On direct systems of implications with graded attributes

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In this paper the problem of defining direct systems of implications in the fuzzy setting is studied. The directness of systems allows a quick computation of the closure operator in cases such as Fuzzy Formal Concept Analysis. Characterizing these properties in algebraic terms is deeply linked to Simplification Logic. After the theoretical results, some thoughts on algorithms to provide direct systems are also considered.

Introduction

This document serves as supplementary material for the paper *On direct systems of implications with graded attributes* submitted to EUSFLAT 2025.

Detailed example

Let us consider a fuzzy formal context $\mathbb{K} = (G, M, I)$, where the set of attributes is $M = \{a, b, c, d, e\}$, and such that the valuation lattice, i.e., the lattice L such that $I \in L^{G\times M}$, is $L = \{0, 0.5, 1\}$, equipped with the Lukasiewicz logical structure. If the degree of element $x \in M$ in a set is $y \in L$, we will denote the element as x_y.

Take the system of implications:

$$\Sigma = \{a_{0.5}\, b_{0.5}\, d \to a\, b\, c\, e, \quad a_{0.5}\, d\, e \to a\, b\, c_{0.5}\}$$

We will use the `DirectSystem` algorithm to construct a direct system Σ_d equivalent to Σ.

Iterations

Iteration 1

We will describe this iteration in detail, the following ones will be more concise.

Derived implications

The algorithm loops over all pairs of implications, computing the *derived implication* when needed (required by the fuzzy exchange condition). In this first iteration, only the two implications are checked, for all $\alpha, \beta \in L$ such that the requirements are met:

Taking

$$\begin{aligned} A \to B &= a_{0.5}\, d\, e \to a\, b\, c_{0.5} \\ C \to D &= a_{0.5}\, b_{0.5}\, d \to a\, b\, c\, e, \end{aligned}$$

the execution of `AddDerived` produces:

- for $\alpha = 0.5$, $\beta = 0.5$, the implication $d_{0.5}\, e_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}$.
- for $\alpha = 1$, $\beta = 0.5$, the implication $a_{0.5}\, d\, e \to b_{0.5}\, c_{0.5}$.
- for $\alpha = 1$, $\beta = 1$, the implication $a_{0.5}\, d\, e \to a\, b\, c$.

Reversing the order in which the implications are considered, i.e.,

$$\begin{aligned} A \to B &= a_{0.5}\, b_{0.5}\, d \to a\, b\, c\, e \\ C \to D &= a_{0.5}\, d\, e \to a\, b\, c_{0.5}, \end{aligned}$$

`AddDerived` provides:

- for $\alpha = 0.5$, $\beta = 0.5$, the implication $d_{0.5} \to a_{0.5}\, b_{0.5}$.
- for $\alpha = 0.5$, $\beta = 1$, the implication $d\, e \to a\, b\, c_{0.5}$
- for $\alpha = 1$, $\beta = 1$, the implication $a_{0.5}\, b_{0.5}\, d \to a\, b\, c_{0.5}$

This produces

$$\begin{aligned} \mathcal{D} = \{ & d_{0.5}\, e_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}, \quad a_{0.5}\, d\, e \to a\, b\, c, \quad d_{0.5} \to a_{0.5}\, b_{0.5}, \\ & d\, e \to a\, b\, c_{0.5}, \quad a_{0.5}\, b_{0.5}\, d \to a\, b\, c_{0.5} \} \end{aligned}$$

Combination phase

The result of applying `Combine` to Σ and $\mathcal{D}$ follows these steps:

- Add the implication $d_{0.5}\, e_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}$ to Σ, since there is no implication in Σ with the same left-hand side.
- Analogously, add the implications $d_{0.5} \to a_{0.5}\, b_{0.5}$ and $d\, e \to a\, b\, c_{0.5}$ to Σ.

- Update $a_{0.5}\, d\, e \to a\, b\, c_{0.5} \in \Sigma$ with $a_{0.5}\, d\, e \to a\, b\, c \in \mathcal{D}$, to obtain the implication $a_{0.5}\, d\, e \to a\, b\, c$.

