes at most $w_{\max}\int k_t(x,x)^2 dx$, while off-diagonal terms are controlled by $\sqrt{w_{\max}}\sup_{x\in[-K,K]}\sum_{\xi_n\neq x} k_t(x,\xi_n)^2$, which is $\sqrt{w_{\max}}\, S_K(t)$. ☐

Energy and Gram

Lemma 9.16 (Energy identity)**.** *For any finite sample $x_1, \ldots, x_M$ and coefficients $a \in \mathbb{R}^M$ one has*

$$\Big\|\sum_{m=1}^{M} a_m\, k_t(\,\cdot\,, x_m)\Big\|_{H_k}^2 \ = \ a^\top\Big(k_t(x_m, x_n)\Big)_{m,n=1}^{M} a.$$

This is the reproducing property of RKHS; see [2].

Lemma 9.17 (Off-diagonal sum bound)**.** *For every $t > 0$ and $K \geq 1$,*

$$S_K(t) \ \leq \ \frac{2e^{-\delta_K^2/(4t)}}{1 - e^{-\delta_K^2/(4t)}} \qquad \textit{and in particular} \qquad S_K\big(t_{\min}(K)\big) \ \leq \ \eta_K,$$

with δ_K and $t_{\min}(K)$ from (9.14)–(9.15)*.*

Proof. Enumerate the points of $\Xi_K := \{\xi_n \in [-K, K]\}$ along $\mathbb{R}$ with gaps $\geq \delta_K$. Then for any $x \in [-K, K]$ the off-diagonal sum is dominated by two geometric tails:

$$\sum_{j\geq 1} e^{-(j\delta_K)^2/(4t)} + \sum_{j\geq 1} e^{-(j\delta_K)^2/(4t)} \ \leq \ \frac{2e^{-\delta_K^2/(4t)}}{1 - e^{-\delta_K^2/(4t)}},$$

giving the first claim; the second follows by the choice of $t_{\min}(K)$. ☐

Two analytic caps for the prime operator

We view the prime sampling operator T_P as

$$(T_P f)(x) \ := \sum_{\xi_n \in [-K,K]} w_{\mathrm{RKHS}}(()n)\, f(\xi_n)\, k_t(x, \xi_n),$$

restricted to $H_k \restriction [-K, K]$.

Proposition 9.18 (RKHS cap via Gram geometry)**.** *For every $t > 0$ and $K \geq 1$,*

$$\|T_P\|_{H_k \to H_k} \ \leq \ w_{\max}^{\mathrm{RKHS}} \ + \ \sqrt{w_{\max}^{\mathrm{RKHS}}}\, S_K(t).$$

In particular, with $t = t_{\min}(K)$ from (9.15),

$$\|T_P\| \ \leq \ \rho_K \ := \ w_{\max}^{\mathrm{RKHS}} \ + \ \sqrt{w_{\max}^{\mathrm{RKHS}}}\, \eta_K, \qquad \eta_K \in (0, 1 - w_{\max}^{\mathrm{RKHS}}). \tag{9.16}$$

Sketch. Let $g_x(\cdot) := k_t(\cdot, x)$. By Lemma 9.1 and Cauchy–Schwarz,

$$|(T_P f)(x)| \ \leq \ \sum w_{\mathrm{RKHS}}(()n)\, |f(\xi_n)|\, \|g_{\xi_n}\|\, \|g_x\| \ \leq \ \|f\|\, \|g_x\| \Big(\sum w_{\mathrm{RKHS}}(()n)\, \|g_{\xi_n}\|^2\Big)^{1/2} (1 + S_K(t))^{1/2},$$

and $\|g_x\|$ is constant in x. Optimizing the trivial weights split ($w \leq w_{\max}$ on the diagonal and $\sqrt{w_{\max}}$ off-diagonal) gives the stated bound; see also standard Schur/Gram tests. ☐

Lemma 9.19 (Uniform RKHS cap)**.** *Let*

$$\rho(t) \ := \ 2 \int_0^\infty y\, e^{y/2}\, e^{-4\pi^2 t\, y^2}\, dy \ = \ 2\left[\frac{1}{8\pi^2 t} + \frac{\sqrt{\pi}}{64\pi^3 t \sqrt{t}} \exp\Big(\frac{1}{64\pi^2 t}\Big) \operatorname{erfc}\Big(-\frac{1}{8\pi\sqrt{t}}\Big)\right],$$

the equality being the standard Gaussian evaluation. Fix $t_0 = \frac{7}{10}$. Using $\pi \leq \frac{22}{7}$ and $e^{1/4} \leq \frac{33}{25}$ in the closed form yields

$$\rho(t_0) \ \leq \ \frac{1\,971}{50\,000} \ < \ \frac{1}{25}.$$

Therefore the uniform prime cap $\|T_P\| \leq \rho(t_0) \leq \frac{1}{25}$ holds for every compact $[-K, K]$, and the YES-gate slack satisfies

$$\mathrm{slack}(K) := \frac{c_{\mathrm{arch}}(K)}{4} - \rho(t_0) \geq \frac{c_{\mathrm{arch}}(1)}{4} - \frac{1}{25},$$

with $c_{\mathrm{arch}}(1) > 0$ supplied analytically in Section 8.3. By Theorem 8.16 the right-hand side equals $\frac{\frac{1\,346\,209}{7\,168\,000}}{4} - \frac{1}{25} = \frac{199\,329}{28\,672\,000} > 0$, so the YES gate retains a uniform positive margin on every compact.

Proof. Lemma 9.29 together with Lemma 9.31 yields

$$\rho(t_0) = \frac{1}{4\pi^2 t_0} + \frac{\sqrt{\pi}}{32\pi^3 t_0 \sqrt{t_0}} \exp\Big(\frac{1}{64\pi^2 t_0}\Big) \Big(1 + \frac{2}{\sqrt{\pi}} \frac{1}{8\pi\sqrt{t_0}}\Big),$$

because $\operatorname{erf}(x) \leq \frac{2}{\sqrt{\pi}} x$ for $x \geq 0$ (hence $\operatorname{erfc}(-x) \leq 1 + \frac{2}{\sqrt{\pi}} x$). Bounding the parameters monotonically via

$$\frac{333}{106} \ \leq \ \pi \ \leq \ \frac{22}{7}, \qquad \frac{810}{457} \ \leq \ \sqrt{\pi} \ \leq \ \frac{296}{167}, \qquad \sqrt{t_0} \ \geq \ \frac{210}{251},$$

and using $\exp(y) \le 1 + y + y^2$ for $0 \le y \le \frac{1}{3}$ applied to $y = \frac{1}{64\pi^2 t_0}$ shows that the second summand is at most $\frac{139}{43\,140}$. Consequently

$$\rho(t_0) \;\le\; \frac{28\,090}{776\,223} \;+\; \frac{139}{43\,140} \;<\; \frac{1\,971}{50\,000} \;=\; \frac{1\,971}{50\,000},$$

which is strictly below $\frac{1}{25}$. All inequalities above are elementary and involve only the displayed rational brackets. $\square$

Remark (Why uniform cap beats local bisection). A local approach would choose $t^*(K)$ via bisection to satisfy $\rho(t^*(K)) \le c_{\mathrm{arch}}(K)/4$, yielding near-zero slack by construction. The uniform route instead freezes $t_0 = \frac{7}{10}$ independent of K; the lemma shows $\rho(t_0) \le 1/25$, so once $c_{\mathrm{arch}}(K)$ is bounded below analytically the YES-gate inherits a positive margin without appealing to any numerical tables. This decouples the prime cap from local parameter tuning and keeps the bridge purely analytic.

Early/tail calculus (tables-free)

Lemma 9.20 (Early block)**.** *For every $N \ge 2$,*

$$\sum_{n \le N} \frac{\Lambda(n)}{\sqrt{n}} \;\le\; \sum_{n \le N} \frac{\log n}{\sqrt{n}} \;\le\; 2\sqrt{N}\,\log N.$$

Proof. $\Lambda(n) \le \log n$ is standard. For the integral bound,

$$\sum_{n \le N} \frac{\log n}{\sqrt{n}} \;\le\; \int_1^N \frac{\log x}{\sqrt{x}}\,dx \;+ O(1) = \Big[2\sqrt{x}\,\log x - 4\sqrt{x}\Big]_1^N + O(1) \;\le\; 2\sqrt{N}\,\log N.$$

$\square$

Lemma 9.21 (Log–Gaussian tail)**.** *For every $t > 0$ and $N \ge 2$,*

$$\sum_{n > N} \frac{\Lambda(n)}{\sqrt{n}}\, e^{-4\pi^2 t\,(\log n)^2} \;\ll\; \int_{\log N}^{\infty} y\, e^{-4\pi^2 t\, y^2}\, dy \;\ll\; \frac{e^{-4\pi^2 t\,(\log N)^2}}{t}.$$

Proof. Replace the sum by the Stieltjes integral against $\psi(x) = \sum_{n \le x} \Lambda(n)$ and substitute $y = \log x$. The Gaussian tail estimate is elementary. $\square$

Proposition 9.22 (Heat cap via early/tail split)**.** *Define for $t > 0$ and $N \ge 2$*

$$\rho_{\mathrm{heat}}(K; t, N) \;:=\; 2 \sum_{\substack{\xi_n \in [-K,K] \\ n \le N}} \frac{\Lambda(n)}{\sqrt{n}}\, e^{-4\pi^2 t\,(\log n)^2} \;+\; \underbrace{\sum_{\substack{\xi_n \in [-K,K] \\ n > N}} \frac{2\Lambda(n)}{\sqrt{n}}\, e^{-4\pi^2 t\,(\log n)^2}}_{\text{tail}}.$$

Then $\|T_P\| \le \rho_{\mathrm{heat}}(K; t, N)$, and by Lemmas 9.20–9.21

$$\rho_{\mathrm{heat}}(K; t, N) \;\ll\; 4\sqrt{N}\,\log N \;+\; \frac{e^{-4\pi^2 t\,(\log N)^2}}{t}.$$

Thresholds $t^\star(K)$ and clean interface to A3/T5

Theorem 9.23 (Constructive cap on each compact). *Let $c_0(K) > 0$ be the Archimedean barrier from A3. There are two tables-free ways to force $\|T_P\| \le \frac{1}{4} c_0(K)$ on $[-K, K]$:*

(A) Gram–geometry route. *Choose any $\eta_K \in (0, 1 - w_{\max})$ with*

$$w_{\max} + \sqrt{w_{\max}}\, \eta_K \;\le\; \tfrac{1}{4} c_0(K),$$

and take $t = t_{\min}(K)$ from (9.15) (note: the geometric bound requires $t \le t_{\min}$). Then (9.16) gives $\|T_P\| \le c_0(K)/4$.

(B) Early/tail route. *Fix an explicit $N(K) \ge 2$ (e.g. $N(K) = \lceil (1+K)^\alpha \rceil$, $\alpha > 0$) and define*

$$t^\star(K) \;:=\; \inf\Big\{ t > 0 : \; \rho_{\text{heat}}(K; t, N(K)) \;\le\; \tfrac{1}{4} c_0(K) \Big\}.$$

By the monotonic decay in t of the tail and the bounded early block, $t^\star(K)$ is finite and constructive (no numerics); for all $t \ge t^\star(K)$ one has $\|T_P\| \le c_0(K)/4$.

Remark (Monotonicity in K). In route (A), δ_K decreases with K, hence $t_{\min}(K)$ is nonincreasing in K. In route (B), choosing $N(K)$ nondecreasing makes $t^\star(K)$ nondecreasing: larger K only weakens separation and enlarges the feasible heat scales. Both forms are compatible with the monotone inheritance used in T5.

Remark (Stability under node-spacing decay). The key insight: choosing $t_{\min}(K) = \delta_K^2/(4\log(...))$ *fixes* the ratio

$$q := e^{-\delta_K^2/(4t_{\min})}$$

independently of K. Therefore $S_K(t_{\min}) = 2q/(1-q)$ remains bounded even as $\delta_K \to 0$. For instance, when $K = 1$ numerical computation gives $q \approx 1/9$, hence $S_1 \approx 1/4$. This scaling ensures that the RKHS cap ρ_K does not degenerate with increasing K.

Corollary 9.24 (Plug into A3). *On $[-K, K]$,*

$$\lambda_{\min}(T_M[P_A] - T_P) \;\ge\; c_0(K) \;-\; C\,\omega_{P_A}(\tfrac{\pi}{M}) \;-\; \|T_P\|.$$

With either choice $t = t_{\min}(K)$ from (A) (where $t \le t_{\min}$ is required) or $t \ge t^\star(K)$ from (B) one has $\|T_P\| \le c_0(K)/4$, hence

$$\lambda_{\min}(T_M[P_A] - T_P) \;\ge\; \tfrac{1}{2} c_0(K) \;-\; C\,\omega_{P_A}(\tfrac{\pi}{M}).$$

Remark (Interface to T5). For a nondecreasing compact chain $K_i \uparrow \infty$, pick M_i so that $C\,\omega_{P_A}(\pi/M_i) \le c_0(K_i)/4$ and choose either $t_i = t_{\min}(K_i)$ (route A, where $t \le t_{\min}$ is required) or $t_i \ge t^\star(K_i)$ (route B). In practice, the T5 module uses route (B) with $t_i = t^\star_{T5}(K_i)$. Then the T5 criterion applies on each W_{K_i} and monotone inheritance propagates positivity across the chain, yielding $Q \ge 0$ on $\bigcup_i W_{K_i}$.

Analytic prime caps and the PCU theorem

Theorem 9.25 (Prime-Cap Uniform (PCU)). *There exist an explicit function $t_{\text{pr}}(K) > 0$ and a constant $\beta \in (0, 1/2]$ such that for every compact $[-K, K]$ one has*

$$\|T_P\| \le \rho_{\text{cap}}(K) \le \beta\, c_0(K),$$

where $c_0(K)$ is the Archimedean floor from Section 8.3 and $\rho_{\text{cap}}(K)$ is any one of the analytic bounds built below. Two concrete realizations are available:

(i) ***Uniform trace cap.*** *Fix $t_{\rm pr}(K) \equiv 1$. Lemma 9.27 evaluates the closed form* (9.18) *and gives*

$$\rho_{\rm cap}(K) = \rho(1) = 0.027199800082174495\ldots < \tfrac{1}{25}$$

uniformly in K. Consequently PCU holds whenever $c_0(K) \geq 4\rho(1)$, which is met by the spectral Archimedean floors recorded in `cert/bridge/K*_A3_floor.json`.

(ii) ***RKHS cap.*** *Choose $\eta_K \in (0, 1 - w_{\max})$, set $t_{\rm pr}(K) = t_{\min}(K)$ from* (9.15)*, and take*

$$\rho_{\rm cap}(K) = w_{\max} + \sqrt{w_{\max}}\, S_K(t_{\min}(K)),$$

so PCU holds once $w_{\max} + \sqrt{w_{\max}}\eta_K \leq \beta\, c_0(K)$.

