SERIES EXPANSIONS

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** Taylor's Series

$$\boxed{f(x) = f(a) + \frac{(x-a)f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!} + \cdots \frac{(x-a)^n f^n(a)}{n!} + \cdots}$$

→ in power of $(x-a)$ or $x=a$

** Maclaurin's Series

$$\boxed{f(x) = f(0) + \frac{(x)f'(0)}{1!} + \frac{(x)^2 f''(0)}{2!} + \frac{(x)^3 f'''(0)}{3!} + \cdots \frac{(x)^n f^n(0)}{n!} + \cdots}$$

→ in power of $(a=0)$ or $(x=0)$

** Maclaurin's Series based Examples :-

① Expansion of e^x at $x=0$

→ means solving by maclaurins series methods. $\{x=0\}$ or $\{a=0\}$

$$\begin{array}{ll} f(x) = e^x & f(a) = e^0 = 1 \\ f'(x) = e^x & f'(a) = e^0 = 1 \\ f''(x) = e^x & f''(a) = e^0 = 1 \\ f'''(x) = e^x & f'''(a) = e^0 = 1 \end{array} \Bigg\} \; a=0$$

$$f(x) = f(0) + \frac{(x)f'(0)}{1!} + \frac{(x)^2 f''(0)}{2!} + \frac{(x)^3 f'''(0)}{3!} + \cdots$$

$$e^x = 1 + \frac{(x)(1)}{1!} + \frac{(x)^2(1)}{2!} + \frac{(x)^3}{3!}(1) + \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{Ans}$$

(2) Expansion of e^{-x} :-

→ replacing x by $-x$ in the e^x series ;

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-x} = 1 + \frac{(-x)}{1!} + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{(x)^2}{2!} - \frac{(x)^3}{3!} + \dots \quad \text{Ans}$$

{Note → we can solve by basic method using maclaurin's formula}

(3) Expansion of e^{ax} :-

→ replacing x by ax in the e^x series ;

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{ax} = 1 + \frac{ax}{1!} + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \dots \quad \text{Ans}$$

(4) Expansion of $\sin x$:-

$f(x) = \sin x$ $\qquad$ $f(0) = \sin 0 = 0$

$f'(x) = \cos x$ $\qquad$ $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ $\qquad$ $f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x$ $\qquad$ $f'''(0) = -\cos 0 = -1$

$f^{iv}(x) = \sin x$ $\qquad$ $f^{iv}(0) = \sin 0 = 0$

$$f(x) = f(0) + \frac{(x)f'(0)}{1!} + \frac{(x)^2 f''(0)}{2!} + \frac{(x)^3 f'''(0)}{3!} + \dots$$

$$\sin x = 0 + \frac{(x)(1)}{1!} + \frac{(x)^2(0)}{2!} + \frac{(x)^3(-1)}{3!} + \frac{(x)^4(0)}{4!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \text{Ans}$$

⑤ Expansion of $\cos x$:-

$f(x) = \cos x \qquad f(0) = \cos 0 = 1$

$f'(x) = -\sin x \qquad f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \qquad f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x \qquad f'''(0) = \sin 0 = 0$

$f^{iv}(x) = \cos x \qquad f^{iv}(0) = \cos 0 = 1$

$$\Rightarrow \cos x = 1 + \frac{(x)(0)}{1!} + \frac{(x)^2(-1)}{2!} + \frac{(x)^3(0)}{3!} + \frac{(x)^4(1)}{4!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots \quad \text{Ans}$$

⑥ Expansion of $\tan x$:-

$f(x) = \tan x \qquad f(0) = \tan 0 = 0$

$f'(x) = \sec^2 x \qquad f'(0) = \sec^2 0 = 1$

$f''(x) = 2(\sec x)(\sec x \tan x) = 2\sec^2 x \tan x \qquad f''(0) = 2\sec^2 0 . \tan 0 = 2(1)(0) = 0$

$f'''(x) = 2(\tan x)\{2\sec^2 x \tan x\} + 2(\sec^2 x)\{\sec^2 x\}$

$= 4\tan^2 x \sec^2 x + 2\sec^4 x$

$f'''(0) = 4\tan^2 0 . \sec^2 0 + 2\sec^4 0 = 4(0)(1) + 2(1) = 2$

$$\Rightarrow \tan x = f(0) + \frac{(x)f'(0)}{1!} + \frac{(x)^2 f''(0)}{2!} + \frac{(x)^3 f'''(0)}{3!} + \cdots$$

$$\tan x = 0 + \frac{(x)(1)}{1!} + \frac{(x)^2(0)}{2!} + \frac{(x)^3(2)}{3!} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad \text{Ans}$$

⑦ Expansion of sinhx :-

hyperbolic function
$\begin{Bmatrix} \sinh x \\ \cosh x \end{Bmatrix}$

$$\Rightarrow \sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\Rightarrow \sinh x = \frac{1}{2}\left[e^{x} - e^{-x}\right]$$

$$\Rightarrow \sinh x = \frac{1}{2}\left[\{e^{x}\} - \{e^{-x}\}\right]$$

$$= \frac{1}{2}\left[\left\{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right\} - \left\{1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \dots\right\}\right]$$

$$= \frac{1}{2}\left[2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \dots\right]$$

$$\Rightarrow \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad \text{As}$$

