

MA 771: Introduction to Dynamical Systems  
Lecture Notes

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# Chapter 1

## Basic Concepts

Dynamical systems refer to systems that evolve over time. A simple example is an ordinary differential equation (ODE), such as

$$\frac{dx}{dt} = g(x)$$

where  $g : \mathbb{R}^n \rightarrow \mathbb{R}$ . Some of the questions we can ask about ODE's include:

- Which points are stable/fixed?
- When do orbits tend to an attracting periodic cycle?
- When are orbits chaotic?

However, the class of dynamical systems is much bigger than ODE's. In particular, in the above example, the variable  $t$ , representing time, changes *continuously*. In this course we will focus mainly on *discrete* dynamical systems.

**Definition 1.0.1.** Let  $X$  be a topological space. A *discrete* dynamical system is a pair  $(X, f)$  where  $f : X \rightarrow X$  is a self-map.

We are interested in the function  $f$  and its iterates  $f^{\circ n} = f \circ f^{\circ(n-1)}$  for  $n \in \mathbb{N}$ . In other words, the quantity that is varying in a *discrete* fashion is the number of iterations of  $f$ .

*Example 1.0.2. Linear maps* Let  $X = \mathbb{R}^n$  and  $f : x \mapsto Ax$  be a linear map.

*Example 1.0.3. Rotations of the Circle*  $X = \mathbb{S}^1$  and  $f(x) = e^{2\pi i \theta} x$  for some  $\theta \in \mathbb{R}$ .

*Example 1.0.4. Logistic Family* Let  $X = \mathbb{R}$  and fix  $k \in \mathbb{R}_{>0}$ . Then the family of maps  $f_k : X \rightarrow X$  given by  $x \mapsto kx(1 - x)$  is called the *logistic family*.

Notice that we are not assuming any conditions on  $f$  such as continuity.

### 1.1 Orbits and Periodic Points

**Definition 1.1.1.** Given a dynamical system  $(X, f)$ ,  $x_0 \in X$ , the sequence

$$x_0, f(x_0), f^{\circ 2}(x_0), \dots, f^{\circ n}(x_0), \dots$$

is called the *forward orbit* of  $x_0$ .

The *reverse orbit* of  $x_0$  is the set  $\{x \in X : f^{\circ n}(x) = x_0 \text{ for some } n \in \mathbb{N}\}$ .

A *fixed point*  $x \in X$  is a point such that  $f(x) = x$ . The set of fixed points of  $f$  is denoted  $\text{Fix}(f)$ . A *periodic point* is a point  $x$  such that  $f^{\circ n}(x) = x$  for some  $n \in \mathbb{N}$ , in other words, a point in  $\text{Fix}(f^{\circ n})$  for some  $n$ .

Any  $n \in \mathbb{N}$  such that  $f^{\circ n}(x) = x$  is said to be a *period* of  $x$ . The smallest period  $n$  is called the *exact* period of  $x$ .

## 1.2 Examples

### Linear Maps of $\mathbb{R}$

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be linear. We know that  $f$  is of the form  $f(x) = mx + b$  where  $m \in \mathbb{R}_{\neq 0}$  and  $b \in \mathbb{R}$ . Note that

$$f^{\circ n}(x) = m^n x + b(m^{n-1} + m^{n-2} + \cdots + m + 1) = m^n x + b \frac{m^n - 1}{m - 1}$$

- If  $m \neq \pm 1$ , then

$$\begin{aligned} |m| < 1 &\implies \forall x \in \mathbb{R}, f^{\circ n}(x) \rightarrow \frac{b}{1 - m} \text{ as } n \rightarrow \infty \\ |m| > 1 &\implies \forall x \in \mathbb{R}, f^{\circ n}(x) \rightarrow \infty \text{ as } n \rightarrow \infty \end{aligned}$$

- If  $m = 1$ , then  $f(x) = x + b$  is a translation and all orbits tend to  $\infty$
- If  $m = -1$ , then note that  $f^{\circ 2}(x) = -(-x + b) + b = x$ , and thus all the odd iterates are equal to  $f$ , and all the even iterates are equal to the identity.

### Circle Maps

*Example 1.2.1.* For any rotation  $f(x) = e^{2\pi i \theta} x$  of the circle  $\mathbb{S}^1$ , we have  $\text{Fix}(f) = \emptyset$  if  $\theta \notin \mathbb{N}$ , and  $\text{Fix}(f) = \mathbb{S}^1$  otherwise.

**Definition 1.2.2.** Fix an integer  $m > 1$ , and identify  $\mathbb{S}^1$  with  $\mathbb{R}/\mathbb{Z}$ . The *expanding map*  $E_m : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is defined as

$$E_m(x) = mx \pmod{1}$$

*Remark 1.2.3.*  $E_m$  is expanding in the following sense: if  $\alpha, \beta \in \mathbb{S}^1$  and  $d_{\mathbb{S}^1}(\alpha, \beta) < \frac{1}{m}$ , then

$$d_{\mathbb{S}^1}(\alpha, \beta) = m \cdot d_{\mathbb{S}^1}(E_m(\alpha), E_m(\beta)).$$

See Figure 1.1 for the graphs of  $E_m$  for  $m = 2, 3$ .

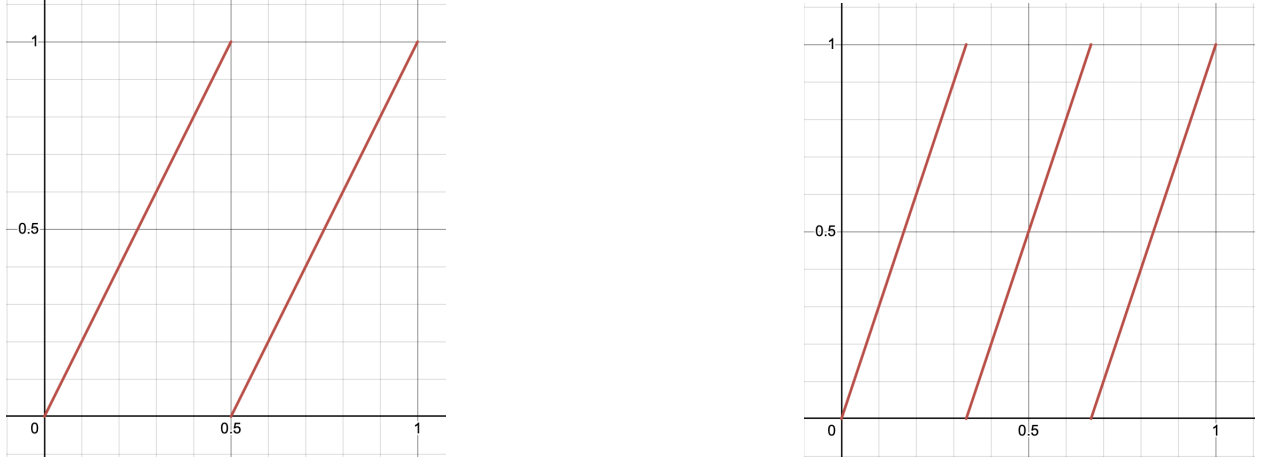


Figure 1.1: The graphs of the expanding maps  $E_2$  (left) and  $E_3$  (right) on the interior of  $\mathbb{S}^1$ , represented by the interval  $(0,1)$ .

Note that  $\phi$  is a fixed point of  $E_m$  if and only if  $m\phi - \phi \in \mathbb{Z}$ . In other words, there exists  $n \in \mathbb{Z}$  such that

$$\begin{aligned} m\phi - \phi &= n \\ \iff \phi &= \frac{n}{m-1} \end{aligned}$$

Similarly,  $\phi$  is a periodic point of  $E_m$  of period dividing  $k$  if and only if there exists  $n \in \mathbb{Z}$  such that

$$\begin{aligned} m^k \phi - \phi &= n \\ \iff \phi &= \frac{n}{m^k - 1} \end{aligned}$$

In other words,

$$\begin{aligned} \text{Fix}(E_m) &= \left\{ 0, \frac{1}{m-1}, \frac{2}{m-1}, \dots, \frac{m-2}{m-1} \right\} \\ \text{Fix}(E_m^{\circ k}) &= \left\{ 0, \frac{1}{m^k-1}, \frac{2}{m^k-1}, \dots, \frac{m^k-2}{m^k-1} \right\} \end{aligned}$$

## Torus Endomorphisms

Given  $n \in \mathbb{N}$ , the  $n$ -torus is the space  $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n = \mathbb{R}^n / \sim$  where  $x \sim y$  if  $x - y \in \mathbb{Z}^n$ . For  $x \in \mathbb{R}^n$ , we let  $[x]$  denote the equivalence class of  $x$  in  $\mathbb{T}^n$ .

**Definition 1.2.4.** Let  $A$  be an  $n \times n$  matrix whose entries are in  $\mathbb{Z}$ . Then  $A$  induces the torus endomorphism  $T_A : \mathbb{T}^n \rightarrow \mathbb{T}^n$  given by

$$T_A([x]) = [Ax] \text{ for } [x] \in \mathbb{T}^n$$

*Exercise 1.2.5.* Show that  $T_A$  as given above is well-defined: that is, for any two vectors  $v, w \in \mathbb{R}^n$ , if  $v - w \in \mathbb{Z}^n$ , then  $Av - Aw \in \mathbb{Z}^n$

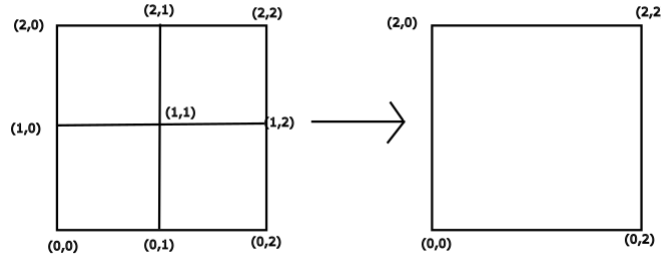


Figure 1.2: An illustration of the torus endomorphism  $T_A : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$  for  $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

*Example 1.2.6.* Let  $m, k \in \mathbb{Z}$  and consider the matrix  $A = \begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix}$ . Consider the map  $T_A$  on  $\mathbb{T}^2$ : this acts as two independent expanding maps: expansion by a factor of  $m$  in the  $x$ -direction, and expansion by a factor of  $k$  in the  $y$ -direction (see Figure 1.2 which illustrates the case  $m = k = 2$ ). Can you show in general that the degree of such a map is  $d = mk$ ? In other words,  $T_A$  is a  $d : 1$  map of  $\mathbb{T}^2$ .

**Definition 1.2.7.** A torus endomorphism  $T_A$  is said to be an *automorphism* if it is invertible.

*Exercise 1.2.8.* (This is also on HW 1) Show that  $T_A$  is invertible if and only if  $A^{-1}$  has integer entries, which in turn is equivalent to  $\det A = \pm 1$ .

**Proposition 1.2.9.** Let  $T_A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$  be a torus automorphism with no eigenvalues of modulus 1. Then the periodic points of  $T_A$  are all the points with rational coordinates.

*Proof.* (periodic  $\implies$  rational):

Let  $[x] = [(x_1, x_2, \dots, x_n)] \in \mathbb{T}^n$  be a periodic point of period  $q$  for some  $q \in \mathbb{N}$ . Then  $T_A^{oq}([x]) = [A^q x] = [x]$ . That is, there exists a vector  $y \in \mathbb{Z}^n$  such that

$$\begin{aligned} A^q x &= x + y \\ \implies A^q x - x &= y \\ \implies (A^q - \text{Id})x &= y \end{aligned}$$

Since  $A$  has no eigenvalues of modulus 1, the matrix  $A^q$  has no eigenvalues of modulus 1. This means that the matrix  $A^q - \text{Id}$  is invertible. So

$$x = (A^q - \text{Id})^{-1}y$$

Since  $y$  has integer coordinates and the matrix  $(A^q - \text{Id})^{-1}$  has rational entries,  $x$  has rational coordinates.

(rational  $\implies$  periodic):

Suppose  $x$  has rational coordinates, we can assume that all the coordinates have a common denominator. In other words,  $x = (\frac{p_1}{r}, \frac{p_2}{r}, \dots, \frac{p_n}{r})$  for some integers  $p_i, r$  with  $r \neq 0$ . Given a  $q \in \mathbb{N}$ , since  $A$  has integer entries,  $A^q x = (\frac{p'_1}{r}, \frac{p'_2}{r}, \dots, \frac{p'_n}{r})$  for some integers  $p'_1, \dots, p'_n$ .

Note that there are only finitely many points in  $\mathbb{T}^n$  with rational coordinates with a common denominator  $r$ . In other words, the set  $\{T_A^{\circ q}([x]) : q \in \mathbb{N}\}$  is finite.

Thus, there exist  $q_1 < q_2 \in \mathbb{N}$  such that  $T_A^{\circ q_1}([x]) = T_A^{\circ q_2}([x])$ . Since  $T_A$  is an automorphism, this means that  $T_A^{\circ(q_2 - q_1)}([x]) = [x]$ .  $\square$

## 1.3 Stable Behavior: The Contraction Principle

In this section we will look at maps on subsets of  $\mathbb{R}^n$  which satisfy a criterion for all orbits converging to a fixed point.

### Global Contractions

**Definition 1.3.1.** A map  $f$  of a subset  $X$  of  $\mathbb{R}^n$  is said to be *Lipschitz-continuous* with Lipschitz constant  $\lambda$ , or  $\lambda$ -*Lipschitz* if

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for any  $x, y \in X$ .

The map  $f$  is said to be a *contraction* if

$$d(f(x), f(y)) < d(x, y) \quad \forall x, y \in X.$$

It is said to be a  $\lambda$ -*contraction* for  $\lambda < 1$  if

$$d(f(x), f(y)) \leq \lambda d(x, y) \quad \forall x, y \in X.$$

*Remark 1.3.2.* If a map  $f$  is Lipschitz-continuous, then we define

$$Lip(f) := \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}$$

*Example 1.3.3.*  $f(x) = \sqrt{x}$  defines a contraction on  $I = [1, \infty)$ . What is  $Lip(f)$ ?

**Theorem 1.3.4** (Contraction Principle in  $\mathbb{R}^n$ ). *Let  $X \subset \mathbb{R}^n$  be closed and  $f : X \rightarrow X$  be a  $\lambda$ -contraction. Then  $f$  has a unique fixed point  $x_0$  and  $d(f^{\circ n}(x), x_0) = \lambda^n d(x, x_0)$  for every  $x \in X$ .*

*Proof.* We have

$$d(f^{\circ n}(x), f^{\circ n}(y)) \leq \lambda^n d(x, y)$$



for all  $x, y \in X$ . But this also means that for any  $x \in X$ , we have

$$d(f^{\circ(n+1)}(x), f^{\circ n}(x)) \leq \lambda^n d(f(x), x)$$

$$\begin{aligned} d(f^{\circ m}(x), f^{\circ n}(x)) &\leq d(f^{\circ m}(x), f^{\circ(m-1)}(x)) + d(f^{\circ(m-1)}(x), f^{\circ(m-2)}(x)) + \cdots d(f^{\circ(n+1)}(x), f^{\circ n}(x)) \\ &\leq (\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^n) d(f(x), x) \\ &\leq \frac{\lambda^n(1 - \lambda^{m-n})}{1 - \lambda} d(f(x), x) \\ &\leq \frac{\lambda^n}{1 - \lambda} d(f(x), x) \end{aligned}$$

In other words, the orbit of  $x$  is a Cauchy sequence. Since  $X$  is closed,  $\lim_{n \rightarrow \infty} f^{\circ n}(x) = x_0$  is a point of  $X$ , and

$$f(x_0) = f(\lim_{n \rightarrow \infty} f^{\circ n}(x)) = \lim_{n \rightarrow \infty} f^{\circ(n+1)}(x) = x_0$$

□

*Remark 1.3.5.* Given a sequence  $(y_n)_{n \geq 0}$  in a metric space  $(Y, d)$ , we say that  $y_n \rightarrow y \in Y$  *exponentially* if there exist constants  $A > 0$  and  $0 < \lambda < 1$  such that

$$d(y_n, y) \leq A\lambda^n d(y_0, y)$$

Note that in the above situation, the orbit under  $f$  of  $x$  converges exponentially to  $x_0$  (here  $A = 1$ ).