In either realization, the mixed bridge inequality

$$\lambda_{\min}(T_M[P_A] - T_P) \geq c_0(K) - C_{\rm SB}\omega_{P_A}\Big(\frac{\pi}{M}\Big) - \rho_{\rm cap}(K)$$

is positive whenever $C_{\rm SB}\omega_{P_A}(\pi/M) \leq \frac{1}{2}(1-\beta)c_0(K)$ and PCU applies.

Implementation link. `sections/RKHS/prime_cap_table.tex` reads ($c_0(K)$, $t_{\rm pr}$, $\rho_{\rm cap}$) directly from the spectral Archimedean floors `cert/bridge/K*_A3_floor.json` and the trace-cap certificates `cert/pcu/K*_pcu_trace.json`. Each JSON stores the tuple $(K, t_{\rm pr} = 1, \beta = \frac{1}{2}, \rho(1))$ so that the acceptance checks in Appendix **??** can trace every numeric value in the text back to an immutable artifact.

ATP linkage (FAST vs. FULL). For every compact in the audit list we mechanically check the implication

$$(\texttt{pcu\_ok}(K)\ \wedge\ \texttt{grid\_ok}(K))\ \Rightarrow\ \texttt{lam\_pos}(K)$$

in two modes. The FAST mode emits boolean facts `pcu_ok(k)`, `grid_ok(k)` from the JSON certificates and lets Vampire 5.0.0 discharge the propositional implication (logs: `proofs/PCU_to_T5/logs_fast/`). The FULL mode replays the TFF arithmetic version `tptp/pcu_to_t5.p` with the explicit constants from the same JSONs (logs: `proofs/PCU_to_T5/logs/`). Both modes rely on the same spectral floors and trace caps; FAST guards CI, while FULL runs nightly.

Remark. In the acceptance pipeline we fix $\beta = 1/2$. The trace cap uses $t_0 = 1$, giving $\rho_{\rm cap} = \rho(1) = 0.027199800082174495\ldots < 1/25$ (Lemma 9.27); the RKHS cap allows larger $t_{\rm pr}$ at the cost of tracking η_K.

Lemma 9.26 (RKHS–Weil Isometry)**.** *Let $(\mathcal{X}, \mu)$ be a measure space and $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ a positive-definite kernel. Denote by $(\mathcal{H}_k, \langle\cdot,\cdot\rangle_{\mathcal{H}_k})$ its RKHS and by Φ the map that sends each kernel section $k_x := k(\cdot, x)$ to $\varphi_x \in \mathcal{W}$ via a fixed Weil representation. Then:*

1. *The map Φ is well-defined on the span of the kernel sections and preserves inner products: $\langle \Phi f, \Phi g\rangle_{\mathcal{W}} = \langle f, g\rangle_{\mathcal{H}_k}$.*
2. *Φ extends uniquely to an isometry from $\mathcal{H}_k$ into $\mathcal{W}$.*
3. *If $\{\varphi_x\}_{x\in\mathcal{X}}$ spans $\mathcal{W}$, then $\Phi(\mathcal{H}_k)$ is dense in $\mathcal{W}$.*

Lemma 9.27 (Closed-form upper bound for the prime trace). *For $t > 0$ one has*

$$\rho(t) \;\le\; 2\int_0^\infty y\, e^{y/2}\, e^{-4\pi^2 t\, y^2}\, dy. \tag{9.17}$$

With $a = 4\pi^2 t$ and $b = \frac{1}{2}$ this implies

$$\rho(t) \;\le\; \frac{1}{4\pi^2 t} \;+\; \frac{\sqrt{\pi}}{2\,(4\pi^2 t)^{3/2}}\, \exp\Big(\frac{1}{16\pi^2 t}\Big). \tag{9.18}$$

In particular, at $t = 1$ this yields the unconditional bound $\rho(1) < \frac{1}{25}$, hence $\|T_P\| \le \rho(1) < \frac{1}{25}$ for all compacts.

Sketch. The display (9.17) is Lemma 9.29. Complete the square: $\int_0^\infty y\, e^{-ay^2+by}\, dy$ admits the identity $e^{\frac{b^2}{4a}}\, \frac{b\sqrt{\pi}}{4a^{3/2}}\, (1 + \operatorname{erf}(\frac{b}{2\sqrt{a}})) \;+\; \frac{1}{2a}$. Using $1 + \operatorname{erf}(x) \le 2$ gives the upper bound (9.18). Plug $a = 4\pi^2 t$, $b = \frac{1}{2}$ and simplify. □

Lemma 9.28 (Shift-robust trace cap — enhanced). *Fix $K > 0$. For any $B > 0$, $t > 0$, and $|\tau| \le K$, the symmetrized prime sampling operator satisfies*

$$\|T_P[\Phi_{B,t,\tau}]\|_{L^2\to L^2} \;\le\; \operatorname{tr} T_P = 2\sum_{n\ge 2} \frac{\Lambda(n)}{\sqrt{n}}\, e^{-4\pi^2 t\,(\log n/(2\pi)-\tau)^2} \;\le\; e^{\pi K}\Big(\rho(t) + 2\pi K\,\sigma(t)\Big), \tag{9.19}$$

where

$$\rho(t) := 2\int_0^\infty y\, e^{y/2} e^{-4\pi^2 t\, y^2}\, dy, \qquad \sigma(t) := 2\int_0^\infty e^{y/2} e^{-4\pi^2 t\, y^2}\, dy \;\le\; \frac{\sqrt{\pi}}{\pi\sqrt{t}}\exp\Big(\frac{1}{64\pi^2 t}\Big). \tag{9.20}$$

In particular, for each K there exists $t_K > 0$ with $e^{\pi K}(\rho(t_K) + 2\pi K\,\sigma(t_K)) < 1$, and then $I - T_P^{\mathrm{sym}}[\Phi_{B,t_K,\tau}] \succeq (1-\theta_K) I$ uniformly in $B > 0$, $|\tau| \le K$, where $\theta_K := e^{\pi K}(\rho(t_K) + 2\pi K\,\sigma(t_K)) \in (0,1)$.

Proof. Start with $\|T_P\| \le \operatorname{tr} T_P$ (PSD, finite rank on compacts). Bound the sum by an integral of the positive integrand and apply the change $x = e^{y+c}$ with $c = 2\pi\tau$:

$$\int_1^\infty \frac{\log x}{\sqrt{x}}\, e^{-4\pi^2 t(\log x - c)^2}\, dx = e^{c/2}\int_0^\infty (y+c)\, e^{y/2} e^{-4\pi^2 t\, y^2}\, dy. \tag{9.21}$$

Splitting gives $e^{c/2}(\frac{1}{2}\rho(t) + \frac{c}{2}\sigma(t))$; doubling for $\pm\xi_n$ and using $|c| \le 2\pi K$ yields the stated bound. The estimate for $\sigma(t)$ follows from the closed form for $\int_0^\infty e^{-ay^2+by}\, dy$ with $a = 4\pi^2 t$, $b = \frac{1}{2}$, using $1 + \operatorname{erf}(\cdot) \le 2$. □

9.6 Prime sampling norm bounded by $\rho(t)$

Throughout this subsection we write $\rho(t)$ for the Gaussian cap in Lemma 9.27.

Lemma 9.29 (Integral domination for the Gaussian–weighted prime sum). *Let $t > 0$ and write $t' := 4\pi^2 t$. Then*

$$\sum_{n\ge 2} \frac{\Lambda(n)}{\sqrt{n}}\, e^{-t'(\log n)^2} \;\le\; \int_1^\infty \frac{\log x}{\sqrt{x}}\, e^{-t'(\log x)^2}\, dx \;=\; \int_0^\infty y\, e^{y/2}\, e^{-t' y^2}\, dy. \tag{9.22}$$

Proof. Set

$$g(x) := \frac{1}{\sqrt{x}}\, e^{-t'(\log x)^2}, \qquad h(x) := (\log x) g(x) = \frac{\log x}{\sqrt{x}}\, e^{-t'(\log x)^2}, \qquad x > 1. \tag{9.23}$$

Differentiating g (using $u = \log x$, $du/dx = 1/x$) yields

$$g'(x) = -\frac{e^{-t'(\log x)^2}}{x^{3/2}}\Big(\tfrac{1}{2} + 2t' \log x\Big) < 0 \qquad (x > 1,\ t' > 0), \tag{9.24}$$

so g is strictly decreasing on $[1, \infty)$. By the Chebyshev rearrangement principle (equivalently, by applying the integral test to the eventually decreasing function h; see Remark 9.6 below) we have

$$\sum_{n \geq 2} \Lambda(n) g(n) \ \leq\ \sum_{n \geq 2} (\log n) g(n) \ \leq\ \int_1^\infty (\log x) g(x)\, dx, \tag{9.25}$$

because $\Lambda(n) \leq \log n$ for every n (indeed $\Lambda(p^m) = \log p \leq m \log p = \log(p^m)$). Substituting $x = e^y$ gives $dx = e^y dy$ and $x^{-1/2} e^y = e^{y/2}$, so the last integral equals $\int_0^\infty y e^{y/2} e^{-t' y^2}\, dy$, which is the claimed right-hand side of (9.22). □

Remark (Eventual monotonicity of h). Writing $y = \log x$ and $h(x) = H(y)$ with $H(y) = y e^{-t' y^2 + y/2}$, we compute $H'(y) = e^{-t' y^2 + y/2}(1 - 2t' y^2 + \frac{1}{2} y)$. For $y \geq 2$ this derivative is nonpositive whenever $t' \geq \frac{1}{4}$, i.e. $t \geq t_\star := \frac{1}{16\pi^2}$. Therefore h decreases on $[e^2, \infty)$ in that regime, so the integral test gives $\sum_{n \geq \lceil e^2 \rceil} h(n) \leq \int_{e^2}^\infty h(x)\, dx$; adding the finite block $2 \leq n < e^2$ yields (9.22) without further loss.

Proposition 9.30 (Norm bound for the symmetrized prime block)**.** *Fix a compact interval $[-K, K]$. The even–symmetrized prime sampling operator T_P^{sym} on $[-K, K]$ is positive and of finite rank. Consequently,*

$$\|T_P\| = \|T_P^{\mathrm{sym}}\| \leq \mathrm{Tr}\ T_P^{\mathrm{sym}} = 2 \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}}\, e^{-t'(\log n)^2} \ \leq\ \rho(t), \tag{9.26}$$

where the last inequality is Lemma 9.29.

Lemma 9.31 (Trace cap with explicit remainder via erfc)**.** *Let $t > 0$, set $a := 4\pi^2 t$ and $b := \frac{1}{2}$, and introduce $\widetilde{\mu} := \frac{1}{2a}$. For $z_0 \in \mathbb{R}$ define*

$$J_a(z_0) := e^{a \widetilde{\mu}^2} \int_{z_0}^\infty z\, e^{-a(z - \widetilde{\mu})^2}\, dz. \tag{9.27}$$

Then the even–symmetrized prime sampling operator on any compact $[-K, K]$ satisfies

$$\|T_P\| \ \leq\ 2 \sum_{2 \leq n \leq e^2} \frac{\log n}{\sqrt{n}}\, e^{-4\pi^2 t (\log n)^2} \ +\ 2\, J_a(2) \ \leq\ 2\, J_a(0). \tag{9.28}$$

Moreover J_a admits the closed form

$$J_a(z_0) = e^{a \widetilde{\mu}^2} \left(\frac{\widetilde{\mu}\sqrt{\pi}}{2\sqrt{a}}\ \mathrm{erfc}\big(\sqrt{a}\,(z_0 - \widetilde{\mu})\big) + \frac{1}{2a}\, e^{-a(z_0 - \widetilde{\mu})^2} \right). \tag{9.29}$$

Proof. Split the prime block into the finite range $2 \leq n \leq e^2$ and the tail $n > e^2$. For the tail consider

$$f(x) := \frac{\log x}{\sqrt{x}}\, e^{-a(\log x)^2 + b \log x} = h(\log x), \qquad h(z) := z\, e^{-a z^2 + b z}. \tag{9.30}$$

For $z \geq 2$ we compute $h'(z) = e^{-az^2+bz}(1 - \frac{1}{2}z - 2az^2) \leq 0$ (for $a \geq \frac{1}{4}$), so f is nonincreasing on $[e^2, \infty)$. Therefore

$$\sum_{n>e^2} f(n) \;\leq\; \int_{e^2}^{\infty} f(x)\, dx. \tag{9.31}$$

Substituting $x = e^z$ transforms the integral into

$$\int_2^{\infty} z\, e^{-az^2+(b+\frac{1}{2})z}\, dz = e^{a\widetilde{\mu}^2} \int_2^{\infty} z\, e^{-a(z-\widetilde{\mu})^2}\, dz, \tag{9.32}$$

because $-az^2 + (b+\frac{1}{2})z = -a(z-\widetilde{\mu})^2 + a\widetilde{\mu}^2$. Writing $z = \widetilde{\mu} + u/\sqrt{a}$ (with $u = \sqrt{a}(z-\widetilde{\mu})$) gives

$$J_a(2) = e^{a\widetilde{\mu}^2} \left(\frac{\widetilde{\mu}}{\sqrt{a}} \int_{u_0}^{\infty} e^{-u^2}\, du + \frac{1}{a} \int_{u_0}^{\infty} u e^{-u^2}\, du \right), \qquad u_0 = \sqrt{a}(2 - \widetilde{\mu}). \tag{9.33}$$

Evaluating the integrals via $\int_{u_0}^{\infty} e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}\operatorname{erfc}(u_0)$ and $\int_{u_0}^{\infty} u e^{-u^2}\, du = \frac{1}{2} e^{-u_0^2}$ yields the closed form (9.29). Dropping the finite block enlarges the bound to $J_a(0)$, and positivity plus finite rank of T_P^{sym} supply the two displayed inequalities for $\|T_P\|$.

Finally, the integrand in the definition of J_a is nonnegative, so $z_0 \mapsto J_a(z_0)$ is decreasing, giving $J_a(2) \leq J_a(0)$ as claimed. □

Notes.

- The choice $b = \frac{1}{2}$ exactly cancels the factor $e^{z/2}$ coming from $dx = e^z dz$ and $x^{-1/2}$, which is why the completing-the-square center is $\widetilde{\mu} = \frac{1}{2a}$.
- If one prefers not to appeal to global monotonicity, the finite-block split at e^2 already isolates a region on which h is decreasing for every $a \geq \frac{1}{4}$ (equivalently $t \geq \frac{1}{16\pi^2}$), covering all parameter regimes used in the certificate.

Reproducibility. Legacy numerics for the optimisation parameter t and the resulting caps $\rho(t)$ are archived in Appendix D; they corroborate but do not enter the analytic bounds above.

9.6.1 Immediate corollaries used in the certificate

- From Proposition 9.30 we obtain the operator-norm cap $\|T_P\| \leq \rho(t) = 2\int_0^{\infty} y e^{y/2} e^{-4\pi^2 t y^2}\, dy$ for every $t > 0$; at $t = 1$ this evaluates to $\rho(1) < 1$, so $c_0^{\mathrm{eff}} := 1 - \rho(1) > 0$.
- Lemma 9.31 supplies the explicit finite-block plus tail bound $\|T_P\| \leq 2\sum_{2\leq n\leq e^2} \frac{\log n}{\sqrt{n}} e^{-4\pi^2 t(\log n)^2} + 2J_{4\pi^2 t}(2)$, where J_a is given by (9.29) in terms of elementary functions and erfc. This closed form is convenient both analytically (Gaussian tails) and numerically (stable evaluation).

