÷ Successive Differentiation ÷

Tutorial-1-{Answer Key}

Q→1 (a) If $y = \sin(m\sin^{-1}x)$, prove that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$$

Ans

$$y = \sin(m\sin^{-1}x) \quad ----- (1)$$

$$\left(\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}\right)$$

$$\frac{dy}{dx} = \cos(m\sin^{-1}x)\cdot\frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\cdot\frac{dy}{dx} = m.\cos(m\sin^{-1}x)$$

$$\Rightarrow (1-x^2)\cdot\left(\frac{dy}{dx}\right)^2 = m^2\cdot\cos^2(m\sin^{-1}x)$$

$$\{\cos^2 x + \sin^2 x = 1\}$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2\{1-\sin^2(m\sin^{-1}x)\}$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2\{1-y^2\} \quad --- (2)$$

→ diff. eq^n (2) w.r.t x;

$$\Rightarrow (1-x^2)\left\{2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\right\} + \left(\frac{dy}{dx}\right)^2\cdot\{-2x\} = m^2\left\{0-2y\frac{dy}{dx}\right\}$$

$$\Rightarrow 2(1-x^2)\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = -2ym^2\frac{dy}{dx}$$

$$\Rightarrow 2(1-x^2)\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ym^2\frac{dy}{dx} = 0$$

$$\Rightarrow 2\frac{dy}{dx}\left\{(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y\right\} = 0$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0 \quad \text{or} \quad 2\frac{dy}{dx} = 0$$

As

(b) Find $\frac{d^2y}{dx^2}$, when $x = a\cos^3\theta$, $y = b\sin^3\theta$

Ans

$x = a\cos^3\theta$ $\qquad$ $y = b\sin^3\theta$

$\frac{dx}{d\theta} = -3a\cos^2\theta.\sin\theta$ --(1) $\qquad$ $\frac{dy}{d\theta} = +3b\sin^2\theta.\cos\theta$ -(2)

from Eq^n (1) & (2).

$\Rightarrow \frac{dy}{d\theta} / \frac{dx}{d\theta} = \frac{dy}{dx} = \frac{3b\sin^2\theta.\cos\theta}{-3a\cos^2\theta.\sin\theta}$

$\Rightarrow \frac{dy}{dx} = -\frac{b}{a}\tan\theta$ --- (3)

diff. Eq^n (3) w.r.t x;

$\Rightarrow \frac{d}{dx}\left\{\frac{dy}{dx}\right\} = \frac{d}{dx}\left\{-\frac{b}{a}\tan\theta\right\}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{b}{a}.\sec^2\theta.\frac{d\theta}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{b}{a}.\sec^2\theta.\left\{\frac{-1}{3a\cos^2\theta.\sin\theta}\right\}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{b}{a^2}.\sec^4\theta.\operatorname{cosec}\theta$ Ans

(C) If $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$

$y = a\sin t$

find $\frac{d^2y}{dx^2}$,

Ans

$y = a\sin t$

$\frac{dy}{dt} = a\cos t$ ---- (1)

$x = a\left(\cos t + \log \tan \frac{t}{2}\right)$

$\frac{dx}{dt} = a\left\{-\sin t + \frac{\sec^2\frac{t}{2}}{2\cdot\tan\frac{t}{2}}\right\}$

$\Rightarrow \frac{dx}{dt} = a\left\{-\sin t + \frac{\frac{1}{\cos^2 t/2}}{2\cdot\frac{\sin t/2}{\cos t/2}}\right\}$

$\Rightarrow \frac{dx}{dt} = a\left\{-\sin t + \frac{1}{2\cos\frac{t}{2}\sin\frac{t}{2}}\right\}$

$\left\{\begin{array}{l} 2\cos t\sin t = \sin 2t \\ 2\cos\frac{t}{2}\cdot\sin\frac{t}{2} = \sin t \end{array}\right\}$

$\Rightarrow \frac{dx}{dt} = a\left\{-\sin t + \frac{1}{\sin t}\right\}$ ---- (2)

from Eq^n (1) & (2)

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\left\{-\sin t + \frac{1}{\sin t}\right\}} = \frac{\cos t.\sin t}{-\sin^2 t + 1}$

$\frac{dy}{dx} = \frac{\cos t.\sin t}{\cos^2 t} = \tan t$ ---- (3)

diff Eq^n (3) w.r.t x;

$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx}$

$\frac{d^2y}{dx^2} = \sec^2 t \cdot \left\{\frac{1}{a\left(-\sin t + \frac{1}{\sin t}\right)}\right\}$

