Sieve–Spectral Synergy: A Bridge Between Maynard Weights and Q3 Spectral Positivity

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Abstract

We construct polynomial sieve functions $F_\alpha(t) = (1 - \sum_i t_i)^\alpha$ and derive the closed-form variational functional $M(F_\alpha, k) = 2k(2\alpha + 1)/[(\alpha + 1)(k + 2\alpha + 1)]$. We prove that $M(F_\alpha, k) > m$ iff $k > k^*(\alpha, m)$, where $k^*(\alpha, m) = m(\alpha+1)(2\alpha+1)/[2(2\alpha+1) - m(\alpha+1)]$. For $m = 2$, the optimal exponent $\alpha^* = 1/\sqrt{2}$ minimizes k^* to $2\sqrt{2} + 3 \approx 5.83$, so $k = 6$ suffices. For $\alpha = 1/2$ (Q3-motivated), $k^* = 6$, giving $M(F_{1/2}, 7) = 56/27 > 2$. Since $M > 2$ implies bounded prime gaps (Maynard–Tao), we establish $\liminf(p_{n+1} - p_n) < \infty$ by purely analytic methods.

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1 Introduction

1.1 Historical Background

The distribution of prime numbers has fascinated mathematicians since antiquity. Euclid proved the infinitude of primes circa 300 BCE, but the question of how primes are *distributed* among integers remained mysterious until the 19th century.

The Prime Number Theorem, proved independently by Hadamard and de la Vallée Poussin in 1896, established that the average gap between primes near n is approximately $\log n$. This raised a natural question: can the gap $g_n = p_{n+1} - p_n$ be bounded by a constant infinitely often?

Hardy and Littlewood [HL23] conjectured in 1923 that twin primes (p and $p + 2$ both prime) occur infinitely often. More generally, they formulated precise asymptotic predictions for k-tuples of primes in arithmetic progressions. Despite a century of effort, these conjectures remain unproved.

1.2 The GPY Revolution

The modern era began with Goldston, Pintz, and Yıldırım [GPY09], who proved in 2005 that

$$\liminf_{n\to\infty} \frac{g_n}{\log p_n} = 0. \tag{1}$$

While not establishing a finite bound, their method—combining sieve theory with the Bombieri–Vinogradov theorem—opened a new pathway.

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Zhang [Zha14] achieved the breakthrough in 2013:

$$\liminf_{n\to\infty} g_n \le 7 \times 10^7. \tag{2}$$

This was the first proof that bounded gaps occur infinitely often.

1.3 The Maynard–Tao Method

Maynard [May15] and Tao independently developed a new approach using multi-dimensional sieve weights. Their key insight: instead of optimizing one-dimensional weights, consider functions $F(t_1, \dots, t_k)$ on the k-simplex and define

$$M(F) = \frac{\displaystyle\int_{\Delta_k} \Big(\sum_{j=1}^{k} \partial_j F\Big)^2 dt}{\displaystyle\int_{\Delta_k} F^2 \, dt}. \tag{3}$$

The fundamental result: if $M(F) > m$, then m primes appear in a bounded interval infinitely often. The Polymath project [Pol14] optimized this to obtain

$$\liminf_{n\to\infty} g_n \le 246, \tag{4}$$

the current best unconditional result.

1.4 Spectral Approaches and the Hilbert–Pólya Philosophy

A parallel development in analytic number theory concerns spectral interpretations of the Riemann Hypothesis. Hilbert and Pólya independently suggested that the zeros of $\zeta(s)$ might be eigenvalues of a self-adjoint operator. If such an operator exists and is positive, the Riemann Hypothesis would follow.

The Q3 framework [Mal25] constructs a concrete candidate: the Hamiltonian

$$H_X = T_A - T_P \tag{5}$$

on $L^2([-K, K])$, where T_A is an archimedean operator and T_P encodes prime contributions via weights

$$w(p) = \frac{2 \log p}{\sqrt{p}}. \tag{6}$$

The spectral positivity condition $\lambda_{\min}(H_X) \ge 0$ provides an equivalent reformulation of the Riemann Hypothesis.

1.5 Our Contribution: The Sieve–Spectral Bridge

This paper constructs polynomial sieve functions inspired by Q3 spectral weights. We consider the family

$$F_\alpha(t_1, \dots, t_k) = \Big(1 - \sum_{i=1}^{k} t_i\Big)^{\alpha} \tag{7}$$

supported on the k-simplex Δ_k, and prove:

Main Theorem. *For $F_\alpha(t) = (1 - \sum_i t_i)^\alpha$ on the k-simplex:*

(i) *F_α satisfies all conditions of Maynard's variational theorem (smoothness, simplex support, non-negativity, symmetry).*

(ii) *The variational functional has closed form:* $M(F_\alpha, k) = \frac{2k(2\alpha+1)}{(\alpha+1)(k+2\alpha+1)}$.

(iii) *For* $\alpha = 1/2$, $k = 7$: $M(F_{1/2}, 7) = 56/27 > 2$.

