2/12

CONSTRUCTING TOPOLOGIES FROM FUNCTIONS & MORE

Exercise: Let $\{T_\alpha\}_{\alpha \in I}$ be a collection of topologies on a set X. Then their union $\mathcal{A} = \bigcup_\alpha T_\alpha$ is the sub-basis for a topology on X.

Subspace Topology

Def: Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Let $i_A : A \longrightarrow X$ be the standard inclusion $i_A(x) = x$. The "subspace topology on A" is $i_A^{-1}(\mathcal{T})$.

We say $B \subseteq A$ is "open in A" if B is open in the subspace topology, and "closed in A" if $A \setminus B$ is open in the subspace topology.

Prop: In the setting above, $B \subseteq A$ is open $\Leftrightarrow$ $B = U \cap A$ for some open set $U \subseteq X$.

Eg: 1. If $(X, \mathcal{T})$ is a top. space and $A \subseteq X$ is open, then $B \subseteq A$ is open in $A \Leftrightarrow B$ is open in X.

Proof:

Rmk: A is both open and closed in A.

2. In $(\mathbb{R}, d)$, for $A = [a, b)$,
The open sets of A are unions of three kinds of sets:
(i) open intervals (c, d) where $a \leq c < d \leq b$.
(ii) intervals of the form $[a, c)$ where $a < c < b$
(iii) intervals of the form (c, b) where $a < c < b$.

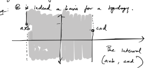
3.

Dictionary Order Topology on $\mathbb{R}^2$

Def: Define the following order on $\mathbb{R} \times \mathbb{R}$:
$a \times b < c \times d$ if either ($*$) $a < c$
or ($*$) $a = c$ and $b < d$.

The interval $(a \times b, c \times d) = \{e \times f : a \times b < e \times f < c \times d\}$.
Let $\mathcal{B} = \{(a \times b, c \times d) : a \times b < c \times d\}$. The dictionary order topology on $\mathbb{R}^2$ is the topology generated by $\mathcal{B}$.

Exercise: $\mathcal{B}$ is indeed a basis for a topology.



The interval $(a \times b, c \times d)$

Product and Box Topology

Exercise: Let $\{\mathcal{T}_\alpha\}_{\alpha \in I}$ be a collection of topologies on a set X. Then $\bigcup_\alpha \mathcal{T}_\alpha$ is a sub-basis for a topology $\mathcal{T}$ on X. $\mathcal{T}$ is the smallest topology containing $\bigcup_\alpha \mathcal{T}_\alpha$.

Def: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces
(For finite products) and $\pi_X : X \times Y \longrightarrow X$, $\pi_Y : X \times Y \longrightarrow Y$ be the maps $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$.
The product topology $\mathcal{T}$ on $X \times Y$ is the smallest topology containing both $\pi_X^{-1}(\mathcal{T}_X)$ and $\pi_Y^{-1}(\mathcal{T}_Y)$.

Prop: Sets of the form $U \times V$ where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ form a basis for $\mathcal{T}$.

Eg: 1. Consider the product topology $\mathcal{T}$ on $\mathbb{R} \times \mathbb{R}$ generated by two copies of the standard topologies on $\mathbb{R}$.
A basis for $\mathcal{T}$ is given by sets of the form $U \times V$ where U, V are open in $\mathbb{R}$.
However, $\because U, V$ are both disjoint unions of open intervals,
$U \times V$ is a disjoint union of sets of the form $(a, b) \times (c, d)$.
[rectangular regions].
In other words, $\mathcal{T}$ is the standard topology on $\mathbb{R}^2$.

2. The product topology on $\mathbb{R}^n$ [w.r.t. n copies of the std. topology on $\mathbb{R}$] is the standard topology on $\mathbb{R}^n$.
(Proof is similar, but we will not do this rigorously).

Proof of prop:

Def: Given a collection of sets $\{X_\alpha\}_{\alpha \in \Lambda}$,
$$X = \prod_{\alpha \in \Lambda} X_\alpha = \{x : \Lambda \longrightarrow \bigcup_{\alpha \in \Lambda} X_\alpha \text{ s.t. } x(\alpha) \in X_\alpha \ \forall \alpha\}$$
every $x \in X$ can be thought of as a tuple of points $x = (x_\alpha)_{\alpha \in \Lambda}$ where $x_\alpha = x(\alpha)$.

Def: Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in \Lambda}$ be a collection of topological spaces, and for each α let $\pi_\alpha : X \longrightarrow X_\alpha$ be the projection $x \longmapsto x_\alpha$.

The product topology on X is the smallest topology containing $\bigcup_\alpha \pi_\alpha^{-1}(\mathcal{T}_\alpha)$.

Prop: The product topology is generated by sets of the form $\prod_{\alpha \in \Lambda} U_\alpha$ where U_α is open in X_α $\forall \alpha \in \Lambda$, and $U_\alpha = X_\alpha$ for all but finitely many values of α.

Def: The "box topology" on X is the topology generated by sets of the form $\prod_{\alpha \in \Lambda} U_\alpha$ where $U_\alpha \in \mathcal{T}_\alpha$ $\forall \alpha \in \Lambda$.

Rmk: In a finite product, product and box topologies coincide. In an infinite product,
(eg) $\prod_{n \in \mathbb{N}} \mathbb{R}$, product topology is generally strictly coarser than the box topology.

2/17

Exercise: Let $(X, \mathcal{T})$ be a topological space and $f : X \longrightarrow Y$ be a function. Then $\{V \subseteq Y : f^{-1}(V) \in \mathcal{T}\}$ is a topology on Y.

Rmk: This is not $\{f(U) \subseteq Y : U \in \mathcal{T}\}$. So we will call this topology $f_* \mathcal{T}$ and not $f(\mathcal{T})$ as you might expect.

Quotient Topology

Def: Let $(X, \mathcal{T})$ be a top. space, X^* be a partition of X into disjoint subsets whose union is X.
Consider the function $p : X \longrightarrow X^*$
s.t. $p(x) \in X^*$ is the unique subset containing x.
The quotient topology on X^* is $p_* \mathcal{T}$:
$U \subseteq X^*$ is open $\Leftrightarrow p^{-1}(U)$ is open.

Eg:

Prop: $p_* \mathcal{T}$ is the largest topology on X^* for which p is continuous.

Metric Topology – revisited

Prop: Let X be a set and $d : X \times X \longrightarrow [0, \infty)$ be a metric.
For each $x \in X$ let $d_x : X \longrightarrow [0, \infty)$
$$d_x(y) = d(x, y)$$
Let $\mathcal{T}$ be the (std) subspace topology on $[0, \infty)$.
Then the metric topology $\mathcal{T}_d$ is the smallest topology containing $\bigcup_{x \in X} d_x^{-1}(\mathcal{T})$

Proof:
