Implications and logic

The knowledge stored in a formal context can also be represented as a set of implications, which are expressions of the form $A \Rightarrow B$ where A and B are sets of attributes or items, indicating that, for every object in which the set of attributes A is present, also B is present. This interpretation is similar to the one defined in data mining/machine learning over the so-named association rules. The confidence (a well-known estimator of the rules' quality) has value 1 in all the implications.

For instance $\left\{{}^{0.5}/\mathrm{P1}\right\} \Rightarrow \{\mathrm{P4}\}$ is a valid implication in the previous example, having the following interpretation: when the attribute P1 has degree at least 0.5 then we have P4 with degree 1.

The *Duquenne-Guigues basis* of implications (Guigues and Duquenne, 1986) is a set of valid implications from which all other valid implications can be deduced. The Duquenne-Guigues basis in our example is given by:

$$
\begin{array}{lrcl}
1: & \varnothing & \Rightarrow & \left\{{}^{0.5}/\mathrm{P2},\ {}^{0.5}/\mathrm{P4}\right\} \\
2: & \left\{{}^{0.5}/\mathrm{P2},\ \mathrm{P4}\right\} & \Rightarrow & \{\mathrm{P2}\} \\
3: & \left\{\mathrm{P2},\ {}^{0.5}/\mathrm{P4}\right\} & \Rightarrow & \{\mathrm{P4}\} \\
4: & \left\{\mathrm{P2},\ {}^{0.5}/\mathrm{P3},\ \mathrm{P4}\right\} & \Rightarrow & \{\mathrm{P3}\} \\
5: & \left\{{}^{0.5}/\mathrm{P1},\ {}^{0.5}/\mathrm{P2},\ {}^{0.5}/\mathrm{P4}\right\} & \Rightarrow & \{\mathrm{P2},\ \mathrm{P4}\} \\
6: & \left\{{}^{0.5}/\mathrm{P1},\ \mathrm{P2},\ \mathrm{P3},\ \mathrm{P4}\right\} & \Rightarrow & \{\mathrm{P1}\}
\end{array}
$$

In Cordero et al. (2002), the simplification logic, denoted as SL_{FD}, was introduced as a method to manipulate implications (functional dependencies or if-then rules), removing redundancies or computing closures of attributes. This logic is equivalent to Armstrong's Axioms (Armstrong, 1974), which are well known from the 80s in databases, artificial intelligence, formal concept analysis, and others. The axiomatic system of this logic considers reflexivity as the axiom scheme

$$\texttt{[Ref]} \quad \frac{A \supseteq B}{A \Rightarrow B}$$

together with the following inference rules called fragmentation, composition and simplification, respectively, which are equivalent to the classical Armstrong's axioms of augmentation and, more importantly, transitivity.

$$\texttt{[Frag]} \ \frac{A \Rightarrow B \cup C}{A \Rightarrow B} \qquad \texttt{[Comp]} \ \frac{A \Rightarrow B,\ C \Rightarrow D}{A \cup C \Rightarrow B \cup D} \qquad \texttt{[Simp]} \ \frac{A \Rightarrow B,\ C \Rightarrow D}{A \cup (C \smallsetminus B) \Rightarrow (D \smallsetminus B)}$$

The main advantage of SL_{FD} with respect to Armstrong's Axioms is that the inference rules may be considered as equivalence rules, (see the work by Mora et al. (2012) for further details and proofs), that is, given a set of implications Σ, the application of the equivalences transforms it into an equivalent set. In the package presented in this paper, we develop the following equivalences:

1. Fragmentation Equivalency **[FrEq]**: $\{A \Rightarrow B\} \equiv \{A \Rightarrow B \smallsetminus A\}$.
2. Composition Equivalency **[CoEq]**: $\{A \Rightarrow B, A \Rightarrow C\} \equiv \{A \Rightarrow B {\cup} C\}$.
3. Simplification Equivalency **[SiEq]**: If $A \subseteq C$, then
$$\{A \Rightarrow B, C \Rightarrow D\} \equiv \{A \Rightarrow B, A \cup (C \smallsetminus B) \Rightarrow D \smallsetminus B\}$$
4. Right-Simplification Equivalency **[rSiEq]**: If $A \subseteq D$, then
$$\{A \Rightarrow B, C \Rightarrow B \cup D\} \equiv \{A \Rightarrow B, C \Rightarrow D\}$$

Usually, many areas, the implications have always atomic attributes on the right-hand side. We emphasize that this logic can manage *aggregated* implications, i.e. the implications' consequents do not have to be singletons. This represents an increase of the logic efficiency.

This logic removes attribute redundancies in some of the implications in the Duquenne-Guigues basis presented before. Particularly, the implications with numbers 2, 3, 4, 5 and 6 are simplified to:

$$
\begin{array}{lrcl}
2: & \{\mathrm{P4}\} & \Rightarrow & \{\mathrm{P2}\} \\
3: & \{\mathrm{P2}\} & \Rightarrow & \{\mathrm{P4}\} \\
4: & \left\{{}^{0.5}/\mathrm{P3},\ \mathrm{P4}\right\} & \Rightarrow & \{\mathrm{P3}\} \\
5: & \left\{{}^{0.5}/\mathrm{P1}\right\} & \Rightarrow & \{\mathrm{P4}\} \\
6: & \left\{{}^{0.5}/\mathrm{P1},\ \mathrm{P3}\right\} & \Rightarrow & \{\mathrm{P1}\}
\end{array}
$$

One of the primary uses of a set of implications is computing the closure of a set of attributes, the maximal fuzzy set that we can arrive at from these attributes using the given implications.