MA 771: Introduction to Dynamical Systems Lecture Notes

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Contents

Chapter 1

Basic Concepts

Dynamical systems refer to systems that evolve over time. A simple example is an ordinary differential equation (ODE), such as

$$\frac{dx}{dt} = g(x)$$

where $g : \mathbb{R}^n \longrightarrow \mathbb{R}$. Some of the questions we can ask about ODE's include:

- Which points are stable/fixed?
- When do orbits tend to an attracting periodic cycle?
- When are orbits chaotic?

However, the class of dynamical systems is much bigger than ODE's. In particular, in the above example, the variable t, representing time, changes *continuously*. In this course we will focus mainly on *discrete* dynamical systems.

Definition 1.0.1. Let X be a topological space. A *discrete* dynamical system is a pair (X, f) where $f : X \longrightarrow X$ is a self-map.

We are interested in the function f and its iterates $f^{\circ n} = f \circ f^{\circ(n-1)}$ for $n \in \mathbb{N}$. In other words, the quantity that is varying in a *discrete* fashion is the number of iterations of f.

Example 1.0.2. **Linear maps** Let $X = \mathbb{R}^n$ and $f : x \mapsto Ax$ be a linear map.

Example 1.0.3. **Rotations of the Circle** $X = \mathbb{S}^1$ and $f(x) = e^{2\pi i\theta}x$ for some $\theta \in \mathbb{R}$.

Example 1.0.4. **Logistic Family** Let $X = \mathbb{R}$ and fix $k \in \mathbb{R}_{>0}$. Then the family of maps $f_k : X \to X$ given by $x \mapsto kx(1-x)$ is called the *logististic family*.

Notice that we are not assuming any conditions on f such as continuity.

1.1 Orbits and Periodic Points

Definition 1.1.1. Given a dynamical system (X, f), $x_0 \in X$, the sequence

$$x_0, f(x_0), f^{\circ 2}(x_0), \cdots, f^{\circ n}(x_0), \cdots$$

is called the *forward orbit* of x_0.
The *reverse orbit of* x_0 is the set $\{x \in X : f^{\circ n}(x) = x_0 \text{ for some } n \in \mathbb{N}\}$.

A *fixed point* $x \in X$ is a point such that $f(x) = x$. The set of fixed points of f is denoted $\mathrm{Fix}(f)$. A *periodic point* is a point x such that $f^{\circ n}(x) = x$ for some $n \in \mathbb{N}$, in other words, a point in $\mathrm{Fix}(f^{\circ n})$ for some n.

Any $n \in \mathbb{N}$ such that $f^{\circ n}(x) = x$ is said to be a *period* of x. The smallest period n is called the *exact* period of x.

1.2 Examples

Linear Maps of $\mathbb{R}$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be linear. We know that f is of the form $f(x) = mx + b$ where $m \in \mathbb{R}_{\neq 0}$ and $b \in \mathbb{R}$. Note that

$$f^{\circ n}(x) = m^n x + b\big(m^{n-1} + m^{n-2} + \cdots + m + 1\big) = m^n x + b\frac{m^n - 1}{m - 1}$$

- If $m \neq \pm 1$, then

$$\begin{aligned} |m| < 1 &\implies \forall x \in \mathbb{R}, f^{\circ n}(x) \to \frac{b}{1-m} \text{ as } n \to \infty \\ |m| > 1 &\implies \forall x \in \mathbb{R}, f^{\circ n}(x) \to \infty \text{ as } n \to \infty \end{aligned}$$

- If $m = 1$, then $f(x) = x + b$ is a translation and all orbits tend to ∞
- If $m = -1$, then note that $f^{\circ 2}(x) = -(-x + b) + b = x$, and thus all the odd iterates are equal to f, and all the even iterates are equal to the identity.

Circle Maps

Example 1.2.1. For any rotation $f(x) = e^{2\pi i\theta}x$ of the circle $\mathbb{S}^1$, we have $\mathrm{Fix}(f) = \emptyset$ if $\theta \notin \mathbb{N}$, and $\mathrm{Fix}(f) = \mathbb{S}^1$ otherwise.

Definition 1.2.2. Fix an integer $m > 1$, and identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$. The *expanding map* $E_m : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is defined as

$$E_m(x) = mx \text{ (mod) } 1$$

Remark 1.2.3. E_m is expanding in the following sense: if $\alpha, \beta \in \mathbb{S}^1$ and $d_{\mathbb{S}^1}(\alpha, \beta) < \frac{1}{m}$, then

$$d_{\mathbb{S}_1}(\alpha, \beta) = m \cdot d_{\mathbb{S}^1}\Big(E_m(\alpha), E_m(\beta)\Big).$$

See Figure 1.1 for the graphs of E_m for $m = 2, 3$.

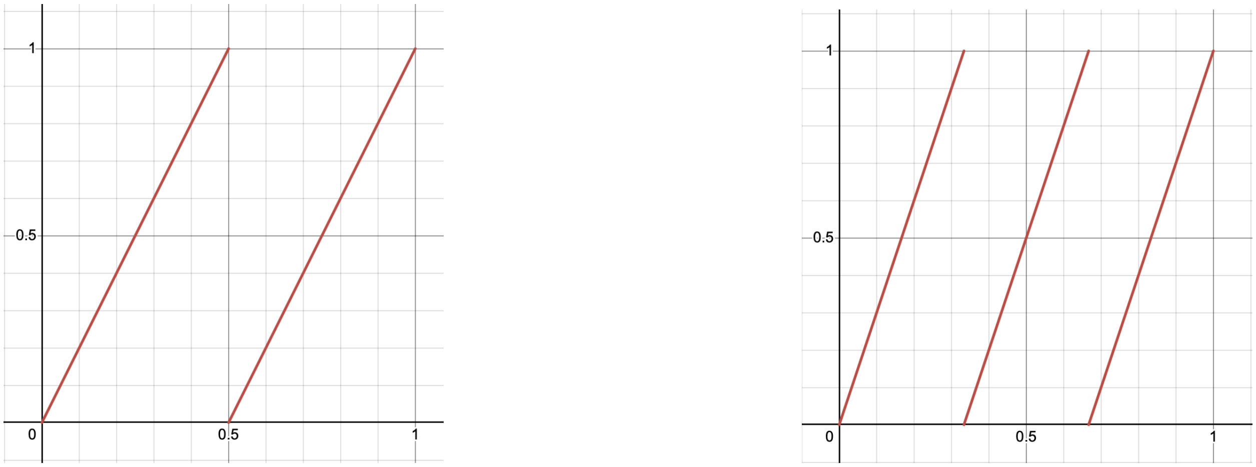

Figure 1.1: The graphs of the expanding maps E_2 (left) and E_3 (right) on the interior of $\mathbb{S}^1$, represented by the interval $(0,1)$.

Note that ϕ is a fixed point of E_m if and only if $m\phi - \phi \in \mathbb{Z}$. In other words, there exists $n \in \mathbb{Z}$ such that

$$\begin{aligned} m\phi - \phi &= n \\ \iff \phi &= \frac{n}{m-1} \end{aligned}$$

Similarly, ϕ is a periodic point of E_m of period dividing k if and only if there exists $n \in \mathbb{Z}$ such that

$$\begin{aligned} m^k\phi - \phi &= n \\ \iff \phi &= \frac{n}{m^k-1} \end{aligned}$$

In other words,

$$\begin{aligned} \operatorname{Fix}(E_m) &= \left\{\frac{1}{m-1}, \frac{2}{m-1} \cdots, \frac{m-2}{m-1}\right\} \\ \operatorname{Fix}\left(E_m^{\circ k}\right) &= \left\{\frac{1}{m^k-1}, \frac{2}{m^k-1} \cdots, \frac{m^k-2}{m^k-1}\right\} \end{aligned}$$

Torus Endomorphisms

Given $n \in \mathbb{N}$, the $n-$torus is the space $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \mathbb{R}^n/\sim$ where $x \sim y$ if $x - y \in \mathbb{Z}^n$. For $x \in \mathbb{R}^n$, we let $[x]$ denote the equivalence class of x in $\mathbb{T}^n$.