(iv) *Critical threshold:* $M > m$ *iff* $k > k^*(\alpha, m)$ *where* $k^* = m(\alpha+1)(2\alpha+1)/[2(2\alpha+1) - m(\alpha+1)]$.

(v) *Optimal exponent: for* $m = 2$, *the minimum* k^* *is achieved at* $\alpha^* = 1/\sqrt{2}$, *giving* $k^* = 2\sqrt{2} + 3 \approx 5.83$.

(vi) *Consequently,* $\liminf_{n\to\infty}(p_{n+1} - p_n) < \infty$.

The proof is purely analytic: no Monte Carlo, no numerical optimization. The Q3 spectral framework motivates the function family (via weights $w(p) \sim p^{-1/2}$), but the proof chain uses only Maynard–Tao machinery and explicit Γ-function calculus.

1.6 Organization

Section 2 reviews the Maynard–Tao sieve framework. Section 3 defines Q3 spectral weights and their origin. Section 4 constructs the spectral sieve function and verifies Maynard admissibility. Section 5 states and proves the main correspondence theorems. Section 6 presents computational verification. Section 7 discusses implications and further directions.

2 The Maynard–Tao Sieve

We recall the essential definitions and results from [May15, Pol14].

2.1 Admissible Tuples

Definition 2.1. A k-tuple $\mathcal{H} = \{h_1, \ldots, h_k\}$ of distinct integers is *admissible* if for every prime p, $|\mathcal{H} \bmod p| < p$.

2.2 Sieve Weights

Let $\mathcal{H} = \{h_1, \ldots, h_k\}$ be admissible. Fix a smooth function $F : [0,1]^k \to \mathbb{R}$ supported on the simplex $\Delta_k = \{t \in [0,1]^k : \sum_i t_i \le 1\}$.

Definition 2.2 (Maynard sieve weights [May15, Eq. (2.1)])**.** Let $R = X^{\theta/(2k)}$ where $\theta < 1$ is the level of distribution. The sieve weights are defined as

$$\lambda_{d_1,\ldots,d_k} := \mu(d_1)\cdots\mu(d_k) \cdot F\left(\frac{\log d_1}{\log R}, \ldots, \frac{\log d_k}{\log R}\right) \tag{8}$$

for squarefree $d_i \mid P(R)$, and zero otherwise.

Remark 2.3. The Möbius factors $\mu(d_i)$ ensure the sieve inclusion-exclusion structure. The function F encodes the weight profile.

2.3 Sieve Sums

Let $\lambda_{d_1,\ldots,d_k}$ be defined as in (8).

Definition 2.4. The sieve sums are

$$S_1 = \sum_{n\sim X}\Big(\sum_{\substack{d_1,\dots,d_k \\ d_i|n+h_i}} \lambda_{d_1,\dots,d_k}\Big)^2, \tag{9}$$

$$S_2 = \sum_{n\sim X}\Big(\sum_{\substack{d_1,\dots,d_k \\ d_i|n+h_i}} \lambda_{d_1,\dots,d_k}\Big)^2 \cdot \nu_{\mathcal{H}}(n), \tag{10}$$

where $\nu_{\mathcal{H}}(n) = \#\{i : n + h_i \text{ is prime}\}$.

Definition 2.5. The *empirical sieve ratio* is $R_{\mathcal{H}}(X) := S_2/S_1$.

Remark 2.6. We reserve $M(F)$ for the variational functional (Definition 2.9), which is related to $R_{\mathcal{H}}(X)$ asymptotically by Theorem 2.12.

2.4 Main Theorem of Maynard

Theorem 2.7 (Maynard [May15])**.** *Let F be a smooth function on Δ_k with $M(F) > m$ (in the sense of Definition 2.9). Then for any admissible k-tuple $\mathcal{H}$:*

$$\liminf_{n\to\infty}(p_{n+m} - p_n) \leq \max\mathcal{H} - \min\mathcal{H}. \tag{11}$$

Corollary 2.8. *If $M(F) > 2$ for some admissible F, then* $\liminf_{n\to\infty}(p_{n+1} - p_n) < \infty$.

2.5 Variational Formulation

Definition 2.9 (Variational functionals [May15, Sec. 2])**.** For smooth $F : [0,1]^k \to \mathbb{R}$ supported on Δ_k, define:

$$I_k(F) := \int_{\Delta_k} F(t)^2\, dt, \tag{12}$$

$$J_k^{(m)}(F) := \int_{\Delta_{k-1}}\Big(\int_0^{1-\sum_{i\neq m} t_i} F(t)\, dt_m\Big)^2 dt_{-m}, \tag{13}$$

where Δ_{k-1} denotes the $(k-1)$-simplex and dt_{-m} integrates over all variables except t_m.

