

MA 771: Introduction to Dynamical Systems

Lecture Notes

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1 Basic concepts

Dynamical systems refer to systems that evolve over time. A simple example is an ordinary differential equation (ODE), such as

$$\frac{dx}{dt} = g(x)$$

where $g : \mathbb{R}^n \rightarrow \mathbb{R}$. Some of the questions we can ask about ODE's include:

- Which points are stable/fixed?
- When do orbits tend to an attracting periodic cycle?
- When are orbits chaotic?

However, the class of dynamical systems is much bigger than ODE's. In particular, in the above example, the variable t , representing time, changes *continuously*. In this course we will focus mainly on *discrete* dynamical systems.

Definition 1.1. Let X be a topological space. A *discrete* dynamical system is a pair (X, f) where $f : X \rightarrow X$ is a self-map.

We are interested in the function f and its iterates $f^{\circ n} = f \circ f^{\circ(n-1)}$ for $n \in \mathbb{N}$. In other words, the quantity that is varying in a *discrete* fashion is the number of iterations of f .

Example 1.2. Linear maps Let $X = \mathbb{R}^n$ and $f : x \mapsto Ax$ be a linear map.

Example 1.3. Rotations of the circle $X = \mathbb{S}^1$ and $f(x) = e^{2\pi i \theta} x$ for some $\theta \in \mathbb{R}$.

Example 1.4. Logistic family Let $X = \mathbb{R}$ and fix $k \in \mathbb{R}_{>0}$. Then the family of maps $f_k : X \rightarrow X$ given by $x \mapsto kx(1 - x)$ is called the *logistic family*.

Notice that we are not assuming any conditions on f such as continuity.

1.1 Orbits and periodic points

Definition 1.5. Given a dynamical system (X, f) , $x_0 \in X$, the sequence

$$x_0, f(x_0), f^{\circ 2}(x_0), \dots, f^{\circ n}(x_0), \dots$$

is called the *forward orbit* of x_0 .

The *reverse orbit* of x_0 is the set $\{x \in X : f^{\circ n}(x) = x_0 \text{ for some } n \in \mathbb{N}\}$.

A *fixed point* $x \in X$ is a point such that $f(x) = x$. The set of fixed points of f is denoted $\text{Fix}(f)$. A *periodic point* is a point x such that $f^{\circ n}(x) = x$ for some $n \in \mathbb{N}$, in other words, a point in $\text{Fix}(f^{\circ n})$ for some n .

Any $n \in \mathbb{N}$ such that $f^{\circ n}(x) = x$ is said to be a *period* of x . The smallest period n is called the *exact* period of x .

1.2 Examples

1.2.1 Linear maps of \mathbb{R}

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be linear. We know that f is of the form $f(x) = mx + b$ where $m \in \mathbb{R}_{\neq 0}$ and $b \in \mathbb{R}$. Note that

$$f^{\circ n}(x) = m^n x + b(m^{n-1} + m^{n-2} + \dots + m + 1) = m^n x + b \frac{m^n - 1}{m - 1}$$

- If $m \neq \pm 1$, then

$$|m| < 1 \implies \forall x \in \mathbb{R}, f^{\circ n}(x) \rightarrow \frac{b}{1 - m} \text{ as } n \rightarrow \infty$$

$$|m| > 1 \implies \forall x \in \mathbb{R}, f^{\circ n}(x) \rightarrow \infty \text{ as } n \rightarrow \infty$$

- If $m = 1$, then $f(x) = x + b$ is a translation and all orbits tend to ∞
- If $m = -1$, then note that $f^{\circ 2}(x) = -(-x + b) + b = x$, and thus all the odd iterates are equal to f , and all the even iterates are equal to the identity.