⑧ Expansion of Coshx :-

$$\Rightarrow \cosh x = \frac{e^{x} + e^{-x}}{2} = \frac{1}{2}\left[\left\{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right\} + \left\{1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right\}\right]$$

$$\cosh x = \frac{1}{2}\left[2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \dots\right]$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \quad \text{As}$$

⑨ Expansion of log(1+x) :-

$f(x) = \log(1+x)$

$f'(x) = \frac{1}{1+x}$

$f''(x) = -\frac{1}{(1+x)^2}$

$f'''(x) = \frac{2}{(1+x)^3}$

$f^{IV}(x) = \frac{-6}{(1+x)^4}$

$f(0) = \log(1+0) = \log 1 = 0$

$f'(0) = \frac{1}{1+0} = 1$

$f''(0) = -\frac{1}{(1+0)^2} = -1$

$f'''(0) = \frac{2}{(1+0)^3} = 2$

$f^{IV}(0) = \frac{-6}{(1+0)^4} = -6$

$$f(x) = f(0) + \frac{(x)(f'(0))}{1!} + \frac{(x)^2 f''(x)}{2!} + \frac{(x)^3 f'''(x)}{3!} + \dots$$

$$\log(1+x) = 0 + \frac{(x)(1)}{1!} + \frac{(x)^2(-1)}{2!} + \frac{(x)^3(2)}{3!} + \frac{(x)^4(-6)}{4!} + \dots$$

$$\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots \quad \underline{\underline{\text{Ans}}}$$

(10) Expansion of $\log(1-x)$;

→ replacing x with $-x$ from $\log(1+x)$ series ;

$$\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \dots \quad \underline{\underline{\text{As}}}$$

$\boxed{Q}$ → Expansion of xe^x upto fourth order when $x=0$

$\underline{\underline{\text{Ans}}}$ $f(x) = xe^x = y$

$f(0) = xe^x = (0)e^0 = 0 = y$

$f'(x) = (x)\{e^x\} + (e^x)\{1\}$

$f'(0) = y + e^x = 0 + e^0 = 1 = y'$

$f'(x) = y + e^x$

$f''(0) = 1 + e^0 = 1 + 1 = 2 = y''$

$f''(x) = y' + e^x$

$f'''(0) = y'' + e^x = 2 + e^0 = 1 + 2 = 3 = y'''$

$f'''(x) = y'' + e^x$

$f^{iv}(0) = y''' + e^x = 3 + 1 = 4 = y^{iv}$

$f^{iv}(x) = y''' + e^x$

$$\Rightarrow xe^x = 0 + \frac{(x)(1)}{1!} + \frac{(x)^2(2)}{2!} + \frac{(x)^3(3)}{3!} + \frac{(x)^4(4)}{4!}$$

$$xe^x = x + \frac{x^2}{1} + \frac{x^3}{2} + \frac{x^4}{6}. \quad \underline{\underline{\text{As}}}$$

(series should stop at 4th term)

** Binominal series :- {used for solving inverse trignometric function}

$\rightarrow$ $f(x) = (1+x)^m$ $\qquad f(0) = (1+0)^m = 1$

$f'(x) = m(1+x)^{m-1}$ $\qquad f'(0) = m(1+0)^{m-1} = m$

$f''(x) = (m)(m-1)(1+x)^{m-2}$ $\qquad f''(0) = (m)(m-1) \quad (1+0)^{m-2} = (m)(m-1)$

$\vdots$

$f^n(x) = (m)(m-1)(m-2)\ldots(1+x)^{m-n}$ $\qquad f^n(0) = (m)(m-1)(m-2)\ldots$

$$f(x) = f(0) + \frac{(x)f'(0)}{1!} + \frac{(x)^2 f''(0)}{2!} + \frac{(x)^3 f'''(0)}{3!} + \ldots$$

$$(1+x)^m = 1 + \frac{(x)(m)}{1!} + \frac{(x)^2(m)(m-1)}{2!} + \frac{(x)^3(m)(m-1)(m-2)}{3!} + \ldots$$

similarly

$$(1-x)^m = 1 - \frac{(x)(m)}{1!} + \frac{(x)^2(m)(m-1)}{2!} - \frac{(x)^3(m)(m-1)(m-2)}{3!} + \ldots$$

$\boxed{Q}$ Expansion of $\tan^{-1}x$:-

$\Rightarrow$ $f(x) = \tan^{-1}x$

$f'(x) = \dfrac{1}{1+x^2} = (1+x^2)^{-1}$ comparing with $(1+x)^m$

$$f'(x) = 1 + \frac{(x^2)(-1)}{1!} + \frac{(x^2)^2(-1)(-2)}{2!} + \frac{(x^2)^3(-1)(-2)(-3)}{3!} + \ldots$$

$$f'(x) = 1 - \frac{x^2}{1} + \frac{2x^4}{2} - \frac{6x^6}{6} + \ldots$$

$$f'(x) = 1 - x^2 + x^4 - x^6 + \ldots$$

Now integrating both side w.r.t x ;

$$\int f'(x)\,dx = \int \{1 - x^2 + x^4 - x^6 \ldots\} dx$$

$$= \quad f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$$ Ans

$\boxed{Q}$ Expansion of $\cot^{-1}x$;

$$f(x) = \cot^{-1}x$$

$$f'(x) = \frac{-1}{(1+x^2)} = -(1+x^2)^{-1} = -[1 - x^2 + x^4 - x^6 \ldots]$$

$$f'(x) = -1 + x^2 - x^4 + x^6 \ldots$$

$$f(x) = -x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \ldots\ldots$$ Ans