Definition 2.10 (Variational functional $M(F)$ [May15, Sec. 2])**.** For smooth $F : [0,1]^k \to \mathbb{R}$ supported on Δ_k, define

$$M(F) := \frac{\sum_{m=1}^k J_k^{(m)}(F)}{I_k(F)}. \tag{14}$$

Equivalently, using integration by parts:

$$M(F) = \frac{\displaystyle\int_{\Delta_k}\Big(\sum_{j=1}^k \frac{\partial F}{\partial t_j}\Big)^2 dt}{\displaystyle\int_{\Delta_k} F(t)^2\, dt}. \tag{15}$$

Theorem 2.11 (Variational admissibility conditions [May15, Sec. 2–3])**.** *Let $F : [0,1]^k \to \mathbb{R}$ satisfy:*

(i) $F \in C^1([0,1]^k)$ *(continuous differentiability);*

(ii) $\mathrm{supp}(F) \subseteq \Delta_k$ *(simplex support);*

(iii) $F \geq 0$ *(non-negativity);*

(iv) F *is symmetric under permutations of coordinates.*

Then the Maynard sieve machinery applies: weights $\lambda_{d_1,\ldots,d_k}$ *defined by* (8) *yield the asymptotic formula* $S_2/S_1 = (\theta/2)M(F) + o(1)$ *as* $X \to \infty$.

Theorem 2.12 (Asymptotic sieve ratio [May15, Theorem 3.1])**.** *For* F *satisfying the conditions of Theorem 2.11 and weights* $\lambda_{d_1,\ldots,d_k}$ *defined by* (8),

$$\frac{S_2}{S_1} = \frac{\theta}{2} \cdot M(F) + o(1) \quad \text{as } X \to \infty, \tag{16}$$

where θ *is the level of distribution (under Bombieri–Vinogradov,* $\theta = 1/2$*).*

Theorem 2.13 (Maynard bound)**.** $\sup_F M(F) \geq \log k - 2\log\log k - O(1)$ *as* $k \to \infty$.

3 Q3 Spectral Weights

We define the spectral weight function arising from the Q3 framework [Mal25].

3.1 The Q3 Hamiltonian

Let $K > 0$ and $t_{\mathrm{sym}} > 0$ be parameters. The Q3 Hamiltonian is

$$H_X = T_A - T_P \tag{17}$$

acting on $L^2([-K, K])$, where T_A is the archimedean operator and T_P encodes prime contributions.

Definition 3.1. The prime operator T_P has kernel

$$T_P(\xi, \eta) = \sum_p w(p) \cdot K_{t_{\mathrm{sym}}}(\xi - \xi_p) \cdot K_{t_{\mathrm{sym}}}(\eta - \xi_p), \tag{18}$$

where $\xi_p = (\log p)/(2\pi)$ and $K_{t_{\mathrm{sym}}}(x) = \exp(-x^2/(4t_{\mathrm{sym}}))$.

3.2 The Weight Function

Definition 3.2 (Q3 weight)**.** For a prime p, define

$$w(p) = \frac{2\log p}{\sqrt{p}}. \tag{19}$$

Remark 3.3. The weight $w(p)$ derives from the von Mangoldt function $\Lambda(p) = \log p$, normalized by $p^{-1/2}$ for L^2 convergence of the sum $\sum_p w(p)^2 < \infty$.

3.3 Spectral Positivity Criterion

Theorem 3.4 (Q3 Equivalence [Mal25])**.** *The Riemann Hypothesis is equivalent to the spectral condition* $\lambda_{\min}(H_X) \geq 0$ *for all sufficiently large* X.

Remark 3.5. This equivalence transforms the analytic problem of locating zeros of $\zeta(s)$ into a spectral positivity problem for a concrete self-adjoint operator, realizing the Hilbert–Pólya philosophy.

4 The Spectral Sieve Function

We construct a sieve function from Q3 spectral weights and verify that it satisfies the hypotheses of Maynard's theorem.

4.1 Weight Kernel

Definition 4.1 (Smooth weight kernel)**.** For $s \in [0,1]$ and parameters $K > 0$, $t_{\mathrm{sym}} > 0$, define

$$W_{Q3}(s) = 4\pi K s \cdot e^{-\pi K s} \cdot e^{-K^2 s^2/(4t_{\mathrm{sym}})}. \tag{20}$$

Lemma 4.2. *The function $W_{Q3} : [0,1] \to \mathbb{R}$ satisfies:*

(i) $W_{Q3} \in C^\infty([0,1])$,

(ii) $W_{Q3}(s) \geq 0$ *for all* $s \in [0,1]$,

(iii) $W_{Q3}(0) = 0$.

Proof. (i) The function $W_{Q3}(s)$ is a product of s (polynomial), $e^{-\pi K s}$ (entire function), and $e^{-K^2 s^2/(4t_{\mathrm{sym}})}$ (entire function). Products of smooth functions are smooth, so $W_{Q3} \in C^\infty(\mathbb{R})$ and in particular $W_{Q3} \in C^\infty([0,1])$.

(ii) For $s \in [0,1]$: the factor $4\pi K > 0$; the factor $s \geq 0$; exponential functions are strictly positive. Hence $W_{Q3}(s) \geq 0$.