The contraction principle applies to  $\lambda$ -contractions defined on complete metric spaces.

**Theorem 1.3.6** (Contraction Principle for complete metric spaces). *Let  $(X, d)$  be a complete metric space and  $f : X \rightarrow X$  be a  $\lambda$ -contraction. Then there exists a unique fixed point  $x_0 \in X$  such that the orbits under  $f$  of all points  $x \in X$  converge exponentially to  $x_0$ .*

*Example 1.3.7* (Rabbits; due to Fibonacci). Say we record the number of rabbits in a forest starting January (month 0) of a given year. The Fibonacci model for rabbit population growth is as follows:

Letting  $b_n$  denote the number (in hundreds) of rabbits at the beginning of month  $n$ , we assume

$$\begin{aligned} b_0 &= 1 \\ b_1 &= 2 \\ b_n &= b_{n-1} + b_{n-2} \text{ for } n \geq 2. \end{aligned}$$

Then it is expected that the rabbit population growth rate stabilises as  $n \rightarrow \infty$ . That is, there exists  $a \in (0, \infty)$  such that  $\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} = b$ . This means that in the long term, the rabbit population grows approximately *exponentially*, by a rough factor of  $b$  each month.

To prove the existence of a value  $a$  as required, we let  $a_n = \frac{b_{n+1}}{b_n}$ . Note that

$$a_{n+1} = \frac{b_{n+2}}{b_{n+1}} = \frac{b_{n+1} + b_n}{b_{n+1}} = 1 + \frac{b_n}{b_{n+1}} = 1 + \frac{1}{a_n}$$

Letting  $g(x) = 1 + \frac{1}{x}$ , we see that

$$a_{n+1} = g^{\circ n}(a_0) = g^{\circ n}(2) \text{ for all } n \geq 0$$

*Claim:* There exists a closed interval  $I \subset \mathbb{R}$  such that  $a_0 = 2$  such that

- $g(I) \subset I$
- $g$  is a  $\lambda$ -contraction on  $I$  for some  $\lambda \in (0, 1)$ , and
- $a_0 = 2 \in I$

If we can prove this claim, then by the contraction principle,  $a$  can be recovered as the unique fixed point of  $g$  in  $I$ .

*Proof of claim:* The function  $g$  is decreasing on  $(0, \infty)$ , and has the horizontal asymptote  $y = 1$ . Note that  $g'(x) = \frac{-1}{x^2}$ .

This means  $\forall x \in [c, \infty)$  where  $c > 1$ ,

$$\begin{aligned} |g'(x)| &= \frac{1}{x^2} \leq \frac{1}{c^2} < 1 \\ \implies |g(x) - g(y)| &\leq \frac{1}{c^2} |x - y| \text{ for all } x, y \in [c, \infty) \end{aligned}$$

In other words, for all  $c > 1$ ,  $g : [c, \infty) \rightarrow \mathbb{R}$  is a  $\lambda$ -contraction with  $\lambda = \frac{1}{c^2}$ . Also note  $g$  has a unique positive fixed point  $x_0$ : we can find it by solving the equation  $g(x) = x$ .

$$\begin{aligned} g(x) &= x \\ \implies 1 + \frac{1}{x} &= x \\ \implies x^2 - x + 1 &= 0 \\ \implies x &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

So  $x_0 = \frac{1+\sqrt{5}}{2}$ . Note that  $\frac{3}{2} < x_0 < 2$ .

Let  $I = [\frac{3}{2}, 2]$ , Then we have

$$g(2) = \frac{3}{2} \text{ and } g(3/2) = \frac{5}{3} < 2.$$

we see that  $g(I) \subset I$ . By the above discussion,  $g$  is a  $\lambda$ -contraction on  $I$  (with  $\lambda = (2/3)^2$ ), and thus the orbit under  $g$  of  $a_0 = 2$  converges to  $a = x_0 = \frac{1+\sqrt{5}}{2}$ .

*Remark 1.3.8.* The choice of  $I$  is not unique: for any  $c \in (1, 3/2]$ , we have  $g[c, 2] = [3/2, g(c)] \subset [c, 2]$ , and  $g$  is a  $\lambda$  contraction on  $[c, 2]$  with  $\lambda = \frac{1}{c^2}$ . I made a small mistake in class by saying  $c$  can be in  $[1, x_0]$ : can you see why  $c \in (3/2, x_0]$  won't work?

## Local Contractions

**Proposition 1.3.9.** *Let  $f$  be a continuously differentiable map of  $\mathbb{R}^n$  with a fixed point  $x_0$  where  $\|Df_{x_0}\| < 1$ . Then there is a closed neighborhood  $U$  of  $x_0$  such that  $f(U) \subset U$  and  $f$  is a contraction on  $U$ .*

To do this we will need the following exercise and proposition:

*Exercise 1.3.10.* Given a linear map  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , recall that

$$\|A\| = \sup_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}$$

Prove that  $A \mapsto \|A\|$  is a continuous function from  $\mathbb{R}^{m \times n}$  to  $\mathbb{R}$ .

**Proposition 1.3.11.** *Let  $V \subset \mathbb{R}^n$  be a closed disk and let  $f : V \rightarrow \mathbb{R}^m$  be continuous, with continuous derivative on the interior of  $V$ . Suppose there exists  $M > 0$  such that  $\|Df_x\| \leq M$  for all  $x$  in the interior of  $V$ . Then*

$$d(f(x), f(y)) \leq Md(x, y) \quad \forall x, y \in V$$

*Proof.* Given  $x, y \in \mathbb{R}^n$ , let  $g : [0, 1] \rightarrow \mathbb{R}^m$  be the function

$$g(t) = f((1-t)x + ty)$$

Then the mean value theorem states that for some  $c \in (0, 1)$ ,

$$d(g(0), g(1)) \leq \|g'(c)\|$$

From this we get

$$\begin{aligned} d(f(x), f(y)) &= d(g(0), g(1)) \leq \|g'(c)\| \\ &= \|Df_{(1-c)x+cy}(y-x)\| \leq \|Df_{(1-c)x+cy}\| \cdot d(x, y) \\ &\leq Md(x, y) \end{aligned}$$

□

*Proof of Proposition 1.3.9.* The function  $f$  is  $C^1$  implies that  $x \mapsto Df$  is continuous. By Exercise 4, the composition  $x \mapsto Df_x \mapsto \|Df_x\|$  is continuous. Fix a point  $\lambda \in (\|Df_{x_0}\|, 1)$ . Then there exists a small closed ball  $U = \overline{B(x_0, \delta)}$  around  $x_0$  on which  $\|Df_x\| \leq \lambda < 1$ .

By Proposition 1.3.11, if  $x, y \in U$ , then  $d(f(x), f(y)) \leq \lambda d(x, y)$ . Moreover, for all  $x \in U$ , we have

$$d(f(x), x_0) = d(f(x), f(x_0)) \leq \lambda d(x, x_0) \leq \lambda \delta < \delta.$$

This shows that  $f(U) \subset U$ , and  $f$  is a  $\lambda$ -contraction on  $U$ .

□

*Suggested Reading* 1.3.12. • [3, Section 2.2]

## 1.4 Fractals

In this section we will define fractals and introduce self-similarity. We will also give an idea of their connection with dynamical systems with some examples.

### The Cantor Set

The simplest example of a fractal is the ternary cantor set.

**Definition 1.4.1.** Let  $I = [0, 1]$ . Inductively define closed subsets  $C_n \subset I$  for  $n \geq 0$  as follows:

$$\begin{aligned} C_0 &= [0, 1] \\ C_1 &= C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \frac{C_0}{3} \cup \left(\frac{C_0}{3} + \frac{2}{3}\right) \\ C_n &= \frac{C_{n-1}}{3} \cup \left(\frac{C_{n-1}}{3} + \frac{2}{3}\right) = C_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) \text{ for all } n \geq 2 \end{aligned}$$

It can be shown that the set  $C_n$  is the disjoint union of  $2^n$  closed intervals, each of length  $\frac{1}{3^n}$ . The ternary Cantor set is defined as

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$$

We look at some of the properties of  $\mathcal{C}$ .

1.  **$\mathcal{C}$  is closed**, since it is the intersection of closed sets
2.  **$\mathcal{C}$  is compact**, since it is a closed subset of a compact set.
3.  **$\mathcal{C}$  is non-empty.**

For example, the points 0 and 1 belong to all the sets  $C_n$ , so they also belong to  $\mathcal{C}$ .

4.  **$\mathcal{C}$  is uncountable.**

We will show this by giving an explicit description of  $\mathcal{C}$ .

**Definition 1.4.2.** Given a number  $x \in [0, 1]$ , a base 3 (or *ternary*) expansion for  $x$  is a sequence  $.\alpha_1\alpha_2\alpha_3\cdots$  with  $\alpha_n \in \{0, 1, 2\}$  for all  $n \in \mathbb{N}$  such that

$$x = \frac{\alpha_1}{3} + \frac{\alpha_2}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$$

The decimal point before the  $\alpha$ 's indicate that the number is less than or equal to 1. More generally, for any real number  $y \in \mathbb{R}$ . The ternary expansion of  $y$  is a sequence  $\beta_{m-1}\beta_{m-2}\cdots\beta_0.\alpha_1\alpha_2\alpha_3\cdots$  with  $\beta_i \in \{0, 1, 2\}$  for all  $i \in \{0, 1, 2, \cdots m-1\}$  and  $\alpha_n \in \{0, 1, 2\}$  for all  $n \in \mathbb{N}$ , such that

$$y = \sum_{i=0}^{m-1} \beta_i \cdot 3^i + \sum_{n=0}^{\infty} \frac{\alpha_n}{3^n}$$

Note that the ternary expansion of a number is not unique. For example,

$$\frac{1}{3} = \sum_{n=1}^{\infty} \frac{2}{3^n}$$

So  $.100000\cdots$  and  $.022222\cdots$  are both ternary expansions for  $\frac{1}{3}$ . Similarly,  $.200000\cdots$  and  $.122222\cdots$  are both ternary expansions for  $\frac{2}{3}$ .

*Exercise 1.4.3.* Every number  $x \in \mathbb{R}$  has only finitely many ternary expansions.

*Remark 1.4.4.* If  $x$  has a ternary expansion  $.\alpha_1\alpha_2\alpha_3\cdots$ , then  $\frac{x}{3}$  has a ternary expansion  $.0\alpha_1\alpha_2\cdots$ .

**Proposition 1.4.5.**

$\mathcal{C} = \{x \in [0, 1] : \text{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \text{ for } x \text{ with } \alpha_n \in \{0, 2\} \text{ for all } n \in \mathbb{N}\}$

*Proof.* We prove this by induction. Note that

$$\mathcal{C}_1 = \{x \in [0, 1] : \text{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \text{ for } x \text{ with } \alpha_1 \in \{0, 2\}\}$$

Using the recursive formula for  $\mathcal{C}_n$ , it is easy to show that

$$\mathcal{C}_N = \{x \in [0, 1] : \text{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \text{ for } x \text{ with } \alpha_1, \alpha_2, \dots, \alpha_N \in \{0, 2\}\}$$

Since every  $x$  has only finitely many ternary expansions, for  $x$  to be in all the  $\mathcal{C}_n$ 's, there exists at least one ternary expansion which satisfies the condition  $\alpha_n \in \{0, 2\}$  for all  $n \in \mathbb{N}$ .  $\square$

Since every sequence in  $\{0, 2\}^{\mathbb{N}}$  can be realized as the ternary expansion of a distinct number  $x \in [0, 1]$ , and the set  $\{0, 2\}^{\mathbb{N}}$  is uncountable, we see that  $\mathcal{C}$  is uncountable.

**5.  $\mathcal{C}$  is perfect.**

**Definition 1.4.6.** Let  $Y$  be a topological space. A subset  $X \subset Y$  is said to be perfect if it is closed in  $Y$  and has no isolated points.

**Proposition 1.4.7.** For every  $x \in \mathcal{C}$ , there exists a sequence  $(x_n)$  of distinct points with  $x_n \in \mathcal{C}$  and  $x_n \rightarrow x$ .

*Proof.* Given  $\epsilon > 0$ , we will exhibit a point  $x_N \neq x$  such that  $|x_N - x| < \epsilon$  and  $x_N \in \mathcal{C}$ . Choose  $N \in \mathbb{N}$  such that  $\frac{2}{3^N} < \epsilon$ . This ensures that the interval of radius  $\frac{1}{3^N}$  centered at  $x$  is contained in the open ball  $B_\epsilon(x)$ . Let  $\tilde{I}$  be the component interval of  $\mathcal{C}_N$  that contains  $x$ . The above condition implies that  $\tilde{I} \subset B_\epsilon(x)$ .

In  $\mathcal{C}_{N+1}$ , the middle third of  $\tilde{I}$  is deleted, and we get two component intervals  $\tilde{I}_0$  and  $\tilde{I}_1$ . Without loss of generality, assume  $x \in \tilde{I}_0$ . Then pick a point  $y \in \mathcal{C} \cap \tilde{I}_1$ . Note that  $y \neq x$  by this choice and since  $\tilde{I}_1 \subset B_\epsilon(x)$ , we have  $|y - x| < \epsilon$ . Therefore we can set  $x_N = y$ .  $\square$

## 6. $\mathcal{C}$ is totally disconnected.

**Definition 1.4.8.** A topological space  $X$  is said to be totally disconnected if its only non-empty connected subsets are singletons.

**Proposition 1.4.9.** *If  $F \subset \mathcal{C}$  is non-empty and connected, then  $F = \{x\}$  for some point  $x \in \mathcal{C}$ .*

*Proof.* Suppose  $x, y \in \mathcal{C}$  are two distinct points in  $F$ . WLOG, assume  $x < y$ . Pick  $N \in \mathbb{N}$  such that  $\frac{1}{3^{N-1}} < |x - y|$ . Then,  $x$  and  $y$  are contained in distinct components of  $C_N$ . So there exists  $z \in (x, y)$  such that  $z \notin \mathcal{C}$ . Let  $A = F \cap [0, z)$  and  $B = F \cap (z, 1]$ . Note that  $A \cup B = F$ . Also note that the closures of  $A$  and  $B$  don't intersect. This contradicts the fact that  $F$  is connected.  $\square$

## 7. $\mathcal{C}$ has Lebesgue measure 0.

Let  $\mu$  denote Lebesgue measure. The set  $\mathcal{C}_n$  is the union of  $2^n$  disjoint intervals, each of length  $3^{-n}$ . Therefore, we have  $\mu(\mathcal{C}_n) = \frac{2^n}{3^n}$ . Since  $\mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3 \supset \cdots$  and  $\mathcal{C}$  is the intersection of the  $\mathcal{C}_n$ , we have

$$\mu(\mathcal{C}) = \lim_{n \rightarrow \infty} \mu(\mathcal{C}_n) = 0$$

The following theorem is the main result of this section. For a proof see [2].

