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METRIC SPACES

Defn: A metric space is a pair (X,d) where X is a set, and $d: X\times X \longrightarrow \mathbb{R}$ is a function satisfying:

(i) $d(x,y)\geq 0 \quad \forall x,y\in X$

(ii) $d(x,y)=0 \iff x=y \quad \forall x,y\in X$

(iii) $d(x,y)=d(y,x) \quad \forall x,y\in X$

(iv) $d(x,z)\leq d(x,y)+d(y,z) \quad \forall x,y,z\in X$ ↳ "Triangle inequality"

Examples:

1. $(\mathbb{R},d)$ where $d(x,y)=\begin{cases} x-y, & x\geq y\\ y-x & x<y\end{cases}$

2. $(\mathbb{R}^2,d)$
$$d\big(\underset{x}{(x_1,x_2)},\underset{y}{(y_1,y_2)}\big)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}.$$
In both 1 and 2, $d(x,y)$ is the length of the line segment bound between x and y.

3. For any set X, define $d: X\times X\longrightarrow \mathbb{R}$ as
$$d(x,y)=\begin{cases}0, & x=y\\ 1, & x\neq y.\end{cases}$$
d clearly satisfies (i), (ii), (iii).
(iv): $\forall x,y,z$, $d(x,y)+d(y,z)=A\in\{0,1,2\}$.
If $A=0$, then $x=y=z$. So $d(x,z)=0$.
If $A=1$, then either $x=y\neq z$, or $x\neq y=z$. So $d(x,z)=A$.
If $A=2$, then $d(x,z)\leq 1<A$.

4. Let $G=(V,E)$ be a graph, and consider $d: V\times V\longrightarrow\mathbb{R}$ given by
$d(u,v)=$ no. of edges in the shortest path joining u and v.
Is d a metric?
Answer: No, in fact d is not well-defined: it is possible that there is no path between two edges.
Fix: Assume G is a "connected" graph (ie, any two vertices $u,v\in V$ can be joined by a path).
d satisfies (i), (ii), (iii) in the definition of a metric.
Triangle inequality: if the shortest path $u\to v$ has m edges, and " " " $v\to w$ has n edges, then combining the paths, we get a path $u\to w$ with $m+n$ edges. So the shortest path $u\to w$ has $\leq m+n$ edges.

Pseudo-metric spaces:

Defn: Any function $d: X\times X\longrightarrow\mathbb{R}$ that satisfies (i), (iii), (iv) but (ii) fails: we still assume $d(x,x)=0 \ \forall x\in X$, but $\exists x\neq y$ s.t. $d(x,y)=0$, is called a pseudo-metric on X.

Constructing metric spaces from pseudo-metric spaces:

Let $(\tilde{X},\tilde{d})$ be a pseudo-metric space.
Let $\sim=\{(x,y): d(x,y)=0\}$.
$\sim$ is an equivalence relation on $\tilde{X}$:
(i) reflexive, ∵ $d(x,x)=0 \ \forall x\in\tilde{X}$ ("since")
(ii) symmetric, ∵ $d(x,y)=d(y,x) \ \forall x,y\in\tilde{X}$
(iii) transitive: if $(x,y),(y,z)\in\sim$, then $d(x,y)=d(y,z)=0$.
Δ inequality $\Rightarrow d(x,z)=0 \Rightarrow (x,z)\in\sim$.
[Note: For any relation R, we say xRy if $(x,y)\in R$].
Let $X=\tilde{X}/\sim \;=\{[x]: x\in\tilde{X}\}$.
Define $d: X\times X\longrightarrow\mathbb{R}$ as $d([x],[y])=\tilde{d}(x,y)$.
d is well-defined: if $[x']=[x]$ and $[y']=[y]$, then
$$\tilde{d}(x',y')\leq\tilde{d}(x',x)+\tilde{d}(x,y)+\tilde{d}(y,y')=\tilde{d}(x,y).$$
Similarly $\tilde{d}(x,y)\leq\tilde{d}(x',y')$.
Therefore $\tilde{d}(x,y)=\tilde{d}(x',y')$.

