Homework 3

MA 771

Due March 6 2025

Please scan or type your homework and hand/email it to me by 5pm on the due date.

1. - Consider $F(x) := x + \frac{1}{4\pi}\sin(2\pi x)$ on the real line. Decide whether F is the lift of a circle homeomorphism, and, if so, decide whether that homeomorphism is orientation-preserving. If it is, determine the rotation number.
   - Show that there exists a point $x_0 \in \mathbb{S}^1$ such that the closure of its orbit under the expanding map E_3 is the set $\{[x] \in \mathbb{S}^1 : x \in \mathcal{C}\}$ where $\mathcal{C} \subset [0,1]$ is the ternary Cantor set.

2. Let A be the annulus $\mathbb{R}/\mathbb{Z} \times [0,1]$. Consider the linear twist $T : A \to A$ given by

$$T([x],t) = ([x+t],t) \text{ for all } ([x],t) \in A$$

Prove that T has the following property of *partial topological mixing*: Let $U, V \subseteq \mathbb{R}/\mathbb{Z}$ be non-empty open sets. Then there exists $N \in N$ such that

$$T^{\circ n}(U \times [0,1]) \cap (V \times [0,1]) \neq \emptyset \text{ for any } n \geq N.$$

3. Let f be a topologically mixing map on a metric space (X,d). Recall that the *diameter* of X is defined as $\sup\{d(x,y) : x,y \in X\} \in [0,\infty]$. Prove that any real number $\Delta \geq 0$ strictly less than the diameter of X is a sensitivity constant for f.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove the following.
   - Suppose I and J are closed intervals such that $I \subset J$ and $J \subset f(I)$. Then show that f has a fixed point in I.
   - Suppose $A_0, A_1, ..., A_N$ are closed intervals and $A_{n+1} \subset f(A_n)$ for $n \in \{0,1,\cdots,N-1\}$. Then for each $i \in \{0,1,\cdots,N-1\}$, there exists $J_i \subset A_0$ such that $f^{\circ n}(J_i) \subset A_n$ for $n \in \{0,1,\cdots,i\}$ and $f^{\circ(i+1)}(J_i) = A_{i+1}$.

5. Consider a 2×2 integer matrix A without eigenvalues of absolute value 1 and with $|\det A| > 1$. Prove that the non-invertible torus endomorphism $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ is hyperbolic and topologically mixing.