The value of the variable `change` returned by `Combine` is `true` since there have been modifications to Σ.

Result of the iteration

Σ after this iteration:

$$\Sigma = \{a_{0.5}\, b_{0.5}\, d \to a\, b\, c\, e, \quad a_{0.5}\, d\, e \to a\, b\, c, \quad d_{0.5}\, e_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}, \quad d_{0.5} \to a_{0.5}\, b_{0.5}, \quad d\, e \to a\, b\, c_{0.5}\}$$

Iteration 2

Now, we will summarise the steps, for the sake of readability.

Derived implications

$$\mathcal{D} = \{d_{0.5}\, e_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}, \quad d\, e_{0.5} \to a\, b\, c\, e, \quad a_{0.5}\, d\, e \to a\, b\, c, \quad b\, c_{0.5} \to a\, c\, d, \quad d_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}\, e_{0.5}, \quad d\, e \to a\, b\, c, \quad a_{0.5}\, b_{0.5}\, d \to a\, b\, c\}$$

Combination phase

- Update $d_{0.5} \to a_{0.5}\, b_{0.5}$ with $d_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}\, e_{0.5}$ to obtain $d_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}\, e_{0.5}$.
- Update $d\, e \to a\, b\, c_{0.5}$ with $d\, e \to a\, b\, c$ to obtain $d\, e \to a\, b\, c$.
- Add $d\, e_{0.5} \to a\, b\, c\, e$ to Σ.

Therefore, the variable `change` is again `true` and a new iteration is needed.

Result of the iteration

Σ after this iteration:

$$\Sigma = \{a_{0.5}\, b_{0.5}\, d \to a\, b\, c\, e, \quad a_{0.5}\, d\, e \to a\, b\, c, \quad d_{0.5}\, e_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}, \quad d_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}\, e_{0.5}, \quad d\, e \to a\, b\, c, \quad d\, e_{0.5} \to a\, b\, c\, e\}.$$

Iteration 3

Derived implications

$$\mathcal{D} = \{d_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}\, e_{0.5}, \quad d \to a\, b\, c\, e, \quad d_{0.5}\, e_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}, \quad d\, e_{0.5} \to a\, b\, c\, e,$$
$$d\, e \to a\, b\, c, \quad a_{0.5}\, b_{0.5}\, d \to a\, b\, c\, e_{0.5}, \quad a_{0.5}\, d\, e \to a\, b\, c\}$$

Combination phase

- Add $d \to a\, b\, c\, e$ to Σ.

For this reason, `change` is set to `true`.

Result of the iteration

The implication system obtained so far is

$$\Sigma = \{a_{0.5}\, b_{0.5}\, d \to a\, b\, c\, e, \quad a_{0.5}\, d\, e \to a\, b\, c, \quad d_{0.5}\, e_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}, \quad d_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}\, e_{0.5},$$
$$d\, e \to a\, b\, c, \quad d\, e_{0.5} \to a\, b\, c\, e, \quad d \to a\, b\, c\, e\}$$

Iteration 4

Derived implications

$$\mathcal{D} = \{d_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}\, e_{0.5}, \quad a_{0.5}\, b_{0.5}\, d \to a\, b\, c\, e_{0.5}, \quad d\, e \to a\, b\, c,$$
$$d_{0.5}\, e_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}, \quad d\, e_{0.5} \to a\, b\, c\, e, \quad a_{0.5}\, d\, e \to a\, b\, c\}$$

Combination phase

It can be observed that $\mathcal{D} \subseteq \Sigma$, thus at this point, `change` is set to `false` and there is no need to iterate further.

Final result

This is the direct system returned by the algorithm:

$$\Sigma_d = \{a_{0.5}\, b_{0.5}\, d \to a\, b\, c\, e, \quad a_{0.5}\, d\, e \to a\, b\, c, \quad d_{0.5}\, e_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}, \quad d_{0.5} \to a_{0.5}\, b_{0.5}\, c_{0.5}\, e_{0.5},$$
$$d\, e \to a\, b\, c, \quad d\, e_{0.5} \to a\, b\, c\, e, \quad d \to a\, b\, c\, e\}$$