

Homework 1

MA 771

Due February 6 2025

Please scan or type your homework and hand/email it to me by 5pm on the due date.

1. Given a real number $\lambda > 0$, find the largest closed interval $I = [a, b]$ on the real line such that $f : I \rightarrow \mathbb{R}$ given by $f(x) = x^2$ is a λ -contraction.
2. Show that the following statements are equivalent:
 - the endomorphism of the torus $T_A : \mathbb{T}^n \rightarrow \mathbb{T}^n$ is invertible
 - $x \in \mathbb{Z}^n$ if and only if $Ax \in \mathbb{Z}^n$, for every $x \in \mathbb{Z}^n$
 - $|\det A| = 1$.
3. Let $m > 1$ be an integer and let $E_m : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be the expanding map $x \mapsto mx \pmod{1}$.
 - Prove that the set of all periodic points of E_m is dense in \mathbb{S}^1 .
 - A point $x \in \mathbb{S}^1$ is pre-periodic under E_m if $y = E_m^{\circ k}(x)$ is a periodic point for some $k \geq 0$. Show that the set of pre-periodic points under E_m is \mathbb{Q}/\mathbb{Z} .
4. Suppose that (X, d) is a compact metric space and $f : X \rightarrow X$ is such that

$$d(f(x), f(y)) < d(x, y)$$

for any $x \neq y$. Prove that f has a unique fixed point $x_0 \in X$ and $\lim_{n \rightarrow \infty} f^{\circ n}(x) = x_0$ for any $x \in X$.

5. Prove that the square Sierpinski carpet is the set of points in the unit square of which at least one coordinate can be represented by a ternary (i.e., base 3) expansion without the digit 1.