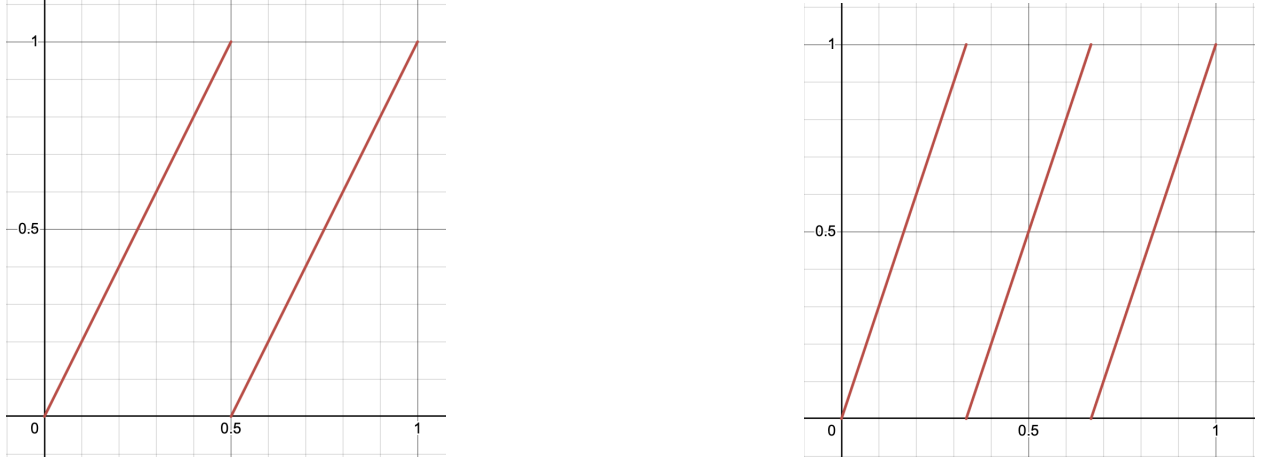


Figure 1: The graphs of the expanding maps E_2 (left) and E_3 (right) on the interior of \mathbb{S}^1 , represented by the interval $(0,1)$.

1.2.2 Circle maps

Example 1.6. For any rotation $f(x) = e^{2\pi i\theta}x$ of the circle \mathbb{S}^1 , we have $\text{Fix}(f) = \emptyset$ if $\theta \notin \mathbb{N}$, and $\text{Fix}(f) = \mathbb{S}^1$ otherwise.

Definition 1.7. Fix an integer $m > 1$, and identify \mathbb{S}^1 with \mathbb{R}/\mathbb{Z} . The *expanding map* $E_m : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is defined as

$$E_m(x) = mx \pmod{1}$$

Remark 1.8. E_m is expanding in the following sense: if $\alpha, \beta \in \mathbb{S}^1$ and $d_{\mathbb{S}^1}(\alpha, \beta) < \frac{1}{m}$, then

$$d_{\mathbb{S}^1}(\alpha, \beta) = m \cdot d_{\mathbb{S}^1}(E_m(\alpha), E_m(\beta)).$$

See Figure 1 for the graphs of E_m for $m = 2, 3$.

Note that ϕ is a fixed point of E_m if and only if $m\phi - \phi \in \mathbb{Z}$. In other words, there exists $n \in \mathbb{Z}$ such that

$$\begin{aligned} m\phi - \phi &= n \\ \iff \phi &= \frac{n}{m-1} \end{aligned}$$

Similarly, ϕ is a periodic point of E_m of period dividing k if and only if there exists $n \in \mathbb{Z}$ such that

$$\begin{aligned} m^k\phi - \phi &= n \\ \iff \phi &= \frac{n}{m^k - 1} \end{aligned}$$

In other words,

$$\begin{aligned}\text{Fix}(E_m) &= \left\{ \frac{1}{m-1}, \frac{2}{m-1} \cdots, \frac{m-2}{m-1} \right\} \\ \text{Fix}(E_m^{\circ k}) &= \left\{ \frac{1}{m^k-1}, \frac{2}{m^k-1} \cdots, \frac{m^k-2}{m^k-1} \right\}\end{aligned}$$

1.2.3 Torus endomorphisms

Given $n \in \mathbb{N}$, the n -torus is the space $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n = \mathbb{R}^n / \sim$ where $x \sim y$ if $x - y \in \mathbb{Z}^n$. For $x \in \mathbb{R}^n$, we let $[x]$ denote the equivalence class of x in \mathbb{T}^n .