**Theorem 1.4.10 (Brouwer).** *Let  $Y \neq \emptyset$  be a complete metric space. If  $Y$  is compact, perfect and totally disconnected, then it is homeomorphic to  $\mathcal{C}$ .*

An easy corollary, for example, is that  $\mathcal{C}$  is homeomorphic to  $\mathcal{C} \times \mathcal{C}$ .

*Suggested Reading* 1.4.11. • [3, Section 2.7.1]

## Dynamical Systems on the Cantor Set

Consider the map  $f : [0, 1] \times [0, 1]$  defined as  $f(x) = \frac{x}{3}$ . It is easy to see that  $f$  is a contraction and  $f(\mathcal{C}) \subset \mathcal{C}$ , and the unique fixed point in  $\mathcal{C}$  is  $x = 0$ . Note that for every  $x \in \mathcal{C}$ , there exists a neighborhood  $U$  of  $x$  such that  $f : U \rightarrow f(U)$  is a homeomorphism.

Also note that  $f$  induces the shift  $.a_1a_2a_3 \cdots \rightarrow .0a_1a_2a_3 \cdots$  on ternary expansions.

**Definition 1.4.12.** A topological space  $X$  is said to be *self-similar*, or to have the *rescaling property*, if there exists a contraction  $f : X \rightarrow X$  such that for every  $x \in X$  and neighborhood  $U$  of  $x$ , there exists a neighborhood  $V \subset U$  of  $x$  such that  $f : V \rightarrow f(V)$  is a homeomorphism.

*Remark 1.4.13.* This is actually equivalent to saying that every  $x \in X$  has a neighborhood  $U$  such that  $f : U \rightarrow f(U)$  is a homeomorphism. Also note that the term self similar is used in different ways in the literature; we will see by and by that this definition is not extensive enough.

*Exercise 1.4.14.* Show that the function  $f(x) = 1 - \frac{x}{3}$  leaves  $\mathcal{C}$  invariant, and is a contraction. Describe the induced operation on ternary expansions, and find the unique fixed point of  $f$  in  $\mathcal{C}$ .

*Exercise 1.4.15.* Show that the function  $f(x) = \frac{x+2}{3} \pmod{1}$  leaves  $\mathcal{C}$  invariant, and is a contraction. Describe the induced operation on ternary expansions, and find the unique fixed point of  $f$  in  $\mathcal{C}$ .

## The Square Sierpinski Carpet

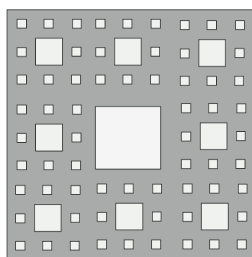


Figure 1.3: The set  $\mathcal{J}_3$  of the Sierpinski carpet construction

**Definition 1.4.16.** Let  $J = [0, 1] \times [0, 1]$  be the unit square. Define

$$\begin{aligned}\mathcal{J}_0 &= J \\ \mathcal{J}_1 &= \mathcal{J}_0 \setminus \left( \frac{1}{3}, \frac{2}{3} \right) \times \left( \frac{1}{3}, \frac{2}{3} \right) \\ \mathcal{J}_n &= \mathcal{J}_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \bigcup_{\ell=0}^{3^{n-1}-1} \left( \frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right) \times \left( \frac{3\ell+1}{3^n}, \frac{3\ell+2}{3^n} \right)\end{aligned}$$

The square Sierpinski carpet is the set  $\mathcal{J} = \bigcap_{n=0}^{\infty} \mathcal{J}_n$ .

*Exercise 1.4.17.* Prove that the Sierpinski carpet is self-similar.

## The Sierpinski Triangle

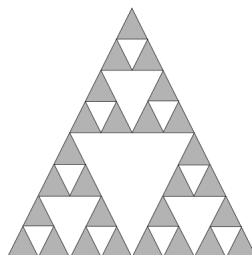


Figure 1.4: The set  $\Delta_2$  of the Sierpinski triangle construction

This set is similar to the Sierpinski carpet. Start with an equilateral triangle  $\Delta_0$  of side length 1, with one side horizontal. Let  $\Delta_1$  be  $\Delta_0$  minus its central equilateral triangle,  $\Delta_2$  be  $\Delta_1$  minus its three smaller central equilateral triangles and so on. Define the Sierpinski triangle

$$\Delta = \bigcap_{n=0}^{\infty} \Delta_n$$

*Exercise 1.4.18.* Prove that the Sierpinski triangle is self-similar.

*Exercise 1.4.19.* Prove that  $\mathcal{J}$  and  $\Delta$  both have infinite perimeter and finite area (Lebesgue measure).

*Exercise 1.4.20.* Prove that neither  $\mathcal{J}$  nor  $\Delta$  is homeomorphic to  $\mathcal{C}$ .

*Suggested Reading 1.4.21.* • [3, Section 2.7.2]

## Cantor Sets and Logistic Maps

Consider the logistic function  $f(x) = 5x(1-x)$  on  $\mathbb{R}$ . Note that this function has one critical point at  $x = \frac{1}{2}$ , and is symmetric around this point in the sense that

$$f(x) = f(1-x) \text{ for all } x \in \mathbb{R}$$

We make the following series of observations:

1. The graph of  $f$  is a downward drawn parabola, and its roots are  $x = 0, 1$ .
2. If  $x > 1$ , then  $f(x) < 0$ .
3. If  $x < 0$ , then  $f(x) < x$  and  $|f(f(x)) - f(x)| > |f(x) - x|$ .

The points (2) and (3) show that if  $x \notin [0, 1]$ , then  $f^{on}(x) \rightarrow -\infty$ . This leads to the following dichotomy:

For every  $x \in \mathbb{R}$ , exactly one of the following is true:

- either  $f^{on}(x) \in [0, 1]$  for all  $n \in \mathbb{N}$ , or
- $f^{om}(x) \notin [0, 1]$  for some  $m \in \mathbb{N}$ , and thus,  $f^{on}(x) \rightarrow -\infty$  as  $n \rightarrow \infty$ .

Therefore, the set  $B$  of points  $x \in \mathbb{R}$  such the orbit  $(f^{on}(x))_{n \geq 0}$  is bounded, is the set of points  $x$  such that  $f^{on}(x) \in [0, 1]$  for all  $n$ . In other words,

$$B = \bigcap_{n=0}^{\infty} (f^{on})^{-1}[0, 1]$$

**Proposition 1.4.22.**  *$B$  is a Cantor set (i.e., it is homeomorphic to  $\mathcal{C}$ ).*

We will prove this in the next chapter.



## 1.5 Topological Conjugacy

**Definition 1.5.1.** Let  $X, Y$  be topological spaces and suppose  $f : X \rightarrow X$  and  $g : Y \rightarrow Y$  be dynamical systems. Then  $(X, f)$  and  $(Y, g)$  are said to be *topologically conjugate* if there exists a homeomorphism  $\varphi : X \rightarrow Y$  such that

$$g \circ \varphi = \varphi \circ f$$

In other words,  $\varphi$  is a homeomorphism that makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow f & & \downarrow g \\ X & \xrightarrow{\varphi} & Y \end{array}$$

Note that for all  $n \in \mathbb{N}$ ,

$$g^{on} = (\varphi \circ f \circ \varphi^{-1})^{on} = (\varphi \circ f \circ \varphi^{-1}) \circ (\varphi \circ f \circ \varphi^{-1}) \circ \cdots \circ (\varphi \circ f \circ \varphi^{-1}) = \varphi \circ f^{on} \circ \varphi^{-1}$$

In particular,  $\varphi$  maps the  $f$ -orbit of  $x$  to the  $g$ -orbit of  $\varphi(x)$  for every  $x \in X$ . So topological conjugacy is a form of equivalence between two dynamical systems.

### Examples

*Example 1.5.2.* The map  $\varphi(x) = \frac{-1}{2}x + \frac{1}{2}$  conjugates the dynamical systems  $f(x) = x^2$  and  $g(x) = 2x(1 - x)$  on  $\mathbb{R}$ . Since  $\varphi$  is linear, we say that  $(\mathbb{R}, f)$  and  $(\mathbb{R}, g)$  are *linearly/affinely* conjugate.

*Exercise 1.5.3.* Prove that every quadratic polynomial  $f : \mathbb{C} \rightarrow \mathbb{C}$  is affine conjugate to a polynomial of the form  $z^2 + c$  for a unique  $c \in \mathbb{C}$ .

*Exercise 1.5.4.* Let  $W$  and  $V$  be vector spaces over  $\mathbb{R}$ , and suppose  $A : W \rightarrow W$  and  $B : V \rightarrow V$  are linear maps. Show that if  $A$  and  $B$  are conjugate as linear maps, i.e., there exists an invertible linear map  $L : W \rightarrow V$  such that  $LA = BL$ , then they are also topologically conjugate in the sense defined in the previous section.

### Logistic Map Revisited

We now revert back to our previous discussion of Cantor sets and the logistic map  $f(x) = 5x(1 - x)$ .

Let  $\Sigma = \{0, 1\}^{\mathbb{N}} = \{s = s_1s_2\cdots | s_i \in \{0, 1\} \forall i \in \mathbb{N}\}$  and  $\sigma : \Sigma \rightarrow \Sigma$  be the map  $s_1s_2s_3\cdots \mapsto s_2s_3\cdots$ . We will equip  $\Sigma$  with a topology under which  $\sigma$  is continuous.

**Theorem 1.5.5.** *Given  $\mu \neq 0$ , let  $g_\mu(x) = \mu x(1 - x)$ , and let  $B_\mu \subset \mathbb{R}$  be the set of points with bounded orbits under  $g_\mu$ .*

*For  $\mu > 4$ , the dynamical systems  $(B_\mu, g_\mu)$  and  $(\Sigma, \sigma)$  are topologically conjugate.*

**Theorem 1.5.6.** *The set  $\Sigma$  is homeomorphic to the ternary Cantor set  $\mathcal{C}$ .*

These theorems imply Proposition 1.4.22. In the next chapter, we will prove Theorem 1.5.6, and Theorem 1.5.5 for the smaller range  $\mu > 2 + \sqrt{5} > 4$ .

# Chapter 2

## Symbolic Dynamics

To prove Theorems 1.5.5 and 1.5.6, we will need the powerful machinery of symbolic dynamics. In the next section we will introduce its basic concepts.

### 2.1 Sequences over a finite alphabet

Let  $(X, d_X)$  be a metric space, and  $A \subset X$  be a finite set with  $|A| \geq 2$ .

**Definition 2.1.1.** The set of sequences with alphabet  $A$  is denoted  $\Sigma_A$ . In other words,

$$\Sigma_A = A^{\mathbb{N}} = \{s = s_1 s_2 s_3 \cdots \mid s_j \in A \forall j \in \mathbb{N}\}$$

#### Topology on the Space of Sequences

We define a metric on  $\Sigma_A$  as follows: for all  $s, t \in \Sigma_A$ , we let

$$d(s, t) = \sum_{j=1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}} \quad (2.1)$$

**Proposition 2.1.2.** *The function  $d : \Sigma_A \times \Sigma_A \longrightarrow \mathbb{R}$  is well-defined.*

*Proof.* We need to show that for all  $s, t \in \Sigma_A$ , the infinite series given above converges. Let  $M = \max_{p, q \in A} d_X(p, q)$ .

$$\begin{aligned} d(s, t) &= \sum_{j=1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}} \\ &\leq \sum_{j=1}^{\infty} \frac{M}{|A|^{j-1}} = M \sum_{j=1}^{\infty} \frac{1}{|A|^{j-1}} \\ &= \frac{M}{1 - \frac{1}{|A|}} \\ &< \infty \end{aligned}$$

□

**Proposition 2.1.3.** *The function defined by Equation 2.1 is a metric on  $\Sigma_A$ .*

*Proof.* We need to show the following:

1. for all  $s, t \in \Sigma_A$ ,  $d(s, t) \geq 0$  ;
2. for all  $s, t \in \Sigma_A$ ,  $d(s, t) = 0 \iff s = t$ ;
3. for all  $s, t, r \in \Sigma_A$ ,  $d(s, r) \leq d(s, t) + d(t, r)$ .

We will see these one by one.

1. is clear since every term in the infinite sequence defining  $d(s, t)$  is non-negative.
2. is clear since if  $d(s, t) = 0$ , then  $d_X(s_j, t_j) = 0$  for all  $j \in \mathbb{N}$ , which implies  $s_j = t_j$  for all  $j \in \mathbb{N}$ .
3. for every  $j \in \mathbb{N}$ ,  $d_X(s_j, r_j) \leq d_X(s_j, t_j) + d_X(t_j, r_j)$ . This immediately shows (3).

□

The metric  $d$  induces a topology on  $\Sigma_A$ . We will see some properties of this topology in the remaining section.

*Remark 2.1.4.* By scaling the metric  $d_X$  if necessary, from now on we assume without loss of generality that  $M = \max_{p, q \in A} d_X(p, q) = 1$ .

**Proposition 2.1.5.** *Suppose  $s, t \in \Sigma_A$  satisfy  $s_j = t_j$  for  $j = 1, 2, \dots, N$ . Then*

$$d(s, t) < \frac{1}{|A|^{N-1}(|A| - 1)} \leq \frac{1}{|A|^{N-1}}$$

*Proof.* The second inequality follows directly since  $\frac{1}{|A|-1} \leq 1$ .  
Since  $s_j = t_j$  for  $j = 1, \dots, N$ ,

$$\begin{aligned} d(s, t) &= \sum_{j=1}^N \frac{d_X(s_j, t_j)}{|A|^{j-1}} + \sum_{j=N+1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}} \\ &= \sum_{j=N+1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}} \\ &\leq \sum_{j=N+1}^{\infty} \frac{1}{|A|^{j-1}} \\ &= \frac{1}{|A|^N} \frac{1}{1 - \frac{1}{|A|}} = \frac{|A|}{|A|^N(|A| - 1)} \\ &= \frac{1}{|A|^{N-1}(|A| - 1)} \end{aligned}$$

□

**Proposition 2.1.6.** *There exists a constant  $\ell = \ell(A) > 0$  such that if  $s, t \in \Sigma_A$  satisfy  $d(s, t) < \frac{\ell}{|A|^{N-1}}$ , then  $s_j = t_j$  for  $j = 1, 2, \dots, N$ .*

*Proof.* Let  $\ell = \min_{\substack{p, q \in A \\ p \neq q}} d_X(p, q)$ . We will prove the contrapositive. Given  $s, t \in \Sigma_A$ , if  $s_j \neq t_j$  for some  $j \in \{1, 2, \dots, N\}$ , then

$$d(s, t) \geq \frac{d_X(s_j, t_j)}{|A|^{j-1}} \geq \frac{\ell}{|A|^{j-1}} \geq \frac{\ell}{|A|^{N-1}}$$

□

## 2.2 Shift Operator on Sequences

**Definition 2.2.1.** The *shift operator*  $\sigma : \Sigma_A \longrightarrow \Sigma_A$  is defined as

$$\sigma(s_1 s_2 s_3 \dots) = s_2 s_3 s_4 \dots \text{ for all } s = s_1 s_2 s_3 \dots \in \Sigma_A \quad (2.2)$$

**Proposition 2.2.2.** *The map  $\sigma$  is surjective and uniformly continuous.*

*Proof.* Given  $s \in \Sigma_A$ , for any  $a \in A$ ,  $\sigma(as_1 s_2 s_3 \dots) = s$ . Therefore  $\sigma$  is surjective. To show it is uniformly continuous, we will exhibit for a given  $\epsilon > 0$ , a constant  $\delta > 0$  such that for all  $s, t \in \Sigma_A$ ,  $d(s, t) < \delta \implies d(\sigma(s), \sigma(t)) < \epsilon$ .