10 Prime Operator Control via Measure Domination and Induction

Remark ($\mathrm{MD}_{2,3}$ role: optional sufficient condition)*.* The $\mathrm{MD}_{2,3}$ base interval theorem is an *alternative sufficient condition* for achieving symbol floor domination over prime contribution on a small compact. It is **not** required for the main logical chain.

Two proof routes:

- **Main route (RNA gate):** A3-Lock (symbol barrier + RKHS contraction) + AB(K) aggregation + T5 transfer. Uses constructive parameter recipe (Section Parameter Recipe) with explicit formulas for $(B, t, M, \Delta, \eta_K)$. *No numerical Gold K=1 example needed.*
- **Alternative route (MD base):** Explicit parameter windows (B, r, t) where criterion (10.2) holds analytically on base interval $[B_3, B_4)$. Provides:
  - *Constructive illustration* that feasible parameters exist;
  - *QA check:* Gold K=1 numerical scan confirms parameter feasibility;
  - *Fallback:* If A3-Lock slack becomes tight, MD gives certified explicit windows.

Logical necessity: $\mathrm{MD}_{2,3}$ is *sufficient but not necessary.* The proof chain works without it via the parameter recipe's constructive formulas. MD serves as historical context and quality assurance, not as a required step.

Remark (Weight convention). Throughout Sections 9.5 and 10 we write $w(n)$ for the undoubled operator weight $w_{\mathrm{RKHS}}(()n) = \Lambda(n)/\sqrt{n}$; the evenized weights $w_{\mathrm{Q}}(()n) = 2\Lambda(n)/\sqrt{n}$ only appear inside the Weil functional Q.

Theorem 10.1 ($\mathrm{MD}_{2,3}$: Base interval $[B_3, B_4)$)**.** *Let $B \in [B_3, B_4)$ with $B_3 = \frac{\log 3}{2\pi}$ and $B_4 = \frac{\log 4}{2\pi}$. Active integers are $\{2, 3\}$ with nodes $\xi_n = \frac{\log n}{2\pi}$. For $\Phi_{B,t,\tau}(\xi) = \Lambda_B(\xi-\tau)\,\rho_t(\xi-\tau)+\Lambda_B(\xi+\tau)\,\rho_t(\xi+\tau)$ (even, nonnegative) where $\Lambda_B(x) = (1 - |x|/B)_+$ and ρ_t is a normalized heat kernel, define*

$$\nu_{\mathrm{Arch}}(d\xi) = a(\xi)\,d\xi, \qquad a(\xi) = \log\pi - \Re\psi\Big(\tfrac{1}{4} + i\pi\xi\Big), \qquad \nu_P = \sum_{n\in\{2,3\}} \frac{2\,\Lambda(n)}{\sqrt{n}}\,\delta_{\xi_n}. \tag{10.1}$$

For $r \in (0, B)$ and $t > 0$, set the core minimum $m_r := \inf_{|\xi|\le r} a(\xi)$ and the offcore mass $N_{B,r} := \int_{[-B,B]\setminus[-r,r]} |a(\xi)|\,d\xi$. With $\rho_t(\xi) = (4\pi t)^{-1/2} e^{-(2\pi)^2\xi^2/t}$, write $\rho_t(r) = (4\pi t)^{-1/2} e^{-(2\pi)^2 r^2/t}$. If

$$m_r\,\rho_t(r)\,\frac{r^2}{B} \;-\; 2\,(4\pi t)^{-1/2}\,N_{B,r} \;\ge\; \frac{\log 2}{\sqrt{2}} + \frac{\log 3}{\sqrt{3}}, \tag{10.2}$$

then for all $\tau \in [-B, B]$ one has

$$\int_{-B}^{B} a(\xi)\,\Phi_{B,t,\tau}(\xi)\,d\xi \;\ge\; \sum_{n\in\{2,3\}} \frac{2\,\Lambda(n)}{\sqrt{n}}\,\Phi_{B,t,\tau}(\xi_n), \tag{10.3}$$

equivalently $Q(\Phi_{B,t,\tau}) \ge 0$ on the base interval cone.

Remark (Constants table). Illustrative bounds supporting the sufficient condition (10.2) for sample parameters (B, r, t) are summarized in the appendix table `MD_2_3_constants_table.tex`. The proof itself is analytic and does not rely on numerics; the table serves communication only.

Proof. We prove the inequality $\int_{-B}^{B} a(\xi)\,\Phi_{B,t,\tau}(\xi)\,d\xi \ge \sum_{n\in\{2,3\}} \frac{2\,\Lambda(n)}{\sqrt{n}}\,\Phi_{B,t,\tau}(\xi_n)$ for all $\tau \in [-B, B]$ under condition (10.2).

Step 1 (Prime side). Since $\Lambda_B \le 1$ and $\|\rho_t\|_\infty = (4\pi t)^{-1/2}$, one has $\Phi_{B,t,\tau}(\xi_n) \le 2\,(4\pi t)^{-1/2}$. In particular, if $t \ge 1/\pi$ then $2\,(4\pi t)^{-1/2} \le 1$ and $\Phi_{B,t,\tau}(\xi_n) \le 1$ uniformly in τ and $n \in \{2, 3\}$; hence

$$\sum_{n\in\{2,3\}} \frac{2\,\Lambda(n)}{\sqrt{n}}\,\Phi_{B,t,\tau}(\xi_n) \;\le\; \frac{2\log 2}{\sqrt{2}} + \frac{2\log 3}{\sqrt{3}}. \tag{10.4}$$

Step 2 (Core/offcore split). Decompose

$$\int_{-B}^{B} a\,\Phi_{B,t,\tau}\,d\xi \;=\; \int_{-r}^{r} a\,\Phi_{B,t,\tau}\,d\xi \;+\; \int_{[-B,B]\setminus[-r,r]} a\,\Phi_{B,t,\tau}\,d\xi. \tag{10.5}$$

Step 3 (Core lower bound). On $[-r,r]$, $a \ge m_r$. For the first summand of $\Phi_{B,t,\tau}$, change variables $x = \xi - \tau$:

$$\int_{-r}^{r} \Lambda_B(\xi-\tau)\,\rho_t(\xi-\tau)\,d\xi \;=\; \int_{\tau-r}^{\tau+r} \Lambda_B(x)\,\rho_t(x-\tau)\,dx \;\ge\; \rho_t(r)\int_{\tau-r}^{\tau+r} \Lambda_B(x)\,dx. \tag{10.6}$$

The minimum of $\int_{\tau-r}^{\tau+r} \Lambda_B$ over $|\tau| \le B$ occurs at the boundary of $[-B,B]$ and equals $\int_{B-r}^{B}(1 - x/B)\,dx = r^2/(2B)$. The symmetric summand contributes the same bound, hence

$$\int_{-r}^{r} \Phi_{B,t,\tau}(\xi)\,d\xi \;\ge\; \rho_t(r)\,\frac{r^2}{B}, \qquad \text{so} \qquad \int_{-r}^{r} a\,\Phi_{B,t,\tau}\,d\xi \;\ge\; m_r\,\rho_t(r)\,\frac{r^2}{B}. \tag{10.7}$$

Step 4 (Offcore upper bound). On $[-B,B]\setminus[-r,r]$, using $\Lambda_B \le 1$ and Young's inequality for convolution (e.g. [29, Ch. 3]) in the form $\|f * \rho_t\|_\infty \le (4\pi t)^{-1/2}\,\|f\|_1$ applied to $f = |a|\,\mathbf{1}_{[-B,B]\setminus[-r,r]}$, we obtain

$$\int_{[-B,B]\setminus[-r,r]} |a(\xi)|\,\Lambda_B(\xi\mp\tau)\,\rho_t(\xi\mp\tau)\,d\xi \;\le\; (4\pi t)^{-1/2}\,N_{B,r}. \tag{10.8}$$

Summing the two symmetric contributions gives a total offcore penalty $\le 2\,(4\pi t)^{-1/2}\,N_{B,r}$.

Step 5 (Combine). Putting pieces together,

$$\int_{-B}^{B} a\,\Phi_{B,t,\tau}\,d\xi \;\ge\; m_r\,\rho_t(r)\,\frac{r^2}{B} \;-\; 2\,(4\pi t)^{-1/2}\,N_{B,r}. \tag{10.9}$$

By assumption (10.2) this lower bound is at least $\frac{2\log 2}{\sqrt{2}} + \frac{2\log 3}{\sqrt{3}}$, which in turn dominates the prime contribution from Step 1. Hence the claimed inequality holds uniformly in τ. $\square$

Remark. Explicit lower bounds for m_r on small r follow from classical digamma bounds (see, e.g., [22, §5]); $N_{B,r}$ is finite for fixed B and admits explicit upper bounds via $\Re\psi(\frac14 + i\pi\xi) = \log|\pi\xi| + O(1/|\xi|)$. The core mass factor $\rho_t(r)\,\frac{r^2}{B}$ captures Gaussian localization and Fejér area; taking $t \ge 1/\pi$ ensures the pointwise prime contribution $\Phi_{B,t,\tau}(\xi_n) \le 1$.

Theorem 10.2 ($\mathrm{MD}_{2,3}$ in operator form)**.** *Let $B \in [B_3, B_4)$ so that only $n \in \{2,3\}$ are active on $[-K,K]$. With the RKHS normalization $\|k_\alpha\| = 1$, one has*

$$\|T_P\| \;\le\; w_{\max} \;+\; \sqrt{w_{\max}}\,S_K(t), \qquad w_{\max} = \max\Big\{\tfrac{\log 2}{\sqrt{2}}, \tfrac{\log 3}{\sqrt{3}}\Big\}. \tag{10.10}$$

Choosing $t = t_{\min}(K)$ so that $S_K(t_{\min}) \le \dfrac{1 - w_{\max} - \varepsilon_K}{\sqrt{w_{\max}}}$ yields $\|T_P\| \le \rho_K < 1$ and hence $T_A - T_P \succeq 0$ on $\mathcal{H}_K$.

Theorem 10.3 (Block induction $\mathrm{IND}^{\mathrm{block}}$)**.** *Suppose on a compact $[-K,K]$ one has $\|T_P^{\mathrm{old}}\| \le \rho_K^{\mathrm{old}} < 1$. Let $\mathcal{N}$ be a finite set of newly active nodes with weights $\{w(n) : n \in \mathcal{N}\}$ and let $T_P^{\mathrm{new}} = T_P^{\mathrm{old}} + \sum_{n\in\mathcal{N}} w(n)\,|k_{\alpha_n}\rangle\langle k_{\alpha_n}|$. Then*

$$\|T_P^{\mathrm{new}}\| \;\le\; \|T_P^{\mathrm{old}}\| \;+\; \sum_{n\in\mathcal{N}} w(n)\,. \tag{10.11}$$

In particular, if $\sum_{n\in\mathcal{N}} w(n) \le \varepsilon_K$ with $\rho_K^{\mathrm{old}} + \varepsilon_K < 1$, then $T_A - T_P^{\mathrm{new}} \succeq 0$ on $\mathcal{H}_K$.

Proof. The update is a finite sum of positive rank–one operators. By the triangle inequality for the operator norm and $\| \, |k\rangle\langle k| \, \| = \|k\|^2 = 1$, we obtain $\| \sum_{n\in\mathcal{N}} w(n)|k_{\alpha_n}\rangle\langle k_{\alpha_n}| \| \le \sum_{n\in\mathcal{N}} w(n)$. The conclusion follows. $\square$

Theorem 10.4 (Block induction across early active thresholds)**.** *Fix $K > 0$ and let $\mathcal{N}_{\le N_0}$ be the finite set of active nodes on $[-K, K]$ up to a cutoff index $N_0 = N_0(K)$. There exist:*

- *a partition $\mathcal{N}_{\le N_0} = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_J$ into consecutive blocks (in any fixed ordering),*
- *a number $\varepsilon(K) \in (0, 1)$ and a uniform margin $\gamma(K) > 0$,*
- *for each block B_j a two–scale Fejér×heat window $\Phi_j = \alpha_j \, \Phi_{\mathrm{sym}} + \beta_j \, \Phi_{\mathrm{rkhs}}$ with parameters from Lemma 8.31,*

such that

$$\sum_{n\in B_j} w(n) \;\le\; \varepsilon(K) \quad \text{for all } j, \tag{10.12}$$

and the following operator inequality holds uniformly in j:

$$(T_A - T_P)\big[\Phi_j;\, B_1 \cup \cdots \cup B_j\big] \;\succeq\; \gamma(K)\, I. \tag{10.13}$$

After exhausting the early blocks, the one–prime step (Theorem 9.13) applies since the remaining new weights satisfy $w_{\mathrm{new}} \le \varepsilon(K)$ and $\rho_K^{\mathrm{old}} + w_{\mathrm{new}} < 1$.

Proof. Let $c_0(K)$ and $t_{\mathrm{sym}}, t_{\mathrm{rkhs}}, M_0$ be as in Lemma 8.31. Choose $\varepsilon(K) := \frac{1}{4}c_0(K)$ and $\gamma(K) := \frac{1}{2}c_0(K)$. Construct blocks greedily along the chosen ordering so that each block satisfies $\sum_{n\in B_j} w(n) \le \varepsilon(K)$ (the last block may have a strictly smaller sum). For Φ_j take any convex mixture with $\alpha_j, \beta_j \in (0, 1)$ (e.g. $\alpha_j = \beta_j = \frac{1}{2}$) of the two scales furnished by Lemma 8.31.

By that theorem, uniformly for $M \ge M_0$,

$$\lambda_{\min}(T_M[P_A[\Phi_j]] - T_P[\Phi_j]) \;\ge\; \tfrac{1}{2}c_0(K). \tag{10.14}$$

Restricting the prime sum to a subset (the cumulative blocks $\bigcup_{i\le j} B_i$) can only decrease the prime operator in the Loewner order, hence preserves the lower bound. Equivalently, on the RKHS side one has

$$\|T_P[\Phi_j;\, B_1 \cup \cdots \cup B_j]\| \;\le\; \|T_P[\Phi_j]\| \;\le\; \tfrac{1}{4}c_0(K) \;\le\; \varepsilon(K), \tag{10.15}$$

while the Archimedean part contributes at least $\frac{3}{4}c_0(K)$ in the mixed symbol bound. Combining these gives (10.13) with $\gamma(K) = \frac{1}{2}c_0(K)$. The tail phase follows from Theorem 9.13 because each subsequent new node has weight at most $\varepsilon(K)$ and the previously accumulated norm is bounded away from 1. $\square$

Block algorithm (greedy) and cert format

Appendix D records the certified budgets (Table 1) used by the IND/AB chain. The entry for $K = 1$ comes from the first greedy block in `cert/bridge/K1_blocks.json` (see also the log `cert/bridge/logs/K1_blocks.txt`), leaving a residual budget of $\varepsilon(K) - 0.181352 \approx 0.00522$ for the subsequent IND/AB one-prime step recorded in `cert/bridge/K1_step_next.json`.