Definition 1.2.4. Let A be an $n \times n$ matrix whose entries are in $\mathbb{Z}$. Then A induces the *torus endomorphism* $T_A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ given by

$$T_A([x]) = [Ax] \text{ for } [x] \in \mathbb{T}^n$$

Exercise 1.2.5. Show that T_A as given above is well-defined: that is, for any two vectors $v, w \in \mathbb{R}^n$, if $v - w \in \mathbb{Z}^n$, then $Av - Aw \in \mathbb{Z}^n$

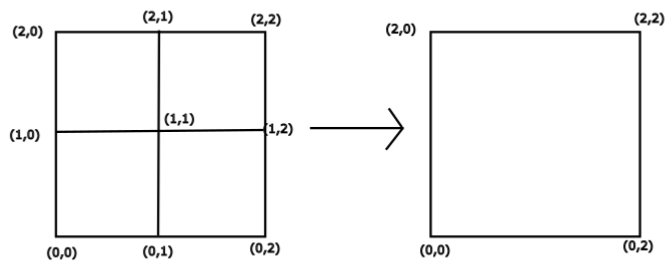

Figure 1.2: An illustration of the torus endomorphism $T_A : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ for $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Example 1.2.6. Let $m, k \in \mathbb{Z}$ and consider the matrix $A = \begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix}$. Consider the map T_A on $\mathbb{T}^2$: this acts as two independent expanding maps: expansion by a factor of m in the $x-$direction, and expansion by a factor of k in the $y-$direction (see Figure 1.2 which illustrates the case $m = k = 2$). Can you show in general that the degree of such a map is $d = mk$? In other words, T_A is a $d : 1$ map of $\mathbb{T}^2$.

Definition 1.2.7. A torus endomorphism T_A is said to be an *automorphism* if it is invertible.

Exercise 1.2.8. (This is also on HW 1) Show that T_A is invertible if and only if A^{-1} has integer entries, which in turn is equivalent to $\det A = \pm 1$.

Proposition 1.2.9. *Let $T_A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ be a torus automorphism with no eigenvalues of modulus 1. Then the periodic points of T_A are all the points with rational coordinates.*

Proof. (periodic $\implies$ rational):
Let $[x] = [(x_1, x_2, \cdots, x_n)] \in \mathbb{T}^n$ be a periodic point of period q for some $q \in \mathbb{N}$. Then $T_A^{\circ q}([x]) = [A^q x] = [x]$. That is, there exists a vector $y \in \mathbb{Z}^n$ such that

$$\begin{aligned} A^q x &= x + y \\ \implies A^q x - x &= y \\ \implies (A^q - \mathrm{Id})x &= y \end{aligned}$$

Since A has no eigenvalues of modulus 1, the matrix A^q has no eigenvalues of modulus 1. This means that the matrix $A^q - \mathrm{Id}$ is invertible. So

$$x = (A^q - \mathrm{Id})^{-1} y$$

Since y has integer coordinates and the matrix $(A^q - \mathrm{Id})^{-1}$ has rational entries, x has rational coordinates.

(rational $\implies$ periodic):
Suppose x has rational coordinates, we can assume that all the coordinates have a common denominator. In other words, $x = \left(\frac{p_1}{r}, \frac{p_2}{r}, \cdots, \frac{p_n}{r}\right)$ for some integers p_i, r with $r \neq 0$. Given a $q \in \mathbb{N}$, since A has integer entries, $A^q x = \left(\frac{p'_1}{r}, \frac{p'_2}{r}, \cdots, \frac{p'_n}{r}\right)$ for some integers $p'_1, \cdots, p'_n$.

Note that there are only finitely many points in $\mathbb{T}^n$ with rational coordinates with a common denominator r. In other words, the set $\{T_A^{\circ q}([x]) : q \in \mathbb{N}\}$ is finite.

Thus, there exist $q_1 < q_2 \in \mathbb{N}$ such that $T_A^{\circ q_1}([x]) = T_A^{\circ q_2}([x])$. Since T_A is an automorphism, this means that $T_A^{\circ(q_2-q_1)}([x]) = [x]$. $\square$

1.3 Stable Behavior: The Contraction Principle

In this section we will look at maps on subsets of $\mathbb{R}^n$ which satisfy a criterion for all orbits converging to a fixed point.

Global Contractions

Definition 1.3.1. A map f of a subset X of $\mathbb{R}^n$ is said to be *Lipschitz-continuous* with Lipschitz constant λ, or λ-*Lipschitz* if

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for any $x, y \in X$.
The map f is said to be a *contraction* if

$$d(f(x), f(y)) < d(x, y) \qquad \forall x, y \in X.$$

It is said to be a $\lambda-$*contraction* for $\lambda < 1$ if

$$d(f(x), f(y)) \leq \lambda d(x, y) \qquad \forall x, y \in X.$$

Remark 1.3.2. If a map f is Lipschitz-continuous, then we define

$$Lip(f) := \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}$$

Example 1.3.3. $f(x) = \sqrt{x}$ defines a contraction on $I = [1, \infty)$. What is $Lip(f)$?

Theorem 1.3.4 (Contraction Principle in $\mathbb{R}^n$)**.** *Let $X \subset \mathbb{R}^n$ be closed and $f : X \longrightarrow X$ be a $\lambda-$contraction. Then f has a unique fixed point x_0 and $d(f^{\circ n}(x), x_0) = \lambda^n d(x, x_0)$ for every $x \in X$.*

Proof. We have

$$d(f^{\circ n}(x), f^{\circ n}(y)) \leq \lambda^n d(x, y)$$

for all $x, y \in X$. But this also means that for any $x \in X$, we have

$$d(f^{\circ(n+1)}(x), f^{\circ n}(x)) \leq \lambda^n d(f(x), x)$$

$$\begin{aligned}
d(f^{\circ m}(x), f^{\circ n}(x)) &\leq d(f^{\circ m}(x), f^{\circ(m-1)}(x)) + d(f^{\circ(m-1)}(x), f^{\circ(m-2)}(x)) + \cdots d(f^{\circ(n+1)}(x), f^{\circ n}(x)) \\
&\leq \Big(\lambda^{m-1} + \lambda^{m-2} + \cdots \lambda^n\Big) d(f(x), x) \\
&\leq \frac{\lambda^n(1-\lambda^{m-n})}{1-\lambda} d(f(x), x) \\
&\leq \frac{\lambda^n}{1-\lambda} d(f(x), x)
\end{aligned}$$

In other words, the orbit of x is a Cauchy sequence. Since X is closed, $\lim_{n\to\infty} f^{\circ n}(x) = x_0$ is a point of X, and

$$f(x_0) = f(\lim_{n\to\infty} f^{\circ n}(x)) = \lim_{n\to\infty} f^{\circ(n+1)}(x_0) = x_0$$

$\square$

Remark 1.3.5. Given a sequence $(y_n)_{n\geq 0}$ in a metric space (Y, d), we say that $y_n \to y \in Y$ *exponentially* if there exist constants $A > 0$ and $0 < \lambda < 1$ such that

$$d(y_n, y) \leq A\lambda^n d(y_0, y)$$

Note that in the above situation, the orbit under f of x converges exponentially to x_0 (here $A = 1$).

The contraction principle applies to $\lambda-$contractions defined on complete metric spaces.