$\boxed{Q}$ Expansion of $\sin^{-1}x$;

$$f(x) = \sin^{-1}x$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$$

$$f'(x) = 1 - \frac{(x^2)(-\frac{1}{2})}{1!} + \frac{(x^2)^2(-\frac{1}{2})(-\frac{3}{2})}{2!} - \frac{(x^2)^3(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} + \ldots$$

$$f'(x) = 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{15x^6}{48} + \ldots$$

$$f(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{15x^7}{326} + \ldots$$ Ans

$\boxed{Q}$ Expansion of $\cos^{-1}x$;

$$f(x) = \cos^{-1}x$$

$$f'(x) = \frac{-1}{\sqrt{1-x^2}} = -(1-x^2)^{-1/2} = -\left\{1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{15x^6}{48} + \ldots\right\}$$

$$f'(x) = -1 - \frac{x^2}{2} - \frac{3x^4}{8} - \frac{15x^6}{48} - - -.$$

$$f(x) = -x - \frac{x^3}{6} - \frac{3x^5}{40} - \frac{15x^7}{326} - - - -$$ Ans

** Taylor's Series based Examples :-

Q Express $2x^3 + 7x^2 + x - 6$ in ascending power of $(x-2)$

Ans $x - 2 = 0$ comparing $x - a = 0$ $(a = 2)$

$f(x) = 2x^3 + 7x^2 + x - 6$ $\qquad f(a) = 2(2)^3 + 7(2)^2 + 2 - 6 = 40$

$f'(x) = 6x^2 + 14x + 1$ $\qquad f'(2) = 6(2)^2 + 14(2) + 1 = 53$

$f''(x) = 12x + 14$ $\qquad f''(2) = 12(2) + 14 = 38$

$f'''(x) = 12$ $\qquad f'''(2) = 12$

$$f(x) = f(a) + \frac{(x-a)f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!} + - - - -$$

$$f(x) = 40 + \frac{(x-2)(53)}{1!} + \frac{(x-2)^2(38)}{2!} + \frac{(x-2)^3(38)}{3!} + \frac{(x-2)^3(12)}{3!}$$

$$f(x) = 40 + 53(x-2) + 19(x-2)^2 + \frac{38(x-3)^2}{6} + \frac{12(x-2)^4}{24} - - -$$

$$f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$ Ans.

Q Expand $49 + 69x + 42x^2 + 11x^3 + x^4$ in power of $(x+2)$

Ans $x + 2 = 0$ comparing $x - a = 0$ $(a = -2)$

$f(x) = 49 + 69x + 42x^2 + 11x^3 + x^4$ $\quad f(-2) = 7$

$f'(x) = 69 + 84x + 33x^2 + 4x^3$ $\quad f'(-2) = 1$

$f''(-2) = 0$

$f''(x) = 84 + 66x + 12x^2$ $\quad f'''(-2) = 18$

$f'''(x) = 66 + 24x$ $\quad f^{IV}(-2) = 24$

$f^{IV}(x) = 24$

$$f(x) = 7 + \frac{(x+2)(1)}{1!} + \frac{(x+2)^2(0)}{2!} + \frac{(x+2)^3(18)}{3!} + \frac{(x+2)^4(24)}{4!} + .$$

$$f(x) = 7 + (x+2) + 3(x+2)^3 + (x+4)^4 \quad \underline{\underline{As}}$$

$\boxed{Q \to}$ Expand $f(x) = (x+1)^3 + 5(x+1)^2 + 10(x+1) + 1$ powers of $(x-1)$

$\underline{\underline{Ans}}$ $\quad (x-1) = 0 \qquad (a=1)$

$f(x) = (x+1)^3 + 5(x+1)^2 + 10(x+1) + 1$ $\quad f(1) = 49$

$f'(x) = 3(x+1)^2 + 10(x+1) + 10$ $\quad f'(1) = 42$

$f''(x) = 6(x+1) + 10$ $\quad f''(1) = 22$

$f'''(x) = 6$ $\quad f'''(1) = 6$

$$f(x) = 49 + \frac{(x-1)(42)}{1!} + \frac{(x-1)^2(22)}{2!} + \frac{(x-1)^3(6)}{3!}$$

$$f(x) = 49 + 42(x-1) + 11(x-1)^2 + (x-1)^3 \quad \underline{\underline{As}}$$

Q $f(x) = \log_e x = \ln x$ upto four order at $x=2$. also find $f(3)$.

