MA 771: Introduction to Dynamical Systems
Lecture Notes

Malavika Mukundan

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Contents

Chapter 1

Basic Concepts

Dynamical systems refer to systems that evolve over time. A simple example is an ordinary differential equation (ODE), such as

$$\frac{dx}{dt} = g(x)$$

where $g : \mathbb{R}^n \longrightarrow \mathbb{R}$. Some of the questions we can ask about ODE's include:

- Which points are stable/fixed?
- When do orbits tend to an attracting periodic cycle?
- When are orbits chaotic?

However, the class of dynamical systems is much bigger than ODE's. In particular, in the above example, the variable t, representing time, changes *continuously*. In this course we will focus mainly on *discrete* dynamical systems.

Definition 1.0.1. Let X be a topological space. A *discrete* dynamical system is a pair (X, f) where $f : X \longrightarrow X$ is a self-map.

We are interested in the function f and its iterates $f^{\circ n} = f \circ f^{\circ(n-1)}$ for $n \in \mathbb{N}$. In other words, the quantity that is varying in a *discrete* fashion is the number of iterations of f.

Example 1.0.2. **Linear maps** Let $X = \mathbb{R}^n$ and $f : x \mapsto Ax$ be a linear map.

Example 1.0.3. **Rotations of the Circle** $X = \mathbb{S}^1$ and $f(x) = e^{2\pi i\theta}x$ for some $\theta \in \mathbb{R}$.

Example 1.0.4. **Logistic Family** Let $X = \mathbb{R}$ and fix $k \in \mathbb{R}_{>0}$. Then the family of maps $f_k : X \to X$ given by $x \mapsto kx(1-x)$ is called the *logististic family*.

Notice that we are not assuming any conditions on f such as continuity.

1.1 Orbits and Periodic Points

Definition 1.1.1. Given a dynamical system (X, f), $x_0 \in X$, the sequence

$$x_0, f(x_0), f^{\circ 2}(x_0), \cdots, f^{\circ n}(x_0), \cdots$$

is called the *forward orbit* of x_0.
The *reverse orbit of* x_0 is the set $\{x \in X : f^{\circ n}(x) = x_0 \text{ for some } n \in \mathbb{N}\}$.

A *fixed point* $x \in X$ is a point such that $f(x) = x$. The set of fixed points of f is denoted $\mathrm{Fix}(f)$. A *periodic point* is a point x such that $f^{\circ n}(x) = x$ for some $n \in \mathbb{N}$, in other words, a point in $\mathrm{Fix}(f^{\circ n})$ for some n.

Any $n \in \mathbb{N}$ such that $f^{\circ n}(x) = x$ is said to be a *period* of x. The smallest period n is called the *exact* period of x.

1.2 Examples

Linear Maps of $\mathbb{R}$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be linear. We know that f is of the form $f(x) = mx + b$ where $m \in \mathbb{R}_{\neq 0}$ and $b \in \mathbb{R}$. Note that

$$f^{\circ n}(x) = m^n x + b\big(m^{n-1} + m^{n-2} + \cdots + m + 1\big) = m^n x + b\frac{m^n - 1}{m - 1}$$

- If $m \neq \pm 1$, then

$$\begin{aligned} |m| < 1 &\implies \forall x \in \mathbb{R}, f^{\circ n}(x) \to \frac{b}{1-m} \text{ as } n \to \infty \\ |m| > 1 &\implies \forall x \in \mathbb{R}, f^{\circ n}(x) \to \infty \text{ as } n \to \infty \end{aligned}$$

- If $m = 1$, then $f(x) = x + b$ is a translation and all orbits tend to ∞
- If $m = -1$, then note that $f^{\circ 2}(x) = -(-x + b) + b = x$, and thus all the odd iterates are equal to f, and all the even iterates are equal to the identity.

Circle Maps

Example 1.2.1. For any rotation $f(x) = e^{2\pi i\theta}x$ of the circle $\mathbb{S}^1$, we have $\mathrm{Fix}(f) = \emptyset$ if $\theta \notin \mathbb{N}$, and $\mathrm{Fix}(f) = \mathbb{S}^1$ otherwise.

Definition 1.2.2. Fix an integer $m > 1$, and identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$. The *expanding map* $E_m : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is defined as

$$E_m(x) = mx \text{ (mod) } 1$$

Remark 1.2.3. E_m is expanding in the following sense: if $\alpha, \beta \in \mathbb{S}^1$ and $d_{\mathbb{S}^1}(\alpha, \beta) < \frac{1}{m}$, then

$$d_{\mathbb{S}_1}(\alpha, \beta) = m \cdot d_{\mathbb{S}^1}\Big(E_m(\alpha), E_m(\beta)\Big).$$

See Figure 1.1 for the graphs of E_m for $m = 2, 3$.

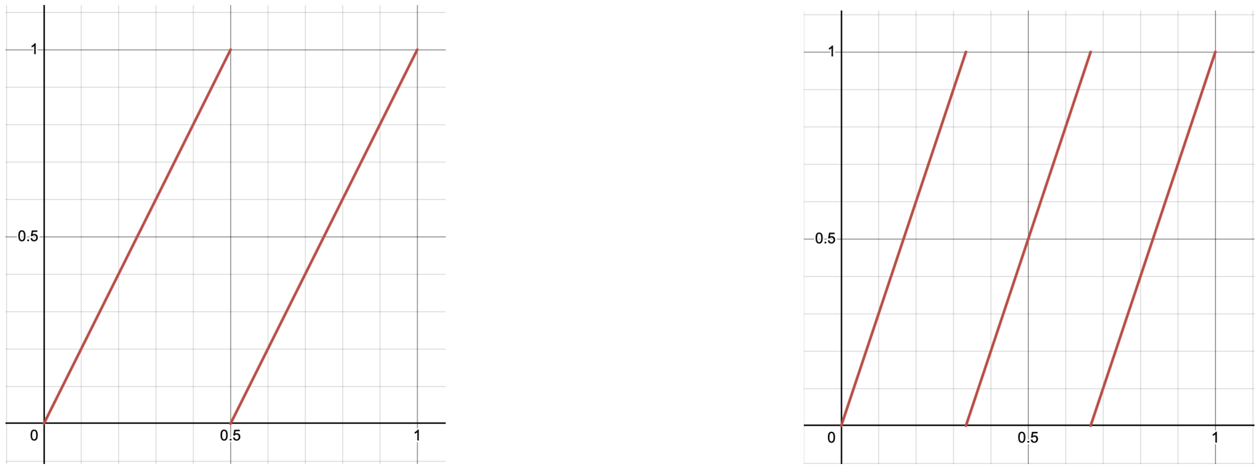

Figure 1.1: The graphs of the expanding maps E_2 (left) and E_3 (right) on the interior of $\mathbb{S}^1$, represented by the interval $(0,1)$.

Note that ϕ is a fixed point of E_m if and only if $m\phi - \phi \in \mathbb{Z}$. In other words, there exists $n \in \mathbb{Z}$ such that

$$\begin{aligned} m\phi - \phi &= n \\ \iff \phi &= \frac{n}{m-1} \end{aligned}$$

Similarly, ϕ is a periodic point of E_m of period dividing k if and only if there exists $n \in \mathbb{Z}$ such that

$$\begin{aligned} m^k\phi - \phi &= n \\ \iff \phi &= \frac{n}{m^k-1} \end{aligned}$$

In other words,

$$\begin{aligned} \mathrm{Fix}(E_m) &= \left\{0, \frac{1}{m-1}, \frac{2}{m-1} \cdots, \frac{m-2}{m-1}\right\} \\ \mathrm{Fix}\left(E_m^{\circ k}\right) &= \left\{0, \frac{1}{m^k-1}, \frac{2}{m^k-1} \cdots, \frac{m^k-2}{m^k-1}\right\} \end{aligned}$$

Torus Endomorphisms

Given $n \in \mathbb{N}$, the $n-$torus is the space $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \mathbb{R}^n/\sim$ where $x \sim y$ if $x - y \in \mathbb{Z}^n$. For $x \in \mathbb{R}^n$, we let $[x]$ denote the equivalence class of x in $\mathbb{T}^n$.

Definition 1.2.4. Let A be an $n \times n$ matrix whose entries are in $\mathbb{Z}$. Then A induces the *torus endomorphism* $T_A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ given by

$$T_A([x]) = [Ax] \text{ for } [x] \in \mathbb{T}^n$$

Exercise 1.2.5. Show that T_A as given above is well-defined: that is, for any two vectors $v, w \in \mathbb{R}^n$, if $v - w \in \mathbb{Z}^n$, then $Av - Aw \in \mathbb{Z}^n$

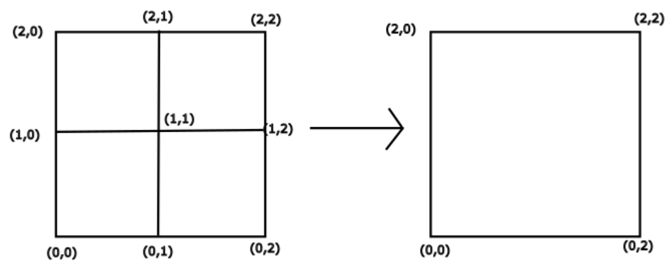

Figure 1.2: An illustration of the torus endomorphism $T_A : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ for $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Example 1.2.6. Let $m, k \in \mathbb{Z}$ and consider the matrix $A = \begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix}$. Consider the map T_A on $\mathbb{T}^2$: this acts as two independent expanding maps: expansion by a factor of m in the $x-$direction, and expansion by a factor of k in the $y-$direction (see Figure 1.2 which illustrates the case $m = k = 2$). Can you show in general that the degree of such a map is $d = mk$? In other words, T_A is a $d : 1$ map of $\mathbb{T}^2$.

Definition 1.2.7. A torus endomorphism T_A is said to be an *automorphism* if it is invertible.

Exercise 1.2.8. (This is also on HW 1) Show that T_A is invertible if and only if A^{-1} has integer entries, which in turn is equivalent to $\det A = \pm 1$.

Proposition 1.2.9. *Let $T_A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ be a torus automorphism with no eigenvalues of modulus* 1. *Then the periodic points of T_A are all the points with rational coordinates.*

Proof. (periodic $\implies$ rational):
Let $[x] = [(x_1, x_2, \cdots, x_n)] \in \mathbb{T}^n$ be a periodic point of period q for some $q \in \mathbb{N}$. Then $T_A^{\circ q}([x]) = [A^q x] = [x]$. That is, there exists a vector $y \in \mathbb{Z}^n$ such that

$$\begin{aligned} A^q x &= x + y \\ \implies A^q x - x &= y \\ \implies (A^q - \mathrm{Id})x &= y \end{aligned}$$

Since A has no eigenvalues of modulus 1, the matrix A^q has no eigenvalues of modulus 1. This means that the matrix $A^q - \mathrm{Id}$ is invertible. So

$$x = (A^q - \mathrm{Id})^{-1} y$$

Since y has integer coordinates and the matrix $(A^q - \mathrm{Id})^{-1}$ has rational entries, x has rational coordinates.

(rational $\implies$ periodic):
Suppose x has rational coordinates, we can assume that all the coordinates have a common denominator. In other words, $x = \left(\frac{p_1}{r}, \frac{p_2}{r}, \cdots, \frac{p_n}{r}\right)$ for some integers p_i, r with $r \neq 0$. Given a $q \in \mathbb{N}$, since A has integer entries, $A^q x = \left(\frac{p_1'}{r}, \frac{p_2'}{r}, \cdots, \frac{p_n'}{r}\right)$ for some integers $p_1', \cdots, p_n'$.

Note that there are only finitely many points in $\mathbb{T}^n$ with rational coordinates with a common denominator r. In other words, the set $\{T_A^{\circ q}([x]) : q \in \mathbb{N}\}$ is finite.

Thus, there exist $q_1 < q_2 \in \mathbb{N}$ such that $T_A^{\circ q_1}([x]) = T_A^{\circ q_2}([x])$. Since T_A is an automorphism, this means that $T_A^{\circ(q_2 - q_1)}([x]) = [x]$. □

1.3 Stable Behavior: The Contraction Principle

In this section we will look at maps on subsets of $\mathbb{R}^n$ which satisfy a criterion for all orbits converging to a fixed point.

Global Contractions

Definition 1.3.1. A map f of a subset X of $\mathbb{R}^n$ is said to be *Lipschitz-continuous* with Lipschitz constant λ, or λ-*Lipschitz* if

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for any $x, y \in X$.
The map f is said to be a *contraction* if

$$d(f(x), f(y)) < d(x, y) \qquad \forall x, y \in X.$$

It is said to be a $\lambda-$*contraction* for $\lambda < 1$ if

$$d(f(x), f(y)) \leq \lambda d(x, y) \qquad \forall x, y \in X.$$

Remark 1.3.2. If a map f is Lipschitz-continuous, then we define

$$Lip(f) := \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}$$

Example 1.3.3. $f(x) = \sqrt{x}$ defines a contraction on $I = [1, \infty)$. What is $Lip(f)$?

Theorem 1.3.4 (Contraction Principle in $\mathbb{R}^n$)**.** *Let $X \subset \mathbb{R}^n$ be closed and $f : X \longrightarrow X$ be a $\lambda-$contraction. Then f has a unique fixed point x_0 and $d(f^{\circ n}(x), x_0) = \lambda^n d(x, x_0)$ for every $x \in X$.*

Proof. We have

$$d(f^{\circ n}(x), f^{\circ n}(y)) \leq \lambda^n d(x, y)$$

for all $x, y \in X$. But this also means that for any $x \in X$, we have

$$d(f^{\circ(n+1)}(x), f^{\circ n}(x)) \leq \lambda^n d(f(x), x)$$

$$\begin{aligned} d(f^{\circ m}(x), f^{\circ n}(x)) &\leq d(f^{\circ m}(x), f^{\circ(m-1)}(x)) + d(f^{\circ(m-1)}(x), f^{\circ(m-2)}(x)) + \cdots d(f^{\circ(n+1)}(x), f^{\circ n}(x)) \\ &\leq \left(\lambda^{m-1} + \lambda^{m-2} + \cdots \lambda^n\right) d(f(x), x) \\ &\leq \frac{\lambda^n(1-\lambda^{m-n})}{1-\lambda} d(f(x), x) \\ &\leq \frac{\lambda^n}{1-\lambda} d(f(x), x) \end{aligned}$$

In other words, the orbit of x is a Cauchy sequence. Since X is closed, $\lim_{n\to\infty} f^{\circ n}(x) = x_0$ is a point of X, and

$$f(x_0) = f(\lim_{n\to\infty} f^{\circ n}(x)) = \lim_{n\to\infty} f^{\circ(n+1)}(x_0) = x_0$$

□

Remark 1.3.5. Given a sequence $(y_n)_{n\geq 0}$ in a metric space (Y, d), we say that $y_n \to y \in Y$ *exponentially* if there exist constants $A > 0$ and $0 < \lambda < 1$ such that

$$d(y_n, y) \leq A\lambda^n d(y_0, y)$$

Note that in the above situation, the orbit under f of x converges exponentially to x_0 (here $A = 1$).

The contraction principle applies to $\lambda-$contractions defined on complete metric spaces.

Theorem 1.3.6 (Contraction Principle for complete metric spaces)**.** *Let (X, d) be a complete metric space and $f : X \longrightarrow X$ be a $\lambda-$contraction. Then there exists a unique fixed point $x_0 \in X$ such that the orbits under f of all points $x \in X$ converge exponentially to x_0.*

Example 1.3.7 (Rabbits; due to Fibonacci). Say we record the number of rabbits in a forest starting January (month 0) of a given year. The Fibonacci model for rabbit population growth is as follows:
Letting b_n denote the number (in hundreds) of rabbits at the beginning of month n, we assume

$$\begin{aligned} b_0 &= 1 \\ b_1 &= 2 \\ b_n &= b_{n-1} + b_{n-2} \text{ for } n \geq 2. \end{aligned}$$

Then it is expected that the rabbit population growth rate stabilises as $n \to \infty$. That is, there exists $a \in (0, \infty)$ such that $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = b$. This means that in the long term, the rabbit population grows approximately *exponentially*, by a rough factor of b each month.

To prove the existence of a value a as required, we let $a_n = \frac{b_{n+1}}{b_n}$. Note that

$$a_{n+1} = \frac{b_{n+2}}{b_{n+1}} = \frac{b_{n+1} + b_n}{b_{n+1}} = 1 + \frac{b_n}{b_{n+1}} = 1 + \frac{1}{a_n}$$

Letting $g(x) = 1 + \frac{1}{x}$, we see that

$$a_{n+1} = g^{\circ n}(a_0) = g^{\circ n}(2) \text{ for all } n \geq 0$$

Claim: There exists a closed interval $I \subset \mathbb{R}$ such that $a_0 = 2$ such that

- $g(I) \subset I$
- g is a λ-contraction on I for some $\lambda \in (0, 1)$, and
- $a_0 = 2 \in I$

If we can prove this claim, then by the contraction principle, a can be recovered as the unique fixed point of g in I.

Proof of claim: The function g is decreasing on $(0, \infty)$, and has the horizontal asymptote $y = 1$. Note that $g'(x) = \frac{-1}{x^2}$.

This means $\forall x \in [c, \infty)$ where $c > 1$,

$$|g'(x)| = \frac{1}{x^2} \leq \frac{1}{c^2} < 1$$
$$\implies |g(x) - g(y)| \leq \frac{1}{c^2}|x - y| \text{ for all } x, y \in [c, \infty)$$

In other words, for all $c > 1$, $g : [c, \infty) \longrightarrow \mathbb{R}$ is a $\lambda-$contraction with $\lambda = \frac{1}{c^2}$. Also note g has a unique positive fixed point x_0: we can find it by solving the equation $g(x) = x$.

$$g(x) = x$$
$$\implies 1 + \frac{1}{x} = x$$
$$\implies x^2 - x + 1 = 0$$
$$\implies x = \frac{1 \pm \sqrt{5}}{2}$$

So $x_0 = \frac{1+\sqrt{5}}{2}$. Note that $\frac{3}{2} < x_0 < 2$.
Let $I = [\frac{3}{2}, 2]$, Then we have

$$g(2) = \frac{3}{2} \text{ and } g(3/2) = \frac{5}{3} < 2.$$

we see that $g(I) \subset I$. By the above discussion, g is a $\lambda-$contraction on I (with $\lambda = (2/3)^2$), and thus the orbit under g of $a_0 = 2$ converges to $a = x_0 = \frac{1+\sqrt{5}}{2}$.

Remark 1.3.8. The choice of I is not unique: for any $c \in (1, 3/2]$, we have $g[c, 2] = [3/2, g(c)] \subset [c, 2]$, and g is a λ contraction on $[c, 2]$ with $\lambda = \frac{1}{c^2}$. I made a small mistake in class by saying c can be in $[1, x_0]$: can you see why $c \in (3/2, x_0]$ won't work?