Definition 1.9. Let A be an $n \times n$ matrix whose entries are in \mathbb{Z} . Then A induces the *torus endomorphism* $T_A : \mathbb{T}^n \rightarrow \mathbb{T}^n$ given by

$$T_A([x]) = [Ax] \text{ for } [x] \in \mathbb{T}^n$$

Exercise 2. Show that T_A as given above is well-defined: that is, for any two vectors $v, w \in \mathbb{R}^n$, if $v - w \in \mathbb{Z}^n$, then $Av - Aw \in \mathbb{Z}^n$

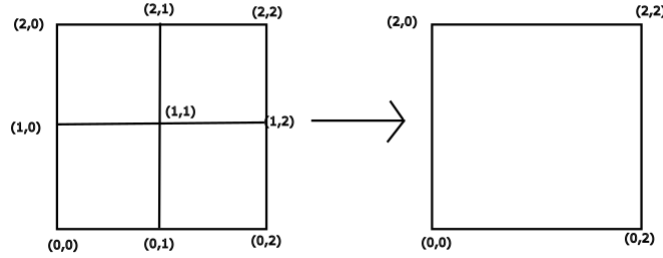


Figure 2: An illustration of the torus endomorphism $T_A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ for $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Example 2.1. Let $m, k \in \mathbb{Z}$ and consider the matrix $A = \begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix}$. Consider the map T_A on \mathbb{T}^2 : this acts as two independent expanding maps: expansion by a factor of m in the x -direction, and expansion by a factor of k in the y -direction (see Figure 2 which illustrates the case $m = k = 2$). Can you show in general that the degree of such a map is $d = mk$? In other words, T_A is a $d : 1$ map of \mathbb{T}^2 .

Definition 2.2. A torus endomorphism T_A is said to be an *automorphism* if it is invertible.

Exercise 3. (This is also on HW 1) Show that T_A is invertible if and only if A^{-1} has integer entries, which in turn is equivalent to $\det A = \pm 1$.

Proposition 3.1. Let $T_A : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be a torus automorphism with no eigenvalues of modulus 1. Then the periodic points of T_A are all the points with rational coordinates.

Proof. (periodic \implies rational):

Let $[x] = [(x_1, x_2, \dots, x_n)] \in \mathbb{T}^n$ be a periodic point of period q for some $q \in \mathbb{N}$. Then $T_A^{\circ q}([x]) = [A^q x] = [x]$. That is, there exists a vector $y \in \mathbb{Z}^n$ such that

$$\begin{aligned} A^q x &= x + y \\ \implies A^q x - x &= y \\ \implies (A^q - \text{Id})x &= y \end{aligned}$$

Since A has no eigenvalues of modulus 1, the matrix A^q has no eigenvalues of modulus 1. This means that the matrix $A^q - \text{Id}$ is invertible. So

$$x = (A^q - \text{Id})^{-1}y$$

Since y has integer coordinates and the matrix $(A^q - \text{Id})^{-1}$ has rational entries, x has rational coordinates.

(rational \implies periodic):

Suppose x has rational coordinates, we can assume that all the coordinates have a common denominator. In other words, $x = (\frac{p_1}{r}, \frac{p_2}{r}, \dots, \frac{p_n}{r})$ for some integers p_i, r with $r \neq 0$. Given a $q \in \mathbb{N}$, since A has integer entries, $A^q x = (\frac{p'_1}{r}, \frac{p'_2}{r}, \dots, \frac{p'_n}{r})$ for some integers p'_1, \dots, p'_n .

Note that there are only finitely many points in \mathbb{T}^n with rational coordinates with a common denominator r . In other words, the set $\{T_A^{\circ q}([x]) : q \in \mathbb{N}\}$ is finite.

Thus, there exist $q_1 < q_2 \in \mathbb{N}$ such that $T_A^{\circ q_1}([x]) = T_A^{\circ q_2}([x])$. Since T_A is an automorphism, this means that $T_A^{\circ(q_2 - q_1)}([x]) = [x]$. \square

3.1 Stable Behavior: The Contraction Principle

In this section we will look at maps on subsets of \mathbb{R}^n which satisfy a criterion for all orbits converging to a fixed point.

3.1.1 Global contraction

Definition 3.2. A map f of a subset X of \mathbb{R}^n is said to be *Lipschitz-continuous* with Lipschitz constant λ , or λ -*Lipschitz* if

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for any $x, y \in X$.