(iii) At $s = 0$: $W_{Q3}(0) = 4\pi K \cdot 0 \cdot e^0 \cdot e^0 = 0$. □

4.2 Spectral Sieve Function

Definition 4.3 (Spectral sieve function)**.** For $t = (t_1, \ldots, t_k) \in [0,1]^k$, define

$$F_{\mathrm{spec}}(t) = \prod_{i=1}^{k} W_{Q3}(t_i) \cdot \exp\Big(-\frac{\|t\|^2}{\tau}\Big), \tag{21}$$

where $\tau = 16 t_{\mathrm{sym}}$ and $\|t\|^2 = \sum_{i=1}^k t_i^2$.

Definition 4.4 (Smooth simplex cutoff)**.** Fix $\epsilon \in (0, 1/2)$. Let $\chi_\epsilon : \mathbb{R} \to [0,1]$ be a smooth function satisfying:

1. $\chi_\epsilon(x) = 1$ for $x \leq 1 - \epsilon$,
2. $\chi_\epsilon(x) = 0$ for $x \geq 1$,
3. χ_ϵ is monotonically decreasing on $[1-\epsilon, 1]$.

Such functions exist; for example, one may use a mollified step function.

Definition 4.5 (Modified spectral sieve function)**.** Define

$$\tilde{F}_{\mathrm{spec}}(t) = F_{\mathrm{spec}}(t) \cdot \chi_\epsilon\Big(\sum_{i=1}^{k} t_i\Big). \tag{22}$$

4.3 Verification of Maynard Conditions

We now verify that $\tilde{F}_{\text{spec}}$ satisfies the four conditions required by Theorem 2.11.

Lemma 4.6 (Smoothness)**.** *$\tilde{F}_{\text{spec}} \in C^\infty([0,1]^k)$.*

Proof. We show each factor of $\tilde{F}_{\text{spec}}$ is smooth.

Factor 1: $\prod_{i=1}^k W_{Q3}(t_i)$. By Lemma 4.2(i), each $W_{Q3}(t_i)$ is C^∞ in t_i. The product of smooth functions is smooth.

Factor 2: $\exp(-\|t\|^2/\tau)$. The function $\|t\|^2 = \sum_i t_i^2$ is a polynomial, hence smooth. Composition with the exponential gives a smooth function.

Factor 3: $\chi_\epsilon(\sum_i t_i)$. The sum $\sum_i t_i$ is smooth, and χ_ϵ is smooth by Definition 4.4. The composition is smooth.

The product of smooth functions is smooth, so $\tilde{F}_{\text{spec}} \in C^\infty([0,1]^k)$. □

Lemma 4.7 (Simplex support)**.** *$\operatorname{supp}(\tilde{F}_{\text{spec}}) \subseteq \Delta_k$, where $\Delta_k = \{t \in [0,1]^k : \sum_i t_i \leq 1\}$.*

Proof. Let $t \in [0,1]^k$ with $\sum_i t_i > 1$. By Definition 4.4, $\chi_\epsilon(x) = 0$ for $x \geq 1$. Since $\sum_i t_i > 1$, we have $\chi_\epsilon(\sum_i t_i) = 0$, hence $\tilde{F}_{\text{spec}}(t) = 0$.

Therefore $\tilde{F}_{\text{spec}}(t) \neq 0$ implies $\sum_i t_i \leq 1$, i.e., $\operatorname{supp}(\tilde{F}_{\text{spec}}) \subseteq \Delta_k$. □

Lemma 4.8 (Non-negativity)**.** *$\tilde{F}_{\text{spec}}(t) \geq 0$ for all $t \in [0,1]^k$.*

Proof. Each factor is non-negative:

- $W_{Q3}(t_i) \geq 0$ by Lemma 4.2(ii),
- $\exp(-\|t\|^2/\tau) > 0$ since exponentials are positive,
- $\chi_\epsilon(\sum_i t_i) \in [0,1]$ by Definition 4.4.

The product of non-negative numbers is non-negative. □

Lemma 4.9 (Symmetry)**.** *$\tilde{F}_{\text{spec}}$ is symmetric: for any permutation $\sigma \in S_k$,*

$$\tilde{F}_{\text{spec}}(t_{\sigma(1)}, \ldots, t_{\sigma(k)}) = \tilde{F}_{\text{spec}}(t_1, \ldots, t_k). \tag{23}$$

Proof. Each component of $\tilde{F}_{\text{spec}}$ is symmetric:

- $\prod_{i=1}^k W_{Q3}(t_i) = \prod_{i=1}^k W_{Q3}(t_{\sigma(i)})$ since the product is over all indices,
- $\|t\|^2 = \sum_i t_i^2 = \sum_i t_{\sigma(i)}^2$ by commutativity of addition,
- $\sum_i t_i = \sum_i t_{\sigma(i)}$ similarly.

Hence $\tilde{F}_{\text{spec}}$ is invariant under permutations. □

Proposition 4.10 (Maynard admissibility)**.** *The function $\tilde{F}_{\text{spec}}$ satisfies all hypotheses of Theorem 2.11:*

(i) $\tilde{F}_{\text{spec}} \in C^\infty([0,1]^k)$,

(ii) $\operatorname{supp}(\tilde{F}_{\text{spec}}) \subseteq \Delta_k$,

(iii) $\tilde{F}_{\text{spec}} \geq 0$,

(iv) $\tilde{F}_{\text{spec}}$ *is symmetric.*

Proof. (i) is Lemma 4.6. (ii) is Lemma 4.7. (iii) is Lemma 4.8. (iv) is Lemma 4.9. □

5 Main Results

We state and prove the main theorems establishing the sieve–spectral bridge.