Local Contractions

Proposition 1.3.9. *Let f be a continuously differentiable map of $\mathbb{R}^n$ with a fixed point x_0 where $||Df_{x_0}|| < 1$. Then there is a closed neighborhood U of x_0 such that $f(U) \subset U$ and f is a contraction on U.*

To do this we will need the following exercise and proposition:

Exercise 1.3.10. Given a linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, recall that

$$||A|| = \sup_{||x|| \neq 0} \frac{||Ax||}{||x||}$$

Prove that $A \mapsto ||A||$ is a continuous function from $\mathbb{R}^{m \times n}$ to $\mathbb{R}$.

Proposition 1.3.11. *Let $V \subset \mathbb{R}^n$ be a closed disk and let $f : V \longrightarrow \mathbb{R}^m$ be continuous, with continuous derivative on the interior of V. Suppose there exists $M > 0$ such that $||Df_x|| \leq M$ for all x in the interior of V. Then*

$$d(f(x), f(y)) \leq Md(x, y) \qquad \forall x, y \in V$$

Proof. Given $x, y \in \mathbb{R}^n$, let $g : [0, 1] \longrightarrow \mathbb{R}^m$ be the function

$$g(t) = f((1 - t)x + ty)$$

Then the mean value theorem states that for some $c \in (0, 1)$,

$$d(g(0), g(1)) \leq ||g'(c)||$$

From this we get

$$\begin{aligned} d(f(x), f(y)) &= d(g(0), g(1)) \leq ||g'(c)|| \\ &= ||Df_{(1-c)x+cy}(y - x)|| \leq ||Df_{(1-c)x+cy}|| \cdot d(x, y) \\ &\leq Md(x, y) \end{aligned}$$

□

Proof of Proposition 1.3.9. The function f is C^1 implies that $x \mapsto Df$ is continuous. By Exercise 4, the composition $x \mapsto Df_x \mapsto ||Df_x||$ is continuous. Fix a point $\lambda \in \Big(||Df_{x_0}||, 1\Big)$. Then there exists a small closed ball $U = \overline{B(x_0, \delta)}$ around x_0 on which $||Df_x|| \leq \lambda < 1$.

By Proposition 1.3.11, if $x, y \in U$, then $d(f(x), f(y)) \leq \lambda d(x, y)$. Moreover, for all $x \in U$, we have

$$d(f(x), x_0) = d(f(x), f(x_0)) \leq \lambda d(x, x_0) \leq \lambda\delta < \delta.$$

This shows that $f(U) \subset U$, and f is a $\lambda-$contraction on U.

□

Suggested Reading 1.3.12.

- [3, Section 2.2]

1.4 Fractals

In this section we will define fractals and introduce self-similarity. We will also give an idea of their connection with dynamical systems with some examples.

The Cantor Set

The simplest example of a fractal is the ternary cantor set.

Definition 1.4.1. Let $I = [0,1]$. Inductively define closed subsets $C_n \subset I$ for $n \geq 0$ as follows:

$$\mathcal{C}_0 = \Big[0,1\Big]$$
$$\mathcal{C}_1 = \mathcal{C}_0 \setminus \left(\frac{1}{3},\frac{2}{3}\right) = \frac{\mathcal{C}_0}{3} \cup \left(\frac{\mathcal{C}_0}{3}+\frac{2}{3}\right)$$
$$\mathcal{C}_n = \frac{\mathcal{C}_{n-1}}{3} \cup \left(\frac{\mathcal{C}_{n-1}}{3}+\frac{2}{3}\right) = \mathcal{C}_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n},\frac{3k+2}{3^n}\right) \text{ for all } n \geq 2$$

It can be shown that the set $\mathcal{C}_n$ is the disjoint union of 2^n closed intervals, each of length $\frac{1}{3^n}$. The ternary Cantor set is defined as

$$\mathcal{C} = \bigcap_{n=0}^{\infty} \mathcal{C}_n$$

We look at some of the properties of $\mathcal{C}$.

1. $\mathcal{C}$ **is closed**, since it is the intersection of closed sets
2. $\mathcal{C}$ **is compact**, since it is a closed subset of a compact set.
3. $\mathcal{C}$ **is non-empty.**
   For example, the points 0 and 1 belong to all the sets $\mathcal{C}_n$, so they also belong to $\mathcal{C}$.
4. $\mathcal{C}$ **is uncountable.**
   We will show this by giving an explicit description of $\mathcal{C}$.

   Definition 1.4.2. Given a number $x \in [0,1]$, a base 3 (or *ternary*) expansion for x is a sequence $.\alpha_1\alpha_2\alpha_3\cdots$ with $\alpha_n \in \{0,1,2\}$ for all $n \in \mathbb{N}$ such that

   $$x = \frac{\alpha_1}{3} + \frac{\alpha_2}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$$

   The decimal point before the α's indicate that the number is less than or equal to 1. More generally, for any real number $y \in \mathbb{R}$. The ternary expansion of y is a sequence $\beta_{m-1}\beta_{m-2}\cdots\beta_0.\alpha_1\alpha_2\alpha_3\cdots$ with $\beta_i \in \{0,1,2\}$ for all $i \in \{0,1,2,\cdots m-1\}$ and $\alpha_n \in \{0,1,2\}$ for all $n \in \mathbb{N}$, such that

   $$y = \sum_{i=0}^{m-1} \beta_i \cdot 3^i + \sum_{n=0}^{\infty} \frac{\alpha_n}{3^n}$$

Note that the ternary expansion of a number is not unique. For example,

$$\frac{1}{3} = \sum_{n=1}^{\infty} \frac{2}{3^n}$$

So $.100000\cdots$ and $.0222222\cdots$ are both ternary expansions for $\frac{1}{3}$. Similarly, $.200000\cdots$ and $.1222222\cdots$ are both ternary expansions for $\frac{2}{3}$.

Exercise 1.4.3. Every number $x \in \mathbb{R}$ has only finitely many ternary expansions.

Remark 1.4.4. If x has a ternary expansion $.\alpha_1\alpha_2\alpha_3\cdots$, then $\frac{x}{3}$ has a ternary expansion $.0\alpha_1\alpha_2\cdots$

Proposition 1.4.5.

$$\mathcal{C} = \{x \in [0,1] : \textit{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \textit{ for } x \textit{ with } \alpha_n \in \{0,2\} \textit{ for all } n \in \mathbb{N}\}$$

Proof. We prove this by induction. Note that

$$\mathcal{C}_1 = \{x \in [0,1] : \text{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \text{ for } x \text{ with } \alpha_1 \in \{0,2\}\}$$

Using the recursive formula for $\mathcal{C}_n$, it is easy to show that

$$\mathcal{C}_N = \{x \in [0,1] : \text{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \text{ for } x \text{ with } \alpha_1, \alpha_2, \cdots, \alpha_N \in \{0,2\}\}$$

Since every x has only finitely many ternary expansions, for x to be in all the $\mathcal{C}_n$'s, there exists at least one ternary expansion which satisfies the condition $\alpha_n \in \{0,2\}$ for all $n \in \mathbb{N}$. □

Since every sequence in $\{0,2\}^{\mathbb{N}}$ can be realized as the ternary expansion of a distinct number $x \in [0,1]$, and the set $\{0,2\}^{\mathbb{N}}$ is uncountable, we see that $\mathcal{C}$ is uncountable.

5. **$\mathcal{C}$ is perfect.**

Definition 1.4.6. Let Y be a topological space. A subset $X \subset Y$ is said to be perfect if it is closed in Y and has no isolated points.

Proposition 1.4.7. *For every $x \in \mathcal{C}$, there exists a sequence (x_n) of distinct points with $x_n \in \mathcal{C}$ and $x_n \to x$.*

Proof. Given $\epsilon > 0$, we will exhibit a point $x_N \neq x$ such that $|x_N - x| < \epsilon$ and $x_N \in \mathcal{C}$. Choose $N \in \mathbb{N}$ such that $\frac{2}{3^N} < \epsilon$. This ensures that the interval of radius $\frac{1}{3^N}$ centered at x is contained in the open ball $B_\epsilon(x)$. Let $\widetilde{I}$ be the component interval of $\mathcal{C}_N$ that contains x. The above condition implies that $\widetilde{I} \subset B_\epsilon(x)$.

In $\mathcal{C}_{N+1}$, the middle third of $\widetilde{I}$ is deleted, and we get two component intervals $\widetilde{I}_0$ and $\widetilde{I}_1$. Without loss of generality, assume $x \in \widetilde{I}_0$. Then pick a point $y \in \mathcal{C} \cap \widetilde{I}_1$. Note that $y \neq x$ by this choice and since $\widetilde{I}_1 \subset B_\epsilon(x)$, we have $|y - x| < \epsilon$. Therefore we can set $x_N = y$. □

6. $\mathcal{C}$ **is totally disconnected.**

   Definition 1.4.8. A topological space X is said to be totally disconnected if its only non-empty connected subsets are singletons.

   Proposition 1.4.9. *If $F \subset \mathcal{C}$ is non-empty and connected, then $F = \{x\}$ for some point $x \in \mathcal{C}$.*

   Proof. Suppose $x, y \in \mathcal{C}$ are two distinct points in F. WLOG, assume $x < y$. Pick $N \in \mathbb{N}$ such that $\frac{1}{3^{N-1}} < |x - y|$. Then, x and y are contained in distinct components of C_N. So there exists $z \in (x, y)$ such that $z \notin \mathcal{C}$. Let $A = F \cap [0, z)$ and $B = F \cap (z, 1]$. Note that $A \cup B = F$. Also note that the closures of A and B don't intersect. This contradicts the fact that F is connected. $\square$

7. $\mathcal{C}$ **has Lebesgue measure** 0**.**
   Let μ denote Lebesgue measure. The set $\mathcal{C}_n$ is the union of 2^n disjoint intervals, each of length 3^{-n}. Therefore, we have $\mu(\mathcal{C}_n) = \frac{2^n}{3^n}$. Since $\mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3 \supset \cdots$ and $\mathcal{C}$ is the intersection of the $\mathcal{C}_n$, we have

$$\mu(\mathcal{C}) = \lim_{n \to \infty} \mu(\mathcal{C}_n) = 0$$

The following theorem is the main result of this section. For a proof see [2].

Theorem 1.4.10 (Brouwer)**.** *Let $Y \neq \emptyset$ be a complete metric space. If Y is compact, perfect and totally disconnected, then it is homeomorphic to $\mathcal{C}$.*

An easy corollary, for example, is that $\mathcal{C}$ is homeomorphic to $\mathcal{C} \times \mathcal{C}$.

Suggested Reading 1.4.11.

- [3, Section 2.7.1]

Dynamical Systems on the Cantor Set

Consider the map $f : [0, 1] \times [0, 1]$ defined as $f(x) = \frac{x}{3}$. It is easy to see that f is a contraction and $f(\mathcal{C}) \subset \mathcal{C}$, and the unique fixed point in $\mathcal{C}$ is $x = 0$. Note that for every $x \in \mathcal{C}$, there exists a neighborhood U of x such that $f : U \longrightarrow f(U)$ is a homeomorphism.
Also note that f induces the shift $.\alpha_1\alpha_2\alpha_3 \cdots \to .0\alpha_1\alpha_2\alpha_3 \cdots$ on ternary expansions.

Definition 1.4.12. A topological space X is said to be *self-similar*, or to have the *rescaling property*, if there exists a contraction $f : X \longrightarrow X$ such that for every $x \in X$ and neighborhood U of x, there exists a neighborhood $V \subset U$ of x such that $f : V \longrightarrow f(V)$ is a homeomorphism.

Remark 1.4.13. This is actually equivalent to saying that every $x \in X$ has a neghborhood U such that $f : U \longrightarrow f(U)$ is a homeomorphism. Also note that the term self similar is used in different ways in the literature; we will see by and by that this definition is not extensive enough.

Exercise 1.4.14. Show that the function $f(x) = 1 - \frac{x}{3}$ leaves $\mathcal{C}$ invariant, and is a contraction. Describe the induced operation on ternary expansions, and find the unique fixed point of f in $\mathcal{C}$.

Exercise 1.4.15. Show that the function $f(x) = \frac{x+2}{3} \pmod 1$ leaves $\mathcal{C}$ invariant, and is a contraction. Describe the induced operation on ternary expansions, and find the unique fixed point of f in $\mathcal{C}$.

The Square Sierpinksi Carpet

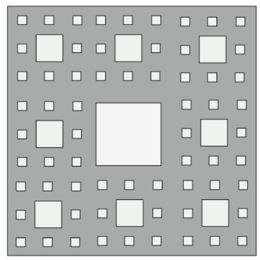

Figure 1.3: The set $\mathcal{J}_3$ of the Sierpinski carpet construction

Definition 1.4.16. Let $J = [0,1] \times [0,1]$ be the unit square. Define

$$\begin{aligned}
\mathcal{J}_0 &= J \\
\mathcal{J}_1 &= \mathcal{J}_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) \times \left(\frac{1}{3}, \frac{2}{3}\right) \\
\mathcal{J}_n &= \mathcal{J}_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \bigcup_{\ell=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) \times \left(\frac{3\ell+1}{3^n}, \frac{3\ell+2}{3^n}\right)
\end{aligned}$$

The square Sierpinski carpet is the set $\mathcal{J} = \bigcap_{n=0}^{\infty} \mathcal{J}_n$.

Exercise 1.4.17. Prove that the Sierpinski carpet is self-similar.

The Sierpinski Triangle

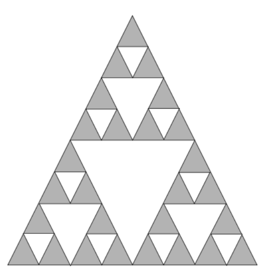

Figure 1.4: The set Δ_2 of the Sierpinski triangle construction

This set is similar to the Sierpinski carpet. Start with an equilateral triangle Δ_0 of side length 1, with one side horizontal. Let Δ_1 be Δ_0 minus its central equilateral triangle, Δ_2 be Δ_1 minus its three smaller central equilateral triangles and so on.
Define the Sierpinski triangle

$$\Delta = \bigcap_{n=0}^{\infty} \Delta_n$$

Exercise 1.4.18. Prove that the Sierpinski triangle is self-similar.

Exercise 1.4.19. Prove that $\mathcal{J}$ and Δ both have infinite perimeter and finite area (Lebesgue measure).

Exercise 1.4.20. Prove that neither $\mathcal{J}$ nor Δ is homeomorphic to $\mathcal{C}$.

Suggested Reading 1.4.21.

- [3, Section 2.7.2]

Cantor Sets and Logistic Maps

Consider the logistic function $f(x) = 5x(1-x)$ on $\mathbb{R}$. Note that this function has one critical point at $x = \frac{1}{2}$, and is symmetric around this point in the sense that

$$f(x) = f(1-x) \text{ for all } x \in \mathbb{R}$$

We make the following series of observations:

1. The graph of f is a downward drawn parabola, and its roots are $x = 0, 1$.
2. If $x > 1$, then $f(x) < 0$.
3. If $x < 0$, then $f(x) < x$ and $|f(f(x)) - f(x)| > |f(x) - x|$.

The points (2) and (3) show that if $x \notin [0,1]$, then $f^{\circ n}(x) \to -\infty$. This leads to the following dichotomy:
For every $x \in \mathbb{R}$, exactly one of the following is true:

- either $f^{\circ n}(x) \in [0,1]$ for all $n \in \mathbb{N}$, or
- $f^{\circ m}(x) \notin [0,1]$ for some $m \in \mathbb{N}$, and thus, $f^{\circ n}(x) \to -\infty$ as $n \to \infty$.

Therefore, the set B of points $x \in \mathbb{R}$ such the orbit $(f^{\circ n}(x))_{n \geq 0}$ is bounded, is the set of points x such that $f^{\circ n}(x) \in [0,1]$ for all n. In other words,

$$B = \bigcap_{n=0}^{\infty} (f^{\circ n})^{-1}[0,1]$$

Proposition 1.4.22. *B is a Cantor set (i.e., it is homeomorphic to $\mathcal{C}$).*

We will prove this in the next chapter.

1.5 Topological Conjugacy

Definition 1.5.1. Let X, Y be topological spaces and suppose $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$ be dynamical systems. Then (X, f) and (Y, g) are said to be *topologically conjugate* if there exists a homeomorphism $\varphi : X \longrightarrow Y$ such that

$$g \circ \varphi = \varphi \circ f$$

In other words, φ is a homeomorphism that makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \big\downarrow f & & \big\downarrow g \\ X & \xrightarrow{\varphi} & Y \end{array}$$

Note that for all $n \in \mathbb{N}$,

$$g^{\circ n} = (\varphi \circ f \circ \varphi^{-1})^{\circ n} = (\varphi \circ f \circ \varphi^{-1}) \circ (\varphi \circ f \circ \varphi^{-1}) \circ \cdots \circ (\varphi \circ f \circ \varphi^{-1}) = \varphi \circ f^{\circ n} \circ \varphi^{-1}$$

In particular, φ maps the $f-$orbit of x to the $g-$orbit of $\varphi(x)$ for every $x \in X$. So topological conjugacy is a form of equivalence between two dynamical systems.

Examples

Example 1.5.2. The map $\varphi(x) = \frac{-1}{2}x + \frac{1}{2}$ conjugates the dynamical systems $f(x) = x^2$ and $g(x) = 2x(1-x)$ on $\mathbb{R}$. Since φ is linear, we say that $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$ are *linearly/affinely* conjugate.

Exercise 1.5.3. Prove that every quadratic polynomial $f : \mathbb{C} \longrightarrow \mathbb{C}$ is affine conjugate to a polynomial of the form $z^2 + c$ for a unique $c \in \mathbb{C}$.

Exercise 1.5.4. Let W and V be vector spaces over $\mathbb{R}$, and suppose $A : W \longrightarrow W$ and $B : V \longrightarrow V$ are linear maps. Show that if A and B are conjugate as linear maps, ie, there exists an invertible linear map $L : W \longrightarrow V$ such that $LA = BL$, then they are also topologically conjugate in the sense defined in the previous section.

Logistic Map Revisited

We now revert back to our previous discussion of Cantor sets and the logistic map $f(x) = 5x(1-x)$.

Let $\Sigma = \{0,1\}^{\mathbb{N}} = \{s = s_1 s_2 \cdots | s_i \in \{0,1\} \forall i \in \mathbb{N}\}$ and $\sigma : \Sigma \to \Sigma$ be the map $s_1 s_2 s_3 \cdots \mapsto s_2 s_3 \cdots$. We will equip Σ with a topology under which σ is continuous.

Theorem 1.5.5. *Given $\mu \neq 0$, let $g_\mu(x) = \mu x(1-x)$, and let $B_\mu \subset \mathbb{R}$ be the set of points with bounded orbits under g_μ.*

For $\mu > 4$, the dynamical systems (B_μ, g_μ) and (Σ, σ) are topologically conjugate.

Theorem 1.5.6. *The set Σ is homeomorphic to the ternary Cantor set $\mathcal{C}$.*

These theorems imply Proposition 1.4.22. In the next chapter, we will prove Theorem 1.5.6, and Theorem 1.5.5 for the smaller range $\mu > 2 + \sqrt{5} > 4$.