Fix  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that  $\frac{1}{|A|^{N-1}} < \epsilon$ . Let  $\delta = \frac{\ell}{|A|^N}$ , where  $\ell$  is the constant from Proposition 2.1.6. Then, we have  $s_j = t_j$  for  $j = 1, 2, \dots, N+1$ . Let  $\underline{s} = \sigma(s)$  and  $\underline{t} = \sigma(t)$ . Note that  $\underline{s}_j = s_{j+1}$  and  $\underline{t}_j = t_{j+1}$  for all  $j \in \mathbb{N}$ . The above condition implies that  $\underline{s}_j = \underline{t}_j$  for  $j = 1, 2, \dots, N$ . Therefore by Proposition 2.1.5, we have  $d(\underline{s}, \underline{t}) < \frac{1}{|A|^{N-1}} < \epsilon$ . □

## Periodic Sequences

**Definition 2.2.3.** For  $m \in \mathbb{N}$ , define

$$\text{Per}_m(\sigma) = \{s \in \Sigma_A \mid \sigma^m(s) = s\}$$

In other words,  $\text{Per}_m(\sigma)$  is the set of sequences whose period under  $\sigma$  divides  $m$ . Also define  $\text{Per}(\sigma)$  to be the set of sequences periodic under  $\sigma$ .

*Remark 2.2.4.* The following properties of periodic sequences are immediate.

1.

$$\text{Per}(\sigma) = \bigcup_{m=1}^{\infty} \text{Per}_m(\sigma)$$

2. Given a finite word  $w = s_1 s_2 \dots s_n$  with  $s_i \in A$  for all  $i$ , we let  $\overline{w}$  denote the infinite word  $s_1 s_2 \dots s_n s_1 s_2 \dots s_n s_1 s_2 \dots s_n \dots$  formed by repeating the finite block  $w$ . Given  $m \in \mathbb{N}$ ,

$$\text{Per}_m(\sigma) = \{\overline{s_1 s_2 \dots s_m} : s_j \in A \text{ for } j = 1, 2, \dots, m\}.$$

This shows that

$$|\text{Per}_m(\sigma)| = |A|^m$$

3. If  $m < n$  and  $m|n$ , then

$$\text{Per}_m(\sigma) \subsetneq \text{Per}_n(\sigma)$$

**Proposition 2.2.5.**  $\text{Per}(\sigma)$  is dense in  $\Sigma_A$ .

*Proof.* Given  $s \in \Sigma_A$  and  $\epsilon > 0$ , pick  $N \in \mathbb{N}$  such that  $\frac{1}{|A|^{N-1}} < \epsilon$ . Then by Proposition 2.1.5, the sequence  $t \in \text{Per}(\sigma)$  given by

$$t = s_1 s_2 \cdots s_N s_1 s_2 \cdots s_N s_1 s_2 \cdots = \overline{s_1 s_2 \cdots s_N}$$

satisfies

$$d(s, t) < \frac{1}{|A|^{N-1}} < \epsilon$$

□

*Suggested Reading* 2.2.6. • [1, Section 1.6]

## Logistic Maps Conjugate to the Shift

In this section we will establish Theorem 1.5.5 for the family of maps  $g_\mu(x) = \mu x(1 - x)$  where  $\mu > 2 + \sqrt{5}$ . The proof for the full range  $\mu > 4$  uses techniques from complex analysis, so we will see this later.

Recall the definition of the set  $B_\mu$ : this is the set of points  $x$  with bounded orbit under the map  $g_\mu$ . Just as we did for  $\mu = 5$ , we will show that for a range of  $\mu$  values, the set  $B_\mu \subset [0, 1]$ .

**Proposition 2.2.7.** When  $\mu > 1$ , for  $x \notin [0, 1]$ ,  $g_\mu^{on}(x) \rightarrow -\infty$  as  $n \rightarrow \infty$ .

*Proof.* First, observe that the graph of  $g_\mu$  is a downward drawn parabola which intersects the  $x$ -axis at the two roots  $x = 0, 1$ .

If  $x < 0$ , then  $g_\mu(x) = \mu x - \mu x^2 < \mu x < x$ . So the terms of the orbit  $g_\mu^{on}(x)$  become more and more negative as  $n$  increases. Now we show that the monotone decreasing sequence  $x, g_\mu(x), g_\mu^2(x), \dots$  does not stay bounded. Suppose to the contrary, then there exists  $p < 0$  such that  $g_\mu^{on}(x) \rightarrow p$ . On the one hand we have  $g_\mu^{o(n+1)}(x) \rightarrow g_\mu(p) < p$ , but on the other hand, the sequence  $(g_\mu^{o(n+1)}(x))_{n \geq 0}$ , as a tail of the sequence  $(g_\mu^{on}(x))_{n \geq 0}$ , also converges to  $p$ . This proves that  $g_\mu^{o(n+1)}(x) \rightarrow -\infty$ .

If  $x > 1$ , then  $g_\mu(x) < 0$ . By the discussion above,  $g_\mu^{on}(x) \rightarrow -\infty$ . □

**Proposition 2.2.8.** For  $\mu > 1$ ,

$$B_\mu = \bigcap_{n \geq 0} (g_\mu^{on})^{-1}[0, 1]$$

If  $1 < \mu \leq 4$ , then  $B_\mu = [0, 1]$ .

*Proof.* The previous proposition shows that  $B_\mu \subseteq [0, 1]$  for all  $\mu > 1$ , and moreover, that  $x \in B_\mu$  if and only if  $g_\mu^{on}(x) \in [0, 1]$  for all  $n \in \mathbb{N}$ .

In other words,

$$B_\mu = \bigcap_{n \geq 0} \{x \in \mathbb{R} : g_\mu^{on}(x) \in [0, 1]\} = \bigcap_{n \geq 0} (g_\mu^{on})^{-1}[0, 1].$$

Note that  $x = \frac{1}{2}$  is the unique point where  $g_\mu$  reaches its maximum, and  $g_\mu(\frac{1}{2}) = \frac{\mu}{4}$ .

Thus, if  $1 < \mu \leq 4$ , then since  $\frac{\mu}{4} \leq 1$ , we have

$$\begin{aligned} g_\mu[0, 1] &\subseteq [0, 1] \\ \implies [0, 1] &\subseteq g_\mu^{-1}[0, 1] \end{aligned}$$

Since we know that  $g_\mu^{-1}[0, 1] \subseteq [0, 1]$ , this shows that  $B_\mu = [0, 1]$ . □

Thus the interesting structure of  $B_\mu$  occurs when  $\mu > 4$ .

**Proposition 2.2.9.** *Fix  $\mu > 4$ . Let  $c_\mu = \sqrt{\frac{1}{4} - \frac{1}{\mu}}$ , and define the disjoint intervals  $I_0 = [0, \frac{1}{2} - c_\mu]$  and  $I_1 = [\frac{1}{2} + c_\mu, 1]$ . Then*

$$g_\mu^{-1}[0, 1] = I_0 \cup I_1$$

*Proof.* Solving  $g_\mu(x) = 1$ , we get

$$\begin{aligned} \mu x - \mu x^2 &= 1 \\ \implies \mu x^2 - \mu x + 1 &= 0 \\ \implies x &= \frac{\mu \pm \sqrt{\mu^2 - 4\mu}}{2\mu} \\ &= \frac{1}{2} \pm \sqrt{\frac{\mu^2 - 4\mu}{4\mu^2}} \\ &= \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{\mu}} \\ &= \frac{1}{2} \pm c_\mu \end{aligned}$$

Note that  $g_\mu[0, 1] = [0, \frac{\mu}{4}]$ . Since  $\frac{1}{2}$  is the point where  $g_\mu$  is maximum, and the graph of  $g_\mu$  is symmetric about the vertical line  $x = \frac{1}{2}$ , we get

$$g_\mu\left(\frac{1}{2} - c_\mu, \frac{1}{2} + c_\mu\right) = \left(1, \frac{\mu}{4}\right]$$

Thus, we have, for  $I_0$  and  $I_1$  as above, that

$$g_\mu(I_0 \cup I_1) = \left[0, \frac{\mu}{4}\right] \setminus \left(1, \frac{\mu}{4}\right] = [0, 1]$$

□

Note that the intervals  $I_0$  and  $I_1$  above are disjoint. Note that  $g_\mu(I_0) = g_\mu(I_1) = [0, 1]$ , so  $(g_\mu^{\circ 2})^{-1}[0, 1] = g_\mu^{-1}(I_0 \cup I_1) = I_{00} \cup I_{01} \cup I_{10} \cup I_{11}$ . here  $I_{00} \cup I_{01} =$ .

**Definition 2.2.10.** We introduce the notation  $\Sigma^0$  for the set of finite non-empty words over the alphabet  $\{0, 1\}$ . Formally,

$$\Sigma^0 = \bigcup_{N \geq 1} \{w = s_1 s_2 \cdots s_N \mid s_i \in \{0, 1\} \text{ for } i = 1, 2, \dots, N\} = \bigcup_{N \geq 1} \{0, 1\}^N$$

We let  $\ell(w)$  denote the length of the finite word  $w$ .

**Definition 2.2.11.** Given  $w = s_1 s_2 \cdots s_N \in \Sigma^0$ , define the set  $I_w \subseteq [0, 1]$  as follows:

$$\begin{aligned} I_w &= \{x \in [0, 1] : x \in I_{s_1}, g_\mu(x) \in I_{s_2}, \dots, g_\mu^{\circ(n-1)}(x) \in I_{s_n}\} \\ &= \bigcap_{j=1}^N \{x \in [0, 1] : g_\mu^{\circ(j-1)}(x) \in I_{s_j}\} \\ &= \bigcap_{j=1}^N (g_\mu^{\circ(j-1)})^{-1}(I_{s_j}) \end{aligned}$$

**Proposition 2.2.12.** *The collection of intervals  $\{I_w : w \in \Sigma^0\}$  satisfies*

1. *Given  $w = s_1 \cdots s_N \in \Sigma^0$  and a symbol  $s_{N+1} \in \{0, 1\}$ , we have  $I_{ws_{N+1}} \subseteq I_w$ .*
2. *Given  $w = s_1 \cdots s_N \in \Sigma^0$  and a symbol  $s_0 \in \{0, 1\}$ , the map  $g_\mu$  maps  $I_{s_0 w}$  homeomorphically onto  $I_w$ .*
3.  *$I_w$  is a closed interval of non-zero length for all  $w \in \Sigma^0$ .*
4. *Given distinct words  $w_1, w_2 \in \{0, 1\}^N$ ,  $I_{w_1} \cap I_{w_2} = \emptyset$ .*
5. *for all  $N \in \mathbb{N}$ ,  $(g_\mu^{\circ N})^{-1}[0, 1] = \bigcup_{w \in \{0, 1\}^N} I_w$ .*

*Proof.* We will show them one by one.

1. Note that  $I_{ws_{N+1}} = I_w \cap \{x : g_\mu^{\circ N}(x) \in I_{s_{N+1}}\} \subseteq I_w$ .
2. Note that  $I_{s_0 w} = \{x : x \in I_{s_0} \text{ and } g_\mu(x) \in I_w\}$ . In other words,  $I_{s_0 w}$  is the full pre-image of  $I_w$  in either  $I_0$  or  $I_1$ , depending on the value of  $s_0$ . Since  $g_\mu$  is monotonic on both  $I_0$  and  $I_1$ , and  $g_\mu(I_0) = g_\mu(I_1) = [0, 1]$ , we get that  $g_\mu(I_{s_0 w}) = [0, 1]$ , and that the mapping is a homeomorphism.
3. By definition,  $I_w$  is an intersection of closed sets, and by the previous point, by induction on  $\ell(w)$ , it is easy to see that it is a closed interval with non-zero length.
4. Without loss of generality suppose the  $j$ th entry of  $w_1$  and  $w_2$  are 0 and 1 respectively. Then  $g_\mu^{\circ(i-1)} I_{w_1} \subseteq I_0$  and  $g_\mu^{\circ(i-1)} I_{w_2} \subseteq I_1$ . Since for a point  $x$  we cannot have  $g_\mu^{\circ(i-1)}$  in both  $I_0$  and  $I_1$ , this shows that  $I_{w_1} \cap I_{w_2} = \emptyset$ .

5. We do this by inducting on  $N$ . When  $N = 1$ ,  $g_\mu^{-1}[0, 1] = I_0 \cup I_1$ .

Induction hypothesis: the statement is true for  $N$ .

Induction step: for  $N + 1$ ,

$$\begin{aligned}
(g_\mu^{\circ(N+1)})^{-1}[0, 1] &= g_\mu^{-1}\left((g_\mu^{\circ N})^{-1}[0, 1]\right) = g_\mu^{-1}\left(\bigcup_{w \in \{0,1\}^N} I_w\right) \\
&= \bigcup_{w \in \{0,1\}^N} g_\mu^{-1}I_w \\
&= \bigcup_{w \in \{0,1\}^N} (I_0 \cap g_\mu^{-1}I_w) \cup (I_1 \cap g_\mu^{-1}I_w) \quad (\text{since } g_\mu^{-1}I_w \subseteq I_0 \cup I_1) \\
&= \bigcup_{w \in \{0,1\}^N} I_{0w} \cup I_{1w} \quad (\text{by the proof of point (2)}) \\
&= \bigcup_{w \in \{0,1\}^{N+1}} I_w
\end{aligned}$$

□

From here, let  $\Sigma = \Sigma_{\{0,1\}}$ .

**Proposition 2.2.13.** *Fix  $\mu > 2 + \sqrt{5}$ . Given any  $s_1 s_2 \cdots =: s \in \Sigma$ , there exists a unique point  $x_s \in B_\mu$  such that*

$$\bigcap_{N=1}^{\infty} I_{s_1 s_2 \cdots s_N} = \{x_s\}$$

*Proof.* We first show that it suffices to prove the following claim:

*Claim 1.* There exists a constant  $\lambda > 1$  such that for all  $N \in \mathbb{N}$  and all  $w \in \{0, 1\}^N$ ,

$$\text{diam}(I_w) \leq \frac{1}{\lambda^{N-1}} \cdot \text{diam}(I_0)$$

Proposition 2.2.12 implies  $I_{s_1} \supseteq I_{s_1 s_2} \supseteq I_{s_1 s_2 s_3} \cdots$ . Therefore, the infinite intersection  $\text{diam}\left(\bigcap_{N=1}^{\infty} I_{s_1 s_2 \cdots s_N}\right)$  is a non-empty closed set. By Claim 1,  $\text{diam}\left(\bigcap_{N=1}^{\infty} I_{s_1 s_2 \cdots s_N}\right) = 0$ , so this infinite intersection is a singleton  $\{x_s\}$ . This point  $x_s$  is in  $B_\mu$  since  $g_\mu^{\circ N}(x_s) \in I_{s_N} \subset [0, 1]$  for all  $N \in \mathbb{N}$ .

Now it is left for us to prove claim 1. We first show the following:

*Claim 2.*  $|g'_\mu(x)| > \sqrt{\mu^2 - 4\mu} > 1$  for all  $x$  in the interiors of  $I_0$  and  $I_1$ .