The map f is said to be a *contraction* if

$$d(f(x), f(y)) < d(x, y) \quad \forall x, y \in X.$$

It is said to be a λ -*contraction* for $\lambda < 1$ if

$$d(f(x), f(y)) \leq \lambda d(x, y) \quad \forall x, y \in X.$$

Remark 3.3. If a map f is Lipschitz-continuous, then we define

$$Lip(f) := \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}$$

Example 3.4. $f(x) = \sqrt{x}$ defines a contraction on $I = [1, \infty)$. What is $Lip(f)$?

Theorem 3.5 (Contraction Principle in \mathbb{R}^n). *Let $X \subset \mathbb{R}^n$ be closed and $f : X \rightarrow X$ be a λ -contraction. Then f has a unique fixed point x_0 and $d(f^{\circ n}(x), x_0) = \lambda^n d(x, x_0)$ for every x .*

Proof. We have

$$d(f^{\circ n}(x), f^{\circ n}(y)) \leq \lambda^n d(x, y)$$

for all $x, y \in X$. But this also means that for any $x \in X$, we have

$$d(f^{\circ(n+1)}(x), f^{\circ n}(x)) \leq \lambda^n d(f(x), x)$$

$$\begin{aligned} d(f^{\circ m}(x), f^{\circ n}(x)) &\leq d(f^{\circ m}(x), f^{\circ(m-1)}(x)) + d(f^{\circ(m-1)}(x), f^{\circ(m-2)}(x)) + \dots + d(f^{\circ(n+1)}(x), f^{\circ n}(x)) \\ &\leq \left(\lambda^{m-1} + \lambda^{m-2} + \dots + \lambda^n \right) d(f(x), x) \\ &\leq \frac{\lambda^n (1 - \lambda^{m-n})}{1 - \lambda} d(f(x), x) \\ &\leq \frac{\lambda^n}{1 - \lambda} d(f(x), x) \end{aligned}$$

In other words, the orbit of x is a Cauchy sequence. Since X is closed, $\lim_{n \rightarrow \infty} f^{\circ n}(x) = x_0$ is a point of X , and

$$f(x_0) = f\left(\lim_{n \rightarrow \infty} f^{\circ n}(x)\right) = \lim_{n \rightarrow \infty} f^{\circ(n+1)}(x) = x_0$$

□

Remark 3.6. Given a sequence $(y_n)_{n \geq 0}$ in a metric space (Y, d) , we say that $y_n \rightarrow y \in Y$ *exponentially* if there exist constants $A > 0$ and $0 < \lambda < 1$ such that

$$d(y_n, y) \leq A \lambda^n d(y_0, y)$$

Note that in the above situation, the orbit under f of x converges exponentially to x_0 (here $A = 1$).

The contraction principle applies to λ -contractions defined on complete metric spaces.

Theorem 3.7 (Contraction Principle for complete metric spaces). *Let (X, d) be a complete metric space and $f : X \rightarrow X$ be a λ -contraction. Then there exists a unique fixed point $x_0 \in X$ such that the orbits under f of all points $x \in X$ converge exponentially to x_0 .*

Example 3.8 (Rabbits; due to Fibonacci). Say we record the number of rabbits in a forest starting January (month 0) of a given year. The Fibonacci model for rabbit population growth is as follows:

Letting b_n denote the number (in hundreds) of rabbits at the beginning of month n , we assume

$$\begin{aligned} b_0 &= 1 \\ b_1 &= 2 \\ b_n &= b_{n-1} + b_{n-2} \text{ for } n \geq 2. \end{aligned}$$

Then it is expected that the rabbit population growth rate stabilises as $n \rightarrow \infty$. That is, there exists $a \in (0, \infty)$ such that $\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} = a$. This means that in the long term, the rabbit population grows approximately *exponentially*, by a rough factor of a each month.