Theorem 1.3.6 (Contraction Principle for complete metric spaces)**.** *Let (X, d) be a complete metric space and $f : X \longrightarrow X$ be a $\lambda-$contraction. Then there exists a unique fixed point $x_0 \in X$ such that the orbits under f of all points $x \in X$ converge exponentially to x_0.*

Example 1.3.7 (Rabbits; due to Fibonacci). Say we record the number of rabbits in a forest starting January (month 0) of a given year. The Fibonacci model for rabbit population growth is as follows:
Letting b_n denote the number (in hundreds) of rabbits at the beginning of month n, we assume

$$\begin{aligned}
b_0 &= 1 \\
b_1 &= 2 \\
b_n &= b_{n-1} + b_{n-2} \text{ for } n \geq 2.
\end{aligned}$$

Then it is expected that the rabbit population growth rate stabilises as $n \to \infty$. That is, there exists $a \in (0, \infty)$ such that $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = b$. This means that in the long term, the rabbit population grows approximately *exponentially*, by a rough factor of b each month.

To prove the existence of a value a as required, we let $a_n = \frac{b_{n+1}}{b_n}$. Note that

$$a_{n+1} = \frac{b_{n+2}}{b_{n+1}} = \frac{b_{n+1} + b_n}{b_{n+1}} = 1 + \frac{b_n}{b_{n+1}} = 1 + \frac{1}{a_n}$$

Letting $g(x) = 1 + \frac{1}{x}$, we see that

$$a_{n+1} = g^{\circ n}(a_0) = g^{\circ n}(2) \text{ for all } n \geq 0$$

Claim: There exists a closed interval $I \subset \mathbb{R}$ such that $a_0 = 2$ such that

- $g(I) \subset I$
- g is a λ-contraction on I for some $\lambda \in (0, 1)$, and
- $a_0 = 2 \in I$

If we can prove this claim, then by the contraction principle, a can be recovered as the unique fixed point of g in I.

Proof of claim: The function g is decreasing on $(0, \infty)$, and has the horizontal asymptote $y = 1$. Note that $g'(x) = \frac{-1}{x^2}$.

This means $\forall x \in [c, \infty)$ where $c > 1$,

$$|g'(x)| = \frac{1}{x^2} \leq \frac{1}{c^2} < 1$$
$$\implies |g(x) - g(y)| \leq \frac{1}{c^2}|x - y| \text{ for all } x, y \in [c, \infty)$$

In other words, for all $c > 1$, $g : [c, \infty) \longrightarrow \mathbb{R}$ is a $\lambda-$contraction with $\lambda = \frac{1}{c^2}$. Also note g has a unique positive fixed point x_0: we can find it by solving the equation $g(x) = x$.

$$g(x) = x$$
$$\implies 1 + \frac{1}{x} = x$$
$$\implies x^2 - x + 1 = 0$$
$$\implies x = \frac{1 \pm \sqrt{5}}{2}$$

So $x_0 = \frac{1+\sqrt{5}}{2}$. Note that $\frac{3}{2} < x_0 < 2$.
Let $I = [\frac{3}{2}, 2]$, Then we have

$$g(2) = \frac{3}{2} \text{ and } g(3/2) = \frac{5}{3} < 2.$$

we see that $g(I) \subset I$. By the above discussion, g is a $\lambda-$contraction on I (with $\lambda = (2/3)^2$), and thus the orbit under g of $a_0 = 2$ converges to $a = x_0 = \frac{1+\sqrt{5}}{2}$.
Remark 1.3.8. The choice of I is not unique: for any $c \in (1, 3/2]$, we have $g[c, 2] = [3/2, g(c)] \subset [c, 2]$, and g is a λ contraction on $[c, 2]$ with $\lambda = \frac{1}{c^2}$. I made a small mistake in class by saying c can be in $[1, x_0]$: can you see why $c \in (3/2, x_0]$ won't work?

Local Contractions

Proposition 1.3.9. *Let f be a continuously differentiable map of $\mathbb{R}^n$ with a fixed point x_0 where $||Df_{x_0}|| < 1$. Then there is a closed neighborhood U of x_0 such that $f(U) \subset U$ and f is a contraction on U.*

To do this we will need the following exercise and proposition:

Exercise 1.3.10. Given a linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, recall that

$$||A|| = \sup_{||x|| \neq 0} \frac{||Ax||}{||x||}$$

Prove that $A \mapsto ||A||$ is a continuous function from $\mathbb{R}^{m \times n}$ to $\mathbb{R}$.

Proposition 1.3.11. *Let $V \subset \mathbb{R}^n$ be a closed disk and let $f : V \longrightarrow \mathbb{R}^m$ be continuous, with continuous derivative on the interior of V. Suppose there exists $M > 0$ such that $||Df_x|| \leq M$ for all x in the interior of V. Then*

$$d(f(x), f(y)) \leq M d(x, y) \qquad \forall x, y \in V$$

Proof. Given $x, y \in \mathbb{R}^n$, let $g : [0, 1] \longrightarrow \mathbb{R}^m$ be the function

$$g(t) = f((1-t)x + ty)$$

Then the mean value theorem states that for some $c \in (0, 1)$,

$$d(g(0), g(1)) \leq ||g'(c)||$$

From this we get

$$\begin{aligned} d(f(x), f(y)) &= d(g(0), g(1)) \leq ||g'(c)|| \\ &= ||Df_{(1-c)x+cy}(y - x)|| \leq ||Df_{(1-c)x+cy}|| \cdot d(x, y) \\ &\leq M d(x, y) \end{aligned}$$

□

Proof of Proposition 1.3.9. The function f is C^1 implies that $x \mapsto Df$ is continuous. By Exercise 4, the composition $x \mapsto Df_x \mapsto ||Df_x||$ is continuous. Fix a point $\lambda \in \Big(||Df_{x_0}||, 1\Big)$. Then there exists a small closed ball $U = \overline{B(x_0, \delta)}$ around x_0 on which $||Df_x|| \leq \lambda < 1$.

By Proposition 1.3.11, if $x, y \in U$, then $d(f(x), f(y)) \leq \lambda d(x, y)$. Moreover, for all $x \in U$, we have

$$d(f(x), x_0) = d(f(x), f(x_0)) \leq \lambda d(x, x_0) \leq \lambda\delta < \delta.$$

This shows that $f(U) \subset U$, and f is a $\lambda-$contraction on U.

□

1.4 Fractals

In this section we will define fractals and introduce self-similarity. We will also give an idea of their connection with dynamical systems with some examples.

The Cantor Set

The simplest example of a fractal is the ternary cantor set.

Definition 1.4.1. Let $I = [0,1]$. Inductively define closed subsets $C_n \subset I$ for $n \geq 0$ as follows:

$$\mathcal{C}_0 = \Big[0,1\Big]$$
$$\mathcal{C}_1 = \mathcal{C}_0 \setminus \Big(\frac{1}{3},\frac{2}{3}\Big) = \frac{\mathcal{C}_0}{3} \cup \Big(\frac{\mathcal{C}_0}{3}+\frac{2}{3}\Big)$$
$$\mathcal{C}_n = \frac{\mathcal{C}_{n-1}}{3} \cup \Big(\frac{\mathcal{C}_{n-1}}{3}+\frac{2}{3}\Big) = \mathcal{C}_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \Big(\frac{3k+1}{3^n},\frac{3k+2}{3^n}\Big) \text{ for all } n \geq 2$$

It can be shown that the set $\mathcal{C}_n$ is the disjoint union of 2^n closed intervals, each of length $\frac{1}{3^n}$. The ternary Cantor set is defined as

$$\mathcal{C} = \bigcap_{n=0}^{\infty} \mathcal{C}_n$$

We look at some of the properties of $\mathcal{C}$.

1. $\mathcal{C}$ **is closed**, since it is the intersection of closed sets
2. $\mathcal{C}$ **is compact**, since it is a closed subset of a compact set.
3. $\mathcal{C}$ **is non-empty.**
   For example, the points 0 and 1 belong to all the sets $\mathcal{C}_n$, so they also belong to $\mathcal{C}$.
4. $\mathcal{C}$ **is uncountable.**
   We will show this by giving an explicit description of $\mathcal{C}$.

