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METRIC SPACES

A metric space is a pair (X, d) where X is a set, and

$d : X \times X \longrightarrow \mathbb{R}$ is a function satisfying:

(i) $d(x,y) \geq 0 \quad \forall x, y \in X$

(ii) $d(x,y) = 0 \iff x = y \quad \forall x, y \in X$

(iii) $d(x,y) = d(y,x) \quad \forall x, y \in X$

(iv) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x, y, z \in X$

↪ "Triangle inequality"

Examples:

1. $(\mathbb{R}, d)$ where

$$d(x,y) = \begin{cases} x-y, & x \geq y \\ y-x & x < y \end{cases}$$

2. $(\mathbb{R}^2, d)$

$$d(\underbrace{(x_1, x_2)}_{x}, \underbrace{(y_1, y_2)}_{y}) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}.$$

In both 1 and 2, $d(x,y)$ is the length of the line segment bound between x and y.

3. For any set X, define

$d : X \times X \longrightarrow \mathbb{R}$ as

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}.$$

d clearly satisfies (i), (ii), (iii).

(iv): $\forall x, y, z$, $d(x,y) + d(y,z) = A \in \{0, 1, 2\}$.

If $A = 0$, then $x = y = z$. So $d(x,z) = 0$.

If $A = 1$, then either $x = y \neq z$, or $x \neq y = z$. So $d(x,z) = A$.

If $A = 2$, then $d(x,z) \leq 1 < A$.

4. Let $G = (V, E)$ be a graph, and consider

$d : V \times V \longrightarrow \mathbb{R}$ given by

$d(u,v)$ = no. of edges in the shortest path joining u and v.

Is d a metric?

Answer: No, in fact d is not well-defined: it is possible that there is no path between two edges.

Fix: Assume G is a "connected" graph (ie, any two vertices $u, v \in V$ can be joined by a path).

d satisfies (i), (ii), (iii) in the definition of a metric.

Triangle inequality: if the shortest path $u \to v$ has m edges, and " " " $v \to w$ has n edges, then combining the paths, we get a path $u \longrightarrow w$ with $m+n$ edges.

So the shortest path $u \to w$ has $\leq m+n$ edges.

Pseudo-metric spaces:

Any function $d : X \times X \longrightarrow \mathbb{R}$ that satisfies (i), (iii), (iv) but (ii) fails: we still assume $d(x,x) = 0 \ \forall x \in X$, but $\exists x \neq y$ s.t. $d(x,y) = 0$

is called a pseudo-metric on X.

Constructing metric spaces from pseudo-metric spaces:

Let $(\tilde{X}, \tilde{d})$ be a pseudo-metric space.

Let $\sim = \{(x,y) : d(x,y) = 0\}$.

$\sim$ is an equivalence relation on $\tilde{X}$:

(i) reflexive, $\because$ ("since") $d(x,x) = 0 \ \forall x \in \tilde{X}$

(ii) symmetric, $\because d(x,y) = d(y,x) \ \forall x, y \in \tilde{X}$

(iii) transitive: if $(x,y), (y,z) \in \sim$, then $d(x,y) = d(y,z) = 0$.

Δ inequality $\Rightarrow d(x,z) = 0 \Rightarrow (x,z) \in \sim$.

[Note: For any relation R, we say xRy if $(x,y) \in R$].

Let $X = \tilde{X}/\sim = \{[x] : x \in \tilde{X}\}$.

Define $d : X \times X \longrightarrow \mathbb{R}$ as

$$d([x],[y]) = \tilde{d}(x,y).$$

d is well-defined: if $[x'] = [x]$ and $[y'] = [y]$, then

$$\tilde{d}(x',y') \leq \tilde{d}(x',x) + \tilde{d}(x,y) + \tilde{d}(y,y') = \tilde{d}(x,y).$$

Similarly $\tilde{d}(x,y) \leq \tilde{d}(x',y')$.

Therefore $\tilde{d}(x,y) = \tilde{d}(x',y')$.

Exercise: Check that d is a metric on X.

Eg: Consider the pseudo-metric space $(\mathbb{N}, d')$ where

$$d'(x,y) = \begin{cases} 0, & x-y \text{ is even} \\ 1, & x-y \text{ is odd} \end{cases}.$$

Note: $d'(x,y) \geq 0 \ \forall x, y \in \mathbb{N}$, and $d'(x,y) = d'(y,x)$.

$d'(x,x) = 0 \ \forall x \in \mathbb{N}$.

and $d'(x,z) \leq d'(x,y) + d'(y,z) \ \forall x, y, z \in \mathbb{N}$

[this can only fail if RHS $= 0$. In that case $x-y$, $y-z$ are even, so $x-z$ is even, so LHS $= 0$].

The construction described above gives a metric space $(\mathbb{N}/\sim, d)$

Note that $2 \sim 4 \sim 6 \sim 8 \sim \cdots$

and $1 \sim 3 \sim 5 \sim 7 \sim \cdots$

but

So $\mathbb{N}/\sim = \{[1], [2]\}$.

$d([1],[2]) = d'(1,2) = 1$

Metrics on $\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n\}$

(i) Euclidean Metric:

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$$

(ii) $d'((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = d(x_1,y_1) + d(x_2,y_2) + \cdots + d(x_n,y_n)$

[d is the Euclidean metric on $\mathbb{R}$]

(iii) $d''((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max\limits_{i \in \{1, \ldots, n\}} d(x_i, y_i)$

Next class we will show all three are metrics.