

Metric Spaces

Def: A metric space is a pair (X, d) where X is a set, and $d: X \times X \rightarrow \mathbb{R}$ is a function satisfying:

- $d(x, y) \geq 0 \quad \forall x, y \in X$
- $d(x, y) = 0 \iff x = y \quad \forall x, y \in X$
- $d(x, y) = d(y, x) \quad \forall x, y \in X$
- $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$
 (Triangle inequality)

Examples:

- (\mathbb{R}, d) where $d(x, y) = \begin{cases} x-y & x \geq y \\ y-x & x < y \end{cases}$
- (\mathbb{R}^2, d) where $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
 In both 1 and 2, $d(x, y)$ is the length of the line segment bound between x and y .
- For any set X , define $d: X \times X \rightarrow \mathbb{R}$ as $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$
 d satisfies (i), (ii), (iii).
 (iv): $\forall x, y, z, d(x, y) + d(y, z) = A \in \{0, 1, 2\}$.
 If $A = 0$, then $x = y = z$. So $d(x, z) = 0$.
 If $A = 1$, then either $x = y \neq z$, or $x \neq y = z$.
 So $d(x, z) = 1 = A$.
 If $A = 2$, then $d(x, z) \leq 1 < A$.
- Let $G = (V, E)$ be a graph, and consider $d: V \times V \rightarrow \mathbb{R}$ given by $d(u, v)$ = no. of edges in the shortest path joining u and v .
 Is d a metric?
Answer: No, in fact d is not well-defined: it is possible that there is no path between two edges.
 Fix: Assume G is a "connected" graph (i.e. any two vertices $u, v \in V$ can be joined by a path).
 d satisfies (i), (ii), (iii) in the definition of a metric.
 Triangle inequality: if the shortest path $u \rightarrow v$ has m edges, and $v \rightarrow w$ has n edges, then combining the paths, we get a path $u \rightarrow w$ with $m+n$ edges.
 So the shortest path $u \rightarrow w$ has $\leq m+n$ edges.

Pseudo-metric spaces

Def: Any function $d: X \times X \rightarrow \mathbb{R}$ that satisfies (i), (ii), (iv) but (iii) fails: we still assume $d(x, x) = 0 \quad \forall x \in X$, but $\exists x \neq y$ s.t. $d(x, y) = 0$.
 is called a pseudo-metric on X .

Constructing metric spaces from pseudo-metric spaces

let (X, \tilde{d}) be a pseudo-metric space.
 let $\sim = \{(x, y) : d(x, y) = 0\}$.
 \sim is an equivalence relation on X :
 (i) reflexive, "0" $d(x, x) = 0 \quad \forall x \in X$
 (ii) symmetric, "0" $d(x, y) = d(y, x) \quad \forall x, y \in X$
 (iii) transitive: if $(x, y), (y, z) \in \sim$, then $d(x, y) = d(y, z) = 0$.
 A inequality $\implies d(x, z) = 0 \implies (x, z) \in \sim$.
 [Note: For any relation R , we say $x R y$ if $(x, y) \in R$.]
 Let $X = \tilde{X} / \sim = \{[x] : x \in \tilde{X}\}$.
 Define $d: X \times X \rightarrow \mathbb{R}$ as $d([x], [y]) = \tilde{d}(x, y)$.
 d is well-defined: if $[x] = [x']$ and $[y] = [y']$, then $\tilde{d}(x', y) \leq \tilde{d}(x', x) + \tilde{d}(x, y) + \tilde{d}(y, y') = \tilde{d}(x, y)$.
 Similarly $\tilde{d}(x, y) \leq \tilde{d}(x, y') + \tilde{d}(y, y')$.
 Therefore $\tilde{d}(x, y) = \tilde{d}(x', y')$.
Exercise: Check that d is a metric on X .

Eg: Consider the pseudo-metric space (\mathbb{N}, d') where $d'(x, y) = \begin{cases} 0 & x-y \text{ is even} \\ 1 & x-y \text{ is odd} \end{cases}$.
 Note: $d'(x, y) \geq 0 \quad \forall x, y \in \mathbb{N}$, and $d'(x, y) = d'(y, x)$.
 $d'(x, x) = 0 \quad \forall x \in \mathbb{N}$.
 and $d'(x, z) \leq d'(x, y) + d'(y, z) \quad \forall x, y, z \in \mathbb{N}$
 [this can only fail if x, y, z are even, so that case $x-y, y-z$ are even, so $x-z$ is even, so LHS = 0].
 The construction described above gives a metric space $(\mathbb{N}/\sim, d)$.
 Note that $2 \sim 4 \sim 6 \sim \dots$ and $1 \sim 3 \sim 5 \sim \dots$
 but no odd number is related to an even number.
 so $\mathbb{N}/\sim = \{[2], [1]\}$.
 $d([2], [2]) = d([1], [1]) = 0$.

Metrics on \mathbb{R}^n $= \{(x_1, \dots, x_n) : x_i \in \mathbb{R} \text{ for } i=1, 2, \dots, n\}$

- Euclidean Metric: $d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$
- $d((x_1, \dots, x_n), (y_1, \dots, y_n)) = d(x_1, y_1) + d(x_2, y_2) + \dots + d(x_n, y_n)$
 [d is the Euclidean metric on \mathbb{R}]
- $d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max_{1 \leq i \leq n} d(x_i, y_i)$

Next class we will show all three are metrics.

(i) Euclidean metric d:

it is clear that $d(x, y) \geq 0$, $d(x, x) = 0$ and $d(x, y) = d(y, x)$.
 If $d(x, y) = 0$, then every term in the sum $\sum_{i=1}^n (x_i - y_i)^2$ is 0. So $x_i = y_i \quad \forall i$.
 $\implies x = y$.

Δ inequality:
 given $x, y, z \in \mathbb{R}^n$, we need $\sqrt{\sum_{i=1}^n (x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^n (x_i - y_i)^2} + \sqrt{\sum_{i=1}^n (y_i - z_i)^2}$

Lemma 1: $\forall u, v \in \mathbb{R}^n$, $\left(\sum_{i=1}^n u_i v_i\right)^2 \leq \left(\sum_{i=1}^n u_i^2\right) \left(\sum_{i=1}^n v_i^2\right)$ ("Cauchy-Schwarz inequality")

Proof of Lemma 1: (There are several proofs, here is a fun one)

let $t \in \mathbb{R}$ and $w_t = u - tv \in \mathbb{R}^n$. Then $0 \leq (d(w_t))^2 = \sum_{i=1}^n (u_i - tv_i)^2 = \sum_{i=1}^n (u_i^2 + t^2 v_i^2 - 2tv_i u_i) = t^2 \sum_{i=1}^n v_i^2 - 2t \left(\sum_{i=1}^n u_i v_i\right) + \sum_{i=1}^n u_i^2$.
 Letting $A = \sum_{i=1}^n v_i^2$, $B = -2 \sum_{i=1}^n u_i v_i$, $C = \sum_{i=1}^n u_i^2$, $At^2 + Bt + C$.
 We have $At^2 + Bt + C \geq 0 \quad \forall t \in \mathbb{R}$.
 This implies the discriminant $B^2 - 4AC \leq 0$.
 $\implies (-2 \sum_{i=1}^n u_i v_i)^2 - 4 \left(\sum_{i=1}^n v_i^2\right) \left(\sum_{i=1}^n u_i^2\right) \leq 0$.
 $\implies 4 \left(\sum_{i=1}^n u_i v_i\right)^2 \leq 4 \left(\sum_{i=1}^n v_i^2\right) \left(\sum_{i=1}^n u_i^2\right)$. QED

Proof of Δ inequality for d using Lemma 1:

Want $(d(x, z))^2 \leq (d(x, y) + d(y, z))^2$.
 Note: $\sum_{i=1}^n (x_i - z_i)^2 = \sum_{i=1}^n (x_i - y_i + y_i - z_i)^2 = \sum_{i=1}^n (x_i - y_i)^2 + \sum_{i=1}^n (y_i - z_i)^2 + 2 \sum_{i=1}^n (x_i - y_i)(y_i - z_i)$

let $u = x - y$, $v = y - z$. Then $(d(x, z))^2 = \sum_{i=1}^n (x_i - z_i)^2 \leq \sum_{i=1}^n u_i^2 + \sum_{i=1}^n v_i^2 + 2 \sum_{i=1}^n u_i v_i$.
 $\leq \sum_{i=1}^n u_i^2 + \sum_{i=1}^n v_i^2 + 2 \left(\sum_{i=1}^n u_i^2\right)^{1/2} \left(\sum_{i=1}^n v_i^2\right)^{1/2}$ [by lemma 1]
 $= A^2 + B^2 + 2AB = (A+B)^2$ [where $A = \left(\sum_{i=1}^n u_i^2\right)^{1/2}$, $B = \left(\sum_{i=1}^n v_i^2\right)^{1/2}$]
 $= \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{1/2} + \left(\sum_{i=1}^n (y_i - z_i)^2\right)^{1/2} = d(x, y) + d(y, z)$

(ii) d': "Taxicab metric"

It is clear that d' satisfies all properties of a metric, including the triangle inequality, $\therefore \forall x, y, z, d'(x, z) \leq d'(x, y) + d'(y, z)$.

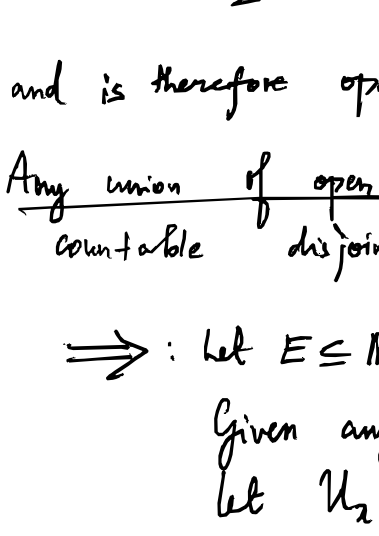
(iii) d''

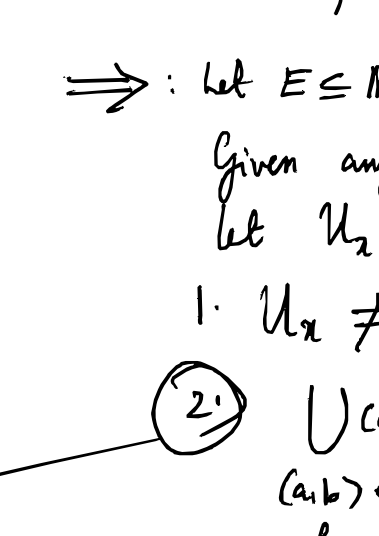
Same holds for d'' .
 Note about Δ inequality:
 $x_i, d(x_i, z_i) \leq d(x_i, y_i) + d(y_i, z_i) \leq \max\{d(x_i, y_i), d(y_i, z_i)\} = d(x, y) + d(y, z)$.
 So $d(x, z) = \max_i d(x_i, z_i) \leq d(x, y) + d(y, z)$.

Open and closed sets

let (X, d) be a metric space.
Def: given $x \in X$ and $\epsilon > 0$, a real number, the " ϵ -ball centered at x " is the set $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$

Eg: In (\mathbb{R}, d) , Note that $d(x, y) = |x - y|$.
 $B_d(x, \epsilon) = \{y \in \mathbb{R} : |x - y| < \epsilon\} = \{y \in \mathbb{R} : -\epsilon < x - y < \epsilon\} = (x - \epsilon, x + \epsilon)$.

In (\mathbb{R}^2, d) , $B_d(x, \epsilon) = \{y \in \mathbb{R}^2 : \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < \epsilon$.


* In (\mathbb{R}^2, d) , $B_d(x, \epsilon)$ is the ϵ -dimensional sphere (without boundary) of radius ϵ centered at x .
 * In (\mathbb{R}^2, d') , $B_d(x, \epsilon) = \{y \in \mathbb{R}^2 : |x_1 - y_1| + |x_2 - y_2| < \epsilon\}$.


