MA 771: Introduction to Dynamical Systems
Lecture Notes

Malavika Mukundan

Spring 2025

Contents

1 Basic concepts

Dynamical systems refer to systems that evolve over time. A simple example is an ordinary differential equation (ODE), such as

$$\frac{dx}{dt} = g(x)$$

where $g : \mathbb{R}^n \longrightarrow \mathbb{R}$. Some of the questions we can ask about ODE's include:

- Which points are stable/fixed?
- When do orbits tend to an attracting periodic cycle?
- When are orbits chaotic?

However, the class of dynamical systems is much bigger than ODE's. In particular, in the above example, the variable t, representing time, changes *continuously*. In this course we will focus mainly on *discrete* dynamical systems.

Definition 1.1. Let X be a topological space. A *discrete* dynamical system is a pair (X, f) where $f : X \longrightarrow X$ is a self-map.

We are interested in the function f and its iterates $f^{\circ n} = f \circ f^{\circ(n-1)}$ for $n \in \mathbb{N}$. In other words, the quantity that is varying in a *discrete* fashion is the number of iterations of f.

Example 1.2. **Linear maps** Let $X = \mathbb{R}^n$ and $f : x \mapsto Ax$ be a linear map.

Example 1.3. **Rotations of the circle** $X = \mathbb{S}^1$ and $f(x) = e^{2\pi i\theta}x$ for some $\theta \in \mathbb{R}$.

Example 1.4. **Logistic family** Let $X = \mathbb{R}$ and fix $k \in \mathbb{R}_{>0}$. Then the family of maps $f_k : X \to X$ given by $x \mapsto kx(1-x)$ is called the *logististic family*.

Notice that we are not assuming any conditions on f such as continuity.

1.1 Orbits and periodic points

Definition 1.5. Given a dynamical system (X, f), $x_0 \in X$, the sequence

$$x_0, f(x_0), f^{\circ 2}(x_0), \cdots, f^{\circ n}(x_0), \cdots$$

is called the *forward orbit* of x_0.
The *reverse orbit of* x_0 is the set $\{x \in X : f^{\circ n}(x) = x_0 \text{ for some } n \in \mathbb{N}\}$.

A *fixed point* $x \in X$ is a point such that $f(x) = x$. The set of fixed points of f is denoted $\mathrm{Fix}(f)$. A *periodic point* is a point x such that $f^{\circ n}(x) = x$ for some $n \in \mathbb{N}$, in other words, a point in $\mathrm{Fix}(f^{\circ n)}$ for some n.

Any $n \in \mathbb{N}$ such that $f^{\circ n}(x) = x$ is said to be a *period* of x. The smallest period n is called the *exact* period of x.

1.2 Examples

1.2.1 Linear maps of $\mathbb{R}$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be linear. We know that f is of the form $f(x) = mx + b$ where $m \in \mathbb{R}_{\neq 0}$ and $b \in \mathbb{R}$. Note that

$$f^{\circ n}(x) = m^n x + b\big(m^{n-1} + m^{n-2} + \cdots + m + 1\big) = m^n x + b\frac{m^n - 1}{m - 1}$$

- If $m \neq \pm 1$, then

$$|m| < 1 \implies \forall x \in \mathbb{R}, f^{\circ n}(x) \to \frac{b}{1-m} \text{ as } n \to \infty$$
$$|m| > 1 \implies \forall x \in \mathbb{R}, f^{\circ n}(x) \to \infty \text{ as } n \to \infty$$

- If $m = 1$, then $f(x) = x + b$ is a translation and all orbits tend to ∞

- If $m = -1$, then note that $f^{\circ 2}(x) = -(-x + b) + b = x$, and thus all the odd iterates are equal to f, and all the even iterates are equal to the identity.

