MA 771: Introduction to Dynamical Systems
Lecture Notes

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Contents

Chapter 1

Basic Concepts

Dynamical systems refer to systems that evolve over time. A simple example is an ordinary differential equation (ODE), such as

$$\frac{dx}{dt} = g(x)$$

where $g : \mathbb{R}^n \longrightarrow \mathbb{R}$. Some of the questions we can ask about ODE's include:

- Which points are stable/fixed?
- When do orbits tend to an attracting periodic cycle?
- When are orbits chaotic?

However, the class of dynamical systems is much bigger than ODE's. In particular, in the above example, the variable t, representing time, changes *continuously*. In this course we will focus mainly on *discrete* dynamical systems.

Definition 1.0.1. Let X be a topological space. A *discrete* dynamical system is a pair (X, f) where $f : X \longrightarrow X$ is a self-map.

We are interested in the function f and its iterates $f^{\circ n} = f \circ f^{\circ(n-1)}$ for $n \in \mathbb{N}$. In other words, the quantity that is varying in a *discrete* fashion is the number of iterations of f.

Example 1.0.2. **Linear maps** Let $X = \mathbb{R}^n$ and $f : x \mapsto Ax$ be a linear map.

Example 1.0.3. **Rotations of the Circle** $X = \mathbb{S}^1$ and $f(x) = e^{2\pi i\theta}x$ for some $\theta \in \mathbb{R}$.

Example 1.0.4. **Logistic Family** Let $X = \mathbb{R}$ and fix $k \in \mathbb{R}_{>0}$. Then the family of maps $f_k : X \to X$ given by $x \mapsto kx(1-x)$ is called the *logististic family*.

Notice that we are not assuming any conditions on f such as continuity.

1.1 Orbits and Periodic Points

Definition 1.1.1. Given a dynamical system (X, f), $x_0 \in X$, the sequence

$$x_0, f(x_0), f^{\circ 2}(x_0), \cdots, f^{\circ n}(x_0), \cdots$$

is called the *forward orbit* of x_0.
The *reverse orbit of* x_0 is the set $\{x \in X : f^{\circ n}(x) = x_0 \text{ for some } n \in \mathbb{N}\}$.

A *fixed point* $x \in X$ is a point such that $f(x) = x$. The set of fixed points of f is denoted $\mathrm{Fix}(f)$. A *periodic point* is a point x such that $f^{\circ n}(x) = x$ for some $n \in \mathbb{N}$, in other words, a point in $\mathrm{Fix}(f^{\circ n})$ for some n.

Any $n \in \mathbb{N}$ such that $f^{\circ n}(x) = x$ is said to be a *period* of x. The smallest period n is called the *exact* period of x.

1.2 Examples

Linear Maps of $\mathbb{R}$

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be linear. We know that f is of the form $f(x) = mx + b$ where $m \in \mathbb{R}_{\neq 0}$ and $b \in \mathbb{R}$. Note that

$$f^{\circ n}(x) = m^n x + b\big(m^{n-1} + m^{n-2} + \cdots + m + 1\big) = m^n x + b\frac{m^n - 1}{m - 1}$$

- If $m \neq \pm 1$, then

$$\begin{aligned} |m| < 1 &\implies \forall x \in \mathbb{R}, f^{\circ n}(x) \to \frac{b}{1-m} \text{ as } n \to \infty \\ |m| > 1 &\implies \forall x \in \mathbb{R}, f^{\circ n}(x) \to \infty \text{ as } n \to \infty \end{aligned}$$

- If $m = 1$, then $f(x) = x + b$ is a translation and all orbits tend to ∞
- If $m = -1$, then note that $f^{\circ 2}(x) = -(-x + b) + b = x$, and thus all the odd iterates are equal to f, and all the even iterates are equal to the identity.

Circle Maps

Example 1.2.1. For any rotation $f(x) = e^{2\pi i\theta}x$ of the circle $\mathbb{S}^1$, we have $\mathrm{Fix}(f) = \emptyset$ if $\theta \notin \mathbb{N}$, and $\mathrm{Fix}(f) = \mathbb{S}^1$ otherwise.

Definition 1.2.2. Fix an integer $m > 1$, and identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$. The *expanding map* $E_m : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is defined as

$$E_m(x) = mx \text{ (mod) } 1$$

Remark 1.2.3. E_m is expanding in the following sense: if $\alpha, \beta \in \mathbb{S}^1$ and $d_{\mathbb{S}^1}(\alpha, \beta) < \frac{1}{m}$, then

$$d_{\mathbb{S}_1}(\alpha, \beta) = m \cdot d_{\mathbb{S}^1}\Big(E_m(\alpha), E_m(\beta)\Big).$$

See Figure 1.1 for the graphs of E_m for $m = 2, 3$.

