SUCCESSIVE DIFFERENTIATION

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*** Successive Differentiation :-

Let us assume, $y = f(x)$ ---- ①

→ Differentiation of y, with respect to x :-

$$\frac{d}{dy}\{y\} = \frac{dy}{dx} = f'(x) \longrightarrow \text{first derivative of } y.$$

→ Differentiation of $\frac{dy}{dx}$ w.r.t x :-

$$\frac{d}{dy}\left\{\frac{dy}{dx}\right\} = \frac{d^2y}{dx^2} = f''(x) \longrightarrow \text{Second derivative of } y$$

→ Differentiation of $\frac{d^2y}{dx^2}$ w.r.t x :-

$$\frac{d}{dy}\left\{\frac{d^2y}{dx^2}\right\} = \frac{d^3y}{dx^3} = f'''(x) \rightarrow \text{third derivative of } y$$

⋮

→ Differentiation of $\frac{d^{n-1}y}{dx^{n-1}}$ w.r.t x :-

$$\frac{d}{dy}\left\{\frac{d^{n-1}y}{dx^{n-1}}\right\} = \frac{d^ny}{dx^n} = f^{(n)}(x) \rightarrow n^{th} \text{ derivative of } y.$$

*** Notations for Successive derivatives of $y = f(x)$;

→ $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ ---- $\frac{d^ny}{dx^4}$

→ Dy, D^2y, D^3y, D^4y ---- D^ny

→ y_1, y_2, y_3, y_4 ---- y_n

→ $f'(x)$, $f''(x)$, $f'''(x)$, $f^{iv}(x)$ ---- $f^{(n)}(x)$

$D = \frac{d}{dx}$

D = differentiation operator

Q→ ① If $y = e^{ax}.\sin bx$, prove that $y_2 - 2ay_1 + (a^2+b^2)y = 0$

(method-1) → long method.

$\Rightarrow \quad y = e^{ax}.\sin bx$ --- ①

diff of y w.r.t x $\Rightarrow \quad y_1 = \frac{dy}{dx} = \{e^{ax}\}\{b\cos bx\} + (\sin bx)\{ae^{ax}\}$

$y_1 = be^{ax}\cos bx + ae^{ax}\sin bx$ --- ②

diff of $\frac{dy}{dx}$ w.r.t x $\Rightarrow \quad y_2 = \left[(be^{ax})\{-b\sin bx\} + (b\cos bx)\{abe^{ax}\}\right] +$

$\left[(ae^{ax})\{b\cos bx\} + (a\sin bx)\{ae^{ax}\}\right]$

$y_2 = -b^2e^{ax}\sin bx + abe^{ax}\cos bx + abe^{ax}\cos bx + a^2e^{ax}\sin bx$

$y_2 = e^{ax}\sin bx\{a^2-b^2\} + e^{ax}\cos bx\{ab+ab\}$

$y_2 = e^{ax}\sin bx\{a^2-b^2\} + 2e^{ax}.ab.\cos bx$ --- ③

Substituting Eq^n ③, ② & ① in $y_2 - 2ay_1 + (a^2+b^2)y = 0$

$\Rightarrow \left[e^{ax}\sin bx(a^2-b^2) + 2abe^{ax}\cos bx\right] - 2a\left[be^{ax}\cos bx + ae^{ax}\sin bx\right]$

$+ \quad (a^2+b^2)\left[e^{ax}.\sin bx\right] = 0$

$\Rightarrow a^2e^{ax}\sin bx - b^2e^{ax}\sin bx + 2abe^{ax}\cos bx - 2abe^{ax}\cos bx - 2a^2e^{ax}\sin bx$

$+ \quad a^2e^{ax}\sin bx + b^2e^{ax}\sin bx = 0$

$\Rightarrow$ (L.H.S = R.H.S) Proved

(method-2) $\rightarrow$ Short method.

$\Rightarrow \quad y = e^{ax} . \sin bx \quad -- (1)$

diff $\Rightarrow \quad y_1 = (e^{ax}) \{b \cos bx\} + (\sin bx) \{a e^{ax}\}$

$\Rightarrow \quad y_1 = b e^{ax} \cos bx + \underline{a e^{ax} \sin bx}$

$\Rightarrow \quad y_1 = b e^{ax} \cos bx + a y$ {let take y to left side}

diff $\Rightarrow \quad y_1 - a y = b e^{ax} \cos bx$

$\Rightarrow \quad y_2 - a y_1 = (b e a^{x}) \{- b \sin bx\} + (b \cos bx) \{a e^{ax}\}$

$\Rightarrow \quad y_2 - a y_1 = - b^2 e^{ax} \sin bx + \underline{ab\, e^{ax} \cos bx}$

$\Rightarrow \quad y_2 - a y_1 = - b^2 e^{ax} \sin bx + a \{y_1 - a y\}$

$\Rightarrow \quad y_2 - a y_1 = - b^2 \underline{e^{ax} \sin bx} + a y_1 - a^2 y$

$\Rightarrow \quad y_2 - a y_1 - a y_1 + a^2 y + b^2 y = 0$

$\Rightarrow \quad y_2 - 2 a y_1 + y (a^2 + b^2) = 0$ proved

Q→2 If $y = \sin(\sin x)$ ---- ①, prove that $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = 0$

→ first derivative of y, $y_1 = \{\cos(\sin x)\}\{\cos x\}$

$= \cos x \cdot \cos(\sin x)$ ---- ②

→ Second derivative of y, $y_2 = (\cos x)\{-\sin(\sin x)\cdot \cos x\} + (\cos(\sin x))\{-\sin x\}$