To prove the existence of a value a as required, we let $a_n = \frac{b_{n+1}}{b_n}$. Note that

$$a_{n+1} = \frac{b_{n+2}}{b_{n+1}} = \frac{b_{n+1} + b_n}{b_{n+1}} = 1 + \frac{b_n}{b_{n+1}} = 1 + \frac{1}{a_n}$$

Letting $g(x) = 1 + \frac{1}{x}$, we see that

$$a_{n+1} = g^{\circ n}(a_0) = g^{\circ n}(2) \text{ for all } n \geq 0$$

Claim: There exists a closed interval $I \subset \mathbb{R}$ such that $a_0 = 2$ such that

- $g(I) \subset I$
- g is a λ -contraction on I for some $\lambda \in (0, 1)$, and
- $a_0 = 2 \in I$

If we can prove this claim, then by the contraction principle, a can be recovered as the unique fixed point of g in I .

Proof of claim: The map The function g is decreasing on $(0, \infty)$, and has the horizontal asymptote $y = 1$. Note that $g'(x) = -\frac{1}{x^2}$

This means $\forall x \in [c, \infty)$ where $c > 1$,

$$\begin{aligned} |g'(x)| &= \frac{1}{x^2} \leq \frac{1}{c^2} < 1 \\ \implies |g(x) - g(y)| &\leq \frac{1}{c^2} |x - y| \text{ for all } x, y \in [c, \infty) \end{aligned}$$

In other words, for all $c > 1$, $g : [c, \infty) \rightarrow \mathbb{R}$ is a λ -contraction with $\lambda = \frac{1}{c^2}$. Also note g has a unique positive fixed point x_0 : we can find it by solving the equation $g(x) = x$.

$$\begin{aligned} g(x) &= x \\ \implies 1 + \frac{1}{x} &= x \\ \implies x^2 - x + 1 &= 0 \\ \implies x &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

So $x_0 = \frac{1+\sqrt{5}}{2}$. Note that $\frac{3}{2} < x_0 < 2$.
 Let $I = [\frac{3}{2}, 2]$, Then we have

$$g(2) = \frac{3}{2} \text{ and } g(3/2) = \frac{5}{3} < 2.$$

we see that $g(I) \subset I$. By the above discussion, g is a λ -contraction on I (with $\lambda = (2/3)^2$), and thus the orbit under g of $a_0 = 2$ converges to $a = x_0 = \frac{1+\sqrt{5}}{2}$.

Remark 3.9. The choice of I is not unique: for any $c \in (1, 3/2]$, we have $g[c, 2] = [3/2, g(c)] \subset [c, 2]$, and g is a λ contraction on $[c, 2]$ with $\lambda = \frac{1}{c^2}$. I made a small mistake in class by saying c can be in $[1, x_0]$: can you see why $c \in (3/2, x_0]$ won't work?

3.1.2 Local contraction

Proposition 3.10. *Let f be a continuously differentiable map of \mathbb{R}^n with a fixed point x_0 where $\|Df_{x_0}\| < 1$. Then there is a closed neighborhood U of x_0 such that $f(U) \subset U$ and f is a contraction on U .*

To do this we will need the following exercise and proposition:

Exercise 4. Given a linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$, recall that

$$\|A\| = \sup_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}$$

Prove that $A \mapsto \|A\|$ is a continuous function from $\mathbb{R}^{m \times n}$ to \mathbb{R} .

Proposition 4.1. *Let $V \subset \mathbb{R}^n$ be a closed disk and let $f : V \rightarrow \mathbb{R}^m$ be continuous, with continuous derivative on the interior of V . Suppose there exists $M > 0$ such that $\|Df_x\| \leq M$ for all x in the interior of V . Then*

$$d(f(x), f(y)) \leq Md(x, y) \quad \forall x, y \in V$$

Proof. Given $x, y \in \mathbb{R}^n$, let $g : [0, 1] \rightarrow \mathbb{R}^m$ be the function

$$g(t) = f((1-t)x + ty)$$

Then the mean value theorem states that for some $c \in (0, 1)$,

$$d(g(0), g(1)) \leq \|g'(c)\|$$

From this we get

$$\begin{aligned} d(f(x), f(y)) &= d(g(0), g(1)) \leq \|g'(c)\| \\ &= \|Df_{(1-c)x+cy}(y-x)\| \leq \|Df_{(1-c)x+cy}\| \cdot d(x, y) \\ &\leq Md(x, y) \end{aligned}$$

□

Proof of Proposition 3.10. The function f is C^1 implies that $x \mapsto Df$ is continuous. By Exercise 4, the composition $x \mapsto Df_x \mapsto \|Df_x\|$ is continuous. Fix a point $\lambda \in (\|Df_{x_0}\|, 1)$. Then there exists a small closed ball $U = \overline{B(x_0, \delta)}$ around x_0 on which $\|Df_x\| \leq \lambda < 1$.