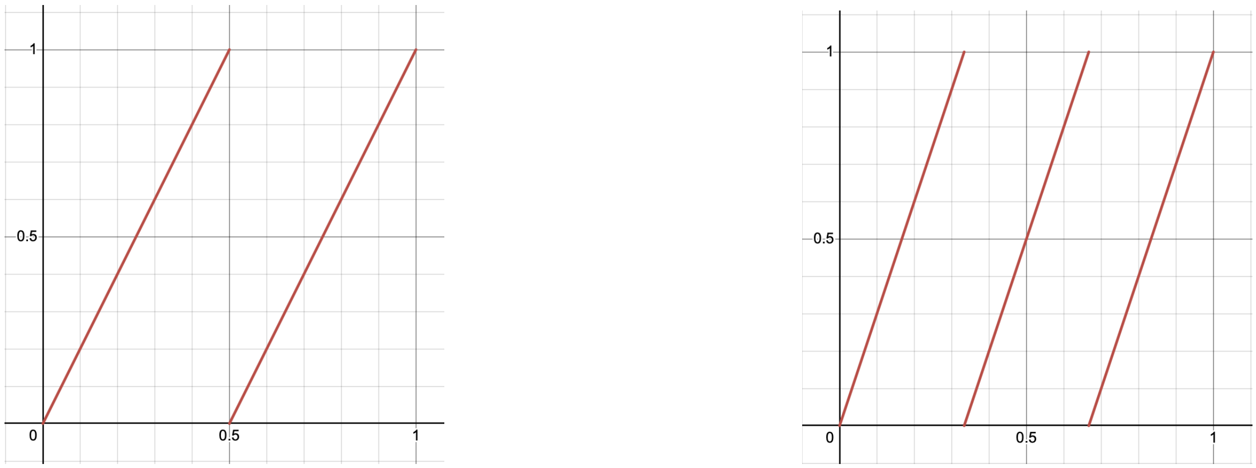

Figure 1.1: The graphs of the expanding maps E_2 (left) and E_3 (right) on the interior of $\mathbb{S}^1$, represented by the interval $(0, 1)$.

Note that ϕ is a fixed point of E_m if and only if $m\phi - \phi \in \mathbb{Z}$. In other words, there exists $n \in \mathbb{Z}$ such that

$$\begin{aligned} m\phi - \phi &= n \\ \iff \phi &= \frac{n}{m-1} \end{aligned}$$

Similarly, ϕ is a periodic point of E_m of period dividing k if and only if there exists $n \in \mathbb{Z}$ such that

$$\begin{aligned} m^k\phi - \phi &= n \\ \iff \phi &= \frac{n}{m^k-1} \end{aligned}$$

In other words,

$$\begin{aligned} \mathrm{Fix}(E_m) &= \left\{\frac{1}{m-1}, \frac{2}{m-1} \cdots, \frac{m-2}{m-1}\right\} \\ \mathrm{Fix}\left(E_m^{\circ k}\right) &= \left\{\frac{1}{m^k-1}, \frac{2}{m^k-1} \cdots, \frac{m^k-2}{m^k-1}\right\} \end{aligned}$$

Torus Endomorphisms

Given $n \in \mathbb{N}$, the $n-$torus is the space $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \mathbb{R}^n/\sim$ where $x \sim y$ if $x - y \in \mathbb{Z}^n$. For $x \in \mathbb{R}^n$, we let $[x]$ denote the equivalence class of x in $\mathbb{T}^n$.

Definition 1.2.4. Let A be an $n \times n$ matrix whose entries are in $\mathbb{Z}$. Then A induces the *torus endomorphism* $T_A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ given by

$$T_A([x]) = [Ax] \text{ for } [x] \in \mathbb{T}^n$$

Exercise 1.2.5. Show that T_A as given above is well-defined: that is, for any two vectors $v, w \in \mathbb{R}^n$, if $v - w \in \mathbb{Z}^n$, then $Av - Aw \in \mathbb{Z}^n$

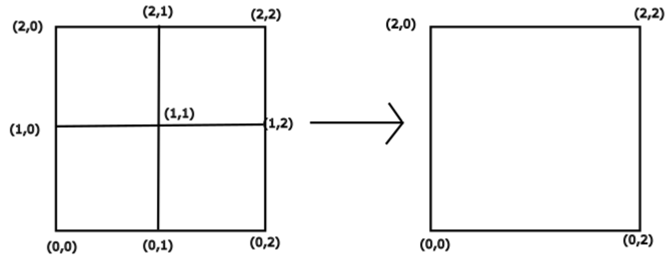

Figure 1.2: An illustration of the torus endomorphism $T_A : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ for $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Example 1.2.6. Let $m, k \in \mathbb{Z}$ and consider the matrix $A = \begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix}$. Consider the map T_A on $\mathbb{T}^2$: this acts as two independent expanding maps: expansion by a factor of m in the $x-$direction, and expansion by a factor of k in the $y-$direction (see Figure 1.2 which illustrates the case $m = k = 2$). Can you show in general that the degree of such a map is $d = mk$? In other words, T_A is a $d : 1$ map of $\mathbb{T}^2$.

Definition 1.2.7. A torus endomorphism T_A is said to be an *automorphism* if it is invertible.

Exercise 1.2.8. (This is also on HW 1) Show that T_A is invertible if and only if A^{-1} has integer entries, which in turn is equivalent to $\det A = \pm 1$.

Proposition 1.2.9. *Let $T_A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ be a torus automorphism with no eigenvalues of modulus* 1*. Then the periodic points of T_A are all the points with rational coordinates.*

Proof. (periodic $\implies$ rational):
Let $[x] = [(x_1, x_2, \cdots, x_n)] \in \mathbb{T}^n$ be a periodic point of period q for some $q \in \mathbb{N}$. Then $T_A^{\circ q}([x]) = [A^q x] = [x]$. That is, there exists a vector $y \in \mathbb{Z}^n$ such that

$$\begin{aligned} A^q x &= x + y \\ \implies A^q x - x &= y \\ \implies (A^q - \mathrm{Id})x &= y \end{aligned}$$

Since A has no eigenvalues of modulus 1, the matrix A^q has no eigenvalues of modulus 1. This means that the matrix $A^q - \mathrm{Id}$ is invertible. So

$$x = (A^q - \mathrm{Id})^{-1} y$$

Since y has integer coordinates and the matrix $(A^q - \mathrm{Id})^{-1}$ has rational entries, x has rational coordinates.

(rational $\implies$ periodic):
Suppose x has rational coordinates, we can assume that all the coordinates have a common denominator. In other words, $x = \left(\frac{p_1}{r}, \frac{p_2}{r}, \cdots, \frac{p_n}{r}\right)$ for some integers p_i, r with $r \neq 0$. Given a $q \in \mathbb{N}$, since A has integer entries, $A^q x = \left(\frac{p'_1}{r}, \frac{p'_2}{r}, \cdots, \frac{p'_n}{r}\right)$ for some integers $p'_1, \cdots, p'_n$.