Ans $f(x) = \ln x$ $\qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x}$ $\qquad f'(2) = \frac{1}{2}$

$f''(x) = -\frac{1}{x^2}$ $\qquad f''(2) = -\frac{1}{4}$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(2) = \frac{2}{8} = \frac{1}{4}$

$f^{IV}(x) = \frac{-6}{x^4}$ $\qquad f^{IV}(2) = \frac{-6}{16} = \frac{-3}{8}$

$$f(x) = \ln 2 + \frac{(x-2)\{+\frac{1}{2}\}}{1!} + \frac{(x-2)^2(-\frac{1}{4})}{2!} + \frac{(x-2)^3(\frac{1}{4})}{3!} + \frac{(x-2)^4(-\frac{3}{8})}{4!}$$

$$\log_e x = \ln 2 + \frac{(x-2)}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} - \frac{(x-2)^4}{64}$$

now $f(3) = \log 3$

$$\log_e 3 = \ln 2 + \left(\frac{3-2}{2}\right) - \left(\frac{3-2}{8}\right)^2 + \left(\frac{3-2}{24}\right)^3 - \frac{(3-2)^4}{64}$$

$$\log_e 3 = \ln 2 + \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} = 1.094188$$ Ans

Q→ Expand $\cos x$ about $x = \pi$; $(a=\pi)$

$f(x) = \cos x$ $\qquad f(a) = f(\pi) = \cos\pi = -1$

$f'(x) = -\sin x$ $\qquad f'(\pi) = 0$

$f''(x) = -\cos x$ $\qquad f''(\pi) = 1$

$f'''(x) = \sin x$ $\qquad f'''(\pi) = 0$

$f^{IV}(x) = \cos x$ $\qquad f^{IV}(\pi) = -1$

$$\cos x = -1 + \frac{(x-\pi)(0)}{1!} + \frac{(x-\pi)^2(1)}{2!} + \frac{(x-\pi)^3(0)}{3!} + \frac{(x-\pi)^4(-1)}{4!} + \cdots$$

$$\cos x = -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} + \cdots \quad \text{As}$$

$\boxed{Q\to}$ obtain $\tan^{-1}x$ in powers of $(x-1)$;

Ans $(a=1)$

$f(x) = \tan^{-1}x$ $\qquad$ $f(1) = \tan^{-1}(1) = \tan^{-1}(\tan\frac{\pi}{4}) = \frac{\pi}{4}$

$f'(x) = \frac{1}{(1+x^2)}$ $\qquad$ $f'(1) = \frac{1}{1+1} = \frac{1}{2}$

$f''(x) = \frac{-2x}{(1+x^2)^2}$ $\qquad$ $f''(1) = \frac{-2(1)}{(1+1)^2} = -\frac{1}{2}$

$f'''(x) = \frac{-2\{1-3x^2\}}{(1+x^2)^3}$ $\qquad$ $f'''(1) = \frac{-2\{1-3\}}{(2)^3} = \frac{1}{2}$

$$\Rightarrow f(x) = \frac{\pi}{4} + \frac{(x-1)(\frac{1}{2})}{1!} + \frac{(x-1)^2(-\frac{1}{2})}{2!} + \frac{(x-1)^3(\frac{1}{2})}{3!} + \cdots$$

$$\Rightarrow \tan^{-1}x = \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} + \cdots \quad \text{As}$$

$\boxed{Q\to}$ obtain e^x in the powers of $(x+1)$. find $f(1)$

$f(x) = e^x$ $\qquad$ $f(-1) = e^{-1}$

$f'(x) = e^x$ $\qquad$ $f'(-1) = e^{-1}$

$f''(x) = e^x$ $\qquad$ $f''(-1) = e^{-1}$

$$f(x) = e^{-1} + \frac{(x+1)e^{-1}}{1!} + \frac{(x+1)^2 e^{-1}}{2!} + \frac{(x+1)^3 e^{-1}}{3!} + \cdots$$

$$e^x = e^{-1} + \frac{(x+1)e^{-1}}{1!} + \frac{(x+1)^2 e^{-1}}{2} + \frac{(x+1)^3 e^{-1}}{6} + \cdots$$

$$f(1) = e^1 = e^{-1} + \frac{(2)(e^{-1})}{1} + \frac{(2)^2 e^{-1}}{2} + \frac{(2)^3 e^{-1}}{6} + \cdots$$

$$e = 2.329903 \quad \underline{\underline{\text{Ans}}}$$

$\boxed{Q \rightarrow}$ Expand log(sin x) in the powers of $(x-2)$

$f(x) = \log \sin x$ $\qquad\qquad f(a) = \log \sin(2)$

$f'(x) = \left(\frac{1}{\sin x}\right) \cos x = \cot x$ $\qquad\qquad f'(a) = \cot(2)$

$f''(x) = -\operatorname{cosec}^2 x$ $\qquad\qquad f''(a) = -\operatorname{cosec}^2(2)$

$f'''(x) = (-2 \operatorname{cosec} x)(-\operatorname{cosec} x . \cot x)$ $\qquad\qquad f'''(a) = 2 \operatorname{cosec}^2(2) . \cot(2)$

$= 2 \operatorname{cosec}^2 x . \cot$

$$\log(\sin x) = \log \sin x + \frac{(x-2)\{\cot(2)\}}{1!} + \frac{(x-2)^2\{-\operatorname{cosec}^2(2)\}}{2!} + \frac{(x-2)^3\{2\operatorname{cosec}^2(2).\cot(2)\}}{3!} + \cdots$$

$\underline{\underline{\text{Ans}}}$

$\boxed{Q \rightarrow}$ Expand $\sqrt{x}$ in the powers of $(x-a)$.