By Proposition 4.1, if $x, y \in U$, then $d(f(x), f(y)) \leq \lambda d(x, y)$. Moreover, for all $x \in U$, we have

$$d(f(x), x_0) = d(f(x), f(x_0)) \leq \lambda d(x, x_0) \leq \lambda \delta < \delta.$$

This shows that $f(U) \subset U$, and f is a λ -contraction on U . □

4.1 Fractals

In this section we will define fractals and introduce self-similarity. We will also give an idea of their connection with dynamical systems with some examples.

4.1.1 The Cantor Set

The simplest example of a fractal is the ternary cantor set.

Definition 4.2. Let $I = [0, 1]$. Inductively define closed subsets $C_n \subset I$ for $n \geq 0$ as follows:

$$\begin{aligned} C_0 &= [0, 1] \\ C_1 &= C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \frac{C_0}{3} \cup \left(\frac{C_0}{3} + \frac{2}{3}\right) \\ C_n &= \frac{C_{n-1}}{3} \cup \left(\frac{C_{n-1}}{3} + \frac{2}{3}\right) = C_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) \text{ for all } n \geq 2 \end{aligned}$$

It can be shown that the set C_n is the disjoint union of 2^n closed intervals, each of length $\frac{1}{3^n}$. The ternary Cantor set is defined as

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$$

We look at some of the properties of \mathcal{C} are clear:

We look at some additional properties one by one.

1. **\mathcal{C} is closed**, since it is the intersection of closed sets
2. **\mathcal{C} is compact**, since it is a closed subset of a compact set.
3. **\mathcal{C} is non-empty.**

For example, the points 0 and 1 belong to all the sets C_n , so they also belong to \mathcal{C} .

4. **\mathcal{C} is uncountable.**

We will show this by giving an explicit description of \mathcal{C} .

Definition 4.3. Given a number $x \in [0, 1]$, a base 3 (or *ternary*) expansion for x is a sequence $.\alpha_1\alpha_2\alpha_3\cdots$ with $\alpha_n \in \{0, 1, 2\}$ for all $n \in \mathbb{N}$ such that

$$x = \frac{\alpha_1}{3} + \frac{\alpha_2}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$$

The decimal point before the α 's indicate that the number is less than or equal to 1. More generally, for any real number $y \in \mathbb{R}$. The ternary expansion of y is a sequence $\beta_{m-1}\beta_{m-2}\cdots\beta_0.\alpha_1\alpha_2\alpha_3\cdots$ with $\beta_i \in \{0, 1, 2\}$ for all $i \in \{0, 1, 2, \dots, m-1\}$ and $\alpha_n \in \{0, 1, 2\}$ for all $n \in \mathbb{N}$, such that

$$y = \sum_{i=0}^{m-1} \beta_i \cdot 3^i + \sum_{n=0}^{\infty} \frac{\alpha_n}{3^n}$$

Note that the ternary expansion of a number is not unique. For example,

$$\frac{1}{3} = \sum_{n=1}^{\infty} \frac{2}{3^n}$$

So $.100000\cdots$ and $.022222\cdots$ are both ternary expansions for $\frac{1}{3}$. Similarly, $.200000\cdots$ and $.122222\cdots$ are both ternary expansions for $\frac{2}{3}$.

Exercise 5. Every number $x \in \mathbb{R}$ has only finitely many ternary expansions.

Remark 5.1. If x has ternary expansion $\alpha_1\alpha_2\alpha_3\cdots$, then $\frac{x}{3}$ has ternary expansion $0\alpha_1\alpha_2\cdots$

Proposition 5.2.