Note that there are only finitely many points in $\mathbb{T}^n$ with rational coordinates with a common denominator r. In other words, the set $\{T_A^{\circ q}([x]) : q \in \mathbb{N}\}$ is finite.

Thus, there exist $q_1 < q_2 \in \mathbb{N}$ such that $T_A^{\circ q_1}([x]) = T_A^{\circ q_2}([x])$. Since T_A is an automorphism, this means that $T_A^{\circ(q_2 - q_1)}([x]) = [x]$. □

1.3 Stable Behavior: The Contraction Principle

In this section we will look at maps on subsets of $\mathbb{R}^n$ which satisfy a criterion for all orbits converging to a fixed point.

Global Contractions

Definition 1.3.1. A map f of a subset X of $\mathbb{R}^n$ is said to be *Lipschitz-continuous* with Lipschitz constant λ, or λ-*Lipschitz* if

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for any $x, y \in X$.
The map f is said to be a *contraction* if

$$d(f(x), f(y)) < d(x, y) \qquad \forall x, y \in X.$$

It is said to be a $\lambda-$*contraction* for $\lambda < 1$ if

$$d(f(x), f(y)) \leq \lambda d(x, y) \qquad \forall x, y \in X.$$

Remark 1.3.2. If a map f is Lipschitz-continuous, then we define

$$Lip(f) := \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}$$

Example 1.3.3. $f(x) = \sqrt{x}$ defines a contraction on $I = [1, \infty)$. What is $Lip(f)$?

Theorem 1.3.4 (Contraction Principle in $\mathbb{R}^n$)**.** *Let $X \subset \mathbb{R}^n$ be closed and $f : X \longrightarrow X$ be a $\lambda-$contraction. Then f has a unique fixed point x_0 and $d(f^{\circ n}(x), x_0) = \lambda^n d(x, x_0)$ for every $x \in X$.*

Proof. We have

$$d(f^{\circ n}(x), f^{\circ n}(y)) \leq \lambda^n d(x, y)$$

for all $x, y \in X$. But this also means that for any $x \in X$, we have

$$d(f^{\circ(n+1)}(x), f^{\circ n}(x)) \leq \lambda^n d(f(x), x)$$

$$\begin{aligned} d(f^{\circ m}(x), f^{\circ n}(x)) &\leq d(f^{\circ m}(x), f^{\circ(m-1)}(x)) + d(f^{\circ(m-1)}(x), f^{\circ(m-2)}(x)) + \cdots d(f^{\circ(n+1)}(x), f^{\circ n}(x)) \\ &\leq \left(\lambda^{m-1} + \lambda^{m-2} + \cdots \lambda^n\right) d(f(x), x) \\ &\leq \frac{\lambda^n(1-\lambda^{m-n})}{1-\lambda} d(f(x), x) \\ &\leq \frac{\lambda^n}{1-\lambda} d(f(x), x) \end{aligned}$$

In other words, the orbit of x is a Cauchy sequence. Since X is closed, $\lim_{n\to\infty} f^{\circ n}(x) = x_0$ is a point of X, and

$$f(x_0) = f(\lim_{n\to\infty} f^{\circ n}(x)) = \lim_{n\to\infty} f^{\circ(n+1)}(x_0) = x_0$$

□

Remark 1.3.5. Given a sequence $(y_n)_{n\geq 0}$ in a metric space (Y, d), we say that $y_n \to y \in Y$ *exponentially* if there exist constants $A > 0$ and $0 < \lambda < 1$ such that

$$d(y_n, y) \leq A\lambda^n d(y_0, y)$$

Note that in the above situation, the orbit under f of x converges exponentially to x_0 (here $A = 1$).

The contraction principle applies to $\lambda-$contractions defined on complete metric spaces.

Theorem 1.3.6 (Contraction Principle for complete metric spaces)**.** *Let (X, d) be a complete metric space and $f : X \longrightarrow X$ be a $\lambda-$contraction. Then there exists a unique fixed point $x_0 \in X$ such that the orbits under f of all points $x \in X$ converge exponentially to x_0.*

Example 1.3.7 (Rabbits; due to Fibonacci). Say we record the number of rabbits in a forest starting January (month 0) of a given year. The Fibonacci model for rabbit population growth is as follows:
Letting b_n denote the number (in hundreds) of rabbits at the beginning of month n, we assume

$$\begin{aligned} b_0 &= 1 \\ b_1 &= 2 \\ b_n &= b_{n-1} + b_{n-2} \text{ for } n \geq 2. \end{aligned}$$

Then it is expected that the rabbit population growth rate stabilises as $n \to \infty$. That is, there exists $a \in (0, \infty)$ such that $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = b$. This means that in the long term, the rabbit population grows approximately *exponentially*, by a rough factor of b each month.

To prove the existence of a value a as required, we let $a_n = \frac{b_{n+1}}{b_n}$. Note that

$$a_{n+1} = \frac{b_{n+2}}{b_{n+1}} = \frac{b_{n+1} + b_n}{b_{n+1}} = 1 + \frac{b_n}{b_{n+1}} = 1 + \frac{1}{a_n}$$

Letting $g(x) = 1 + \frac{1}{x}$, we see that

$$a_{n+1} = g^{\circ n}(a_0) = g^{\circ n}(2) \text{ for all } n \geq 0$$

Claim: There exists a closed interval $I \subset \mathbb{R}$ such that $a_0 = 2$ such that

- $g(I) \subset I$
- g is a λ-contraction on I for some $\lambda \in (0, 1)$, and
- $a_0 = 2 \in I$

If we can prove this claim, then by the contraction principle, a can be recovered as the unique fixed point of g in I.