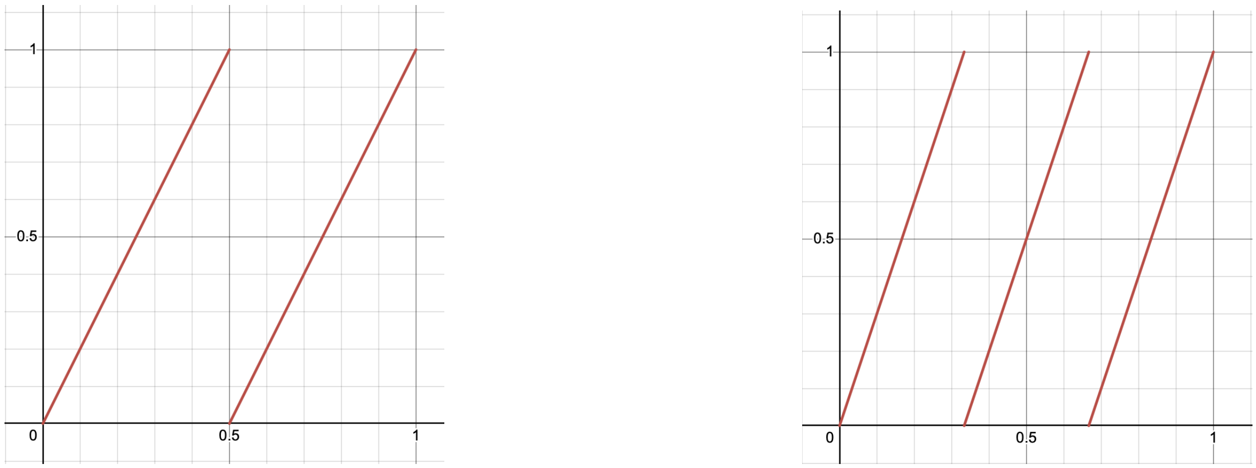

Figure 1: The graphs of the expanding maps E_2 (left) and E_3 (right) on the interior of $\mathbb{S}^1$, represented by the interval $(0, 1)$.

1.2.2 Circle maps

Example 1.6. For any rotation $f(x) = e^{2\pi i\theta}x$ of the circle $\mathbb{S}^1$, we have $\text{Fix}(f) = \emptyset$ if $\theta \notin \mathbb{N}$, and $\text{Fix}(f) = \mathbb{S}^1$ otherwise.

Definition 1.7. Fix an integer $m > 1$, and identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$. The *expanding map* $E_m : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is defined as

$$E_m(x) = mx \pmod 1$$

Remark 1.8. E_m is expanding in the following sense: if $\alpha, \beta \in \mathbb{S}^1$ and $d_{\mathbb{S}^1}(\alpha, \beta) < \frac{1}{m}$, then

$$d_{\mathbb{S}_1}(\alpha, \beta) = m \cdot d_{\mathbb{S}^1}\Big(E_m(\alpha), E_m(\beta)\Big).$$

See Figure 1 for the graphs of E_m for $m = 2, 3$.

Note that ϕ is a fixed point of E_m if and only if $m\phi - \phi \in \mathbb{Z}$. In other words, there exists $n \in \mathbb{Z}$ such that

$$\begin{aligned} m\phi - \phi &= n \\ \iff \phi &= \frac{n}{m-1} \end{aligned}$$

Similarly, ϕ is a periodic point of E_m of period dividing k if and only if there exists $n \in \mathbb{Z}$ such that

$$\begin{aligned} m^k\phi - \phi &= n \\ \iff \phi &= \frac{n}{m^k-1} \end{aligned}$$

In other words,

$$\mathrm{Fix}(E_m) = \left\{\frac{1}{m-1}, \frac{2}{m-1} \cdots, \frac{m-2}{m-1}\right\}$$
$$\mathrm{Fix}\left(E_m^{\circ k}\right) = \left\{\frac{1}{m^k-1}, \frac{2}{m^k-1} \cdots, \frac{m^k-2}{m^k-1}\right\}$$

1.2.3 Torus endomorphisms

Given $n \in \mathbb{N}$, the $n-$torus is the space $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \mathbb{R}^n/\sim$ where $x \sim y$ if $x - y \in \mathbb{Z}^n$. For $x \in \mathbb{R}^n$, we let $[x]$ denote the equivalence class of x in $\mathbb{T}^n$.