   Definition 1.4.2. Given a number $x \in [0,1]$, a base 3 (or *ternary*) expansion for x is a sequence $.\alpha_1\alpha_2\alpha_3\cdots$ with $\alpha_n \in \{0,1,2\}$ for all $n \in \mathbb{N}$ such that

   $$x = \frac{\alpha_1}{3} + \frac{\alpha_2}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$$

   The decimal point before the α's indicate that the number is less than or equal to 1. More generally, for any real number $y \in \mathbb{R}$. The ternary expansion of y is a sequence $\beta_{m-1}\beta_{m-2}\cdots\beta_0.\alpha_1\alpha_2\alpha_3\cdots$ with $\beta_i \in \{0,1,2\}$ for all $i \in \{0,1,2,\cdots m-1\}$ and $\alpha_n \in \{0,1,2\}$ for all $n \in \mathbb{N}$, such that

   $$y = \sum_{i=0}^{m-1} \beta_i \cdot 3^i + \sum_{n=0}^{\infty} \frac{\alpha_n}{3^n}$$

Note that the ternary expansion of a number is not unique. For example,

$$\frac{1}{3} = \sum_{n=1}^{\infty} \frac{2}{3^n}$$

So $.100000\cdots$ and $.0222222\cdots$ are both ternary expansions for $\frac{1}{3}$. Similarly, $.200000\cdots$ and $.1222222\cdots$ are both ternary expansions for $\frac{2}{3}$.

Exercise 1.4.3. Every number $x \in \mathbb{R}$ has only finitely many ternary expansions.

Remark 1.4.4. If x has a ternary expansion $.\alpha_1\alpha_2\alpha_3\cdots$, then $\frac{x}{3}$ has a ternary expansion $.0\alpha_1\alpha_2\cdots$

Proposition 1.4.5.

$$\mathcal{C} = \{x \in [0,1] : \textit{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \textit{ for } x \textit{ with } \alpha_n \in \{0,2\} \textit{ for all } n \in \mathbb{N}\}$$

Proof. We prove this by induction. Note that

$$\mathcal{C}_1 = \{x \in [0,1] : \text{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \text{ for } x \text{ with } \alpha_1 \in \{0,2\}\}$$

Using the recursive formula for $\mathcal{C}_n$, it is easy to show that

$$\mathcal{C}_N = \{x \in [0,1] : \text{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \text{ for } x \text{ with } \alpha_1, \alpha_2, \cdots, \alpha_N \in \{0,2\}\}$$

Since every x has only finitely many ternary expansions, for x to be in all the $\mathcal{C}_n$'s, there exists at least one ternary expansion which satisfies the condition $\alpha_n \in \{0,2\}$ for all $n \in \mathbb{N}$. □

Since every sequence in $\{0,2\}^{\mathbb{N}}$ can be realized as the ternary expansion of a distinct number $x \in [0,1]$, and the set $\{0,2\}^{\mathbb{N}}$ is uncountable, we see that $\mathcal{C}$ is uncountable.

5. **$\mathcal{C}$ is perfect.**

Definition 1.4.6. Let Y be a topological space. A subset $X \subset Y$ is said to be perfect if it is closed in Y and has no isolated points.

Proposition 1.4.7. *For every $x \in \mathcal{C}$, there exists a sequence (x_n) of distinct points with $x_n \in \mathcal{C}$ and $x_n \to x$.*

Proof. Given $\epsilon > 0$, we will exhibit a point $x_N \neq x$ such that $|x_N - x| < \epsilon$ and $x_N \in \mathcal{C}$. Choose $N \in \mathbb{N}$ such that $\frac{2}{3^N} < \epsilon$. This ensures that the interval of radius $\frac{1}{3^N}$ centered at x is contained in the open ball $B_\epsilon(x)$. Let $\widetilde{I}$ be the component interval of $\mathcal{C}_N$ that contains x. The above condition implies that $\widetilde{I} \subset B_\epsilon(x)$.

In $\mathcal{C}_{N+1}$, the middle third of $\widetilde{I}$ is deleted, and we get two component intervals $\widetilde{I}_0$ and $\widetilde{I}_1$. Without loss of generality, assume $x \in \widetilde{I}_0$. Then pick a point $y \in \mathcal{C} \cap \widetilde{I}_1$. Note that $y \neq x$ by this choice and since $\widetilde{I}_1 \subset B_\epsilon(x)$, we have $|y - x| < \epsilon$. Therefore we can set $x_N = y$. □

6. $\mathcal{C}$ **is totally disconnected.**

   Definition 1.4.8. A topological space X is said to be totally disconnected if its only non-empty connected subsets are singletons.

   Proposition 1.4.9. *If $F \subset \mathcal{C}$ is non-empty and connected, then $F = \{x\}$ for some point $x \in \mathcal{C}$.*

   Proof. Suppose $x, y \in \mathcal{C}$ are two distinct points in F. WLOG, assume $x < y$. Pick $N \in \mathbb{N}$ such that $\frac{1}{3^{N-1}} < |x - y|$. Then, x and y are contained in distinct components of C_N. So there exists $z \in (x, y)$ such that $z \notin \mathcal{C}$. Let $A = F \cap [0, z)$ and $B = F \cap (z, 1]$. Note that $A \cup B = F$. Also note that the closures of A and B don't intersect. This contradicts the fact that F is connected. $\square$

7. $\mathcal{C}$ **has Lebesgue measure** 0**.**
   Let μ denote Lebesgue measure. The set $\mathcal{C}_n$ is the union of 2^n disjoint intervals, each of length 3^{-n}. Therefore, we have $\mu(\mathcal{C}_n) = \frac{2^n}{3^n}$. Since $\mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3 \supset \cdots$ and $\mathcal{C}$ is the intersection of the $\mathcal{C}_n$, we have

$$\mu(\mathcal{C}) = \lim_{n \to \infty} \mu(\mathcal{C}_n) = 0$$

The following theorem is the main result of this section. For a proof see [1].

Theorem 1.4.10 (Brouwer)**.** *Let $Y \neq \emptyset$ be a complete metric space. If Y is compact, perfect and totally disconnected, then it is homeomorphic to $\mathcal{C}$.*

An easy corollary, for example, is that $\mathcal{C}$ is homeomorphic to $\mathcal{C} \times \mathcal{C}$.

Dynamical Systems on the Cantor Set

Consider the map $f : [0, 1] \times [0, 1]$ defined as $f(x) = \frac{x}{3}$. It is easy to see that f is a contraction and $f(\mathcal{C}) \subset \mathcal{C}$, and the unique fixed point in $\mathcal{C}$ is $x = 0$. Note that for every $x \in \mathcal{C}$, there exists a neighborhood U of x such that $f : U \longrightarrow f(U)$ is a homeomorphism.
Also note that f induces the shift $.\alpha_1\alpha_2\alpha_3 \cdots \to .0\alpha_1\alpha_2\alpha_3 \cdots$ on ternary expansions.

Definition 1.4.11. A topological space X is said to be *self-similar*, or to have the *rescaling property*, if there exists a contraction $f : X \longrightarrow X$ such that for every $x \in X$ and neighborhood U of x, there exists a neighborhood $V \subset U$ of x such that $f : V \longrightarrow f(V)$ is a homeomorphism.

Remark 1.4.12. This is actually equivalent to saying that every $x \in X$ has a neghborhood U such that $f : U \longrightarrow f(U)$ is a homeomorphism. Also note that the term self similar is used in different ways in the literature; we will see by and by that this definition is not extensive enough.

Exercise 1.4.13. Show that the function $f(x) = 1 - \frac{x}{3}$ leaves $\mathcal{C}$ invariant, and is a contraction. Describe the induced operation on ternary expansions, and find the unique fixed point of f in $\mathcal{C}$.

Exercise 1.4.14. Show that the function $f(x) = \frac{x+2}{3} \pmod 1$ leaves $\mathcal{C}$ invariant, and is a contraction. Describe the induced operation on ternary expansions, and find the unique fixed point of f in $\mathcal{C}$.

