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TOPOLOGICAL SPACES

Def: Let X be a set. A "topology on X" is is a set $\mathcal{T} \subseteq P(X)$ such that

(i) $\phi, X \in \mathcal{T}$

(ii) For every $\mathcal{U} \subseteq \mathcal{T}$, the union $\bigcup_{E \in \mathcal{U}} E \in \mathcal{T}$

i.e. the union of elements in any sub-collection of $\mathcal{T}$ is an element of $\mathcal{T}$

(iii) For every finite $\mathcal{U} \subseteq \mathcal{T}$, the intersection $\bigcap_{E \in \mathcal{U}} E \in \mathcal{T}$,

i.e., the intersection of elements in any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A topological space is an ordered pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X.

Def: Given a topological space $(X, \mathcal{T})$ and $U \subseteq X$, U is said to be open if $U \in \mathcal{T}$, and closed if $X \setminus U \in \mathcal{T}$

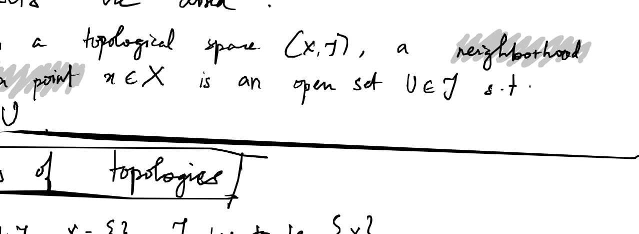
Thus, the union of an arbitrary collection of open sets, and the intersection of finite collections of open sets, are open.

Note: Arbitrary intersections & finite unions of closed sets are closed.

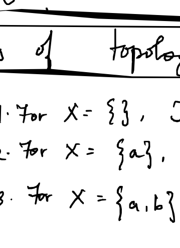
Def Given a topological space $(X, \mathcal{T})$, a neighborhood of a point $x \in X$ is an open set $U \in \mathcal{T}$ s.t. $x \in U$

Examples of topologies

1. For $X = \{\}$, $\mathcal{T}$ has to be $\{X\}$.
2. For $X = \{a\}$, $\mathcal{T} = \{\phi, X\}$.
3. For $X = \{a, b\}$, $\mathcal{T}_1 = \{\phi, X\}$,
   $\mathcal{T}_2 = \{\phi, \{a\}, X\}$
   $\mathcal{T}_3 = \{\phi, \{b\}, X\}$,
   $\mathcal{T}_4 = \{\phi, \{a\}, \{b\}, X\}$
   are all possible topologies.
4. For $X = \{a, b, c\}$,



are possible topologies.



is not [Reason:

See Eg. 1 Ch 2 in Munkres for other topologies on X.

5. Let (X, d) be a metric space. Recall that $U \subseteq X$ is said to be open if every $x \in U$ has some $\varepsilon > 0$ s.t. $B_d(x, \varepsilon) \subseteq U$.
   By a theorem proved in class last time,
   $\mathcal{T}_d = \{U \subseteq X : U \text{ is open}\}$ is a topology on X,
   $\mathcal{T}_d$ is called the topology induced by the metric d.

Def:

- ★ The topology $\mathcal{T} = \{\phi, X\}$ is called the trivial/indiscrete topology.
- ★ The collection $\mathcal{T}$ of all subsets of X is called the discrete topology on X. Note that $\mathcal{T} = \mathcal{T}_d$ where d is the discrete metric on X.
- ★ The collection $\mathcal{T} = \{E : E = \phi \text{ or } X \setminus E \text{ is countable}\}$ is a topology, aka countable complement/cocountable topology.

Proof:

- ★ The topology $\mathcal{T} = \{E : E = \phi \text{ or } X \setminus E \text{ is finite}\}$ is called the finite complement/cofinite topology

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Basis for a topology

Defn: Let X be a set. A "basis" for a topology on X is a collection $\mathcal{B}$ of subsets of X called "basis elements", s.t.

(i) for every $x \in X$, $\exists B \ni x$ s.t. $B \subseteq \mathcal{B}$

(ii) if $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $\exists B \in \mathcal{B}$ s.t. $x \in B$

Defn:
Given $\mathcal{B}$ as above, the topology $\mathcal{T}$ generated by $\mathcal{B}$ is the collection of all sets $U \subseteq X$ satisfying the property that $\forall x \in U$, $\exists B \in \mathcal{B}$ $x \in B$ and $B \subseteq U$. [This implies $\mathcal{B} \subseteq \mathcal{T}$].

Prop: $\mathcal{T}$ as above is indeed a topology on X.

Proof: (i)

(ii)

(iii)

Lemma 1: Let $\mathcal{T}$ be the topology on X generated by a basis $\mathcal{B}$. Then U is open (ie, $U \in \mathcal{T}$) $\Leftrightarrow \exists \mathcal{B}' \subseteq \mathcal{B}$ s.t. $U = \bigcup_{B \in \mathcal{B}'} B$

Proof:

Eg.:

1. Let (X, d) be a metric space. Then the collection of all ε-balls $\mathcal{B}$ is a basis for the topology $\mathcal{T}_d$.
2. Let $X = \mathbb{R}^2$, and $\mathcal{B}$ be the set of circular regions [interior of circles] in $\mathbb{R}^2$. Then $\mathcal{B}$ is a basis for a topology on $\mathbb{R}^2$.

   Proof:

3. Let d be the Euclidean metric on $\mathbb{R}^2$. Then $\mathcal{T}_d$ is the topology generated by $\mathcal{B}$ as in the above example.
4. Let $X = \mathbb{R}^2$ and $\mathcal{B}'$ be the set of all rectangular regions [interiors of rectangles]. Then $\mathcal{B}'$ is a basis that generates the same topology as $\mathcal{B}$ (namely $\mathcal{T}_d$).
5. Let $X = \mathbb{R}$ and $\mathcal{B} = \{(a, b) : a < b\}$. Then $\mathcal{B}$ generates a topology on $\mathbb{R}$ which coincides with the one induced by the Euclidean metric $d(x, y) = |x - y|$. This is called the standard topology on $\mathbb{R}$

Proof:

Lemma 2: Let $(X, \mathcal{T})$ be a topological space. Suppose $\mathcal{B} \subseteq P(X)$ is such that for each open set $U \subseteq X$ and each $x \in U$, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$.

Proof:

Comparing topologies

Defn:
Let $\mathcal{T}, \mathcal{T}'$ be two topologies on a set X.
If $\mathcal{T} \subseteq \mathcal{T}'$ [ie, every set open in $\mathcal{T}$ is open in $\mathcal{T}'$], we say $\mathcal{T}'$ is finer/larger than $\mathcal{T}$, and $\mathcal{T}$ is coarser/smaller than $\mathcal{T}'$.
If $\mathcal{T} \subsetneq \mathcal{T}'$, then we say $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, and $\mathcal{T}$ is strictly coarser than $\mathcal{T}'$

If either $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T}' \subseteq \mathcal{T}$, we say $\mathcal{T}, \mathcal{T}'$ are comparable.

Lemma: Let $\mathcal{B}, \mathcal{B}'$ be bases for topologies $\mathcal{T}, \mathcal{T}'$ resp. on a set X. Then TFAE:

(i) $\mathcal{T}'$ is finer than $\mathcal{T}$

(ii) $\forall x \in X$, $B \in \mathcal{B}$ s.t. $x \in B$, $\exists B' \in \mathcal{B}'$ s.t. $B' \subseteq B$.

Proof: (i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (i)

Examples:

1. Let $X = \mathbb{R}$ and $\mathcal{B}' = \{[a, b) \mid a < b\}$.
   Then $\mathcal{B}'$ is a basis for a topology $\mathcal{T}'$ on $\mathbb{R}$ called the lower limit topology.
   [We often use the notation $\mathbb{R}_\ell$ for the topological space $(\mathbb{R}, \mathcal{T}')$].

   Prop: $\mathbb{R}_\ell$ is strictly finer than $\mathbb{R}$ with the standard topology $\mathcal{T}$.
   [ie, $\mathcal{T}' \supsetneq \mathcal{T}$]

   Proof:

2. Let $K = \{1/n : n \in \mathbb{N}\}$, and
   $\mathcal{B}'' = \{(a, b) : a < b\} \cup \{(a, b) \setminus K : a < b\}$
   Then $\mathcal{B}''$ generates a topology $\mathcal{T}''$ on $\mathbb{R}$.
   and we let $\mathbb{R}_K = (\mathbb{R}, \mathcal{T}'')$ ["K-topology"]

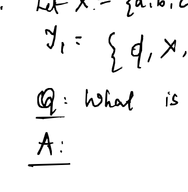
   Prop: $\mathbb{R}_K$ is strictly finer than $\mathbb{R}$ with std. topology $\mathcal{T}$.
   [ie, $\mathcal{T}'' \supsetneq \mathcal{T}$].

   Proof:

3. **Def** Given two metrics d and d' on X, we say that $d \sim d'$ if $\mathcal{T}_d = \mathcal{T}_{d'}$ [ie, a set is open in (X, d) iff it is open in (X, d')].

Examples:
1. For any (X, d), $d \sim \bar{d}$, where $\bar{d}$ is the standard bounded metric induced by d.
2. Consider $(\mathbb{R}^2, d)$ and $(\mathbb{R}^2, d')$ [d' - Taxicab metric].
   We claim $d \sim d'$.

Sketch of proof: Any ε-ball in $(\mathbb{R}^2, d)$ [disk] contains a δ-ball in $(\mathbb{R}^2, d')$ [diamond] for suitable $\delta > 0$ with the same center.



Conversely, any ε-ball in $(\mathbb{R}^2, d')$ contains a δ-ball in $(\mathbb{R}^2, d)$ with the same center.

4. Let $X = \{a, b, c\}$,
   $\mathcal{T}_1 = \{\phi, X, \{a\}, \{b\}\}$ and $\mathcal{T}_2 = \{\phi, X, \{a\}, \{b, c\}\}$

   Q: What is the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$?

   A:

   Q: What is the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$?

   A:

Def: Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$.
A basis of neighborhoods at x is a collection $\mathcal{U}_x \subseteq \mathcal{T}$ s.t. for any neighborhood U of X, $\exists B \in \mathcal{U}_x$ s.t. $B \in U$.

Eg: In any metric space (X, d), $\mathcal{U}_x$ can be taken to be the collection of all ε-balls centered at x.

Def: A topological space (X, d) is said to be first countable if every point $x \in X$ has a countable basis of neighborhoods.

Eg: In $(\mathbb{R}^n, d)$, given any $x \in \mathbb{R}^n$ and a neighborhood $U \ni x$, $\exists \varepsilon \in \mathbb{Q}$ s.t. $B_d(x, \varepsilon) \subseteq U$. Thus $\mathcal{U}_x = \{B_d(x, \varepsilon) : \varepsilon \in \mathbb{Q}\}$ is a countable basis of neighborhoods at x.

Def: Let X be a set. A subbasis for a topology on X is a collection $\mathcal{S} \subseteq P(X)$ s.t.
$\bigcup_{B \in \mathcal{S}} B = X$.
The topology generated by $\mathcal{S}$ is the collection of all unions of finite intersections of elements of $\mathcal{S}$.

Exercise: Find a subbasis for the standard topology on $\mathbb{R}$.
