

Homework 1

MA 564

Due February 3, 2026

1. (a) Prove that there cannot exist an uncountable collection of pairwise disjoint open intervals in \mathbb{R} .
(b) Recall that a real number is called *algebraic* if it is the root of a polynomial with integer coefficients. Prove that the set of algebraic real numbers \mathbb{A} is countable.
(c) Given two non-empty sets A and B , B is said to have greater cardinality than A if there exists an injection from A to B , and no injection from B to A . Construct a sequence of infinite sets $(A_n)_{n \in \mathbb{N}}$ so that A_{n+1} has greater cardinality than A_n for all $n \in \mathbb{N}$.
2. (a) Let $C(\mathbb{R})$ denote the set of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Define a map $I : C(\mathbb{R}) \rightarrow C(\mathbb{R})$ as follows:

$$\forall f \in C(\mathbb{R}), \quad I(f)(x) = \int_0^x f(t) dt$$

Is I one-to-one? onto? (No justification needed).

- (b) Let $f : X \rightarrow X$ be a function, and let $f^{\circ n} : X \rightarrow X$ be f composed with itself n times. Suppose for every $x \in X$, there exists $n \in \mathbb{N}$ (possibly dependent on x) such that $f^{\circ n}(x) = x$. Then prove that f is a bijection.
3. Prove the *Cantor-Schroeder-Bernstein Theorem*: Given sets A and B , if $f : A \rightarrow B$ and $g : B \rightarrow A$ are injective maps, then there exists a bijection from A to B (hint: partition A into subsets based on if the cardinality of $g^{-1}(x)$ is odd, even or infinite).
4. Suppose (X, d) is a metric space.
 - (a) Define $d'(x, y) = \min\{1, d(x, y)\}$ for every $x, y \in X$. Prove that d' is a metric on X and that $d \sim d'$.
 - (b) Define $d''(x, y) = \frac{d(x, y)}{1 + d(x, y)}$ for every $x, y \in X$. Prove that d'' is a metric on X and that $d \sim d''$.
5. Let p be a prime number. Define $|\cdot|_p : \mathbb{Q} \rightarrow \mathbb{Q}$ as follows:

$$|0|_p = 0.$$

For every $x \neq 0$ in \mathbb{Q} , x can be uniquely written as $\frac{p^k m}{n}$, where $k, m, n \in \mathbb{Z}$ and p does not divide m or n . Let $|x|_p = \frac{1}{p^k}$.

- (a) Show that $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.
 - (b) Prove that $|\cdot|_p$ is a norm on \mathbb{Q} .
 - (c) Let d_p be the metric induced by the norm $|\cdot|_p$: i.e., $d_p(x, y) = |x - y|_p$ for all $x, y \in \mathbb{Q}$. Calculate $d_2(2^n, 0)$ for all $n \in \mathbb{N}$. What is $\lim_{n \rightarrow \infty} d_2(2^n, 0)$?
 - (d) Give a specific example of x, y, z, p for which $d_p(x, z) < \max\{d_p(x, y), d_p(y, z)\}$.
6. Denote a point in $p \in \mathbb{R}^2$ as $p = (p_x, p_y)$. Consider $d_v : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$d_v(p, q) = \begin{cases} 1 & p_x \neq q_x \text{ or } |p_y - q_y| \geq 1 \\ |p_y - q_y| & p_x = q_x \text{ and } |p_y - q_y| < 1 \end{cases}$$

Prove that d_v is a metric on \mathbb{R}^2 and induces a topology that is finer than the standard topology.

7. For $n \in \mathbb{N}$, let $V^n = \{0, 1\}^n = \{(a_1, \dots, a_n) : a_i \in \{0, 1\} \forall i\}$. That is, V^n is the set of all words of length n over $\{0, 1\}$. Consider $D_H : V^n \times V^n \rightarrow \mathbb{R}$ given by defining $D_H(a, b)$ to be the number of positions where a differs from b (i.e., the cardinality of $\{i : a_i \neq b_i\}$).
- (a) Show that D_H is a metric on V^n . This is called the *Hamming distance* between words.
 - (b) Can you extend D_H to a metric on $\bigcup_{n \in \mathbb{N}} V^n$?
8. Show that the topologies of \mathbb{R}_ℓ and \mathbb{R}_K are not comparable.
9. (a) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on a set X . Show that $\bigcap_\alpha \mathcal{T}_\alpha$ is a topology on X . Is $\bigcup_\alpha \mathcal{T}_\alpha$ a topology on X ?
- (b) Let \mathcal{A} be a basis for a topology on a set X . Show that the topology generated by \mathcal{A} is the intersection of all topologies on X that contain \mathcal{A} .
10. Let X be an uncountable set, and let \mathcal{T} be the cofinite topology on X (i.e., $A \in \mathcal{T} \iff X \setminus A$ is a finite set). Show that no point $x \in X$ has a countable basis of neighborhoods in \mathcal{T} (this would then imply that the topological space (X, \mathcal{T}) is not first countable).