Chapter 2

Symbolic Dynamics

To prove Theorems 1.5.5 and 1.5.6, we will need the powerful machinery of symbolic dynamics. In the next section we will introduce its basic concepts.

2.1 Sequences over a finite alphabet

Let (X, d_X) be a metric space, and $A \subset X$ be a finite set with $|A| \geq 2$.

Definition 2.1.1. The set of sequences with alphabet A is denoted Σ_A. In other words,

$$\Sigma_A = A^{\mathbb{N}} = \{s = s_1 s_2 s_3 \cdots | s_j \in A \forall j \in \mathbb{N}\}$$

Topology on the Space of Sequences

We define a metric on Σ_A as follows: for all $s, t \in \Sigma_A$, we let

$$d(s,t) = \sum_{j=1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}} \tag{2.1}$$

Proposition 2.1.2. *The function $d : \Sigma_A \times \Sigma_A \longrightarrow \mathbb{R}$ is well-defined.*

Proof. We need to show that for all $s, t \in \Sigma_A$, the infinite series given above converges. Let $M = \max_{p,q \in A} d_X(p, q)$.

$$\begin{aligned} d(s,t) &= \sum_{j=1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}} \\ &\leq \sum_{j=1}^{\infty} \frac{M}{|A|^{j-1}} = M \sum_{j=1}^{\infty} \frac{1}{|A|^{j-1}} \\ &= \frac{M}{1 - \frac{1}{|A|}} \\ &< \infty \end{aligned}$$

□

Proposition 2.1.3. *The function defined by Equation 2.1 is a metric on Σ_A.*

Proof. We need to show the following:

1. for all $s, t \in \Sigma_A$, $d(s,t) \geq 0$;
2. for all $s, t \in \Sigma_A$, $d(s,t) = 0 \iff s = t$;
3. for all $s, t, r \in \Sigma_A$, $d(s,r) \leq d(s,t) + d(t,r)$.

We will see these one by one.

1. is clear since every term in the infinite sequence defining $d(s,t)$ is non-negative.
2. is clear since if $d(s,t) = 0$, then $d_X(s_j, t_j) = 0$ for all $j \in \mathbb{N}$, which implies $s_j = t_j$ for all $j \in \mathbb{N}$.
3. for every $j \in \mathbb{N}$, $d_X(s_j, r_j) \leq d_X(s_j, t_j) + d_X(t_j, r_j)$. This immediately shows (3).

□

The metric d induces a topology on Σ_A. We will see some properties of this topology in the remaining section.

Remark 2.1.4. By scaling the metric d_X if necessary, from now on we assume without loss of generality that $M = \max_{p,q \in A} d_X(p,q) = 1$.

Proposition 2.1.5. *Suppose $s, t \in \Sigma_A$ satisfy $s_j = t_j$ for $j = 1, 2, \cdots, N$. Then*

$$d(s,t) < \frac{1}{|A|^{N-1}(|A|-1)} \leq \frac{1}{|A|^{N-1}}$$

Proof. The second inequality follows directly since $\frac{1}{|A|-1} \leq 1$.
Since $s_j = t_j$ for $j = 1, \cdots, N$,

$$\begin{aligned}
d(s,t) &= \sum_{j=1}^{N} \frac{d_X(s_j,t_j)}{|A|^{j-1}} + \sum_{j=N+1}^{\infty} \frac{d_X(s_j,t_j)}{|A|^{j-1}} \\
&= \sum_{j=N+1}^{\infty} \frac{d_X(s_j,t_j)}{|A|^{j-1}} \\
&\leq \sum_{j=N+1}^{\infty} \frac{1}{|A|^{j-1}} \\
&= \frac{1}{|A|^N} \frac{1}{1 - \frac{1}{|A|}} = \frac{|A|}{|A|^N(|A|-1)} \\
&= \frac{1}{|A|^{N-1}(|A|-1)}
\end{aligned}$$

□

Proposition 2.1.6. *There exists a constant $\ell = \ell(A) > 0$ such that if $s, t \in \Sigma_A$ satisfy $d(s,t) < \frac{\ell}{|A|^{N-1}}$, then $s_j = t_j$ for $j = 1, 2, \cdots, N$.*

Proof. Let $\ell = \min_{\substack{p,q\in A\\ p\neq q}} d_X(p,q)$. We will prove the contrapositive. Given $s, t \in \Sigma_A$, if $s_j \neq t_j$ for some $j \in \{1, 2, \cdots, N\}$, then

$$d(s,t) \geq \frac{d_X(s_j,t_j)}{|A|^{j-1}} \geq \frac{\ell}{|A|^{j-1}} \geq \frac{\ell}{|A|^{N-1}}$$

□

2.2 Shift Operator on Sequences

Definition 2.2.1. The *shift operator* $\sigma : \Sigma_A \longrightarrow \Sigma_A$ is defined as

$$\sigma(s_1 s_2 s_3 \cdots) = s_2 s_3 s_4 \cdots \text{ for all } s = s_1 s_2 s_3 \cdots \in \Sigma_A \tag{2.2}$$

Proposition 2.2.2. *The map σ is surjective and uniformly continuous.*

Proof. Given $s \in \Sigma_A$, for any $a \in A$, $\sigma(a s_1 s_2 s_3 \cdots) = s$. Therefore σ is surjective.
To show it is uniformly continuous, we will exhibit for a given $\epsilon > 0$, a constant $\delta > 0$ such that for all $s, t \in \Sigma_A$, $d(s,t) < \delta \implies d(\sigma(s), \sigma(t)) < \epsilon$.

Fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{|A|^{N-1}} < \epsilon$. Let $\delta = \frac{\ell}{|A|^N}$, where ℓ is the constant from Proposition 2.1.6. Then, we have $s_j = t_j$ for $j = 1, 2, \cdots, N+1$. Let $\sigma(s) = \underline{s}$ and $\underline{t} = \sigma(t)$. Note that $\underline{s}_j = s_{j+1}$ and $\underline{t}_j = t_{j+1}$ for all $j \in \mathbb{N}$. The above condition implies that $\underline{s}_j = \underline{t}_j$ for $j = 1, 2, \cdots, N$. Therefore by Proposition 2.1.5, we have $d(\underline{s}, \underline{t}) < \frac{1}{|A|^{N-1}} < \epsilon$. □

Periodic Sequences

Definition 2.2.3. For $m \in \mathbb{N}$, define

$$\mathrm{Per}_m(\sigma) = \{s \in \Sigma_A | \sigma^{\circ m}(s) = s\}$$

In other words, $\mathrm{Per}_m(\sigma)$ is the set of sequences whose period under σ divides m.
Also define $\mathrm{Per}(\sigma)$ to be the set of sequences periodic under σ.

Remark 2.2.4. The following properties of periodic sequences are immediate.

1. $$\mathrm{Per}(\sigma) = \bigcup_{m=1}^{\infty} \mathrm{Per}_m(\sigma)$$

2. Given a finite word $w = s_1 s_2 \cdots s_n$ with $s_i \in A$ for all i, we let $\overline{w}$ denote the infinite word $s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n \cdots$ formed by repeating the finite block w.
   Given $m \in \mathbb{N}$,
   $$\mathrm{Per}_m(\sigma) = \{\overline{s_1 s_2 \cdots s_m} : s_j \in A \text{ for } j = 1, 2, \cdots, m\}.$$
   This shows that
   $$|\mathrm{Per}_m(\sigma)| = |A|^m$$

3. If $m < n$ and $m|n$, then
$$\mathrm{Per}_m(\sigma) \subsetneq \mathrm{Per}_n(\sigma)$$

Proposition 2.2.5. $\mathrm{Per}(\sigma)$ *is dense in* Σ_A.

Proof. Given $s \in \Sigma_A$ and $\epsilon > 0$, pick $N \in \mathbb{N}$ such that $\frac{1}{|A|^{N-1}} < \epsilon$. Then by Proposition 2.1.5, the sequence $t \in \mathrm{Per}(\sigma)$ given by

$$t = s_1 s_2 \cdots s_N s_1 s_2 \cdots s_N s_1 s_2 \cdots = \overline{s_1 s_2 \cdots s_N}$$

satisfies
$$d(s,t) < \frac{1}{|A|^{N-1}} < \epsilon$$

□

Suggested Reading 2.2.6. • [1, Section 1.6]

Logistic Maps Conjugate to the Shift

In this section we will establish Theorem 1.5.5 for the family of maps $g_\mu(x) = \mu x(1-x)$ where $\mu > 2+\sqrt{5}$. The proof for the full range $\mu > 4$ uses techniques from complex analysis, so we will see this later.

Recall the definition of the set B_μ: this is the set of points x with bounded orbit under the map g_μ. Just as we did for $\mu = 5$, we will show that for a range of μ values, the set $B_\mu \subset [0,1]$.

Proposition 2.2.7. *When* $\mu > 1$, *for* $x \notin [0,1]$, $g_\mu^{\circ n}(x) \to -\infty$ *as* $n \to \infty$.

Proof. First, observe that the graph of g_μ is a downward drawn parabola which intersects the x-axis at the two roots $x = 0, 1$.

If $x < 0$, then $g_\mu(x) = \mu x - \mu x^2 < \mu x < x$. So the terms of the orbit $g_\mu^{\circ n}(x)$ become more and more negative as n increases. Now we show that the monotone decreasing sequence $x, g_\mu(x), g_\mu^{\circ 2}(x), \cdots$ does not stay bounded. Suppose to the contrary, then there exists $p < 0$ such that $g_\mu^{\circ n}(x) \to p$. On the one hand we have $g_\mu^{\circ(n+1)}(x) \to g_\mu(p) < p$, but on the other hand, the sequence $(g_\mu^{\circ(n+1)}(x))_{n\geq 0}$, as a tail of the sequence $(g_\mu^{\circ n}(x))_{n\geq 0}$, also converges to p. This proves that $g_\mu^{\circ(n+1)}(x) \to -\infty$.

If $x > 1$, then $g_\mu(x) < 0$. By the discussion above, $g_\mu^{\circ n}(x) \to -\infty$. □

Proposition 2.2.8. *For* $\mu > 1$,

$$B_\mu = \bigcap_{n\geq 0} (g_\mu^{\circ n})^{-1}[0,1]$$

If $1 < \mu \leq 4$, *then* $B_\mu = [0,1]$.

Proof. The previous proposition shows that $B_\mu \subseteq [0,1]$ for all $\mu > 1$, and moreover, that $x \in B_\mu$ if and only if $g_\mu^{\circ n}(x) \in [0,1]$ for all $n \in \mathbb{N}$.
In other words,
$$B_\mu = \bigcap_{n \geq 0} \{x \in \mathbb{R} : g_\mu^{\circ n}(x) \in [0,1]\} = \bigcap_{n \geq 0} (g_\mu^{\circ n})^{-1}[0,1].$$
Note that $x = \frac{1}{2}$ is the unique point where g_μ reaches its maximum, and $g_\mu\left(\frac{1}{2}\right) = \frac{\mu}{4}$.

Thus, if $1 < \mu \leq 4$, then since $\frac{\mu}{4} \leq 1$, we have
$$\begin{aligned} & g_\mu[0,1] \subseteq [0,1] \\ \implies & [0,1] \subseteq g_\mu^{-1}[0,1] \end{aligned}$$

Since we know that $g_\mu^{-1}[0,1] \subseteq [0,1]$, this shows that $B_\mu = [0,1]$. $\square$

Thus the interesting structure of B_μ occurs when $\mu > 4$.

Proposition 2.2.9. *Fix $\mu > 4$. Let $c_\mu = \sqrt{\frac{1}{4} - \frac{1}{\mu}}$, and define the disjoint intervals $I_0 = \left[0, \frac{1}{2} - c_\mu\right]$ and $I_1 = \left[\frac{1}{2} + c_\mu, 1\right]$. Then*
$$g_\mu^{-1}[0,1] = I_0 \cup I_1$$

Proof. Solving $g_\mu(x) = 1$, we get
$$\begin{aligned} \mu x - \mu x^2 &= 1 \\ \implies \mu x^2 - \mu x + 1 &= 0 \\ \implies x &= \frac{\mu \pm \sqrt{\mu^2 - 4\mu}}{2\mu} \\ &= \frac{1}{2} \pm \sqrt{\frac{\mu^2 - 4\mu}{4\mu^2}} \\ &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{\mu}} \\ &= \frac{1}{2} \pm c_\mu \end{aligned}$$
Note that $g_\mu[0,1] = \left[0, \frac{\mu}{4}\right]$. Since $\frac{1}{2}$ is the point where g_μ is maximum, and the graph of g_μ is symmetric about the vertical line $x = \frac{1}{2}$, we get
$$g_\mu\left(\frac{1}{2} - c_\mu, \frac{1}{2} + c_\mu\right) = \left(1, \frac{\mu}{4}\right]$$
Thus, we have, for I_0 and I_1 as above, that
$$g_\mu(I_0 \cup I_1) = \left[0, \frac{\mu}{4}\right] \setminus \left(1, \frac{\mu}{4}\right] = [0,1]$$
$\square$

Note that the intervals I_0 and I_1 above are disjoint. Note that $g_\mu(I_0) = g_\mu(I_1) = [0, 1]$, so $(g_\mu^{\circ 2})^{-1}[0, 1] = g_\mu^{-1}(I_0 \cup I_1) = I_{00} \cup I_{01} \cup I_{10} \cup I_{11}$. here $I_{00} \cup I_{01} =$.

Definition 2.2.10. We introduce the notation Σ^0 for the set of finite non-empty words over the alphabet $\{0, 1\}$. Formally,

$$\Sigma^0 = \bigcup_{N \geq 1} \{w = s_1 s_2 \cdots s_N | s_i \in \{0, 1\} \text{ for } i = 1, 2, \cdots, N\} = \bigcup_{N \geq 1} \{0, 1\}^N$$

We let $\ell(w)$ denote the length of the finite word w.

Definition 2.2.11. Given $w = s_1 s_2 \cdots s_N \in \Sigma^0$, define the set $I_w \subseteq [0, 1]$ as follows:

$$\begin{aligned} I_w &= \{x \in [0, 1] : x \in I_{s_1}, g_\mu(x) \in I_{s_2}, \cdots, g_\mu^{\circ(n-1)}(x) \in I_{s_n}\} \\ &= \bigcap_{j=1}^{N} \{x \in [0, 1] : g_\mu^{\circ(j-1)}(x) \in I_{s_j}\} \\ &= \bigcap_{j=1}^{N} (g_\mu^{\circ(j-1)})^{-1}(I_{s_j}) \end{aligned}$$

Proposition 2.2.12. *The collection of intervals $\{I_w : w \in \Sigma^0\}$ satisfies*

1. *Given $w = s_1 \cdots s_N \in \Sigma^0$ and a symbol $s_{N+1} \in \{0, 1\}$, we have $I_{ws_{N+1}} \subseteq I_w$.*
2. *Given $w = s_1 \cdots s_N \in \Sigma^0$ and a symbol $s_0 \in \{0, 1\}$, the map g_μ maps $I_{s_0 w}$ homeomorphically onto I_w.*
3. *I_w is a closed interval of non-zero length for all $w \in \Sigma^0$.*
4. *Given distinct words $w_1, w_2 \in \{0, 1\}^N$, $I_{w_1} \cap I_{w_2} = \emptyset$.*
5. *for all $N \in \mathbb{N}$, $(g_\mu^{\circ N})^{-1}[0, 1] = \bigcup_{w \in \{0,1\}^N} I_w$.*

Proof. We will show them one by one.

1. Note that $I_{ws_{N+1}} = I_w \cap \{x : g_\mu^{\circ N}(x) \in I_{s_{N+1}}\} \subseteq I_w$.
2. Note that $I_{s_0 w} = \{x : x \in I_{s_0} \text{ and } g_\mu(x) \in I_w\}$. In other words, $I_{s_0 w}$ is the full pre-image of I_w in either I_0 or I_1, depending on the value of s_0. Since g_μ is monotonic on both I_0 and I_1, and $g_\mu(I_0) = g_\mu(I_1) = [0, 1]$, we get that $g_\mu(I_{s_0 w}) = [0, 1]$, and that the mapping is a homeomorphism.
3. By definition, I_w is an intersection of closed sets, and by the previous point, by induction on $\ell(w)$, it is easy to see that it is a closed interval with non-zero length.
4. Without loss of generality suppose the jth entry of w_1 and w_2 are 0 and 1 respectively. Then $g_\mu^{\circ(i-1)} I_{w_1} \subseteq I_0$ and $g_\mu^{\circ(i-1)} I_{w_2} \subseteq I_1$. Since for a point x we cannot have $g_\mu^{\circ(i-1)}$ in both I_0 and I_1, this shows that $I_{w_1} \cap I_{w_2} = \emptyset$.

5. We do this by inducting on N. When $N=1$, $g_\mu^{-1}[0,1] = I_0 \cup I_1$.
Induction hypothesis: the statement is true for N.
Induction step: for $N+1$,

$$\begin{aligned}
(g_\mu^{\circ(N+1)})^{-1}[0,1] &= g_\mu^{-1}\Big((g_\mu^{\circ N})^{-1}[0,1]\Big) = g_\mu^{-1}\Big(\bigcup_{w\in\{0,1\}^N} I_w\Big) \\
&= \bigcup_{w\in\{0,1\}^N} g_\mu^{-1} I_w \\
&= \bigcup_{w\in\{0,1\}^N} (I_0 \cap g_\mu^{-1} I_w) \cup (I_1 \cap g_\mu^{-1} I_w) && \text{(since } g_\mu^{-1} I_w \subseteq I_0 \cup I_1\text{)} \\
&= \bigcup_{w\in\{0,1\}^N} I_{0w} \cup I_{1w} && \text{(by the proof of point (2))} \\
&= \bigcup_{w\in\{0,1\}^{N+1}} I_w
\end{aligned}$$

□

From here, let $\Sigma = \Sigma_{\{0,1\}}$.

Proposition 2.2.13. *Fix $\mu > 2+\sqrt{5}$. Given any $s_1 s_2 \cdots =: s \in \Sigma$, there exists a unique point $x_s \in B_\mu$ such that*

$$\bigcap_{N=1}^{\infty} I_{s_1 s_2 \cdots s_N} = \{x_s\}$$

Proof. We first show that it suffices to prove the following claim:
Claim 1. There exists a constant $\lambda > 1$ such that for all $N \in \mathbb{N}$ and all $w \in \{0,1\}^N$,

$$\operatorname{diam}(I_w) \leq \frac{1}{\lambda^{N-1}} \cdot \operatorname{diam}(I_0)$$

Proposition 2.2.12 implies $I_{s_1} \supseteq I_{s_1 s_2} \supseteq I_{s_1 s_2 s_3} \cdots$. Therefore, the infinite intersection diam $\left(\bigcap_{N=1}^{\infty} I_{s_1 s_2 \cdots s_N}\right)$ is a non-empty closed set. By Claim 1, diam $\left(\bigcap_{N=1}^{\infty} I_{s_1 s_2 \cdots s_N}\right) = 0$, so this infinite intersection is a singleton $\{x_s\}$. This point x_s is in B_μ since $g_\mu^{\circ N}(x_s) \in I_{s_N} \subset [0,1]$ for all $N \in \mathbb{N}$.