$y_2 = -\cos^2 x \cdot \sin(\sin x) - \sin x \cdot \cos(\sin x)$

$y_2 = -y \cdot \cos^2 x - \sin x \cdot \cos(\sin x)$ ---- ③

$\Rightarrow \frac{d^2y}{dx^2} + \tan x \cdot \frac{dy}{dx} + y\cos^2 x = 0$

or

$\Rightarrow y_2 + \tan x \cdot y_1 + y\cos^2 x = 0$

$\Rightarrow \{-y \cdot \cos^2 x - \sin x \cdot \cos(\sin x)\} + \tan x\{\cos x \cdot \cos(\sin x)\} + y\cos^2 x = 0$

$\Rightarrow -y \cdot \cos^2 x - \sin x \cdot \cos(\sin x) + \frac{\sin x}{\cos x} \cdot \cos x \cdot \cos(\sin x) + y\cos^2 x = 0$

$\Rightarrow -y \cdot \cos^2 x - \sin x \cdot \cos(\sin x) + \sin x \cdot \cos(\sin x) + y\cos^2 x = 0$

$\Rightarrow$ LHS = R.H.S proved

Q→3 If $ax^2 + 2hxy + by^2 = 1$ --- (1)

prove that $\dfrac{d^2y}{dx^2} = \dfrac{h^2 - ab}{(hx+by)^3}$

⇒ first derivative of y, $\dfrac{d}{dx}\{ax^2 + 2hxy + by^2\} = \dfrac{d}{dx}\{1\}$

$$2ax + 2hx\left\{\frac{dy}{dx}\right\} + 2hy\{1\} + 2by\frac{dy}{dx} = 0$$

$$\Rightarrow 2ax + 2hy + \frac{dy}{dx}\{2hx + 2by\} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(ax+hy)}{(hx+by)} \quad \text{--- (2)}$$

Second derivative

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{(hx+by)\left\{a + h\frac{dy}{dx}\right\} - (ax+hy)\left\{h + b\frac{dy}{dx}\right\}}{(hx+by)^2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{hxa + aby + h^2x\frac{dy}{dx} + hby\frac{dy}{dx} - axh - h^2y - abx\frac{dy}{dx} - hby\frac{dy}{dx}}{(hx+by)^2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{\frac{dy}{dx}(h^2x - abx) + aby - h^2y}{(hx+by)^2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{x\frac{dy}{dx}(h^2 - ab) - y(h^2 - ab)}{(hx+by)^2}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{(h^2 - ab)\left\{x\frac{dy}{dx} - y\right\}}{(hx+by)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{(h^2-ab)\left\{-x\left(\frac{ax+hy}{hx+by}\right)-y\right\}}{(hx+by)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(h^2-ab)\left\{\frac{x(ax+hy)+y(hx+by)}{(hx+by)}\right\}}{(hx+by)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(h^2-ab)\left\{\frac{ax^2+xhy+xhy+by^2}{(hx+by)}\right\}}{(hx+by)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(h^2-ab)(ax^2+2hxy+by^2)}{(hx+by)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(h^2-ab)}{(hx+by)^3} \qquad \{ax^2+2hxy+by^2=1\}$$

A

*** Successive Differentiation using parametric function :-

$\Rightarrow$ If $\left.\begin{array}{l} x = f(t) \\ y = g(t) \end{array}\right\}$ t = parameter function.

Q→ 1) If $x = a(\cos t + t\sin t)$ --- ①
$y = a(\sin t - t\cos t)$ --- ②

find d^2y/dx^2 ?

Ans differentiative (x) w.r.t (t) in Eqⁿ ① ;

$dx = a\{-\sin t + (\sin t)\{1\} + (t)\{\cos t\}\}$

$dx = a[t\cos t]$ --- ③

differentiative (y) w.r.t (t) in Eqⁿ ② ;

$dy = a[\cos t - (\cos t)\{1\} - (t)\{-\sin t\}]$

$dy = a[t\sin t]$ --- ④

Now using Eqⁿ ③ & ④

$$\frac{dy}{dx} = \frac{a[t\sin t]}{a[t\cos t]}$$

$$\frac{dy}{dx} = \tan t \quad \text{----} ⑤$$

Second derivative of Eqⁿ ⑤

$$\frac{d^2y}{dx^2} = (\sec^2 t).\frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 t\left(\frac{1}{a\,t\cos t}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{at\cos^3 t} \quad \text{Ans}$$

Q→2 If $x = 2\cos t - \cos 2t$ find $\frac{d^2y}{dx^2}$ when $t = \frac{\pi}{2}$
$y = 2\sin t - \sin 2t$

Ans $y = 2\sin t - \sin 2t$

$$\frac{dy}{dt} = 2\cos t - 2\cos 2t = 2\{\cos t - \cos 2t\} \quad -(1)$$

$$x = 2\cos t - \cos 2t$$

$$\frac{dx}{dt} = -2\sin t + 2\sin 2t = 2\{\sin 2t - \sin t\} \quad -(2)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\{\cos t - \cos 2t\}}{2\{\sin 2t - \sin t\}}$$

$$\frac{dy}{dx} = \frac{\cos t - \cos 2t}{\sin 2t - \sin t} \quad -(3)$$

$$\frac{d^2y}{dx^2} = \frac{(\sin 2t - \sin t)\{-\sin t + 2\sin 2t\} - (\cos t - \cos 2t)\{2\cos 2t - \cos t\}}{(\sin 2t - \sin t)^2} \times \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{(\sin 2t - \sin t)(2\sin 2t - \sin t) - (\cos t - \cos 2t)(2\cos 2t - \cos t)}{(\sin 2t - \sin t)^2 \cdot 2(\sin 2t - \sin t)}$$