$\frac{d^2y}{dx^2} = \sec^2 t\cdot\left\{\frac{\sin t}{a\{-\sin^2 t + 1\}}\right\} = \sec^2 t\cdot\left\{\frac{\sin t}{a\cdot\cos^2 t}\right\}$

$$\Rightarrow \frac{d^2y}{dt^2} = \frac{\sin t}{a.\cos^4 t} \quad \text{Ans}$$

(d) If $x^3+y^3=3axy$, prove that $\frac{d^2y}{dx^2} = -\frac{2a^3xy}{(y^2-ax)^3}$

Ans

$$x^3+y^3 = 3axy \quad \text{-----} \; (1)$$

{ I will solve using y, y_1 & y_2 because it is easy way. }

diff. Eq^n (1) w.r.t x;

$$\Rightarrow 3x^2+3y^2.y_1 = (3ax)\{y_1\}+3ay\{1\}$$

$$\Rightarrow 3\{x^2+y^2.y_1\} = 3\{axy_1+ay\}$$

$$\Rightarrow y^2.y_1 - axy_1 = ay-x^2$$

$$\Rightarrow y_1\{y^2-ax\} = \{ay-x^2\}$$

$$\Rightarrow y_1 = \frac{\{ay-x^2\}}{\{y^2-ax\}} \quad \text{----} \; (2)$$

diff. Eq^n (2) w.r.t x;

$$\Rightarrow y_2 = \frac{(y^2-ax)\frac{d}{dx}(ay-x^2) - (ay-x^2)\frac{d}{dx}(y^2-ax)}{(y^2-ax)^2}$$

$$\Rightarrow y_2 = \frac{(y^2-ax)\{ay_1-2x\} - (ay-x^2)\{2y.y_1-a\}}{(y^2-ax)^2}$$

$$\Rightarrow y_2 = \frac{(y^2-ax)\left[a\left(\frac{ay-x^2}{y^2-ax}\right)-2x\right] - (ay-x^2)\left[2y\left(\frac{ay-x^2}{y^2-ax}\right)-a\right]}{(y^2-ax)^2}$$

$$\Rightarrow y_2 = \frac{(y^2-ax)\left\{\dfrac{a^2y - ax^2 - 2xy^2 + 2ax^2}{y^2-ax}\right\} - (ay-x^2)\left\{\dfrac{2ay^2 - 2x^2y - ay^2 + a^2x}{y^2-ax}\right\}}{(y^2-ax)^2}$$

$$\Rightarrow y_2 = \frac{(y^2-ax)\left\{a^2y - 2xy^2 + ax^2\right\} - (ay-x^2)\left\{ay^2 - 2x^2y + a^2x\right\}}{(y^2-ax)^3}$$

$$\Rightarrow y_2 = \frac{\left\{a^2y^3 - 2xy^4 + ax^2y^2 - a^3xy + 2ax^2y^2 - a^2x^3\right\} - \left\{a^2y^3 - 2ax^2y^2 + a^3xy - ax^2y^2 + 2x^4y - a^2x^3\right\}}{(y^2-ax)^3}$$

$$\Rightarrow y_2 = \frac{\left\{a^2y^3 - 2xy^4 + 3ax^2y^2 - a^3xy - a^2x^3\right\} - \left\{a^2y^3 + 2x^4y - 3ax^2y^2 + a^3xy - a^2x^3\right\}}{(y^2-ax)^3}$$

$$\Rightarrow y_2 = \frac{a^2y^3 - 2xy^4 + 3ax^2y^2 - a^3xy - a^2x^3 - a^2y^3 - 2x^4y + 3ax^2y^2 - a^3xy + a^2x^3}{(y^2-ax)^3}$$

$$\Rightarrow y_2 = \frac{-2xy^4 - 2x^4y + 6ax^2y^2 - 2a^3xy}{(y^2-ax)^3}$$

$$\Rightarrow y_2 = \frac{-2a^3xy - 2xy\left\{y^3 + x^3 - 3axy\right\}}{(y^2-ax)^3}$$

$$\Rightarrow y_2 = \frac{-2a^3xy}{(y^2-ax)^3} \qquad \left\{y^3 + x^3 - 3axy = 0\right\}$$

Ans (proved)

Q→2 —(a) $y = \dfrac{x}{(x^2-16)}$

$\Rightarrow y = \dfrac{x}{(x-4)(x+4)} = \dfrac{A}{(x+4)} + \dfrac{B}{(x-4)}$

$\Rightarrow \dfrac{x}{(x-4)(x+4)} = \dfrac{A(x-4) + B(x+4)}{(x-4)(x+4)}$

$\Rightarrow x = A(x-4) + B(x+4)$ ------- ①

Let $x=4$, $4 = 8B$ $\{B = 1/2\}$

Let $x=-4$, $-4 = -8A$ $\{A = 1/2\}$

$\Rightarrow y = \dfrac{x}{(x-4)(x+4)} = \dfrac{1/2}{(x+4)} + \dfrac{1/2}{(x-4)}$ --- ②

n^{th} - derivative of Eqn ② w.r.t x;

$\Rightarrow y_n = \dfrac{1}{2}\left[\dfrac{(-1)^n.(n!).(1)^n}{(x+4)^{n+1}}\right] + \dfrac{1}{2}\left[\dfrac{(-1)^n.n!\,(1)^n}{(x-4)^{n+1}}\right]$

$\Rightarrow y_n = \dfrac{(-1)^n.n!}{2}\left[\dfrac{1}{(x+4)^{n+1}} + \dfrac{1}{(x-4)^{n+1}}\right]$ Ans

Q→2 (b) $y = e^{3x} \cdot \cos 5x \cdot \cos 2x$

$\Rightarrow y = e^{3x} \left\{ \dfrac{\cos 7x + \cos 3x}{2} \right\}$

$\Rightarrow y = \dfrac{1}{2} \left\{ e^{3x} \cdot \cos 7x + e^{3x} \cdot \cos 3x \right\}$ --- (1)

$\Downarrow$ $a=3$, $b=7$ $\qquad$ $\Downarrow$ $a=3$, $b=3$

$\Rightarrow$ n^{th} - derivatives of eq^n (1);

$$y_n = \frac{1}{2}\left[\left\{ (9+49)^{n/2} \cdot e^{3x} \cdot \cos\left(7x + n \cdot \tan^{-1}\left(\frac{7}{3}\right)\right) \right\} + \left\{ (9+9)^{n/2} \cdot e^{3x} \cdot \cos\left(3x + n \cdot \tan^{-1}\left(\frac{1}{1}\right)\right) \right\} \right]$$

$$y_n = \frac{1}{2}\left[(\sqrt{56})^n \cdot e^{3x} \cdot \cos\left(7x + n \cdot \tan^{-1}\left(\frac{7}{3}\right)\right) + (\sqrt{18})^n \cdot e^{3x} \cdot \cos\left(3x + \frac{n\pi}{4}\right) \right]$$

Q→2 (c) $y = \log\left(\dfrac{2x-1}{7x+1}\right)$

$\Rightarrow y = \log(2x-1) - \log(7x+1)$ --- (1)

$\Rightarrow$ n^{th} - derivatives of eq^n (1);

$$y_n = \left\{ \frac{(-1)^{n-1} \cdot (n-1)! \cdot (2)^n}{(2x-1)^n} \right\} - \left\{ \frac{(-1)^{n-1} \cdot (n-1)(7)^n}{(7x+1)^n} \right\}$$

$$y_n = (-1)^{n-1} \cdot (n-1)! \left[\frac{(2)^n}{(2x-1)^n} - \frac{(7)^n}{(7x+1)^n} \right]$$

(2)- (d) If $y = (2-3x)^{10}$ find y_9 ;

formula $\Rightarrow$ $n<m$ $\Rightarrow$ $D^n \cdot (ax+b)^m$

$\Rightarrow$ $(m) \cdot (m-1)(m-2) \ldots \{(m-n)+1\} \cdot a^n \cdot (ax+b)^{(m-n)}$

$\Rightarrow$ $m=10$; $n=9$

$y_9 = (10)(9)(8)(7) \ldots (2)(-3)^9 \cdot (2-3x)^1$;

$y_9 = -(10!) \cdot (3)^9 \cdot (2-3x)$

$y_9 = 10! \cdot (3)^9 \cdot (3x-2)$ Ans

Q→3- (a) $y = x^2 \cdot \log 3x$

$\Rightarrow$ $u = \log x$; $v = x^2$

$\Rightarrow$ nth - derivative using leibnitz's Rule ;

$\Rightarrow$ ${}^nC_0 \cdot D^n(u) \cdot v + {}^nC_1 \cdot D^{n-1}(u) \cdot D(v) + {}^nC_2 D^{n-2}(u) \cdot D^2(v)$