The Square Sierpinksi Carpet

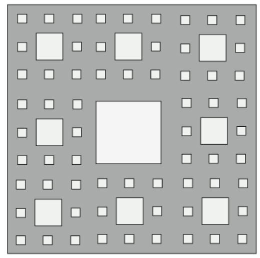

Figure 1.3: The set $\mathcal{J}_3$ of the Sierpinski carpet construction

Definition 1.4.15. Let $J = [0,1] \times [0,1]$ be the unit square. Define

$$\begin{aligned}
\mathcal{J}_0 &= J \\
\mathcal{J}_1 &= \mathcal{J}_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) \times \left(\frac{1}{3}, \frac{2}{3}\right) \\
\mathcal{J}_n &= \mathcal{J}_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \bigcup_{\ell=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) \times \left(\frac{3\ell+1}{3^n}, \frac{3\ell+2}{3^n}\right)
\end{aligned}$$

The square Sierpinski carpet is the set $\mathcal{J} = \bigcap_{n=0}^{\infty} \mathcal{J}_n$.

Exercise 1.4.16. Prove that the Sierpinski carpet is self-similar.

The Sierpinski Triangle

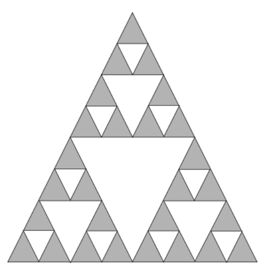

Figure 1.4: The set Δ_2 of the Sierpinski triangle construction

This set is similar to the Sierpinski carpet. Start with an equilateral triangle Δ_0 of side length 1, with one side horizontal. Let Δ_1 be Δ_0 minus its central equilateral triangle, Δ_2 be Δ_1 minus its three smaller central equilateral triangles and so on.
Define the Sierpinski triangle

$$\Delta = \bigcap_{n=0}^{\infty} \Delta_n$$

Exercise 1.4.17. Prove that the Sierpinski triangle is self-similar.

Exercise 1.4.18. Prove that $\mathcal{J}$ and Δ both have infinite perimeter and finite area (Lebesgue measure).

Exercise 1.4.19. Prove that neither $\mathcal{J}$ nor Δ is homeomorphic to $\mathcal{C}$.

Cantor Sets and Logistic Maps

Consider the logistic function $f(x) = 5x(1-x)$ on $\mathbb{R}$. Note that this function has one critical point at $x = \frac{1}{2}$, and is symmetric around this point in the sense that

$$f(x) = f(1-x) \text{ for all } x \in \mathbb{R}$$

We make the following series of observations:

1. The graph of f is a downward drawn parabola, and its roots are $x = 0, 1$.
2. If $x > 1$, then $f(x) < 0$.
3. If $x < 0$, then $f(x) < x$ and $|f(f(x)) - f(x)| > |f(x) - x|$.

The points (2) and (3) show that if $x \notin [0,1]$, then $f^{\circ n}(x) \to -\infty$. This leads to the following dichotomy:
For every $x \in \mathbb{R}$, exactly one of the following is true:

- either $f^{\circ n}(x) \in [0,1]$ for all $n \in \mathbb{N}$, or
- $f^{\circ m}(x) \notin [0,1]$ for some $m \in \mathbb{N}$, and thus, $f^{\circ n}(x) \to -\infty$ as $n \to \infty$.

Therefore, the set B of points $x \in \mathbb{R}$ such the orbit $(f^{\circ n}(x))_{n \geq 0}$ is bounded, is the set of points x such that $f^{\circ n}(x) \in [0,1]$ for all n. In other words,

$$B = \bigcap_{n=0}^{\infty} (f^{\circ n})^{-1}[0,1]$$

Proposition 1.4.20. *B is a Cantor set (i.e., it is homeomorphic to $\mathcal{C}$).*

We will prove this in the next chapter.

1.5 Topological Conjugacy

Definition 1.5.1. Let X, Y be topological spaces and suppose $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$ be dynamical systems. Then (X, f) and (Y, g) are said to be *topologically conjugate* if there exists a homeomorphism $\varphi : X \longrightarrow Y$ such that

$$g \circ \varphi = \varphi \circ f$$

In other words, φ is a homeomorphism that makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

Note that for all $n \in \mathbb{N}$,

$$g^{\circ n} = (\varphi \circ f \circ \varphi^{-1})^{\circ n} = (\varphi \circ f \circ \varphi^{-1}) \circ (\varphi \circ f \circ \varphi^{-1}) \circ \cdots \circ (\varphi \circ f \circ \varphi^{-1}) = \varphi \circ f^{\circ n} \circ \varphi^{-1}$$

In particular, φ maps the $f-$orbit of x to the $g-$orbit of $\varphi(x)$ for every $x \in X$. So topological conjugacy is a form of equivalence between two dynamical systems.

Examples

Example 1.5.2. The map $\varphi(x) = \frac{-1}{2}x + \frac{1}{2}$ conjugates the dynamical systems $f(x) = x^2$ and $g(x) = 2x(1-x)$ on $\mathbb{R}$. Since φ is linear, we say that $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$ are *linearly/affinely* conjugate.

Exercise 1.5.3. Prove that every quadratic polynomial $f : \mathbb{C} \longrightarrow \mathbb{C}$ is affine conjugate to a polynomial of the form $z^2 + c$ for a unique $c \in \mathbb{C}$.

Exercise 1.5.4. Let W and V be vector spaces over $\mathbb{R}$, and suppose $A : W \longrightarrow W$ and $B : V \longrightarrow V$ are linear maps. Show that if A and B are conjugate as linear maps, ie, there exists an invertible linear map $L : W \longrightarrow V$ such that $LA = BL$, then they are also topologically conjugate in the sense defined in the previous section.

Logistic Map Revisited

We now revert back to our previous discussion of Cantor sets and the logistic map $f(x) = 5x(1-x)$.

Let $\Sigma = \{0,1\}^{\mathbb{N}} = \{s = s_1 s_2 \cdots | s_i \in \{0,1\} \forall i \in \mathbb{N}\}$ and $\sigma : \Sigma \to \Sigma$ be the map $s_1 s_2 s_3 \cdots \mapsto s_2 s_3 \cdots$. We will equip Σ with a topology under which σ is continuous.

Theorem 1.5.5. *Given $\mu \neq 0$, let $g_\mu(x) = \mu x(1-x)$, and let $B_\mu \subset \mathbb{R}$ be the set of points with bounded orbits under g_μ.*

For $\mu > 4$, the dynamical systems (B_μ, g_μ) and (Σ, σ) are topologically conjugate.

Theorem 1.5.6. *The set Σ is homeomorphic to the ternary Cantor set $\mathcal{C}$.*

These theorems imply Proposition 1.4.20. In the next chapter, we will prove Theorem 1.5.6, and Theorem 1.5.5 for the smaller range $\mu > 2 + \sqrt{5} > 4$.

Chapter 2

Symbolic Dynamics

To prove Theorems 1.5.5 and 1.5.6, we will need the powerful machinery of symbolic dynamics. In the next section we will introduce its basic concepts.

2.1 Sequences over a finite alphabet

Let (X, d_X) be a metric space, and $A \subset X$ be a finite set with $|A| \geq 2$.

Definition 2.1.1. The set of sequences with alphabet A is denoted Σ_A. In other words,

$$\Sigma_A = A^{\mathbb{N}} = \{s = s_1 s_2 s_3 \cdots | s_j \in A \forall j \in \mathbb{N}\}$$

Topology on the Space of Sequences

We define a metric on Σ_A as follows: for all $s, t \in \Sigma_A$, we let

$$d(s,t) = \sum_{j=1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}} \tag{2.1}$$

Proposition 2.1.2. *The function $d : \Sigma_A \times \Sigma_A \longrightarrow \mathbb{R}$ is well-defined.*

Proof. We need to show that for all $s, t \in \Sigma_A$, the infinite series given above converges. Let $M = \max_{p,q \in A} d_X(p, q)$.

$$\begin{aligned} d(s,t) &= \sum_{j=1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}} \\ &\leq \sum_{j=1}^{\infty} \frac{M}{|A|^{j-1}} = M \sum_{j=1}^{\infty} \frac{1}{|A|^{j-1}} \\ &= \frac{M}{1 - \frac{1}{|A|}} \\ &< \infty \end{aligned}$$

$\square$

Proposition 2.1.3. *The function defined by Equation 2.1 is a metric on Σ_A.*

Proof. We need to show the following:

1. for all $s, t \in \Sigma_A$, $d(s,t) \geq 0$;
2. for all $s, t \in \Sigma_A$, $d(s,t) = 0 \iff s = t$;
3. for all $s, t, r \in \Sigma_A$, $d(s,r) \leq d(s,t) + d(t,r)$.