Proof of claim: The function g is decreasing on $(0, \infty)$, and has the horizontal asymptote $y = 1$. Note that $g'(x) = \frac{-1}{x^2}$.

This means $\forall x \in [c, \infty)$ where $c > 1$,

$$|g'(x)| = \frac{1}{x^2} \leq \frac{1}{c^2} < 1$$
$$\implies |g(x) - g(y)| \leq \frac{1}{c^2}|x - y| \text{ for all } x, y \in [c, \infty)$$

In other words, for all $c > 1$, $g : [c, \infty) \longrightarrow \mathbb{R}$ is a $\lambda-$contraction with $\lambda = \frac{1}{c^2}$. Also note g has a unique positive fixed point x_0: we can find it by solving the equation $g(x) = x$.

$$g(x) = x$$
$$\implies 1 + \frac{1}{x} = x$$
$$\implies x^2 - x + 1 = 0$$
$$\implies x = \frac{1 \pm \sqrt{5}}{2}$$

So $x_0 = \frac{1+\sqrt{5}}{2}$. Note that $\frac{3}{2} < x_0 < 2$.
Let $I = [\frac{3}{2}, 2]$, Then we have

$$g(2) = \frac{3}{2} \text{ and } g(3/2) = \frac{5}{3} < 2.$$

we see that $g(I) \subset I$. By the above discussion, g is a $\lambda-$contraction on I (with $\lambda = (2/3)^2$), and thus the orbit under g of $a_0 = 2$ converges to $a = x_0 = \frac{1+\sqrt{5}}{2}$.

Remark 1.3.8. The choice of I is not unique: for any $c \in (1, 3/2]$, we have $g[c, 2] = [3/2, g(c)] \subset [c, 2]$, and g is a λ contraction on $[c, 2]$ with $\lambda = \frac{1}{c^2}$. I made a small mistake in class by saying c can be in $[1, x_0]$: can you see why $c \in (3/2, x_0]$ won't work?

Local Contractions

Proposition 1.3.9. *Let f be a continuously differentiable map of $\mathbb{R}^n$ with a fixed point x_0 where $||Df_{x_0}|| < 1$. Then there is a closed neighborhood U of x_0 such that $f(U) \subset U$ and f is a contraction on U.*

To do this we will need the following exercise and proposition:

Exercise 1.3.10. Given a linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, recall that

$$||A|| = \sup_{||x|| \neq 0} \frac{||Ax||}{||x||}$$

Prove that $A \mapsto ||A||$ is a continuous function from $\mathbb{R}^{m\times n}$ to $\mathbb{R}$.

Proposition 1.3.11. *Let $V \subset \mathbb{R}^n$ be a closed disk and let $f : V \longrightarrow \mathbb{R}^m$ be continuous, with continuous derivative on the interior of V. Suppose there exists $M > 0$ such that $||Df_x|| \leq M$ for all x in the interior of V. Then*

$$d(f(x), f(y)) \leq M d(x, y) \qquad \forall x, y \in V$$

Proof. Given $x, y \in \mathbb{R}^n$, let $g : [0, 1] \longrightarrow \mathbb{R}^m$ be the function

$$g(t) = f((1-t)x + ty)$$

Then the mean value theorem states that for some $c \in (0, 1)$,

$$d(g(0), g(1)) \leq ||g'(c)||$$

From this we get

$$\begin{aligned} d(f(x), f(y)) &= d(g(0), g(1)) \leq ||g'(c)|| \\ &= ||Df_{(1-c)x+cy}(y-x)|| \leq ||Df_{(1-c)x+cy}|| \cdot d(x, y) \\ &\leq M d(x, y) \end{aligned}$$

□

Proof of Proposition 1.3.9. The function f is C^1 implies that $x \mapsto Df$ is continuous. By Exercise 4, the composition $x \mapsto Df_x \mapsto ||Df_x||$ is continuous. Fix a point $\lambda \in \Big(||Df_{x_0}||, 1\Big)$. Then there exists a small closed ball $U = \overline{B(x_0, \delta)}$ around x_0 on which $||Df_x|| \leq \lambda < 1$.

By Proposition 1.3.11, if $x, y \in U$, then $d(f(x), f(y)) \leq \lambda d(x, y)$. Moreover, for all $x \in U$, we have

$$d(f(x), x_0) = d(f(x), f(x_0)) \leq \lambda d(x, x_0) \leq \lambda\delta < \delta.$$

This shows that $f(U) \subset U$, and f is a $\lambda-$contraction on U.

□

1.4 Fractals

In this section we will define fractals and introduce self-similarity. We will also give an idea of their connection with dynamical systems with some examples.

The Cantor Set

The simplest example of a fractal is the ternary cantor set.

Definition 1.4.1. Let $I = [0,1]$. Inductively define closed subsets $C_n \subset I$ for $n \geq 0$ as follows:

$$\begin{aligned}
\mathcal{C}_0 &= \Big[0,1\Big] \\
\mathcal{C}_1 &= \mathcal{C}_0 \setminus \left(\frac{1}{3},\frac{2}{3}\right) = \frac{\mathcal{C}_0}{3} \cup \left(\frac{\mathcal{C}_0}{3}+\frac{2}{3}\right) \\
\mathcal{C}_n &= \frac{\mathcal{C}_{n-1}}{3} \cup \left(\frac{\mathcal{C}_{n-1}}{3}+\frac{2}{3}\right) = \mathcal{C}_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n},\frac{3k+2}{3^n}\right) \text{ for all } n \geq 2
\end{aligned}$$

It can be shown that the set $\mathcal{C}_n$ is the disjoint union of 2^n closed intervals, each of length $\frac{1}{3^n}$. The ternary Cantor set is defined as

$$\mathcal{C} = \bigcap_{n=0}^{\infty} \mathcal{C}_n$$

We look at some of the properties of $\mathcal{C}$.