$\mathcal{C} = \{x \in [0, 1] : \text{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \text{ for } x \text{ with } \alpha_n \in \{0, 2\} \text{ for all } n \in \mathbb{N}\}$

Proof. We prove this by induction. Note that

$$\mathcal{C}_1 = \{x \in [0, 1] : \text{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \text{ for } x \text{ with } \alpha_1 \in \{0, 2\}\}$$

Using the recursive formula for \mathcal{C}_n , it is easy to show that

$$\mathcal{C}_N = \{x \in [0, 1] : \text{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \text{ for } x \text{ with } \alpha_1, \alpha_2, \dots, \alpha_N \in \{0, 2\}\}$$

Since every x has only finitely many ternary expansions, for x to be in all the \mathcal{C}_n 's, there exists at least one ternary expansion which satisfies the condition $\alpha_n \in \{0, 2\}$ for all $n \in \mathbb{N}$. \square

Since every sequence in $\{0, 2\}^{\mathbb{N}}$ can be realized as the ternary expansion of a distinct number $x \in [0, 1]$, and the set $\{0, 2\}^{\mathbb{N}}$ is uncountable, we see that \mathcal{C} is uncountable.

5. \mathcal{C} is perfect.

Definition 5.3. Let Y be a topological space. A subset $X \subset Y$ is said to be perfect if it is closed in Y and has no isolated points.

Proposition 5.4. *For every $x \in \mathcal{C}$, there exists a sequence (x_n) of distinct points with $x_n \in \mathcal{C}$ and $x_n \rightarrow x$.*

Proof. Given $\epsilon > 0$, we will exhibit a point $x_N \neq x$ such that $|x_N - x| < \epsilon$ and $x_N \in \mathcal{C}$. Choose $N \in \mathbb{N}$ such that $\frac{2}{3^N} < \epsilon$. This ensures that the interval of radius $\frac{1}{3^N}$ centered at x is contained in the open ball $B_\epsilon(x)$. Let \tilde{I} be the component interval of \mathcal{C}_N that contains x . The above condition implies that $\tilde{I} \subset B_\epsilon(x)$.

In \mathcal{C}_{N+1} , the middle third of \tilde{I} is deleted, and we get two component intervals \tilde{I}_0 and \tilde{I}_1 . Without loss of generality, assume $x \in \tilde{I}_0$. Then pick a point $y \in \mathcal{C} \cap \tilde{I}_1$. Note that $y \neq x$ by this choice and since $\tilde{I}_1 \subset B_\epsilon(x)$, we have $|y - x| < \epsilon$. Therefore we can set $x_N = y$. \square

6. \mathcal{C} is totally disconnected.

Definition 5.5. A topological space X is said to be totally disconnected if its only non-empty connected subsets are singletons.

Proposition 5.6. *If $F \subset \mathcal{C}$ is non-empty and connected, then $F = \{x\}$ for some point $x \in \mathcal{C}$.*

Proof. Suppose $x, y \in \mathcal{C}$ are two distinct points in F . WLOG, assume $x < y$. Pick $N \in \mathbb{N}$ such that $\frac{1}{3^{N-1}} < |x - y|$. Then, x and y are contained in distinct components of \mathcal{C}_N . So there exists $z \in (x, y)$ such that $z \notin \mathcal{C}$. Let $A = F \cap [0, z)$ and $B = F \cap (z, 1]$. Note that $A \cup B = F$. Also note that the closures of A and B don't intersect. This contradicts the fact that F is connected. \square

7. \mathcal{C} has Lebesgue measure 0.

Let μ denote Lebesgue measure. The set \mathcal{C}_n is the union of 2^n disjoint intervals, each of length 3^{-n} . Therefore, we have $\mu(\mathcal{C}_n) = \frac{2^n}{3^n}$. Since $\mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3 \supset \dots$ and \mathcal{C} is the intersection of the \mathcal{C}_n , we have

$$\mu(\mathcal{C}) = \lim_{n \rightarrow \infty} \mu(\mathcal{C}_n) = 0$$

The following theorem is the main result of this section. For a proof see [1].

Theorem 5.7 (Brouwer). *Let $Y \neq \emptyset$ be a complete metric space. If Y is compact, perfect and totally disconnected, then it is homeomorphic to \mathcal{C} .*

An easy corollary, for example, is that \mathcal{C} is homeomorphic to $\mathcal{C} \times \mathcal{C}$.