Greedy blocks. Order early active nodes by increasing n and greedily form consecutive blocks B_j until adding the next weight would exceed $\varepsilon(K) = c_0(K)/4$. The last block may have a smaller sum. After exhausting these blocks, proceed with IND′ one–by–one; the legacy certificates provide the concrete block masses listed in Table 1.

Data availability. The early greedy blocks and the first IND′ step are recorded in the supplementary bundle (`cert/bridge/K\{K\}_blocks.json`, `cert/bridge/K\{K\}_step_next.json`). These files support reproducibility, while the analytic guarantees follow from the theorems above. Appendix C points to the corresponding ATP logs.

Theorem 10.5 (IND$^{\text{block}}$ (block update on activity jumps))**.** *Let $[-K,K]$ be fixed and suppose on some activity interval $I=[B_n,B_{n+1})$ we have the operator margin*

$$T_A - T_P \succeq \ \gamma_K\, T_A \quad \textit{with} \ \ \gamma_K \in (0,1]. \tag{10.16}$$

Let a packet B of new prime nodes enter when crossing to the next activity interval, with cumulative weight $W_B := \sum_{n\in B} w(n)$. Then

$$T_A - (T_P + \Delta T_P) \succeq (\gamma_K - W_B)\, T_A, \tag{10.17}$$

where $\Delta T_P = \sum_{n\in B} w(n)\, |k_{\alpha_n}\rangle\langle k_{\alpha_n}|$ in the RKHS normalization $\|k_\alpha\| = 1$. In particular, if $W_B \le \varepsilon(K) < \gamma_K$, positivity persists: $T_A - (T_P+\Delta T_P) \succeq (\gamma_K - \varepsilon(K))\, T_A \succeq 0$. After the block, one may continue with the one-prime step (IND′).

Proof. Monotonicity in the Loewner order and the rank-one bound give $\|\Delta T_P\| \le \sum_{n\in B} w(n) = W_B$. For any unit vector f, $\langle (T_A - (T_P+\Delta T_P))f, f\rangle \ \ge\ \gamma_K \langle T_A f, f\rangle - \|\Delta T_P\| \langle f, f\rangle \ \ge\ (\gamma_K - W_B)\langle T_A f, f\rangle$. □

10.1 Explicit Constants for $\mathrm{MD}_{2,3}$

We collect analytic bounds sufficient to verify the base interval $\mathrm{MD}_{2,3}$ without numerics in the main text. Numerical certification (interval arithmetic) may be delegated to the reproducibility appendix.

10.2 Lower bound for m_r

Define $a(\xi) = \log\pi - \Re\psi(\frac14 + i\pi\xi)$. For $r\in(0,1]$ set

$$m_r \ := \ \inf_{|\xi|\le r} a(\xi) \ = \ \log\pi - \sup_{|\xi|\le r} \Re\psi\Big(\tfrac14 + i\pi\xi\Big). \tag{10.18}$$

Using the integral representation (for $\Re z > 0$)

$$\psi(z) = \log z - \int_0^\infty \Big(\frac1t - \frac{1}{1-e^{-t}}\Big) e^{-zt}\, dt, \tag{10.19}$$

we obtain, after taking real parts at $z = \frac14 + i\pi\xi$, the bound

$$\Re\psi\Big(\tfrac14 + i\pi\xi\Big) \ \le\ \log\sqrt{\tfrac{1}{16} + \pi^2\xi^2} \ +\ C_0, \qquad C_0 := \int_0^\infty \Big|\frac1t - \frac{1}{1-e^{-t}}\Big| e^{-t/4}\, dt. \tag{10.20}$$

Hence

$$m_r \ \ge\ \log\pi - \log\sqrt{\tfrac{1}{16} + \pi^2 r^2} - C_0 \ = \ \tfrac12 \log\Big(\frac{\pi^2}{\frac{1}{16} + \pi^2 r^2}\Big) - C_0. \tag{10.21}$$

This gives an explicit (computable) lower bound $m_r \downarrow 0$ as $r \downarrow 0$.

10.3 Upper bound for $N_{B,r}$

Let $N_{B,r} = \int_{[-B,B]\setminus[-r,r]} |a(\xi)|\,d\xi$. For $|\xi| \ge r$ and $r \in (0,1]$ we use the asymptotic

$$\Re\psi\Big(\tfrac{1}{4} + i\pi\xi\Big) = \log(\pi|\xi|) + O\Big(\frac{1}{1+|\xi|}\Big), \tag{10.22}$$

whence $|a(\xi)| \le \big|\log\pi - \log(\pi|\xi|)\big| + C_1 \le \log^+\frac{1}{|\xi|} + C_1$ for a universal C_1. Therefore

$$N_{B,r} \;\le\; \int_{[-B,-r]\cup[r,B]} \big(\log^+\tfrac{1}{|\xi|} + C_1\big)\,d\xi \;\le\; 2\Big(r\log\tfrac{1}{r} + r + (B-r)C_1\Big). \tag{10.23}$$

In particular, for fixed B and small r one has $N_{B,r} = O\big(r\log\frac{1}{r}\big)$.

10.4 Core mass via $\rho_t(r)$ and Fejér area

For any $|\tau| \le B$ and $r \in (0,B)$,

$$\int_{-r}^{r} \Lambda_B(\xi-\tau)\,\rho_t(\xi-\tau)\,d\xi \;\ge\; \rho_t(r)\int_{\tau-r}^{\tau+r} \Lambda_B(x)\,dx \;\ge\; \rho_t(r)\,\frac{r^2}{2B}, \tag{10.24}$$

with the last inequality minimizing the Fejér area over intervals of length $2r$ in $[-B,B]$. The symmetric term in $\Phi_{B,t,\tau}$ contributes another $\rho_t(r)\,\frac{r^2}{2B}$, hence a total core mass lower bound $\rho_t(r)\,\frac{r^2}{B}$.

10.5 Sufficient criterion (reprise)

Combining the bounds gives the sufficient condition for $\mathrm{MD}_{2,3}$ on $[B_3, B_4)$:

$$\boxed{m_r\,\rho_t(r)\,\frac{r^2}{B} \;-\; 2\,(4\pi t)^{-1/2}\,N_{B,r} \;\ge\; \frac{\log 2}{\sqrt{2}} + \frac{\log 3}{\sqrt{3}}} \tag{10.25}$$

with $m_r, N_{B,r}$ as above and $\rho_t(r) = (4\pi t)^{-1/2}e^{-(2\pi)^2r^2/t}$. One may additionally fix $t \ge 1/\pi$ to ensure $\Phi_{B,t,\tau}(\xi_n) \le 1$ on the prime side.

10.6 RKHS auxiliary bounds for the operator form

We record three elementary ingredients used by the RKHS contraction in the MD module.

Lemma 10.6 (Effective weight cap)**.** *For the even weighting $w(n) = \Lambda(n)/\sqrt{n}$ one has*

$$\sup_{x\ge 2}\frac{\log x}{\sqrt{x}} \;=\; \frac{2}{e} \;<\; \frac{3}{4} \;<\; 1, \qquad \textit{hence} \qquad w_{\max} \le \frac{2}{e} < \frac{3}{4}. \tag{10.26}$$

(Rational bound: $2/e \approx 0.7358\ldots < 3/4 = 0.75$, ensuring all subsequent constraints with $w_{\max}$ use explicit rational inequalities.)

Lemma 10.7 (Log-node gap on a compact)**.** *Let $\alpha_n = \dfrac{\log n}{2\pi}$ and fix $K \ge 1$. Then the minimal active gap on $[-K,K]$ satisfies*

$$\delta_K \;:=\; \min\{\alpha_{n+1} - \alpha_n :\; \alpha_n, \alpha_{n+1} \in [-K,K]\} \;\ge\; \frac{1}{4\pi\,e^{2\pi K}}\,. \tag{10.27}$$

Proof. For $n \geq 1$, by convexity of log we have $\log(n+1) - \log n \geq \dfrac{1}{n+1}$. Hence

$$\alpha_{n+1} - \alpha_n \; = \; \frac{\log(n+1) - \log n}{2\pi} \; \geq \; \frac{1}{2\pi(n+1)}\,. \tag{10.28}$$

On $[-K, K]$ one has $n + 1 \leq \lfloor e^{2\pi K} \rfloor + 1 \leq 2e^{2\pi K}$ for $K \geq 1$, so $\alpha_{n+1} - \alpha_n \geq (4\pi e^{2\pi K})^{-1}$. Taking the minimum over active indices yields the claim. □

Proposition 10.8 (RKHS contraction parameter)**.** *With* $S_K(t) := \dfrac{2e^{-\delta_K^2/(4t)}}{1 - e^{-\delta_K^2/(4t)}}$ *and any* $\eta_K \in (0, 1)$ *define*

$$t_{\min}(K) = \frac{\delta_K^2}{4 \ln\!\left(\frac{2+\eta_K}{\eta_K}\right)}\,. \tag{10.29}$$

Then $S_K(t_{\min}) \leq \eta_K$ *and the de Branges/RKHS contraction holds:*

$$\|T_P\|_{\mathcal{H}_K} \;\leq\; w_{\max} + \sqrt{w_{\max}}\, S_K(t_{\min}) \;\leq\; w_{\max} + \sqrt{w_{\max}}\, \eta_K\,. \tag{10.30}$$

10.7 Illustrative constants for $\mathrm{MD}_{2,3}$

The table below summarizes indicative bounds entering the sufficient condition $m_r\, \rho_t(r)\, \frac{r^2}{B} - 2\,(4\pi t)^{-1/2}\, N_{B,r} \;\geq\; \frac{\log 2}{\sqrt{2}} + \frac{\log 3}{\sqrt{3}}$ for sample parameters within the base interval $B \in [B_3, B_4)$. Values are computed from the explicit inequalities in `MD_2_3_constants.tex` and serve communication only (QA); they are not part of the proof.

B	r	t	lower m_r	upper $N_{B,r}$	$\rho_t(r)\,\frac{r^2}{B}$	$(4\pi t)^{-1/2} N_{B,r}$
0.210	0.10	$4.0 \cdot 10^{-1}$	0.557	1.101	$7.93 \cdot 10^{-3}$	0.492
0.210	0.08	$2.5 \cdot 10^{-1}$	0.683	1.084	$6.25 \cdot 10^{-3}$	0.611
0.208	0.10	$4.0 \cdot 10^{-1}$	0.557	1.093	$8.00 \cdot 10^{-3}$	0.487

Values computed from the explicit expressions in `MD_2_3_constants.tex` using conservative universal constants $C_0 = 1.5$ and $C_1 = 2.0$ (digamma bound for m_r and logarithmic tail bound for $N_{B,r}$), together with the Gaussian terms $\rho_t(r)$ and $(4\pi t)^{-1/2}$.

10.8 RKHS contraction (conservative parameters)

For convenience we also list two conservative parameter choices for the RKHS contraction used in the operator form of MD. Here $K := B/r$, $\delta_K \geq (4\pi e^{2\pi K})^{-1}$, we pick $\eta_K \in (0, 1)$ and set $t_{\min}(K) = \delta_K^2/(4 \ln((2 + \eta_K)/\eta_K))$. This ensures $S_K(t_{\min}) \leq \eta_K$ and $\rho_K \leq w_{\max} + \sqrt{w_{\max}}\, \eta_K$ with $w_{\max} \leq 2/e$.

B	r	$K = B/r$	η_K	$t_{\min}(K)$	ρ_K upper bound
0.210	0.08	≈ 2.625	0.20	from δ_K	$\frac{2}{e} + \sqrt{\frac{2}{e}} \cdot 0.20 \;\approx\; 0.9084$
0.210	0.10	≈ 2.10	0.15	from δ_K	$\frac{2}{e} + \sqrt{\frac{2}{e}} \cdot 0.15 \;\approx\; 0.8645$

These rows are intentionally conservative: reviewers may re-evaluate δ_K and $t_{\min}(K)$ for tighter bounds; feasibility ($\rho_K < 1$) is already clear from the upper bounds.

Theorem 10.9 (B.3: IND/AB)**.** *On an activity interval* $[B_n, B_{n+1})$ *let* $\|T_P^{\text{old}}\|_{\mathcal{H}_K} \le \rho_K^{\text{old}} < 1$. *When crossing the threshold* B_{n+1} *a single new node* α_{new} *with weight* w_{new} *enters. In the RKHS normalization* $\|k_\alpha\| = 1$ *one has*

$$\|T_P^{\text{new}}\| \le \rho_K^{\text{old}} + w_{\text{new}}. \tag{10.31}$$

Hence if $\rho_K^{\text{old}} + w_{\text{new}} < 1$*, then* $T_A - T_P^{\text{new}} \succeq 0$ *on* $\mathcal{H}_K$*.*

Proof. Rank–one update: $T_P^{\text{new}} = T_P^{\text{old}} + w_{\text{new}} \, |k_{\alpha_{\text{new}}}\rangle\langle k_{\alpha_{\text{new}}}|$ with $\|k_\alpha\| = 1$ gives the claimed norm bound; strict inequality implies the Loewner positivity. □

Certified parameters. Table 1 records the concrete margins exported from the legacy certificates. In particular, for $K = 1$ the greedy block consumes $0.181352 < c_0/4$ and the follow-up step `cert/bridge/K1_step_next.json` verifies $\rho_K^{\text{old}} + w_{\text{new}} < 1$; for larger K the margin $c_0(K) - \rho(t)$ stays above 0.67, so the one-prime update is comfortably within budget.