Def: let (X, d) be a metric space, and $E \subseteq X$.
 E is said to be "open", if for every $x \in E$, $\exists \epsilon > 0$ (possibly dependent on x) such that $B_d(x, \epsilon) \subseteq E$.
 A set $F \subseteq X$ is "closed" if $X \setminus F$ is open.

Examples

1. Let (X, d) be a metric space where $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$.
 Then every $E \subseteq X$ is both open and closed.
Proof: Note: $\forall x \in X, B_d(x, \epsilon) = \{x\}$, $0 < \epsilon \leq 1$.
 Fix $0 < \epsilon \leq 1$.
 Given $E \subseteq X$ and any $x \in E$, $B_d(x, \epsilon) = \{x\} \subseteq E$.
 $\therefore E$ is open.
 For any $F \subseteq X$, $X \setminus F$ is open, so F is closed.
 The metric d above is called the "discrete metric" on X .

2. Every ϵ -ball in a metric space (X, d) is open.

let $E = B_d(x, \epsilon)$. Given $y \in E$, we want to show $\exists r > 0$ s.t. $B_d(y, r) \subseteq E$.
 let $d = d(x, y)$, and pick $r < \epsilon - d$, $\epsilon - d$ is actually always > 0 .
 Then $\forall z \in B_d(y, r)$, we have $d(x, z) \leq d(x, y) + d(y, z) < d + r < d + \epsilon - d = \epsilon \implies z \in B_d(x, \epsilon)$.

3. Every singleton set $\{x\}$ in a metric space (X, d) is closed.

Proof: Given $y \in X \setminus \{x\}$, let $\epsilon = d(x, y)$.
 Then $z \in B_d(y, \epsilon)$ implies $B_d(x, \epsilon) \subseteq X \setminus \{x\}$.
 So $X \setminus \{x\}$ is open.
 So $\{x\}$ is closed.

2.1 4. In (\mathbb{R}, d) , $E \subseteq \mathbb{R}$ is open $\iff E$ is a countable union of pairwise disjoint open intervals.

Proof: \Leftarrow : 1. Every open interval $(a, b) = (x - \epsilon, x + \epsilon)$ for $x = \frac{a+b}{2}$, $\epsilon = \frac{b-a}{2}$.
 and is therefore open.
 Reason: given $\{I_\alpha\}_{\alpha \in I}$ 2. Any union of open intervals (in particular a countable disjoint union) is an open set.
 of open intervals and $x \in \bigcup I_\alpha$, \implies let $E \subseteq \mathbb{R}$ be open.
 Take $\epsilon > 0$ s.t. $(x - \epsilon, x + \epsilon) \subseteq I_\alpha \subseteq \bigcup I_\alpha$ let $U_\alpha = \{ (a, b) \subseteq E \text{ s.t. } x \in (a, b) \}$
 so $\bigcup I_\alpha$ is open. 1. $U_\alpha \neq \emptyset$, is uncountable.
 (2) $\bigcup (a, b) = (a_0, b_0)$
 where $a_0 = \inf \{a : (a, b) \in U_\alpha\}$
 $b_0 = \sup \{b : (a, b) \in U_\alpha\}$.

Proof: \Leftarrow is clear since $(a, b) \subseteq (a_0, b_0) \quad \forall (a, b) \in U_\alpha$.
 \Rightarrow Exercise: Show $\forall y \in (a_0, b_0)$, $\exists (a, b) \in U_\alpha$ s.t. $y \in (a, b)$.
 1. $\exists (a_1, b_1), (a_2, b_2) \in U_\alpha$ s.t. $a_0 < a_1 < y < b_2 < b_0$.
 2. $(a_1, b_1) \cup (a_2, b_2)$ is in fact an open interval in E .
 let $I_\alpha = (a_0, b_0)$. Note that I_α is the largest interval in E containing x , and that $E = \bigcup_{x \in E} I_x$.

Remark: for $x \neq y \in E$, either $I_x \cap I_y = \emptyset$ or $I_x = I_y$.
 This follows from below:
 if $I_x \cap I_y \neq \emptyset$, then $I_x \cup I_y$ is an interval in E containing x , [is $I_x \cup I_y \in U_\alpha$]
 $\implies I_x \cup I_y \subseteq I_x \implies I_x \supseteq I_y$
 $\implies I_x = I_y$.