$f(x) = \sqrt{x} = (x)^{1/2}$ $\qquad\qquad f(a) = \sqrt{a}$

$f'(x) = \frac{1}{2}(x)^{\frac{1}{2}-1} = \frac{1}{2}(x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ $\qquad\qquad f'(a) = \frac{1}{2\sqrt{a}}$

$f''(x) = -\frac{1}{4}(x)^{-3/2} = -\frac{1}{4x.\sqrt{x}}$ $\qquad\qquad f''(a) = -\frac{1}{4a\sqrt{a}}$

$f'''(x) = \frac{3}{8}(x)^{5/2} = \frac{3}{8x^2.\sqrt{x}}$ $\qquad\qquad f'''(a) = \frac{3}{8a^2\sqrt{a}}$

$$\sqrt{x} = \sqrt{a} + \frac{(x-a)\left\{\frac{1}{2\sqrt{a}}\right\}}{1!} + \frac{(x-a)^2\left\{-\frac{1}{4a\sqrt{a}}\right\}}{2!} + \frac{(x-a)^3\left\{\frac{3}{8a^2\sqrt{a}}\right\}}{3!} + \cdots$$

$$\sqrt{x} = \sqrt{a} + \frac{(x-a)}{2\sqrt{a}} - \frac{(x-a)^2}{8a\sqrt{a}} + \frac{3(x-a)^3}{48a^2\sqrt{a}} + \cdots \quad \text{Ans}$$

*** Find the value based on taylor's series :-

Q→ Using taylor's series $f(x) = \log_e x$ in the power of $(x-1)$. hence find the value of $\log_e(1.1)$.

Ans $a = 1$

$f(x) = \log_e(x)$ $\qquad$ $f(a) = \log_e 1 = 0$

$f'(x) = \frac{1}{x}$ $\qquad$ $f'(a) = \frac{1}{1} = 1$

$f''(x) = -\frac{1}{x^2}$ $\qquad$ $f''(a) = -\frac{1}{1} = -1$

$f'''(x) = \frac{2}{x^3}$ $\qquad$ $f'''(a) = \frac{2}{1} = 2$

$f^{IV}(x) = \frac{-6}{x^4}$ $\qquad$ $f^{IV}(a) = -\frac{6}{1} = -6$

$$\log_e x = 0 + \frac{(x-1)\{1\}}{1!} + \frac{(x-1)^2\{-1\}}{2!} + \frac{(x-1)^3\{2\}}{3!} + \frac{(x-1)^4\{-6\}}{4!} + \cdots$$

$$\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

Now for finding $\log_e(1.1)$ subtitute $x = 1.1$ in above Eqn.

$$\log_e(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - \frac{(1.1-1)^4}{4} + \cdots$$

$$\log_e(1.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \cdots$$

$\log_e(1.1) = 0.095308$ (measured value)

$= 0.095310$ (calculator value) Ans

Q→ Expand $\sin x$ in power of $(x-\frac{\pi}{2})$. hence find the $\sin 91^\circ$

$(a=\frac{\pi}{2})$

Ans $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{IV}(x) = \sin x$

$f(a) = \sin\frac{\pi}{2} = 1$

$f'(a) = \cos\frac{\pi}{2} = 0$

$f''(a) = -\sin\frac{\pi}{2} = -1$

$f'''(a) = -\cos\frac{\pi}{2} = 0$

$f^{IV}(a) = \sin\frac{\pi}{2} = 1$

$$\sin x = 1 + \frac{(x-\frac{\pi}{2})(0)}{1!} + \frac{(x-\frac{\pi}{2})^2(-1)}{2!} + \frac{(x-\frac{\pi}{2})^3(0)}{3!} + \frac{(x-\frac{\pi}{2})^4(1)}{4!} + \cdots$$

$$\sin x = 1 - \frac{(x-\frac{\pi}{2})^2}{2} + \frac{(x-\frac{\pi}{2})^4}{24} + \cdots\cdots$$

Now finding $\sin 91^\circ$ substitute $x = 91^\circ = 91 \times \frac{\pi}{180} = 1.587$

$\frac{\pi}{2} = \frac{3.14}{2} = 1.57$

* { this conversion is necessary because 91° is in degree, so for substitution in series conversion need in radian. }

$(x-\frac{\pi}{2}) = (1.587 - 1.57) = 0.017$

$$\sin 91^\circ = 1 - \frac{(0.017)^2}{2} + \frac{(0.017)^4}{24} + \cdots$$

ans = 0.017

$= 1 - \text{ans}^\wedge 2 (1 \lrcorner 2) + \text{ans}^\wedge 4 (1 \lrcorner 24) \cdots$ {calculator method}

$\sin 91^\circ = 0.999855$ **Ans**

Q→ Expand $f(x) = \tan x$ in power of $(x-\frac{\pi}{4})$, find $\tan 46^\circ$.