Now it is left for us to prove claim 1. We first show the following:
Claim 2. $|g_\mu'(x)| > \sqrt{\mu^2 - 4\mu} > 1$ for all x in the interiors of I_0 and I_1.

Proof of Claim 2. If x is in the interior of I_0 or I_1, then

$$\begin{aligned}
|g'(x)| &= |\mu(1-2x)| = 2\mu\left|\frac{1}{2} - x\right| \\
&> 2\mu c_\mu = 2\mu \cdot \sqrt{\frac{1}{4} - \frac{1}{\mu}} \\
&= \sqrt{\mu^2 - 4\mu} > \sqrt{(2+\sqrt{5})^2 - 4(2+\sqrt{5})} \quad \text{(since } \mu \mapsto \mu^2 - 4\mu \text{ is increasing for } \mu > 2\text{)} \\
&= \sqrt{4+5-8} = 1
\end{aligned}$$

$\blacksquare$

Thus for all $x \in I_0 \cup I_1$, we have $|g_\mu'(x)| \geq \sqrt{\mu^2 - 4\mu} > 1$.

Proof of Claim 1. Let $\lambda = \sqrt{\mu^2 - 4\mu}$. Given $w \in \{0,1\}^N$ with $w = s_1 s_2 s_3 \cdots s_N$, note that since $g_\mu^{\circ(N-1)} : I_w \longrightarrow I_{s_1}$ is a diffeomorphism, looking at the inverse map $f = (g_\mu^{\circ(N-1)})^{-1}$ and using the fact that $|f'(x)| = \frac{1}{\left|\left(g_\mu^{\circ(N-1)}\right)'(f^{-1}(x))\right|}$, for all $x, y \in I_{s_1}$,

$$|f(x) - f(y)| \leq |f'(x)||x - y| \leq \frac{1}{\lambda^{N-1}}|x - y|$$

Thus,

$$\operatorname{diam}(I_w) \leq \frac{1}{\lambda^{N-1}} \operatorname{diam}(I_{s_1}) = \frac{1}{\lambda^{N-1}} \operatorname{diam}(I_0)$$

$\blacksquare$

This finishes the proof of the proposition. $\square$

Definition 2.2.14. Fix $\mu > 2 + \sqrt{5}$. Define a map $\varphi : \Sigma \longrightarrow B_\mu$ by setting $\varphi(s) = x_s$ for all $s \in \Sigma$.

Proposition 2.2.15. *φ is a homeomorphism.*

Proof. *φ is injective:* If $s \neq t$, choose $N \in \mathbb{N}$ such that $s_N \neq t_N$. Since $\varphi(s) = x_s \in I_{s_1 s_2 \cdots s_N}$ and $\varphi(t) = x_t \in I_{t_1 \cdots t_N}$, and by the condition $s_N \neq t_N$ we have $I_{s_1 \cdots s_N} \cap I_{t_1 \cdots t_N} = \emptyset$, we must have $\varphi(s) \neq \varphi(t)$.

φ is surjective: If $x \in B_\mu$, for all $n \in \mathbb{N}$, let $s_n = 0$ if $g_\mu^{\circ(n-1)}(x) \in I_0$ and $s_n = 1$ if $g_\mu^{\circ(n-1)}(x) \in I_1$. Then it is easy to check that $\varphi(s_1 s_2 s_3 \cdots) = x$.

φ is continuous: Given $s \in \Sigma$ and $\epsilon > 0$, since the diameter of I_w tends to 0 as $\ell(w) \to \infty$, choose $N \in \mathbb{N}$ such that $I_{s_1 s_2 \cdots s_N} \subset B_\epsilon(x)$. Then set $\delta = \frac{1}{2^N}$. By Proposition 2.1.6, if $d(s,t) < \delta$, then $t_j = s_j$ for $j = 1, 2, \cdots N$. Thus $\varphi(t) \in I_{t_1 \cdots t_N} = I_{s_1 \cdots s_N}$, and by our assumption on N, we have $|\varphi(t) - \varphi(s)| < \epsilon$.

φ^{-1} is continuous: Given $x \in B_\mu$ and $\epsilon > 0$, let $s = \varphi^{-1}(x)$. Choose $N \in \mathbb{N}$ such that $\frac{1}{2^{N-1}} < \epsilon$, and choose $\delta > 0$ such that $B_\delta(x) \subset I_{s_1 s_2 \cdots s_N}$. Then, for any $y \in B_\delta(x) \cap B_\mu$, the sequence $t = \varphi^{-1}(y)$ satisfies $t_j = s_j$ for $j = 1, 2, \cdots, N$. By Proposition 2.1.5, we know that $d(s,t) < \frac{1}{2^{N-1}} < \epsilon$. $\square$

Proposition 2.2.16. *φ conjugates σ to g_μ.*

Proof. For all $s \in \Sigma$,

$$\begin{aligned} \varphi(s) &\in \bigcap_{N=1}^{\infty} I_{s_1 \cdots s_N} \\ \Longrightarrow \quad g_\mu(\varphi(s)) &\in \bigcap_{N=1}^{\infty} g_\mu(I_{s_1 \cdots s_N}) = \bigcap_{N=1}^{\infty} I_{s_2 s_3 \cdots s_N} \\ &= \{\varphi(\sigma(s))\} \end{aligned}$$

In other words, $g_\mu \circ \varphi = \varphi \circ \sigma$. $\square$

Propositions 2.2.15 and 2.2.16 together prove Theorem 1.5.5.

Now let A be a finite alphabet with $|A| \geq 2$

Proposition 2.2.17. *The space Σ_A is a complete metric space.*

Proof. We already know that Σ_A is a metric space. To see that it is complete, we need to show that every Cauchy sequence converges.

Let $(s^n)_{n\geq 0}$ be a Cauchy sequence. Note that each s^n is a sequence of the form $s_1^n s_2^n s_3^n \cdots$ with $s_j^n \in A$ for all $j \in \mathbb{N}$.

Claim 1. For every $j \in \mathbb{N}$, the terms s_j^n are eventually constant as $n \to \infty$.

Proof of Claim 1. Fix j. Let $\ell = \min_{\substack{p,q\in A \\ p\neq q}} d_X(p,q)$. Since $(s^n)_{n\geq 0}$ is Cauchy, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(s^n, s^m) < \frac{\ell}{|A|^{j-1}}$.

By Proposition 2.1.6, we have $s_j^n = s_j^m$ for all $m, n \geq N$. $\blacksquare$

Due to Claim 1, we can define for every j the symbol $s_j = \lim_{n\to\infty} s_j^n$. Since the sequence $(s_j^n)_{n\geq 0}$ is eventually constant, $s_j \in A$. Consider the sequence $s \in \Sigma_A$ given by $s = s_1 s_2 s_3 \cdots$.

Claim 2. The Cauchy sequence $(s^n)_{n\geq 0}$ converges to s.

Proof of Claim 2. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{|A|^{N-1}} < \epsilon$. By Claim 1, we can choose $M \in \mathbb{N}$ such that for all $n \geq M$, we have

$$s_1^n s_2^n \cdots s_N^n = s_1 s_2 \cdots s_N$$

By Proposition 2.1.5, for $n \geq M$ we have $d(s, s^n) \leq \frac{1}{|A|^{N-1}} < \epsilon$. $\blacksquare$

$\square$

Proposition 2.2.18. *1. Σ_A has bounded diameter.*

2. Σ_A is totally bounded: that is, given any $\epsilon > 0$, it can be covered by finitely many $\epsilon-$balls.

Proof. 1. For all $s, t \in \Sigma_A$, we have $d(s,t) = \sum_{j=1}^{\infty} \frac{d_X(s_j,t_j)}{|A|^{j-1}} \leq \sum_{j=1}^{\infty} \frac{1}{|A|^{j-1}} \leq \frac{1}{1-\frac{1}{|A|}}$.

2. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{|A|^{N-1}} < \epsilon$. Consider the set of words $W_N = A^N$ (these are the finite words of length N over A). Since A is finite, W_N is finite. Fix any element $a \in A$, and consider the finite set of sequences $S = \{waaa\cdots | w \in W_N\} \subset \Sigma_A$.

   Given any $t \in \Sigma_A$, there exists $s = waaaa \cdots \in S$ such that $w = t_1 t_2 \cdots t_N$. Thus

$$d(s,t) < \frac{1}{|A|^{\ell-1}} < \epsilon$$

by Proposition 2.1.5 and our choice of N.

Thus the finite collection of $\epsilon-$balls $\{B_\epsilon(s) : s \in S\}$ covers Σ_A.

$\square$

Corollary 2.2.19. *Σ_A is compact.*

Remark 2.2.20. In class I claimed that bounded diameter of Σ_A is sufficient for compactness; this is in fact not true, and uniform boundedness is necessary.

Proof. A metric space is compact if and only if it is complete and totally bounded. $\square$

Proposition 2.2.21. *Σ_A is perfect.*

We will prove this proposition by showing that there exists an element $s \in \Sigma_A$ such that every $t \in \Sigma_A$ can be approximated by elements of the form $\sigma^{\circ n}(s)$.

Suggested Reading 2.2.22.

- [1, Section 1.5]
- [3, Section 7.3.4, Section 7.4.3]

Topological Transitivity

Definition 2.2.23. Let X be a topological space and $f : X \longrightarrow X$ be a dynamical system on X.

- If f is not invertible, it is said to be topologically transitive if there exists a point $x_0 \in X$ such that the orbit $(f^{\circ n}(x_0))_{n \geq 0}$ is dense in X.
- If f is invertible, it is said to be topologically transitive if there exists a point $x_0 \in X$ such that the *grand orbit* $(f^{\circ n}(x_0))_{n \in \mathbb{Z}}$, which is the union of the forward and backward orbits of x_0, is dense in X.

Proposition 2.2.24. *The shift operator $\sigma : \Sigma_A \longrightarrow \Sigma_A$ is topologically transitive.*

Corollary 2.2.25. *Σ_A is perfect.*

Proof of Proposition 2.2.24. Since σ is not invertible, we need to prove the existence of a dense orbit $(\sigma^{\circ n}(s))_{n \geq 0}$. We will do this constructively by defining s.

1. For $N \in \mathbb{N}$, let $W_N = A^N$ be the set of finite words of length N over the alphabet A. Since W_N is finite, we may enumerate all the words of W_N as $w_1, w_2, \cdots w_r$, and form a master word $\widetilde{w}_1 = w_1 w_2 w_3 \cdots w_r$.

   For example, if $A = \{0, 1\}$ and $N = 1$, we have $W_1 = \{0, 1\}$ and we can take $\widetilde{w}_1 = 10$. Similarly, $W_2 = \{00, 01, 10, 11\}$ and we can take $\widetilde{w}_2 = 00011011$.

2. Define $s \in \Sigma_A$ as

   $$s = \widetilde{w}_1 \widetilde{w}_2 \widetilde{w}_3 \cdots$$

   For example when $A = \{0, 1\}$, with $\widetilde{w}_1$ and $\widetilde{w}_2$ as above, we have $s = 1000011011 \cdots$.

To see that s has a dense orbit under σ, given any $t \in \Sigma_A$ and $\epsilon > 0$, we will show that there exists $m \in \mathbb{N}$ such that $d(\sigma^{\circ m}(s), t) < \epsilon$.

Choose $N \in \mathbb{N}$ such that $\frac{1}{|A|^{N-1}} < \epsilon$. The finite word $t_1 t_2 \cdots t_N$ is a sub-string of the master word $\widetilde{w}_N$. By definition of s, there exists $m \in \mathbb{N}$ such that

$$\sigma^{\circ m}(s) = t_1 t_2 \cdots t_N \cdots$$

By Proposition 2.1.5 and our choice of N, we have

$$d(\sigma^{\circ m}(s), t) < \frac{1}{|A|^{N-1}} < \epsilon$$

□

Proposition 2.2.26. *For $\mu > 2 + \sqrt{5}$, the set B_μ is totally disconnected.*

Proof. This proof is left as an exercise to the reader; it follows the same lines as Proposition 1.4.5, point 6. We will need to use Claim 1 from Proposition 2.2.13. □

Corollary 2.2.27. *The shift space $\Sigma = \Sigma_{\{0,1\}}$ is totally disconnected.*

Proof. By Proposition 2.2.15, we know Σ is homeomorphic to B_μ for $\mu > 2 + \sqrt{5}$. By Proposition 2.2.26, the statement follows. □

Proposition 2.2.17 and Corollaries 2.2.19, 2.2.25 and 2.2.27 show that Σ satisfies all the conditions of Theorem 1.4.10. Thus we get that Σ is homeomorphic to the ternary Cantor set $\mathcal{C}$, and thereby finish the proof of Theorem 1.5.6.

Chapter 3

Low-Dimensional Dynamics

3.1 Basic Concepts

Throughout this section we assume that X is a topological space and $f : X \to X$ is a continuous map.

Definition 3.1.1. Given $x \in X$,

- The orbit of x under f is the sequence $(f^{\circ n}(x))_{n\geq 0}$
- The grant orbit of x under f is the set $\{z \in X | f^{\circ m}(z) = f^{\circ n}(x) \text{ for some } m, n \in \mathbb{N}\}$
- A bi-infinite orbit for x under f is a sequence $(x_n)_{n\in\mathbb{Z}}$, where $x_0 = x$, and $x_{n+1} = f(x_n)$ for all $n \in \mathbb{Z}$.

Remark 3.1.2. A point x can have more than one bi-infinite orbit.
The grand orbit of x contains its orbit and any bi-infinite orbit. If f is a homeomorphism, then there is only one bi-infinite orbit for x, which is the whole grand orbit.

Definition 3.1.3. Given $m \in \mathbb{N}$,

$$\mathrm{Per}_m(f) = \{x \in X : f^{\circ m}(x) = x\}$$

This is the set of periodic points under f whose period under f divides m.

$$\mathrm{Per}(f) = \bigcup_{m=1}^{\infty} \mathrm{Per}_m(f)$$

This is the set of periodic points under f.

Definition 3.1.4. Let $A \subseteq X$. A is said to be forward invariant under f if $f(A) \subseteq A$, and backward invariant under f if $f^{-1}(A) \subseteq A$.

Transitivity, Mixing and Chaos

We restate the definition of topological transitivity here.

Definition 3.1.5.
- If f is not invertible, it is said to be topologically transitive if there exists a point $x_0 \in X$ such that the orbit $(f^{\circ n}(x_0))_{n \geq 0}$ is dense in X.
- If f is invertible, it is said to be topologically transitive if there exists a point $x_0 \in X$ such that the *grand orbit* $(f^{\circ n}(x_0))_{n \in \mathbb{Z}}$, which is the union of the forward and backward orbits of x_0, is dense in X.

Definition 3.1.6. Suppose f is a homeomorphism. It is said to be *minimal* if the grand orbit of every point is dense in X.

Remark 3.1.7. Minimality $\implies$ Topological Transitivity.

Definition 3.1.8. f is said to be *chaotic* if it is topologically transitive and $\mathrm{Per}(f)$ is dense in X.

Definition 3.1.9. f is said to be *topologically mixing* if for every pair of non-empty open sets $U, V \subseteq X$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $f^{\circ}(U) \cap V \neq \emptyset$.

3.2 Circle Maps

We can represent the circle $\mathbb{S}^1$ in two different ways:

- As the set $\mathbb{R}/\mathbb{Z} = \{[x] : x \in \mathbb{R}\}$, where $[x] = [y]$ iff $x - y \in \mathbb{Z}$.
- As the set $\{e^{2\pi i x} : [x] \in \mathbb{R}/\mathbb{Z}\} \subset \mathbb{C}$, i.e., the visual representation of $\mathbb{R}/\mathbb{Z}$ on the complex plane.

Definition 3.2.1. The *arc length* metric on $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ is given by

$$d_{\mathbb{S}^1}([x], [y]) = \min\{|x' - y'| : x', y' \in \mathbb{R}, x' \in [x], y' \in [y]\} \text{ for all } x, y \in \mathbb{S}^1$$

Exercise 3.2.2. Prove that $d_{\mathbb{S}^1}([x], [y])$ is the length of the shorter arc formed by $e^{2\pi i x}$ and $e^{2\pi i y}$ on the unit circle in the complex plane.

Rotations

Definition 3.2.3. Let $\alpha \in \mathbb{R}/\mathbb{Z}$. The rotation map $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ is defined as

$$R_\alpha([x]) = [x + \alpha] \qquad \text{(additive representation)}$$
$$R_\alpha(e^{2\pi i x}) = e^{2\pi i \alpha} e^{2\pi i x} = e^{2\pi i(\alpha + x)} \qquad \text{(multiplicative representation)}$$

Note that every rotation R_α is a homeomorphism of $\mathbb{S}^1$; its inverse is $R_{-\alpha}$. Every rotation is in fact an isometry in the metric $d_{\mathbb{S}^1}$. That is, $d_{\mathbb{S}^1}(R_\alpha(x), R_\alpha(y)) = d_{\mathbb{S}^1}(x, y)$ for all $x, y \in \mathbb{S}^1$.

These maps exhibit widely different behavior when $\alpha \in \mathbb{Q}/\mathbb{Z}$ vs. when $\alpha \notin \mathbb{Q}/\mathbb{Z}$.

Proposition 3.2.4. • *If $\alpha \in \mathbb{Q}/\mathbb{Z}$, then* $\mathrm{Per}(R_\alpha) = \mathbb{S}^1$.

• *If α is irrational, then* $\mathrm{Per}(R_\alpha) = \emptyset$.

Proof. • If α is rational, it is of the form $\alpha = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ and $n \neq 0$. Thus

$$R_\alpha^{\circ n}([x]) = [x + n\alpha] = [x + m] = [x] \text{ for all } [x] \in \mathbb{R}/\mathbb{Z}$$

In other words, $R_\alpha^{\circ n} = \mathrm{id}_{\mathbb{S}^1}$. Thus $\mathrm{Per}_n(R_\alpha) = \mathbb{S}^1$, which implies $\mathrm{Per}(R_\alpha) = \mathbb{S}^1$.

• If α is irrational and $[x] \in \mathbb{S}^1$ is a periodic point of R_α of period n for some $n \in \mathbb{N}$, then $[x + n\alpha] = [x]$, which implies $n\alpha \in \mathbb{Z}$. But this means α has a rational representative, which is a contradiction.

□

Proposition 3.2.5. *Let $f : X \to X$ be an open map. Then if f is topologically transitive, there exist no pair of disjoint non-empty open sets U and V such that $f(U) \subseteq U$ and $f(V) \subseteq V$.*

We will prove Proposition 3.2.5 later on. However, since R_α is a homeomorphism, and thus open, using this proposition we will show the following.