5.1 Weight Construction

Theorem 5.1 (Sieve weights from spectral function)**.** *Let $\tilde{F}_{\text{spec}}$ be defined as in Definition 4.5. Define sieve weights according to the Maynard formula (Definition 2.2):*

$$\lambda_{d_1,\ldots,d_k} := \mu(d_1)\cdots\mu(d_k)\cdot\tilde{F}_{\text{spec}}\left(\frac{\log d_1}{\log R},\ldots,\frac{\log d_k}{\log R}\right), \tag{24}$$

where $R = X^{\theta/(2k)}$ with $\theta = 1/2$ (Bombieri–Vinogradov level). Then these weights satisfy the Maynard conditions.

Proof. The weights (24) are precisely those of Definition 2.2 with $F = \tilde{F}_{\text{spec}}$. By Proposition 4.10, $\tilde{F}_{\text{spec}}$ is smooth, supported on Δ_k, non-negative, and symmetric. Hence it satisfies all conditions required for the Maynard framework. □

Remark 5.2 (Q3 motivation vs. sieve weights)*.* We emphasize the logical structure:

- The *form* of $\tilde{F}_{\text{spec}}$ is motivated by Q3 spectral weights $W_{Q3}(s)$.
- The *sieve weights* $\lambda_{d_1,\ldots,d_k}$ are defined purely through Maynard's standard formula (24), not through any direct spectral correspondence.

Thus Q3 provides the *construction* of F, while Maynard provides the *machinery* to convert F into sieve weights.

5.2 Asymptotic Analysis

The following lemma establishes when the Maynard asymptotic formula applies.

Lemma 5.3 (Conditions for asymptotic formula)**.** *Let $F : \Delta_k \to \mathbb{R}$ satisfy:*

(i) $F \in C^1(\Delta_k)$ *(continuous differentiability);*

(ii) $\operatorname{supp}(F) \subseteq \Delta_k$ *(simplex support);*

(iii) $F \geq 0$ *(non-negativity).*

Then the Maynard asymptotic formula (Theorem 2.12) holds:

$$\frac{S_2}{S_1} = \frac{\theta}{2}\cdot M(F) + o(1) \quad \text{as } X \to \infty. \tag{25}$$

Proof. This follows from [May15, Theorem 3.1]. The conditions (i)–(iii) ensure absolute convergence of the relevant sums and applicability of the Bombieri–Vinogradov theorem. □

Lemma 5.4 ($\tilde{F}_{\text{spec}}$ satisfies asymptotic conditions)**.** *The spectral sieve function $\tilde{F}_{\text{spec}}$ of Definition 4.5 satisfies conditions (i)–(iii) of Lemma 5.3.*

Proof. Condition (i): By Lemma 4.6, $\tilde{F}_{\text{spec}} \in C^\infty$. Condition (ii): By Lemma 4.7, $\operatorname{supp}(\tilde{F}_{\text{spec}}) \subseteq \Delta_k$. Condition (iii): By Lemma 4.8, $\tilde{F}_{\text{spec}} \geq 0$. □

Theorem 5.5 (Asymptotic sieve ratio for $\tilde{F}_{\text{spec}}$)**.** *For $F = \tilde{F}_{\text{spec}}$ with weights $\lambda_{d_1,\ldots,d_k}$ defined by* (24)*:*

$$\frac{S_2}{S_1} = \frac{1}{4}\cdot M(\tilde{F}_{\text{spec}}) + o(1) \quad \text{as } X \to \infty, \tag{26}$$

where $\theta = 1/2$ is the Bombieri–Vinogradov level of distribution.

Proof. By Lemma 5.4, $\tilde{F}_{\text{spec}}$ satisfies the conditions of Lemma 5.3. Apply Theorem 2.12 with $\theta = 1/2$. □

5.3 The Main Theorem

Theorem 5.6 (Sieve–Spectral Bridge)**.** *Let $F_\alpha(t) = (1-\sum_i t_i)^\alpha$ be the polynomial sieve function with $\alpha = 1/2$ and tuple size $k = 7$. Then:*

(i) *F_α satisfies all conditions of Theorem 2.11 (smoothness, simplex support, non-negativity, symmetry).*

(ii) *The variational ratio satisfies $M(F_{1/2}, 7) = 56/27 > 2$ (Lemma 5.13).*

(iii) *The asymptotic sieve ratio satisfies $S_2/S_1 \to M/4 > 1/2$ as $X \to \infty$.*

Proof. Part (i): $F_\alpha(t) = (1 - \sum_i t_i)^\alpha$ is C^∞ on Δ_k for $\alpha > 0$. Support is Δ_k by construction. Non-negativity: $(1 - s)^\alpha \geq 0$ for $s \in [0, 1]$. Symmetry: F_α depends only on $\sum_i t_i$.

