

Homework 2

MA 771

Due February 20 2025

Please scan or type your homework and hand/email it to me by 5pm on the due date.

1. Let Σ_A be the space of sequences over a finite alphabet $A \subset X$, and let

$$d(s, t) = \sum_{j=1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}}$$

be the usual metric on A . The topology on Σ_A induced by d is called the *standard cylinder topology*.

- For $s \in \Sigma_A$ and $N \in \mathbb{N}$, define the *cylinder set* $U(s, N) = \{t \in \Sigma_A : s_j = t_j \text{ for } j = 1, 2, \dots, N\}$. Prove that the collection of cylinder sets $\{U(s, N) : s \in \Sigma_A, N \in \mathbb{N}\}$ forms a basis for the above topology on Σ_A .
 - Prove the left shift operator $\sigma : \Sigma_A \longrightarrow \Sigma_A$ is topologically mixing.
2. Show that $f(x) = 2x(1 - x)$ and $g(x) = 5x(1 - x)$ are not topologically conjugate.
 3. Prove that if $f, g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ are expanding maps of degree 2, then the dynamical systems (\mathbb{S}^1, f) and (\mathbb{S}^1, g) are topologically conjugate.
 4. Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with real entries. Suppose the linear map $x \mapsto Ax$ is a contraction on \mathbb{R}^n (i.e., $\|A\| < 1$), prove that

$$\text{tr } A = \sum_{j=1}^n a_{jj} < n.$$

5. Let (X, d) be a metric space and suppose $f : X \longrightarrow X$ is an isometry (i.e., $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$). Show that f is not topologically mixing.
6. Let G be a metrizable compact topological group. Let $g_0 \in G$ be an element for which the translation $L_{g_0} : h \mapsto g_0 \cdot h$ is topologically transitive.
 - Prove that the grand orbit $(L_{g_0}^n(h))_{n \in \mathbb{Z}}$ is dense in G for every $h \in G$.
 - Use the above to show that the cyclic subgroup of G generated by g_0 is dense in G .

- Prove that G is Abelian.

Hint: The multiplication map on G is continuous. Using everything you did above, prove that for any $g, h \in G$, the elements $g \cdot h$ and $h \cdot g$ are arbitrarily close to each other in the metric of G .