*Proof of Claim 2.* If  $x$  is in the interior of  $I_0$  or  $I_1$ , then

$$\begin{aligned}
|g'(x)| &= |\mu(1 - 2x)| = 2\mu \left| \frac{1}{2} - x \right| \\
&> 2\mu c_\mu = 2\mu \cdot \sqrt{\frac{1}{4} - \frac{1}{\mu}} \\
&= \sqrt{\mu^2 - 4\mu} > \sqrt{(2 + \sqrt{5})^2 - 4(2 + \sqrt{5})} \quad (\text{since } \mu \mapsto \mu^2 - 4\mu \text{ is increasing for } \mu > 2) \\
&= \sqrt{4 + 5 - 8} = 1
\end{aligned}$$



■

Thus for all  $x \in I_0 \cup I_1$ , we have  $|g'_\mu(x)| \geq \sqrt{\mu^2 - 4\mu} > 1$ .

*Proof of Claim 1.* Let  $\lambda = \sqrt{\mu^2 - 4\mu}$ . Given  $w \in \{0, 1\}^N$  with  $w = s_1 s_2 s_3 \cdots s_N$ , note that since  $g_\mu^{\circ(N-1)} : I_w \rightarrow I_{s_1}$  is a diffeomorphism, looking at the inverse map  $f = (g_\mu^{\circ(N-1)})^{-1}$  and using the fact that  $|f'(x)| = \frac{1}{|(g_\mu^{\circ(N-1)})'(f^{-1}(x))|}$ , for all  $x, y \in I_{s_1}$ ,

$$|f(x) - f(y)| \leq |f'(x)| |x - y| \leq \frac{1}{\lambda^{N-1}} |x - y|$$

Thus,

$$\text{diam}(I_w) \leq \frac{1}{\lambda^{N-1}} \text{diam}(I_{s_1}) = \frac{1}{\lambda^{N-1}} \text{diam}(I_0)$$

■

This finishes the proof of the proposition. □

**Definition 2.2.14.** Fix  $\mu > 2 + \sqrt{5}$ . Define a map  $\varphi : \Sigma \rightarrow B_\mu$  by setting  $\varphi(s) = x_s$  for all  $s \in \Sigma$ .

**Proposition 2.2.15.**  $\varphi$  is a homeomorphism.

*Proof.*  $\varphi$  is injective: If  $s \neq t$ , choose  $N \in \mathbb{N}$  such that  $s_N \neq t_N$ . Since  $\varphi(s) = x_s \in I_{s_1 s_2 \cdots s_N}$  and  $\varphi(t) = x_t \in I_{t_1 \cdots t_N}$ , and by the condition  $s_N \neq t_N$  we have  $I_{s_1 \cdots s_N} \cap I_{t_1 \cdots t_N} = \emptyset$ , we must have  $\varphi(s) \neq \varphi(t)$ .

$\varphi$  is surjective: If  $x \in B_\mu$ , for all  $n \in \mathbb{N}$ , let  $s_n = 0$  if  $g_\mu^{\circ(n-1)}(x) \in I_0$  and  $s_n = 1$  if  $g_\mu^{\circ(n-1)}(x) \in I_1$ . Then it is easy to check that  $\varphi(s_1 s_2 s_3 \cdots) = x$ .

$\varphi$  is continuous: Given  $s \in \Sigma$  and  $\epsilon > 0$ , since the diameter of  $I_w$  tends to 0 as  $\ell(w) \rightarrow \infty$ , choose  $N \in \mathbb{N}$  such that  $I_{s_1 s_2 \cdots s_N} \subset B_\epsilon(x)$ . Then set  $\delta = \frac{1}{2^N}$ . By Proposition 2.1.6, if  $d(s, t) < \delta$ , then  $t_j = s_j$  for  $j = 1, 2, \dots, N$ . Thus  $\varphi(t) \in I_{t_1 \cdots t_N} = I_{s_1 \cdots s_N}$ , and by our assumption on  $N$ , we have  $|\varphi(t) - \varphi(s)| < \epsilon$ .

$\varphi^{-1}$  is continuous: Given  $x \in B_\mu$  and  $\epsilon > 0$ , let  $s = \varphi^{-1}(x)$ . Choose  $N \in \mathbb{N}$  such that  $\frac{1}{2^{N-1}} < \epsilon$ , and choose  $\delta > 0$  such that  $B_\delta(x) \subset I_{s_1 s_2 \cdots s_N}$ . Then, for any  $y \in B_\delta(x) \cap B_\mu$ , the sequence  $t = \varphi^{-1}(y)$  satisfies  $t_j = s_j$  for  $j = 1, 2, \dots, N$ . By Proposition 2.1.5, we know that  $d(s, t) < \frac{1}{2^{N-1}} < \epsilon$ . □

**Proposition 2.2.16.**  $\varphi$  conjugates  $\sigma$  to  $g_\mu$ .

*Proof.* For all  $s \in \Sigma$ ,

$$\begin{aligned} \varphi(s) &\in \bigcap_{N=1}^{\infty} I_{s_1 \cdots s_N} \\ \implies g_\mu(\varphi(s)) &\in \bigcap_{N=1}^{\infty} g_\mu(I_{s_1 \cdots s_N}) = \bigcap_{N=1}^{\infty} I_{s_2 s_3 \cdots s_N} \\ &= \{\varphi(\sigma(s))\} \end{aligned}$$

In other words,  $g_\mu \circ \varphi = \varphi \circ \sigma$ . □

Propositions 2.2.15 and 2.2.16 together prove Theorem 1.5.5.

Now let  $A$  be a finite alphabet with  $|A| \geq 2$

**Proposition 2.2.17.** *The space  $\Sigma_A$  is a complete metric space.*

*Proof.* We already know that  $\Sigma_A$  is a metric space. To see that it is complete, we need to show that every Cauchy sequence converges.

Let  $(s^n)_{n \geq 0}$  be a Cauchy sequence. Note that each  $s^n$  is a sequence of the form  $s_1^n s_2^n s_3^n \cdots$  with  $s_j^n \in A$  for all  $j \in \mathbb{N}$ .

*Claim 1.* For every  $j \in \mathbb{N}$ , the terms  $s_j^n$  are eventually constant as  $n \rightarrow \infty$ .

*Proof of Claim 1.* Fix  $j$ . Let  $\ell = \min_{\substack{p, q \in A \\ p \neq q}} d_X(p, q)$ . Since  $(s^n)_{n \geq 0}$  is Cauchy, there exists  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ ,  $d(s^n, s^m) < \frac{\ell}{|A|^{j-1}}$ .

By Proposition 2.1.6, we have  $s_j^n = s_j^m$  for all  $m, n \geq N$ . ■

Due to Claim 1, we can define for every  $j$  the symbol  $s_j = \lim_{n \rightarrow \infty} s_j^n$ . Since the sequence  $(s_j^n)_{n \geq 0}$  is eventually constant,  $s_j \in A$ . Consider the sequence  $s \in \Sigma_A$  given by  $s = s_1 s_2 s_3 \cdots$ .

*Claim 2.* The Cauchy sequence  $(s^n)_{n \geq 0}$  converges to  $s$ .

*Proof of Claim 2.* Given  $\epsilon > 0$ , choose  $N \in \mathbb{N}$  such that  $\frac{1}{|A|^{N-1}} < \epsilon$ . By Claim 1, we can choose  $M \in \mathbb{N}$  such that for all  $n \geq M$ , we have

$$s_1^n s_2^n \cdots s_N^n = s_1 s_2 \cdots s_N$$

By Proposition 2.1.5, for  $n \geq M$  we have  $d(s, s^n) \leq \frac{1}{|A|^{N-1}} < \epsilon$ . ■

□

**Proposition 2.2.18.** 1.  $\Sigma_A$  has bounded diameter.

2.  $\Sigma_A$  is totally bounded: that is, given any  $\epsilon > 0$ , it can be covered by finitely many  $\epsilon$ -balls.

*Proof.* 1. For all  $s, t \in \Sigma_A$ , we have  $d(s, t) = \sum_{j=1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}} \leq \sum_{j=1}^{\infty} \frac{1}{|A|^{j-1}} \leq \frac{1}{1 - \frac{1}{|A|}}$ .

2. Given  $\epsilon > 0$ , choose  $N \in \mathbb{N}$  such that  $\frac{1}{|A|^{N-1}} < \epsilon$ . Consider the set of words  $W_N = A^N$  (these are the finite words of length  $N$  over  $A$ ). Since  $A$  is finite,  $W_N$  is finite. Fix any element  $a \in A$ , and consider the finite set of sequences  $S = \{waaa \cdots \mid w \in W_N\} \subset \Sigma_A$ .

Given any  $t \in \Sigma_A$ , there exists  $s = waaaa \cdots \in S$  such that  $w = t_1 t_2 \cdots t_N$ . Thus

$$d(s, t) < \frac{1}{|A|^{\ell-1}} < \epsilon$$

by Proposition 2.1.5 and our choice of  $N$ .

Thus the finite collection of  $\epsilon$ -balls  $\{B_\epsilon(s) : s \in S\}$  covers  $\Sigma_A$ . □

**Corollary 2.2.19.**  $\Sigma_A$  is compact.

*Remark 2.2.20.* In class I claimed that bounded diameter of  $\Sigma_A$  is sufficient for compactness; this is in fact not true, and uniform boundedness is necessary.

*Proof.* A metric space is compact if and only if it is complete and totally bounded. □

**Proposition 2.2.21.**  $\Sigma_A$  is perfect.

We will prove this proposition by showing that there exists an element  $s \in \Sigma_A$  such that every  $t \in \Sigma_A$  can be approximated by elements of the form  $\sigma^{on}(s)$ .

*Suggested Reading* 2.2.22. • [1, Section 1.5]

- [3, Section 7.3.4, Section 7.4.3]

## Topological Transitivity

**Definition 2.2.23.** Let  $X$  be a topological space and  $f : X \rightarrow X$  be a dynamical system on  $X$ .

- If  $f$  is not invertible, it is said to be topologically transitive if there exists a point  $x_0 \in X$  such that the orbit  $(f^{on}(x_0))_{n \geq 0}$  is dense in  $X$ .
- If  $f$  is invertible, it is said to be topologically transitive if there exists a point  $x_0 \in X$  such that the *grand orbit*  $(f^{on}(x_0))_{n \in \mathbb{Z}}$ , which is the union of the forward and backward orbits of  $x_0$ , is dense in  $X$ .

**Proposition 2.2.24.** The shift operator  $\sigma : \Sigma_A \rightarrow \Sigma_A$  is topologically transitive.

**Corollary 2.2.25.**  $\Sigma_A$  is perfect.

*Proof of Proposition 2.2.24.* Since  $\sigma$  is not invertible, we need to prove the existence of a dense orbit  $(\sigma^{on}(s))_{n \geq 0}$ . We will do this constructively by defining  $s$ .

1. For  $N \in \mathbb{N}$ , let  $W_N = A^N$  be the set of finite words of length  $N$  over the alphabet  $A$ . Since  $W_N$  is finite, we may enumerate all the words of  $W_N$  as  $w_1, w_2, \dots, w_r$ , and form a master word  $\tilde{w}_1 = w_1 w_2 w_3 \dots w_r$ .

For example, if  $A = \{0, 1\}$  and  $N = 1$ , we have  $W_1 = \{0, 1\}$  and we can take  $\tilde{w}_1 = 10$ . Similarly,  $W_2 = \{00, 01, 10, 11\}$  and we can take  $\tilde{w}_2 = 00011011$ .

2. Define  $s \in \Sigma_A$  as

$$s = \tilde{w}_1 \tilde{w}_2 \tilde{w}_3 \dots$$

For example when  $A = \{0, 1\}$ , with  $\tilde{w}_1$  and  $\tilde{w}_2$  as above, we have  $s = 1000011011 \dots$ .

To see that  $s$  has a dense orbit under  $\sigma$ , given any  $t \in \Sigma_A$  and  $\epsilon > 0$ , we will show that there exists  $m \in \mathbb{N}$  such that  $d(\sigma^{om}(s), t) < \epsilon$ .

Choose  $N \in \mathbb{N}$  such that  $\frac{1}{|A|^{N-1}} < \epsilon$ . The finite word  $t_1 t_2 \cdots t_N$  is a sub-string of the master word  $\tilde{w}_N$ . By definition of  $s$ , there exists  $m \in \mathbb{N}$  such that

$$\sigma^{om}(s) = t_1 t_2 \cdots t_N \cdots$$

By Proposition 2.1.5 and our choice of  $N$ , we have

$$d(\sigma^{om}(s), t) < \frac{1}{|A|^{N-1}} < \epsilon$$

□

**Proposition 2.2.26.** *For  $\mu > 2 + \sqrt{5}$ , the set  $B_\mu$  is totally disconnected.*

*Proof.* This proof is left as an exercise to the reader; it follows the same lines as Proposition 1.4.5, point 6. We will need to use Claim 1 from Proposition 2.2.13. □

**Corollary 2.2.27.** *The shift space  $\Sigma = \Sigma_{\{0,1\}}$  is totally disconnected.*

*Proof.* By Proposition 2.2.15, we know  $\Sigma$  is homeomorphic to  $B_\mu$  for  $\mu > 2 + \sqrt{5}$ . By Proposition 2.2.26, the statement follows. □

Proposition 2.2.17 and Corollaries 2.2.19, 2.2.25 and 2.2.27 show that  $\Sigma$  satisfies all the conditions of Theorem 1.4.10. Thus we get that  $\Sigma$  is homeomorphic to the ternary Cantor set  $\mathcal{C}$ , and thereby finish the proof of Theorem 1.5.6.

# Chapter 3

## Low-Dimensional Dynamics

### 3.1 Basic Concepts

Throughout this section we assume that  $X$  is a topological space and  $f : X \rightarrow X$  is a continuous map.

**Definition 3.1.1.** Given  $x \in X$ ,

- The orbit of  $x$  under  $f$  is the sequence  $(f^{on}(x))_{n \geq 0}$
- The grand orbit of  $x$  under  $f$  is the set  $\{z \in X \mid f^{om}(z) = f^{on}(x) \text{ for some } m, n \in \mathbb{N}\}$
- A bi-infinite orbit for  $x$  under  $f$  is a sequence  $(x_n)_{n \in \mathbb{Z}}$ , where  $x_0 = x$ , and  $x_{n+1} = f(x_n)$  for all  $n \in \mathbb{Z}$ .

*Remark 3.1.2.* A point  $x$  can have more than one bi-infinite orbit.

The grand orbit of  $x$  contains its orbit and any bi-infinite orbit. If  $f$  is a homeomorphism, then there is only one bi-infinite orbit for  $x$ , which is the whole grand orbit.

**Definition 3.1.3.** Given  $m \in \mathbb{N}$ ,

$$\text{Per}_m(f) = \{x \in X : f^{om}(x) = x\}$$

This is the set of periodic points under  $f$  whose period under  $f$  divides  $m$ .

$$\text{Per}(f) = \bigcup_{m=1}^{\infty} \text{Per}_m(f)$$

This is the set of periodic points under  $f$ .

**Definition 3.1.4.** Let  $A \subseteq X$ .  $A$  is said to be forward invariant under  $f$  if  $f(A) \subseteq A$ , and backward invariant under  $f$  if  $f^{-1}(A) \subseteq A$ .

## Transitivity, Mixing and Chaos

We restate the definition of topological transitivity here.

**Definition 3.1.5.** • If  $f$  is not invertible, it is said to be topologically transitive if there exists a point  $x_0 \in X$  such that the orbit  $(f^{on}(x_0))_{n \geq 0}$  is dense in  $X$ .