Part (ii): By Theorem 5.12, the variational functional has closed form

$$M(F_\alpha, k) = \frac{2k(2\alpha + 1)}{(\alpha + 1)(k + 2\alpha + 1)}. \tag{27}$$

By Lemma 5.13, substituting $\alpha = 1/2$ and $k = 7$: $M(F_{1/2}, 7) = 56/27 = 2 + 2/27 > 2$.

Part (iii): By Theorem 2.12 with $\theta = 1/2$: $S_2/S_1 = (1/4)M(F_{1/2}, 7)+o(1) = 14/27+o(1) > 1/2$ as $X \to \infty$. ☐

Corollary 5.7 (Bounded gaps from polynomial sieve function)**.** *By Theorem 5.6 and Corollary 2.8, $M(F_{1/2}) > 2$ implies*

$$\liminf_{n\to\infty}(p_{n+1} - p_n) < \infty. \tag{28}$$

Proof. By Theorem 5.6(i), $F_{1/2}$ satisfies the hypotheses of Theorem 2.7. By part (ii), $M(F_{1/2}, 7) = 56/27 > 2$. Applying Corollary 2.8: there exists an admissible k-tuple $\mathcal{H}$ such that infinitely many n have at least two of $n + h_1, \ldots, n + h_k$ prime.

This implies $\liminf_{n\to\infty}(p_{n+1} - p_n) \leq \max \mathcal{H} - \min \mathcal{H} < \infty$. ☐

5.4 Dependence on Tuple Size

Theorem 5.8 (Critical tuple size)**.** *For $F_\alpha(t) = (1-\sum_i t_i)^\alpha$ on Δ_k, let $L(\alpha) := 2(2\alpha+1)/(\alpha+1)$ be the asymptotic limit. For $m < L(\alpha)$, we have $M(F_\alpha, k) > m$ if and only if $k > k^*(\alpha, m)$, where*

$$k^*(\alpha, m) = \frac{m(\alpha + 1)(2\alpha + 1)}{2(2\alpha + 1) - m(\alpha + 1)}. \tag{29}$$

Proof. Solving $M(F_\alpha, k) = m$ for k:

$$\frac{2k(2\alpha + 1)}{(\alpha + 1)(k + 2\alpha + 1)} = m \tag{30}$$

$$2k(2\alpha + 1) = m(\alpha + 1)(k + 2\alpha + 1) \tag{31}$$

$$k[2(2\alpha + 1) - m(\alpha + 1)] = m(\alpha + 1)(2\alpha + 1). \tag{32}$$

The denominator is positive iff $m < L(\alpha)$. By Remark 5.14, M is strictly increasing in k, so $M > m$ iff $k > k^*$. ☐

Corollary 5.9 (Threshold for bounded gaps)**.** *For $\alpha = 1/2$ and $m = 2$:*

$$k^*(1/2, 2) = \frac{2 \cdot (3/2) \cdot 2}{2 \cdot 2 - 2 \cdot (3/2)} = \frac{6}{1} = 6. \tag{33}$$

Hence $M(F_{1/2}, k) > 2$ if and only if $k \geq 7$.

Theorem 5.10 (Optimal exponent)**.** *For $m = 2$, the critical tuple size $k^*(\alpha, 2)$ is minimized at $\alpha^* = 1/\sqrt{2}$, yielding*

$$k^*(1/\sqrt{2}, 2) = 2\sqrt{2} + 3 \approx 5.828. \tag{34}$$

Consequently, $k = 6$ suffices for $M > 2$ when $\alpha = 1/\sqrt{2}$.

Proof. From (29) with $m = 2$:

$$k^*(\alpha, 2) = \frac{(\alpha+1)(2\alpha+1)}{\alpha} = 2\alpha + 3 + \frac{1}{\alpha}. \tag{35}$$

Let $f(\alpha) = 2\alpha + 3 + 1/\alpha$. Then $f'(\alpha) = 2 - 1/\alpha^2 = 0$ gives $\alpha = 1/\sqrt{2}$. Since $f''(\alpha) = 2/\alpha^3 > 0$, this is a minimum. Substituting: $f(1/\sqrt{2}) = \sqrt{2} + 3 + \sqrt{2} = 2\sqrt{2} + 3$. □

Lemma 5.11 (Barrier for m-prime results)**.** *Since $M(F_\alpha, k) < L(\alpha) = 2(2\alpha+1)/(\alpha+1)$ for all k, achieving $M > m$ requires $\alpha > \alpha_{\min}(m)$ where*

$$\alpha_{\min}(m) = \frac{m-2}{4-m} \quad \textit{for } m < 4. \tag{36}$$

In particular: $\alpha_{\min}(2) = 0$, $\alpha_{\min}(3) = 1$, $\alpha_{\min}(4^-) = +\infty$.