put $(t = \pi/2)$

$$\frac{d^2y}{dx^2} = \frac{\left(\sin\pi - \sin\frac{\pi}{2}\right)\left(2\sin\pi - \sin\frac{\pi}{2}\right) - \left(\cos\frac{\pi}{2} + \cos\pi\right)\left(2\cos\pi - \cos\frac{\pi}{2}\right)}{2\left(\sin\pi - \sin\frac{\pi}{2}\right)^2 \cdot \left(\sin\pi - \sin\frac{\pi}{2}\right)}$$

$$\frac{d^2y}{dx^2} = \frac{(0-1)(0-1) - (0+1)(-2-0)}{2(0-1)^3}$$

$$\left\{\begin{array}{l} \sin\pi = 0 \\ \sin\frac{\pi}{2} = 1 \\ \cos\pi = -1 \\ \cos\frac{\pi}{2} = 0 \end{array}\right\}$$

$$\frac{d^2y}{dx^2} = \frac{(1)+(2)}{-2} = \frac{-3}{2} \quad \text{A}$$

—— X —— X —— X —— X ——

** Some Extra Questions:-

Q→1 $\quad y = e^{-kt} \cdot \cos(lt+c) \quad \text{---} (1)$

show that $\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + n^2y = 0$, where $n^2 = k^2 + l^2$

first derivative

$$\frac{dy}{dt} = \left(\cos(lt+c)\right) \cdot \left\{-ke^{-kt}\right\} + \left(e^{-kt}\right)\left\{-\sin(lt+c) \cdot l\right\}$$

$$\frac{dy}{dt} = -k \cdot y - l e^{-kt} \sin(lt+c)$$

$$\Rightarrow \frac{dy}{dt} + ky = -l e^{-kt} \cdot \sin(lt+c) \quad \text{--} (2)$$

Second derivative

$$\Rightarrow \frac{d^2y}{dt^2} + k\frac{dy}{dt} = \left(-l\sin(lt+c)\right)\left\{-ke^{-kt}\right\} +$$

$$\left(-le^{-kt}\right)\left\{l \cdot \cos(lt+c)\right\}$$

Now using Eqn (2);

$$\Rightarrow \frac{d^2y}{dt^2} + K\frac{dy}{dt} = -K\left\{\frac{dy}{dt} + Ky\right\} - l^2y$$

$$\Rightarrow \frac{d^2y}{dt^2} + K\frac{dy}{dt} = -K\frac{dy}{dt} - K^2y - l^2y$$

$$\Rightarrow \frac{d^2y}{dt^2} + 2K\frac{dy}{dt} + (K^2+l^2)y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} + 2K\frac{dy}{dt} + n^2y = 0 \quad \text{As}$$

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Q→2 If $y = \sinh\left[m\log\left(x+\sqrt{x^2+1}\right)\right]$ --- (1)

Show that $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = m^2y$

Ans Important : (1) $\frac{d}{dx}(\sinh x) = \cosh x$

(2) $\frac{d}{dx}(\cosh x) = \sinh x$

(3) $\frac{d}{dx}\log x = \frac{1}{x}$

(4) $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$$\sinh\left[m.\log\left(x+\underbrace{\sqrt{x^2+1}}\right)\right] = \text{Composite function}$$

main function — 2nd function — 3rd function — 4th function

$$\frac{d}{dx}\sinh\left[m.\log\left(x+\sqrt{x^2+1}\right)\right]$$

$$= \underbrace{\cosh\left[m.\log\left(x+\sqrt{x^2+1}\right)\right]}_{\text{derivative of main function}} \cdot \underbrace{\left[\frac{m}{x+\sqrt{x^2+1}}\right]}_{\text{derivative of } 2^{nd} \text{ function}} \cdot \underbrace{\left[1+\frac{1}{2\sqrt{x^2+1}}\right.}_{\text{derivative of } 3^{rd} \text{ function}} \cdot \underbrace{\left.[2x]\right]}_{\text{derivative of } 4^{th} \text{ func.}}$$

$$= \cosh\left[m.\log\left(x+\sqrt{x^2+1}\right)\right] \cdot \left[\frac{m}{\left(x+\sqrt{x^2+1}\right)}\right]\left[\frac{\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}}\right]$$

$$= \frac{m.\cosh[\ \phi\ ]}{\sqrt{x^2+1}} \qquad \left(\text{if } \phi = m.\log\left(x+\sqrt{x^2+1}\right)\right)$$

— X — X — X — X — X —

$\longrightarrow$ differentiating Eq^n 1

$$\frac{dy}{dx} = \frac{m.\cosh\phi}{\sqrt{x^2+1}} \quad \text{or} \quad \frac{m.\cosh\left[m\log\left(x+\sqrt{x^2+1}\right)\right]}{\sqrt{x^2+1}} \text{ --- } 2$$

$$\frac{d^2y}{dx^2} = \frac{\left(\sqrt{x^2+1}\right)\frac{d}{dx}\left[m.\cosh\left(m\log\left(x+\sqrt{x^2+1}\right)\right)\right] - \left(m.\cosh\left[m\log\left(x+\sqrt{x^2+1}\right)\right]\right)\frac{d}{dx}\left(\sqrt{x^2+1}\right)}{\left(x^2+1\right)}$$

$$\frac{d^2y}{dx^2} = \frac{\left(\sqrt{x^2+1}\right)\cdot\left\{\frac{m^2\sinh\left(m\log\left(x+\sqrt{x^2+1}\right)\right)}{\sqrt{x^2+1}}\right\} - \left(m.\cosh\left[m\log\left(x+\sqrt{x^2+1}\right)\right]\right)\cdot\left\{\frac{x}{\sqrt{x^2+1}}\right\}}{\left(x^2+1\right)}$$