$(a=\frac{\pi}{4})$, $f(x) = \tan x$

$f(x) = \tan x$ $\qquad\qquad$ $f(a) = \tan\frac{\pi}{4} = 1$

$f'(x) = \sec^2 x$ $\qquad\qquad$ $f'(a) = \sec^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$

$f''(x) = (2\sec x)(\sec x \tan x)$ $\qquad\qquad$ $f''(a) = 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 2(2)(1) = 4$

$= 2\sec^2 x \tan x$. $\qquad\qquad$ $f'''(a) = 4(2)(1) + 2(4) = 16$

$f'''(x) = (2\tan x)\{2\sec^2 x \tan x\} + (2\sec^2 x)\{\sec^2 x\}$

$= 4\sec^2 x \tan^2 x + 2\sec^4 x$

$$\tan x = (1) + \frac{(x-\frac{\pi}{4})(2)}{1!} + \frac{(x-\frac{\pi}{4})^2(4)}{2!} + \frac{(x-\frac{\pi}{4})^3(16)}{3!} + \cdots$$

$$\tan x = 1 + 2(x-\tfrac{\pi}{4}) + 2(x-\tfrac{\pi}{4})^2 + \tfrac{8}{3}(x-\tfrac{\pi}{4})^3 + \cdots$$

$\tan 46^\circ$ means $x = 46^\circ = \frac{46\times\pi}{180} = 0.80285$

$\frac{\pi}{4} = \frac{3.14}{4} = 0.785$

$(x-\frac{\pi}{4}) = (0.80285 - 0.785) = (0.01785)$

$\tan 46^\circ = 1 + 2(0.01785) + 2(0.01785)^2 + \frac{8}{3}(0.01785)^3 \cdots$

$\tan 46^\circ = 1.03623$ Ans

*** taylor's series if function will be $f(a+h)$:

$$\boxed{f(a+h) = f(a) + \frac{(h)f'(a)}{1!} + \frac{(h)^2 f''(a)}{2!} + \frac{(h)^3 f'''(a)}{3!} + \cdots}$$

$\boxed{Q\to 1}$ find the Expansion of $\tan(x+\frac{\pi}{4})$ in ascending power of x upto x^4 & find $\tan 43^\circ$

Ans $f(a+h) = \tan(x+\frac{\pi}{4})$

$$\begin{cases} a = \pi/4 \\ h = x \\ f(x) = \tan x \end{cases}$$

$f(x) = \tan x = y$

$y' = \sec^2 x$

$y'' = (2\sec x)(\sec x.\tan x)$

$= 2\sec^2 x \tan x$

$y'' = 2(y')(y)$

$y''' = (2y)\{y''\} + (2y')\{y'\} = 2yy'' + 2(y')^2$

$y^{IV} = (2y)\{y'''\} + (2y'')\{y'\} + 4y'y''$

$y(a) = y(\frac{\pi}{4}) = \tan\frac{\pi}{4} = 1$

$y'(a) = \sec^2(\frac{\pi}{4}) = (\sqrt{2})^2 = 2$

$y''(a) = 2(2)(1) = 4$

$y'''(a) = 2(1)(4) + 2(2)^2 = 16$

$y^{IV}(a) = 2(1)(16) + 2(4)(2) + 4(2)(4)$

$= 2(16) + 16 + 32$

$= 32 + 32 + 16 = 80$

{ this type of derivation is used because repeated terms }

$\tan(\frac{\pi}{4}+x) = 1 + \frac{(x)(2)}{1!} + \frac{(x)^2(4)}{2!} + \frac{(x)^3(16)}{3!} + \frac{(x)^4(80)}{4!} + \dots$

$\tan(\frac{\pi}{4}+x) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 \dots$

Now $\tan 43^\circ = \tan(45-2) \approx \tan(\frac{\pi}{4}+x)$

hence $x = -2^\circ = -2 \times \frac{\pi}{180} = -0.0349$

$\frac{\pi}{4} = 0.7853$

$\tan 43^\circ = 1 + 2(-0.0349) + 2(-0.0349)^2 + \frac{8}{3}(-0.0349)^3 + \frac{10}{3}(0.0349)^4 \dots$

$= 1 + 2\text{ Ans} + 2\text{ Ans}^\wedge 2 + \text{Ans}^\wedge 3\,(8 \lrcorner 3) + \text{Ans}^\wedge 4\,(10 \lrcorner 3)$

$\simeq 0.9326$ (approx). Ans

Q→ Express $(x-1)^4 + 2(x-1)^3 + 5(x-1) + 2$ in ascending powers of x.

Ans. $f(a+h) = (x-1)^4 + 2(x-1)^3 + 5(x-1) + 2$

$a = -1$
$h = x$

$f(x) = x^4 + 2x^3 + 5x + 2$ $\qquad f(a) = (-1)^4 + 2(-1)^3 + 5(-1) + 2 = -4$

$f'(x) = 4x^3 + 6x^2 + 5$ $\qquad f'(a) = 4(-1)^3 + 6(-1)^2 + 5 = 7$

$f''(x) = 12x^2 + 12x$ $\qquad f''(a) = 12(-1)^2 + 12(-1) = 0$

$f'''(x) = 24x + 12$ $\qquad f'''(a) = 24(-1) + 12 = -12$

$f^{iv}(x) = 24$ $\qquad f^{iv}(a) = 24$

$$f(a+h) = f(a) + \frac{(h)f'(a)}{1!} + \frac{(h)^2 f''(a)}{2!} + \frac{(h)^3 f'''(a)}{3!}$$

$$f(a+h) = (-4) + \frac{(x)(7)}{1!} + \frac{(x)^2(0)}{2!} + \frac{(x)^3(-12)}{3!} + \frac{(x)^4(24)}{4!}$$

$$f(a+h) = -4 + 7x + 0 - \frac{12x^3}{6} + \frac{24x^4}{24}$$

$$f(a+h) = x^4 - 2x^3 + 7x - 4 \quad \text{Ans}$$

Q→ Express $7 + (x+2) + 3(x+2)^2 + (x+2)^4 - (x+2)^5$ in ascending power of x.