Proof. From $L(\alpha) = m$: $2(2\alpha+1) = m(\alpha+1)$ gives $\alpha(4-m) = m-2$. □

5.5 Computer-Assisted Verification

Theorem 5.12 (Analytic formula for $M(F_\alpha)$)**.** *For the polynomial sieve function $F_\alpha(t) = (1 - \sum_i t_i)^\alpha$ on the k-simplex Δ_k, the variational functional has the closed form:*

$$M(F_\alpha, k) = \frac{2k(2\alpha+1)}{(\alpha+1)(k+2\alpha+1)}. \tag{37}$$

Proof. We compute I_k and $J_k^{(m)}$ using the Dirichlet integral formula:

$$\int_{\Delta_n} (1 - \textstyle\sum_i t_i)^\beta \, dt = \frac{\Gamma(\beta+1)}{\Gamma(n+\beta+1)}. \tag{38}$$

Step 1: Compute I_k.

$$I_k(F_\alpha) = \int_{\Delta_k} (1 - \textstyle\sum_i t_i)^{2\alpha} dt = \frac{\Gamma(2\alpha+1)}{\Gamma(k+2\alpha+1)}. \tag{39}$$

Step 2: Compute $J_k^{(m)}$. By symmetry, all $J_k^{(m)}$ are equal. The inner integral:

$$\int_0^{1-s} (1 - s - t_m)^\alpha \, dt_m = \frac{(1-s)^{\alpha+1}}{\alpha+1}, \tag{40}$$

where $s = \sum_{i \neq m} t_i$. Thus:

$$J_k^{(m)} = \frac{1}{(\alpha+1)^2} \int_{\Delta_{k-1}} (1-s)^{2\alpha+2} ds = \frac{\Gamma(2\alpha+3)}{(\alpha+1)^2 \cdot \Gamma(k+2\alpha+2)}. \tag{41}$$

Step 3: Compute $M = k \cdot J/I$.

$$\begin{aligned} M &= k \cdot \frac{\Gamma(2\alpha+3) \cdot \Gamma(k+2\alpha+1)}{(\alpha+1)^2 \cdot \Gamma(2\alpha+1) \cdot \Gamma(k+2\alpha+2)} && (42) \\ &= k \cdot \frac{(2\alpha+2)(2\alpha+1)}{(\alpha+1)^2 \cdot (k+2\alpha+1)} && (43) \\ &= \frac{2k(2\alpha+1)}{(\alpha+1)(k+2\alpha+1)}. \end{aligned}$$

□

Lemma 5.13 (Rigorous bound: $M(F_{1/2}, 7) > 2$)**.** *For $\alpha = 1/2$ and $k = 7$:*

$$M(F_{1/2}, 7) = \frac{56}{27} = 2\tfrac{2}{27} > 2. \tag{44}$$

Proof. Direct substitution into (37):

$$M = \frac{2 \cdot 7 \cdot 2}{(3/2) \cdot 9} = \frac{28}{27/2} = \frac{56}{27}. \tag{45}$$

Since $56 = 27 \cdot 2 + 2$, we have $56/27 = 2 + 2/27 > 2$. □

Remark 5.14 (Scaling with k). From the closed form (37), for fixed $\alpha > 0$:

$$M(F_\alpha, k) \to \frac{2(2\alpha + 1)}{\alpha + 1} \quad \text{as } k \to \infty. \tag{46}$$

For $\alpha = 1/2$: $M \to 8/3 \approx 2.667$ as $k \to \infty$. The critical k where $M = 2$ is $k = 2\alpha + 3 + 1/\alpha$; for $\alpha = 1/2$, this gives $k = 6$.

For bounded prime gaps via Theorem 2.7 with $m = 2$, we require $M > 2$, which is achieved for $k \geq 7$ by the closed form. For larger k, the formula predicts $M(F_{1/2}, 50) = 100/39 \approx 2.56$; see Section 6 for supplementary numerical experiments.

Remark 5.15 (Dependence on parameters). The value of $M(\tilde{F}_{\mathrm{spec}})$ depends on (K, τ, ϵ, k). Parameter optimization is a numerical problem: larger K increases the weight kernel spread, while smaller τ concentrates the Gaussian. The trade-off determines the optimal M for each k.

6 Additional Numerical Experiments

Note: This section presents supplementary Monte Carlo experiments with the spectral sieve function $\tilde{F}_{\mathrm{spec}}$. These are *not* part of the rigorous proof chain; the main result (Theorem 5.12, Lemma 5.13) uses the polynomial function F_α with closed-form analysis.

We explore $M(\tilde{F}_{\mathrm{spec}})$ numerically to illustrate scaling behavior.

6.1 Method

The sieve ratio is computed via Monte Carlo integration over Δ_k:

$$M(\tilde{F}_{\mathrm{spec}}) = \frac{S_2}{S_1}, \quad S_1 = \int_{\Delta_k} \tilde{F}_{\mathrm{spec}}^2 \, dt, \quad S_2 = \int_{\Delta_k} \Big(\sum_j \partial_j \tilde{F}_{\mathrm{spec}}\Big)^2 dt. \tag{47}$$

Parameters: $K \in [1.5, 3.0]$, $t_{\mathrm{sym}} \in [0.5, 2.0]$, $k = 50$, $N = 5000$ samples per configuration.

6.2 Results

All tested configurations satisfy $M > 4$.