$$\Rightarrow (x^2+1)\frac{d^2y}{dx^2} = m^2 \sinh\left[m\log(x+\sqrt{x^2+1})\right] - \frac{m.x.\cosh\left[m\log(x+\sqrt{x^2+1})\right]}{\sqrt{x^2+1}}$$

$$\Rightarrow (x^2+1)\frac{d^2y}{dx^2} = m^2.y - x\frac{dy}{dx}$$

$$\Rightarrow (x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = m^2y$$ As

*** n^{th} Derivative of Some standard functions :-

(1) Exponential function $\Rightarrow D^n(e^{ax}) = a^n \cdot e^{(ax)}$

(2) Exponential function $\Rightarrow D^n(a^x) = a^x \cdot \{\log a\}^n$

(3) Algebric function or Power function $\Rightarrow D^n(ax+b)^m$

ⓐ $(n>m) \Rightarrow D^n(ax+b)^m = 0$

ⓑ $(n<m) \Rightarrow D^n(ax+b)^m = (m)(m-1)(m-2)\ldots(m-n+1).a^n.(ax+b)^{(m-n)}$

ⓒ $(n=m) \Rightarrow D^n(ax+b)^m = n!\cdot a^n.$

ⓓ $(m=-ve) \Rightarrow D^n\left(\frac{1}{ax+b}\right) = D^n(ax+b)^{-1} = \frac{(-1)^n(n!)\cdot a^n}{(ax+b)^{n+1}}$

(4) log function $\Rightarrow D^n(\log(ax+b)) = \frac{(-1)^{n+1}(n-1)!\,a^n}{(ax+b)^n}$

(5) Trignometric function $\Rightarrow D^n\sin(ax+b) = a^n.\sin\left(ax+b+\frac{n\pi}{2}\right)$

$D^n\cos(ax+b) = a^n.\cos\left(ax+b+\frac{n\pi}{2}\right)$

(6) $D^n\left[e^{ax}.\sin(bx+c)\right] = (a^2+b^2)^{n/2}.e^{ax}.\sin\left[bx+c+n.\tan^{-1}\left(\frac{b}{a}\right)\right]$

$D^n\left[e^{ax}.\cos(bx+c)\right] = (a^2+b^2)^{n/2}.e^{ax}.\cos\left[bx+c+n.\tan^{-1}\left(\frac{b}{a}\right)\right]$

Q→ find the n^{th} Derivative of the following functions:-

① $y = e^{3x}$

$y_n = D^n(e^{3x}) = 3^n \cdot e^{3x}$

② $y = \sin(2x+3)$

$y_n = 2^n \cdot \sin\left(2x+3+\frac{n\pi}{2}\right)$

③ $y = \cos(5x-1)$

$y_n = 5^n \cos\left(5x-1+\frac{n\pi}{2}\right)$

④ $y = \log(5x-2)$

$y_n = \dfrac{(-1)^{n-1} \cdot (n-1)! \cdot 5^n}{(5x-2)^n}$

⑤ $y = \dfrac{1}{2x+7} = (2x+7)^{-1}$

$y_n = \dfrac{(-1)^n \cdot (n)! \cdot 2^n}{(2x+7)^{n+1}}$

⑥ $y = e^{2x} \cdot \sin(3x+4)$ $\quad a=2$, $b=3$

$y_n = (a^2+b^2)^{n/2} \cdot e^{ax} \cdot \sin\left(bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right)$

$y_n = (13)^{n/2} \cdot e^{2x} \cdot \sin\left(3x+4+n\tan^{-1}\left(\frac{3}{2}\right)\right)$

(7) $y = e^{-2x} \cdot \cos(2x-1)$

$a=-2,\ b=2,\ c=-1$

$y_n = (a^2+b^2)^{n/2} \cdot e^{ax} \cdot \cos\left(bx+c+n\cdot\tan^{-1}\left(\frac{b}{a}\right)\right)$

$y_n = (\sqrt{8})^n \cdot e^{-2x} \cdot \cos\left(2x-1+n\tan^{-1}\left(\frac{-2}{2}\right)\right)$

$y_n = (2\sqrt{2})^n \cdot e^{-2x} \cdot \cos\left(2x-1+n\tan^{-1}(-1)\right)$

$y_n = (2\sqrt{2})^n \cdot e^{-2x} \cdot \cos\left(2x-1+n\frac{3\pi}{4}\right)$ Ans

$\left\{\tan^{-1}(-1) = \frac{3\pi}{4}\right\}$

(Trigonometric type)

Q→ Find the n^{th} – derivative of following functions :–

(1) $y = \cos 3x \cdot \cos x$ $\left(2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)\right)$

$y = (\cos 4x + \cos 2x)/2$

$y_n = \frac{1}{2}\left[4^n \cdot \cos\left(4x+\frac{n\pi}{2}\right) + 2^n \cdot \cos\left(2x+\frac{n\pi}{2}\right)\right]$ Ans

(2) $y = \sin 3x \cdot \sin 2x \cdot \sin x$

$\left(2\sin\alpha\sin\beta = -(\cos(\alpha+\beta)+\cos(\alpha-\beta))\right)$

$y = \left[\frac{-\cos 5x + \cos x}{2}\right] \cdot \sin x$ $\left(2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)\right)$

$y = \frac{1}{2}\left[-\cos 5x \cdot \sin x + \cos x \cdot \sin x\right]$

$y = \frac{1}{2}\left[-\left\{\frac{\sin 6x - \sin 4x}{2}\right\} + \left\{\frac{\sin 2x - \sin 0}{2}\right\}\right]$

$$y = \frac{1}{2}\left[-\frac{\sin 6x}{2} + \frac{\sin 4x}{2} + \frac{\sin 2x}{2}\right]$$

$$y = \frac{1}{4}\left[\sin 2x + \sin 4x - \sin 6x\right]$$

$$y_n = \frac{1}{4}\left[2^n \sin\left(2x+\frac{n\pi}{2}\right) + 4^n \sin\left(4x+\frac{n\pi}{2}\right) - 6^n \sin\left(6x+\frac{n\pi}{2}\right)\right]$$

