ChatGPT

Key Results on Prime Gaps and Correlation Lower Bounds

Result (Type)	Functional Focus	Methodology	Source (Year)	Conclusion / Bound
Hardy–Littlewood Prime *k*-Tuple Conjecture (Heuristic)	Twin prime count $S_2(X)=\sum_{n\le X}\Lambda(n)\Lambda(n+2)$ (pair correlation)	Hardy–Littlewood circle method; random primes model	Hardy & Littlewood 1923 [1]	*Predicted:* $S_2(X)\sim 2\,C_2\,X$, where C_2 is the twin prime constant. Implies $E(X)\ge cX^1$ and infinitely many twin primes [1].
Chen's Theorem (Proved)	"Almost twin" primes: p and $p+2$ with $p+2$ at most semiprime	Selberg sieve (combinatorial)	Chen 1973 [2]	Infinitely many primes p such that $p+2$ is prime or a product of 2 primes [2]. (First unconditional partial progress toward twin primes.)
Elliott–Halberstam Conjecture (EH, Conditional)	Primes in arithmetic progressions (level of distribution $=1$)	Analytic & large-sieve methods [3]	Elliott & Halberstam 1970 [3]	Assumes near-optimal prime distribution in APs. *Consequences:* implies bounded prime gaps [4], but by itself *does not* guarantee twin primes [4] (needs uniform control of prime pairs).
Bombieri–Vinogradov Theorem (Proved) *(Base result)*	Primes in arithmetic progressions (level $=1/2$ on average)	Large sieve (Hilbert space inequality)	Bombieri & Vinogradov 1971 [5]	Unconditionally, primes have level $1/2$ distribution in APs (key input for modern gap results [5]). *Setup foundation:* Used in sieve methods but doesn't directly bound $S_2(X)$.

Result (Type)	Functional Focus	Methodology	Source (Year)	Conclusion / Bound
Goldston–Pintz–Yıldırım (GPY) (Proved & Conditional)	Small prime gaps (correlations at variable h)	Weighted sieve (GPY parity method)	Goldston et al. 2005/2009 [6]	**Unconditional:** $\displaystyle \liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\ln p_n}=0$ [7] (infinitely often gaps arbitrarily small relative to average). **With EH:** infinitely many prime pairs with gap ≤ 16 [8] (first conditional bounded-gap result).
Zhang's Theorem (Proved)	Bounded prime gaps exist (some fixed H)	GPY + deep analytic number theory (exponential sums, distribution $\vartheta>1/2$)	Zhang 2014 [9] [10]	Established the *first* finite bound H: infinitely many prime pairs with separation $H\le 70{,}000,000$ [9]. Sparked rapid improvements (Polymath8a lowered to $H=4,680$) [10].
Maynard's Theorem (Proved & Conditional)	Smaller prime gaps (multiple primes in tuples)	Refined GPY sieve (selective multi-prime)	Maynard 2015 [11]	**Unconditional:** $\liminf (p_{n+1}-p_n)\le 600$ [11] (infinitely many gaps ≤ 600). **With EH:** $\liminf (p_{n+1}-p_n)\le 12$ [11] (gap ≤ 12 infinitely often).
Polymath8 Project (Proved & Conditional)	Record small gaps (combining approaches)	Mixed sieve (GPY + Maynard) + optimized variational method	D.H.J. Polymath 2014 [12] [13]	**Unconditional:** improved gap bound to **$H=246$** (infinitely many prime gaps ≤ 246) [12]. **Assuming GEH** (Generalized EH): gap bound improved to **$H=6$** (six) [14]. *(Twin primes $H=2$ remained just beyond reach under GEH [15].)*

Result (Type)	Functional Focus	Methodology	Source (Year)	Conclusion / Bound
Generalized EH for Prime Pairs (GEH-2) (Conjecture)	Correlations of primes (von Mangoldt convolution)	Extended uniformity conjecture (bilinear forms)	Trey Smith 2025 [16] [17]	Assuming **GEH-2** (a strong uniform prime-pair distribution), one deduces $S_2(X)\sim cX$ for some $c>0$ [18]. *In particular, GEH-2 implies infinitely many twin primes* [19] (resolves the twin prime conjecture *conditionally*).

Sources: Key historical conjectures and results are documented in the literature [1] [2] [4]. Breakthrough papers by Goldston–Pintz–Yıldırım [6], Zhang [9], Maynard [11] and the Polymath collaborations [12] provide proven lower bounds of type $E(X)\ge c\cdot X^\delta$ for various prime-gap functionals. Under stronger hypotheses like Elliott–Halberstam (EH) [4] and its generalizations, even tighter bounds (down to gap 2) become reachable in principle [14] [19]. These results and conjectures collectively support the belief that $S_2(X)$ grows without bound (hence twin primes are infinite), though an unconditional inequality $S_2(X)\ge c X^\delta$ (with $\delta>0$) for the *actual* twin prime count remains elusive.

---

[1] [3] [4] [5] [9] [16] [17] [18] [19] A Generalized Elliott-Halberstam Conjecture Implying the Twin Prime Hypothesis
https://arxiv.org/html/2511.14810v1

[2] Chen's theorem - Wikipedia
https://en.wikipedia.org/wiki/Chen%27s_theorem

[6] [7] [8] [math/0508185] Primes in Tuples I
https://arxiv.org/abs/math/0508185

[10] [12] [13] Bounded gaps between primes - Polymath Wiki
https://michaelnielsen.org/polymath/index.php?title=Bounded_gaps_between_primes

[11] Small gaps between primes - Annals of Mathematics
https://annals.math.princeton.edu/2015/181-1/p07

[14] [15] Polymath8b, VII: Using the generalised Elliott-Halberstam hypothesis to enlarge the sieve support yet further | What's new
http://terrytao.wordpress.com/2014/01/28/polymath8b-vii-using-the-generalised-elliott-halberstam-hypothesis-to-enlarge-the-sieve-support-yet-further/