- If  $f$  is invertible, it is said to be topologically transitive if there exists a point  $x_0 \in X$  such that the *grand orbit*  $(f^{on}(x_0))_{n \in \mathbb{Z}}$ , which is the union of the forward and backward orbits of  $x_0$ , is dense in  $X$ .

**Definition 3.1.6.** Suppose  $f$  is a homeomorphism. It is said to be *minimal* if the grand orbit of every point is dense in  $X$ .

*Remark 3.1.7.* Minimality  $\implies$  Topological Transitivity.

**Definition 3.1.8.**  $f$  is said to be *chaotic* if it is topologically transitive and  $\text{Per}(f)$  is dense in  $X$ .

**Definition 3.1.9.**  $f$  is said to be *topologically mixing* if for every pair of non-empty open sets  $U, V \subseteq X$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $f^n(U) \cap V \neq \emptyset$ .

## 3.2 Circle Maps

We can represent the circle  $\mathbb{S}^1$  in two different ways:

- As the set  $\mathbb{R}/\mathbb{Z} = \{[x] : x \in \mathbb{R}\}$ , where  $[x] = [y]$  iff  $x - y \in \mathbb{Z}$ .
- As the set  $\{e^{2\pi ix} : [x] \in \mathbb{R}/\mathbb{Z}\} \subset \mathbb{C}$ , i.e., the visual representation of  $\mathbb{R}/\mathbb{Z}$  on the complex plane.

**Definition 3.2.1.** The *arc length* metric on  $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$  is given by

$$d_{\mathbb{S}^1}([x], [y]) = \min\{|x' - y'| : x', y' \in \mathbb{R}, x' \in [x], y' \in [y]\} \text{ for all } x, y \in \mathbb{S}^1$$

*Exercise 3.2.2.* Prove that  $d_{\mathbb{S}^1}([x], [y])$  is the length of the shorter arc formed by  $e^{2\pi ix}$  and  $e^{2\pi iy}$  on the unit circle in the complex plane.

## Rotations

**Definition 3.2.3.** Let  $\alpha \in \mathbb{R}/\mathbb{Z}$ . The rotation map  $R_\alpha : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is defined as

$$\begin{aligned} R_\alpha([x]) &= [x + \alpha] && \text{(additive representation)} \\ R_\alpha(e^{2\pi ix}) &= e^{2\pi i\alpha} e^{2\pi ix} = e^{2\pi i(\alpha+x)} && \text{(multiplicative representation)} \end{aligned}$$

Note that every rotation  $R_\alpha$  is a homeomorphism of  $\mathbb{S}^1$ ; its inverse is  $R_{-\alpha}$ . Every rotation is in fact an isometry in the metric  $d_{\mathbb{S}^1}$ . That is,  $d_{\mathbb{S}^1}(R_\alpha(x), R_\alpha(y)) = d_{\mathbb{S}^1}(x, y)$  for all  $x, y \in \mathbb{S}^1$ .

These maps exhibit widely different behavior when  $\alpha \in \mathbb{Q}/\mathbb{Z}$  vs. when  $\alpha \notin \mathbb{Q}/\mathbb{Z}$ .

**Proposition 3.2.4.** • If  $\alpha \in \mathbb{Q}/\mathbb{Z}$ , then  $\text{Per}(R_\alpha) = \mathbb{S}^1$ .

• If  $\alpha$  is irrational, then  $\text{Per}(R_\alpha) = \emptyset$ .

*Proof.* • If  $\alpha$  is rational, it is of the form  $\alpha = \frac{m}{n}$ , where  $m, n \in \mathbb{Z}$  and  $n \neq 0$ . Thus

$$R_\alpha^{on}([x]) = [x + n\alpha] = [x + m] = [x] \text{ for all } [x] \in \mathbb{R}/\mathbb{Z}$$

In other words,  $R_\alpha^{on} = \text{id}_{\mathbb{S}^1}$ . Thus  $\text{Per}_n(R_\alpha) = \mathbb{S}^1$ , which implies  $\text{Per}(R_\alpha) = \mathbb{S}^1$ .

• If  $\alpha$  is irrational and  $[x] \in \mathbb{S}^1$  is a periodic point of  $R_\alpha$  of period  $n$  for some  $n \in \mathbb{N}$ , then  $[x + n\alpha] = [x]$ , which implies  $n\alpha \in \mathbb{Z}$ . But this means  $\alpha$  has a rational representative, which is a contradiction. □

**Proposition 3.2.5.** Let  $f : X \rightarrow X$  be an open map. Then if  $f$  is topologically transitive, there exist no pair of disjoint non-empty open sets  $U$  and  $V$  such that  $f(U) \subseteq U$  and  $f(V) \subseteq V$ .

We will prove Proposition 3.2.5 later on. However, since  $R_\alpha$  is a homeomorphism, and thus open, using this proposition we will show the following.

**Proposition 3.2.6.** If  $\alpha \in \mathbb{Q}/\mathbb{Z}$ , then  $R_\alpha$  is not topologically transitive.

*Proof.* We will find a pair of disjoint, invariant non-empty open sets, showing the contrapositive of Proposition 3.2.5. Choose any two points  $u, v \in \mathbb{S}^1$ . We know that  $\alpha$  is of the form  $\frac{m}{n}$  for some  $m, n \in \mathbb{Z}$ ,  $n \neq 0$ .

Choose  $\epsilon > 0$  so that the open intervals  $B_\epsilon(u), R_\alpha(B_\epsilon(u)), \dots, R_\alpha^{o(n-1)}(B_\epsilon(u)), B_\epsilon(v), R_\alpha(B_\epsilon(v)), \dots, R_\alpha^{o(n-1)}(B_\epsilon(v))$  are all pairwise disjoint. Then, letting  $U = \bigcup_{j=0}^{n-1} R_\alpha^{oj}(B_\epsilon(u))$  and  $V = \bigcup_{j=0}^{n-1} R_\alpha^{oj}(B_\epsilon(v))$ , we see that  $U \cap V = \emptyset$ ,  $U$  and  $V$  are non-empty,  $f(U) = U$  and  $f(V) = V$ . □

**Proposition 3.2.7.** If  $\alpha \notin \mathbb{Q}/\mathbb{Z}$ , then  $R_\alpha$  is minimal. This in turn implies it is topologically transitive.

*Proof.* Fix  $x \in \mathbb{S}^1$ . It suffices to show that the grand orbit of  $x$  is dense in  $\mathbb{S}^1$ . We will in fact show the stronger statement that the forward orbit  $(R_\alpha^{on}(x))_{n \geq 0}$  is dense in  $\mathbb{S}^1$ .

Let  $z \in \mathbb{S}^1$  and  $\epsilon > 0$ . We need to exhibit an orbit point  $R_\alpha^{on}(x) \in B_\epsilon(z)$ .

For  $N \geq \lfloor \frac{1}{\epsilon} \rfloor + 1$ , any set of  $N$  points on  $\mathbb{S}^1$  contains at least two points  $u, v$  such that  $d_{\mathbb{S}^1}(u, v) < \epsilon$ .

Let  $S = \{x, R_\alpha(x), \dots, R_\alpha^{o(N-1)}(x)\}$ . Since  $\alpha$  is irrational, these points listed here are distinct, so  $|S| = N$ . By the above statement, there exist  $\ell, k$  with  $0 \leq \ell < k \leq N$  such that  $d_{\mathbb{S}^1}(R_\alpha^{o\ell}(x), R_\alpha^{ok}(x)) < \epsilon$ .

Since  $R_\alpha$  is an isometry, we have  $d_{\mathbb{S}^1}(x, R_\alpha^{o(k-\ell)}(x)) < \epsilon$ .

*Claim.* If  $d_{\mathbb{S}^1}(x, R_\alpha^{on}(x)) < \epsilon$  for some  $n \in \mathbb{N}$ , then for all  $y \in \mathbb{S}^1$ , we have  $d_{\mathbb{S}^1}(y, R_\alpha^{on}(y)) < \epsilon$ .

*Proof of Claim.* We know that  $y = R_{y-x}(x)$ . Thus,

$$\begin{aligned}
d_{\mathbb{S}^1}(y, R_\alpha^{on}(y)) &= d_{\mathbb{S}^1}(R_{y-x}(x), R_\alpha^{on} \circ R_{y-x}(x)) \\
&= d_{\mathbb{S}^1}(R_{y-x}(x), R_{y-x} \circ R_\alpha^{on}(x)) && \text{since } R_\alpha R_\beta = R_\beta R_\alpha \text{ for all } \alpha, \beta \in \mathbb{S}^1 \\
&= d_{\mathbb{S}^1}(x, R_\alpha^{on}(x)) && \text{since all rotations are isometries} \\
&< \epsilon
\end{aligned}$$

■

By the above claim, letting  $y = 0$ , we have  $d_{\mathbb{S}^1}(0, R_\alpha^{o(\ell-k)}(0)) < \epsilon$ . Note that  $d_{\mathbb{S}^1}(0, R_\alpha^{o(\ell-k)}(0)) = |\theta|$ , where  $\theta = [(\ell - k)\alpha] \in \mathbb{S}^1$ .

By this choice of  $\theta$ , for  $M \geq \lfloor \frac{1}{\theta} \rfloor + 1$ , the points  $\{x, R_\theta(x), \dots, R_\theta^{o(M-1)}(x)\}$  split the circle into intervals all of length  $< \epsilon$ . Thus there exists  $n \in \{0, 1, \dots, M-1\}$  such that  $d_{\mathbb{S}^1}(R_\theta^{on}(x), z) < \epsilon$ . Since  $R_\theta^{on}(x) = R_\alpha^{on(\ell-k)}(x)$ , the proposition follows.  $\square$

**Proposition 3.2.8.** *No circle rotation is chaotic.*

*Proof.* By the above series of propositions, rational rotations are not topologically transitive, and irrational rotations have no periodic points. So neither kind of rotations are chaotic.  $\square$

**Proposition 3.2.9.** *No homeomorphism  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is topologically mixing.*

*Proof.* Pick any three distinct points  $x, y, z \in \mathbb{S}^1$ . Then  $\mathbb{S}^1 \setminus \{x, y, z\}$  is the union of three disjoint intervals  $A, B$  and  $C$ . For  $N \in \mathbb{N}$ , let  $\mathcal{B}_N = \{f^{\circ N}(X) \cap Y \mid X, Y \in \{A, B, C\}\}$ . It suffices to show the following claim:

*Claim.* For all  $N \in \mathbb{N}$ , at least one element of  $\mathcal{B}_N$  is empty.

Here is how this claim proves that  $f$  is not topologically mixing: by the claim, for all  $N$ , there are sets  $X_N, Y_N \in \{A, B, C\}$  such that  $f^{\circ N}(X_N) \cap Y_N = \emptyset$ . Then, since  $\{A, B, C\}$  is finite, upto some subsequence,  $X_N$  is constant and  $Y_N$  is constant. So wlog, up to a subsequence  $N_k \rightarrow \infty$ , we can assume  $X_{N_k} = A$  and  $Y_{N_k} = B$ , say. Thus, the claim implies that  $f^{\circ n}(A) \cap B = \emptyset$  for infinitely many natural numbers  $n$ . This means  $f$  is not topologically mixing.

*Proof of Claim.* Fix  $N \in \mathbb{N}$ . Since  $A, B, C$  are pairwise disjoint, and  $f^{\circ N}$  is a homeomorphism for all  $N$ , the intervals  $f^{\circ N}(A), f^{\circ N}(B), f^{\circ N}(C)$  are pairwise disjoint. Suppose  $f^{\circ N}(A)$  intersects  $A, B$  and  $C$ . Then it has to contain one of the intervals - wlog suppose  $f^{\circ N}(A) \supseteq A$ . But this means that  $f^{\circ N}(B) \cap A = f^{\circ N}(C) \cap A = \emptyset$ . Thus not all elements of  $\mathcal{B}_N$  can be simultaneously non-empty.

■

□



## Lifts

**Definition 3.2.10.** Given  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  continuous, a *lift* of  $f$  is a continuous map  $G : \mathbb{R} \rightarrow \mathbb{R}$  that satisfies  $f \circ \pi = \pi \circ G$  for the universal covering map  $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$ .

**Proposition 3.2.11.** Let  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  be continuous.

1. For any  $[x_0] \in \mathbb{S}^1$  represented by some point  $x_0 \in \mathbb{R}$ , let  $y_0 \in \mathbb{R}$  be such that  $f([x_0]) = [y_0]$ . Then there exists a lift  $G$  of  $f$  such that  $G(x_0) = y_0$ .
2. Any two lifts of  $f$  differ by a constant  $C \in \mathbb{Z}$ .
3. Let  $G$  be a lift of  $f$ . Then  $\deg f := G(x+1) - G(x)$  is an integer that is independent of  $x, G$ . This is called the *degree* of  $f$ .

*Proof.* We will prove each point one by one. Let  $A = f^{-1}(y_0) = \{[x_r] = [x_0 + 1] = [x_0], [x_1], [x_2], \dots, [x_{r-1}]\}$  where the points are in counterclockwise order starting at  $[x_0]$ . Let  $y_n = y_0 + n$  for  $n \in \mathbb{Z}$  and consider  $\pi_n = \pi|_{[y_n, y_{n+1})}$ . Note that  $\pi_n$  is a continuous bijection onto  $\mathbb{S}^1$  whose inverse is continuous on  $\mathbb{S}^1 \setminus \{[y_0]\}$ .

1. We show this in the following steps:

- Define  $G$  on  $[x_0, x_1]$  as follows. Starting at  $x_0 \in \mathbb{R}$ , move counterclockwise from  $[x_0]$  on the circle. In this process,  $f(\pi(x)) = f([x])$  starts traveling either clockwise or anticlockwise on the circle from  $[y_0]$ . Let  $\Delta = \pm 1$  depending on whether this movement is clockwise or counterclockwise, and define  $G$  on  $[x_0, x_1]$  as  $G(x_0) = y_0$  and  $G(x) = \pi_{\Delta}^{-1} \circ f \circ \pi(x)$  for all  $x \in (x_0, x_1)$ . Note that at the end of this process,  $\lim_{x \rightarrow (x_1)^-} G(x) = y_0$  or  $y_0 + \Delta$ . In both cases, we uniquely extend  $G$  continuously to  $[x_0, x_1]$ . Note that in this process,  $G$  satisfies  $\pi \circ G = f \circ \pi$ .
- Inductively for some  $i \in \{1, 2, \dots, r-2\}$ , suppose  $G$  is well-defined on  $[x_{i-1}, x_i]$  and satisfies  $f \circ \pi = \pi \circ G$ . We will extend  $G$  to  $[x_i, x_{i+1}]$ . Set  $\Delta = 1$  if  $f$  does not change direction at  $[x_i]$ , or  $\Delta = -1$  otherwise. Extend  $G$  continuously to  $[x_i, x_{i+1}]$  by defining it to be  $G(x) = \pi_{G(x_i)-y_0+\Delta}^{-1} \circ f \circ \pi$  on  $(x_i, x_{i+1})$  and  $G(x_{i+1}) = \lim_{x \rightarrow x_{i+1}^-} G(x)$ . At the end of this process,  $G(x_{i+1}) = G(x_i)$  or  $G(x_i) + \Delta$ . Note that  $G(x_0 + 1) - G(x_0) \in \mathbb{Z}$ . Clearly  $G$  satisfies  $\pi \circ G = f \circ \pi$  on  $[x_0, x_0 + 1]$ .
- This finishes the definition of  $G$  on  $[x_0, x_r] = [x_0, x_0 + 1]$ . Inductively for  $n \in \mathbb{Z}$  with  $n \neq 0$ , for  $x \in [x_0 + n, x_0 + n + 1]$ , let  $G(x) = G(x - n) + n(G(x_0 + 1) - G(x_0))$ ,  $G$  is continuous on  $(x_0 + n, x_0 + n + 1)$  for all  $n \in \mathbb{Z}$ . We will show that it is continuous at each point of the form  $x_0 + n$ .

$$\begin{aligned}
 \lim_{x \rightarrow (x_0+n)^-} G(x) &= \lim_{x \rightarrow (x_0+n)^-} G(x - (n-1)) + (n-1)(G(x_0+1) - G(x_0)) \\
 &= nG(x_0+1) - (n-1)G(x_0) \\
 \lim_{x \rightarrow (x_0+n)^+} G(x) &= \lim_{x \rightarrow (x_0+n)^+} G(x - n) + n(G(x_0+1) - G(x_0)) \\
 &= nG(x_0+1) - (n-1)G(x_0)
 \end{aligned}$$

Also note that  $G(x) - G(x - n) \in \mathbb{Z}$  for all  $n, x$ . This, along with the definition of  $G$  on  $[x_0, x_0 + 1]$ , shows that  $\pi \circ G = f \circ \pi$ .