Ans $f(a+h) = 7 + (x+2) + 3(x+2)^2 + (x+2)^4 - (x+2)^5$

$a = 2$
$h = x$

$f(x) = 7 + x + 3x^2 + x^4 - x^5$ $\qquad f(a) = 5$

$f'(x) = 1 + 6x + 4x^3 - 5x^4$ $\qquad f'(a) = -35$

$f''(x) = 6 + 12x^2 - 20x^3$ $\qquad f''(a) = -108$

$f'''(x) = 24x - 60x^2$ $\qquad f'''(a) = -192$

$f^{iv}(x) = 24 - 120x$ $\qquad$ $f^{iv}(a) = -216$

$f^{v}(x) = -120$ $\qquad$ $f^{v}(a) = -120$

$$f(a+h) = 5 + \frac{(x)(-35)}{1!} + \frac{(x)^2(-108)}{2!} + \frac{(x)^3(-192)}{3!} + \frac{(x)^4(-216)}{4!} + \frac{(x)^5(-120)}{5!}$$

$$f(a+h) = 5 - 35x - \frac{108x^2}{2} - \frac{192x^3}{6} - \frac{216}{24}x^4 - \frac{120}{120}x^5$$

$$f(a+h) = 5 - 35x - 54x^2 - 32x^3 - 9x^4 - x^5 \quad : \text{Ans}$$

Q→ show that $\tan^{-1}(x+h) = \tan^{-1}x + (h\sin\alpha)\frac{\sin\alpha}{1} - (h\sin\alpha)^2\frac{\sin 2\alpha}{2}$

$$+ (h\sin\alpha)^3\frac{\sin 3\alpha}{3} + \ldots$$

Ans $f(x+h) = \tan^{-1}(x+h)$

$f(x) = \tan^{-1}x$ ---- (1)

$f'(x) = \frac{1}{1+x^2}$ ----- (2)

$$f''(x) = \frac{(1+x^2)\{0\} - (1)\{2x\}}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2} \quad ---- (3)$$

$$f'''(x) = \frac{\left((1+x^2)^2\right)\{-2\} - (-2x)\{2(1+x^2)\cdot 2x\}}{(1+x^2)^4}$$

$$= \frac{2(1+x^2)\left[-(1+x^2) + 4x^2\right]}{(1+x^2)^4} = \frac{2\left[-1-x^2+4x^2\right]}{(1+x^2)^3}$$

$$f'''(x) = \frac{2(3x^2-1)}{(1+x^2)^3} \quad ---- (4)$$

Put $x = \cot\alpha$,

$$f'(\cot\alpha) = \frac{1}{1+\cot^2\alpha} = \frac{1}{\operatorname{cosec}^2\alpha} = \sin^2\alpha \quad \text{--- (i)}$$

$$f''(\cot\alpha) = -\frac{2\cot\alpha}{(1+\cot^2\alpha)^2} = \frac{-2\cot\alpha}{(\operatorname{cosec}^2\alpha)^2}$$

$$= -2\sin^2\alpha\sin^2\alpha \cdot \frac{\cos\alpha}{\sin\alpha} = -\sin^2\alpha(2\sin\alpha\cos\alpha)$$

$$= -\sin^2\alpha \cdot \sin 2\alpha \quad \text{---- (ii)}$$

$$f'''(\cot\alpha) = \frac{2\{3\cot^2\alpha - 1\}}{\{1+\cot^2\alpha\}^3} = \frac{2\left\{\frac{3\cos^2\alpha}{\sin^2\alpha} - 1\right\}}{(\operatorname{cosec}^2\alpha)^3}$$

$$= \frac{2\left\{\frac{3\cos^2\alpha - \sin^2\alpha}{\sin^2\alpha}\right\}}{\operatorname{cosec}^6\alpha} = \frac{2\{3\cos^2\alpha - \sin^2\alpha\}}{\operatorname{cosec}^6\alpha \cdot \sin^2\alpha}$$

$$= \frac{2\{3(1-\sin^2\alpha) - \sin^2\alpha\}}{\operatorname{cosec}^4\alpha} = 2\{3 - 4\sin^2\alpha\}\sin^4\alpha$$

$$= 2\{3\sin\alpha - 4\sin^3\alpha\}\sin^3\alpha = 2\sin 3\alpha \cdot \sin^3\alpha \quad \text{--- (iii)}$$

$$f(x+h) = \tan^{-1}x + \frac{h\sin^2\alpha}{1!} + \frac{h^2(-\sin^2\alpha\sin 2\alpha)}{2!} + \frac{h^3(2\sin 3\alpha \cdot \sin^3\alpha)}{3!}$$

$$= \tan^{-1}x + h\sin\alpha\left(\frac{\sin\alpha}{1}\right) - (h\sin\alpha)^2\frac{\sin 2\alpha}{2} + (h\sin\alpha)^3\frac{\sin 3\alpha}{3} + \ldots$$

Ans

Q→ Expand $\log_e\{\tan(x+\frac{\pi}{4})\}$ in power of x.