1. $\mathcal{C}$ **is closed**, since it is the intersection of closed sets
2. $\mathcal{C}$ **is compact**, since it is a closed subset of a compact set.
3. $\mathcal{C}$ **is non-empty.**
   For example, the points 0 and 1 belong to all the sets $\mathcal{C}_n$, so they also belong to $\mathcal{C}$.
4. $\mathcal{C}$ **is uncountable.**
   We will show this by giving an explicit description of $\mathcal{C}$.

   Definition 1.4.2. Given a number $x \in [0,1]$, a base 3 (or *ternary*) expansion for x is a sequence $.\alpha_1\alpha_2\alpha_3\cdots$ with $\alpha_n \in \{0,1,2\}$ for all $n \in \mathbb{N}$ such that

   $$x = \frac{\alpha_1}{3} + \frac{\alpha_2}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}$$

   The decimal point before the α's indicate that the number is less than or equal to 1. More generally, for any real number $y \in \mathbb{R}$. The ternary expansion of y is a sequence $\beta_{m-1}\beta_{m-2}\cdots\beta_0.\alpha_1\alpha_2\alpha_3\cdots$ with $\beta_i \in \{0,1,2\}$ for all $i \in \{0,1,2,\cdots m-1\}$ and $\alpha_n \in \{0,1,2\}$ for all $n \in \mathbb{N}$, such that

   $$y = \sum_{i=0}^{m-1} \beta_i \cdot 3^i + \sum_{n=0}^{\infty} \frac{\alpha_n}{3^n}$$

Note that the ternary expansion of a number is not unique. For example,

$$\frac{1}{3} = \sum_{n=1}^{\infty} \frac{2}{3^n}$$

So $.100000\cdots$ and $.0222222\cdots$ are both ternary expansions for $\frac{1}{3}$. Similarly, $.200000\cdots$ and $.1222222\cdots$ are both ternary expansions for $\frac{2}{3}$.

Exercise 1.4.3. Every number $x \in \mathbb{R}$ has only finitely many ternary expansions.

Remark 1.4.4. If x has a ternary expansion $.\alpha_1\alpha_2\alpha_3\cdots$, then $\frac{x}{3}$ has a ternary expansion $.0\alpha_1\alpha_2\cdots$

Proposition 1.4.5.

$$\mathcal{C} = \{x \in [0,1] : \textit{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \textit{ for } x \textit{ with } \alpha_n \in \{0,2\} \textit{ for all } n \in \mathbb{N}\}$$

Proof. We prove this by induction. Note that

$$\mathcal{C}_1 = \{x \in [0,1] : \text{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \text{ for } x \text{ with } \alpha_1 \in \{0,2\}\}$$

Using the recursive formula for $\mathcal{C}_n$, it is easy to show that

$$\mathcal{C}_N = \{x \in [0,1] : \text{there exists a ternary expansion } .\alpha_1\alpha_2\cdots \text{ for } x \text{ with } \alpha_1, \alpha_2, \cdots, \alpha_N \in \{0,2\}\}$$

Since every x has only finitely many ternary expansions, for x to be in all the $\mathcal{C}_n$'s, there exists at least one ternary expansion which satisfies the condition $\alpha_n \in \{0,2\}$ for all $n \in \mathbb{N}$. □

Since every sequence in $\{0,2\}^{\mathbb{N}}$ can be realized as the ternary expansion of a distinct number $x \in [0,1]$, and the set $\{0,2\}^{\mathbb{N}}$ is uncountable, we see that $\mathcal{C}$ is uncountable.

5. **$\mathcal{C}$ is perfect.**

Definition 1.4.6. Let Y be a topological space. A subset $X \subset Y$ is said to be perfect if it is closed in Y and has no isolated points.

Proposition 1.4.7. *For every $x \in \mathcal{C}$, there exists a sequence (x_n) of distinct points with $x_n \in \mathcal{C}$ and $x_n \to x$.*

Proof. Given $\epsilon > 0$, we will exhibit a point $x_N \neq x$ such that $|x_N - x| < \epsilon$ and $x_N \in \mathcal{C}$. Choose $N \in \mathbb{N}$ such that $\frac{2}{3^N} < \epsilon$. This ensures that the interval of radius $\frac{1}{3^N}$ centered at x is contained in the open ball $B_\epsilon(x)$. Let $\widetilde{I}$ be the component interval of $\mathcal{C}_N$ that contains x. The above condition implies that $\widetilde{I} \subset B_\epsilon(x)$.

In $\mathcal{C}_{N+1}$, the middle third of $\widetilde{I}$ is deleted, and we get two component intervals $\widetilde{I}_0$ and $\widetilde{I}_1$. Without loss of generality, assume $x \in \widetilde{I}_0$. Then pick a point $y \in \mathcal{C} \cap \widetilde{I}_1$. Note that $y \neq x$ by this choice and since $\widetilde{I}_1 \subset B_\epsilon(x)$, we have $|y - x| < \epsilon$. Therefore we can set $x_N = y$. □

6. $\mathcal{C}$ **is totally disconnected.**

   Definition 1.4.8. A topological space X is said to be totally disconnected if its only non-empty connected subsets are singletons.