Definition 1.9. Let A be an $n \times n$ matrix whose entries are in $\mathbb{Z}$. Then A induces the *torus endomorphism* $T_A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ given by

$$T_A([x]) = [Ax] \text{ for } [x] \in \mathbb{T}^n$$

Exercise 2. Show that T_A as given above is well-defined: that is, for any two vectors $v, w \in \mathbb{R}^n$, if $v - w \in \mathbb{Z}^n$, then $Av - Aw \in \mathbb{Z}^n$

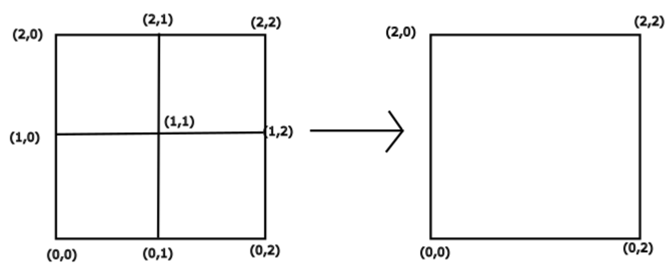

Figure 2: An illustration of the torus endomorphism $T_A : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ for $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Example 2.1. Let $m, k \in \mathbb{Z}$ and consider the matrix $A = \begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix}$. Consider the map T_A on $\mathbb{T}^2$: this acts as two independent expanding maps: expansion by a factor of m in the $x-$direction, and expansion by a factor of k in the $y-$direction (see Figure 2 which illustrates the case $m = k = 2$). Can you show in general that the degree of such a map is $d = mk$? In other words, T_A is a $d : 1$ map of $\mathbb{T}^2$.

Definition 2.2. A torus endomorphism T_A is said to be an *automorphism* if it is invertible.

Exercise 3. (This is also on HW 1) Show that T_A is invertible if and only if A^{-1} has integer entries, which in turn is equivalent to $\det A = \pm 1$.

Proposition 3.1. *Let $T_A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ be a torus automorphism with no eigenvalues of modulus* 1. *Then the periodic points of T_A are all the points with rational coordinates.*

Proof. (periodic $\implies$ rational):
Let $[x] = [(x_1, x_2, \cdots, x_n)] \in \mathbb{T}^n$ be a periodic point of period q for some $q \in \mathbb{N}$. Then $T_A^{\circ q}([x]) = [A^q x] = [x]$. That is, there exists a vector $y \in \mathbb{Z}^n$ such that

$$\begin{aligned} A^q x &= x + y \\ \implies A^q x - x &= y \\ \implies (A^q - \mathrm{Id})x &= y \end{aligned}$$

Since A has no eigenvalues of modulus 1, the matrix A^q has no eigenvalues of modulus 1. This means that the matrix $A^q - \mathrm{Id}$ is invertible. So

$$x = (A^q - \mathrm{Id})^{-1} y$$

Since y has integer coordinates and the matrix $(A^q - \mathrm{Id})^{-1}$ has rational entries, x has rational coordinates.

(rational $\implies$ periodic):
Suppose x has rational coordinates, we can assume that all the coordinates have a common denominator. In other words, $x = \left(\frac{p_1}{r}, \frac{p_2}{r}, \cdots, \frac{p_n}{r}\right)$ for some integers p_i, r with $r \neq 0$. Given a $q \in \mathbb{N}$, since A has integer entries, $A^q x = \left(\frac{p_1'}{r}, \frac{p_2'}{r}, \cdots, \frac{p_n'}{r}\right)$ for some integers $p_1', \cdots, p_n'$.