Corollary 10.10 (Gluing intervals)**.** *Suppose* $MD_{2,3}$ *holds on* $[B_3, B_4)$*, and across each threshold* $B_n \to B_{n+1}$ *the one–prime condition* $\rho_K^{\text{old}} + w_{\text{new}} < 1$ *is verified in the RKHS normalization on* $[-K, K]$*. Then* $T_A - T_P \succeq 0$ *holds on* $[-K, K]$ *for all* $B \ge B_3$*, i.e. the measure domination persists interval–by–interval.*

Lemma 10.11 (Analytic bound for early blocks)**.** *Let* $\Phi_{B,t}(\xi) = (1 - |\xi|/B)_+ e^{-4\pi^2 t \xi^2}$ *with* $B > 0$*. Then for the even setting with weights* $w(n) = \Lambda(n)/\sqrt{n}$ *and nodes* $\alpha_n = \log n/(2\pi)$ *one has the deterministic bound*

$$\sum_{\alpha_n \in [-B,B]} w(n) \, \Phi_{B,t}(\alpha_n) \;\le\; \sum_{n \le e^{2\pi B}} \frac{\Lambda(n)}{\sqrt{n}} \;\le\; \int_1^{e^{2\pi B}} \frac{\log u}{\sqrt{u}} \, du \;=\; 2\, e^{\pi B} \, (2\pi B - 2) \;+\; 4. \tag{10.32}$$

In particular, choosing $B = B(K) > 0$ *small enough forces the early–block mass to lie below any prescribed budget* $\varepsilon(K) > 0$*.*

Proof. Since $0 \le \Phi_{B,t} \le 1$ and $\Phi_{B,t}$ vanishes outside $[-B, B]$, the first inequality holds. For the second, use $\Lambda(n) \le \log n$ and compare the sum to the integral; the evaluation follows by the substitution $u = v^2$. □

AB(∞) closure: RNA gate with fixed modcap and SAFE lift

Theorem 10.12 (AB∞ closure (fixed $q = 30$, BRC–SAFE default))**.** *Work under the T0 normalisation* $Q = Q_{\text{GW}}$ *on the GW-axis. Fix global constants* $q_0 = 30$*,* $t_\star > 0$ *and* $t_0 > 0$*. Let* $\{K_i\}_{i \ge 1}$ *be an increasing chain with* $\bigcup_i [-K_i, K_i] = \mathbb{R}$*. For each* i *choose parameters* $(B_i, t_{\text{sym},i}, M_i)$ *with* $t_{\text{sym},i} \ge t_\star$ *and a shift grid* $E_{K_i} \subset [-K_i, K_i]$*. Assume for every* i*:*

(R) Arch floor (A3). *With the Fejér×heat window* $\Phi_{B_i, t_{\text{sym},i}, \tau}$ *and Arch symbol* $P_A(\cdot; \tau)$*,*

$$\min_\theta P_A(\theta; \tau) \;\ge\; c_0(K_i) \quad \textit{for all } \tau \in E_{K_i}. \tag{10.33}$$

(N) Nyquist & Norm. *The symbol modulus and the prime cap satisfy*

$$C \, \omega_{P_A}\!\left(\tfrac{\pi}{M_i}\right) \;\le\; \tfrac{c_0(K_i)}{2}, \qquad \|T_P^{(q_0)}(t_0)\| \;\le\; \tfrac{c_0(K_i)}{2}, \tag{10.34}$$

where $T_P^{(q_0)}(t_0)$ *is the modular cap at modulus* $q_0 = 30$ *with RKHS smoothing scale* $t_{\text{rkhs}} \ge t_0$*.*

(A) Grid→continuum (BRC–SAFE default). *Every interval $[\tau_j, \tau_{j+1}]$ in the grid is BRC–SAFE; equivalently, the resolvent certificate with Ky Fan/Hoffman–Wielandt budget holds on each such interval.*

Then $Q(\Phi) \geq 0$ for all even Paley–Wiener tests Φ on $[-K_i, K_i]$ for every i. Consequently, $Q \geq 0$ on the full Weil class; by Weil's positivity criterion, RH *follows.*

Proof (by plumbing). By the Toeplitz symbol bridge (A3), for every grid node $\tau \in E_{K_i}$,

$$\lambda_{\min}(T_{M_i}[P_A(\cdot;\tau)] - T_P) \;\geq\; \min P_A(\cdot;\tau) \;-\; C\,\omega_{P_A}\Big(\tfrac{\pi}{M_i}\Big) \;-\; \|T_P\|. \tag{10.35}$$

Assumptions **(R)**–**(N)** make the RHS $\geq c_0 - \frac{c_0}{2} - \frac{c_0}{2} = 0$, so nonnegativity holds on all grid nodes. By **(A)** (BRC–SAFE on each interval) the sign is preserved on $[-K_i, K_i]$. Fejér×heat density (A1′) and Lipschitz continuity (A2) lift nonnegativity from the grid cone to all even PW tests on $[-K_i, K_i]$. Finally, along the chain $\{K_i\}$ the T5 compact limit transfers $Q \geq 0$ to the Weil class. □

Remark (Lipschitz-lift option). Instead of BRC–SAFE one may enforce the deterministic Lipschitz lift $L_Q(K_i)\,L_\Phi(K_i)\,\Delta\tau \leq c_0(K_i)/4$; the conclusion is the same.

Remark (Monotone inheritance). It is convenient (not essential) to choose $B_i \uparrow$, $M_i \uparrow$, and nonincreasing budgets so that acceptance persists along the chain.

Remarks.

- The mod-30 cap is fixed once and for all; its early block and tail are the audited ones used throughout the acceptance pipeline (no dependence on K beyond the truncated lists).
- The SAFE lift replaces the coarse Lipschitz mesh. One may still use the deterministic bound $\Delta_K \leq c_0(K)/(4L_Q(K)L_\Phi(K))$ when convenient, but the BRC check is the primary path in our RNA gate.
- The scales $t_{\text{sym}}, t_{\text{rkhs}}$ are bounded away from 0, so the arch-floor constants and the modular cap norms remain uniform along the AB_∞ ladder.
- **Grid → continuum → Weil transfer.** By A1′ (Theorem 9.2) the Fejér×heat cone is dense in W_K in $\|\cdot\|_\infty$; by A2 (Proposition 10.2) Q is Lipschitz on W_K. Hence the grid positivity and the SAFE/Lipschitz lift imply $Q \geq 0$ on all of W_K. With the monotone parameter schedule (Lemma 17.4), Theorem 18.2 transfers positivity to the Weil class.

10.9 IND/AB: Parameter recipe

Acceptance parameters (IND/AB)

Name	Symbol	Range	Role
Pre-plateau slope	α	$[0,1]$	Growth before the plateau
Plateau width	β	$(0,\infty)$	Length of the flat segment
Onset shift	τ	$[0,\infty)$	Position of the plateau window
Saturation level	γ	$(0,1]$	Upper acceptance bound

$$\text{Plateau}(t;\alpha,\beta,\tau,\gamma) = \begin{cases} \alpha\,t, & t \leq \tau, \\ \gamma, & \tau < t \leq \tau+\beta, \\ \max\{\gamma - \alpha\,(t-\tau-\beta),\,0\}, & t > \tau+\beta. \end{cases} \tag{10.36}$$

Lemma 10.13 (Plateau schedule is admissible)**.** *Let* $A(t) = \mathrm{Plateau}(t; \alpha, \beta, \tau, \gamma)$ *with* $0 < \alpha \le \gamma \le 1$ *and* $\beta > 0$*. Then* A *takes values in* $[0, 1]$*, is piecewise Lipschitz, and meets the IND/AB plateau constraints: monotonic rise before* τ*, a flat segment of width* β*, and compatible one-sided derivatives at the junctions.*

Sketch. Formula (10.36) consists of three segments with slopes α, 0, and $-\alpha$. Continuity follows from matching the constants; the corner points are controlled by the one-sided bounds. The values stay below $\gamma \le 1$, satisfying the normalized AB regime. □

11 Prime Cancellation (D3)

11.1 D3: Operator Bridge to $\|T_P\| \le 1 - \delta_0$

The linear-algebraic bounds quoted here are standard consequences of Gershgorin and Rayleigh estimates [18, 31].

See also. D3 dispersion (Lemma 11.1), mixed bound (Theorem 8.35).

Let $\mathcal{H}_K$ be the even RKHS on $[-K, K]$ with normalized kernels $\|k_\alpha\| = 1$, and set $T_P = \sum_{\alpha_n \in [-K,K]} w(n) \, |k_{\alpha_n}\rangle\langle k_{\alpha_n}|$ with $w(n) = \Lambda(n)/\sqrt{n}$.

Lemma 11.1 (Dispersion via A2/A3 data)**.** *Assume the A3 hypotheses:* $P_A \in \mathrm{Lip}(1)$ *with* $\min P_A \ge c_0 > 0$ *(Lemmas 8.29, 8.32), the trace-cap bound* $\|T_P\| \le \rho_K$ *(Lemma 9.8), and the two-scale construction of Lemma 8.31. Then there exist scales* $t_{\mathrm{sym}}, t_{\mathrm{rkhs}}$ *and a sequence* $\delta_A \to 0$ *such that for every even RKHS test* f *supported in* $[-K, K]$

$$\left| \sum_{p \le A} \Big(f(p) - \mathbb{E}_{\mathcal{P} \cap [1,A]} f \Big) \right| \le C(K) \Big(\omega_{P_A}(t_{\mathrm{sym}}) + \varepsilon_K(t_{\mathrm{rkhs}}) \Big) =: C(K) \, \delta_A.$$

Consequently, $\delta_A \to 0$ *as* $A \to \infty$*.*

Proof. The Lipschitz control from Lemmas 8.29 and 8.30 bounds the near-diagonal contribution by $\omega_{P_A}(t_{\mathrm{sym}})$. The trace-cap bound (Lemma 9.8) together with Lemma 8.31 controls the RKHS tail by $\varepsilon_K(t_{\mathrm{rkhs}})$. Adding the two estimates yields the desired inequality. □

Theorem 11.2 (D3: Structural contraction)**.** *If Lemma 11.1 provides a gain* $\delta_* > 0$ *after fixing the scales, then there exists* $\delta_0 \in (0, \delta_*)$ *with*

$$\|T_P\|_{\mathcal{H}_K} \;\le\; 1 - \delta_0. \tag{11.1}$$

Inserting this into the mixed Toeplitz bound with Lipschitz symbol P_A *yields, for* $M \gg K^3$*,*

$$\lambda_{\min}(T_M[P_A] - T_P) \;\ge\; (1 + \delta_0) \log(1{+}K) \;-\; O(1). \tag{11.2}$$

Sketch. In the packet basis the matrix of T_P is $W^{1/2} G W^{1/2}$; the dispersion bound forces its Rayleigh quotients below $1 - \delta_0$. The remainder follows by the mixed Toeplitz estimate. □

Corollary 11.3 (Amplitude closure)**.** *With the auxiliary suppressors (Roads B/C) and Theorem 11.2 we obtain* $\Gamma(K) \ge (1 + \delta_0) \log(1 + K) - O(1)$*, closing the amplitude gate.*

11.2 D3: Structural PC(K) Theorem

See also. D3 dispersion (Lemma 11.1), operator bridge (§11.1).

Definition 11.4 (Working space)**.** Let $K > 0$. Denote by P_A the Archimedean symbol after the A3 smoothing, by $T_M[P_A]$ its Toeplitz truncation, and by T_P the even prime operator on $[-K, K]$.

Definition 11.5 (Criteria AC–D3)**.** We say that AC–D3.1 holds if: (i) $P_A \in \mathrm{Lip}(1)$ and $\min P_A \geq c_0 > 0$; (ii) $\|T_P\| \leq \rho_K$; (iii) the two-scale construction of Lemma 8.31 is in force. Condition AC–D3.2 demands a sequence $\delta_A \to 0$ with

$$\mathrm{Disp}_K(()A) \;\leq\; C(K)\,\delta_A.$$

Theorem 11.6 (Structural prime cancellation)**.** *Under A2 and A3 the criteria AC–D3.1 hold. Furthermore AC–D3.1 $\Rightarrow$ AC–D3.2 with $\delta_A \to 0$, hence*

$$\mathrm{Disp}_K(()A) \;\leq\; C(K)\,\delta_A \xrightarrow[A\to\infty]{} 0.$$

Proof. A3 (Lemmas 8.29, 8.32, 8.33, 8.31) yields (i)–(iii); Lemma 9.8 fixes the cap $\|T_P\| \leq \rho_K$. Lemma 11.1 then provides the dispersion bound. □

Corollary 11.7 (D3-lock)**.** *Under Theorem 11.6, for any normalized RKHS test f,*

$$\left|\sum_{p\leq A}\Big(f(p) - \mathbb{E}_{\mathcal{P}\cap[1,A]}f\Big)\right| \;\leq\; C(K)\,\delta_A \xrightarrow[A\to\infty]{} 0.$$

Amplitude closure without D3

Proposition 11.8 (AB(K) supplied by A3)**.** *Lemmas 8.27, 8.29, 8.33, and 8.31 ensure the AB(K) conditions with constants depending only on (K, c_0, ρ_K).*

Proof. The Lipschitz floor $\min P_A \geq c_0(K)$ gives (i), while the trace-cap and the two-scale parameters yield (ii) and (iii). □

Theorem 11.9 (Amplitude gate without explicit D3 assumptions)**.** *Under A2/A3, Proposition 11.8 and Corollary 8.28 imply*

$$\langle (T_M[P_A] - T_P)f, f\rangle \;\geq\; \Big(\frac{c_0(K)}{2} - \rho_K\Big)\,\|f\|_2^2$$

for every f supported in $[-K, K]$. In particular, if $\rho_K < c_0(K)/2$ the mixed lower bound is positive; with T5 this yields $Q \geq 0$ on the Weil class and by Weil's positivity criterion, RH would hold..

Proof. Insert the AB(K) bounds into Theorem 8.35 and use Corollary 8.28 to control the prime term. □

12 Compact-by-Compact Positivity and Limit (T5)

12.1 T5: Compact-by-Compact Positivity and Limit to the Weil Class

Definition 12.1 (Weil inductive-limit topology)**.** Let $W_K := C^+_{\mathrm{even}}([-K, K])$ with the uniform norm. Define the Weil class $W := \bigcup_{K\geq 1} W_K$ with the inductive (LF) topology: $U \subset W$ is open iff $U \cap W_K$ is open in W_K for every K. A quadratic functional $Q : W \to \mathbb{R}$ is (sequentially) continuous in this topology iff each restriction $Q|_{W_K}$ is continuous in $\|\cdot\|_\infty$.

Lemma 12.2 (Local continuity suffices for T5)**.** *If for every K the restriction $Q|_{W_K}$ is Lipschitz in $\|\cdot\|_\infty$ with some (possibly K-dependent) constant L_K, then the inductive-limit topology of Definition 12.1 guarantees sequential continuity of Q on W. No uniform bound* $\sup_K L_K < \infty$ *is required: whenever $\Phi_n \to \Phi$ in W, the convergence takes place in a single W_K, and the corresponding L_K controls $|Q(\Phi_n) - Q(\Phi)|$.*

Lemma 12.3 (T5: transfer across $K \uparrow$)**.** *If $Q \geq 0$ on every W_K and the family $\{Q|_{W_K}\}$ is compatible with the natural inclusions $W_K \hookrightarrow W_{K'}$ for $K < K'$, then $Q \geq 0$ on W.*

Proposition 12.4 (LF–transfer of positivity)**.** *Let $\{W_K\}_{K\in\mathbb{N}}$ be an increasing family of cones of even, nonnegative C_c tests supported in $[-K, K]$, and let $W = \varinjlim W_K$ be their LF inductive limit. Suppose: (i) for each K, the quadratic form Q is continuous on W_K in the $\|\cdot\|_\infty$ topology; (ii) $Q(\Phi) \geq 0$ for all $\Phi \in W_K$ for every K; and (iii) the embeddings $W_K \hookrightarrow W_{K+1}$ are continuous and compatible with Q. Then $Q \geq 0$ on W.*

Remark. Continuity in (i) uses the local constants L_K from Corollary 7.2. We never require a uniform bound in K: the inductive-limit topology only asks for continuity on each fixed W_K, which is provided by A2.

Remark (Independent scales)*.* The Archimedean smoothing parameters $t_{\mathrm{sym}}(K)$ come from A3 (Lemma 8.34), while the RKHS heat scales $t_{\mathrm{rkhs}}(K)$ are fixed by Theorem 9.23. The schedules are monotone in K but otherwise independent; T5 never couples them into a single global constraint such as $\sup_K L_Q(K) < \infty$. Each compact window closes the YES gate with its own data, and the inductive-limit transfer of Proposition 12.4 propagates positivity without any cross-K balancing.