We will see these one by one.

1. is clear since every term in the infinite sequence defining $d(s,t)$ is non-negative.
2. is clear since if $d(s,t) = 0$, then $d_X(s_j, t_j) = 0$ for all $j \in \mathbb{N}$, which implies $s_j = t_j$ for all $j \in \mathbb{N}$.
3. for every $j \in \mathbb{N}$, $d_X(s_j, r_j) \leq d_X(s_j, t_j) + d_X(t_j, r_j)$. This immediately shows (3).

□

The metric d induces a topology on Σ_A. We will see some properties of this topology in the remaining section.

Remark 2.1.4. By scaling the metric d_X if necessary, from now on we assume without loss of generality that $M = \max_{p,q \in A} d_X(p,q) = 1$.

Proposition 2.1.5. *Suppose $s, t \in \Sigma_A$ satisfy $s_j = t_j$ for $j = 1, 2, \cdots, N$. Then*

$$d(s,t) < \frac{1}{|A|^{N-1}(|A|-1)} \leq \frac{1}{|A|^{N-1}}$$

Proof. The second inequality follows directly since $\frac{1}{|A|-1} \leq 1$.
Since $s_j = t_j$ for $j = 1, \cdots, N$,

$$\begin{aligned} d(s,t) &= \sum_{j=1}^{N} \frac{d_X(s_j,t_j)}{|A|^{j-1}} + \sum_{j=N+1}^{\infty} \frac{d_X(s_j,t_j)}{|A|^{j-1}} \\ &= \sum_{j=N+1}^{\infty} \frac{d_X(s_j,t_j)}{|A|^{j-1}} \\ &\leq \sum_{j=N+1}^{\infty} \frac{1}{|A|^{j-1}} \\ &= \frac{1}{|A|^N} \frac{1}{1 - \frac{1}{|A|}} = \frac{|A|}{|A|^N(|A|-1)} \\ &= \frac{1}{|A|^{N-1}(|A|-1)} \end{aligned}$$

□

Proposition 2.1.6. *There exists a constant $\ell = \ell(A) > 0$ such that if $s, t \in \Sigma_A$ satisfy $d(s,t) < \frac{\ell}{|A|^{N-1}}$, then $s_j = t_j$ for $j = 1, 2, \cdots, N$.*

Proof. Let $\ell = \min_{\substack{p,q \in A \\ p \neq q}} d_X(p,q)$. We will prove the contrapositive. Given $s, t \in \Sigma_A$, if $s_j \neq t_j$ for some $j \in \{1, 2, \cdots, N\}$, then

$$d(s,t) \geq \frac{d_X(s_j, t_j)}{|A|^{j-1}} \geq \frac{\ell}{|A|^{j-1}} \geq \frac{\ell}{|A|^{N-1}}$$

□

Shift Operator on Sequences

Definition 2.1.7. The *shift operator* $\sigma : \Sigma_A \longrightarrow \Sigma_A$ is defined as

$$\sigma(s_1 s_2 s_3 \cdots) = s_2 s_3 s_4 \cdots \text{ for all } s = s_1 s_2 s_3 \cdots \in \Sigma_A \tag{2.2}$$

Proposition 2.1.8. *The map σ is surjective and uniformly continuous.*

Proof. Given $s \in \Sigma_A$, for any $a \in A$, $\sigma(a s_1 s_2 s_3 \cdots) = s$. Therefore σ is surjective.
To show it is uniformly continuous, we will exhibit for a given $\epsilon > 0$, a constant $\delta > 0$ such that for all $s, t \in \Sigma_A$, $d(s,t) < \delta \implies d(\sigma(s), \sigma(t)) < \epsilon$.

Fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{|A|^{N-1}} < \epsilon$. Let $\delta = \frac{\ell}{|A|^N}$, where ℓ is the constant from Proposition 2.1.6. Then, we have $s_j = t_j$ for $j = 1, 2, \cdots, N+1$. Let $\sigma(s) = \underline{s}$ and $\underline{t} = \sigma(t)$. Note that $\underline{s}_j = s_{j+1}$ and $\underline{t}_j = t_{j+1}$ for all $j \in \mathbb{N}$. The above condition implies that $\underline{s}_j = \underline{t}_j$ for $j = 1, 2, \cdots, N$. Therefore by Proposition 2.1.5, we have $d(\underline{s}, \underline{t}) < \frac{1}{|A|^{N-1}} < \epsilon$. □

Periodic Sequences

Definition 2.1.9. For $m \in \mathbb{N}$, define

$$\mathrm{Per}_m(\sigma) = \{s \in \Sigma_A | \sigma^{\circ m}(s) = s\}$$

In other words, $\mathrm{Per}_m(\sigma)$ is the set of sequences whose period under σ divides m.
Also define $\mathrm{Per}(\sigma)$ to be the set of sequences periodic under σ.

Remark 2.1.10. The following properties of periodic sequences are immediate.

1. $$\mathrm{Per}(\sigma) = \bigcup_{m=1}^{\infty} \mathrm{Per}_m(\sigma)$$

2. Given a finite word $w = s_1 s_2 \cdots s_n$ with $s_i \in A$ for all i, we let $\overline{w}$ denote the infinite word $s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n \cdots$ formed by repeating the finite block w.
   Given $m \in \mathbb{N}$,

   $$\mathrm{Per}_m(\sigma) = \{\overline{s_1 s_2 \cdots s_m} : s_j \in A \text{ for } j = 1, 2, \cdots, m\}.$$

   This shows that

   $$|\mathrm{Per}_m(\sigma)| = |A|^m$$

3. If $m < n$ and $m|n$, then
$$\mathrm{Per}_m(\sigma) \subsetneq \mathrm{Per}_n(\sigma)$$

Proposition 2.1.11. $\mathrm{Per}(\sigma)$ *is dense in* Σ_A.

Proof. Given $s \in \Sigma_A$ and $\epsilon > 0$, pick $N \in \mathbb{N}$ such that $\frac{1}{|A|^{N-1}} < \epsilon$. Then by Proposition 2.1.5, the sequence $t \in \mathrm{Per}(\sigma)$ given by

$$t = s_1 s_2 \cdots s_N s_1 s_2 \cdots s_N s_1 s_2 \cdots = \overline{s_1 s_2 \cdots s_N}$$

satisfies
$$d(s,t) < \frac{1}{|A|^{N-1}} < \epsilon$$

□

Logistic Maps Conjugate to the Shift

In this section we will establish Theorem 1.5.5 for the family of maps $g_\mu(x) = \mu x(1-x)$ where $\mu > 2+\sqrt{5}$. The proof for the full range $\mu > 4$ uses techniques from complex analysis, so we will see this later.
Recall the definition of the set B_μ: this is the set of points x with bounded orbit under the map g_μ. Just as we did for $\mu = 5$, we will show that for a range of μ values, the set $B_\mu \subset [0,1]$.

Proposition 2.1.12. *When* $\mu > 1$, *for* $x \notin [0,1]$, $g_\mu^{\circ n}(x) \to -\infty$ *as* $n \to \infty$.

Proof. First, observe that the graph of g_μ is a downward drawn parabola which intersects the x-axis at the two roots $x = 0, 1$.