Define an equivalence relation \sim on E as follows:
 $x \sim y$ if $I_x = I_y$

Exercise: Check this is an equivalence relation.

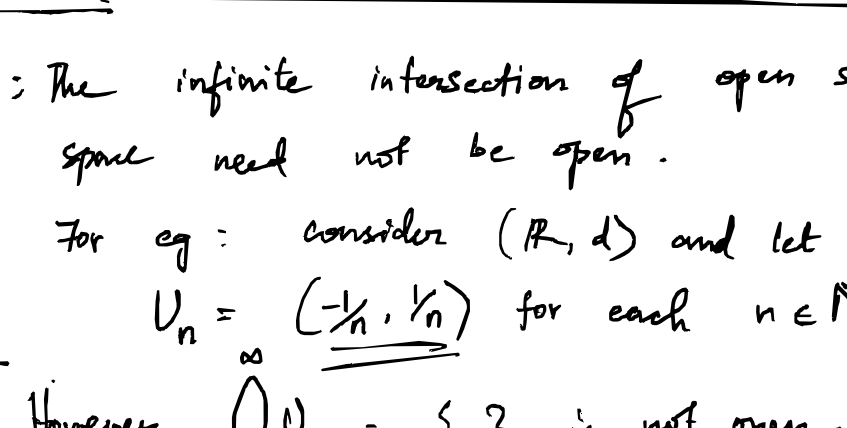
Note: $E = \bigcup_{[x] \in E/\sim} I_x$

$\forall z \in E, \exists q \in E \cap \mathbb{Q}$ s.t. $I_x = I_q$, since every interval has a rational number.
 $\implies E/\sim$ is countable.
 $\implies E$ is a countable disjoint union of intervals.

Q: In (\mathbb{R}^2, d) , is every open set a disjoint union of balls?

A: No.

Counter-examples (without rigorous proof)
 1. $E = \{x \in \mathbb{R}^2 : 1 < d(0, x) < 2\}$



2. $E =$ (a shaded region in R^2) is open, but not a disjoint union of balls.

Theorem: Let (X, d) be a metric space, and let $\mathcal{J} = \{E \subseteq X : E \text{ is open}\}$. Then

- given any $\mathcal{U} \subseteq \mathcal{J}$, the set $\bigcup_{E \in \mathcal{U}} E \in \mathcal{J}$.
 In other words, any union of open sets is open.
- given any finite $\mathcal{U} \subseteq \mathcal{J}$, the set $\bigcap_{E \in \mathcal{U}} E \in \mathcal{J}$.
 i.e. any finite intersection of open sets is open.
- \emptyset, X are open and closed.

Proof: 1. Given \mathcal{U} , $x \in \bigcup_{E \in \mathcal{U}} E$, we know $x \in E$ for some $E \in \mathcal{U}$.
 Take $\epsilon > 0$ s.t. $B_d(x, \epsilon) \subseteq E \subseteq \bigcup_{E \in \mathcal{U}} E$.

2. Given \mathcal{U} , $x \in \bigcap_{E \in \mathcal{U}} E$, we want $\epsilon > 0$ s.t. $B_d(x, \epsilon) \subseteq \bigcap_{E \in \mathcal{U}} E$.
 Write $\mathcal{U} = \{E_1, \dots, E_n\}$. $\therefore E_i$ is open $\forall i$.
 $\exists \epsilon_i > 0$ s.t. $B_d(x, \epsilon_i) \subseteq E_i$. Let $\epsilon = \min_{1 \leq i \leq n} \epsilon_i$.
 Note: $B_d(x, \epsilon) \subseteq E_i \quad \forall i$. So $B_d(x, \epsilon) \subseteq \bigcap_{i=1}^n E_i$.

Remark: The infinite intersection of open sets in a metric space need not be open.

For eg: consider (\mathbb{R}, d) and let $U_n = (-\frac{1}{n}, \frac{1}{n})$ for each $n \in \mathbb{N}$. U_n is open, However $\bigcap_{n=1}^{\infty} U_n = \{0\}$ is not open.

Exercise: Let (X, d) be a metric space and $x \in X$.
 Show that $F = \{y \in X : d(x, y) \leq \epsilon\}$ is closed.

Further examples

Recall: $\text{diam}(X) = \sup_{(x, y) \in X \times X} d(x, y)$

1. **Def:** Let (X, d) be a metric space. The metric d is said to be "bounded" if $\exists M > 0$ s.t. $\text{diam}(X) \leq M$ [equivalently, $\forall x, y \in X, d(x, y) \leq M$].
 Eg: The discrete metric is a bounded metric [diam(X) = 1].
 2. **Def:** Let (X, d) be a metric space. A "bounded metric" is a metric \tilde{d} on X such that $\tilde{d}(x, y) = \min\{1, d(x, y)\}$.
 Then $\tilde{d}(x, y) \leq 1 \quad \forall x, y \in X$, and is thus bounded. We claim \tilde{d} is a metric on X .

Proof: Δ inequality: Want $\tilde{d}(x, z) \leq \tilde{d}(x, y) + \tilde{d}(y, z)$.
 case 1: If $\tilde{d}(x, y) + \tilde{d}(y, z) < 1$, then $\tilde{d}(x, y), \tilde{d}(y, z) < 1$.
 $\implies d(x, y) = \tilde{d}(x, y)$ and $d(y, z) = \tilde{d}(y, z)$.
 $\therefore \tilde{d}(x, z) \leq d(x, z) \leq d(x, y) + d(y, z) = \tilde{d}(x, y) + \tilde{d}(y, z)$.

case 2: If $\tilde{d}(x, y) + \tilde{d}(y, z) \geq 1$.
 $\therefore \tilde{d}(x, z) \leq 1 \leq \tilde{d}(x, y) + \tilde{d}(y, z)$.

\tilde{d} is called the "standard bounded metric" induced by d .

Exercise: Show that $E \subseteq X$ is open in (X, d) iff it is open in (X, \tilde{d}) .

3. **Def:** let V be a vector space over \mathbb{R} or \mathbb{Q} .
 A "norm" on V is a function $\| \cdot \|: V \rightarrow \mathbb{R}$ such that

- $\|x\| \geq 0 \quad \forall x \in V$
- $\|x\| = 0 \iff x = 0$
- $\| \alpha x \| = |\alpha| \|x\|$ for all scalars α
- $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$

Any norm on V induces a metric $d: V \times V \rightarrow \mathbb{R}$ given by $d(x, y) = \|x - y\|$.

Exercise: Check that the above formula does indeed give a metric on V .

Q: Given any vector space V over \mathbb{R} or \mathbb{Q} with a metric d , is the function $x \mapsto d(x, 0)$ a norm on V ?
 A: (i), (ii) are clear. (iii), (iv) may not work.

Continuous functions

Def: let (X, d_X) and (Y, d_Y) be metric spaces and $f: X \rightarrow Y$ be a function. Then f is said to be "continuous" at $x \in X$ if for every $\epsilon > 0$, $\exists \delta > 0$ s.t. $\forall y \in X$, $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$.
 [Equivalently, $f(B_{d_X}(x, \delta)) \subseteq B_{d_Y}(f(x), \epsilon)$]
 f is said to be "continuous" if it is continuous at every $x \in X$.

Eg: 1. For any metric space (X, d) , the map $\text{id}: X \rightarrow X$ is continuous.
 $z \mapsto x$

2. The map $\pi_1: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $\pi_1(x, y) = x$ is continuous (same both \mathbb{R}^2, \mathbb{R} have Euclidean metric).

Proof: 3. The map $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = x + y$ is continuous.

Proof: Given $x = (x_1, x_2) \in \mathbb{R}^2$ consider the unique line through x of slope 1. This line has x -intercept $(0, x_2)$ and y -intercept $(x_1, 0)$ where $f(x) = x_1 + x_2$.
 Consider the lines L_ϵ^+ of slope 1 as in the figure.
 let $\delta = \frac{1}{2} \times$ distance between L_ϵ^+ and L_ϵ^- .
 Then $\forall y$ s.t. $d(x, y) < \delta$, we have $|f(x) - f(y)| < \epsilon$.