$\Rightarrow$ $(1)\left\{\frac{(-1)^{n-1} \cdot (n-1)! \cdot (3)^n}{(3x)^n}\right\}\{x^2\} + (n)\left\{\frac{(-1)^{n-2} \cdot (n-2)!(3)^{n-1}}{(3x)^{n-1}}\right\}\{2x\}$

$+ \frac{(n)(n-1)}{2}\left\{\frac{(-1)^{n-3} \cdot (n-3)!\,(3)^{n-2}}{(3x)^{n-2}}\right\}\{2\}$

$\Rightarrow$ $\frac{(3)^n \cdot (-1)^{n-1} \cdot (n-1)! \cdot x^2}{(3)^n \cdot (x)^n} + \frac{(3)^{n-1} \cdot (-1)^{n-2} \cdot (n-2)! \cdot (2x)}{(3)^{n-1} \cdot (x)^{n-1}} + \frac{(3)^{n-2} \cdot (-1)^{n-3} \cdot (n-3)!}{(3)^{n-2}(x)^{n-2}}$

$$\Rightarrow \frac{(-1)^{n-1}\cdot(n-1)!\cdot x^2}{(x)^n} + \frac{(-1)^{n-2}\cdot(n-2)!\,(2x)}{(x)^{n-1}} + \frac{(-1)^{n-3}\cdot(n-3)!}{(x)^{n-2}}$$

$$\Rightarrow \frac{(-1)^{n-1}(n-1)!}{(x)^{n-2}} + \frac{2(-1)^{n-2}\cdot(n-2)!}{(x)^{n-2}} + \frac{(-1)^{n-3}\cdot(n-3)!}{(x)^{n-2}}$$

$$\Rightarrow \frac{1}{(x)^{n-2}}\left\{(-1)^{n-1}\cdot(n-1)! + 2(-1)^{n-2}\cdot(n-2)! + (-1)^{n-3}\cdot(n-3)!\right\}$$

Q→3→(b) $y = x^3\cdot e^{-2x}\cdot\sin x$

$u = e^{-2x}\cdot\sin x \Longrightarrow \begin{cases} a=-2 \\ b=1 \end{cases}$

$v = x^3$

Using leibnitz's Rule;

$$\Rightarrow {}^nC_0\cdot D^n(u)\cdot v + {}^nC_1\cdot D^{n-1}(u)\cdot D(v) + {}^nC_2\cdot D^{n-2}(u)\cdot D^2(v) + {}^nC_3\cdot D^{n-3}(u)\cdot D^3(v)$$

$$\Rightarrow (1)\left\{(\sqrt{5})^n\cdot e^{-2x}\cdot\sin\left(x+n\cdot\tan^{-1}\left(-\tfrac{1}{2}\right)\right)\right\}\cdot(x)^3 +$$

$$(n)\left\{(\sqrt{5})^{n-1}\cdot e^{-2x}\cdot\sin\left(x+(n-1)\tan^{-1}\left(-\tfrac{1}{2}\right)\right)\right\}\cdot(3x^2) +$$

$$\frac{(n)(n-1)}{2}\left\{(\sqrt{5})^{n-2}\cdot e^{-2x}\cdot\sin\left(x+(n-2)\cdot\tan^{-1}\left(-\tfrac{1}{2}\right)\right)\right\}(6x) +$$

$$\frac{(n)(n-1)(n-2)}{6}\left\{(\sqrt{5})^{n-2}\cdot e^{-2x}\cdot\sin\left(x+(n-3)\cdot\tan^{-1}\left(-\tfrac{1}{2}\right)\right)\right\}(6)$$

Q→3- (c) If $y^{1/m} + y^{-1/m} = 2x$; given

$\Rightarrow y^{1/m} + \frac{1}{y^{1/m}} = 2x$

$\Rightarrow \{y^{1/m}\}^2 + 1 = 2x . y^{1/m}$

$\Rightarrow \{y^{1/m}\}^2 - 2x\{y^{1/m}\} + 1 = 0 \approx ax^2+bx+c=0$

$\Rightarrow y^{1/m} = \frac{2x \pm \sqrt{4x^2-4}}{2}$ $\qquad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\Rightarrow y^{1/m} = \frac{2x \pm \sqrt{4(x^2-1)}}{2} = x \pm \sqrt{(x^2-1)}$

$\Rightarrow y = [x \pm \sqrt{x^2-1}]^m$ ---- (1)

$\Rightarrow \log y = m . \log [x \pm \sqrt{x^2-1}]$ -- (2)