If $x < 0$, then $g_\mu(x) = \mu x - \mu x^2 < \mu x < x$. So the terms of the orbit $g_\mu^{\circ n}(x)$ become more and more negative as n increases. Now we show that the monotone decreasing sequence $x, g_\mu(x), g_\mu^{\circ 2}(x), \cdots$ does not stay bounded. Suppose to the contrary, then there exists $p < 0$ such that $g_\mu^{\circ n}(x) \to p$. On the one hand we have $g_\mu^{\circ(n+1)}(x) \to g_\mu(p) < p$, but on the other hand, the sequence $(g_\mu^{\circ(n+1)}(x))_{n\geq 0}$, as a tail of the sequence $(g_\mu^{\circ n}(x))_{n\geq 0}$, also converges to p. This proves that $g_\mu^{\circ(n+1)}(x) \to -\infty$.

If $x > 1$, then $g_\mu(x) < 0$. By the discussion above, $g_\mu^{\circ n}(x) \to -\infty$. □

Proposition 2.1.13. *For* $\mu > 1$,

$$B_\mu = \bigcap_{n\geq 0} (g_\mu^{\circ n})^{-1}[0,1]$$

If $1 < \mu \leq 4$, *then* $B_\mu = [0,1]$.

Proof. The previous proposition shows that $B_\mu \subseteq [0,1]$ for all $\mu > 1$, and moreover, that $x \in B_\mu$ if and only if $g_\mu^{\circ n}(x) \in [0,1]$ for all $n \in \mathbb{N}$.
In other words,
$$B_\mu = \bigcap_{n\geq 0} \{x \in \mathbb{R} : g_\mu^{\circ n}(x) \in [0,1]\} = \bigcap_{n\geq 0} (g_\mu^{\circ n})^{-1}[0,1].$$

Note that $x = \frac{1}{2}$ is the unique point where g_μ reaches its maximum, and $g_\mu\left(\frac{1}{2}\right) = \frac{\mu}{4}$.

Thus, if $1 < \mu \leq 4$, then since $\frac{\mu}{4} \leq 1$, we have

$$\begin{aligned} & g_\mu[0,1] \subseteq [0,1] \\ \implies & [0,1] \subseteq g_\mu^{-1}[0,1] \end{aligned}$$

Since we know that $g_\mu^{-1}[0,1] \subseteq [0,1]$, this shows that $B_\mu = [0,1]$. □

Thus the interesting structure of B_μ occurs when $\mu > 4$.

Proposition 2.1.14. *Fix $\mu > 4$. Let $c_\mu = \sqrt{\frac{1}{4} - \frac{1}{\mu}}$, and define the disjoint intervals $I_0 = \left[0, \frac{1}{2} - c_\mu\right]$ and $I_1 = \left[\frac{1}{2} + c_\mu, 1\right]$. Then*

$$g_\mu^{-1}[0,1] = I_0 \cup I_1$$

Proof. Solving $g_\mu(x) = 1$, we get

$$\begin{aligned} & \mu x - \mu x^2 = 1 \\ \implies & \mu x^2 - \mu x + 1 = 0 \\ \implies & x = \frac{\mu \pm \sqrt{\mu^2 - 4\mu}}{2\mu} \\ & = \frac{1}{2} \pm \sqrt{\frac{\mu^2 - 4\mu}{4\mu^2}} \\ & = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{\mu}} \\ & = \frac{1}{2} \pm c_\mu \end{aligned}$$

Note that $g_\mu[0,1] = \left[0, \frac{\mu}{4}\right]$. Since $\frac{1}{2}$ is the point where g_μ is maximum, and the graph of g_μ is symmetric about the vertical line $x = \frac{1}{2}$, we get

$$g_\mu\left(\frac{1}{2} - c_\mu, \frac{1}{2} + c_\mu\right) = \left(1, \frac{\mu}{4}\right]$$

Thus, we have, for I_0 and I_1 as above, that

$$g_\mu(I_0 \cup I_1) = \left[0, \frac{\mu}{4}\right] \setminus \left(1, \frac{\mu}{4}\right] = [0,1]$$

□

Note that the intervals I_0 and I_1 above are disjoint. Note that $g_\mu(I_0) = g_\mu(I_1) = [0,1]$, so $(g_\mu^{\circ 2})^{-1}[0,1] = g_\mu^{-1}(I_0 \cup I_1) = I_{00} \cup I_{01} \cup I_{10} \cup I_{11}$. here $I_{00} \cup I_{01} =$.

Definition 2.1.15. We introduce the notation Σ^0 for the set of finite non-empty words over the alphabet $\{0,1\}$. Formally,

$$\Sigma^0 = \bigcup_{N\geq 1}\{w = s_1 s_2 \cdots s_N | s_i \in \{0,1\} \text{ for } i = 1,2,\cdots,N\} = \bigcup_{N\geq 1}\{0,1\}^N$$

We let $\ell(w)$ denote the length of the finite word w.

Definition 2.1.16. Given $w = s_1 s_2 \cdots s_N \in \Sigma^0$, define the set $I_w \subseteq [0,1]$ as follows:

$$\begin{aligned} I_w &= \{x \in [0,1] : x \in I_{s_1}, g_\mu(x) \in I_{s_2}, \cdots, g_\mu^{\circ(n-1)}(x) \in I_{s_n}\} \\ &= \bigcap_{j=1}^{N}\{x \in [0,1] : g_\mu^{\circ(j-1)}(x) \in I_{s_j}\} \\ &= \bigcap_{j=1}^{N}(g_\mu^{\circ(j-1)})^{-1}(I_{s_j}) \end{aligned}$$

Proposition 2.1.17. *The collection of intervals $\{I_w : w \in \Sigma^0\}$ satisfies*

1. *Given $w = s_1 \cdots s_N \in \Sigma^0$ and a symbol $s_{N+1} \in \{0,1\}$, we have $I_{ws_{N+1}} \subseteq I_w$.*
2. *Given $w = s_1 \cdots s_N \in \Sigma^0$ and a symbol $s_0 \in \{0,1\}$, the map g_μ maps $I_{s_0 w}$ homeomorphically onto I_w.*
3. *I_w is a closed interval of non-zero length for all $w \in \Sigma^0$.*
4. *Given distinct words $w_1, w_2 \in \{0,1\}^N$, $I_{w_1} \cap I_{w_2} = \emptyset$.*
5. *for all $N \in \mathbb{N}$, $(g_\mu^{\circ N})^{-1}[0,1] = \bigcup_{w\in\{0,1\}^N} I_w$.*

Proof. We will show them one by one.

1. Note that $I_{ws_{N+1}} = I_w \cap \{x : g_\mu^{\circ N}(x) \in I_{s_{N+1}}\} \subseteq I_w$.
2. Note that $I_{s_0 w} = \{x : x \in I_{s_0} \text{ and } g_\mu(x) \in I_w\}$. In other words, $I_{s_0 w}$ is the full pre-image of I_w in either I_0 or I_1, depending on the value of s_0. Since g_μ is monotonic on both I_0 and I_1, and $g_\mu(I_0) = g_\mu(I_1) = [0,1]$, we get that $g_\mu(I_{s_0 w}) = [0,1]$, and that the mapping is a homeomorphism.
3. By definition, I_w is an intersection of closed sets, and by the previous point, by induction on $\ell(w)$, it is easy to see that it is a closed interval with non-zero length.
4. Without loss of generality suppose the jth entry of w_1 and w_2 are 0 and 1 respectively. Then $g_\mu^{\circ(i-1)} I_{w_1} \subseteq I_0$ and $g_\mu^{\circ(i-1)} I_{w_2} \subseteq I_1$. Since for a point x we cannot have $g_\mu^{\circ(i-1)}$ in both I_0 and I_1, this shows that $I_{w_1} \cap I_{w_2} = \emptyset$.