   Proposition 1.4.9. *If $F \subset \mathcal{C}$ is non-empty and connected, then $F = \{x\}$ for some point $x \in \mathcal{C}$.*

   Proof. Suppose $x, y \in \mathcal{C}$ are two distinct points in F. WLOG, assume $x < y$. Pick $N \in \mathbb{N}$ such that $\frac{1}{3^{N-1}} < |x - y|$. Then, x and y are contained in distinct components of C_N. So there exists $z \in (x, y)$ such that $z \notin \mathcal{C}$. Let $A = F \cap [0, z)$ and $B = F \cap (z, 1]$. Note that $A \cup B = F$. Also note that the closures of A and B don't intersect. This contradicts the fact that F is connected. $\square$

7. $\mathcal{C}$ **has Lebesgue measure** 0**.**
   Let μ denote Lebesgue measure. The set $\mathcal{C}_n$ is the union of 2^n disjoint intervals, each of length 3^{-n}. Therefore, we have $\mu(\mathcal{C}_n) = \frac{2^n}{3^n}$. Since $\mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3 \supset \cdots$ and $\mathcal{C}$ is the intersection of the $\mathcal{C}_n$, we have

$$\mu(\mathcal{C}) = \lim_{n\to\infty} \mu(\mathcal{C}_n) = 0$$

The following theorem is the main result of this section. For a proof see [1].

Theorem 1.4.10 (Brouwer)**.** *Let $Y \neq \emptyset$ be a complete metric space. If Y is compact, perfect and totally disconnected, then it is homeomorphic to $\mathcal{C}$.*

An easy corollary, for example, is that $\mathcal{C}$ is homeomorphic to $\mathcal{C} \times \mathcal{C}$.

Dynamical Systems on the Cantor Set

Consider the map $f : [0, 1] \times [0, 1]$ defined as $f(x) = \frac{x}{3}$. It is easy to see that f is a contraction and $f(\mathcal{C}) \subset \mathcal{C}$, and the unique fixed point in $\mathcal{C}$ is $x = 0$. Note that for every $x \in \mathcal{C}$, there exists a neighborhood U of x such that $f : U \longrightarrow f(U)$ is a homeomorphism.
Also note that f induces the shift $.\alpha_1\alpha_2\alpha_3 \cdots \to .0\alpha_1\alpha_2\alpha_3 \cdots$ on ternary expansions.

Definition 1.4.11. A topological space X is said to be *self-similar*, or to have the *rescaling property*, if there exists a contraction $f : X \longrightarrow X$ such that for every $x \in X$ and neighborhood U of x, there exists a neighborhood $V \subset U$ of x such that $f : V \longrightarrow f(V)$ is a homeomorphism.

Remark 1.4.12. This is actually equivalent to saying that every $x \in X$ has a neghborhood U such that $f : U \longrightarrow f(U)$ is a homeomorphism. Also note that the term self similar is used in different ways in the literature; we will see by and by that this definition is not extensive enough.

Exercise 1.4.13. Show that the function $f(x) = 1 - \frac{x}{3}$ leaves $\mathcal{C}$ invariant, and is a contraction. Describe the induced operation on ternary expansions, and find the unique fixed point of f in $\mathcal{C}$.

Exercise 1.4.14. Show that the function $f(x) = \frac{x+2}{3} \pmod 1$ leaves $\mathcal{C}$ invariant, and is a contraction. Describe the induced operation on ternary expansions, and find the unique fixed point of f in $\mathcal{C}$.

The Square Sierpinksi Carpet

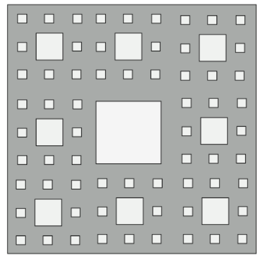

Figure 1.3: The set $\mathcal{J}_3$ of the Sierpinski carpet construction

Definition 1.4.15. Let $J = [0, 1] \times [0, 1]$ be the unit square. Define

$$\begin{aligned}
\mathcal{J}_0 &= J \\
\mathcal{J}_1 &= \mathcal{J}_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) \times \left(\frac{1}{3}, \frac{2}{3}\right) \\
\mathcal{J}_n &= \mathcal{J}_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \bigcup_{\ell=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) \times \left(\frac{3\ell+1}{3^n}, \frac{3\ell+2}{3^n}\right)
\end{aligned}$$

The square Sierpinski carpet is the set $\mathcal{J} = \bigcap_{n=0}^{\infty} \mathcal{J}_n$.

Exercise 1.4.16. Prove that the Sierpinski carpet is self-similar.

The Sierpinski Triangle

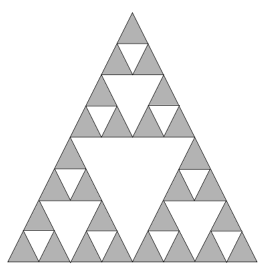

Figure 1.4: The set Δ_2 of the Sierpinski triangle construction

This set is similar to the Sierpinski carpet. Start with an equilateral triangle Δ_0 of side length 1, with one side horizontal. Let Δ_1 be Δ_0 minus its central equilateral triangle, Δ_2 be Δ_1 minus its three smaller central equilateral triangles and so on.
Define the Sierpinski triangle

$$\Delta = \bigcap_{n=0}^{\infty} \Delta_n$$

Exercise 1.4.17. Prove that the Sierpinski triangle is self-similar.

Exercise 1.4.18. Prove that $\mathcal{J}$ and Δ both have infinite perimeter and finite area (Lebesgue measure).

Exercise 1.4.19. Prove that neither $\mathcal{J}$ nor Δ is homeomorphic to $\mathcal{C}$.

Cantor Sets and Logistic Maps

Consider the logistic function $f(x) = 5x(1-x)$ on $\mathbb{R}$. Note that this function has one critical point at $x = \frac{1}{2}$, and is symmetric around this point in the sense that

$$f(x) = f(1-x) \text{ for all } x \in \mathbb{R}$$

We make the following series of observations:

1. The graph of f is a downward drawn parabola, and its roots are $x = 0, 1$.
2. If $x > 1$, then $f(x) < 0$.
3. If $x < 0$, then $f(x) < x$ and $|f(f(x)) - f(x)| > |f(x) - x|$.