Proposition 3.2.6. *If $\alpha \in \mathbb{Q}/\mathbb{Z}$, then R_α is not topologically transitive.*

Proof. We will find a pair of disjoint, invariant non-empty open sets, showing the contrapositive of Porposition 3.2.5. Choose any two points $u, v \in \mathbb{S}^1$. We know that α is of the form $\frac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n \neq 0$.

Choose $\epsilon > 0$ so that the open intervals $B_\epsilon(u), R_\alpha(B_\epsilon(u)), \cdots, R_\alpha^{\circ(n-1)}(B_\epsilon(u))$, $B_\epsilon(v), R_\alpha(B_\epsilon(v)), \cdots, R_\alpha^{\circ(n-1)}(B_\epsilon(v))$ are all pairwise disjoint. Then, letting $U = \bigcup_{j=0}^{n-1} R_\alpha^{\circ j}(B_\epsilon(u))$ and $V = \bigcup_{j=0}^{n-1} R_\alpha^{\circ j}(B_\epsilon(v))$, we see that $U \cap V = \emptyset$, U and V are non-empty, $f(U) = U$ and $f(V) = V$. □

Proposition 3.2.7. *If $\alpha \notin \mathbb{Q}/\mathbb{Z}$, then R_α is minimal. This in turn implies it is topologically transitive.*

Proof. Fix $x \in \mathbb{S}^1$. It suffices to show that the grand orbit of x is dense in $\mathbb{S}^1$. We will in fact show the stronger statement that the forward orbit $(R_\alpha^{\circ n}(x))_{n \geq 0}$ is dense in $\mathbb{S}^1$.

Let $z \in \mathbb{S}^1$ and $\epsilon > 0$. We need to exhibit an orbit point $R_\alpha^{\circ n}(x) \in B_\epsilon(z)$.

For $N \geq \lfloor \frac{1}{\epsilon} \rfloor + 1$, any set of N points on $\mathbb{S}^1$ contains at least two points u, v such that $d_{\mathbb{S}^1}(u, v) < \epsilon$.

Let $S = \{x, R_\alpha(x), \cdots, R_\alpha^{\circ(N-1)}(x)\}$. Since α is irrational, this points listed here are distinct, so $|S| = N$. By the above statement, there exist ℓ, k with $0 \leq \ell < k \leq N$ such that $d_{\mathbb{S}^1}(R_\alpha^{\circ \ell}(x), R_\alpha^{\circ k}(x)) < \epsilon$.

Since R_α is an isometry, we have $d_{\mathbb{S}^1}(x, R_\alpha^{\circ(k-\ell)}(x)) < \epsilon$.

Claim. If $d_{\mathbb{S}^1}(x, R_\alpha^{\circ n}(x)) < \epsilon$ for some $n \in \mathbb{N}$, then for all $y \in \mathbb{S}^1$, we have $d_{\mathbb{S}^1}(y, R_\alpha^{\circ n}(y)) < \epsilon$.

Proof of Claim. We know that $y = R_{y-x}(x)$. Thus,

$$\begin{aligned} d_{\mathbb{S}^1}(y, R_\alpha^{\circ n}(y)) &= d_{\mathbb{S}^1}(R_{y-x}(x), R_\alpha^{\circ n} \circ R_{y-x}(x)) \\ &= d_{\mathbb{S}^1}(R_{y-x}(x), R_{y-x} \circ R_\alpha^{\circ n}(x)) \qquad \text{since } R_\alpha R_\beta = R_\beta R_\alpha \text{ for all } \alpha, \beta \in \mathbb{S}^1 \\ &= d_{\mathbb{S}^1}(x, R_\alpha^{\circ n}(x)) \qquad \text{since all rotations are isometries} \\ &< \epsilon \end{aligned}$$

■

By the above claim, letting $y = 0$, we have $d_{\mathbb{S}^1}(0, R_\alpha^{\circ(\ell-k)}(0)) < \epsilon$.
Note that $d_{\mathbb{S}^1}(0, R_\alpha^{\circ(\ell-k)}(0)) = |\theta|$, where $\theta = [(\ell - k)\alpha] \in \mathbb{S}^1$.

By this choice of θ, for $M \geq \lfloor \frac{1}{\theta} \rfloor + 1$, the points $\{x, R_\theta(x), \cdots, R_\theta^{\circ(M-1)}(x)\}$ split the circle into intervals all of length $< \epsilon$. Thus there exists $n \in \{0, 1, \cdots, M-1\}$ such that $d_{\mathbb{S}^1}(R_\theta^{\circ n}(x), z) < \epsilon$. Since $R_\theta^{\circ n}(x) = R_\alpha^{\circ n(\ell-k)}(x)$, the proposition follows. □

Proposition 3.2.8. *No circle rotation is chaotic.*

Proof. By the above series of propositions, rational rotations are not topologically transitive, and irrational rotations have no periodic points. So neither kind of rotations are chaotic. □

Proposition 3.2.9. *No homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ is topologically mixing.*

Proof. Pick any three distinct points $x, y, z \in \mathbb{S}^1$. Then $\mathbb{S}^1 \setminus \{x, y, z\}$ is the union of three disjoint intervals A, B and C. For $N \in \mathbb{N}$, let $\mathcal{B}_N = \{f^{\circ N}(X) \cap Y | X, Y \in \{A, B, C\}\}$. It suffices to show the following claim:

Claim. For all $N \in \mathbb{N}$, at least one element of $\mathcal{B}_N$ is empty.

Here is how this claim proves that f is not topologically mixing: by the claim, for all N, there are sets $X_N, Y_N \in \{A, B, C\}$ such that $f^{\circ N}(X_N) \cap Y_N = \emptyset$. Then, since $\{A, B, C\}$ is finite, upto some subsequence, X_N is constant and Y_N is constant. So wlog, up to a subsequence $N_k \to \infty$, we can assume $X_{N_k} = A$ and $Y_{N_k} = B$, say. Thus, the claim implies that $f^{\circ n}(A) \cap B = \emptyset$ for infinitely many natural numbers n. This means f is not topologically mixing.

Proof of Claim. Fix $N \in \mathbb{N}$. Since A, B, C are pairwise disjoint, and $f^{\circ N}$ is a homeomorphism for all N, the intervals $f^{\circ N}(A), f^{\circ N}(B), f^{\circ N}(C)$ are pairwise disjoint. Suppose $f^{\circ N}(A)$ intersects A, B and C. Then it has to contain one of the intervals - wlog suppose $f^{\circ N}(A) \supseteq A$. But this means that $f^{\circ N}(B) \cap A = f^{\circ N}(C) \cap A = \emptyset$. Thus not all elements of $\mathcal{B}_N$ can be simultaneously non-empty.

■

□

Lifts

Definition 3.2.10. Given $f : \mathbb{S}^1 \to \mathbb{S}^1$ continuous, a *lift* of f is a continuous map $G : \mathbb{R} \to \mathbb{R}$ that satisfies $f \circ \pi = \pi \circ G$ for the universal covering map $\pi : \mathbb{R} \to \mathbb{S}^1$.

Proposition 3.2.11. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be continuous.*

1. *For any $[x_0] \in \mathbb{S}^1$ represented by some point $x_0 \in \mathbb{R}$, let $y_0 \in \mathbb{R}$ be such that $f([x_0]) = [y_0]$. Then there exists a lift G of f such that $G(x_0) = y_0$.*
2. *Any two lifts of f differ by a constant $C \in \mathbb{Z}$.*
3. *Let G be a lift of f. Then $\deg f := G(x+1) - G(x)$ is an integer that is independent of x, G. This is called the degree of f.*

Proof. We will prove each point one by one. Let $A = f^{-1}(y_0) = \{[x_r] = [x_0 + 1] = [x_0], [x_1], [x_2], \cdots, [x_{r-1}]\}$ where the points are in counterclockwise order starting at $[x_0]$. Let $y_n = y_0 + n$ for $n \in \mathbb{Z}$ and consider $\pi_n = \pi|[y_n, y_{n+1})$. Note that π_n is a continuous bijection onto $\mathbb{S}^1$ whose inverse is continuous on $\mathbb{S}^1 \setminus \{[y_0]\}$.

1. We show this in the following steps:
   - Define G on $[x_0, x_1]$ as follows. Starting at $x_0 \in \mathbb{R}$, move counterclockwise from $[x_0]$ on the circle. In this process, $f(\pi(x)) = f([x])$ starts traveling either clockwise or anticlockwise on the circle from $[y_0]$. Let $\Delta = \pm 1$ depending on whether this movement is clockwise or counterclockwise, and define G on $[x_0, x_1)$ as $G(x_0) = y_0$ and $G(x) = \pi_\Delta^{-1} \circ f \circ \pi(x)$ for all $x \in (x_0, x_1)$. Note that at the end of this process, $\lim_{x \to (x_1)^-} G(x) = y_0$ or $y_0 + \Delta$. In both cases, we uniquely extend G continuously to $[x_0, x_1]$. Note that in this process, G satisfies $\pi \circ G = f \circ \pi$.
   - Inductively for some $i \in \{1, 2, \cdots, r-2\}$, suppose G is well-defined on $[x_{i-1}, x_i]$ and satisfies $f \circ \pi = \pi \circ G$. We will extend G to $[x_i, x_{i+1}]$. Set $\Delta = 1$ if f does not change direction at $[x_i]$, or $\Delta = -1$ otherwise. Extend G continuously to $[x_i, x_{i+1})$ by defining it to be $G(x) = \pi^{-1}_{G(x_i) - y_0 + \Delta} \circ f \circ \pi$ on (x_i, x_{i+1}) and $G(x_{i+1}) = \lim_{x \to x_{i+1}^-} G(x)$. At the end of this process, $G(x_{i+1}) = G(x_i)$ or $G(x_i) + \Delta$. Note that $G(x_0 + 1) - G(x_0) \in \mathbb{Z}$. Clearly G satisfies $\pi \circ G = f \circ \pi$ on $[x_0, x_0 + 1]$.
   - This finishes the definition of G on $[x_0, x_r] = [x_0, x_0 + 1]$. Inductively for $n \in \mathbb{Z}$ with $n \neq 0$, for $x \in [x_0 + n, x_0 + n + 1]$, let $G(x) = G(x - n) + n(G(x_0 + 1) - G(x_0))$, G is continuous on $(x_0 + n, x_0 + n + 1)$ for all $n \in \mathbb{Z}$. We will show that it is continuous at each point of the form $x_0 + n$.

     $$\begin{aligned}\lim_{x \to (x_0+n)^-} G(x) &= \lim_{x \to (x_0+n)^-} G(x - (n-1)) + (n-1)(G(x_0+1) - G(x_0)) \\ &= nG(x_0+1) - (n-1)G(x_0) \\ \lim_{x \to (x_0+n)^+} G(x) &= \lim_{x \to (x_0+n)^+} G(x - n) + n(G(x_0+1) - G(x_0)) \\ &= nG(x_0+1) - (n-1)G(x_0)\end{aligned}$$

     Also note that $G(x) - G(x-n) \in \mathbb{Z}$ for all n, x. This, along with the definition of G on $[x_0, x_0 + 1]$, shows that $\pi \circ G = f \circ \pi$.

2. Let G_1 and G_2 be two lifts of f. Then by definition, for all $x \in \mathbb{R}$, $\pi(G_1(x)) = f(\pi(x)) = \pi(G_2(x))$, and thus $G_1(x) - G_2(x)$ is an integer. Since $G_1 - G_2$ is a continuous function from $\mathbb{R}$ to $\mathbb{Z}$, it is a constant.

3. Let G be a lift of f. We note that the function $\widetilde{G}(x) = G(x+1)$ is also lift of f. By the previous point, $\widetilde{G}(x) - G(x) = G(x+1) - G(x)$ is an integer that is independent of $x \in \mathbb{R}$. To show that it is independent of G, note that for any other lift H, we have $H = G + c$ for some $c \in \mathbb{Z}$, and thus $H(x+1) - H(x) = G(x+1) - G(x)$ for all $x \in \mathbb{R}$.

□

Proposition 3.2.12. *If f is an injective continuous map , then $|\deg f| = 1$ and any lift G is strictly monotone.*

Proof. Let G be a lift of f. First we show that $|\deg f| = 1$. If $\deg f = 0$, then $G(1) = G(0)$. In particular, G is not monotone on $[0,1]$. Thus there exist points $c_1 \neq c_2 \in (0,1)$ such that $G(c_1) = G(c_2)$. In particular, $f([c_1]) = f([c_2])$, while $[c_1] \neq [c_2]$, which also contradicts the injectivity of f. So $\deg f \neq 0$. If $|\deg f| > 1$, then there exists $c \in (0,1)$ such that $|G(c) - G(0)| = 1$. But this shows that $f([c]) = f([0])$, which also contradicts the inductivity of f. This shows that $|\deg f| = 1$.

WLOG assume $\deg f = 1$. Then note that $G(x+1) = G(x) + 1$ for all x. So it suffices to show that G is strictly increasing on $[0,1]$. Since $G : [0,1] \to [G(0), G(1)]$ is continuous, if it is not strictly increasing, there exist points $c_1 < c_2 \in (0,1)$ such that $G(c_1) \geq G(c_2)$. However, there is then a point $c \in (c_2, 1]$ such that $G(c) = G(c_1)$. This implies $f([c]) = f([c_1])$ and contradicts the injectivity of f. □

Remark 3.2.13. If f is injective and continuous, if $\deg f = 1$, then any lift is strictly increasing, and if $\deg f = -1$, then any lift is strictly decreasing.

Proposition 3.2.14. *If $|\deg f| = 1$ and some lift of f is strictly monotone, then f is a homeomorphism.*

Proof. Let G be a strictly monotone lift of f. Note that within $(0,1)$, $G(x) = G(y)$ implies $x = y$. Thus f is injective. Since G is continuous and $|\deg f| = 1$, G maps $[0,1]$ onto some interval of length 1. This shows that $f(\mathbb{S}^1) = \mathbb{S}^1$.
Since G is a strictly monotone continuous map of R, it has a strictly monotone continuous inverse G^{-1}. Then $G^{-1}(x+1) - G^{-1}(x) = \deg f$ for all $x \in G$. This also shows that G^{-1} is the lift of a continuous circle map h, and h satisfies $h \circ f = f \circ h = \mathrm{id}_{\mathbb{S}^1}$. Thus f^{-1} is continuous. □

Suggested Reading 3.2.15.

- [3, Section 4.3.1]

Expanding Maps

Definition 3.2.16. A map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is said to be *expanding* if it is continuously differentiable and $\exists \lambda > 1$ such that $|f'(x)| \geq \lambda$ for all $x \in \mathbb{S}^1$.

Definition 3.2.17. A linear expanding map of $\mathbb{S}^1$ is a map of the form $E_m([x]) = [mx]$ for all $[x] \in \mathbb{S}^1$, where m is an integer ≥ 2.

Exercise 3.2.18. Fix an integer $m \geq 2$.
Show that for $k \in \mathbb{N}$, $\mathrm{Per}_k(E_m) = \{[\frac{i}{m^k-1}] : i = 0, 1, \cdots, m^k - 2\}$.

Proposition 3.2.19. *For any integer $m \geq 2$, the set $\mathrm{Per}(E_m)$ is dense in $\mathbb{S}^1$.*

Proof. Given $\epsilon > 0$, choose $k \in \mathbb{N}$ such that $\frac{1}{m^k-1} < \epsilon$. Then $\mathrm{Per}_k(E_m) \subset \mathrm{Per}(E_m)$ splits the circle into $m^k - 1$ small intervals each of length $\frac{1}{m^k-1} < \epsilon$. $\square$

Proposition 3.2.20. *Any expanding map f of the circle is topologically mixing.*

Proof. Given U, V non-empty open sets in $\mathbb{S}^1$, let $I \subset \pi^{-1}(U)$ be an interval. We will show that $f^{\circ N}(\pi(I)) = \mathbb{S}^1$ for some $N \in \mathbb{N}$. This will imply the topological mixing property.

Let ℓ be the length of I. Since f is expanding, there exists $\lambda > 1$ such that $|f'| \geq \lambda$. Then for any lift G of f, we have $|G'| \geq \lambda$. But this in turn means that the length of $G^{\circ n}(I)$ is greater than $\lambda^n \ell$ for all $n \to \infty$. Pick $N \in \mathbb{N}$ such that $\lambda^n \ell > 1$. In other words, $G^{\circ N}(I)$ contains an interval of length 1. But this means that $f^{\circ N}(\pi(I))$ covers $\mathbb{S}^1$. $\square$

Proposition 3.2.21. *Any expanding map f of the circle is is topologically transitive.*

Before proving this, we show the following proposition and theorem.

Proposition 3.2.22. *Let X be a topological space and $f : X \to X$ be continuous. Then if f has a dense bi-infinite orbit, then for every $\emptyset \neq U, V \subset X$, open, there exists an $N \in \mathbb{Z}$ such that $f^{\circ N}(U) \cap V \neq \emptyset$. Furthermore, if X is perfect, then this N can be chosen in $\mathbb{N}$.*

Proof. Let U, V be as above, and $(x_n)_{n \in \mathbb{Z}}$ be the dense bi-infinite orbit. Then there exists $n \in \mathbb{Z}$ such that $z_n \in V$, and $m \in \mathbb{Z}$ such that $z_m \in U$. Letting $N = n - m$ (this could be ≤ 0), we see that $z_n \in f^{\circ N}(\{z_m\})$, thus $f^{\circ N}(U) \cap V = \emptyset$.

Now assume X is perfect. For the given U, V, if the above $N \geq 0$ we are done. Suppose $N = n - m < 0$. Since z_n is not an isolated point, there exists a subsequence $|n_k| \to \infty$ such that $z_{n_k} \to z_n$ and $z_{n_k} \in V$ for all k.

1. If we can choose $n_k \to +\infty$, then for some k, we have $n_k \geq n$. Letting $\widetilde{N} = n_k - n \geq 0$, we see that $f^{\circ \widetilde{N}}(x_n) = x_{n_k} \in V$. Thus $f^{\circ \widetilde{N}}(U) \cap V \neq \emptyset$.

2. If we have $n_k \to -\infty$, choose $n' = n_k$ with k large enough such that $z_{n_k} = z_{n'} \in V \cap f^{\circ N}(U)$, and $n' < 2n - m$. Then $z' = z_{n'+m-n} := f^{\circ(m-n)}(z'_n) \in f^{\circ N+m-n}(U) = f^{\circ n-m+n-m}(U) \subseteq U$. Moreover, letting $\widetilde{N} = 2n - m - n'$, we see that $\widetilde{N} > 0$ and $f^{\circ \widetilde{N}}(z') = f^{\circ(2n-m-n')}(z') = z_n \in V$. Thus $f^{\circ \widetilde{N}}(U) \cap V \neq \emptyset$.