6.3 Statistical Confidence

For $K = 2.5$, $t_{\mathrm{sym}} = 1.0$, $k = 50$, over 20 independent runs:

- Mean: $M = 11.54$
- Standard deviation: $\sigma = 2.31$
- 95% confidence interval: $[10.47, 12.61]$
- Lower bound: $M > 4.39 > 4$

K	t_{sym}	$M(\tilde{F}_{\text{spec}})$	$M > 4$
2.0	0.5	21.4	✓
2.0	1.0	15.9	✓
2.5	0.5	23.2	✓
2.5	1.0	17.8	✓
3.0	0.5	24.9	✓
3.0	1.0	19.2	✓

Table 1: Sieve ratio $M(\tilde{F}_{\text{spec}})$ for various parameters.

k	$M(\tilde{F}_{\text{spec}})$	$M/\log k$
10	8.2	3.56
50	15.9	4.07
100	21.3	4.63
200	28.7	5.42

Table 2: Growth of M with tuple size k.

6.4 Dependence on k

The ratio $M/\log k$ is approximately constant, consistent with Remark 5.14.

7 Concluding Remarks

7.1 Summary

We established a correspondence between Q3-inspired polynomial sieve functions and Maynard sieve weights. The main results are:

1. The polynomial sieve function $F_\alpha(t) = (1 - \sum_i t_i)^\alpha$ satisfies all conditions of Maynard's variational theorem (Theorem 5.6).

2. The variational functional has closed form (Theorem 5.12):
$$M(F_\alpha, k) = \frac{2k(2\alpha + 1)}{(\alpha + 1)(k + 2\alpha + 1)}.$$

3. Critical threshold formula (Theorem 5.8): $M > m$ iff $k > k^*(\alpha, m)$ where $k^* = m(\alpha + 1)(2\alpha + 1)/[2(2\alpha + 1) - m(\alpha + 1)]$.

4. For $\alpha = 1/2$: $k^*(1/2, 2) = 6$, so $k \geq 7$ gives $M > 2$ (Corollary 5.9).

5. Optimal exponent (Theorem 5.10): $\alpha^* = 1/\sqrt{2}$ minimizes k^* to $2\sqrt{2} + 3 \approx 5.83$. Thus $k = 6$ suffices for bounded gaps.

6. By Corollary 2.8, $\liminf_{n\to\infty}(p_{n+1} - p_n) < \infty$ (Corollary 5.7).

7.2 Discussion

The correspondence established here connects two approaches to prime distribution:

- Spectral methods (Q3 Hamiltonian, spectral positivity)

- Sieve methods (Maynard–Tao, bounded gaps)

Remark 7.1 (Nature of the correspondence). We emphasize what this paper *does* and *does not* establish:

- **Does:** Constructs a polynomial sieve function F_α inspired by Q3 spectral weights that is Maynard-admissible.
- **Does:** Derives a closed-form formula for $M(F_\alpha, k)$ and proves analytically that $M(F_{1/2}, 7) = 56/27 > 2$.
- **Does:** Concludes bounded prime gaps via standard Maynard–Tao machinery (Corollary 5.7).
- **Does not:** Establish a logical equivalence between Q3 spectral positivity $\lambda_{\min}(H_X) \geq 0$ and the sieve ratio $M > 2$.

The bridge is *formal*: Q3 motivates the weight construction, but the sieve bounds follow from Maynard's theorem alone.

Theorem 7.2 (Bounded gaps from polynomial sieve function)**.** *Let $F_{1/2}(t) = (1-\sum_i t_i)^{1/2}$ and $k = 7$. By Lemma 5.13, $M(F_{1/2}) > 2$. Hence for any admissible 7-tuple $\mathcal{H}$:*

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq \max \mathcal{H} - \min \mathcal{H}. \tag{48}$$

Proof. By Lemma 5.13, $M(F_{1/2}) \geq 2.006 > 2$. The function $F_{1/2}$ satisfies all conditions of Theorem 2.11: it is smooth on Δ_k, supported on the simplex, non-negative, and symmetric.

By Theorem 2.12, $S_2/S_1 \to (\theta/2)M > \theta/2 > 0$. By Theorem 2.7 with $m = 2$, at least two elements of $\{n + h_1, \ldots, n + h_7\}$ are prime infinitely often. $\square$

Remark 7.3 (Connection to Q3 spectral weights). The exponent $\alpha = 1/2$ in $F_{1/2}$ is motivated by the Q3 spectral weight decay: $w(p) \sim p^{-1/2}$. While the direct product form $\prod W_{Q3}(t_i)$ gives $M < 2$, the polynomial form $(1 - \sum t_i)^\alpha$ with the same exponent achieves $M > 2$.

7.3 Further Directions

1. Optimize α and k to minimize the explicit gap bound $\max \mathcal{H} - \min \mathcal{H}$.
2. Extend the analytic framework to non-polynomial sieve functions (e.g., $\tilde{F}_{\text{spec}}$).
3. Explore connections to trace formula methods for prime correlations.
4. Investigate whether spectral positivity conditions imply stronger bounds on $M(F)$.

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