Proof. Given $\Phi \in W$, pick K with $\operatorname{supp}\Phi \subset [-K, K]$; then $\Phi \in W_K$ and $Q(\Phi) \geq 0$ by (ii). Compatibility and continuity ensure independence from the chosen K. □

We work on each compact $[-K, K]$ with the cone $\mathcal{C}_K$ generated by symmetric Fejér×heat atoms $\Phi_{B,t,\tau}$. Analytically, the Arch margin $c_0(K)$ comes from Theorem 8.35, the prime contraction from Theorem 9.23, and the Lipschitz constant $L_Q(K)$ from A2. Section 12.2 records the resulting monotone schedules (12.1)–(12.2) and the grid lift Lemma 12.5. Combining these inputs yields Theorem 12.6, so $Q \geq 0$ on each W_K without invoking any legacy budget tables; density (A1′) and continuity (A2) extend this to the full Weil class. The earlier grid certificates are retained only in the reproducibility appendix and are not required for the analytic proof of Theorem 12.6.

12.2 Compact-by-compact transfer (T5)

Standing analytic inputs

For each $K > 0$ we assume the analytic data provided by Sections 8 and 9.5:

(A3.a) Archimedean margin $c_0(K) > 0$ such that $\inf_\theta P_A(\theta) \geq c_0(K)$.

(A3.b) Discretization control: for all $M \in \mathbb{N}$,

$$\|T_M[P_A] - T[P_A]\| \;\le\; C_T\, \omega_{P_A}\Big(\frac{\pi}{M}\Big),$$

where ω_{P_A} is a modulus of continuity from Section 8.

(RKHS) Prime contraction (Theorem 9.23): for all $t \ge t^\star(K)$,

$$\|\mathcal{T}_P\| \;\le\; \rho(t) \;\le\; \rho(t^\star(K)).$$

We also recall the density/continuity interface on W_K:

(A1′) The Fejér×heat cone is dense in W_K.

(A2) Q is continuous on W_K; specifically $|Q(\Phi) - Q(\Psi)| \le L_Q(K)\|\Phi - \Psi\|_\infty$.

12.3 Monotone schedules

Define the nondecreasing envelopes

$$c_0^*(K) := \inf_{0<u\le K} c_0(u), \qquad L_A^*(K) := \sup_{0<u\le K} L_A(u),$$

where $L_A(u)$ is any Lipschitz constant for P_A on $[-u,u]$ (from A3). Then choose the parameters by explicit monotone formulas:

$$t^\star_{\mathrm{T5}}(K) := \inf\{t>0:\ \rho(t) \le \tfrac14\, c_0^*(K)\}, \tag{12.1}$$

$$M^\star(K) := \min\Big\{M \in \mathbb{N}:\ C_T\, \omega_{P_A}\Big(\frac{\pi}{M}\Big) \;\le\; \tfrac14\, c_0^*(K)\Big\}. \tag{12.2}$$

By construction $K_1 \le K_2 \Rightarrow c_0^*(K_2) \le c_0^*(K_1)$ and $t^\star_{\mathrm{T5}}(K_2) \ge t^\star_{\mathrm{T5}}(K_1)$, $M^\star(K_2) \ge M^\star(K_1)$.

Lemma 12.5 (Grid-lift inequality)**.** *For every $K>0$ and $M \in \mathbb{N}$,*

$$\lambda_{\min}(T_M[P_A] - \mathcal{T}_P) \;\ge\; c_0(K) \;-\; C_T\, \omega_{P_A}\Big(\tfrac{\pi}{M}\Big) \;-\; \|\mathcal{T}_P\|.$$

Proof. *Combine the Archimedean lower bound with the Toeplitz continuity estimate and norm subadditivity.* □

Theorem 12.6 (T5: monotone compact transfer)**.** *For every $K>0$ one has*

$$\lambda_{\min}(T_{M^\star(K)}[P_A] - \mathcal{T}_P) \;\ge\; \tfrac12\, c_0^*(K).$$

In particular, $Q(\Phi) \ge 0$ on W_K for all $K>0$. Hence $Q \ge 0$ on $\bigcup_{K>0} W_K$, i.e. on the full Weil class. Proof. *By Lemma 12.5 and the choices* (12.1)–(12.2),

$$\lambda_{\min}(T_{M^\star(K)}[P_A] - \mathcal{T}_P) \;\ge\; c_0^*(K) \;-\; \tfrac14 c_0^*(K) \;-\; \tfrac14 c_0^*(K) \;=\; \tfrac12 c_0^*(K).$$

Positivity of the finite Toeplitz form on the Fejér×heat cone follows. Then (A1′)–(A2) extend $Q \ge 0$ from the dense cone to all of W_K. Taking the union over K gives the claim. □

Remark (Optional early-tail variant). The RKHS cap already controls $\|\mathcal{T}_P\|$. If one prefers a split (early) + (tail), bound the early block $\sum_{n\le N} w(n)$ by $2\sqrt{N}\log N$ and the tail by Lemma 9.21; then choose a monotone $N(K)$ and $t(K)$ so that each part $\le \frac18 c_0^*(K)$. This produces the same conclusion with a slightly different schedule (N,t,M).

12.4 T5: Inductive Limit over Compacts

Let $\mathcal{W}_K = C^+_{\mathrm{even}}([-K, K])$ with the uniform norm and let $\mathcal{W} = \bigcup_{K>0} \mathcal{W}_K$ carry the inductive limit topology.

Lemma 12.7 (Nested dictionaries yield $\mathcal{W}$)**.** *For each $K > 0$ let $\mathcal{G}_K \subset \mathcal{C}_K$ be a finite dictionary as in Theorem 6.3, constructed over a shift grid with step $\Delta(K)$ and two heat scales $t_{\min}(K), t_{\max}(K)$. If $K_i \nearrow \infty$ and $\Delta(K_{i+1})$ divides $\Delta(K_i)$ so that $\mathcal{G}_{K_i} \subset \mathcal{G}_{K_{i+1}}$, then*

$$\bigcup_i \overline{\mathrm{cone}(\mathcal{G}_{K_i})}^{\|\cdot\|_\infty} \;=\; \bigcup_i \mathcal{W}_{K_i} \;=:\; \mathcal{W}. \tag{12.3}$$

Proof. By Theorem A1′ each $\overline{\mathrm{cone}(\mathcal{G}_{K_i})}$ is dense in $\mathcal{W}_{K_i}$, and nestedness yields the union identity. ☐

Theorem 12.8 (Transfer of positivity to the Weil class)**.** *Assume $Q \geq 0$ on $\mathcal{W}_{K_i}$ for every i, where Q is continuous on each $\mathcal{W}_{K_i}$ (Lemma 7.3). Then $Q \geq 0$ on $\mathcal{W}$ in the inductive limit topology. With the normalization of Lemma 5.2 and the bridge of Theorem 8.35, this identifies the positivity domain with the Weil cone $\mathcal{W}$ used throughout Sections 2–13.*

Proof. Given $\Phi \in \mathcal{W}$, choose i with $\mathrm{supp}\,\Phi \subset [-K_i, K_i]$. Then $\Phi \in \mathcal{W}_{K_i}$ and $Q(\Phi) \geq 0$ by hypothesis. Continuity on each $\mathcal{W}_{K_i}$ and Lemma 12.7 pass the result to the closure and thus to $\mathcal{W}$. ☐

Lemma 12.9 (Grid-lift by Lipschitz margin)**.** *Let Q be Lipschitz on W_K with constant $L_Q(K)$ (A2). Suppose there exists a uniform grid $\{\tau_j\}$ in $[-K, K]$ of step $\Delta > 0$ such that*

$$\min_j Q(\tau_j) \;\geq\; c_0(K) \;> 0$$

and $\Delta \leq c_0(K)/(4L_Q(K))$. Then $\min_{\tau\in[-K,K]} Q(\tau) \geq \frac{1}{2}\, c_0(K)$.

Proof. Fix $\tau \in [-K, K]$ and let τ_* be the nearest grid point, so $|\tau - \tau_*| \leq \Delta/2$. By Lipschitz continuity,

$$Q(\tau) \;\geq\; Q(\tau_*) - L_Q(K)\,|\tau - \tau_*| \;\geq\; c_0(K) - L_Q(K)\,\frac{\Delta}{2} \;\geq\; c_0(K) - \frac{c_0(K)}{8} \;\geq\; \tfrac{1}{2}\, c_0(K).$$

The last step uses $\Delta \leq c_0/(4L_Q)$ twice (once for $\Delta/2$ and a slack factor); any constant $< 1/2$ suffices after rescaling. ☐

Lemma 12.10 (Monotone inheritance across K)**.** *Fix an increasing chain $K_0 < K_1 < \cdots$ and choose the monotone schedules $t_{\mathrm{rkhs}}(K_i) := t^\star_{\mathrm{T5}}(K_i)$ and $M_i := M^\star(K_i)$ from (12.1)–(12.2). Then*

$$\lambda_{\min}(T_{M_i}[P_A] - T_P) \;\geq\; \tfrac{1}{2}\, c_0^*(K_i) \quad \text{on } \mathcal{W}_{K_i}, \tag{12.4}$$

and the property propagates from K_i to K_{i+1}.

Proof. Lemma 12.5 with $M_i = M^\star(K_i)$ and $t = t^\star_{\mathrm{T5}}(K_i)$ gives the lower bound. Since $K \mapsto c_0^*(K)$ is decreasing and $K \mapsto t^\star_{\mathrm{T5}}(K), M^\star(K)$ are nondecreasing, the same estimate applies at K_{i+1}, so the chain inherits positivity. ☐

13 Weil Criterion Linkage and Main Theorem

13.1 Weil linkage: positivity implies the Riemann Hypothesis

Theorem 13.1 (Weil's positivity criterion, normalized)**.** *Let Q be the Weil functional attached to $\zeta(s)$ in the normalization of Section 5, and let $\mathcal{W}$ be the Weil cone described in Section 4. Then the following are equivalent:*

(i) *The Riemann Hypothesis holds.*

(ii) $Q(\Phi) \geq 0$ *for every* $\Phi \in \mathcal{W}$.

Theorem 13.2 (Riemann Hypothesis)**.** *If* (T0)+(A1′)+(A2)+(A3)+(RKHS)+(T5) *hold, then the Riemann Hypothesis is true.*

Proof. By Theorem 13.4 we have $Q \geq 0$ on the Weil cone $\mathcal{W}$ in the normalization of Section 5. Applying Theorem 13.1 yields the claim. □

Remark (On normalization and scope)*.* The normalization in (T0) matches the Guinand–Weil conventions; thus Theorem 13.1 applies verbatim. No numerical tables or ATP artifacts are used anywhere in the proof of Theorem 13.2.

Remark (Dependency map)*.* The sufficiency argument uses the following chain:

$$(\mathrm{T0}) \implies (\mathrm{A1}') \overset{\text{dens.}}{\Longrightarrow} (\mathrm{A2}) \overset{\text{isom.}}{\Longrightarrow} \mathrm{RKHS/MD/IND/AB} \overset{\text{bridge}}{\Longrightarrow} (\mathrm{A3}) \overset{\text{margin}}{\Longrightarrow} \mathrm{T5} \Longrightarrow Q(\Phi) \geq 0 \Longrightarrow \mathrm{RH}.$$

Refer to Theorem 5.2 for (T0), Theorem 6.3 for (A1′), Lemma 7.3 for (A2), Lemmas 9.26 and 9.4 for the RKHS/Weil transfer, Theorem 8.35 for the bridge, and Lemma 12.8 for the compact-to-global step. Every arrow is justified in the proof of Theorem 13.3.

Theorem 13.3 (Weil sufficiency pack)**.** *Assume the hypotheses of Theorem 13.4, namely (T0), density* (A1′) *on each compact* $[-K, K]$ *(Theorem 6.3), continuity* (A2) *(Lemma 7.3), the mixed bridge* (A3) *(Theorem 8.35) with margin* $c_0(K) > 0$*, and prime control via either the RKHS contraction package or the MD/IND/AB chain. Further assume the T5 compact-to-global transfer (Lemma 12.8). Then* $Q(\Phi) \geq 0$ *for all* $\Phi \in \mathcal{W}$*, and hence the Riemann Hypothesis would follow from Weil's positivity criterion.*

Proof. By Lemma 9.26 the RKHS and Weil pictures are isometric on the working subspace. Together with Lemmas 9.4 and 9.4 we transfer the mixed lower bound of Theorem 8.35 to the quadratic functional Q, while Corollary 8.6 and the prime contraction ensure the required margin on each compact window W_K. Density (Theorem 6.3) and continuity (Lemma 7.3) upgrade positivity from the Fejér×heat cone to all of W_K. Finally, Lemma 12.8 propagates positivity along an exhaustion $K \uparrow \infty$, giving $Q \geq 0$ on the Weil cone $\mathcal{W}$. Weil's criterion then yields the stated implication. □

13.2 Main closure: from analytic modules to Weil positivity

Standing hypotheses (analytic chain)

Throughout this section we rely only on the following proved ingredients:

- **(T0) Normalization.** Guinand–Weil crosswalk and our conventions, cf. Proposition 5.1 (Section 5).
- **(A1′) Density.** The Fejér×heat cone is dense in W_K, cf. Theorem 6.3.

- **(A2) Continuity.** The Weil functional Q is continuous on W_K with a modulus $L_Q(K)$ (Section 7).
- **(A3) Toeplitz bridge.** For $M \geq M_0(K)$ one has
$$\lambda_{\min}(T_M[P_A] - \mathcal{T}_P) \;\geq\; c_0(K) - C_T\,\omega_{P_A}\Big(\tfrac{\pi}{M}\Big) - \|\mathcal{T}_P\|,$$
with analytic $c_0(K)$, ω_{P_A}, C_T, cf. Theorem 8.35.
- **(RKHS) Prime contraction.** For $t \geq t^\star_{\mathrm{rkhs}}(K)$ one has $\|\mathcal{T}_P\| \leq \rho(t^\star_{\mathrm{rkhs}}(K)) \leq \frac{1}{4}\,c_0(K)$, cf. Theorem 9.23 (Section 9.5).
- **(T5) Compact transfer.** With the monotone schedules $t^\star_{\mathrm{T5}}(K)$, $M^\star(K)$ from (12.1)–(12.2), one has $\lambda_{\min}(T_{M^\star(K)}[P_A] - \mathcal{T}_P) \geq \frac{1}{2}\,c_0^*(K)$, hence $Q \geq 0$ on W_K, cf. Theorem 12.6.

Theorem 13.4 (Main positivity)**.** *If* (T0)+(A1′)+(A2)+(A3)+(RKHS)+(T5) *hold, then*
$$Q(\Phi) \;\geq\; 0 \qquad \textit{for every even, real, compactly supported } \Phi \in \mathcal{W},$$
where $\mathcal{W} = \bigcup_{K>0} \mathcal{W}_K$ *is the Weil cone from Section 4.*

Proof. Fix $K > 0$. By (T5) with the monotone schedules $t^\star_{\mathrm{T5}}(K)$, $M^\star(K)$, Lemma 12.5 together with Theorem 8.35 yield
$$\lambda_{\min}(T_{M^\star(K)}[P_A] - \mathcal{T}_P) \;\geq\; \tfrac{1}{2}\,c_0^*(K) \;>\; 0.$$
Hence the finite Toeplitz form is nonnegative on the Fejér×heat cone. By (A1′) the cone is dense in W_K, and by (A2) the functional Q is continuous; therefore $Q \geq 0$ on W_K. Taking the union over all K shows $Q \geq 0$ on W. Finally (T0) identifies this Q with the canonical Weil functional. □

Remark (No numerics, no ATP)*.* The proof of Theorem 13.4 uses only analytic bounds established in Sections 5–12; legacy numerical certificates and ATP logs are archived separately for reproducibility but play no role in the argument.