$\square$

Theorem 3.2.23. *Let X be a complete separable (that is, there is a countable dense subset) metric space with no isolated points. If $f : X \to X$ is a continuous map, then the following four conditions are equivalent:*

1. *f is topologically transitive, i.e., it has a dense orbit.*

2. *f has a dense bi-infinite orbit.*

3. *If $\emptyset \neq U, V \subset X$, then there exists an $N \in \mathbb{N}$ such that $f^{\circ N}(U) \cap V \neq \emptyset$.*

4. *If $\emptyset \neq U, V \subset X$, then there exists an $N \in \mathbb{Z}$ such that $f^{\circ N}(U) \cap V \neq \emptyset$.*

Proof. (1) $\implies$ (2) and (3) $\implies$ (4) are always true.

By Proposition 3.2.22, (2) implies (3).

We will show that separability of X imples (4) implies (2) and (3) implies (1). This will prove the theorem. The proof methods are similar, so we will only do the case (3) implies (1).

Let S be a countable dense set. For every $p \in \mathbb{Q}$ and every $x \in S$,let $U_p(x)$ be the ball of radius $\frac{p}{q}$ centered at x. Consider the collection $\{U_p(x) : x \in S, p \in Q\}$. This collection is countable, and can be enumerated as $\{U_1, U_2, \cdots\}$. Every tail $\{U_N, U_{N+1}, \cdots,\}$ is an open cover of X. Let $U_0 = f^{-1}(U_1)$. By condition (3), there exists $N_1 \in \mathbb{N}$ such that $f^{\circ N_1}(U_1) \cap U_2 \neq \emptyset$. Pick an open ball V_1 of radius < 1 such that $\overline{V_1} \subset U_1 \cap f^{-\circ N_1}(U_2)$.

Then $f^{\circ N_1}(V_1) \cap U_2 \neq \emptyset$. Inductively, for $k \geq 2$, there exists an $N_k \in \mathbb{N}$ such that $f^{\circ N_k}(V_{k-1}) \cap U_{k+1} \neq \emptyset$. Let V_k be an open ball of radius $< \frac{1}{2^{k-1}}$ such that $\overline{V_k} \subset V_{k-1} \cap f^{\circ -N_k}(U_{k+1})$. Then note that $f^{\circ N_k}(\overline{V_k}) \subset U_{k+1}$.

Furthermore, $\overline{V_1} \supset \overline{V_2} \supset \overline{V_3} \cdots$ is a decreasing chain of closed balls whose diameter goes to 0. Thus, $\cap_{k=1}^{\infty} \overline{V_k} = \{x\}$ for a unique point $x \in X$. Let $x_0 = x \in U_1$, and $x_k = f^{\circ k}(x)$ for $k \in \mathbb{N}$. since $N_k \in \mathbb{N}$ for all k, and this gives a dense orbit in X. $\square$

Corollary 3.2.24. *A continuous open map f of a complete, separable, perfect metric space is topologically transitive if and only if there are no two disjoint open nonempty $f-$invariant sets.*

Proof. $\implies$ is obvious, since a dense orbit visits every open set.
$\impliedby$: If $U, V \subset X$ are open, then the sets $W = \bigcup_{n \in Z} f^{\circ n}(U)$ and $O = \bigcup_{n \in Z} f^{\circ n}(V)$ are open because f is an open map, and satisfy $f(W) \subseteq W$, and $f(O) \subseteq O$. Therefore they not disjoint by assumption, so $f^{\circ n}(U) \cap f^{\circ m}(V) \neq \emptyset$ for some $n, m \in \mathbb{Z}$. Then $f^{\circ(n-m)}(U) \cap V \neq \emptyset$ and f is topologically transitive by the above theorem. $\square$

Proof of Proposition 3.2.21. Since f is topologically mixing, it is also topologically transitive by Theorem 3.2.23. $\square$

Suggested Reading 3.2.25.
- [3, Section 7.1.3, Section 7.2.1, Section 7.2.3, Section 7.4.1]

General vs Linear Expanding Maps

Proposition 3.2.26. *If $f : \mathbb{S}^1 \to \mathbb{S}^1$ is an expanding map, then $|\deg f| > 1$ and $\mathrm{Per}_k(f) = |\deg f^k - 1|$ for all $k \in \mathbb{N}$.*

Proof. $|f'| > 1$ implies $|G'| > 1$ for any lift, so, by the Mean-Value Theorem, $|\deg f| = |G(x+1) - G(x)| > 1$. Any iterate $f^{\circ k}$ is also expanding, and note that $G^{\circ k}$ is a lift of $f^{\circ k}$. Furthermore,

$$\forall x \in \mathbb{R}, \qquad G^{\circ k}(x+1) - G^{\circ k}(x) = (G(x+1) - G(x))^k \qquad \forall k \in \mathbb{N}$$

This implies that $\deg(f^{\circ k}) = (\deg f)^{\circ k}$. so it suffices to consider the case $k = 1$. Consider G on the interval $[0, 1]$. Note that

$$\mathrm{Fix}(f) = \{[x] : x \in [0, 1], G(x) - x \in \mathbb{Z}\}$$

The function $g(x) := G(x) - x$ satisfies $g(1) = g(0) + \deg f - 1$, and is also monotone. In other words, $g[0, 1]$ is an interval of length $|g(1) - g(0)| = |\deg f - 1|$. From this it follows that

$$|\deg f - 1| \leq \#\{x \in [0, 1] : G(x) - x \in \mathbb{Z}\} \leq |\deg f - 1| + 1$$

- If $g(0) \notin \mathbb{Z}$, then $g(1) \notin \mathbb{Z}$, and $\#\{x \in [0, 1] : G(x) - x \in \mathbb{Z}\} = |\deg f - 1|$. In this case, all the points in this set map to distinct points on the circle, and thus, $|\mathrm{Fix}(f)| = |\deg f - 1|$.
- If $g(0) \in \mathbb{Z}$, then $g(1) \in \mathbb{Z}$, and $\#\{x \in [0, 1] : G(x) - x \in \mathbb{Z}\} = |\deg f - 1| + 1$. This set now contains $0, 1$, and $|\deg f - 1| - 1$ distinct points in $(0, 1)$. Since $[0] = [1] \in \mathbb{S}^1$, we get $|\mathrm{Fix}(f)| = |\deg f - 1|$.

□

Remark 3.2.27. Note that $|\mathrm{Per}_k(f)| = \mathrm{Per}_k(E_m)|$, where $m = \deg f$.

Theorem 3.2.28. *Let f be an expanding map of $\mathbb{S}^1$. Then f is topologically conjugate to E_m, where $m = \deg f$.*

Corollary 3.2.29. *Any two expanding maps of the circle of the same degree are topologically conjugate.*

We will not be proving this in our lectures, but the general idea is given here.

Definition 3.2.30. Suppose that $g : X \to X$ and $f : Y \to Y$are maps of metric spaces X and Y and that there is a continuous surjective map $h : X \to Y$ such that $h \circ g = f \circ h$. Then g is said to be *semi-conjugate* to g via the *semiconjugacy* h.

Remark 3.2.31. If this h is a homeomorphism, then f and g are topologically conjugate. In this case we will call h a *conjugacy*.

Proposition 3.2.32. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an expanding map with $\deg f = m$. Then f is semi-conjugate to the shift $\sigma : \Sigma_A \to \Sigma_A$ where $A = \{0, 1, \cdots, |m| - 1\} \subset \mathbb{R}$.*

Proof. Let $[y_0] = f([0])$. Note that there exist m intervals $\Delta_0, \Delta_1, \cdots, \Delta_{m-1} \subset \mathbb{S}^1$ such that $f(\Delta_i) = \mathbb{S}^1 \setminus [y_0]$ and the ends of each Δ_i are pre-images of y_0.

Additionally, for each i, j, there exists a unique maximal interval $\Delta_{ij} \subset \Delta_i$ such that $f(\Delta_{ij}) \subset \Delta_j$. Inductively, suppose Δ_w is well-defined for all words w of length n over A.

Given a word $w' = w_1w_2\cdots w_{n+1}$ of length $n+1$ over A, define $\Delta_{w'} \subset \Delta_{w_1}$ to be the unique maximal interval such that $f(\Delta_{w'}) \subset \Delta_{w_2w_3\cdots w_{n+1}}$.

Since f is expanding and scales lengths by at least a factor $\lambda > 1$, given any infinite word $s \in \Sigma_A$, the intersection $\cap_{N=1}^{\infty}\overline{\Delta_{s_1s_2\cdots s_N}}$ is a unique point $x_s \in \mathbb{S}^1$.

Claim. The map $h : \Sigma_A \to \mathbb{S}^1$ given by $s \mapsto x_s$ is surjective and continuous.

Proof of Claim. This is very similar to the proof of Proposition 2.2.15. ■

Note that by construction, $h(\sigma(s)) = f(x_s) = f(h(s))$. This shows that h is a semi-conjugacy between σ and f. □

Idea of Proof of Theorem 3.2.28. Let $g = E_m$. Let $h_f, h_g : \Sigma_A \to \mathbb{S}^1$ be the semi-conjugacies of f and g respectively with σ. Consider the set $H_x = h_g \circ h_f^{-1}(x)$ for any $x \in \mathbb{S}^1$. The claim is that H_x always consists of precisely one point, which we will call $h(x)$. The map $h : \mathbb{S}^1 \to \mathbb{S}^1$ gives the required topological conjugacy between f and g □

Proposition 3.2.33. *Expanding maps of $\mathbb{S}^1$ are chaotic.*

Proof. This follows from Proposition 3.2.21, Theorem 3.2.28 and the fact that the periodic points of linear expanding maps are dense in $\mathbb{S}^1$. □

Suggested Reading 3.2.34. • [3, Section 7.3.1, Section 7.4.1]

Rotation Number

Definition 3.2.35. A homeomorphism f of $\mathbb{S}^1$ is said to be orientation-preserving if $\deg f = 1$ and orientation-reversing if $\deg f = -1$.

Lemma 3.2.36. *If f is an orientation-preserving circle homeomorphism and G a lift, then $G(y) - y \leq G(x) - x + 1$ for all $x, y \in \mathbb{R}$.*

Proof. Ler $k = \lfloor y - x \rfloor$. Then

$$\begin{aligned} G(y) - y &= G(y) + G(x+k) - G(x+k) + (x+k) - (x+k) - y \\ &= (G(x+k) - (x+k)) + (G(y) - G(x+k)) - (y - (x+k)) \end{aligned}$$

Now $G(x+k)-(x+k) = G(x)-x$ and $0 \leq y-(x+k) < 1$ by choice of k. So $G(y)-G(x+k) \leq 1$. Thus the right hand side above is at most $G(x) - x + 1 - 0$. □

Lemma 3.2.37. *Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers such that $a_{m+n} \leq a_n + a_{m+k} + L$ for some k, L. Then the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists and belongs to $\mathbb{R} \cup \{-\infty\}$.*

Proof.

$$a_{m+k} \leq a_m + a_{2k} + L$$

gives

$$a_{m+n} \leq a_n + a_m + a_{2k} + 2L = a_n + a_m + L'$$

where $L' = a_{2k} + 2L$. So wlog, we may take $k = 0$. Let $a = \liminf_{|n|\to\infty} \frac{a_n}{n} \in \mathbb{R} \cup \{-\infty\}$.

If $a < b < c$ and $|n| > \frac{2L'}{(c-b)}$ such that $\frac{a_n}{n} < b$. Then for any ℓ with $|\ell| \geq |n|$ which satisfies $|\ell|(c-b) > \max_{|r|<|n|} a_r$, writing $\ell = nk' + r$ for some $k' \in \mathbb{Z}$, $|r| < |n|$, we have

$$\begin{aligned}\frac{a_\ell}{\ell} &\leq \frac{(k'a_n + a_r + k'L')}{\ell} \\ &\leq \frac{a_n}{n} + \frac{a_r}{\ell} + \frac{L'}{n} < c\end{aligned}$$

Thus, $\limsup \frac{a_\ell}{\ell} \leq c$. Since $c > a$ was arbitrary, this proves the lemma. □

Proposition 3.2.38. *Let f be an orientation-preserving homeomorphism of the circle.*

1. *Let G be a lift of f. Then the following limit exists and is independent of $x \in \mathbb{R}$:*

$$\rho(G) = \lim_{|n|\to\infty} \frac{G^{\circ n}(x) - x}{n}$$

2. *For any other lift $\widetilde{G}$ of f, we have $\rho(G) - \rho(\widetilde{G}) = G - \widetilde{G} \in \mathbb{Z}$.*
3. *$\rho(G)$ is rational if and only if f has a periodic point.*

Proof. 1. *Existence:* Take $x \in \mathbb{R}$ and $a_n = G^{\circ n}(x) - x$ for all $n \in \mathbb{Z}$. Then by Lemma 3.2.36 applied to $f^{\circ m}$ and $G^{\circ m}$, we have $G^{\circ m}(y) - y \leq G^{\circ m}(x) - x + 1$ for all $y \in \mathbb{R}$. Thus, for all $m, n \in \mathbb{Z}$,

$$a_{m+n} = G^{\circ(m+n)}(x) - x = G^{\circ m}(G^{\circ n}(x)) - G^{\circ n}(x) + a_n \leq a_m + 1 + a_n$$

Now, using Lemma 3.2.37, we know that $\lim_{n\to\infty} \frac{a_n}{n} = c^+ \in \mathbb{R} \cup \{-\infty\}$. However,

$$\frac{a_n}{n} = \frac{1}{n}\sum_{i=0}^{n-1} G^{\circ(i+1)}(x) - G^{\circ i}(x) \geq \min_{y\in\mathbb{R}} G(y) - y = \min_{y\in[0,1]} G(y) - y \in \mathbb{R}$$

Thus the limit c^+ is a real number. Similarly, $\lim_{n\to-\infty} \frac{a_n}{n} = c^- \in \mathbb{R}$. It suffices to show that $c^+ = c^-$.

2. Note that there exists $c \in \mathbb{Z}$ such that $\tilde{G} = G + c$. Then for every $x \in \mathbb{R}$ and every $n \in \mathbb{Z}$, $\widetilde{G}^{\circ n}(x) = G^{\circ n}(x) + nc$. Thus $\frac{\widetilde{G}^{\circ n}(x) - x}{n} = \frac{G^{\circ n}(x) - x}{n} + c$. This proves that $\rho(\widetilde{G}) = \rho(G) + c$.

3. If f has a $k-$periodic point $[x_0]$ and G is a lift of f, then $G^{\circ k}(x_0) = x_0 + c$ for some $c \in \mathbb{Z}$, note that $G^{\circ nk}(x_0) = x_0 + nc$ for all $n \in \mathbb{Z}$. Thus $\rho(G) = \lim_{|n|\to\infty} \frac{G^{\circ nk}(x_0) - x_0}{nk} = \frac{c}{k} \in \mathbb{Q}$.
Conversely, if $\rho(G) = \frac{m}{k} \in \mathbb{Q}$, then $\rho(G^{\circ k}) = m$. Suppose $\widetilde{f} = f^{\circ k}$ has no fixed point. Pick $\widetilde{G}$ to be a lift such that $\widetilde{G}(0) \in [0, 1)$. Then $\widetilde{G}(x) - x \notin \mathbb{Z}$ for all $x \in \mathbb{R}$ since $\widetilde{G}(x) - x \in \mathbb{Z}$ would imply that $[x]$ is a fixed point for $\widetilde{f}$. Therefore, $0 < \widetilde{G}(x) - x < 1$ for all $x \in \mathbb{R}$. Since $\widetilde{G} - \mathrm{id}$ is continuous and periodic, it attains its minimum and maximum and therefore there exists a $\delta > 0$ such that

$$0 < \delta \leq \widetilde{G}(x) - x \leq 1 - \delta < 1$$

for all $x \in R$. In particular, we can take $x = \widetilde{G}^{\circ i}(0)$ and use $\widetilde{G}^{\circ n}(0) = \widetilde{G}^{\circ n}(0) - 0 = \sum_{i=0}^{n-1} \widetilde{G}^{\circ(i+1)}(0) - \widetilde{G}^{\circ i}(0)$ to get

$$n\delta \leq \widetilde{G}^{\circ n}(0) \leq (1-\delta)n$$

or

$$\delta \leq \frac{\widetilde{G}^{\circ n}(0)}{n} \leq 1 - \delta$$

As $n \to \infty$, this gives $\rho(F) \neq 0$, which proves the claim by contraposition.

□

Definition 3.2.39. Let f be an orientation-preserving homeomorphism of the circle. The *rotation number* of f is given by $\rho(f) = [\rho(G)] \in \mathbb{S}^1$ where G is any lift of f.

Definition 3.2.40. Let X be a topological space and $f : X \to X$ be a dynamical system. A point $x \in X$ is said to be *heteroclinic* to $p, q \in X$ under f if $\lim_{n\to\infty} f^{\circ n}(x) = p$ and $\lim_{n\to-\infty} f^{\circ n}(x) = q$. If $p = q$, we say x is *homoclinic* under f to p.

Theorem 3.2.41. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving homeomorphism. Suppose $\rho(f) = \frac{p}{q} \in \mathbb{Q}$.*

- *If f has exactly one periodic orbit $A = \{x_1, x_2, \cdots, x_r\}$, then for every point $x \in \mathbb{S}^1 \setminus A$, x is heteroclinic under $f^{\circ q}$ to two points $x_i, x_j \in A$. Furthermore, $x_i = x_j$ if and only if $r = 1$.*
- *If f has more than one periodic orbit, then every non-periodic point x is heteroclinic under $f^{\circ q}$ to two points u and v that are in distinct periodic orbits.*

Theorem 3.2.42 (Poincaré Classification Theorem)**.** *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation-preserving homeomorphism. Suppose $\rho(f) = \rho \notin \mathbb{Q}$. Then there exists a continuous monotone map $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h \circ f = R_\rho \circ h$.*

- *If f is topologically conjugate, then h is a homeomorphism.*
- *If f is not topologically conjugate, then h is not invertible.*

Definition 3.2.43. A map $f : X \to X$ of a metric space is said to exhibit sensitive dependence on initial conditions if there is a $\Delta > 0$, called a *sensitivity constant*, such that for every $x \in X$ and $\epsilon > 0$ there exists a point $y \in X$ with $d(x, y) < \epsilon$ and $d(f^{\circ N}(x), f^{\circ N}(y)) \geq \Delta$ for some $N \in N$.

Theorem 3.2.44. *Chaotic maps exhibit sensitive dependence on initial conditions, except when the entire space consists of a single periodic orbit.*

Suggested Reading 3.2.45.

- [3, Section 4.3.2 - Section 4.3.6]. These sections contain a proof of Theorems 3.2.41 and 3.2.42.

3.3 Linear Maps of the Plane

Before we look at maps on the plane, consider any linear map $f : \mathbb{R} \to \mathbb{R}$. We know that $f(x) = kx + b$ for some $k, b \in \mathbb{R}$. The case $k = 0$ is trivial. We note that if $k = 1$, then all forward and backward orbits tend to $\pm\infty$, with the sign depending on the sign of b.