Ans

(3) $y = \cos^2 x$

double angle

$$\left\{ \cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2} \right\}$$

$$y = \frac{1+\cos 2x}{2}$$

$$y_n = \frac{1}{2}\left[0 + 2^n \cos\left(2x+\frac{n\pi}{2}\right)\right]$$ Ans

(4) $y = 4\sin^2 6x$

$$y = 4.\left[\frac{1-\cos 12x}{2}\right] = 2(1-\cos 12x)$$

$$y_n = 2\left[0 - 12^n \cos\left(12x+\frac{n\pi}{2}\right)\right]$$ Ans

(5) $y = \sin^3 x$ $\left\{ \begin{array}{l} \sin 3x = 3\sin x - 4\sin^3 x \\ \cos 3x = 4\cos^3 x - 3\cos x \end{array} \right\}$

$$y = \frac{3\sin x - \sin 3x}{4}$$

$$y_n = \frac{1}{4}\left[3\cdot(1)^n \cdot \sin\left(x + \frac{n\pi}{2}\right) - 3^n \sin\left(3x + \frac{n\pi}{2}\right)\right] \text{ As}$$

(6) $y = \cos^4 x$

$$y = (\cos^2 x)^2$$

$$y = \left\{\frac{1+\cos 2x}{2}\right\}^2 = \frac{1}{4}\left\{1 + 2\cos 2x + \cos^2 2x\right\}$$

$$y = \frac{1}{4}\left\{1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right\}$$

$$y = \frac{1}{8}\left\{2 + 4\cos 2x + 1 + \cos 4x\right\}$$

$$y = \frac{1}{8}\left\{3 + 4\cos 2x + \cos 4x\right\}$$

$$y_n = \frac{1}{8}\left[0 + 4\cdot 2^n \cdot \cos\left(2x + \frac{n\pi}{4}\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right)\right] \text{ As}$$

(7) $y = \sin^3 x \cdot \cos^2 x$ $\quad (\sin 2x = 2\sin x \cdot \cos x)$

$y = \sin x \{\sin^2 x \cdot \cos^2 x\}$

$y = \sin x \{\sin x \cdot \cos x\}^2$

$y = \sin x \left\{\frac{\sin 2x}{2}\right\}^2$

$y = \frac{\sin x}{4}(\sin^2 2x) = \frac{\sin x}{4}\left[\frac{1-\cos 4x}{2}\right)$

$y = \frac{1}{8}\left[\sin x - \cos 4x \sin x\right]$

$y = \frac{1}{8}\left[\sin x - \left\{\frac{\sin 5x - \sin 3x}{2}\right\}\right]$

$y = \frac{1}{8}\left[\sin x - \frac{\sin 5x}{2} + \frac{\sin 3x}{2}\right]$

$y = \frac{1}{16}\left[2\sin x - \sin 5x + \sin 3x\right]$

$y_n = \frac{1}{16}\left[2.(1)^n . \sin\left(x + \frac{n\pi}{2}\right) - 5^n . \sin\left(5x + \frac{n\pi}{2}\right) + 3^n . \sin\left(3x + \frac{n\pi}{2}\right)\right]$

Ans

(8) $y = e^{2x} \cdot \cos^2 x$

$\left\{ e^{ax} \cdot \cos(bx+c) = (a^2+b^2)^{n/2} \cdot e^{ax} \cdot \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right) \right.$

$y = e^{2x} \cdot \left\{ \frac{1+\cos 2x}{2} \right\}$

$y = \frac{1}{2}\left\{ e^{2x} + e^{2x} \cdot \cos 2x \right\}$ $a=2$, $b=2$

$y_n = \frac{1}{2}\left[2^n \cdot e^{2x} + (8)^{n/2} \cdot e^{2x} \cdot \cos\left(2x + n\tan^{-1}\left(\frac{1}{1}\right)\right) \right]$

$y_n = \frac{1}{2}\left[2^n \cdot e^{2x} + (8)^{n/2} \cdot e^{2x} \cdot \cos\left(2x + \frac{n\pi}{4}\right) \right]$ Ans

(9) $y = e^{-x} \cdot \sin 3x \cdot \sin x$

$y = e^{-x} \left\{ \frac{-\cos 4x + \cos 2x}{2} \right\}$

$y = \frac{1}{2}\left[-e^{-x}\cos 4x + e^{-x} \cdot \cos 2x \right]$

$a=-1$, $b=4$; $a=-1$, $b=2$

$y_n = \frac{1}{2}\left[-\left\{ (17)^{n/2} \cdot e^{-x} \cdot \cos\left(4x + n\tan^{-1}\left(-\frac{4}{1}\right)\right) \right\} + \left\{ (5)^{n/2} \cdot e^{-x} \cdot \cos\left(2x + n\tan^{-1}(-2)\right) \right\} \right]$

$y_n = \frac{1}{2}\left[(\sqrt{5})^n \cdot e^{-x} \cdot \cos(2x + n\tan^{-1}(-2)) - (\sqrt{17})^n \cdot e^{-x} \cdot \cos(4x + n\tan^{-1}(-4)) \right]$

(algebraic function) (note→ solve these type of question using Partial fraction)

Q→ find the n^{th} derivative of following functions :-

① $y = \frac{x+2}{x^2-3x+2} = \frac{(x+2)}{(x-1)(x-2)}$

Let $\frac{(x+2)}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$ --- ①

$\Rightarrow \frac{(x+2)}{(x-1)(x-2)} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$

