INDETERMINATE FORMS

* Some limits of functions which have indeterminate forms, that have no meaning on direct substitution, but it may be a meaningful and have existence of limit after simplification or in other way.

* We will use an important L' Hospital rule for finding such limits, which is an important application of taylor's theorems.

* We shall discuss the indeterminate forms;

a) $\frac{0}{0}$ forms b) $\frac{\infty}{\infty}$ form c) $\infty - \infty$ form

d) $0 \times \infty$ form

e) $0^0, \infty^0, 1^\infty$ forms.

** L' Hospital's Rule for $\left(\frac{0}{0}\right)$ form:

statement: If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \quad g'(a) \neq 0$$

$\Longrightarrow$ Using taylor's Series,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(a) + \frac{(x-a) \cdot f'(a)}{1!} + \frac{(x-a)^2 \cdot f''(a)}{2!} + \frac{(x-a)^3 \cdot f'''(a)}{3!} + \cdots}{g(a) + \frac{(x-a) \cdot g'(a)}{1!} + \frac{(x-a)^2 \cdot g''(a)}{2!} + \frac{(x-a)^3 \cdot g'''(a)}{3!} + \cdots}$$

$$\left\{ \begin{array}{l} \Longrightarrow \lim_{x \to a} f(x) = f(a) = 0 \\ \Longrightarrow \lim_{x \to a} g(x) = g(a) = 0 \end{array} \right\}$$

$$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{(x-a)f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 \cdot f'''(a)}{3!} + \cdots}{\frac{(x-a)\cdot g'(a)}{1!} + \frac{(x-a)^2 g''(a)}{2!} + \frac{(x-a)^3 g'''(a)}{3!} + \cdots}$$

$$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{(x-a)\left\{\frac{f'(a)}{1!} + \frac{(x-a).f''(a)}{2!} + \frac{(x-a)^2 f'''(a)}{3!} + \cdots\right\}}{(x-a)\left\{\frac{g'(a)}{1!} + \frac{(x-a)\cdot g''(a)}{2!} + \frac{(x-a)^2\cdot g'''(a)}{3!} + \cdots\right\}}$$

$$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a) + \frac{(x-a)f''(a)}{2!} + \frac{(x-a)^2\cdot f'''(a)}{3!} + \cdots}{g'(a) + \frac{(x-a)\cdot g''(a)}{2!} + \frac{(x-a)^2\cdot g'''(a)}{3!} + \cdots}$$

$$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

$$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$, hence proved

*** Rule to Evaluate :- $\left\{\frac{0}{0} \text{ form}\right\}$.

1) Check wheather the limit is an indeterminate form.

2) Apply L'Hospital's Rule and Continue Differentiate of $f(x)$ & $g(x)$ till the required value is reached.

*** Shortcut formula :-

1) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ or $\lim_{x\to 0} \frac{x}{\sin x} = 1$

2) $\lim_{x\to 0} \frac{\tan x}{x} = 1$

3) $\lim_{x\to 0} \frac{\sinh x}{x} = 1$

Remember
$\left\{ \lim_{x\to 0} \frac{\sin(K)}{K} = 1 \right\}$

4) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

5) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$

6) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$

7) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

Q→ 1) Evalute, $\lim_{x\to 0} \frac{\tan x - \sin x}{x^2}$ $\left(\frac{0}{0}\right)$

$\Rightarrow$ apply LH Rule, $\lim_{x\to 0} \frac{\sec^2 x - \cos x}{2x}$ $\left(\frac{0}{0}\right)$

$\Rightarrow$ apply LH Rule, $\lim_{x\to 0} \frac{2\cdot\sec^2 x\cdot\tan x + \sin x}{2}$

$\Rightarrow \frac{2(1)(0) + (0)}{2} = 0$ Ans

2) Evalute, $\lim_{x\to 0} \frac{\log(1-x^2)}{\log \cos x}$ $\left(\frac{0}{0}\right)$

$\left\{ \begin{array}{l} \cos 0 = 1 \\ \log 1 = 0 \end{array} \right\}$

$\Rightarrow$ apply LH Rule; $\lim_{x\to 0} \frac{\frac{-2x}{1-x^2}}{\frac{-\sin x}{\cos x}}$

$= \lim_{x\to 0} \frac{2x}{(1-x^2)\cdot\tan x}$

$\left\{ \lim_{x\to 0} \frac{x}{\tan x} = 1 \right\}$

$$= \lim_{x\to 0} \frac{2}{1+x^2} = \frac{2}{1} = 2 \quad \text{Ans}$$

Q→3 ⟶ $\lim_{x\to 0} \frac{x - \tan^{-1}x}{x^3} \quad \left(\frac{0}{0}\right)$

applying LH Rule

$$= \lim_{x\to 0} \frac{1 - \frac{1}{1+x^2}}{3x^2} = \lim_{x\to 0} \frac{\frac{1+x^2-1}{1+x^2}}{3x^2}$$

$$= \lim_{x\to 0} \frac{x^2}{3x^2(1+x^2)} = \lim_{x\to 0} \frac{1}{3(1+x^2)}$$

$$= \frac{1}{3} \quad \text{Ans}$$

Q→4 ⇒ $\lim_{x\to \pi/2} \frac{\log \sin x}{(\pi - 2x)^2} \quad \left(\frac{0}{0}\right)$