Exercise: Check that d is a metric on X.

Eg: Consider the pseudo-metric space $(\mathbb{N},d')$ where
$$d'(x,y)=\begin{cases}0, & x-y \text{ is even}\\ 1, & x-y \text{ is odd}.\end{cases}$$
Note: $d'(x,y)\geq 0 \ \forall x,y\in\mathbb{N}$, and $d'(x,y)=d'(y,x)$.
$d'(x,x)=0 \ \forall x\in\mathbb{N}$,
and $d'(x,z)\leq d'(x,y)+d'(y,z) \quad \forall x,y,z\in\mathbb{N}$
[this can only fail if RHS $=0$. In that case $x-y$, $y-z$ are even, so $x-z$ is even, so LHS $=0$]
The construction described above gives a metric space $(\mathbb{N}/\sim, d)$.
Note that $2\sim4\sim6\sim8\sim\cdots$
and $1\sim3\sim5\sim7\sim\cdots$
but no odd number is related to an even number.
So $\mathbb{N}/\sim=\{[1],[2]\}$.
$d([1],[2])=d'(1,2)=1$

Metrics on $\mathbb{R}^n=\{(x_1,\dots,x_n): x_i\in\mathbb{R} \text{ for } i=1,2,\dots,n\}$

(i) Euclidean Metric:
$$d\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2}$$
(ii) $d'\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big)=d(x_1,y_1)+d(x_2,y_2)+\dots+d(x_n,y_n)$
[d is the Euclidean metric on $\mathbb{R}$]
(iii) $d''\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big)=\max\limits_{i\in\{1,\dots,n\}} d(x_i,y_i)$

Next class we will show all three are metrics.

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(1) Euclidean metric d:
it is clear that $d(x,y)\geq 0$, $d(x,x)=0$ and $d(x,y)=d(y,x)$ $\forall x,y$.
If $d(x,y)=0$, then every term in the sum $\sum_{i=1}^n (x_i-y_i)^2$ is 0. So $x_i=y_i \ \forall i$
$\Rightarrow x=y$.

Δ inequality:
Given $x,y,z\in\mathbb{R}^n$, we need
$$\sqrt{\sum_i(x_i-z_i)^2}\leq\sqrt{\sum_i(x_i-y_i)^2}+\sqrt{\sum_i(y_i-z_i)^2}$$

Lemma 1: $\forall u,v\in\mathbb{R}^n$,
$$\boxed{\Big(\sum_i u_iv_i\Big)^2\leq\Big(\sum_i u_i^2\Big)\cdot\Big(\sum_i v_i^2\Big)} \longrightarrow (\star)$$
↳ "Cauchy-Schwarz inequality"

Proof of Lemma 1: (There are several proofs, here is a fun one)

Let $t\in\mathbb{R}$ and $w_t=u-tv\in\mathbb{R}^n$. Then
$$0\leq\big(d(0,w_t)\big)^2=\sum_i(u_i-tv_i)^2=\sum_i\big(u_i^2+t^2v_i^2-2tu_iv_i\big)$$
$$=t^2\Big(\sum_i v_i^2\Big)-2t\Big(\sum_i u_iv_i\Big)+\sum_i u_i^2.$$
Letting $A=\sum_i v_i^2$, $B=-2\sum_i u_iv_i$, $C=\sum_i u_i^2$, At^2+Bt+C

We have $At^2+Bt+C\geq 0 \ \forall t\in\mathbb{R}$.
This implies the discriminant $B^2-4AC\leq 0$
$$\Rightarrow \Big(-2\sum_i u_iv_i\Big)^2-4\Big(\sum_i u_i^2\Big)\Big(\sum_i v_i^2\Big)\leq 0$$
$$\Rightarrow 4\Big(\sum_i u_iv_i\Big)^2\leq 4\Big(\sum_i u_i^2\Big)\Big(\sum_i v_i^2\Big). \quad \text{QED}$$

Proof of Δ inequality for d using Lemma 1:
Want $(d(x,z))^2\leq(d(x,y)+d(y,z))^2$.
Note:
$$\sum_i(x_i-z_i)^2=\sum_i\big((x_i-y_i)+(y_i-z_i)\big)^2=\sum_i(x_i-y_i)^2+\sum_i(y_i-z_i)^2+2\sum_i(x_i-y_i)(y_i-z_i)$$
Let $u=x-y$, $v=y-z$. Then
$$(d(x,z))^2=\sum_i(x_i-z_i)^2\leq\sum_i u_i^2+\sum_i v_i^2+2\sum_i u_iv_i$$
$$\leq\sum_i u_i^2+\sum_i v_i^2+2\Big(\sum_i u_i^2\Big)^{1/2}\Big(\sum_i v_i^2\Big)^{1/2} \quad [\text{by Lemma 1}]$$
$$=A^2+B^2+2AB=(A+B)^2 \quad \Big[\text{where } A=\Big(\sum_i u_i^2\Big)^{1/2},\ B=\Big(\sum_i v_i^2\Big)^{1/2}\Big]$$
$$=\Big(\Big(\sum_i(x_i-y_i)^2\Big)^{1/2}+\Big(\sum_i(y_i-z_i)^2\Big)^{1/2}\Big)^2=\big(d(x,y)+d(y,z)\big)^2$$

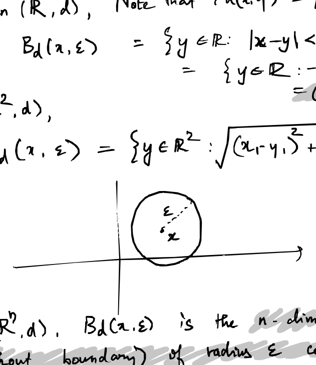
(ii) d': "Taxicab metric"
It is clear that d' satisfies all properties of a metric, including the triangle inequality, ∵ $\forall i$, $d(x_i,z_i)\leq d(x_i,y_i)+d(y_i,z_i)$.

(iii) d''
Same holds for d''.
Note about Δ inequality:
$$\forall i,\quad d(x_i,z_i)\leq d(x_i,y_i)+d(y_i,z_i)\leq\max_j d(x_j,y_j)+\max_j d(y_j,z_j)=d(x,y)+d(y,z).$$
So $d(x,z)=\max_i d(x_i,z_i)\leq d(x,y)+d(y,z)$.

Open and closed sets

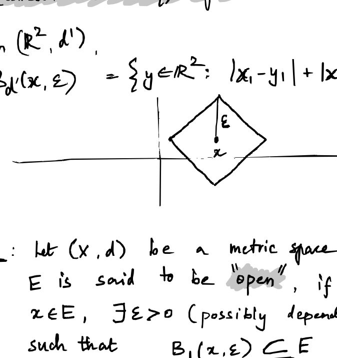
Let (X,d) be a metric space.

Defn: Given $x\in X$ and $\varepsilon>0$, a real number, the "ε-ball centered at x" is the set
$$B_d(x,\varepsilon)=\{y\in X: d(x,y)<\varepsilon\}.$$
Eg: In $(\mathbb{R},d)$, Note that $d(x,y)=|x-y|$.
$$B_d(x,\varepsilon)=\{y\in\mathbb{R}: |x-y|<\varepsilon\}=\{y\in\mathbb{R}: -\varepsilon<x-y<\varepsilon\}=(x-\varepsilon,x+\varepsilon).$$
In $(\mathbb{R}^2,d)$,
$$B_d(x,\varepsilon)=\Big\{y\in\mathbb{R}^2: \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}<\varepsilon$$



★ In $(\mathbb{R}^n,d)$, $B_d(x,\varepsilon)$ is the n-dimensional sphere (without boundary) of radius ε centered at x.

★ In $(\mathbb{R}^2,d')$,
$$B_{d'}(x,\varepsilon)=\{y\in\mathbb{R}^2: |x_1-y_1|+|x_2-y_2|<\varepsilon\}.$$



Defn: Let (X,d) be a metric space, and $E\subseteq X$. E is said to be "open", if for every $x\in E$, $\exists\varepsilon>0$ (possibly dependent on x) such that $B_d(x,\varepsilon)\subseteq E$.
A set $F\subseteq X$ is "closed" if $X\setminus F$ is open.

Examples

1. Let (X,d) be a metric space where
$$d(x,y)=\begin{cases}0 & x=y\\ 1 & x\neq y\end{cases}$$
Then every $E\subseteq X$ is both open and closed.
Proof: Note: $\forall x\in X$, $B_d(x,\varepsilon)=\begin{cases}\{x\}, & 0<\varepsilon\leq 1\\ X & \varepsilon>1\end{cases}$
Fix $0<\varepsilon\leq 1$.
Given $E\subseteq X$ and any $x\in E$, $B_d(x,\varepsilon)=\{x\}\subseteq E$.
∴ E is open.
For any $F\subseteq X$, $X\setminus F$ is open, so F is closed.
The metric d above is called the "discrete metric" on X.

2. Every ε-ball in a metric space (X,d) is open.
Let $E=B_d(x,\varepsilon)$. Given $y\in E$, we want to show $\exists r>0$ s.t. $B_d(y,r)\subseteq E$.
Let $l=d(x,y)$, and pick $r<l,\varepsilon-l$ [$r<\varepsilon-l$ is actually enough?]
Then $\forall z\in B_d(y,r)$, we have
$$d(x,z)\leq d(x,y)+d(y,z)<l+r<l+\varepsilon-l=\varepsilon\Rightarrow z\in B_d(x,\varepsilon)$$

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3. Every singleton set $\{x\}$ in a metric space (X,d) is closed.
Proof: Given $y\in X\setminus\{x\}$, let $\varepsilon=\frac{d(x,y)}{2}$.
Then $x\notin B_d(y,\varepsilon)$, ie, $B_d(y,\varepsilon)\subseteq X\setminus\{x\}$.
So $X\setminus\{x\}$ is open.

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4. In $(\mathbb{R},d)$, $E\subseteq\mathbb{R}$ is open $\iff$ E is a countable union of pairwise disjoint open intervals.

Proof: $\Leftarrow$: 1. Every open interval $(a,b)=(x-\varepsilon,x+\varepsilon)$ for $x=\frac{a+b}{2}$, $\varepsilon=\frac{b-a}{2}$

and is therefore open.

2. Any union of open intervals (in particular a countable disjoint union) is an open set.

Reason: Given $\{I_\alpha\}_{\alpha\in A}$ of open intervals and $x\in\bigcup_\alpha I_\alpha$, take $\varepsilon>0$ s.t. $(x-\varepsilon,x+\varepsilon)\subset I_\alpha\subseteq\bigcup_\alpha I_\alpha$, so $\bigcup_\alpha I_\alpha$ is open.

$\Rightarrow$: Let $E\subseteq\mathbb{R}$ be open.
Given any $x\in E$,
let $\mathcal{U}_x=\{(a,b)\subseteq E \text{ s.t. } x\in(a,b)\}$
1. $\mathcal{U}_x\neq\emptyset$, is uncountable.
2. $$\bigcup_{(a,b)\in\mathcal{U}_x}(a,b)=(a_0,b_0)$$
where $a_0=\inf\{a:(a,b)\in\mathcal{U}_x\}$, $b_0=\sup\{b:(a,b)\in\mathcal{U}_x\}$.

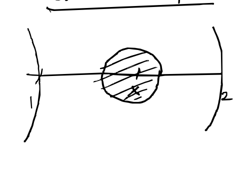
Proof: $\subseteq$ is clear, since $(a,b)\subseteq(a_0,b_0) \ \forall (a,b)\in\mathcal{U}_x$
$\supseteq$: Exercise: Show $\forall y\in(a_0,b_0)$,
1. $\exists (a_1,b_1),(a_2,b_2)\in\mathcal{U}_x$ s.t.
$a_0<a_1<y$
$y<b_2<b_0$

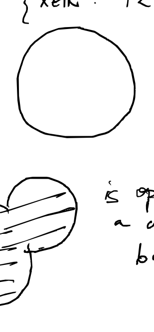
2. $(a_1,b_1)\cup(a_2,b_2)$ is in fact an open interval in E.

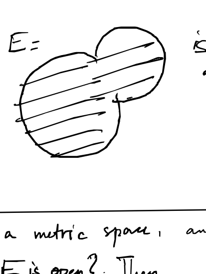
Let $I_x=(a_0,b_0)$. Note that I_x is the largest interval in E containing x, and that
$$E=\bigcup_{x\in E}I_x$$
Remark: for $x\neq y\in E$, either $I_x\cap I_y=\emptyset$ or $I_x=I_y$.
This follows from below:
if $I_x\cap I_y\neq\emptyset$, then $I_x\cup I_y$ is an interval in E containing x, [ie $I_x\cup I_y\in\mathcal{U}_x$]
$\Rightarrow I_x\cup I_y\subseteq I_x\Rightarrow I_x\supseteq I_y$.
$\Rightarrow I_x=I_y$,

Define an equivalence relation $\sim$ on E as follows:
$x\sim y$ if $I_x=I_y$
Exercise: Check this is an equivalence relation.
Note: $$E=\bigsqcup_{[x]\in E/\sim}I_x$$
$\forall x\in E$, $\exists q\in E\cap\mathbb{Q}$ s.t. $I_x=I_q$, since every interval has a rational number.
$\Rightarrow E/\sim$ is countable,
$\Rightarrow E$ is a countable disjoint union of intervals.

Q: In $(\mathbb{R}^2,d)$, is every open set a disjoint union of balls?
A: No.
Counter-examples
1. $E=\{x\in\mathbb{R}^2: 1<d(0,x)<2\}$.





2. $E=$  is open, but not a disjoint union of balls.

Theorem 1: Let (X,d) be a metric space, and let $\mathcal{T}=\{E\subseteq X. \ E \text{ is open}\}$. Then
1. given any $\mathcal{U}\subseteq\mathcal{T}$, the set $\bigcup_{E\in\mathcal{U}}E\in\mathcal{T}$.
In other words, any union of open sets is open.
2. given any finite $\mathcal{U}\subseteq\mathcal{T}$, the set $\bigcap_{E\in\mathcal{U}}E\in\mathcal{T}$
ie, any finite intersection of open sets is open.
3. $\emptyset$, X are open and closed.

Proof: 1. Given $\mathcal{U}$, $x\in\bigcup_{E\in\mathcal{U}}E$, we know $x\in E$ for some $E\in\mathcal{U}$.
Take $\varepsilon>0$ s.t. $B_d(x,\varepsilon)\subseteq E\subseteq\bigcup_{E\in\mathcal{U}}E$
2. Given $\mathcal{U}$, $x\in\bigcap_{E\in\mathcal{U}}E$, we want $\varepsilon>0$ s.t. $B_d(x,\varepsilon)\subset\bigcap_{E\in\mathcal{U}}E$.
Write $\mathcal{U}=\{E_1,\dots,E_n\}$. ∵ E_i is open $\forall i$,
$\exists\varepsilon_i>0$ s.t. $B_d(x,\varepsilon_i)\subseteq E_i$. Let $\varepsilon=\min_{i=1}^n\varepsilon_i$.
Note: $B_d(x,\varepsilon)\subset E_i \ \forall i$. So $B_d(x,\varepsilon)\subset\bigcap_{i=1}^n E_i$

Remark: The infinite intersection of open sets in a metric space need not be open.
For eg: consider $(\mathbb{R},d)$ and let
$U_n=(-1/n,1/n)$ for each $n\in\mathbb{N}$. U_n is open,

However $\bigcap_{n=1}^\infty U_n=\{0\}$ is not open.

Exercise: Let (X,d) be a metric space and $x\in X$. Show that $F=\{y\in X: d(x,y)\leq\varepsilon\}$ is closed.

Further examples

Recall: $\operatorname{diam}(X)=\sup\limits_{(x,y)\in X\times X}(x,y)$

1. **Defn:** Let (X,d) be a metric space. The metric d is said to be bounded if $\exists M\geq 0$ s.t. $\operatorname{diam}(X)\leq M$ [equivalently, $\forall x,y\in X$, $d(x,y)\leq M$].
Eg: 1. The discrete metric is a bounded metric. [$\operatorname{diam}(X)=1$]
$(\mathbb{R},d)$ - unbounded

2. Generating a bounded metric:
Given (X,d), define a new function
$\bar{d}(x,y)=\min\{1,d(x,y)\}$.
Then $\bar{d}(x,y)\leq 1 \ \forall x,y\in X$, and is thus bounded. We claim $\bar{d}$ is a metric on X.

Proof: Δ inequality: Want $\bar{d}(x,z)\leq\bar{d}(x,y)+\bar{d}(y,z)$
case 1: If $\bar{d}(x,y)+\bar{d}(y,z)<1$, then $\bar{d}(x,y),\bar{d}(y,z)<1$.
$\Rightarrow d(x,y)=\bar{d}(x,y)$ and $d(y,z)=\bar{d}(y,z)$
∵ $\bar{d}(x,z)\leq d(x,z)\leq d(x,y)+d(y,z)=\bar{d}(x,y)+\bar{d}(y,z)$
case 2: If $\bar{d}(x,y)+\bar{d}(y,z)\geq 1$
∵ $\bar{d}(x,z)\leq 1\leq\bar{d}(x,y)+\bar{d}(y,z)$,

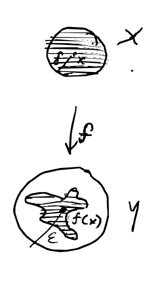
$\bar{d}$ is called the standard bounded metric induced by d.

Exercise: Show that $E\subseteq X$ is open in (X,d) iff it is open in $(X,\bar{d})$.

3. **Def:** Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$.
A norm on V is a function $\|\cdot\|: V\longrightarrow\mathbb{R}$ such that
(i) $\|x\|\geq 0 \ \forall x\in V$
(ii) $\|x\|=0\iff x=0$.
(iii) $\|\alpha x\|=|\alpha|\|x\|$ for all scalars α.
(iv) $\|x+y\|\leq\|x\|+\|y\| \ \forall x,y\in V$.

Any norm on V induces a metric $d: V\times V\longrightarrow\mathbb{R}$ given by $d(x,y)=\|x-y\|$
Exercise: Check that the above formula does indeed give a metric on V.
Q: Given any vector space V over $\mathbb{R}$ or $\mathbb{C}$ with a metric d, is the function $x\longmapsto d(x,0)$ a norm on V?
A: (i), (ii) are clear. (iii), (iv) may not work.

Continuous functions



Def: Let (X,d_X) and (Y,d_Y) be metric spaces and $f: X\longrightarrow Y$ be a function. Then f is said to be continuous at $x\in X$ if for every $\varepsilon>0$, $\exists\delta>0$ s.t. $\forall y\in X$,
$$d_X(x,y)<\delta\Rightarrow d_Y(f(x),f(y))<\varepsilon$$
[Equivalently, $f\big(B_{d_X}(x,\delta)\big)\subseteq B_{d_Y}(f(x),\varepsilon)$]
f is said to be continuous if it is continuous at every $x\in X$.

Eg: 1. For any metric space (X,d), the map
$$id: X\longrightarrow X, \quad x\longmapsto x$$
is continuous.

2. The map $\pi_1:\mathbb{R}^2\longrightarrow\mathbb{R}$ given by $\pi_1((x_1,x_2))=x_1$ is continuous [assume both $\mathbb{R}^2$, $\mathbb{R}$ have Euclidean metric]
Proof:

3. The map $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ given by $f((x_1,x_2))=x_1+x_2$ is continuous.
Proof: Given $x=(x_1,x_2)\in\mathbb{R}^2$, consider the unique line through x of slope 1. This line has x-intercept $(l,0)$ and y-intercept $(0,l)$ where $l=f(x)$.
Consider the lines $L^{\pm}_\varepsilon$ of slope 1 as in the figure.
Let $\delta=\frac{1}{2}\times$ distance between L^+_ε and L^-_ε.
Then $\forall y$ s.t. $d(x,y)<\delta$, we have $|f(x)-f(y)|<\varepsilon$.