5.0.1 Some dynamical systems on the Cantor set

Consider the map $f : [0, 1] \times [0, 1]$ defined as $f(x) = \frac{x}{3}$. It is easy to see that f is a contraction and $f(\mathcal{C}) \subset \mathcal{C}$, and the unique fixed point in \mathcal{C} is $x = 0$. Note that for every $x \in \mathcal{C}$, there exists a neighborhood U of x such that $f : U \rightarrow f(U)$ is a homeomorphism.

Also note that f induces the shift $.a_1a_2a_3\cdots \rightarrow .0a_1a_2a_3\cdots$ on ternary expansions.

Definition 5.8. A topological space X is said to be *self-similar*, or to have the *rescaling property*, if there exists a contraction $f : X \rightarrow X$ such that for every $x \in X$ and neighborhood U of x , there exists a neighborhood $V \subset U$ of x such that $f : V \rightarrow f(V)$ is a homeomorphism.

Remark 5.9. This is actually equivalent to saying that every $x \in X$ has a neighborhood U such that $f : U \rightarrow f(U)$ is a homeomorphism. Also note that the term self similar is used in different ways in the literature; we will see by and by that this definition is not extensive enough.

Exercise 6. Show that the function $f(x) = 1 - \frac{x}{3}$ leaves \mathcal{C} invariant, and is a contraction. Describe the induced operation on ternary expansions, and find the unique fixed point of f in \mathcal{C} .

Exercise 7. Show that the function $f(x) = \frac{x+2}{3} \pmod{1}$ leaves \mathcal{C} invariant, and is a contraction. Describe the induced operation on ternary expansions, and find the unique fixed point of f in \mathcal{C} .

7.0.1 The Square Sierpinski Carpet

Definition 7.1. Let $J = [0, 1] \times [0, 1]$ be the unit square. Define

$$\begin{aligned}\mathcal{J}_0 &= J \\ \mathcal{J}_1 &= \mathcal{J}_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) \times \left(\frac{1}{3}, \frac{2}{3}\right) \\ \mathcal{J}_n &= \mathcal{J}_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \bigcup_{\ell=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) \times \left(\frac{3\ell+1}{3^n}, \frac{3\ell+2}{3^n}\right)\end{aligned}$$

The square Sierpinski carpet is the set $\mathcal{J} = \bigcap_{n=0}^{\infty} \mathcal{J}_n$.

7.0.2 The Sierpinski Triangle

7.0.3 Cantor sets and Logistic maps

Consider the logistic function $f(x) = 5x(1-x)$ on \mathbb{R} . Note that this function has one critical point at $x = \frac{1}{2}$, and is symmetric around this point in the sense that

$$f(x) = f(1-x) \text{ for all } x \in \mathbb{R}$$

We make the following series of observations:

1. The graph of f is a downward drawn parabola, and its roots are $x = 0, 1$.

2. If $x > 1$, then $f(x) < 0$.
3. If $x < 0$, then $f(x) < x$ and $|f(f(x)) - f(x)| > |f(x) - x|$.

The points (2) and (3) show that if $x \notin [0, 1]$, then $f^{on}(x) \rightarrow -\infty$. This leads to the following dichotomy:

For every $x \in \mathbb{R}$, exactly one of the following is true:

- either $f^{on}(x) \in [0, 1]$ for all $n \in \mathbb{N}$, or
- $f^{om}(x) \notin [0, 1]$ for some $m \in \mathbb{N}$, and thus, $f^{on}(x) \rightarrow -\infty$ as $n \rightarrow \infty$.

Therefore, the set B of points $x \in \mathbb{R}$ such the orbit $(f^{on}(x))_{n \geq 0}$ is bounded, is the set of points x such that $f^{on}(x) \in [0, 1]$ for all n . In other words,

$$B = \bigcap_{n=0}^{\infty} (f^{on})^{-1}[0, 1]$$

Proposition 7.2. *B is a Cantor set (i.e., it is homeomorphic to \mathcal{C}).*

Proof.

□

References

- [1] Michael Francis. Two topological uniqueness theorems for spaces of real numbers, 2012.