⇒ applying LH Rule; $\lim_{x\to \pi/2} \frac{\frac{\cos x}{\sin x}}{2(\pi-2x)\cdot(-2)}$

⇒ $\lim_{x\to \pi/2} \frac{\cot x}{-4(\pi - 2x)} \quad \left(\frac{0}{0}\right)$

⇒ applying LH Rule; $\lim_{x\to \pi/2} \frac{-\operatorname{cosec}^2 x}{8}$

$$= -\frac{1}{8} \quad \text{Ans}$$

Q→5 $\lim_{x\to y} \frac{x^y - y^x}{x^x - y^y}$ $\left(\frac{0}{0}\right)$

⟹ applying LH Rule; $\lim_{x\to y} \frac{y x^{y-1} - y^x \cdot \log y}{x^x(1+\log x)}$

$\frac{d}{dx}\{x^x\}$ ⟹ Let $y = x^x$

$\log y = \log (x)^x$

$\log y = x\cdot \log x$

$\frac{1}{y}\frac{dy}{dx} = x\left\{\frac{1}{x}\right\} + \log x\{1\}$

$\frac{dy}{dx} = y\{1+\log x\}$

$\frac{dy}{dx} = x^x\{1+\log x\}$

$\Rightarrow \frac{y\cdot y^{(y-1)} - y^y \cdot \log y}{y^y(1+\log y)} = \frac{y^y - y^y\cdot \log y}{y^y(1+\log y)}$

$\Rightarrow \frac{y^y\{1-\log y\}}{y^y\{1+\log y\}} = \frac{1-\log y}{1+\log y}$ Ans

Q→6 $\lim_{x\to 0} \frac{\cosh x - \cos x}{x\cdot \sin x}$ ← if denominator differentiated then terms will be increasing and it will tough to calculate.

$= \lim_{x\to 0} \frac{\cosh x - \cos x}{x\cdot \sin x}\cdot \frac{x}{x}$ $\left(\lim_{x\to 0}\frac{x}{\sin x} = 1\right)$

$= \lim_{x\to 0} \frac{\cosh x - \cos x}{x^2}$ $\left(\frac{0}{0}\right)$ $\left(\begin{matrix}\cosh 0 = 1\\ \cos 0 = 1\end{matrix}\right)$

applying LH Rule, $\lim_{x\to 0} \frac{\sinh x + \sin x}{2x}$

$= \lim_{x\to 0} \frac{1}{2}\left[\frac{\sinh x}{x} + \frac{\sin x}{x}\right]$

$= \frac{1}{2}[1+1] = \frac{2}{2} = 1$ Ans

Q→7 $\lim_{x\to 0} \frac{e^{x}+e^{-x}-x^{2}-2}{\sin^{2}x-x^{2}}$ $\left(\frac{0}{0}\right)$

$\Rightarrow$ Applying LH Rule, $\lim_{x\to 0} \frac{e^{x}-e^{-x}-2x}{2\sin x\cos x-2x}$

$\Rightarrow \lim_{x\to 0} \frac{e^{x}-e^{-x}-2x}{\sin 2x-2x}$ $\left(\frac{0}{0}\right)$

$\Rightarrow$ applying LH Rule, $\lim_{x\to 0} \frac{e^{x}+e^{-x}-2}{2\cos 2x-2}$ $\left(\frac{0}{0}\right)$

$\Rightarrow$ applying LH Rule, $\lim_{x\to 0} \frac{e^{x}-e^{-x}}{-4\sin 2x}$ $\left(\frac{0}{0}\right)$

$\Rightarrow$ applying LH Rule, $\lim_{x\to 0} \frac{e^{x}+e^{-x}}{-8\cos 2x}$

$\Rightarrow \frac{1+1}{-8(1)} = -\frac{1}{4}$ Ans

Q→8→ $\lim_{x\to 0} \frac{\tan^2 x - x^2}{x^2 \cdot \tan^2 x}$ ← (need to remove $\tan^2 x$)

$\Rightarrow \lim_{x\to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x} \cdot \frac{x^2}{x^2}$

$\Rightarrow \lim_{x\to 0} \frac{\tan^2 x - x^2}{x^4} \quad \left(\frac{0}{0}\right)$

applying LH Rule; $\lim_{x\to 0} \frac{2\cdot \tan x \cdot \sec^2 x - 2x}{4x^3} \quad \left(\frac{0}{0}\right)$

applying LH Rule; $\lim_{x\to 0} \frac{2\sec^4 x + 4\tan^2 x \cdot \sec^2 x - 2}{12x^2} \quad \left(\frac{0}{0}\right)$

applying LH Rule; $\lim_{x\to 0} \frac{8\sec^4 x \cdot \tan x + 4\left[\tan^3 x \cdot 2\sec^2 x + 2\tan x \cdot \sec^4 x\right]}{24x}$