2. Let  $G_1$  and  $G_2$  be two lifts of  $f$ . Then by definition, for all  $x \in \mathbb{R}$ ,  $\pi(G_1(x)) = f(\pi(x)) = \pi(G_2(x))$ , and thus  $G_1(x) - G_2(x)$  is an integer. Since  $G_1 - G_2$  is a continuous function from  $\mathbb{R}$  to  $\mathbb{Z}$ , it is a constant.
3. Let  $G$  be a lift of  $f$ . We note that the function  $\tilde{G}(x) = G(x+1)$  is also lift of  $f$ . By the previous point,  $\tilde{G}(x) - G(x) = G(x+1) - G(x)$  is an integer that is independent of  $x \in \mathbb{R}$ . To show that it is independent of  $G$ , note that for any other lift  $H$ , we have  $H = G + c$  for some  $c \in \mathbb{Z}$ , and thus  $H(x+1) - H(x) = G(x+1) - G(x)$  for all  $x \in \mathbb{R}$ .

□

**Proposition 3.2.12.** *If  $f$  is an injective continuous map, then  $|\deg f| = 1$  and any lift  $G$  is strictly monotone.*

*Proof.* Let  $G$  be a lift of  $f$ . First we show that  $|\deg f| = 1$ . If  $\deg f = 0$ , then  $G(1) = G(0)$ . In particular,  $G$  is not monotone on  $[0, 1]$ . Thus there exist points  $c_1 \neq c_2 \in (0, 1)$  such that  $G(c_1) = G(c_2)$ . In particular,  $f([c_1]) = f([c_2])$ , while  $[c_1] \neq [c_2]$ , which also contradicts the injectivity of  $f$ . So  $\deg f \neq 0$ . If  $|\deg f| > 1$ , then there exists  $c \in (0, 1)$  such that  $|G(c) - G(0)| = 1$ . But this shows that  $f([c]) = f([0])$ , which also contradicts the injectivity of  $f$ . This shows that  $|\deg f| = 1$ .

WLOG assume  $\deg f = 1$ . Then note that  $G(x+1) = G(x) + 1$  for all  $x$ . So it suffices to show that  $G$  is strictly increasing on  $[0, 1]$ . Since  $G : [0, 1] \rightarrow [G(0), G(1)]$  is continuous, if it is not strictly increasing, there exist points  $c_1 < c_2 \in (0, 1)$  such that  $G(c_1) \geq G(c_2)$ . However, there is then a point  $c \in (c_2, 1]$  such that  $G(c) = G(c_1)$ . This implies  $f([c]) = f([c_1])$  and contradicts the injectivity of  $f$ . □

*Remark 3.2.13.* If  $f$  is injective and continuous, if  $\deg f = 1$ , then any lift is strictly increasing, and if  $\deg f = -1$ , then any lift is strictly decreasing.

**Proposition 3.2.14.** *If  $|\deg f| = 1$  and some lift of  $f$  is strictly monotone, then  $f$  is a homeomorphism.*

*Proof.* Let  $G$  be a strictly monotone lift of  $f$ . Note that within  $(0, 1)$ ,  $G(x) = G(y)$  implies  $x = y$ . Thus  $f$  is injective. Since  $G$  is continuous and  $|\deg f| = 1$ ,  $G$  maps  $[0, 1]$  onto some interval of length 1. This shows that  $f(\mathbb{S}^1) = \mathbb{S}^1$ .

Since  $G$  is a strictly monotone continuous map of  $\mathbb{R}$ , it has a strictly monotone continuous inverse  $G^{-1}$ . Then  $G^{-1}(x+1) - G^{-1}(x) = \deg f$  for all  $x \in G$ . This also shows that  $G^{-1}$  is the lift of a continuous circle map  $h$ , and  $h$  satisfies  $h \circ f = f \circ h = \text{id}_{\mathbb{S}^1}$ . Thus  $f^{-1}$  is continuous. □

*Suggested Reading 3.2.15.* • [3, Section 4.3.1]

## Expanding Maps

**Definition 3.2.16.** A map  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is said to be *expanding* if it is continuously differentiable and  $\exists \lambda > 1$  such that  $|f'(x)| \geq \lambda$  for all  $x \in \mathbb{S}^1$ .

**Definition 3.2.17.** A linear expanding map of  $\mathbb{S}^1$  is a map of the form  $E_m([x]) = [mx]$  for all  $[x] \in \mathbb{S}^1$ , where  $m$  is an integer  $\geq 2$ .

*Exercise 3.2.18.* Fix an integer  $m \geq 2$ .

Show that for  $k \in \mathbb{N}$ ,  $\text{Per}_k(E_m) = \{[\frac{i}{m^k-1}] : i = 0, 1, \dots, m^k - 2\}$ .

**Proposition 3.2.19.** For any integer  $m \geq 2$ , the set  $\text{Per}(E_m)$  is dense in  $\mathbb{S}^1$ .

*Proof.* Given  $\epsilon > 0$ , choose  $k \in \mathbb{N}$  such that  $\frac{1}{m^k-1} < \epsilon$ . Then  $\text{Per}_k(E_m) \subset \text{Per}(E_m)$  splits the circle into  $m^k - 1$  small intervals each of length  $\frac{1}{m^k-1} < \epsilon$ .  $\square$

**Proposition 3.2.20.** Any expanding map  $f$  of the circle is topologically mixing.

*Proof.* Given  $U, V$  non-empty open sets in  $\mathbb{S}^1$ , let  $I \subset \pi^{-1}(U)$  be an interval. We will show that  $f^{\circ N}(\pi(I)) = \mathbb{S}^1$  for some  $N \in \mathbb{N}$ . This will imply the topological mixing property.

Let  $\ell$  be the length of  $I$ . Since  $f$  is expanding, there exists  $\lambda > 1$  such that  $|f'| \geq \lambda$ . Then for any lift  $G$  of  $f$ , we have  $|G'| \geq \lambda$ . But this in turn means that the length of  $G^{\circ n}(I)$  is greater than  $\lambda^n \ell$  for all  $n \rightarrow \infty$ . Pick  $N \in \mathbb{N}$  such that  $\lambda^N \ell > 1$ . In other words,  $G^{\circ N}(I)$  contains an interval of length 1. But this means that  $f^{\circ N}(\pi(I))$  covers  $\mathbb{S}^1$ .  $\square$

**Proposition 3.2.21.** Any expanding map  $f$  of the circle is topologically transitive.

Before proving this, we show the following proposition and theorem.

**Proposition 3.2.22.** Let  $X$  be a topological space and  $f : X \rightarrow X$  be continuous. Then if  $f$  has a dense bi-infinite orbit, then for every  $\emptyset \neq U, V \subset X$ , open, there exists an  $N \in \mathbb{Z}$  such that  $f^{\circ N}(U) \cap V \neq \emptyset$ . Furthermore, if  $X$  is perfect, then this  $N$  can be chosen in  $\mathbb{N}$ .

*Proof.* Let  $U, V$  be as above, and  $(x_n)_{n \in \mathbb{Z}}$  be the dense bi-infinite orbit. Then there exists  $n \in \mathbb{Z}$  such that  $z_n \in V$ , and  $m \in \mathbb{Z}$  such that  $z_m \in U$ . Letting  $N = n - m$  (this could be  $\leq 0$ ), we see that  $z_n \in f^{\circ N}(\{z_m\})$ , thus  $f^{\circ N}(U) \cap V \neq \emptyset$ .

Now assume  $X$  is perfect. For the given  $U, V$ , if the above  $N \geq 0$  we are done. Suppose  $N = n - m < 0$ . Since  $z_n$  is not an isolated point, there exists a subsequence  $|n_k| \rightarrow \infty$  such that  $z_{n_k} \rightarrow z_n$  and  $z_{n_k} \in V$  for all  $k$ .

1. If we can choose  $n_k \rightarrow +\infty$ , then for some  $k$ , we have  $n_k \geq n$ . Letting  $\tilde{N} = n_k - n \geq 0$ , we see that  $f^{\circ \tilde{N}}(x_n) = x_{n_k} \in V$ . Thus  $f^{\circ \tilde{N}}(U) \cap V \neq \emptyset$ .
2. If we have  $n_k \rightarrow -\infty$ , choose  $n' = n_k$  with  $k$  large enough such that  $z_{n_k} = z_{n'} \in V \cap f^{\circ N}(U)$ , and  $n' < 2n - m$ . Then  $z' = z_{n'+m-n} := f^{\circ(m-n)}(z'_n) \in f^{\circ N+m-n}(U) = f^{\circ n-m+n-m}(U) \subseteq U$ . Moreover, letting  $\tilde{N} = 2n - m - n'$ , we see that  $\tilde{N} > 0$  and  $f^{\circ \tilde{N}}(z') = f^{\circ(2n-m-n')}(z') = z_n \in V$ . Thus  $f^{\circ \tilde{N}}(U) \cap V \neq \emptyset$ .

$\square$

**Theorem 3.2.23.** Let  $X$  be a complete separable (that is, there is a countable dense subset) metric space with no isolated points. If  $f : X \rightarrow X$  is a continuous map, then the following four conditions are equivalent:

1.  $f$  is topologically transitive, i.e., it has a dense orbit.
2.  $f$  has a dense bi-infinite orbit.
3. If  $\emptyset \neq U, V \subset X$ , then there exists an  $N \in \mathbb{N}$  such that  $f^{\circ N}(U) \cap V \neq \emptyset$ .
4. If  $\emptyset \neq U, V \subset X$ , then there exists an  $N \in \mathbb{Z}$  such that  $f^{\circ N}(U) \cap V \neq \emptyset$ .

*Proof.* (1)  $\implies$  (2) and (3)  $\implies$  (4) are always true.

By Proposition 3.2.22, (2) implies (3).

We will show that separability of  $X$  implies (4) implies (2) and (3) implies (1). This will prove the theorem. The proof methods are similar, so we will only do the case (3) implies (1).

Let  $S$  be a countable dense set. For every  $p \in \mathbb{Q}$  and every  $x \in S$ , let  $U_p(x)$  be the ball of radius  $\frac{p}{q}$  centered at  $x$ . Consider the collection  $\{U_p(x) : x \in S, p \in \mathbb{Q}\}$ . This collection is countable, and can be enumerated as  $\{U_1, U_2, \dots\}$ . Every tail  $\{U_N, U_{N+1}, \dots\}$  is an open cover of  $X$ . Let  $U_0 = f^{-1}(U_1)$ . By condition (3), there exists  $N_1 \in \mathbb{N}$  such that  $f^{\circ N_1}(U_1) \cap U_2 \neq \emptyset$ . Pick an open ball  $V_1$  of radius  $< 1$  such that  $\overline{V_1} \subset U_1 \cap f^{\circ N_1}(U_2)$ .

Then  $f^{\circ N_1}(V_1) \cap U_2 \neq \emptyset$ . Inductively, for  $k \geq 2$ , there exists an  $N_k \in \mathbb{N}$  such that  $f^{\circ N_k}(V_{k-1}) \cap U_{k+1} \neq \emptyset$ . Let  $V_k$  be an open ball of radius  $< \frac{1}{2^{k-1}}$  such that  $\overline{V_k} \subset V_{k-1} \cap f^{\circ N_k}(U_{k+1})$ . Then note that  $f^{\circ N_k}(\overline{V_k}) \subset U_{k+1}$ .

Furthermore,  $\overline{V_1} \supset \overline{V_2} \supset \overline{V_3} \dots$  is a decreasing chain of closed balls whose diameter goes to 0. Thus,  $\bigcap_{k=1}^{\infty} \overline{V_k} = \{x\}$  for a unique point  $x \in X$ . Let  $x_0 = x \in U_1$ , and  $x_k = f^{\circ k}(x)$  for  $k \in \mathbb{N}$ . Since  $N_k \in \mathbb{N}$  for all  $k$ , and this gives a dense orbit in  $X$ .  $\square$

**Corollary 3.2.24.** *A continuous open map  $f$  of a complete, separable, perfect metric space is topologically transitive if and only if there are no two disjoint open nonempty  $f$ -invariant sets.*

*Proof.*  $\implies$  is obvious, since a dense orbit visits every open set.

$\Leftarrow$  : If  $U, V \subset X$  are open, then the sets  $W = \bigcup_{n \in \mathbb{Z}} f^{\circ n}(U)$  and  $O = \bigcup_{n \in \mathbb{Z}} f^{\circ n}(V)$  are open because  $f$  is an open map, and satisfy  $f(W) \subseteq W$ , and  $f(O) \subseteq O$ . Therefore they are not disjoint by assumption, so  $f^{\circ n}(U) \cap f^{\circ m}(V) \neq \emptyset$  for some  $n, m \in \mathbb{Z}$ . Then  $f^{\circ(n-m)}(U) \cap V \neq \emptyset$  and  $f$  is topologically transitive by the above theorem.  $\square$

*Proof of Proposition 3.2.21.* Since  $f$  is topologically mixing, it is also topologically transitive by Theorem 3.2.23.  $\square$

*Suggested Reading* 3.2.25. • [3, Section 7.1.3, Section 7.2.1, Section 7.2.3, Section 7.4.1]

## General vs Linear Expanding Maps

**Proposition 3.2.26.** *If  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is an expanding map, then  $|\deg f| > 1$  and  $\text{Per}_k(f) = |\deg f^k - 1|$  for all  $k \in \mathbb{N}$ .*

*Proof.*  $|f'| > 1$  implies  $|G'| > 1$  for any lift, so, by the Mean-Value Theorem,  $|\deg f| = |G(x+1) - G(x)| > 1$ . Any iterate  $f^{\circ k}$  is also expanding, and note that  $G^{\circ k}$  is a lift of  $f^{\circ k}$ . Furthermore,

$$\forall x \in \mathbb{R}, \quad G^{\circ k}(x+1) - G^{\circ k}(x) = (G(x+1) - G(x))^k \quad \forall k \in \mathbb{N}$$

This implies that  $\deg(f^{\circ k}) = (\deg f)^{\circ k}$ . so it suffices to consider the case  $k = 1$ . Consider  $G$  on the interval  $[0, 1]$ . Note that

$$\text{Fix}(f) = \{[x] : x \in [0, 1], G(x) - x \in \mathbb{Z}\}$$

The function  $g(x) := G(x) - x$  satisfies  $g(1) = g(0) + \deg f - 1$ , and is also monotone. In other words,  $g[0, 1]$  is an interval of length  $|g(1) - g(0)| = |\deg f - 1|$ . From this it follows that

$$|\deg f - 1| \leq \#\{x \in [0, 1] : G(x) - x \in \mathbb{Z}\} \leq |\deg f - 1| + 1$$

- If  $g(0) \notin \mathbb{Z}$ , then  $g(1) \notin \mathbb{Z}$ , and  $\#\{x \in [0, 1] : G(x) - x \in \mathbb{Z}\} = |\deg f - 1|$ . In this case, all the points in this set map to distinct points on the circle, and thus,  $|\text{Fix}(f)| = |\deg f - 1|$ .
- If  $g(0) \in \mathbb{Z}$ , then  $g(1) \in \mathbb{Z}$ , and  $\#\{x \in [0, 1] : G(x) - x \in \mathbb{Z}\} = |\deg f - 1| + 1$ . This set now contains 0, 1, and  $|\deg f - 1| - 1$  distinct points in  $(0, 1)$ . Since  $[0] = [1] \in \mathbb{S}^1$ , we get  $|\text{Fix}(f)| = |\deg f - 1|$ .