5. We do this by inducting on N. When $N = 1$, $g_\mu^{-1}[0,1] = I_0 \cup I_1$.
Induction hypothesis: the statement is true for N.
Induction step: for $N+1$,

$$\begin{aligned}
(g_\mu^{\circ(N+1)})^{-1}[0,1] = g_\mu^{-1}\Big((g_\mu^{\circ N})^{-1}[0,1]\Big) &= g_\mu^{-1}\Big(\bigcup_{w\in\{0,1\}^N} I_w\Big) \\
&= \bigcup_{w\in\{0,1\}^N} g_\mu^{-1} I_w \\
&= \bigcup_{w\in\{0,1\}^N} (I_0 \cap g_\mu^{-1} I_w) \cup (I_1 \cap g_\mu^{-1} I_w) && \text{(since } g_\mu^{-1} I_w \subseteq I_0 \cup I_1\text{)} \\
&= \bigcup_{w\in\{0,1\}^N} I_{0w} \cup I_{1w} && \text{(by the proof of point (2))} \\
&= \bigcup_{w\in\{0,1\}^{N+1}} I_w
\end{aligned}$$

□

From here, let $\Sigma = \Sigma_{\{0,1\}}$.

Proposition 2.1.18. *Fix $\mu > 2+\sqrt{5}$. Given any $s_1s_2\cdots =: s \in \Sigma$, there exists a unique point $x_s \in B_\mu$ such that*

$$\bigcap_{N=1}^{\infty} I_{s_1s_2\cdots s_N} = \{x_s\}$$

Proof. We first show that it suffices to prove the following claim:
Claim 1. There exists a constant $\lambda > 1$ such that for all $N \in \mathbb{N}$ and all $w \in \{0,1\}^N$,

$$\operatorname{diam}(I_w) \leq \frac{1}{\lambda^{N-1}} \cdot \operatorname{diam}(I_0)$$

Proposition 2.1.17 implies $I_{s_1} \supseteq I_{s_1s_2} \supseteq I_{s_1s_2s_3}\cdots$. Therefore, the infinite intersection $\operatorname{diam}\left(\bigcap_{N=1}^{\infty} I_{s_1s_2\cdots s_N}\right)$ is a non-empty closed set. By Claim 1, $\operatorname{diam}\left(\bigcap_{N=1}^{\infty} I_{s_1s_2\cdots s_N}\right) = 0$, so this infinite intersection is a singleton $\{x_s\}$. This point x_s is in B_μ since $g_\mu^{\circ N}(x_s) \in I_{s_N} \subset [0,1]$ for all $N \in \mathbb{N}$.

Now it is left for us to prove claim 1. We first show the following:
Claim 2. $|g_\mu'(x)| > \sqrt{\mu^2 - 4\mu} > 1$ for all x in the interiors of I_0 and I_1.

Proof of Claim 2. If x is in the interior of I_0 or I_1, then

$$\begin{aligned}
|g'(x)| = |\mu(1-2x)| &= 2\mu\left|\frac{1}{2} - x\right| \\
&> 2\mu c_\mu = 2\mu \cdot \sqrt{\frac{1}{4} - \frac{1}{\mu}} \\
&= \sqrt{\mu^2 - 4\mu} > \sqrt{(2+\sqrt{5})^2 - 4(2+\sqrt{5})} \quad \text{(since } \mu \mapsto \mu^2 - 4\mu \text{ is increasing for } \mu > 2\text{)} \\
&= \sqrt{4+5-8} = 1
\end{aligned}$$

$\blacksquare$

Thus for all $x \in I_0 \cup I_1$, we have $|g_\mu'(x)| \geq \sqrt{\mu^2 - 4\mu} > 1$.

Proof of Claim 1. Let $\lambda = \sqrt{\mu^2 - 4\mu}$. Given $w \in \{0,1\}^N$ with $w = s_1 s_2 s_3 \cdots s_N$, note that since $g_\mu^{\circ(N-1)} : I_w \longrightarrow I_{s_1}$ is a diffeomorphism, looking at the inverse map $f = (g_\mu^{\circ(N-1)})^{-1}$ and using the fact that $|f'(x)| = \frac{1}{\left|\left(g_\mu^{\circ(N-1)}\right)'(f^{-1}(x))\right|}$, for all $x, y \in I_{s_1}$,

$$|f(x) - f(y)| \leq |f'(x)||x - y| \leq \frac{1}{\lambda^{N-1}}|x - y|$$

Thus,

$$\operatorname{diam}(I_w) \leq \frac{1}{\lambda^{N-1}} \operatorname{diam}(I_{s_1}) = \frac{1}{\lambda^{N-1}} \operatorname{diam}(I_0)$$

$\blacksquare$

This finishes the proof of the proposition. $\square$

Definition 2.1.19. Fix $\mu > 2 + \sqrt{5}$. Define a map $\varphi : \Sigma \longrightarrow B_\mu$ by setting $\varphi(s) = x_s$ for all $s \in \Sigma$.

Proposition 2.1.20. *φ is a homeomorphism.*

Proof. *φ is injective:* If $s \neq t$, choose $N \in \mathbb{N}$ such that $s_N \neq t_N$. Since $\varphi(s) = x_s \in I_{s_1 s_2 \cdots s_N}$ and $\varphi(t) = x_t \in I_{t_1 \cdots t_N}$, and by the condition $s_N \neq t_N$ we have $I_{s_1 \cdots s_N} \cap I_{t_1 \cdots t_N} = \emptyset$, we must have $\varphi(s) \neq \varphi(t)$.

φ is surjective: If $x \in B_\mu$, for all $n \in \mathbb{N}$, let $s_n = 0$ if $g_\mu^{\circ(n-1)}(x) \in I_0$ and $s_n = 1$ if $g_\mu^{\circ(n-1)}(x) \in I_1$. Then it is easy to check that $\varphi(s_1 s_2 s_3 \cdots) = x$.

φ is continuous: Given $s \in \Sigma$ and $\epsilon > 0$, since the diameter of I_w tends to 0 as $\ell(w) \to \infty$, choose $N \in \mathbb{N}$ such that $I_{s_1 s_2 \cdots s_N} \subset B_\epsilon(x)$. Then set $\delta = \frac{1}{2^N}$. By Proposition 2.1.6, if $d(s,t) < \delta$, then $t_j = s_j$ for $j = 1, 2, \cdots N$. Thus $\varphi(t) \in I_{t_1 \cdots t_N} = I_{s_1 \cdots s_N}$, and by our assumption on N, we have $|\varphi(t) - \varphi(s)| < \epsilon$.

φ^{-1} is continuous: Given $x \in B_\mu$ and $\epsilon > 0$, let $s = \varphi^{-1}(x)$. Choose $N \in \mathbb{N}$ such that $\frac{1}{2^{N-1}} < \epsilon$, and choose $\delta > 0$ such that $B_\delta(x) \subset I_{s_1 s_2 \cdots s_N}$. Then, for any $y \in B_\delta(x) \cap B_\mu$, the sequence $t = \varphi^{-1}(y)$ satisfies $t_j = s_j$ for $j = 1, 2, \cdots, N$. By Proposition 2.1.5, we know that $d(s,t) < \frac{1}{2^{N-1}} < \epsilon$. $\square$

Proposition 2.1.21. *φ conjugates σ to g_μ.*

Proof. For all $s \in \Sigma$,

$$\begin{aligned} \varphi(s) &\in \bigcap_{N=1}^{\infty} I_{s_1 \cdots s_N} \\ \Longrightarrow \; g_\mu(\varphi(s)) &\in \bigcap_{N=1}^{\infty} g_\mu(I_{s_1 \cdots s_N}) = \bigcap_{N=1}^{\infty} I_{s_2 s_3 \cdots s_N} \\ &= \{\varphi(\sigma(s))\} \end{aligned}$$

In other words, $g_\mu \circ \varphi = \varphi \circ \sigma$. $\square$

Bibliography

[1] Michael Francis. Two topological uniqueness theorems for spaces of real numbers, 2012.