

# Homework 1

MA 771

Due February 6 2025

Please scan or type your homework and hand/email it to me by 5pm on the due date.

1. Given a real number  $\lambda > 0$ , find the largest closed interval  $I = [a, b]$  on the real line such that  $f : I \rightarrow \mathbb{R}$  is a  $\lambda$ -contraction.
2. Show that the following statements are equivalent:
  - the endomorphism of the torus  $T_A : \mathbb{T}^n \rightarrow \mathbb{T}^n$  is invertible
  - $x \in \mathbb{Z}^n$  if and only if  $Ax \in \mathbb{Z}^n$ , for every  $x \in \mathbb{Z}^n$
  - $|\det A| = 1$ .
3. Let  $m > 1$  be an integer and let  $E_m : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  be the expanding map  $x \mapsto mx \pmod{1}$ .
  - Prove that the set of all periodic points of  $E_m$  is dense in  $\mathbb{S}^1$ .
  - A point  $x \in \mathbb{S}^1$  is pre-periodic under  $E_m$  if  $y = E_m^{\circ k}(x)$  is a periodic point for some  $k \geq 0$ . Show that the set of pre-periodic points under  $E_m$  is  $\mathbb{Q}/\mathbb{Z}$ .
4. Suppose that  $(X, d)$  is a compact metric space and  $f : X \rightarrow X$  is such that

$$d(f(x), f(y)) < d(x, y)$$

for any  $x \neq y$ . Prove that  $f$  has a unique fixed point  $x_0 \in X$  and  $\lim_{n \rightarrow \infty} f^{\circ n}(x) = x_0$  for any  $x \in X$ .

5. Prove that the square Sierpinski carpet is the set of points in the unit square of which at least one coordinate can be represented by a ternary (i.e., base 3) expansion without the digit 1.