□

*Remark 3.2.27.* Note that  $|\text{Per}_k(f)| = |\text{Per}_k(E_m)|$ , where  $m = \deg f$ .

**Theorem 3.2.28.** *Let  $f$  be an expanding map of  $\mathbb{S}^1$ . Then  $f$  is topologically conjugate to  $E_m$ , where  $m = \deg f$ .*

**Corollary 3.2.29.** *Any two expanding maps of the circle of the same degree are topologically conjugate.*

We will not be proving this in our lectures, but the general idea is given here.

**Definition 3.2.30.** Suppose that  $g : X \rightarrow X$  and  $f : Y \rightarrow Y$  are maps of metric spaces  $X$  and  $Y$  and that there is a continuous surjective map  $h : X \rightarrow Y$  such that  $h \circ g = f \circ h$ . Then  $g$  is said to be *semi-conjugate* to  $f$  via the *semiconjugacy*  $h$ .

*Remark 3.2.31.* If this  $h$  is a homeomorphism, then  $f$  and  $g$  are topologically conjugate. In this case we will call  $h$  a *conjugacy*.

**Proposition 3.2.32.** *Let  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  be an expanding map with  $\deg f = m$ . Then  $f$  is semi-conjugate to the shift  $\sigma : \Sigma_A \rightarrow \Sigma_A$  where  $A = \{0, 1, \dots, |m| - 1\} \subset \mathbb{R}$ .*

*Proof.* Let  $[y_0] = f([0])$ . Note that there exist  $m$  intervals  $\Delta_0, \Delta_1, \dots, \Delta_{m-1} \subset \mathbb{S}^1$  such that  $f(\Delta_i) = \mathbb{S}^1 \setminus [y_0]$  and the ends of each  $\Delta_i$  are pre-images of  $y_0$ .

Additionally, for each  $i, j$ , there exists a unique maximal interval  $\Delta_{ij} \subset \Delta_i$  such that  $f(\Delta_{ij}) \subset \Delta_j$ . Inductively, suppose  $\Delta_w$  is well-defined for all words  $w$  of length  $n$  over  $A$ .

Given a word  $w' = w_1 w_2 \cdots w_{n+1}$  of length  $n+1$  over  $A$ , define  $\Delta_{w'} \subset \Delta_{w_1}$  to be the unique maximal interval such that  $f(\Delta_{w'}) \subset \Delta_{w_2 w_3 \cdots w_{n+1}}$ .

Since  $f$  is expanding and scales lengths by at least a factor  $\lambda > 1$ , given any infinite word  $s \in \Sigma_A$ , the intersection  $\bigcap_{N=1}^{\infty} \overline{\Delta_{s_1 s_2 \cdots s_N}}$  is a unique point  $x_s \in \mathbb{S}^1$ .

*Claim.* The map  $h : \Sigma_A \rightarrow \mathbb{S}^1$  given by  $s \mapsto x_s$  is surjective and continuous.

*Proof of Claim.* This is very similar to the proof of Proposition 2.2.15. ■

Note that by construction,  $h(\sigma(s)) = f(x_s) = f(h(s))$ . This shows that  $h$  is a semi-conjugacy between  $\sigma$  and  $f$ . □

*Idea of Proof of Theorem 3.2.28.* Let  $g = E_m$ . Let  $h_f, h_g : \Sigma_A \rightarrow \mathbb{S}^1$  be the semi-conjugacies of  $f$  and  $g$  respectively with  $\sigma$ . Consider the set  $H_x = h_g \circ h_f^{-1}(x)$  for any  $x \in \mathbb{S}^1$ . The claim is that  $H_x$  always consists of precisely one point, which we will call  $h(x)$ . The map  $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  gives the required topological conjugacy between  $f$  and  $g$ . □

**Proposition 3.2.33.** *Expanding maps of  $\mathbb{S}^1$  are chaotic.*

*Proof.* This follows from Proposition 3.2.21, Theorem 3.2.28 and the fact that the periodic points of linear expanding maps are dense in  $\mathbb{S}^1$ . □

*Suggested Reading* 3.2.34. • [3, Section 7.3.1, Section 7.4.1]

## Rotation Number

**Definition 3.2.35.** A homeomorphism  $f$  of  $\mathbb{S}^1$  is said to be orientation-preserving if  $\deg f = 1$  and orientation-reversing if  $\deg f = -1$ .

**Lemma 3.2.36.** *If  $f$  is an orientation-preserving circle homeomorphism and  $G$  a lift, then  $G(y) - y \leq G(x) - x + 1$  for all  $x, y \in \mathbb{R}$ .*

*Proof.* Let  $k = \lfloor y - x \rfloor$ . Then

$$\begin{aligned} G(y) - y &= G(y) + G(x + k) - G(x + k) + (x + k) - (x + k) - y \\ &= (G(x + k) - (x + k)) + (G(y) - G(x + k)) - (y - (x + k)) \end{aligned}$$

Now  $G(x + k) - (x + k) = G(x) - x$  and  $0 \leq y - (x + k) < 1$  by choice of  $k$ . So  $G(y) - G(x + k) \leq 1$ . Thus the right hand side above is at most  $G(x) - x + 1 - 0$ . □

**Lemma 3.2.37.** *Let  $(a_n)_{n \in \mathbb{Z}}$  be a sequence of real numbers such that  $a_{m+n} \leq a_n + a_{m+k} + L$  for some  $k, L$ . Then the limit  $\lim_{n \rightarrow \infty} \frac{a_n}{n}$  exists and belongs to  $\mathbb{R} \cup \{-\infty\}$ .*

*Proof.*

$$a_{m+k} \leq a_m + a_{2k} + L$$

gives

$$a_{m+n} \leq a_n + a_m + a_{2k} + 2L = a_n + a_m + L'$$

where  $L' = a_{2k} + 2L$ . So wlog, we may take  $k = 0$ . Let  $a = \liminf_{|n| \rightarrow \infty} \frac{a_n}{n} \in \mathbb{R} \cup \{-\infty\}$ .

If  $a < b < c$  and  $|n| > \frac{2L'}{(c-b)}$  such that  $\frac{a_n}{n} < b$ . Then for any  $\ell$  with  $|\ell| \geq |n|$  which satisfies  $|\ell|(c-b) > \max_{|r| < |n|} a_r$ , writing  $\ell = nk' + r$  for some  $k' \in \mathbb{Z}$ ,  $|r| < |n|$ , we have

$$\begin{aligned} \frac{a_\ell}{\ell} &\leq \frac{(k'a_n + a_r + k'L')}{\ell} \\ &\leq \frac{a_n}{n} + \frac{a_r}{\ell} + \frac{L'}{n} < c \end{aligned}$$

Thus,  $\limsup \frac{a_\ell}{\ell} \leq c$ . Since  $c > a$  was arbitrary, this proves the lemma.  $\square$

**Proposition 3.2.38.** *Let  $f$  be an orientation-preserving homeomorphism of the circle.*

1. *Let  $G$  be a lift of  $f$ . Then the following limit exists and is independent of  $x \in \mathbb{R}$ :*

$$\rho(G) = \lim_{|n| \rightarrow \infty} \frac{G^{\circ n}(x) - x}{n}$$

2. *For any other lift  $\tilde{G}$  of  $f$ , we have  $\rho(G) - \rho(\tilde{G}) = G - \tilde{G} \in \mathbb{Z}$ .*

3.  *$\rho(G)$  is rational if and only if  $f$  has a periodic point.*

*Proof.* 1. *Existence:* Take  $x \in \mathbb{R}$  and  $a_n = G^{\circ n}(x) - x$  for all  $n \in \mathbb{Z}$ . Then by Lemma 3.2.36 applied to  $f^{\circ m}$  and  $G^{\circ m}$ , we have  $G^{\circ m}(y) - y \leq G^{\circ m}(x) - x + 1$  for all  $y \in \mathbb{R}$ . Thus, for all  $m, n \in \mathbb{Z}$ ,

$$a_{m+n} = G^{\circ(m+n)}(x) - x = G^{\circ m}(G^{\circ n}(x)) - G^{\circ n}(x) + a_n \leq a_m + 1 + a_n$$

Now, using Lemma 3.2.37, we know that  $\lim_{n \rightarrow \infty} \frac{a_n}{n} = c^+ \in \mathbb{R} \cup \{-\infty\}$ . However,

$$\frac{a_n}{n} = \frac{1}{n} \sum_{i=0}^{n-1} G^{\circ(i+1)}(x) - G^{\circ i}(x) \geq \min_{y \in \mathbb{R}} G(y) - y = \min_{y \in [0,1]} G(y) - y \in \mathbb{R}$$

Thus the limit  $c^+$  is a real number. Similarly,  $\lim_{n \rightarrow -\infty} \frac{a_n}{n} = c^- \in \mathbb{R}$ . It suffices to show that  $c^+ = c^-$ .

2. Note that there exists  $c \in \mathbb{Z}$  such that  $\tilde{G} = G + c$ . Then for every  $x \in \mathbb{R}$  and every  $n \in \mathbb{Z}$ ,  $\tilde{G}^{\circ n}(x) = G^{\circ n}(x) + nc$ . Thus  $\frac{\tilde{G}^{\circ n}(x) - x}{n} = \frac{G^{\circ n}(x) - x}{n} + c$ . This proves that  $\rho(\tilde{G}) = \rho(G) + c$ .

3. If  $f$  has a  $k$ -periodic point  $[x_0]$  and  $G$  is a lift of  $f$ , then  $G^{\circ k}(x_0) = x_0 + c$  for some  $c \in \mathbb{Z}$ , note that  $G^{\circ nk}(x_0) = x_0 + nc$  for all  $n \in \mathbb{Z}$ . Thus  $\rho(G) = \lim_{|n| \rightarrow \infty} \frac{G^{\circ nk}(x_0) - x_0}{nk} = \frac{c}{k} \in \mathbb{Q}$ .

Conversely, if  $\rho(G) = \frac{m}{k} \in \mathbb{Q}$ , then  $\rho(G^{\circ k}) = m$ . Suppose  $\tilde{f} = f^{\circ k}$  has no fixed point. Pick  $\tilde{G}$  to be a lift such that  $\tilde{G}(0) \in [0, 1)$ . Then  $\tilde{G}(x) - x \notin \mathbb{Z}$  for all  $x \in \mathbb{R}$  since  $\tilde{G}(x) - x \in \mathbb{Z}$  would imply that  $[x]$  is a fixed point for  $\tilde{f}$ . Therefore,  $0 < \tilde{G}(x) - x < 1$  for all  $x \in \mathbb{R}$ . Since  $\tilde{G} - \text{id}$  is continuous and periodic, it attains its minimum and maximum and therefore there exists a  $\delta > 0$  such that

$$0 < \delta \leq \tilde{G}(x) - x \leq 1 - \delta < 1$$

for all  $x \in R$ . In particular, we can take  $x = \tilde{G}^{oi}(0)$  and use  $\tilde{G}^{on}(0) = \tilde{G}^{on}(0) - 0 = \sum_{i=0}^{n-1} \tilde{G}^{o(i+1)}(0) - \tilde{G}^{oi}(0)$  to get

$$n\delta \leq \tilde{G}^{on}(0) \leq (1 - \delta)n$$

or

$$\delta \leq \frac{\tilde{G}^{on}(0)}{n} \leq 1 - \delta$$

As  $n \rightarrow \infty$ , this gives  $\rho(F) \neq 0$ , which proves the claim by contraposition.  $\square$

**Definition 3.2.39.** Let  $f$  be an orientation-preserving homeomorphism of the circle. The *rotation number* of  $f$  is given by  $\rho(f) = [\rho(G)] \in \mathbb{S}^1$  where  $G$  is any lift of  $f$ .

**Definition 3.2.40.** Let  $X$  be a topological space and  $f : X \rightarrow X$  be a dynamical system. A point  $x \in X$  is said to be *heteroclinic* to  $p, q \in X$  under  $f$  if  $\lim_{n \rightarrow \infty} f^{on}(x) = p$  and  $\lim_{n \rightarrow -\infty} f^{on}(x) = q$ . If  $p = q$ , we say  $x$  is *homoclinic* under  $f$  to  $p$ .

**Theorem 3.2.41.** Let  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  be an orientation-preserving homeomorphism. Suppose  $\rho(f) = \frac{p}{q} \in \mathbb{Q}$ .

- If  $f$  has exactly one periodic orbit  $A = \{x_1, x_2, \dots, x_r\}$ , then for every point  $x \in \mathbb{S}^1 \setminus A$ ,  $x$  is heteroclinic under  $f^{oq}$  to two points  $x_i, x_j \in A$ . Furthermore,  $x_i = x_j$  if and only if  $r = 1$ .
- If  $f$  has more than one periodic orbit, then every non-periodic point  $x$  is heteroclinic under  $f^{oq}$  to two points  $u$  and  $v$  that are in distinct periodic orbits.

**Theorem 3.2.42** (Poincaré Classification Theorem). Let  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  be an orientation-preserving homeomorphism. Suppose  $\rho(f) = \rho \notin \mathbb{Q}$ . Then there exists a continuous monotone map  $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that  $h \circ f = R_\rho \circ h$ .

- If  $f$  is topologically conjugate, then  $h$  is a homeomorphism.
- If  $f$  is not topologically conjugate, then  $h$  is not invertible.

**Definition 3.2.43.** A map  $f : X \rightarrow X$  of a metric space is said to exhibit sensitive dependence on initial conditions if there is a  $\Delta > 0$ , called a *sensitivity constant*, such that for every  $x \in X$  and  $\epsilon > 0$  there exists a point  $y \in X$  with  $d(x, y) < \epsilon$  and  $d(f^{oN}(x), f^{oN}(y)) \geq \Delta$  for some  $N \in \mathbb{N}$ .

**Theorem 3.2.44.** Chaotic maps exhibit sensitive dependence on initial conditions, except when the entire space consists of a single periodic orbit.

*Suggested Reading* 3.2.45. • [3, Section 4.3.2 - Section 4.3.6]. These sections contain a proof of Theorems 3.2.41 and 3.2.42.



### 3.3 Linear Maps of the Plane

#### Hyperbolicity

**Definition 3.3.1.** Let  $A$  be a linear map of the plane, and  $\{\lambda, \mu\}$  be its set of eigenvalues (these are elements of  $\mathbb{C}$ ).  $A$  said to be *hyperbolic* if  $|\lambda| < 1$ , and  $|\mu| > 1$ .

### 3.4 Torus Maps Revisited

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