The points (2) and (3) show that if $x \notin [0,1]$, then $f^{\circ n}(x) \to -\infty$. This leads to the following dichotomy:
For every $x \in \mathbb{R}$, exactly one of the following is true:

- either $f^{\circ n}(x) \in [0,1]$ for all $n \in \mathbb{N}$, or
- $f^{\circ m}(x) \notin [0,1]$ for some $m \in \mathbb{N}$, and thus, $f^{\circ n}(x) \to -\infty$ as $n \to \infty$.

Therefore, the set B of points $x \in \mathbb{R}$ such the orbit $(f^{\circ n}(x))_{n \geq 0}$ is bounded, is the set of points x such that $f^{\circ n}(x) \in [0,1]$ for all n. In other words,

$$B = \bigcap_{n=0}^{\infty} (f^{\circ n})^{-1}[0,1]$$

Proposition 1.4.20. *B is a Cantor set (i.e., it is homeomorphic to $\mathcal{C}$).*

We will prove this in the next chapter.

Topological Conjugacy

Definition 1.4.21. Let X, Y be topological spaces and suppose $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$ be dynamical systems. Then (X, f) and (Y, g) are said to be *topologically conjugate* if there exists a homeomorphism $\varphi : X \longrightarrow Y$ such that

$$g \circ \varphi = \varphi \circ f$$

In other words, φ is a homeomorphism that makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

Note that for all $n \in \mathbb{N}$,

$$g^{\circ n} = (\varphi \circ f \circ \varphi^{-1})^{\circ n} = (\varphi \circ f \circ \varphi^{-1}) \circ (\varphi \circ f \circ \varphi^{-1}) \circ \cdots \circ (\varphi \circ f \circ \varphi^{-1}) = \varphi \circ f^{\circ n} \circ \varphi^{-1}$$

In particular, φ maps the $f-$orbit of x to the $g-$orbit of $\varphi(x)$ for every $x \in X$. So topological conjugacy is a form of equivalence between two dynamical systems.

Example 1.4.22. The map $\varphi(x) = \frac{-1}{2}x + \frac{1}{2}$ conjugates the dynamical systems $f(x) = x^2$ and $g(x) = 2x(1-x)$ on $\mathbb{R}$. Since φ is linear, we say that $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$ are *linearly/affinely* conjugate. This is stronger than topological conjugacy.

We now revert back to our previous discussion of Cantor sets and the logistic map $f(x) = 5x(1-x)$.

Let $\Sigma = \{0,1\}^{\mathbb{N}} = \{s = s_1 s_2 \cdots | s_i \in \{0,1\} \forall i \in \mathbb{N}\}$ and $\sigma : \Sigma \to \Sigma$ be the map $s_1 s_2 s_3 \cdots \mapsto s_2 s_3 \cdots$. We will equip Σ with a topology under which σ is continuous.

In the next chapter, we will show the following results:

Theorem 1.4.23. *Given $\mu \neq 0$, let $g_\mu(x) = \mu x(1-x)$, and let $B_\mu \subset \mathbb{R}$ be the set of points with bounded orbits under g_μ.*

For $\mu > 4$, the dynamical systems (B_μ, g_μ) and (Σ, σ) are topologically conjugate.

Theorem 1.4.24. *The set Σ is homeomorphic to the ternary Cantor set $\mathcal{C}$.*

These theorems imply Proposition 1.4.20.

Chapter 2

Symbolic Dynamics

To prove Theorems 1.4.23 and 1.4.24, we will need the powerful machinery of symbolic dynamics. In the next section we will introduce its basic concepts.

2.1 Sequences over a finite alphabet

Let (X, d_X) be a metric space, and $A \subset X$ be a finite set with $|A| \geq 2$.

Definition 2.1.1. The set of sequences with alphabet A is denoted Σ_A. In other words,

$$\Sigma_A = A^{\mathbb{N}} = \{s = s_1 s_2 s_3 \cdots | s_j \in A \forall j \in \mathbb{N}\}$$

Topology on the Space of Sequences

We define a metric on Σ_A as follows: for all $s, t \in \Sigma_A$, we let

$$d(s,t) = \sum_{j=1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}} \tag{2.1}$$

Proposition 2.1.2. *The function $d : \Sigma_A \times \Sigma_A \longrightarrow \mathbb{R}$ is well-defined.*

Proof. We need to show that for all $s, t \in \Sigma_A$, the infinite series given above converges. Let $M = \max_{p,q \in A} d_X(p, q)$.

$$\begin{aligned} d(s,t) &= \sum_{j=1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}} \\ &\leq \sum_{j=1}^{\infty} \frac{M}{|A|^{j-1}} = M \sum_{j=1}^{\infty} \frac{1}{|A|^{j-1}} \\ &= \frac{M}{1 - \frac{1}{|A|}} \\ &< \infty \end{aligned}$$

$\square$

Proposition 2.1.3. *The function defined by Equation 2.1 is a metric on Σ_A.*

Proof. We need to show the following:

1. for all $s, t \in \Sigma_A$, $d(s,t) \geq 0$;
2. for all $s, t \in \Sigma_A$, $d(s,t) = 0 \iff s = t$;
3. for all $s, t, r \in \Sigma_A$, $d(s,r) \leq d(s,t) + d(t,r)$.

We will see these one by one.

1. is clear since every term in the infinite sequence defining $d(s,t)$ is non-negative.
2. is clear since if $d(s,t) = 0$, then $d_X(s_j, t_j) = 0$ for all $j \in \mathbb{N}$, which implies $s_j = t_j$ for all $j \in \mathbb{N}$.
3. for every $j \in \mathbb{N}$, $d_X(s_j, r_j) \leq d_X(s_j, t_j) + d_X(t_j, r_j)$. This immediately shows (3).

□

The metric d induces a topology on Σ_A. We will see some properties of this topology in the remaining section.

Remark 2.1.4. By scaling the metric d_X if necessary, from now on we assume without loss of generality that $M = \max_{p,q \in A} d_X(p,q) = 1$.

Proposition 2.1.5. *Suppose $s, t \in \Sigma_A$ satisfy $s_j = t_j$ for $j = 1, 2, \cdots, N$. Then*

$$d(s,t) < \frac{1}{|A|^{N-1}(|A|-1)} \leq \frac{1}{|A|^{N-1}}$$

Proof. The second inequality follows directly since $\frac{1}{|A|-1} \leq 1$.
Since $s_j = t_j$ for $j = 1, \cdots, N$,

$$\begin{aligned}
d(s,t) &= \sum_{j=1}^{N} \frac{d_X(s_j,t_j)}{|A|^{j-1}} + \sum_{j=N+1}^{\infty} \frac{d_X(s_j,t_j)}{|A|^{j-1}} \\
&= \sum_{j=N+1}^{\infty} \frac{d_X(s_j,t_j)}{|A|^{j-1}} \\
&\leq \sum_{j=N+1}^{\infty} \frac{1}{|A|^{j-1}} \\
&= \frac{1}{|A|^N} \frac{1}{1-\frac{1}{|A|}} = \frac{|A|}{|A|^N(|A|-1)} \\
&= \frac{1}{|A|^{N-1}(|A|-1)}
\end{aligned}$$

□

Proposition 2.1.6. *There exists a constant $\ell = \ell(A) > 0$ such that if $s, t \in \Sigma_A$ satisfy $d(s,t) < \frac{\ell}{|A|^{N-1}}$, then $s_j = t_j$ for $j = 1, 2, \cdots, N$.*

Proof. Let $\ell = \min_{\substack{p,q \in A \\ p \neq q}} d_X(p,q)$. We will prove the contrapositive. Given $s, t \in \Sigma_A$, if $s_j \neq t_j$ for some $j \in \{1, 2, \cdots, N\}$, then

$$d(s,t) \geq \frac{d_X(s_j, t_j)}{|A|^{j-1}} \geq \frac{\ell}{|A|^{j-1}} \geq \frac{\ell}{|A|^{N-1}}$$

□

Shift Operator on Sequences

Definition 2.1.7. The *shift operator* $\sigma : \Sigma_A \longrightarrow \Sigma_A$ is defined as

$$\sigma(s_1 s_2 s_3 \cdots) = s_2 s_3 s_4 \cdots \text{ for all } s = s_1 s_2 s_3 \cdots \in \Sigma_A \tag{2.2}$$

Proposition 2.1.8. *The map σ is surjective and uniformly continuous.*

Proof. Given $s \in \Sigma_A$, for any $a \in A$, $\sigma(a s_1 s_2 s_3 \cdots) = s$. Therefore σ is surjective.
To show it is uniformly continuous, we will exhibit for a given $\epsilon > 0$, a constant $\delta > 0$ such that for all $s, t \in \Sigma_A$, $d(s,t) < \delta \implies d(\sigma(s), \sigma(t)) < \epsilon$.

Fix $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{|A|^{N-1}} < \epsilon$. Let $\delta = \frac{\ell}{|A|^N}$, where ℓ is the constant from Proposition 2.1.6. Then, we have $s_j = t_j$ for $j = 1, 2, \cdots, N+1$. Let $\sigma(s) = \underline{s}$ and $\underline{t} = \sigma(t)$. Note that $\underline{s}_j = s_{j+1}$ and $\underline{t}_j = t_{j+1}$ for all $j \in \mathbb{N}$. The above condition implies that $\underline{s}_j = \underline{t}_j$ for $j = 1, 2, \cdots, N$. Therefore by Proposition 2.1.5, we have $d(\underline{s}, \underline{t}) < \frac{1}{|A|^{N-1}} < \epsilon$. □

Periodic Sequences

Definition 2.1.9. For $m \in \mathbb{N}$, define

$$\mathrm{Per}_m(\sigma) = \{s \in \Sigma_A | \sigma^{\circ m}(s) = s\}$$

In other words, $\mathrm{Per}_m(\sigma)$ is the set of sequences whose period under σ divides m.
Also define $\mathrm{Per}(\sigma)$ to be the set of sequences periodic under σ.

Remark 2.1.10. The following properties of periodic sequences are immediate.

1. $$\mathrm{Per}(\sigma) = \bigcup_{m=1}^{\infty} \mathrm{Per}_m(\sigma)$$

2. Given a finite word $w = s_1 s_2 \cdots s_n$ with $s_i \in A$ for all i, we let $\overline{w}$ denote the infinite word $s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n \cdots$ formed by repeating the finite block w.
   Given $m \in \mathbb{N}$,

   $$\mathrm{Per}_m(\sigma) = \{\overline{s_1 s_2 \cdots s_m} : s_j \in A \text{ for } j = 1, 2, \cdots, m\}.$$

   This shows that

   $$|\mathrm{Per}_m(\sigma)| = |A|^m$$

3. If $m < n$ and $m|n$, then

$$\mathrm{Per}_m(\sigma) \subsetneq \mathrm{Per}_n(\sigma)$$

Proposition 2.1.11. $\mathrm{Per}(\sigma)$ *is dense in* Σ_A.

Proof. Given $s \in \Sigma_A$ and $\epsilon > 0$, pick $N \in \mathbb{N}$ such that $\frac{1}{|A|^{N-1}} < \epsilon$. Then by Proposition 2.1.5, the sequence $t \in \mathrm{Per}(\sigma)$ given by

$$t = s_1 s_2 \cdots s_N s_1 s_2 \cdots s_N s_1 s_2 \cdots = \overline{s_1 s_2 \cdots s_N}$$

satisfies

$$d(s,t) < \frac{1}{|A|^{N-1}} < \epsilon$$

$\square$

Bibliography

[1] Michael Francis. Two topological uniqueness theorems for spaces of real numbers, 2012.