Now consider the case $k \neq 1$. In this case f has a unique fixed point at $x = \frac{b}{1-k}$. Then by taking $\varphi : \mathbb{R} \to \mathbb{R}$ be the translation $\varphi(x) = x - \frac{b}{1-k}$, we see that $g = \varphi \circ f \circ \varphi^{-1}(x) = kx$ for all $x \in \mathbb{R}$. In other words, f is linearly conjugate to the map $x \mapsto kx$. Thus we only need to look at the dynamics of g.

1. if $|k| > 1$, then $g^{\circ n}(x) \to \infty$ and $g^{\circ -n}(x) \to 0$ as $n \to \infty$.
2. if $|k| < 1$, then $g^{\circ n}(x) \to 0$ and $g^{\circ -n}(x) \to \infty$ as $n \to \infty$.
3. if $k = -1$, then g is a reflection, and so $g \circ g = \mathrm{id}_{\mathbb{R}}$,

Now consider linear maps on $\mathbb{R}^2$. Any such map is of the form $x \mapsto Ax$ where A is a 2×2 matrix with real entries. Note that it has two eigenvalues λ_1 and λ_2, where both are real or they are a complex conjugate pair.

Proposition 3.3.1. *If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$, then under a change of basis $P : \mathbb{R}^2 \to \mathbb{R}^2$,*

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Proof. Let v_1 and v_2 be a pair of eigenvectors for λ_1 and λ_2 respectively. Then by choosing P so that $P(e_1) = v_1$ and $P(e_2) = v_2$, we see that $P^{-1}AP(e_1) = \lambda_1 e_1$ and $P^{-1}AP(e_2) = \lambda_2 e_2$. Therefore this matrix has the required diagonal form.

□

Proposition 3.3.2. *If $\lambda_1 = \lambda_2 \in \mathbb{R}$, then under a change of basis $P : \mathbb{R}^2 \to \mathbb{R}^2$,*

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_1 \end{bmatrix}$$

where $a = 0$ if λ_1 has two linearly independent eigenvectors, and $a \neq 0$ otherwise.

Proof. Suppose there exist linearly independent eigenvectors v_1 and v_2 for λ_1, we proceed exactly as in the pervious proposition. Now suppose λ_1 only one eigenvector upto scaling, call it v_1. Then pick any basis change P so that $P(e_1) = v_1$. Then the matrix PAP^{-1} has the form

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_1 \end{bmatrix}$$

for some non-zero value a. □

Exercise 3.3.3. For any $a \in \mathbb{R}$ with $a \neq 0$, the matrices $\begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_1 \end{bmatrix}$ and $\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$ are linearly conjugate.

Remark 3.3.4. The above exercise shows that we can assume $a = 1$ in the previous proposition.

Exercise 3.3.5. Show that if λ_1 and λ_2 are a complex conjugate pair, then there exists a unique $\rho > 0$ and $\theta \in \mathbb{R}$ unique upto translation by 2π, and a linear change of coordinates P, such that

$$PAP^{-1} = \rho \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Contracting case

Proposition 3.3.6. *If both eigenvalues of A have absolute value < 1, then A is contracting and all forward orbits tend to 0.*

Proof. This is trivial, since the condition implies that $||A|| < 1$, so we have $||Ax|| \leq ||A||||x||$ for all x. □

Proposition 3.3.7. *If both eigenvalues are real and $|\lambda_1| < |\lambda_2| < 1$, then the curves $|y| = C|x|^{\frac{\log|\lambda_2|}{\log|\lambda_1|}}$ for all $C \in \mathbb{R}$ are invariant under A.*

Proof. By Proposition 3.3.1, we can assume $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Also note that $A^n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda_1^n x \\ \lambda_2^n y \end{bmatrix} = \lambda_1^n \begin{bmatrix} x \\ \left(\frac{\lambda_2}{\lambda_1}\right)^n y \end{bmatrix}$ and the orbit of every vector tends to 0. It suffices to show the following claim.

Claim. The forward orbit of any vector $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ lies on the curve $|y| = C|x|^{\frac{\log|\lambda_2|}{\log|\lambda_1|}}$ for some constant C that depends only on x_0, y_0.

Proof of Claim. Let $C = \frac{|y_0|}{|x_0|^{\frac{\log|\lambda_2|}{\log|\lambda_1|}}}$. We show that $\begin{bmatrix} x \\ y \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ lies on this curve as well.

$$\begin{aligned} \frac{|y|}{|x|^{\frac{\log|\lambda_2|}{\log|\lambda_1|}}} &= \frac{|\lambda_2^n|^{\frac{\log|\lambda_2|}{\log|\lambda_1|}}|y_0|}{|\lambda_1^n||x_0|^{\frac{\log|\lambda_2|}{\log|\lambda_1|}}} \\ &= \frac{|\lambda_1^n||y_0|}{|\lambda_1^n||x_0|^{\frac{\log|\lambda_2|}{\log|\lambda_1|}}} \\ &= C \end{aligned}$$

■

□

Exercise 3.3.8. If both eigenvalues are real and $|\lambda_1| = |\lambda_2| < 1$, then the curves $x = Cy + y\frac{log|y|}{log|\lambda_1|}$ are invariant.

Exercise 3.3.9. If neither eigenvalue is real and are of the form $\rho e^{\pm i\theta}$ with $0 < \rho < 1$, then the polar curves $(r; \varphi)$ where $r = Ce^{-(\theta^{-1}\log\rho)\varphi}$ are invariant.

Remark 3.3.10 (Expanding Case). If $|\lambda_1|, |\lambda_2| > 1$, then A is invertible and A^{-1} is contracting by the above discussion.

Hyperbolic/Saddle Case

Definition 3.3.11. A is said to be *hyperbolic* if $|\lambda_1| < 1$, and $|\lambda_2| > 1$.

Note that if A is hyperbolic both eigenvalues must be real, so A is diagonalizable.

Proposition 3.3.12.

3.4 Linear Maps in higher dimensions

Throughout this section, we let A be an $n \times n$ matrix over $\mathbb{R}$.

Definition 3.4.1. We call the set of eigenvalues of A (a subset of $\mathbb{C}$) the *spectrum* of A and denote it by $\operatorname{sp} A$. We denote the maximal absolute value of an eigenvalue of A by $r(A)$ and call it the spectral radius of A

Definition 3.4.2. A *norm* on $\mathbb{R}^n$ is a function $||.|| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

- $||v|| = 0$ if and only if $v = 0$
- $||cv|| = |c|||v||$ for all $c \in \mathbb{R}$, $v \in \mathbb{R}^n$
- $||u + v|| \leq ||u|| + ||v||$ for all $u, v \in \mathbb{R}^n$.

Given a norm $||.||$ on $\mathbb{R}^n$ and a matrix A, the matrix norm $||A||$ is given by

$$||A|| = \sup_{\substack{v \in \mathbb{R}^n \\ v \neq 0}} \frac{||Av||}{||v||}$$

Definition 3.4.3. Let $|A| = \max_{1 \leq i,j \leq n} |a_{ij}|$.

Lemma 3.4.4. *Let $||.||$ be the usual Euclidean norm on $\mathbb{R}^n$. Then*

$$|A| \leq ||A|| \leq \sqrt{n}|A|$$

Proof.

$$\begin{aligned} ||Av|| &= \sqrt{\sum_{i=1}^{n} \Big(\sum_{j=1}^{n} a_{ij} v_j \Big)^2} \leq |A| \sqrt{\sum_{i=1}^{n} \Big(\sum_{j=1}^{n} v_j \Big)^2} = \sqrt{n}|A|||v|| \\ \implies ||A|| &= \sup_{v} \frac{||Av||}{||v||} \leq \sqrt{n}|A| \end{aligned}$$

On the other hand,

$$\begin{aligned} &|a_{ij}| \leq ||Ae_j|| \leq ||A|| \qquad \forall i, j \\ \implies &|A| \leq ||A|| \end{aligned}$$

$\square$

Proposition 3.4.5. *For every δ, there exists a norm $||.||$ on $\mathbb{R}^n$ with respect to which*

$$||A|| \leq r(A) + \delta$$

Proof. The proof of this proposition is a HW exercise, but it uses the following lemma. By the lemma it suffices to show that there is a a norm for which

$$||A^n|| \leq C\Big(r(A) + \frac{\delta}{2}\Big)^n$$

for all $n \in \mathbb{N}$. $\square$

Lemma 3.4.6. *Consider $\mathbb{R}^n$ with any norm $||||$. If $C, \lambda > 0$ are constants such that $||A^n|| \leq C\lambda^n$ for all $n \in \mathbb{N}$, and if $\mu > \lambda$, then there is a norm $||.||$ on $\mathbb{R}^n$ with respect to which $||A|| \leq \mu$.*

Proof. Choose $n \in \mathbb{N}$ be such that $C\big(\frac{\lambda}{\mu}\big)^n < 1$. Then define

$$||v||_\mu = \sum_{i=0}^{n-1} \frac{||A^i v||}{\mu^i}$$

This does indeed define a norm on $\mathbb{R}^n$, and we have

$$\begin{aligned} ||Av||_\mu = \sum_{i=1}^{n} \frac{||A^i v||}{\mu^{i-1}} &= \mu\Big(||v||_\mu + \frac{||A^n v||}{\mu^n} - ||v||\Big) \\ &\leq \mu||v||_\mu - \Big(1 - C\frac{\lambda^n}{\mu^n}\Big)||v|| \\ &\leq \mu||v||_\mu \end{aligned}$$

Therefore,

$$||A||_\mu = \sup_{\substack{v\in\mathbb{R}^n \\ v\neq 0}} \frac{||Av||_\mu}{||v||_\mu} \leq \mu$$

$\square$

Contracting Case

Definition 3.4.7. Let V be a normed vector space. We call an operator A eventually contracting on V, if, with respect to some choice of norm $||\hat{.}||$, we have $||\hat{A}|| < 1$.

Proposition 3.4.8. *Given a linear map A of $\mathbb{R}^n$ such that all eigenvalues satisfy $|\lambda| < 1$, A is an eventually contracting map of $\mathbb{R}^n$.*

Proof. The given condition implies that $r(A) < 1$. Let $\delta > 0$ be such that $r(A) + \delta < 1$. By Proposition 3.4.5, the statement follows. $\square$

Hyperbolic/Saddle Case in $\mathbb{R}^3$

Proposition 3.4.9. *Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ so that $|\lambda_1|, |\lambda_2| < 1$ and $|\lambda_3| > 1$. Then there exist linear subspaces W_s and W_u, called stable and unstable sets respectively, which satisfy the following properties:*

1. $\mathbb{R}^3 = W_s \oplus W_u$
2. $A(W_s) \subseteq W_s$ *and* $A(W_u) = W_u$
3. *For every* $v \in W_s$, *we have* $A^n v \to 0$ *as* $n \to \infty$
4. *For every* $v \in W_u$, $A^n v \to \infty$ *as* $n \to \infty$, *and restricting* A *to* W_u *and taking the inverse, we have* $A^{-n} v \to 0$ *for all* $v \in W_u$.
5. *For all* $v \notin W_s \cup W_u$, $A^n v \to \infty$ *as* $n \to \infty$.

Proof. There are two mutually disjoint cases:

1. all three eigenvalues are real. In this case A is linearly conjugate to
$$\begin{bmatrix} \lambda_1 & * & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$
2. λ_1, λ_2 are complex conjugates and λ_3 is real. In this case A is linearly conjugate to
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

In both these cases, we can take W_s to be the xy plane and W_u to be the z axis. Using the results about linear maps of $\mathbb{R}^2$ where all eigenvalues are within the unit disk, the result follows. $\square$

General Case

Definition 3.4.10. Let λ be a real eigenvalue of A with multiplicity k. Then the generalized eigenspace of λ is the space

$$E_\lambda = \{v \in \mathbb{R}^n : (A - \lambda \operatorname{Id})^k v = 0\}$$

Definition 3.4.11. Let $\lambda, \overline{\lambda}$ be a pair of complex conjugate eigenvalues of A, and consider the complexification $A : \mathbb{C}^n \to \mathbb{C}^n$ given by $A(iv) = iA(v)$ for all $v \in \mathbb{R}^n$. Then we consider the generalized eigenspaces E_λ and $E_{\overline{\lambda}}$ in $\mathbb{C}^n$ defined as above, and let

$$E_{\lambda,\overline{\lambda}} = \mathbb{R}^n \cap (E_\lambda \bigoplus E_{\overline{\lambda}})$$

Definition 3.4.12. Let Λ be the set of real eigenvalues of A union one eigenvalue out of every complex conjugate pair.

$$E^- = \bigoplus_{\substack{\lambda\in\Lambda\cap\mathbb{R}\\|\lambda|<1}} E_\lambda \oplus \bigoplus_{\substack{\lambda\in\Lambda\setminus\mathbb{R}\\|\lambda|<1}} E_{\lambda,\bar{\lambda}}$$

$$E^+ = \bigoplus_{\substack{\lambda\in\Lambda\cap\mathbb{R}\\|\lambda|>1}} E_\lambda \oplus \bigoplus_{\substack{\lambda\in\Lambda\setminus\mathbb{R}\\|\lambda|>1}} E_{\lambda,\bar{\lambda}}$$

$$E^0 = \bigoplus_{\substack{\lambda\in\Lambda\cap\mathbb{R}\\|\lambda|=1}} E_\lambda \oplus \bigoplus_{\substack{\lambda\in\Lambda\setminus\mathbb{R}\\|\lambda|=1}} E_{\lambda,\bar{\lambda}}$$

It is clear that

$$\mathbb{R}^n = E^+ \oplus E^- \oplus E^0$$

We will not prove the following proposition, but its proof is similar to the arguments we have used in this section.

Proposition 3.4.13. *The spaces E^-, E^+ and E^0 are $A-$invariant. Let $z \in \mathbb{R}^n$.*

1. *If $z \in E^-$, then $Az \in E^-$ and $A^n z \to 0$ as $n \to \infty$.*

2. *$A|E^+$ is invertible, so for every $z \in E^-$, we have $A^{-n}z \to 0$ as $n \to \infty$ and $A^n z \to \infty$ as $n \to \infty$.*

3.5 Hyperbolic Toral Automorphisms

Arnold's Cat Map

Consider the following linear map of $\mathbb{R}^2$:

$$\begin{aligned} L\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2x+y \\ x+y \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

If two vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix}$ represent the same element of the torus $\mathbb{T}^2$, that is, if $\begin{bmatrix} x-x' \\ y-y' \end{bmatrix} \in \mathbb{Z}^2$, then $L\begin{bmatrix} x \\ y \end{bmatrix} - L\begin{bmatrix} x' \\ y' \end{bmatrix} \in \mathbb{Z}^2$, so $L\begin{bmatrix} x \\ y \end{bmatrix}$ and $L\begin{bmatrix} x' \\ y' \end{bmatrix}$ also represent the same element of $\mathbb{T}^2$.

Thus L defines a torus endomorphism $T_L : \mathbb{T}^2 \to \mathbb{T}^2$:

$$T_L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ x+y \end{bmatrix} (\mod 1)$$

Since $\det L = 1$, the map T_L is an automorphism of the torus. We note that $L^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

Exercise 3.5.1. The eigenvalues of L are $\lambda_1 = \frac{3+\sqrt{5}}{2} > 1$ and $\lambda_1^{-1} = \lambda_2 = \frac{3-\sqrt{5}}{2} < 1$.

Figure **??** gives an idea of the action of T_L on the fundamental square $I = \{(x, y) : 0 \leq x < 1, 0 \leq y \leq 1\}$. The lines with arrows are the eigendirections. For any matrix L with determinant ± 1, the map T_L preserves the area of sets on the torus.

Proposition 3.5.2. *Periodic points of* T_L *are dense and* $\mathrm{Per}_n(T_L) = \lambda_1^n + \lambda_2^n - 2$.

Proof. To obtain density we show that points with rational coordinates are periodic points. Let $x, y \in \mathbb{Q}$. Taking the common denominator write $x = \frac{s}{q}, y = \frac{t}{q}$, where $s, t, q \in \mathbb{Z}$. Then

$$T_L \begin{bmatrix} s/q \\ t/q \end{bmatrix} = \begin{bmatrix} \frac{2s+t}{q} \\ \frac{s+t}{q} \end{bmatrix}$$

is a rational point whose coordinates also have denominator q. But there are only q^2 different points on $\mathbb{T}^2$ whose coordinates can be represented as rational numbers with denominator q, and all iterates $T_L^n \begin{bmatrix} \frac{s}{q} \\ \frac{t}{q} \end{bmatrix}, n = 0, 1, 2 \cdots$, belong to that finite set. Thus they must repeat, that is, $T_L^n \begin{bmatrix} \frac{s}{q} \\ \frac{t}{q} \end{bmatrix} = T_L^m \begin{bmatrix} \frac{s}{q} \\ \frac{t}{q} \end{bmatrix}$ for some $n, m \in \mathbb{Z}$. But since T_L is invertible, $T_L^{n-m} \begin{bmatrix} \frac{s}{q} \\ \frac{t}{q} \end{bmatrix} = \begin{bmatrix} \frac{s}{q} \\ \frac{t}{q} \end{bmatrix}$ and $\begin{bmatrix} \frac{s}{q} \\ \frac{t}{q} \end{bmatrix}$ is a periodic point, as required. This gives density.

