Homework 2

MA 564

Due February 17, 2026

1. Let $\Gamma = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$, and let d be the usual Euclidean metric on $\mathbb{R}^2$.

   - For a point $p = (x, y) \in \Gamma$ with $y > 0$, let $\mathcal{B}_p = \{B_d(p, \epsilon) : 0 < \epsilon < y\}$.
   - For a point $p = (x, 0) \in \Gamma$, let
     $\mathcal{B}_p = \{\{p\} \cup A : A$ is a usual open disc in the upper half-plane tangent to the x-axis at $p\}$.

   (a) Let $\mathcal{B} = \bigcup_{p \in \Gamma} \mathcal{B}_p$. Show that $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on Γ. The topological space $(\Gamma, \mathcal{T})$ is called the "Moore Plane".

   (b) Show that $(\Gamma, \mathcal{T})$ is first countable and separable.

   (c) Is $(\Gamma, \mathcal{T})$ second countable?

   (d) Show that the subspace topology on the $x-$axis is the discrete topology (i.e., every subset of the $x-$axis is open in the subspace topology).

2. Prove the following identities for any subsets A and B of a topological space $(X, \mathcal{T})$.

   (a) $\operatorname{Cl}(A \cup B) = \operatorname{Cl}(A) \cup \operatorname{Cl}(B)$

   (b) $\operatorname{Cl} A = A \cup \partial A$

   (c) $\operatorname{Int} A = A \setminus \partial A = X \setminus \operatorname{Cl}(X \setminus A)$

   (d) $\partial A = \operatorname{Cl}(A) \setminus \operatorname{Int} A$

   (e) The sets $\operatorname{Int} A$, ∂A and $\operatorname{Int}(X \setminus A)$ are disjoint and their union is X.

3. A *pre-order* on a set X is a relation that is reflexive and transitive. A set $A \subseteq X$ is called an *upper set* if for every $a \in A$, if $b \in X$ satisfies $a \leq b$ then $b \in A$. A is called a lower set if for every $a \in A$, if $b \in X$ satisfies $b \leq a$ then $b \in A$. Assume that X is endowed with a topology $\mathcal{T}$.

   (a) Show that the relation $\leq_{\mathcal{T}}$ given by $x \leq_{\mathcal{T}} y$ if $x \in \operatorname{Cl}(\{y\})$ is a pre-order on X. This is called the *specialization pre-order* of $\mathcal{T}$.

   (b) Show that all open sets in $(X, \mathcal{T})$ are upper sets with respect to $\leq_{\mathcal{T}}$ and all closed sets are lower sets.

   (c) Given $x \in X$, let $\uparrow x = \{y \in X : x \leq_{\mathcal{T}} y\}$ and $\downarrow x = \{y \in X : y \leq_{\mathcal{T}} x\}$. Show that $\uparrow x$ is the intersection of all neighborhoods of x and that $\downarrow x = \operatorname{Cl}(\{x\})$.

(d) Is $\uparrow x$ always open? Is $\downarrow x$ always closed?

(e) If $X = \{0,1\}$ and $\mathcal{T} = \{\emptyset, \{1\}, \{0,1\}\}$, the pair $(X, \mathcal{T})$ is a topological space called *Sierpinksi space*. Compute $L(\{0\})$ and $L(\{1\})$, and find the specialization pre-order $\leq_{\mathcal{T}}$.

4. Consider the set $Y = [-1,1]$ in $\mathbb{R}$ equipped with the standard topology. Which of the following sets are open in Y, and which are open in $\mathbb{R}$? No justification needed.

$$A = \{x : \tfrac{1}{2} < |x| < 1\} \qquad B = \{x : \tfrac{1}{2} < |x| \leq 1\}$$

$$C = \{x : \tfrac{1}{2} \leq |x| < 1\} \qquad D = \{x : \tfrac{1}{2} \leq |x| \leq 1\}$$

$$E = \{0 < |x| < 1 \text{ and } \tfrac{1}{x} \notin \mathbb{N}\}$$

5. Let $\mathbb{R}_0$ denote $\mathbb{R}$ with the discrete topology and $\mathbb{R}_1$ denote $\mathbb{R}$ with the standard topology. Show that the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_0 \times \mathbb{R}_1$. Compare this topology with the standard topology on $\mathbb{R}^2$.

6. Let $(X, \mathcal{T})$ be a topological space. An *open cover* of X is a collection $\mathcal{U}$ of open sets such that $\bigcup_{E \in \mathcal{U}} E = X$. Any collection $\mathcal{U}' \subseteq \mathcal{U}$ with $\bigcup_{E \in \mathcal{U}'} E = X$ is called a sub-cover of $\mathcal{U}$. The space $(X, \mathcal{T})$ is said to be *Lindelöf* if every open cover has a countable sub-cover.

(a) Show that every second countable space is Lindelöf.

(b) Show that a metric space is Lindelöf if and only if it is second countable.

(c) Prove that the product topology $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not Lindelöf.

7. Let $\mathcal{T}$ be the cofinite topology on $\mathbb{R}$ and let $E = \{2, 4, 6, \cdots\}$ be the set of even natural numbers.

(a) Find $\operatorname{Cl} E$, $L(E)$ and ∂E in the topological space $(\mathbb{R}, \mathcal{T})$.

(b) Is $(\mathbb{R}, \mathcal{T})$ separable?

(c) Show that $(\mathbb{R}, \mathcal{T})$ is not pseudometrizable.

8. Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be metric spaces. Let $\mathcal{B} \subseteq 2^Y$ be a basis for $\mathcal{T}'$ and $\mathscr{S} \subseteq 2^Y$ be a sub-basis for $\mathcal{T}_Y$. Prove or disprove:

(a) $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous if and only if $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.

(b) f as above is continuous if and only if $f^{-1}(B)$ is open for every $B \in \mathscr{S}$.

9. Prove that every infinite T_2 (Hausdorff) space X contains an infinite discrete subspace (i.e, an infinite subset Y for which the subspace topology is the discrete topology on Y).

10. Let X and Y be topological spaces and $f : X \to Y$ be a function. f is said to be open if $f(U)$ is open for every open set U. Consider the map $f : [0, 2\pi) \to \mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ given by $f(\theta) = (\cos\theta, \sin\theta)$. Equip $[0, 2\pi)$ with the subspace topology of the standard topology on $\mathbb{R}$, and $\mathbb{S}^1$ with the subspace topology of the standard topology on $\mathbb{R}^2$. Is the map f continuous? Is it open?