CONTINUOUS FUNCTIONS

Def: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then $f: X \longrightarrow Y$ is said to be continuous if $U = f^{-1}(V)$ is open $\forall\, V \subseteq Y$ open.

Eg: 1. Constant functions between any pair of topological spaces are continuous.

Proof:

2. $f: \mathbb{R} \longrightarrow \mathbb{R}$ with std. topology of the form $f(x) = ax + b$, where $a, b \in \mathbb{R}$ is cts.

3. As proved in an earlier lecture,

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \text{and} \qquad g: \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto x+y \qquad\qquad (x,y) \longmapsto x$$

are continuous.

4. If $(X, \mathcal{T})$ is a discrete top. space, any function $f: X \longrightarrow Y$ is continuous.

Proof:

Prop. Let P and Q be sets and $f: P \longrightarrow Q$ be a function.

Then for any topology $\mathcal{T}$ on Q, the collection $f^{-1}(\mathcal{T}) = \{ f^{-1}(V) : V \in \mathcal{T} \}$ is a topology on P.

Prop: Given top. spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, $f: X \longrightarrow Y$ is continuous $\Longleftrightarrow f^{-1}(\mathcal{T}_Y) \subseteq \mathcal{T}_X$

Proof:

Prop: Given a topological space $(Y, \mathcal{T}_Y)$ and a function $f: X \longrightarrow Y$, the topology $\mathcal{T}_X = f^{-1}(\mathcal{T}_Y)$ is the smallest/coarsest topology on X for which f is continuous.

Proof: