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TOPOLOGICAL SPACES

Def: Let X be a set. A "topology on X" is is a set $\mathcal{T} \subseteq P(X)$ such that
$\overset{\|}{2^X}$

(i) $\phi, X \in \mathcal{T}$

(ii) For every $\mathcal{U} \subseteq \mathcal{T}$, the union $\bigcup_{E \in \mathcal{U}} E \in \mathcal{T}$

i.e. the union of elements in any sub-collection of $\mathcal{T}$ is an element of $\mathcal{T}$

(iii) For every finite $\mathcal{U} \subseteq \mathcal{T}$, the intersection $\bigcap_{E \in \mathcal{U}} E \in \mathcal{T}$,

i.e., the intersection of elements in any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A topological space is an ordered pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X.

Def: Given a topological space $(X, \mathcal{T})$ and $U \subseteq X$, U is said to be open if $U \in \mathcal{T}$, and closed if $X \setminus U \in \mathcal{T}$.

Thus, the union of an arbitrary collection of open sets, and the intersection of finite collections of open sets, are open.

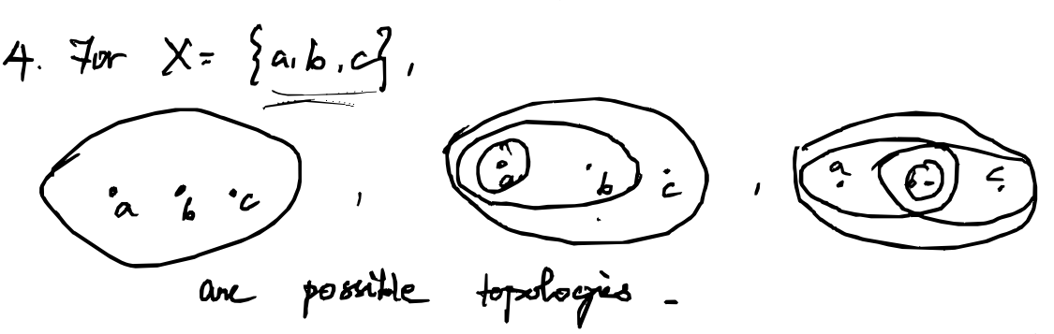
Note: Arbitrary intersections & finite unions of closed sets are closed.

Def Given a topological space $(X, \mathcal{T})$, a neighborhood of a point $x \in X$ is an open set $U \in \mathcal{T}$ s.t. $x \in U$

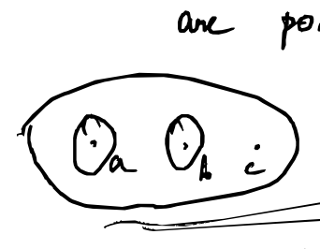
Examples of topologies

1. For $X = \{\}$, $\mathcal{T}$ has to be $\{\{\}\}$.
2. For $X = \{a\}$, $\mathcal{T} = \{\phi, X\}$.
3. For $X = \{a, b\}$, $\mathcal{T}_1 = \{\phi, X\}$,
   $\mathcal{T}_2 = \{\phi, \{a\}, X\}$
   $\mathcal{T}_3 = \{\phi, \{b\}, X\}$,
   $\mathcal{T}_4 = \{\phi, \{a\}, \{b\}, X\} = P(X)$

   are all possible topologies.
4. For $X = \{a, b, c\}$,



are possible topologies.



is not [Reason: $\{a, b\} \notin \mathcal{T}$].

See Eg. 1 Ch 2 in Munkres for other topologies on X.

5. Let (X, d) be a metric space. Recall that $U \subseteq X$ is said to be open if every $x \in U$ has some $\varepsilon > 0$ s.t. $B_d(x, \varepsilon) \subseteq U$.
   By a theorem proved in class last time, $\mathcal{T}_d = \{U \subseteq X : U \text{ is open}\}$ is a topology on X,
   $\mathcal{T}_d$ is called the topology induced by the metric d.

Def:

- ★ The topology $\mathcal{T} = \{\phi, X\}$ is called the trivial/indiscrete topology.
- ★ The collection $\mathcal{T}$ of all subsets of X is called the discrete topology on X. Note that $\mathcal{T} = \mathcal{T}_d$ where d is the discrete metric on X.
- ★ The collection $\mathcal{T} = \{E : E = \phi \text{ or } X \setminus E \text{ is countable}\}$ is a topology, aka countable complement/cocountable topology.

Proof: (i) $E = \phi$, $E = X \in \mathcal{T}$

(ii) unions of open sets are open:

Say we have a collection $\mathcal{U}$ of sets E s.t. $X \setminus E$ is countable

Note: $X \setminus \left(\bigcup_{E \in \mathcal{U}} E\right) = \bigcap_{E \in \mathcal{U}} (X \setminus E)$ is the intersection of countable sets, and is thus countable.

(iii) Given $E_1, \ldots, E_n \in \mathcal{T}$,

Note that $X \setminus \bigcap_{i=1}^{n} E_i = \bigcup_{i=1}^{n} (X \setminus E_i)$ is countable, since any countable union of countable sets is countable.

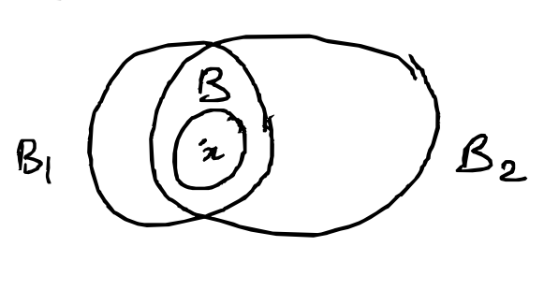
Exercise: ★ The topology $\mathcal{T} = \{E : E = \phi \text{ or } X \setminus E \text{ is finite}\}$ is called the finite complement/cofinite topology.

Basis for a topology

Defn: Let X be a set. A "basis" for a topology on X is a collection $\mathcal{B}$ of subsets of X called "basis elements", s.t.

(i) for every $x \in X$, $\exists B \ni x$ s.t. $B \in \mathcal{B}$

(ii) if $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then $\exists B \in \mathcal{B}$ s.t. $x \in B$ and $B \subset B_1 \cap B_2$.



Defn: Let X be a set.
Given $\mathcal{B}$ as above, the topology $\mathcal{T}$ generated by $\mathcal{B}$ is the collection of all sets $U \subseteq X$ satisfying the property that $\forall x \in U$, $\exists B \in \mathcal{B}$ $x \in B$ and $B \subseteq U$. [This implies $\mathcal{B} \subseteq \mathcal{T}$].

Prop: $\mathcal{T}$ as above is indeed a topology on X.

Proof: (i) $\phi, X \in \mathcal{T}$: $\forall x \in X$, $\exists B_x \subseteq \mathcal{B}$ s.t. $x \in B_x$. $X = \bigcup_x B_x \in \mathcal{T}$.
$\phi \in \mathcal{T}$ tautologically.

(ii) If $\{U_\alpha\}_\alpha \subseteq \mathcal{T}$, let $U = \bigcup_\alpha U_\alpha$. Suppose $x \in U$, then $x \in U_\alpha$ for some α, and so $\exists B \in \mathcal{B}$ s.t.
$x \in B \subseteq U_\alpha \subseteq U$.

(iii) If $\{U_1, \ldots, U_n\} \subseteq \mathcal{T}$, let $U = \bigcap_i U_i$. Suppose $x \in U$,
We want $B \in \mathcal{B}$ s.t. $x \in B \subset U$.
We know $\exists B_i \subset U_i$ containing x for $i = 1, 2, \ldots, n$.
Then $\exists B \in \mathcal{B}$ s.t. $x \in B \subset B_1 \cap B_2$,
$\exists B' \in \mathcal{B}$ s.t. $x \in B' \subset B \cap B_3$
and so on. eventually we get a set $\tilde{B} \in \mathcal{B}$ s.t. $x \in \tilde{B} \subset B_1 \cap B_2 \cap \cdots \cap B_n \subset U$.

Lemma 2: Let $\mathcal{T}$ be the topology on X generated by a basis $\mathcal{B}$. Then U is open (ie, $U \in \mathcal{T}$)
$\iff \exists \mathcal{B}' \subseteq \mathcal{B}$ s.t. $U = \bigcup_{B \in \mathcal{B}'} B$

Proof: Note that $\mathcal{T} = \{U : \forall x \in U, \exists B_x \in \mathcal{B} \text{ s.t. } x \in B_x \subset U\}$

$\Rightarrow$: If $U \in \mathcal{T}$, then $U = \bigcup_{x \in U} B_x$

$\Leftarrow$: If $U = \bigcup_{B \in \mathcal{B}'} B$ for some $\mathcal{B}' \subset \mathcal{B}$,
then for $x \in U$, $\exists B \in \mathcal{B}'$ s.t. $x \in B \subseteq U$.

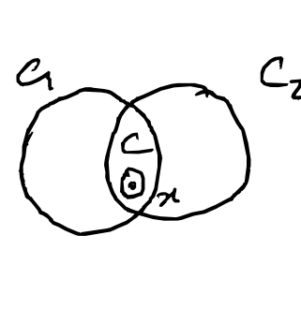
Eg:

1. Let (X, d) be a metric space. Then the collection of all balls $\mathcal{B}$ is a basis for the topology $\mathcal{T}_d$.

Proof: → any point $x \in X$ is contained in a ball.
→ given any balls $B_d(x, \varepsilon)$, $B_d(y, \tilde{\varepsilon})$, and $z \in B_d(x, \varepsilon) \cap B_d(y, \tilde{\varepsilon})$,
$\exists \delta$ s.t. $B_d(z, \varepsilon) \subset B_d(x, \varepsilon) \cap B_d(y, \tilde{\varepsilon})$.
$\therefore \mathcal{B}$ is a basis.

2. Let $X = \mathbb{R}^2$, and $\mathcal{B}$ be the set of circular regions [interior of circles] in $\mathbb{R}^2$. Then $\mathcal{B}$ is a basis for a topology on $\mathbb{R}^2$.

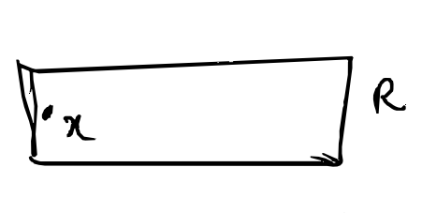
Proof: Let $x \in \mathbb{R}^2$. Clearly $\exists C \in \mathcal{B}$ s.t. $x \in C$.
Given $C_1, C_2 \in \mathcal{B}$ and $x \in C_1 \cap C_2$, $\exists$ $C \in \mathcal{B}$ s.t. $x \in C$ and $C \subseteq C_1 \cap C_2$



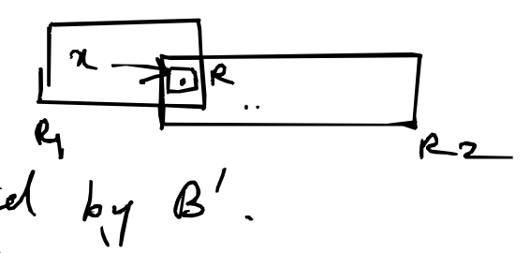
3. Let d be the Euclidean metric on $\mathbb{R}^2$. Then $\mathcal{T}_d$ is the topology generated by $\mathcal{B}$ as in the above example.

4. Let $X = \mathbb{R}^2$ and $\mathcal{B}'$ be the set of all rectangular regions [interiors of rectangles]. Then $\mathcal{B}'$ is a basis that generates the same topology as $\mathcal{B}$ (namely $\mathcal{T}_d$)

To see that $\mathcal{B}'$ is a basis, note that
(i) given $x \in \mathbb{R}^2$, $\exists R \in \mathcal{B}'$ containing x



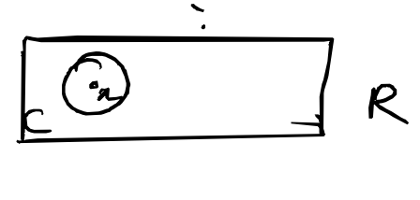
(ii) given any $R_1, R_2 \in \mathcal{B}'$ and $x \in R_1 \cap R_2$, $\exists R \in \mathcal{B}'$ s.t. $x \in R \subset R_1 \cap R_2$.



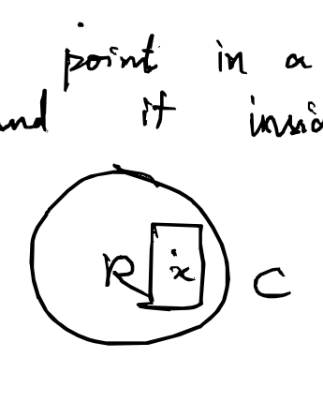
Let $\mathcal{T}'$ be the topology generated by $\mathcal{B}'$.
We will show that $\mathcal{B}' \subseteq \mathcal{T}_d$ and $\mathcal{B} \subseteq \mathcal{T}'$

$$\text{Lemma 2} \Rightarrow \mathcal{T}_d = \mathcal{T}'.$$

To see $\mathcal{B}' \subseteq \mathcal{T}_d$, given $R \in \mathcal{B}'$, $x \in R$, note $\exists C \in \mathcal{B}$ s.t.: $x \in C \subset R$. [any point within a rectangle has a circle around it inside the rectangle].



Similarly, given $C \in \mathcal{B}$ and $x \in C$, $\exists R \in \mathcal{B}'$ s.t. $x \in R \subset C$ [any point in a disk has a rectangle around it inside the disk].



This shows $\mathcal{B} \subseteq \mathcal{T}'$.

5. Let $X = \mathbb{R}$ $\{\mathcal{B} = (a, b) : a < b\}$. Then $\mathcal{B}$ generates a topology on $\mathbb{R}$ which coincides with the one induced by the Euclidean metric $d(x, y) = |x - y|$. This is called the standard topology on $\mathbb{R}$.

Proof: Given any $x \in \mathbb{R}$, $\exists (a, b) \in \mathcal{B}$ s.t. $x \in (a, b)$.

Given $(a, b), (a', b') \in \mathcal{B}$, and $x \in (a, b) \cap (a', b')$,
$\exists \varepsilon$ s.t. $(x - \varepsilon, x + \varepsilon) \subset (a, b) \cap (a', b')$.
$\therefore \mathcal{B}$ is a basis.
$\because$ open intervals are all balls in the Euclidean metric,
the topology generated by $\mathcal{B} = \mathcal{T}_d$.

Lemma 3: Let $(X, \mathcal{T})$ be a topological space. Suppose $\mathcal{B} \subseteq P(X)$ is such that for each open set $U \subseteq X$ and each $x \in U$, $\exists B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$.

Proof:

Comparing topologies

Defn:
Let $\mathcal{T}, \mathcal{T}'$ be two topologies on a set X.
If $\mathcal{T} \subseteq \mathcal{T}'$ [ie, every set open in $\mathcal{T}$ is open in $\mathcal{T}'$], we say $\mathcal{T}'$ is finer/larger than $\mathcal{T}$, and $\mathcal{T}$ is coarser/smaller than $\mathcal{T}'$.

If $\mathcal{T} \subsetneq \mathcal{T}$, then we say $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, and $\mathcal{T}$ is strictly coarser than $\mathcal{T}'$

If either $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T}' \subseteq \mathcal{T}$, we say $\mathcal{T}, \mathcal{T}'$ are comparable.

Lemma: Let $\mathcal{B}, \mathcal{B}'$ be bases for topologies $\mathcal{T}, \mathcal{T}'$ resp. on a set X. Then TFAE:

(i) $\mathcal{T}'$ is finer than $\mathcal{T}$ [not necessarily strictly finer]

(ii) $\forall x \in X$, $B \in \mathcal{B}$ s.t. $x \in B$, $\exists B' \in \mathcal{B}'$ s.t. $B' \subseteq B$.

Proof: (i) ⇒ (ii)

(ii) ⇒ (i)

Examples:

1. Let $X = \mathbb{R}$ and $\mathcal{B}' = \{[a, b) \mid a < b\}$.
   Then $\mathcal{B}'$ is a basis for a topology $\mathcal{T}'$ on $\mathbb{R}$ called the lower limit topology.
   [We often use the notation $\mathbb{R}_\ell$ for the topological space $(\mathbb{R}, \mathcal{T}')$].

   Prop: $\mathbb{R}_\ell$ is strictly finer than $\mathbb{R}$ with the standard topology $\mathcal{T}$.
   [ie, $\mathcal{T}' \supsetneq \mathcal{T}$]

   Proof: $\mathcal{B}'$ is clearly a basis.
   To prove $\mathcal{T}' \supseteq \mathcal{T}$, we will show any $(a, b) \in \mathcal{T}'$.
   Let $N \in \mathbb{N}$ be s.t. $a + 1/N < b$.
   Then $(a, b) = \bigcup_{n \geq N} [a + 1/n, b)$. By Lemma 2, $(a, b) \in \mathcal{T}'$.

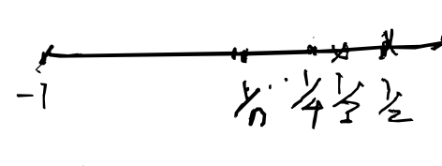
   $\mathcal{T}$ does not contain any elements of $\mathcal{B}'$
   $\therefore \mathcal{T}' \supsetneq \mathcal{T}$.

2. Let $K = \{1/n : n \in \mathbb{N}\}$, and
   $$\mathcal{B}'' = \{(a, b) : a < b\} \cup \{(a, b) \setminus K : a < b\}$$
   Then $\mathcal{B}''$ generates a topology $\mathcal{T}''$ on $\mathbb{R}$.
   and we let $\mathbb{R}_K = (\mathbb{R}, \mathcal{T}'')$ ["K-topology"]

   Proof that $\mathcal{B}''$ is a basis: Clearly any $x \in \mathbb{R}$ is contained in some $B \in \mathcal{B}''$.
   If $B_1, B_2 \in \mathcal{B}''$ then $B_1 \cap B_2$ is either an open interval (a, b) or of the form $(a, b) \setminus K$. So $B_1 \cap B_2 \in \mathcal{B}''$.
   $\therefore \mathcal{B}''$ is a basis.

   Prop: $\mathbb{R}_K$ is strictly finer than $\mathbb{R}$ with std. topology $\mathcal{T}$.
   [ie, $\mathcal{T}'' \supsetneq \mathcal{T}$].

   Proof: $\because \mathcal{B}''$ contains the basis $\mathcal{B} = \{(a, b) : a < b\}$, and so, clearly $\mathcal{T}'' \supseteq \mathcal{T}$.
   To see that $\mathcal{T}'' \supsetneq \mathcal{T}$, note that the set $(-1, 1) \setminus K \in \mathcal{B}''$, so $(-1, 1) \setminus K \in \mathcal{T}''$



   However, $(-1, 1) \setminus K$ contains no open interval around 0, thus $(-1, 1) \setminus K \notin \mathcal{T}$.

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3. ~~Def~~ Given two metrics d and d' on X, we say that $d \sim d'$ if $\mathcal{T}_d = \mathcal{T}_{d'}$ [ie, a set is open in (X, d) iff it is open in (X, d')].

Eg ★ In $\mathbb{R}^2$, we have $d \sim d'$, where d: Euclidean metric, d': Taxi-cab metric.

Proof:

★ Any metric d on a set X is equivalent to the associated standard bounded metric $\bar{d}(x, y) = \min\{1, d(x, y)\}$.

4. Let $X = \{a, b, c\}$,
   $\mathcal{T}_1 = \{\phi, X, \{a\}, \{b\}\}$ and $\mathcal{T}_2 = \{\phi, X, \{a\}, \{b, c\}\}$

   Q: What is the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$?
   A:

   Q: What is the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$?
   A:

Def: Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$.
A basis of neighborhoods at x is a collection $\mathcal{U}_x \subseteq \mathcal{T}$ of neighborhoods U of x [ie, open sets containing x] s.t. for any neighborhood V of x, $\exists U \in \mathcal{U}_x$ s.t. $U \subseteq V$.

Eg (i) Trivial one: $\mathcal{U}_x = \{U \in \mathcal{T} : x \in U\}$

(ii) In any metric space (X, d), $\mathcal{U}_x$ can be taken to be the collection of all balls centered at x.

Def: A topological space (X, d) is said to be first countable if every point $x \in X$ has a countable basis of neighborhoods $\mathcal{U}_x$.

Eg: In $(\mathbb{R}^n, d)$, given any $x \in \mathbb{R}^n$ and a neighborhood $U \ni x$, $\exists \varepsilon \in \mathbb{Q}$ s.t. $B_d(x, \varepsilon) \subseteq U$. Thus $\mathcal{U}_x = \{B_d(x, \varepsilon) : \varepsilon \in \mathbb{Q}\}$ is a countable basis of neighborhoods at x.

Def: Let X be a set. A subbasis for a topology on X is a collection $\mathcal{S} \subseteq P(X)$ s.t.
$$\bigcup_{B \in \mathcal{S}} B = X.$$
The topology generated by $\mathcal{S}$ is the collection of all unions of finite intersections of elements of $\mathcal{S}$.

Exercise: Find a subbasis for the standard topology on $\mathbb{R}$.