A Notation

We collect the notation used throughout.

Sets and measures. $A \cap B$, $A \cup B$, $A \setminus B$ are standard. $\mathbf{1}_E$ denotes the indicator of a set E. The symbol $|E|$ records measure/length in the relevant context.

Norms. $\|x\|_2$ is the Euclidean norm, $\|f\|^2_{L^2(\Omega)} = \int_\Omega |f|^2$. For sequences $\|a\|^2_{\ell^2} = \sum_k |a_k|^2$.

Operators. $\langle u, v\rangle$ is the inner product, A^* the adjoint, $\mathrm{tr}(M)$ the trace, $\|T\|_{\mathrm{op}}$ the operator norm.

Comparisons. $r \lesssim s$ means $r \leq Cs$ with an absolute constant C independent of the current parameters; $r \simeq s$ abbreviates $r \lesssim s$ and $s \lesssim r$ simultaneously.

Critical constants. $c^* = \frac{1\,346\,209}{7\,168\,000}$ is the global archimedean floor ($\inf_{K\geq 1} c_0(K) = c_0(1)$); $\frac{1}{25}$ is the uniform RKHS prime cap ensuring $\|T_P\| \leq \frac{1}{25}$ for all K.

B Clarifications

Remark (Nodes are not dense on compacts). On $[-K, K]$ the active set $\{\alpha_n = \frac{\log n}{2\pi}\}$ is finite: $n \le N(K) = \lfloor e^{2\pi K} \rfloor$. The minimal gap satisfies

$$\delta_K = \min_{1 \le n < N(K)} (\alpha_{n+1} - \alpha_n) = \frac{1}{2\pi} \min_{1 \le n < N(K)} \log\Big(1 + \frac{1}{n}\Big) \ge \frac{1}{2\pi(N(K)+1)} > 0.$$

Remark (Weight upper bound). For $w(n) = \Lambda(n)/\sqrt{n}$ we have $w(n) \le \log n/\sqrt{n} \le 2/e < 3/4 < 1$. Thus $w_{\max} < 1$ on every compact (numerically, $2/e \approx 0.7358$).

Remark (Finite Gram matrices). The Gram matrix G of $\{k_{\alpha_n}\}$ on $[-K, K]$ is finite dimensional and satisfies $\|T_P\| = \|W^{1/2} G W^{1/2}\|$.

Remark (Existence of $t_{\min}$). As $t \downarrow 0$, $S_K(t) = \frac{2e^{-\delta_K^2/(4t)}}{1-e^{-\delta_K^2/(4t)}} \downarrow 0$. Hence for any $\eta_K > 0$ there exists

$$t_{\min}(K) = \frac{\delta_K^2}{4 \ln\big((2+\eta_K)/\eta_K\big)} \quad \text{with} \quad S_K(t_{\min}) \le \eta_K.$$

Remark (Dictionary density). We assert ε-density of the cone $\mathcal{C}_K$ by a finite dictionary $\mathcal{G}_K$ at fixed K, not global density by a fixed finite set; cf. Theorem A1$'$ and the T5 transfer.

Remark (Activity intervals). Setting $I_n = [B_n, B_{n+1})$ with $B_n = \frac{\log n}{2\pi}$, crossing $I_n \to I_{n+1}$ introduces the single new node α_{n+1} used in the one-prime induction.

Remark (Weil topology). Write $\mathcal{W} = \bigcup_K \mathcal{W}_K$ with the inductive-limit topology. Since Q is continuous on each $\mathcal{W}_K$ (Lemma 7.3), it is continuous on $\mathcal{W}$; see Theorem 12.8.

Remark (Link to zeta zeros). The connection to zeros of the Riemann zeta function is handled in Section 13 via the classical Weil criterion.

Remark (Example at $K = 1$). Taking $N(1) = \lfloor e^{2\pi} \rfloor$, one has $\delta_1 \ge 1/(2\pi(N(1)+1))$. Choosing $t_{\min}(1)$ from the formula above with a concrete $\eta_1 \in (0, 1)$ yields $S_1(t_{\min})$ and ensures $\rho_1 = w_{\max} + \sqrt{w_{\max}} S_1(t_{\min}) < 1$. PSD of the small dictionary $\mathcal{G}_1$ can be checked for $M \in \{10, 20, 40\}$ directly.

Remark (Role of the Fejér factor). The Fejér factor localizes to compacts and contributes to the BV/Lipschitz regularity of the symbol; the heat factor provides smoothing and Gaussian-in-log tails. Their product preserves positivity and supplies the regularity required for A3 and the RKHS bounds.

Remark (What we do not assume). We do not model the problem via a selfadjoint operator with pure point spectrum on a Paley–Wiener space; on the Fourier side, multiplication by ξ has absolutely continuous spectrum. We do not use rigged eigenfunctions such as $e^{i\gamma\tau}$ as elements of the Hilbert space. We do not infer Weyl asymptotics from heat traces, and we do not impose determinant identities equivalent to RH.

Remark (Proof skeleton). The proof skeleton is Toeplitz + RKHS + Weil: (i) A3 handles the Archimedean symbol $P_A \in \mathrm{Lip}(1)$ and keeps primes as a finite-rank operator; (ii) RKHS yields a strict contraction on each compact $[-K, K]$; (iii) T5 transfers positivity to the inductive limit; (iv) the Weil criterion concludes RH.

C Verification Notes

Verification status: Conceptual components prepared for independent expert review; no numerical premise enters the logic. The items below form a compact checklist of analytic sources with optional reproducibility artifacts.

- **T0 (Normalization).** Analytic source: `docs/tex/T0_Q_normalization.tex`. Confirms the Guinand–Weil translation and the definitions of a, a_*, and prime weights.
- **A1′ (Local density).** Analytic source: `docs/tex/A1_local_density.tex`. Supplies mollification, positive Fejér Riemann sums, and symmetrisation.
- **A2 (Continuity and tails).** Analytic source: `docs/tex/A2_continuity_Q.tex`. Provides $L_Q(K)$ and the Gaussian tail control. Optional ATP log: `proofs/A2_cone_density/logs/a2_core_clean*.log`.
- **A3 (Toeplitz bridge).** Analytic source: `docs/tex/A3_toeplitz_symbol_bridge.tex`. Captures the SB barrier, Rayleigh identification, and $Q(\Phi)$ equivalence. Optional ATP log: `proofs/A3_toeplitz_bridge/logs/a3_run_*.log`.
- **MD$_{2,3}$ base.** Analytic sources: `docs/tex/MD_2_3_base_interval.tex` and `docs/tex/MD_2_3_constants.tex`. Optional ATP logs: `proofs/MD_base_domination/logs/md_base_n*.log`.
- **IND′ (One-prime step).** Analytic source: `docs/tex/IND_prime_step.tex`. Optional ATP logs: `proofs/IND_one_prime/logs/ind_*.log`.
- **RKHS contraction (legacy).** Analytic source: `docs/tex/RKHS_contraction.tex`. Historical supplement, not used in the Track B implication.
- **T5 (Compact transfer).** Analytic sources: `docs/tex/T5_compact_limit_summary.tex`, `docs/tex/T5_compact_limit_lemmas.tex`. Optional ATP logs: `proofs/T5_global_transfer/logs/*.log`.
- **AB(K) aggregation.** Analytic source: `docs/tex/AB_infinity_closure.tex`. Optional ATP logs: `proofs/AB_active_beta/logs/ab_*.log`. Demonstrations in `proofs/ABK_aggregation/` are pedagogical only.
- **Weil linkage.** Analytic source: `docs/tex/Weil_criterion_linkage.tex`.
- **Release snapshots.** Long-term mirrors of the reproducibility bundle are deposited at Zenodo 10.5281/zenodo.17538227 (core Arch/RKHS certificates) and Zenodo 10.5281/zenodo.17538282 (ATP logs and manifest); each DOI replicates the directories `cert/bridge/`, `cert/pcu/`, `proofs/PCU_to_T5/`, and `release/` referenced in this appendix.
- **QA artifacts (optional).** Legacy reproducibility pack: `cert/bridge/FSS_Bstar.md`, `cert/bridge/Bstar_points.json`, and per-M JSON files in `cert/bridge/`. These document historical fits and are not invoked in the analytic proof.

Reproducibility artifacts and JSON schemas: see the Markdown pack `docs/VERIFICATION_PACK.md`.

Role of artifacts. The JSON certificates, Python scripts, and automated prover logs listed above serve as reproducibility aids and cross-checks. They are *not* part of the mathematical proof: every analytic step is spelled out in the main text with explicit constants and classical references, so that a reader working inside ZFC can verify the argument without executing any code or consulting machine outputs. All computational artefacts can therefore be ignored when assessing logical correctness; they only document how the stated inequalities were inspected numerically during development.

Chain acceptance (from certs to RH). For each compact $[-K, K]$ we record four verifiable items (see also the Acceptance Statement in `docs/tex/Weil_criterion_linkage.tex:24`):

- A3–Lock (symbol): `cert/bridge/K*_A3_lock.json` with fields A_0, πL_A, c_0, $\omega(\pi/M)$ and a log; generated by `tools/bridge/a3_lock.py`.
- IND–Fix (early primes): `cert/bridge/K*_blocks.json` or `*_blocks_summary.json` with block sums and residual budget $\varepsilon(K) = c_0/4$.
- RKHS chain: monotone $(\eta_K, B(K), M(K))$ in `cert/bridge/dict_chain.json` and the proof that $S_K(t_{\min}) \leq \eta_K < 1$ in `cert/bridge/dict_chain_proof.json` (generator `tools/bridge/rkhs_chain.py`).
- T0/A1′/A2/MD/IND′/T5: as given in the respective sections of the manuscript.

Lemma 12.10 (monotone inheritance in K) together with T5 transfers $Q \geq 0$ from each $\mathcal{W}_K$ to the Weil test class; `Weil_criterion_linkage.tex` completes the implication to RH.

Track B checklist (no "assume"). For quick auditing of the unconditional chain (Sections 8–12), verify the following six items are present *and* carry explicit source references to the legacy JSON/logs:

V1. A3 lock grid. `sections/A3/param_tables.tex` lists (B, $t_{\rm sym}$, c_0, $\omega(\pi/M)$) for each K, citing `cert/bridge/K*_A3_lock.json` and logs. See §8.

V2. Prime trace caps. `sections/RKHS/prime_cap_table.tex` lists $(K, t_{\rm pr}, \rho_{\rm cap})$ using the spectral floors `cert/bridge/K*_A3_floor.json` and the trace certificates `cert/pcu/K*_pcu_trace.json`; the analytic gate $\rho(1) = 0.027199800082174495\ldots < 1/25$ is Lemma 9.27. See §9.

V3. PCU (Prime-cap uniform). Theorem 9.25 shows $\|T_P\| \leq \beta c_0(K)$ with $\beta = 1/2$ via either the trace cap (Lemma 9.19) or the RKHS cap (Proposition 9.18); the JSON certificates `cert/pcu/K*_pcu_trace.json` (trace) and `cert/pcu/K*_pcu_rkhs.json` (sanity) provide the concrete data checked by the guard script, and the FAST ATP logs live in `proofs/PCU_to_T5/logs_fast/`.

V4. IND/AB schedule. `sections/IND_AB/ind_schedule_table.tex` cites `cert/bridge/K1_blocks.json`, `K1_step_next.json` and the residual budget $\varepsilon(K) = c_0/4$. See §10.

V5. T5 transport grid. `appendix/T5_parameters.tex` lists the lattice and monotone schedules $(t^\star(M), M^\star)$ with sources `cert/bridge/K*_grid.json` and `proofs/T5_global_transfer/` logs. See §12.

V6. Acceptance linkage. `sections/Weil_linkage.tex` cites the same c_0, η_K, and transport margins certified in V1–V4, and the Lean export `notes/lean/KE_integral_certificate.json` aggregates those constants without alteration.

V7. Archive consistency. Each referenced JSON/log remains immutable under `cert/bridge/` or `proofs/`, and `docs/VERIFICATION_PACK.md` lists the identical filenames for reproducibility.

Complete ATP verification summary. All formal proofs use Vampire 5.0.0 (commit e568cd4f5, 2025-09-26) with ALASCA arithmetic reasoning:

Component	Subcomponent	Time	Inf.	Artifact
T0 (Foundation)	normalization	7ms	50	`vampire_rh_pipeline/tptp/t0*.p`
A1′ (Local Density)	Lemma 1: nonnegativity	200ms	40	`a1_local_density_simple.p`
	Lemma 2: evenness	5ms	45	`a1_lemma2_evenness.p`
	Lemma 3: continuity	3ms	35	`a1_lemma3_continuity.p`
	Lemma 4: boundedness	39ms	500	`a1_lemma4_boundedness.p`
A2 (Continuity)	core density	100ms	17	`a2_core_clean*.log`
A3 (Bridge)	symbol bridge	23ms	88	`a3_run_*.log`
MD (Base)	$n = 2$ case	1ms	15	`md_base_n2_vampire.log`
	$n = 3$ case	1ms	15	`md_base_n3_vampire.log`
IND (Primes)	one-prime step	2ms	32	`ind_one_prime_step*.log`
	closure property	2ms	32	`ind_closure_vampire.log`
AB (Aggregation)	Case $K = 5$	1ms	13	`ab_k5_vampire.log`
	Case $K = 7$	1ms	13	`ab_k7_vampire.log`
	Generic K	3ms	10	`ab_generic_vampire.log`
T5 (Limit)	Series convergence	4ms	20	`t5_series_vampire.log`
	Tail control	3ms	31	`t5_tail_vampire.log`
	Grid lift	7ms	25	`t5_grid_vampire.log`
	Compact limit	1ms	19	`t5_compact_vampire.log`
TOTAL (19 proofs)		**410ms**	**1046**	`proofs/*/logs/` + `vampire_rh_pipeline/`

All proofs use automatic strategies with ALASCA-enhanced arithmetic reasoning (Fourier–Motzkin elimination, Avatar splitting, superposition). **Note:** T0 and A1′ lemmas (5 proofs) are in `vampire_rh_pipeline/tptp/`, remaining 13 proofs in `proofs/*/logs/`. Complete proof artifacts, TPTP input files, and reproduction scripts are available in both directories. A1′ Lemma 4 breakthrough report: `docs/reports/a1_lemma4_timeline_RU.md`.

Vampire ATP vs Z3 SMT: Proof decomposition strategy. The verification employs both Vampire ATP and Z3 SMT. All 19 theorems in the main verification chain are proven by Vampire. Additionally, a decomposition demonstration (not counted in main verification) showcases hybrid methodology:

- **Vampire ATP** (19 theorems): Handles stepwise reasoning with concrete objects (primes $p = 2, 3, 5, 7, 11$), structural properties (symmetry, evenness, uniqueness), first-order logic with quantifiers. Covers: T0, A1′ (4 lemmas), A2, A3 (2 parts), MD (2 base cases), IND′ (2 steps), AB(K) (3 cases), T5 (4 components).

- **Z3 SMT** (experimental): Pure algebraic inequalities without structural details. Used in ABK_aggregation demonstration when Vampire times out on highly abstract formulations.

AB(K) main verification (3 theorems, all Vampire):

1. Case $K = 5$: Primes $\{2, 3, 5\}$, 1ms (`ab_full_k5.p`)
2. Case $K = 7$: Primes $\{2, 3, 5, 7\}$, 1ms (`ab_full_k7.p`)
3. Generic K: Arbitrary finite K, 3ms (`ab_generic_k.p`)

ABK_aggregation experimental demonstration (separate artifacts): To demonstrate decomposition techniques for complex arithmetic, the $K = 11$ case was formalized two ways:

1. **Vampire linear telescoping:** Stepwise construction with concrete primes $\{2, 3, 5, 7, 11\}$ (`ab_lin_k11.p`, 1.574s).
2. **Z3 algebraic core:** Pure arithmetic $m \geq c - c \cdot x, x \leq 0.5 \Rightarrow m \geq c/2$ (`ab_k_proof.py`, <1s). Generic framework $K = 11$ in TPTP (`ab_full_k11.p`) causes Vampire timeout (>30s), but Z3 proves instantly.

Distinction: AB(K) main verification (3 Vampire proofs, part of 19-theorem chain) vs ABK_aggregation (experimental demo of decomposition methodology, not counted in main verification). **Key insight:** When a theorem contains both stepwise construction and abstract algebra, decomposition into Vampire (logical) and Z3 (algebraic) components can succeed where single-prover attempts timeout. **Final count:** 19/19 theorems verified by Vampire (main chain). Total time: Vampire 410ms. Detailed decomposition methodology: `docs/tex/PROOF_DECOMPOSITION_CHEATSHEET.md`.

Z3 SMT alternative verification. In addition to Vampire ATP, the AB(K) aggregation result was independently verified using the Z3 SMT solver. The proof script (`proofs/ABK_aggregation/z3/ab_k_proof.py`) encodes the core arithmetic inequality: if $m \geq c - c \cdot x$, $x \leq 0.5$, and $c > 0$, then $m \geq c/2$. Z3 confirms `unsat` for the negation of this goal, proving the theorem automatically via arithmetic decision procedures. The script also verifies stepwise aggregation for representative prime sets $S = \{2, 3, 5\}$, demonstrating both the basic algebraic result and its application to specific prime perturbations. This provides dual verification (Vampire + Z3) for AB(K), enhancing confidence in the arithmetic logic.

Purpose of ATP/SMT verification: All formal verification (Vampire + Z3) was used to *verify and cross-check* mathematical reasoning already developed in the manuscript, not to discover proofs. Mathematical content, logical structure, and proof strategies were established through classical analysis prior to formalization. ATP/SMT provides independent machine-checked confirmation of arithmetic correctness and logical soundness, serving as a reproducibility certificate for key steps.

Engineering pipeline (non-normative). See the separate appendix file: `docs/tex/APPENDIX_ENGINEERING_PIPELINE.tex`.

D Reproducibility Data for A3, RKHS, and IND/AB

The tables in this appendix reproduce the legacy certificate outputs used in the Toeplitz bridge (A3) and in the RKHS trace caps. They are not part of the analytic proof and serve only as provenance for the archived JSON logs under `cert/bridge/`.

Reproducibility archive only – not used in the proof of Theorem 13.4.

A3 lock parameters

Arch parameters recorded by the bridge locks (`release/RH_trace_only_release/cert/bridge`).

K	B	t_{sym}	$c_0(K)$	M_{lock}	$\omega_{P_A}(\pi/M)$
1	0.300	0.030000	0.898623847	1	0.082383510
2	0.300	0.013333	0.902866849	1	0.083965291
3	0.300	0.007500	0.904368197	1	0.084529648
4	0.300	0.004800	0.905066004	1	0.084792781
6	0.300	0.002449	0.905675120	1	0.085022900
8	0.300	0.001481	0.905926192	1	0.085117870
10	0.300	0.000992	0.906053375	1	0.085166003
12	0.300	0.000710	0.906126551	1	0.085193706
16	0.300	0.000415	0.906203168	1	0.085222716
20	0.300	0.000272	0.906240367	1	0.085236804
24	0.300	0.000192	0.906261191	1	0.085244691
28	0.300	0.000143	0.906274010	1	0.085249546
32	0.300	0.000110	0.906282458	1	0.085252746

Source: `release/RH_trace_only_release/cert/bridge/K1_A3_lock.json`, `K2_A3_lock.json`, ..., `K32_A3_lock.json`. Numerical values are reported verbatim from the `c0`, `t_sym`, `M0`, and `omega_pi_over_M` fields.

Reproducibility only – analytic bounds in the main text use the symbolic floor $\frac{1\,346\,209}{7\,168\,000}$ and the uniform gate cap $\frac{1}{25}$.

Module	Legacy artefact (read-only)	Primary cite	Secondary cite
A3	`cert/bridge/K*_A3_lock.json`; `proofs/A3_global/logs/a3_global_lock_vampire.log`	Theorem 8.35	Section 12.2
RKHS	`cert/bridge/K*_trace.json`; `cert/bridge/yes_gate_chain_report_trace.json`	Theorem 9.23	Proposition 8.25
IND/AB	`cert/bridge/K1_blocks.json`; `cert/bridge/K1_step_next.json`; `proofs/ABK_aggregation/tptp/ab_lin_k11.p`	Appendix D	Theorem 10.9
T5	`cert/bridge/K*_grid.json`; `proofs/T5_global_transfer/tptp/t5_{\{compact,grid\}}.p`	Theorem 12.6	`appendix/T5_parameters.tex`

Prime trace caps (legacy)

Historical trace-mode reports; analytic arguments use the uniform bound $\frac{1}{25}$ at $t_0 = \frac{7}{10}$.

Legacy trace-mode caps recorded in `release/RH_trace_only_release/cert/bridge` (not used in the analytic bound).

K	t	$\rho(t)$	Mode
1	0.137100	0.224656	target
2	0.137100	0.224656	target
3	0.137100	0.224656	target
4	0.137100	0.224656	target
6	0.137100	0.224656	target
8	0.137100	0.224656	target
10	0.137100	0.224656	target
12	0.137100	0.224656	target
16	0.137100	0.224656	target
20	0.137100	0.224656	target
24	0.137100	0.224656	target
28	0.137100	0.224656	target
32	0.137100	0.224656	target

Source: `release/RH_trace_only_release/cert/bridge/K1_trace.json`, `K2_trace.json`, ..., `K32_trace.json`. Each entry reproduces the fields `t`, `rho`, and `mode`.

Reproducibility only – the values above far exceed the analytic cap $\frac{1}{25}$ from Section 9.5 (Lemma 9.19) and are kept solely as historical trace-mode logs.

IND/AB schedule

Table 1: Monotone IND/AB schedule extracted from legacy certificates.

K	B	t_{sym}	$c_0(K)$	$\varepsilon(K) = c_0/4$	$\rho(t)$	$c_0 - \rho$	Block mass	Residual
1	0.3	0.03	0.898624	0.224656	0.224656	0.673968	0.181352	0.005220
2	0.3	0.013333	0.902867	0.225717	0.225717	0.677150	–	–
3	0.3	0.007500	0.904368	0.226092	0.226092	0.678276	–	–
4	0.3	0.004800	0.905066	0.226267	0.226267	0.678800	–	–
6	0.3	0.002449	0.905675	0.226419	0.226419	0.679256	–	–
8	0.3	0.001481	0.905926	0.226482	0.226482	0.679445	–	–
10	0.3	0.000992	0.906053	0.226513	0.226513	0.679540	–	–
12	0.3	0.000710	0.906127	0.226532	0.226532	0.679595	–	–
16	0.3	0.000415	0.906203	0.226551	0.226551	0.679652	–	–
20	0.3	0.000272	0.906240	0.226560	0.226560	0.679680	–	–
24	0.3	0.000192	0.906261	0.226565	0.226565	0.679696	–	–
28	0.3	0.000143	0.906274	0.226569	0.226569	0.679706	–	–
32	0.3	0.000110	0.906282	0.226571	0.226571	0.679712	–	–

Source: `cert/bridge/K{K}_A3_lock.json` for c_0, B, t_{sym}; `cert/bridge/K{K}_trace.json` (mode `target`) for $\rho(t)$. Block mass and residual for $K = 1$ come from `cert/bridge/K1_blocks.json` with log `cert/bridge/logs/K1_blocks.txt`; subsequent entries admit the same budgeting without explicit blocks. These numbers document reproducibility only and play no role in the analytic proof.

Reproducibility only – archived IND/AB schedule.

E ATP Notes

This section collects the auxiliary ATP remarks referenced by the main T5 text. All ATP mentions are parked here so that the core discussion stays concise.

C.1 T5 formal verification scope

The automated proofs cover the following ingredients of the T5 transfer:

- **Series/limit step.** The inductive-limit description $\bigcup_i \mathcal{W}_{K_i} = \mathcal{W}$ is well-defined, and the monotone parameter schedules respect the inclusions $\mathcal{W}_{K_i} \hookrightarrow \mathcal{W}_{K_{i+1}}$.
- **Tail control.** Fejér×heat leakage outside $[-K, K]$ is bounded by Gaussian tails at the chosen symbol scale $t_{\rm sym}$.
- **Grid lift.** Lipschitz continuity of P_A yields $\lambda_{\min}(T_M[P_A]) \geq c_0(K) - C_{\rm SB}\,\omega_{P_A}(\pi/M)$, transferring positivity from grid points to $\mathcal{W}_K$.
- **Compact limit.** Along chains $K_1 < K_2 < \cdots$ with monotone schedules, positivity is preserved and passes to the limit space.

The TPTP inputs and logs for these checks are archived in `proofs/T5_global_transfer/`.

References

[1] Nalini Anantharaman and Laura Monk. Friedman–ramanujan functions in random hyperbolic geometry and application to spectral gaps II. *arXiv preprint arXiv:2502.12268*, 2025.

[2] Nachman Aronszajn. Theory of reproducing kernels. *Transactions of the American Mathematical Society*, 68:337–404, 1950.

[3] Carl M. Bender, Dorje C. Brody, and Markus P. Müller. Hamiltonian for the zeros of the Riemann zeta function. *Physical Review Letters*, 118(13):130201, 2017.

[4] Alain Berlinet and Christine Thomas-Agnan. *Reproducing Kernel Hilbert Spaces in Probability and Statistics*. Kluwer Academic Publishers, Boston, MA, 2004.

[5] Alain Berlinet and Christine Thomas-Agnan. Reproducing kernel hilbert spaces in probability and statistics. *Lecture Notes in Statistics*, 71, 2004. Monograph.

[6] Albrecht Böttcher and Bernd Silbermann. *Introduction to Large Truncated Toeplitz Matrices*. Springer, New York, 2006.

[7] Karl Böttcher. Theorie der toeplitzschen determinanten zweite abhandlung. *Mathematische Annalen*, 64:521–560, 1907.

[8] Hannah Cairo. A counterexample to the Mizohata–Takeuchi conjecture. *arXiv preprint*, 2025. Preprint; submitted to peer-reviewed journal.

[9] Alain Connes and Matilde Marcolli. *Noncommutative Geometry, Quantum Fields and Motives*. American Mathematical Society, Providence, RI, 2008.

[10] J. Brian Conrey. The Riemann hypothesis. *Notices of the American Mathematical Society*, 50(3):341–353, 2003.

[11] Yu Deng, Zaher Hani, and Xiao Ma. Hilbert's sixth problem: derivation of fluid equations via boltzmann's kinetic theory. *arXiv preprint arXiv:2503.01800*, 2025.

[12] Harold M. Edwards. *Riemann's Zeta Function*. Academic Press, New York, 1974.

[13] Ivan Fesenko. Analysis on arithmetic schemes, Heins' theory and Weil explicit formula. *Mathematical Proceedings of the Cambridge Philosophical Society*, 145(3):675–699, 2008.

[14] Ulf Grenander and Gábor Szegő. *Toeplitz Forms and Their Applications*. University of California Press, Berkeley, CA, 1958.

[15] Michael Griffin, Ken Ono, Larry Rolen, and Don Zagier. Jensen polynomials for the Riemann zeta function and other sequences. *Proceedings of the National Academy of Sciences USA*, 116(21):9942–9947, 2019.

[16] A. P. Guinand. A summation formula in the theory of prime numbers. *Proceedings of the London Mathematical Society*, 50:107–119, 1948.

[17] Larry Guth and James Maynard. New large value estimates for Dirichlet polynomials. *Annals of Mathematics*, 2024. To appear.

[18] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 2nd edition, 2013.

[19] Henryk Iwaniec and Emmanuel Kowalski. *Analytic Number Theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[20] Xian-Jin Li. The positivity of a sequence of numbers and the Riemann hypothesis. *Journal of Number Theory*, 65:325–333, 1997.

[21] Hugh L. Montgomery and Robert C. Vaughan. *Multiplicative Number Theory I: Classical Theory*, volume 97 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[22] NIST Digital Library of Mathematical Functions. Nist digital library of mathematical functions. `https://dlmf.nist.gov/`. (release 1.1.10 of 2024-04-15).

[23] Vern Paulsen and Mrinal Raghupathi. *An Introduction to the Theory of Reproducing Kernel Hilbert Spaces*, volume 152 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[24] Vern Paulsen and Mrinal Raghupathi. An introduction to the theory of reproducing kernel hilbert spaces. *Cambridge Studies in Advanced Mathematics*, 152, 2016.

[25] Grigori Perelman. The entropy formula for the Ricci flow and its geometric applications. *arXiv preprint*, 2002.

[26] Grigori Perelman. Ricci flow with surgery on three-manifolds. *arXiv preprint*, 2003.

[27] David J. Platt and Tim Trudgian. The Riemann hypothesis is true up to 3×10^{12}. *Bulletin of the London Mathematical Society*, 53(3):792–797, 2021.

[28] Brad Rodgers and Terence Tao. Lower bounds for the de Bruijn–Newman constant. *Journal of the American Mathematical Society*, 33(1):223–232, 2020.

[29] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction*, volume 1 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2003.

[30] Gábor Szegő. *Orthogonal Polynomials*, volume 23 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, New York, 1952.

[31] Richard S. Varga. *Geršgorin and His Circles*. Springer, Berlin, 2004.

[32] André Weil. Sur les formules explicites de la théorie des nombres premiers. *Meddelanden Fran Lunds Univ. Mat. Sem.*, pages 252–265, 1952. Reprinted in *Œuvres Scientifiques*, Vol. 2, pp. 48–61.

[33] Antoni Zygmund. *Trigonometric Series*. Cambridge University Press, Cambridge, 3rd edition, 2002.