Note that there are only finitely many points in $\mathbb{T}^n$ with rational coordinates with a common denominator r. In other words, the set $\{T_A^{\circ q}([x]) : q \in \mathbb{N}\}$ is finite.

Thus, there exist $q_1 < q_2 \in \mathbb{N}$ such that $T_A^{\circ q_1}([x]) = T_A^{\circ q_2}([x])$. Since T_A is an automorphism, this means that $T_A^{\circ(q_2 - q_1)}([x]) = [x]$. $\square$

3.1 The Contraction Principle

In this section we will look at maps on subsets of $\mathbb{R}^n$ which satisfy a criterion for all orbits converging to a fixed point.

3.1.1 Global contraction

Definition 3.2. A map f of a subse tX of $\mathbb{R}^n$ is said to be *Lipschitz-continuous* with Lipschitz constant λ, or λ-*Lipschitz* if

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for any $x, y \in X$.
The map f is said to be a *contraction* or a $\lambda-$*contraction* if $\lambda < 1$.

Remark 3.3. If a map f is Lipschitz-continuous, then we define

$$Lip(f) := \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}$$

Example 3.4. $f(x)=\sqrt{x}$ defines a contraction on $I = [1, \infty)$. What is $Lip(f)$?

Proposition 3.5. ***(Contraction Principle)*** *Let $X \subset \mathbb{R}^n$ be closed and $f : X \longrightarrow X$ a $\lambda-$contraction. Then f has a unique fixed point x_0 and $d(f^{\circ n}(x), x_0) = \lambda^n d(x, x_0)$ for every x.*

Proof. We have

$$d(f^{\circ n}(x), f^{\circ n}(y)) \leq \lambda^n d(x, y)$$

for all $x, y \in X$. But this also means that for any $x \in X$, we have

$$d(f^{\circ (n+1)}(x), f^{\circ n}(x)) \leq \lambda^n d(f(x), x)$$

$$\begin{aligned} d(f^{\circ m}(x), f^{\circ n}(x)) &\leq d(f^{\circ m}(x), f^{\circ (m-1)}(x)) + d(f^{\circ (m-1)}(x), f^{\circ (m-2)}(x)) + \cdots d(f^{\circ (n+1)}(x), f^{\circ n}(x)) \\ &\leq \left(\lambda^{m-1} + \lambda^{m-2} + \cdots \lambda^n\right) d(f(x), x) \\ &\leq \frac{\lambda^n(1-\lambda^{m-n})}{1-\lambda} d(f(x), x) \\ &\leq \frac{\lambda^n}{1-\lambda} d(f(x), x) \end{aligned}$$

In other words, the orbit of x is a Cauchy sequence. Since X is closed, $\lim_{n\to\infty} f^{\circ n}(x) = x_0$ is a point of X, and

$$f(x_0) = f(\lim_{n\to\infty} f^{\circ n}(x)) = \lim_{n\to\infty} f^{\circ (n+1)}(x_0) = x_0$$

□

3.1.2 Local contraction

Proposition 3.6. *Let f be a continuously differentiable map of $\mathbb{R}^n$ with a fixed point x_0 where $||Df_{x_0}|| < 1$. Then there is a closed neighborhood U of x_0 such that $f(U) \subset U$ and f is a contraction on U.*

Since Df is continuous, there exists a small closed ball $U = \overline{B(x_0, \eta)}$ around x_0 on which $||Df_x|| \leq \lambda < 1$. If $x, y \in U$, then $d(f(x), f(y)) \leq \lambda d(x, y)$. So f is a contraction on U. Furthermore, taking $y = x_0$ shows that if $x \in U$, then $d(f(x), x_0) = d(f(x), f(x_0)) \leq \lambda d(x, x_0) \leq \lambda\eta < \eta$, and hence $f(x) \in U$.