Next we show that points with rational coordinates are the only periodic points for T_L. Write $T_L^n \begin{bmatrix} \frac{s}{q} \\ \frac{t}{q} \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} (\mod 1)$, where $a, b, c, d \in \mathbb{Z}$. If $T_L^n \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$\begin{aligned} ax + by &= x + k \\ cx + dy &= y + l \end{aligned}$$

for some $k, l \in \mathbb{Z}$. Since 1 is not an eigenvalue for L^n, we can solve for (x, y):

$$\begin{aligned} x &= \frac{(d-1)k - bl}{(a-1)(d-1) - cb} \\ y &= \frac{(a-1)l - ck}{(a-1)(d-1) - cb} \end{aligned}$$

Thus $x, y \in \mathbb{Q}$. Now we calculate $\mathrm{Per}_n(T_L)$. The map

$$G = T_L^n - \mathrm{id} : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} (a-1)x + by \\ cx + (d-1)y \end{bmatrix} (\mod 1)$$

is a well-defined noninvertible map of the torus onto itself. As before, if T_L^n fixes $\begin{bmatrix} x \\ y \end{bmatrix}$, then then $(a-1)x + by$ and $cx + (d-1)y$ are integers;
hence $G \begin{bmatrix} x \\ y \end{bmatrix} = 0(\mod 1)$, that is, the fixed points of T_L^n are exactly the preimages of the

point $\begin{bmatrix}0\\0\end{bmatrix}$ under G. Modulo 1 these are exactly the points of $\mathbb{Z}^2$ in $(L^n - \mathrm{id})([0,1) \times [0,1))$. We presently show that their number is given by the area of $(L^n - \mathrm{id})([0,1) \times [0,1))$, which is $|\det(L^n - \mathrm{id})| = |(\lambda_1^n - 1)(\lambda_2^n - 1)| = \lambda_1^n + \lambda_2^n - 2$. $\square$

Lemma 3.5.3. *The area of a parallelogram with integer vertices is the number of lattice points it contains, where points on the edges are counted as half, and all vertices count as a single point.*

Proof. Denote the area of the parallelogram by A. Adding the number of lattice points it contains in the prescribed way gives an integer N, which is the same for any translate of the parallelogram. Now consider the canonical tiling of the plane by copies of this parallelogram translated by integer multiples of the edges. Denote by l the longest diagonal. The area of the tiles can be determined in a backward way by determining how many tiles lie in the square $[0, n) \times [0, n)$ for $n > 2l$. Those that lie inside cover the smaller square $[l, n - l) \times [l, n - l)$ completely, so there are at least

$$\frac{(n-2l)^2}{A} \geq \frac{n^2}{A} \cdot \left(1 - \frac{4l}{n}\right)$$

Since any tile that intersects the square is contained in $[l, n - l) \times [l, n - l)$, there are at most

$$\frac{(n-2l)^2}{A} = \frac{n^2}{A} \cdot \left(1 + \frac{4l}{n}\left(1 + \frac{\ell}{n}\right)\right) < \frac{n^2}{A}\left(1 + \frac{6\ell}{n}\right)$$

The number n^2 of integer points in the square is at least the number of points in tiles in the square and at most the number of points in tiles that intersect the square. Therefore

$$N \cdot \frac{n^2}{A} \cdot \left(1 - \frac{4l}{n}\right) \leq n^2 \leq N \cdot \frac{n^2}{A} \cdot \left(1 + \frac{6l}{n}\right)$$

And for all $n > 2l$,

$$\left(1 - \frac{4l}{n}\right) \leq \frac{N}{A} \leq \left(1 + \frac{6l}{n}\right)$$

This shows $N = A$.

$\square$

The eigenvectors corresponding to the first eigenvalue belong to the line $y = \frac{\sqrt{5}-1}{2}x$. The family of lines parallel to it is invariant under L, and L uniformly expands distances on those lines by a factor λ_1. Similarly, there is an invariant family of contracting lines $y = \frac{\sqrt{5}-1}{2}x + c$ where c is a constant.

In additive notation let $\gamma = [\gamma_1, \cdots, \gamma_n]^T \in \mathbb{T}^n$. Consider the natural multidimen- sional generalization of rotations given by the translation

$$T_\gamma \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 + \gamma_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n + \gamma_n \end{bmatrix} \pmod 1$$

If all coordinates of the vector γ are rational numbers, say $\gamma_i = \frac{p_i}{q_i}$ with relatively prime p_i and q_i for each $i = 1, ..., n$, then T_γ is periodic. Its minimal period is the least common multiple of the denominators $q_1, ..., q_n$. However, unlike the cases of the circle, minimality is not the only alternative to periodicity. For example, if $n = 2$ and $\gamma = [\alpha, 0]^T$, where α is an irrational number, then the torus $\mathbb{T}^2$ splits into a family of invariant circles $x^2 = c$, and every orbit stays on one of these circles and fills it densely.

Definition 3.5.4. A set $A \subset \mathbb{R}$ is said to be rationally independent if $x_1, ..., x_n \in A$ and $[k_1, ..., k_n] \in \mathbb{Z}^{n+1} \setminus \{0\}$ imply $\sum_i k_i \gamma_i \neq 0$.

Proposition 3.5.5. *The automorphism T_L is topologically mixing.*

Fix open sets $U, V \subset \mathbb{T}^2$. The $L-$invariant family of lines $\frac{\sqrt{5}-1}{2}x + c$ projects to $\mathbb{T}^2$ as an T_L-invariant family of orbits of the linear flow T^t_ω with irrational slope $\omega = \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix}$. By Proposition 5.1.3, this flow is minimal. Thus the projection of each line is everywhere dense on the torus, and hence U contains a piece J of an expanding line; furthermore, for any $\epsilon > 0$, there exists $T = T(\epsilon)$ and a segment of an expanding line of length T that intersects any ϵ-ball on the torus. Since all segments of a given length are translations of one another, this property holds for all segments. Now take ϵ such that V contains an ϵ-ball and $N \in \mathbb{N}$ such that $f^{\circ N}(J)$ has length at least T. Then $f^{\circ n}(J) \cap V \neq \emptyset$ for $n \geq N$ and thus $f^{\circ n}(U) \cap V \neq \emptyset$ for $n \geq N$.

Corollary 3.5.6. *The automorphism T_L is chaotic.*

Proof. Combine Propositions 3.5.2, 3.5.5 and Theorem 3.2.23. $\square$

3.6 Coding

The Smale Horseshoe

We now describe Smale's original "horseshoe". Let Δ be a rectangle in $\mathbb{R}^2$ and $f : \Delta \to \mathbb{R}^2$ a diffeomorphism of Δ onto its image such that the intersection $\Delta \cap f(\Delta)$ consists of two "horizontal" rectangles Δ_0 and Δ_1 and the restriction of f to the components $\Delta^i := f^{-1}(\Delta_i)$, $i = 0, 1$, is a hyperbolic linear map, contracting in the vertical direction and expanding in the horizontal direction.

This implies that the sets Δ^0 and Δ^1 are "vertical" rectangles. One of the simplest ways to achieve this effect is to bend into a "horseshoe"(see figure above), or rather into the shape of a permanent magnet, although this method produces some inconveniences with orientation. Another way, which is better from the point of view of orientation, is to bend roughly into a paperclip shape (see above). If the horizontal and vertical rectangles lie strictly inside Δ, then the maximal invariant subset $\Lambda = \bigcap_{n=-\infty}^{\infty} f^{\circ -n}(\Delta)$ is contained in the interior of Δ.

Definition 3.6.1. Let $\Omega = \{\bar{s} = \cdots s_{-2}s_{-1}.s_0s_1s_2 \cdots : s_i \in \{0,1\} \forall i \in \mathbb{Z}\}$ be the space of bi-infinite sequences over $\{0, 1\}$. Define the shift map $\sigma : \Omega \to \Omega$ by setting

$$\sigma(\cdots s_{-2}s_{-1}.s_0s_1s_2 \cdots) = \cdots s_{-1}s_0.s_1s_2 \cdots$$

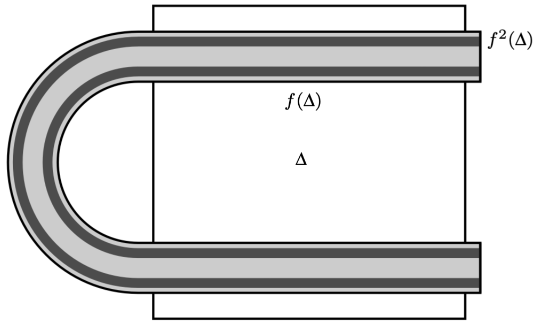

Figure 3.1: The horseshoe: the light grey set is $f(\Delta)$ and the dark grey set is $f^{\circ 2}(\Delta)$

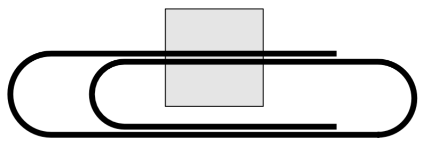

Figure 3.2: The paperclip

Just like the space $\Sigma = \Sigma_{\{0,1\}}$ of one-sided infinite sequences, Ω has a metric given by

$$d_\Omega(\overline{s}, \overline{t}) = \sum_{n\in\mathbb{Z}} \frac{d(s_n, t_n)}{2^{|n|}}$$

It can be shown that Ω is homeomorphic to the Cantor set, and that σ is continuous with respect to the topology induced by this metric. Additionally, in this case σ is invertible.

Proposition 3.6.2. *$f : \Lambda \to \Lambda$ is topologically conjugate to $\sigma : \Omega \to \Omega$.*

Proof. We use Δ^0 and Δ^1 as the"pieces" in the coding construction and start with positive iterates. The intersection $\Delta \cap f(\Delta) \cap f^{\circ 2}(\Delta)$ consists of four thin horizontal rectangles: $\Delta_{ij} = \Delta_i \cap f(\Delta_j) = f(\Delta^i) \cap f^{\circ 2}(\Delta^j)$, $i, j \in \{0, 1\}$.

Continuing inductively, one sees that $\cap_{i=0}^n f^{\circ i}(\Delta)$ consists of $2n$ thin disjoint horizontal rectangles whose heights are exponentially decreasing with n. Each such rectangle has the form $\Delta_w = \cap_{i=1}^n f^{\circ i}(\Delta^{w_i})$, where $w = w_1 w_2 \cdots w_n$ is a finite word over $\{0, 1\}$.

Each infinite intersection $\cap_{n=1}^\infty f^{\circ n}(\Delta^{w_n})$, $w_n \in \{0, 1\}$, is a horizontal segment, and the intersection $\cap_{n=1}^\infty f^{\circ n}(\Delta)$ is the product of the horizontal segment with a Cantor set in the vertical direction.

Similarly, one defines and studies vertical rectangles $\Delta^{w_0,\ldots,w_{-n}} = \cap_{i=0}^n f^{\circ(-i)}(\Delta^{w_{-i}})$, the vertical segments $\cap_{n=0}^\infty f^{\circ(-n)}(\Delta^{w_{-n}})$, and the set $\cap_{n=0}^\infty f^{\circ(-n)}(\Delta)$, which is the product of a segment in the vertical direction with a Cantor set in the horizontal direction. The desired invariant set= $\cap_{n=-\infty}^\infty f^{\circ(-n)}(\Delta)$ is the product of two Cantor sets and hence is a Cantor set itself, and the map

$$h : \Sigma \to \Lambda$$

given by

$$h(\cdots s_{-2}s_{-1}s_0.s_1s_2s_3\cdots) = \cap_{n=-\infty}^\infty f^{\circ(-n)}(\Delta^{s_n})$$

is a homeomorphism that conjugates σ and f. $\square$

Suggested Reading 3.6.3. [3, Section 7.4.4]

3.7 Maps on the real line

Sharkovsky's Theorem

3.8 Topological Entropy

Throughout this section, we let (X, d) be a compact metric space.

Remark 3.8.1. The following observation holds for every $r > 0$. There exists a finite set $E \subset X$ such that $\bigcup_{x \in E} B_d(x, r) \supset X$.

Capacity of a Compact Metric Space

Definition 3.8.2. Given $r > 0$, the $r-$capacity of X w.r.t. the metric d, denoted $S_d(r)$, is defined as

$$\mathrm{S}_d(r) = \left\{ |E| : |E| < \infty, \bigcup_{x \in E} B_d(x, r) \supset X \right\}$$

We observe that the $r-$capacity is a natural number.

Definition 3.8.3. The box dimension of X is given as

$$\text{bdim } X = \lim_{r \to 0} -\frac{\log S_d(r)}{\log r}$$

Example 3.8.4. When $X = [0, 1]$, $S_d(r) \approx \frac{1}{2r}$.

Example 3.8.5. When $X = [0, 1]^2$,

Now let $f : X \to X$ be a continuous dynamical system.

Definition 3.8.6. For $n \in \mathbb{N}$, the metric d_n^f is the maximum separation between orbit points truncated upto length $n-1$. In other words,

$$d_n^f(x, y) = \max_{0 \leq i \leq n-1} d(f^{\circ i}(x), f^{\circ i}(y)) \qquad \forall x, y \in X$$

Definition 3.8.7. The *topological entropy* of f is the following double limit:

$$h_{top}(f) := h_d(f) := \lim_{r \to 0^+} \limsup_n \frac{1}{n} \log S_{d_n^f}(r)$$

To show that this limit exists, we will give two alternative definitions of topological entropy and prove that they are all equivalent.

Separated Sets

Definition 3.8.8. Let $r > 0$ be a real number. A set $E \subset X$ is said to be $r-$separated w.r.t. the metric d if for every $x, y \in E$ with $x \neq y$, we have

$$d(x, y) \geq r$$

We let

$$\mathrm{Sep}_d(r) = \max\{|E| : E \subset X \text{ is } r-\text{separated}\}$$

Covers of Bounded Diameter

Definition 3.8.9. Let $r > 0$ be a real number. We define

$$\mathrm{Cov}_d(r) = \min\{|E| : E = \{U_1, \cdots, U_k\} \text{ is a finite open cover of X, with } \mathrm{diam}_d(U_i) < r \text{ for all } i\}$$

Proposition 3.8.10. *Let $f : X \to X$ be a continuous dynamical system and $r > 0$ be real. Then for every $n \in \mathbb{N}$, we have*

$$\mathrm{Cov}_{d_n^f}(2r) \leq S_{d_n^f}(r) \leq \mathrm{Sep}_{d_n^f}(r) \leq \mathrm{Cov}_{d_n^f}(r)$$

Proof. Fix $n \in \mathbb{N}$. Let A be a finite set of minimal cardinality such that

$$\bigcup_{y \in A} B_{d_n^f}(y, r) \supset X,$$

where $B_{d_n^f}(y, r)$ denotes balls centered at y with d_n^f-radius less than r. Then these balls have d_n^f-diameter less than $2r$, and since they cover X,

$$\mathrm{Cov}_{d_n^f}(2r) \leq \mathrm{S}_{d_n^f}(r),$$

proving the first inequality. Next, let B be an r-separated set of maximum cardinality w.r.t. metric d_n^f. So we cannot add any more points to B such that it still has the separated property. Then for every $x \in X \setminus B$ and every $y \in B$, the inequality $d_n^f(x, y) \geq r$ cannot hold. Therefore, for all $x \in X$ we can choose $y \in B$, such that $d_n^f(x, y) < r$. By definition of $r-$capacity, this means that

$$\mathrm{S}_{d_n^f}(r) \leq |B| = \mathrm{Sep}_{d_n^f}(r),$$

proving the second inequality.

Finally, consider B again and let C be a covering of X of minimal cardinality with sets of d_n^f-diameter less than r. Thus,

$$|C| = \mathrm{Cov}_{d_n^f}(r)$$

holds.

For the last inequality, we want to show that $|B| \leq |C|$. Suppose to the contrary that $|B| > |C|$. Then there exists an element of C which contains more than one point of B. In other words, there exists a set of d_n^f-diameter less than r which contains more than one point of B. Thus, there are two points in B which are less than r apart in the d_n^f-metric, contradicting the fact that B is r-separated. □

Corollary 3.8.11.

Suggested Reading 3.8.12.

3.8.13 Examples

Suggested Reading 3.8.14.

Chapter 4

Ergodic Theory

Basic Definitions

Let X be a set.

Definition 4.0.1. A collection of subsets $\mathscr{U}$ of X is said to be a $\sigma-$*algebra* if it is closed under complements and countable unions. The elements of $\mathscr{U}$ are called measurable sets.

A *measure* on $\mathscr{U}$ is a function $\mu : \mathscr{U} \to [0, \infty]$ satisfying the $\sigma-$additivity property, which requires that for any countable union of pairwise disjoint sets $U_i \in \mathscr{U}$ for $i \in \mathbb{N}$, we have

$$\mu\Big(\bigcup_{i\in\mathbb{N}} U_i\Big) = \sum_{i\in\mathbb{N}} \mu(U_i)$$

We call $(X, \mathscr{U}, \mu)$ a *measure space*. If $\mu(X) = 1$, we call μ a *probability measure* on X. If $\mu(X)$ is finite, then $\frac{\mu}{\mu(X)}$ is a probability measure.

Definition 4.0.2. If X is a topological space, the $\sigma-$algebra $\mathscr{U}$ generated by te open sets of X is called its *Borel* $\sigma-$algebra, and with any measure μ on $\mathscr{U}$, we call $(X, \mathscr{U}, \mu)$ a *Borel measure space*.

Let $(X, \mathscr{U}, \mu)$ be a measure space.

Definition 4.0.3. A set $A \in \mathscr{U}$ is called a *null set* if $\mu(A) = 0$. A set whose complement is a null set is called a *full set*, or said to have full measure in X.

Let $(X, \mathscr{U}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be measure space and $T : X \to Y$ be a function.

Definition 4.0.4. T is said to be *measurable* if for every $B \in \mathscr{B}$, we have $T^{-1}(B) \in \mathscr{U}$.

T is said to be *measure-preserving* if for every $B \in \mathscr{B}$, we have $\mu(T^{-1}(B)) = \nu(B)$.

If $(Y, \mathscr{B}, \nu) = (X, \mathscr{U}, \mu)$ and T is measure-preserving, we say the measure μ is $T-$invariant.

We denote by λ the Lebesgue measure on $\mathbb{R}^n$. This is a Borel measure: it assigns to each hypercube its Euclidean volume. So for $n = 1$, the Lebesgue measure of any interval is its Euclidean length.

We say that a property holds almost everywhere (a.e.) or $\mu-$ a.e. in X if it holds for all points in a set $A \in \mathscr{U}$ of full measure in X.

4.0.5 Poincaré Recurrence Theorem

Theorem 4.0.6. *Let T be a measure-preserving transformation of a probability space $(X, \mathscr{U}, \mu)$. If $A \in \mathscr{U}$. Then*

1. *For a.e. $x \in A$, there is some $n \in \mathbb{N}$ such that $T^{\circ n}(x) \in A$.*
2. *Consequently, for a.e. $x \in A$, there are infinitely many $k \in \mathbb{N}$ for which $T^{\circ k}(x) \in A$.*

Proof. Let

$$B = \{x \in A | T^{\circ k}(x) \notin A \text{ for all } k \in \mathbb{N}\} = A \setminus \bigcap_{k \in \mathbb{N}} T^{-\circ k}(A).$$

Then clearly, $B \in \mathscr{U}$. We know that every $k \in \mathbb{N}$, the set $T^{-\circ k}(B)$ is measurable and has the same measure as B.

Claim. For distinct integers $k, \ell \geq 0$, we have

$$T^{-\circ k}(B) \cap T^{-\circ \ell}(B) = \emptyset$$

Proof. WLOG suppose $k > \ell$. Suppose $x \in T^{-\circ k}(B) \cap T^{-\circ \ell}(B)$. Then since x is in the first set, we have $T^{\circ k}(x) \in B \subset A$. So $T^{\circ k}(x) \in A$. On the other hand, we have $T^{\circ \ell}(x) \in B$, and by definition of B, for any $n > 0$, we must have

$$T^{\circ n} \circ T^{\circ \ell}(x) \notin A.$$

But for $n = k - \ell$, this gives $T^{\circ (n+\ell)}(x) = T^{\circ k}(x) \notin A$, which is a contradiction. ■

By the $\sigma-$additivity property, we have

$$\mu\Big(\bigcup_{k \in \mathbb{N}} T^{-\circ k}(B)\Big) = \sum_{k=1}^{\infty} \mu(T^{-\circ k}(B)) = \sum_{k=1}^{\infty} \mu(B)$$

Since $\mu(X) < \infty$, the left side of the above equation is finite. This forces $\mu(B) = 0$. By definition of B, every point in $A \setminus B$ returns to A eventually. This proves the first point.

□

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