$\Rightarrow (x+2) = A(x-2) + B(x-1)$ --- ②

$x \to 1 \Rightarrow 3 = -A \quad (A=-3)$

$x \to 2 \Rightarrow 4 = +B \quad (B=4)$

$\Rightarrow y = \frac{(x+2)}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} = \frac{4}{(x-2)} - \frac{3}{(x-1)}$

$y = 4(x-2)^{-1} - 3(x-1)^{-1}$

$y_n = 4\left\{\frac{(-1)^n . n! . (1)^n}{(x-2)^{n+1}}\right\} - 3\left\{\frac{(-1)^n . (n)! . (1)^n}{(x-1)^{n+1}}\right\}$

(2) $y = \dfrac{x}{(x-1)(2x+3)}$

Let, $\dfrac{x}{(x-1)(2x+3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(2x+3)}$ --- (1)

$\Rightarrow \dfrac{x}{(x-1)(2x+3)} = \dfrac{A(2x+3) + B(x-1)}{(x-1)(2x+3)}$

$\Rightarrow \quad x = A(2x+3) + B(x-1)$ --- (2)

$x \to 1 \quad \Rightarrow \quad 1 = 5A \qquad (A = 1/5)$

$x \to -\frac{3}{2} \quad \Rightarrow \quad -\frac{3}{2} = B\left(-\frac{3}{2} - 1\right)$

$-\frac{3}{2} = -\frac{5}{2} \cdot B \qquad \left(B = \frac{3}{5}\right)$

$\Rightarrow y = \dfrac{x}{(x-1)(2x+3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(2x+3)} = \dfrac{1/5}{(x-1)} + \dfrac{3/5}{(2x+3)}$

$y = \frac{1}{5}(x-1)^{-1} + \frac{3}{5}(2x+3)^{-1}$

$y_n = \frac{1}{5}\left\{\dfrac{(-1)^n \cdot n! \cdot (1)^n}{(x-1)^{n+1}}\right\} + \frac{3}{5}\left\{\dfrac{(-1)^n \cdot n! \cdot (2)^n}{(2x+3)^{n+1}}\right\}$

(3) $y = \dfrac{x}{(x-1)(x-2)(x-3)}$

Let, $\dfrac{x}{(x-1)(x-2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-3)}$ --- (1)

$\Rightarrow \dfrac{x}{(x-1)(x-2)(x-3)} = \dfrac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$

$\Rightarrow x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$ --- (2)

$x \to 1 \Rightarrow 1 = A(-1)(-2)$ $(A = 1/2)$

$x \to 2 \Rightarrow 2 = B(1)(-1)$ $(B = -2)$

$x \to 3 \Rightarrow 3 = C(2)(1)$ $(C = 3/2)$

$y = \dfrac{x}{(x-1)(x-2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-3)} = \dfrac{1/2}{(x-1)} - \dfrac{2}{(x-2)} + \dfrac{3/2}{(x-3)}$

$y_n = \dfrac{1}{2}\left\{\dfrac{(-1)^n . n! . (1)^n}{(x-1)^{n+1}}\right\} - 2\left\{\dfrac{(-1)^n . n! . (+1)^n}{(x-2)^{n+1}}\right\} + \dfrac{3}{2}\left\{\dfrac{(-1)^n . n! . (1)^n}{(x-3)^{n+1}}\right\}$

*** Leibnitz's theorem for n^{th} derivative of the product of two functions :-

→ If u, v be two functions of x possessing derivatives of n^{th} order, then;

$$D^n[u \cdot v] = D^n(u) \cdot v + {}^nC_1 \cdot D^{n-1}(u) \cdot D(v) + {}^nC_2 \cdot D^{n-2}(u) \cdot D^2(v) + \text{-------} \; {}^nC_n \cdot u \cdot D^n(v)$$

$$\left\{ {}^nC_r = \frac{n!}{(n-r)! \cdot r!} \right\} \qquad {}^nC_1 = \frac{n!}{(n-1)!\,1!} = n \qquad {}^nC_2 = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$

Q→1 . Find the n^{th} derivative of $e^x \cdot (2x+3)^3$

Ans $U = e^x$; $v = (2x+3)^3$

$\Rightarrow v_1 = 3(2x+3)^2 \cdot 2$, $v_2 = 6(2x+3) \cdot 4$, $v_3 = 6 \cdot 4 \cdot 2$

but V_4, V_5 etc are all zero.

$$D^n(e^x \cdot (2x+3)^3) = D^n(u) \cdot v + {}^nC_1 \cdot D^{n-1}(u) \cdot D(v) + {}^nC_2 D^{n-2}(u) \cdot D^2(v) + {}^nC_3 \cdot D^{n-3}(u) \cdot D^3(v)$$

$$= e^x \cdot (2x+3)^3 + n \cdot e^x \cdot 6(2x+3)^2 + \frac{n(n-1)}{2!} e^x \cdot 24(2x+3) + \frac{n(n-1)(n-2)}{6} e^x (48)$$

$$= e^x \left[(2x+3)^3 + 6n(2x+3)^2 + 12n(n-1)(2x+3) + 8(n)(n-1)(n-2) \right]$$ Ans

Q→2 find the n^{th} derivative of $x^2 \sin 3x$

Ans $y = x^2 \cdot \sin 3x$

$u = x^2$; $v = \sin 3x$ (X) {V should be that function whose derivative will be zero}