$= \lim_{x\to 0} \frac{16\sec^4 x \cdot \tan x + 8\tan^3 x \cdot \sec^2 x}{24x} \quad \left(\frac{0}{0}\right)$

instead of applying LH Rule, I will sepearte denominator;

$= \lim_{x\to 0} \left\{ \frac{16\sec^4 x \cdot \tan x}{24x} + \frac{8\tan x \cdot \tan^2 x \cdot \sec^2 x}{24x} \right\}$

$= \lim_{x\to 0} \left\{ \frac{16\sec^4 x}{24} + \frac{8\tan^2 x \cdot \sec^2 x}{24} \right\}$

$= \frac{16}{24} + 0 = \frac{2}{3}$ Ans

** L'Hospital's Rule for the $\left(\frac{\infty}{\infty}\right)$ form;

If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

→ hence, we can extend the rule for the $\left(\frac{\infty}{\infty}\right)$ form by diff. the numerator and denominator separately as many times as would be necessary.

$$\Rightarrow \frac{\infty}{\infty} = \frac{1/0}{1/0} = \frac{0}{0}$$

Q→1 $\lim\limits_{x\to 0} \frac{\log x}{\cot x}$ $\left(\frac{\infty}{\infty}\right)$. $(\log 0 = \infty)$

⇒ applying LH Rule; $\lim\limits_{x\to 0} \frac{1/x}{-\operatorname{cosec}^2 x}$

$$= \lim_{x\to 0} -\frac{\sin^2 x}{x} = \lim_{x\to 0} -\sin x \qquad \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$= 0$ Ans

Q→2 $\lim\limits_{x\to 0} \frac{\log(\sin x)}{\log(\tan x)}$ $\left(\frac{\infty}{\infty}\right)$

applying LH Rule; $\lim\limits_{x\to 0} \frac{\frac{\cos x}{\sin x}}{\frac{\sec^2 x}{\tan x}}$

$$= \lim_{x\to 0} \frac{\cos x \cdot \tan x}{\sin x \cdot \sec^2 x} = \lim_{x\to 0} \frac{\tan x}{\tan x \cdot \sec^2 x}$$

$$= \lim_{x\to 0} \frac{1}{\sec^2 x} = \lim_{x\to 0} \cos^2 x = 1 \quad \text{Ans}$$

Q→3 $$\lim_{x\to\infty} \frac{x^n}{e^x} \quad \left(\frac{\infty}{\infty}\right)$$

applying LH Rule; $\lim_{x\to\infty} \frac{n(x)^{n-1}}{e^x} \quad \left(\frac{\infty}{\infty}\right)$

applying LH Rule; $\lim_{x\to\infty} \frac{(n)(n-1)(x)^{n-2}}{e^x} \quad \left(\frac{\infty}{\infty}\right)$

- - - - - - - - - - - - - - - -

applying n^{th} times LH Rule;

$$= \lim_{x\to\infty} \frac{(n)(n-1)(n-2) \text{----} (3)(2)(1)}{e^x}$$

$$= \lim_{x\to\infty} \frac{n!}{e^x} = \frac{n!}{\infty} = 0 \quad \text{Ans}$$

Q→4 $$\lim_{x\to\infty} \frac{x(\log x)^3}{1+x+x^2} \quad \left(\frac{\infty}{\infty}\right)$$

applying LH Rule; $\lim_{x\to\infty} \frac{(\log x)^3\{1\} + (x)\left\{3(\log x)^2 \cdot \left(\frac{1}{x}\right)\right\}}{1+2x}$

$$= \lim_{x\to\infty} \frac{(\log x)^3 + 3(\log x)^2}{1+2x} \quad \left(\frac{\infty}{\infty}\right)$$

applying LH Rule; $\lim_{x\to\infty} \frac{3[\log x)^2\left(\frac{1}{x}\right) + 6(\log x)\left(\frac{1}{x}\right)}{2}$

$$= \lim_{x\to\infty} \frac{3(\log x)^2 + 6(\log x)}{2x} \quad \left(\frac{\infty}{\infty}\right)$$

applying LH Rule; $\lim_{x\to\infty} \frac{6(\log x)(\frac{1}{x}) + 6(\frac{1}{x})}{2}$

$$= \lim_{x\to\infty} \frac{6\log x + 6}{2x} \quad \left(\frac{\infty}{\infty}\right)$$

applying LH Rule; $\lim_{x\to\infty} \frac{\frac{6}{x}}{2}$

$$= \lim_{x\to\infty} \frac{3}{x} = 0$$ Ans