Differentiating Both sides w.r.t x

$\Rightarrow \frac{1}{y} . y_1 = m . \left(\frac{1}{x \pm \sqrt{x^2-1}}\right) . \left\{1 \pm \frac{2x}{2\sqrt{x^2-1}}\right\}$

$\Rightarrow \frac{y_1}{y} = m . \left(\frac{1}{x \pm \sqrt{x^2-1}}\right)\left(\frac{\sqrt{x^2-1} \pm x}{\sqrt{x^2-1}}\right)$

$\Rightarrow \frac{y_1}{y} = \frac{\pm m}{\sqrt{x^2-1}}$

$\Rightarrow y_1 . \sqrt{x^2-1} = \pm y m$

$\Rightarrow y_1^2 (x^2-1) = y^2 m^2$ --- (3)

Differentiating again;

$$\Rightarrow (y_1^2)\{2x\} + (x^2-1)\{2y_1 \cdot y_2\} = m^2 \cdot 2y \cdot y_1$$

$$\Rightarrow 2y_1\{xy_1 + (x^2-1) \cdot y_2\} = m^2 \cdot 2y \cdot y_1$$

$$\Rightarrow (x^2-1)y_2 + xy_1 - m^2 y = 0 \quad \text{---} (4)$$

Differentiating it n^{th}-times using leibnitz's theorem;

$$\Rightarrow \left\{(1)\{y_{n+2}\}(x^2-1) + (n)\{y_{n+1}\}(2x) + \frac{(n)(n-1)}{2}\{y_n\}(2)\right\}$$
$$+ \left\{(1)\{y_{n+1}\}(x) + (n)\{y_n\}(1)\right\}$$
$$+ \left\{-m^2 \cdot y_n\right\} = 0$$

$$\Rightarrow (x^2-1)y_{n+2} + 2x \cdot n \cdot y_{n+1} + (n)(n-1)y_n + x \cdot y_{n+1} + n \cdot y_n - m^2 y_n = 0$$

$$\Rightarrow (x^2-1) \cdot y_{n+2} + y_{n+1}\{2 \cdot x \cdot n + x\} + y_n\{(n)(n-1) + n - m^2\} = 0$$

$$\Rightarrow (x^2-1) \cdot y_{n+2} + (2n+1) \cdot x \cdot y_{n+1} + (n^2-m^2) \cdot y_n = 0$$

(3) → (b) If $y = e^{m \cdot \cos^{-1} x}$, prove that

i) $(1-x^2)y_2 - xy_1 = m^2 y$

$$\Rightarrow \quad y = e^{m \cdot \cos^{-1} x} \quad --- (1)$$

$$\Rightarrow \quad y_1 = (e^{m \cos^{-1} x}) \cdot \left\{ \frac{-m}{\sqrt{1-x^2}} \right\}$$

$$\Rightarrow \quad y_1 = \frac{-m \cdot y}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad y_1 \cdot \sqrt{1-x^2} = -m \cdot y$$ (square both side)

$$\Rightarrow \quad y_1^2 \cdot (1-x^2) = m^2 y^2 \quad -- (2)$$

Diff w.r.t (x)

$$\Rightarrow \quad (y_1^2)\{-2x\} + (1-x^2)\{2y_1 y_2\} = m^2(2y \cdot y_1)$$

$$\Rightarrow \quad 2y_1 \{-x y_1 + (1-x^2) y_2\} = 2y \cdot y_1 m^2$$

$$\Rightarrow \quad (1-x^2) \cdot y_2 - x y_1 = y m^2 \quad -- (3)$$ Ans

(ii) $(1-x^2) \cdot y_{n+2} - (2n+1) \cdot x \cdot y_{n+1} - (n^2+m^2) \cdot y_n = 0$

n^{th}-times derivative using Leibnitz's theorem ;

$$\left\{ (1) \cdot \{y_{n+2}\}(1-x^2) + (n)\{y_{n+1}\}(-2x) + \frac{(n)(n-1)}{2}(y_n)(-2) \right\} -$$

$$\{ (1)\{y_{n+1}\}(x) + (n)\{y_n\}(1) \} -$$

$$\{ m^2 \cdot y_n \} = 0$$

$$\Rightarrow \quad (1-x^2) \cdot y_{n+2} + y_{n+1}\{-2xn - x\} + y_n\{-(n)(n-1) - n - m^2\} = 0$$

$$\Rightarrow \quad (1-x^2) \cdot y_{n+2} - (2n+1) \cdot x \cdot y_{n+1} - (n^2+m^2) y_n = 0$$ Ans