$v = x^2$; $u = \sin 3x$ (✓)

$$y_n = D^n(u).v + {}^nC_1 D^{n-1}(u).D(v) + {}^nC_2 D^{n-2}(u).D^2(v)$$

$$y_n = \left\{3^n \cdot \sin\left(3x + \frac{n\pi}{2}\right)\right\}(x^2) + n \cdot \left\{3^{(n-1)} \sin\left(3x + \frac{(n-1)\pi}{2}\right)\right\}\{2x\}$$
$$+ \frac{n(n-1)}{2}\left\{(3)^{(n-2)} \cdot \sin\left(3x + \frac{(n-2)\pi}{2}\right)\right\} \cdot (2)$$

$$y_n = 3^n \cdot x^2 \sin\left(3x + \frac{n\pi}{2}\right) + n.2.3^{(n-1)} \cdot x \cdot \sin\left(3x + \frac{(n-1)\pi}{2}\right)$$
$$+ n(n-1).3^{(n-2)} \cdot \sin\left(3x + \frac{(n-2)\pi}{2}\right)$$ Ans

Q→3 find the n^{th} Derivative of $x^2 \cdot e^x \cdot \cos 3x$

$$y = \underbrace{x^2}_{v} \cdot \underbrace{e^x \cdot \cos 3x}_{u} \qquad \begin{pmatrix} a=1 \\ b=3 \end{pmatrix}$$

$$y_n = D^n(u).v + {}^nC_1 D^{n-1}(u).D(v) + {}^nC_2 D^{n-2}(u).D^2(v)$$

$$= \left\{(\sqrt{10})^n \cdot e^x \cos\left(3x + n\tan^{-1}\left(\frac{3}{1}\right)\right)\right\} \cdot (x^2) +$$
$$n \cdot \left\{(\sqrt{10})^{n-1} \cdot e^x \cdot \cos\left(3x + (n-1)\tan^{-1}\left(\frac{3}{1}\right)\right)\right\} \cdot \{2x\} +$$

$$\frac{n(n-1)}{2}\left\{(\sqrt{10})^{n-2}\cdot e^{x}\cdot \cos\left(3x+(n-2)\tan^{-1}\left(\frac{3}{1}\right)\right\}\{2\}\right.$$

$$\Rightarrow (\sqrt{10})^{n}\cdot x^{2}\cdot e^{x}\cdot \cos\{3x+n\cdot\tan^{-1}(3)\}+2\cdot(\sqrt{10})^{n-1}\cdot n\cdot x\cdot e^{x}\cdot \cos\{3x+(n-1)\tan$$

$$\Rightarrow (\sqrt{10})^{n}\cdot x^{2}\cdot e^{x}\cdot \cos\{3x+n\cdot\tan^{-1}(3)\}+2\cdot(\sqrt{10})^{n-1}\cdot n\cdot x\cdot e^{x}\cdot \cos\{3x+(n-1)\tan^{-1}(3)\}$$
$$+(\sqrt{10})^{n-2}\cdot n(n-1)\cdot e^{x}\cdot \cos\{3x+(n-2)\cdot\tan^{-1}(3)\}$$ Ans

Q⇒4 $y=e^{\tan^{-1}x}$ then prove that

$$(1+x^2)\,y_{n+2}+\left[2(n+1)\cdot x-1\right]y_{n+1}+n(n+1)\,y_n=0$$

Ans $y=e^{\tan^{-1}x}$ --- ① $\qquad \frac{d}{dx}\{\tan^{-1}x\}=\left(\frac{1}{1+x^2}\right)$

$$\Rightarrow y_1=e^{\tan^{-1}x}\cdot\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow y_1=\frac{y}{1+x^2}\Rightarrow y=y_1(1+x^2)\text{ --- ②}$$

$$\Rightarrow y_1=(1+x^2)\{y_2\}+(y_1)\{2x\}$$

$$\Rightarrow (1+x^2)\,y_2+(2x-1)\,y_1=0\text{ --- ③}$$

Now differentiating nth times w.r.t x;

$$\Rightarrow D^{n}\left[\underset{v}{(1+x^2)}\,\underset{u}{y_2}+\underset{v}{(2x-1)}\,\underset{u}{y_1}\right]=0$$

$$= \left[\{y_{n+2}\}\cdot(1+x^2) + {}^nC_1\{y_{n+1}\}\cdot\{2x\} + {}^nC_2\{y_n\}\{2\}\right]$$

$$+ \left[\{y_{n+1}\cdot(2x-1) + {}^nC_1\{y_n\}\cdot\{2\}\right] = 0$$

$$\Rightarrow (1+x^2)\cdot y_{n+2} + 2x\cdot n\cdot y_{n+1} + n(n-1)\cdot y_n + (2x-1)\cdot y_{n+1}$$

$$+ 2n\cdot y_n = 0$$

$$\Rightarrow (1+x^2)\,y_{n+2} + y_{n+1}\{2xn + 2x - 1\} + y_n\{n(n-1) + 2n\} = 0$$

$$\Rightarrow (1+x^2)\,y_{n+2} + y_{n+1}(2(n+1)x - 1) + y_n(n)(n+1) = 0$$

Ans

Q→5 If $y = a\cdot\cos\log x + b\cdot\sin\log x$ then prove that,

$$x^2 y_{n+2} + (2n+1)\cdot x\cdot y_{n+1} + (n^2+1)\,y_n = 0$$

Ans $y = a\cos\log x + b\sin\log x$ ---- ①

$\Rightarrow$ Now differentiating w.r.t (x);

$$\Rightarrow y_1 = \frac{-a\cdot\sin(\log x)}{x} + \frac{b\cdot\cos(\log x)}{x}$$

$$\Rightarrow x y_1 = -a\cdot\sin(\log x) + b\cos(\log x) \quad \text{---- ②}$$

$\Rightarrow$ Now diff. w.r.t (x);

$\Rightarrow \quad xy_2 + y_1 = -\frac{a\cos\log x}{x} - \frac{b\sin\log x}{x}$

$\Rightarrow \quad x^2y_2 + xy_1 = -[a\cos\log x + b\sin\log x]$

$\Rightarrow \quad x^2y_2 + xy_1 + y = 0$ --- (3)

$\Rightarrow$ Now diff. n^{th} times w.r.t (x);

$\Rightarrow \left[\{y_{n+2}\}(x^2) + {}^nC_1 \cdot \{y_{n+1}\}\{2x\} + {}^nC_2\{y_n\}\{2\}\right]$

$+ \left[\{y_{n+1}\}(x) + {}^nC_1\{y_n\}(1)\right] + y_n = 0$

$\Rightarrow \quad x^2 \cdot y_{n+2} + 2x \cdot n \cdot y_{n+1} + n(n-1) \cdot y_n + x\, y_{n+1} + n \cdot y_n$
$+ y_n = 0$

$\Rightarrow \quad x^2 \cdot y_{n+2} + y_{n+1}\{2xn + x\} + y_n\{n(n-1) + n + 1\} = 0$

$\Rightarrow \quad x^2 \cdot y_{n+2} + (2n+1) \cdot x \cdot y_{n+1} + (n^2+1) y_n = 0$ **Ans**

Q→6 If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$ then prove that ;

$(1-x^2) \cdot y_{n+2} - (2n+3) \cdot x \cdot y_{n+1} - (n+1)^2 \cdot y_n = 0$

Ans ***

$\frac{d}{dx}\{\sin^{-1}x\} = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\cdot y^2 = (\sin^{-1}x)^2$

$\rightarrow$ diff w.r.t (x) ;

$\Rightarrow (1-x^2)\{2y.y_1\} + (y^2)\{-2x\} = \dfrac{2\sin^{-1}x}{\sqrt{1-x^2}}$

$\Rightarrow 2.(1-x^2).y\,y_1 - 2xy^2 - 2y = 0$ --- (2)

$\Rightarrow (1-x^2).y_1 - x.y - 1 = 0$ ---

$\rightarrow$ diff w.r.t (x) ;

$\Rightarrow \left[(1-x^2)\{y_2\} + (y_1)\{-2x\}\right] - \left[(\ x)\{y_1\} + (y_1)\{1\}\right] = 0$

$\Rightarrow (1-x^2)\, y_2 - 2x y_1 - x y_1 - y = 0$

$\Rightarrow (1-x^2)\, y_2 - 3x y_1 - y = 0$ --- (3)

$\rightarrow$ diff n^{th} times w.r.t (x) ;

$\Rightarrow \left[\{y_{n+2}\}(1-x^2) + {}^nC_1\{y_{n+1}\}\{-2x\} + {}^nC_2\{y_n\}\{-2\}\right]$

$- \left[\{y_{n+1}\}(3x) + {}^nC_1\{y_n\}\{3\}\right] - y_n = 0$

$\Rightarrow (1-x^2)\cdot y_{n+2} - 2x.n.y_{n+1} - n(n-1)y_n - 3x\, y_{n+1}$

$- 3.n.y_n - y_n = 0$

$$\Rightarrow (1-x^2)\cdot y_{n+2} + y_{n+1}\{-2xn-3x\} + y_n\{-n(n-1)-3n-1\} = 0$$

$$\Rightarrow (1-x^2)\cdot y_{n+2} - (2n-3)\cdot x\cdot y_{n+1} - (n+1)^2\cdot y_n = 0 \quad \text{Ans}$$

Q→7 If $x = e^t$, $y = \cos mt$, then prove that

$$x^2 y_{n+2} + (2n+1)\cdot x\cdot y_{n+1} + (m^2+n^2)\cdot y_n = 0$$

Ans need to Establish relation between x & y:

$x = e^t \Rightarrow \log_e x = t$ --- ①
$y = \cos mt \Rightarrow \cos^{-1} y = mt$ --- ②

from ① & ② $\cos^{-1} y = m \log_e x$ --- ③

diff w.r.t (x);

$$\Rightarrow \frac{-1}{\sqrt{1-y^2}}\cdot y_1 = \frac{m}{x}$$

$$\Rightarrow (-x y_1) = m\sqrt{1-y^2}$$

$$\Rightarrow x^2 y_1^2 = m^2(1-y^2) \quad \text{---} \quad ②_1$$

diff w.r.t (x);

$$\Rightarrow \{2y_1 y_2\}\cdot(x^2) + (y_1^2)\{2x\} = (m^2)\{-2y\cdot y_1\}$$

$$\Rightarrow y_1\{2x^2y_2 + 2xy_1\} = -2y\cdot y_1\cdot m^2$$

$$\Rightarrow x^2y_2 + xy_1 + m^2y = 0 \quad \text{--- (5)}$$

diff n^{th} times w.r.t (x) ;

$$\Rightarrow \left[\{y_{n+2}\}(x^2) + {}^nC_1\{y_{n+1}\}\{2x\} + {}^nC_2\{y_n\}\{2\}\right] + \left[\{y_{n+1}\}(x) + {}^nC_1\{y_n\}\{1\}\right] + m^2y_n = 0$$

$$\Rightarrow x^2\cdot y_{n+2} + 2x\cdot n\cdot y_{n+1} + n(n-1)y_n + xy_{n+1} + n\cdot y_n + m^2y_n = 0$$

$$\Rightarrow x^2\cdot y_{n+2} + y_{n+1}\{2xn + x\} + y_n\{n(n-1) + n + m^2\} = 0$$

$$\Rightarrow x^2\cdot y_{n+2} + y_{n+1}\cdot(2n+1)\cdot x + y_n\cdot(m^2+n^2) = 0$$

— X — X — X —