Ans $f(a+h) = \log_e\{\tan(x+\frac{\pi}{4})\}$

$a = \frac{\pi}{4}$

$h = x$

$f(x) = \log_e \tan x$

$\Rightarrow$ $f(x) = \log_e \tan x$

$f'(x) = \frac{\sec^2 x}{\tan x} = \frac{1+\tan^2 x}{\tan x}$

$f'(x) = \cot x + \tan x$

$f''(x) = -\operatorname{cosec}^2 x + \sec^2 x$

$f'''(x) = 2\operatorname{cosec}^2 x \cot x + 2\sec^2 x \tan x$

$f(a) = \log_e \tan\frac{\pi}{4} = \log 1 = 0$

$f'(a) = \cot\frac{\pi}{4} + \tan\frac{\pi}{4} = 1+1 = 2$

$f''(a) = -\operatorname{cosec}^2\frac{\pi}{4} + \sec^2\frac{\pi}{4} = -2+2 = 0$

$f'''(a) = 2\operatorname{cosec}^2\frac{\pi}{4}\cot\frac{\pi}{4} + 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4}$

$= 2(2)(1) + 2(2)(1) = 8$

$$\log_e\{\tan(x+\tfrac{\pi}{4})\} = 0 + \frac{(x)(2)}{1!} + \frac{(x)^2(0)}{2!} + \frac{(x)^3(8)}{3!} + \cdots$$

$$\log_e\{\tan(x+\tfrac{\pi}{4})\} = 2x + \frac{4}{3}x^3 + \cdots \quad \text{Ans}$$

Q→ Expand $\log_e \cos(x+\frac{\pi}{4})$ & find $\log_e \cos 48^\circ$.

Ans $f(a+h) = \log_e \cos(x+\frac{\pi}{4})$

$a = \frac{\pi}{4}$, $x = h$, $f(x) = \log_e \cos x$.

$f(x) = \log_e \cos x$

$f'(x) = -\tan x$

$f''(x) = -\sec^2 x$

$f'''(x) = -2\sec^2 x \tan x$

$f^{iv}(x) = -[4\sec^2 x \tan^2 x + 2\sec^4 x]$

$f(a) = \log_e \cos\frac{\pi}{4} = -\log_e\sqrt{2}$

$f'(a) = -1$

$f''(a) = -2$

$f'''(a) = -2(2)(1) = -4$

$f^{iv}(a) = -[4(2)(1) + 2(2)^2] = -16$

$$\log_e\{\tan(x+\tfrac{\pi}{4})\} = -\log\sqrt{2} + \frac{(x)(-1)}{1!} + \frac{(x)^2(-2)}{2!} + \frac{(x)^3(-4)}{3!} + \frac{(x)^4(-16)}{4!} + \cdots$$

$$\log_e\{\cos(x+\tfrac{\pi}{4})\} = -\log_e\sqrt{2} - x - x^2 - \tfrac{2}{3}x^3 - \tfrac{2}{3}x^4 - - - -$$

$$\log_e \cos 48^\circ = \log_e\{\cos(45^\circ+3^\circ)\} \approx \log_e\{\cos(\tfrac{\pi}{4}+x)\}$$

$$x = 3^\circ \times \frac{\pi}{180} = 0.0523$$

$$\log_e \cos 48^\circ = -\log_e\sqrt{2} - (0.0523) - (0.0523)^2 - \tfrac{2}{3}(0.0523)^3 - \tfrac{2}{3}(0.0523)^4 + \ldots$$

$$= -0.402 \text{ approx}$$

Ans

$\boxed{Q\to}$ find $\sqrt{10}$ correct upto 4 decimal places by using taylor's series.

Ans let $f(a+h) = \sqrt{a+h}$ $\quad a = 9$ [take closest square value]

$\sqrt{10} = \sqrt{9+1} = \sqrt{a+h}$ $\quad h = x = 1$

$f(x) = \sqrt{x}$

$$f(x) = \sqrt{x} = (x)^{1/2} \qquad f(a) = \sqrt{9} = 3$$

$$f'(x) = +\frac{1}{2\sqrt{x}} \qquad f'(a) = \frac{1}{(2)(3)} = \frac{1}{6}$$

$$f''(x) = -\frac{1}{4}\frac{1}{x.\sqrt{x}} = -\frac{1}{4(x)^{3/2}} \qquad f''(a) = -\frac{1}{4(9)(3)} = -\frac{1}{108}$$

$$f'''(x) = \frac{3}{8}\times\frac{1}{(x)^{5/2}} = \frac{3}{8x^2.\sqrt{x}} \qquad f'''(a) = \frac{3}{8(81)(3)} = \frac{1}{648}$$

$$\sqrt{a+h} = 3 + \frac{(x)(\frac{1}{6})}{1!} + \frac{(x)^2(-\frac{1}{108})}{2!} + \frac{(x)^3(\frac{1}{648})}{3!} + - - -$$

$$\sqrt{a+h} = 3 + \frac{x}{6} - \frac{x^2}{(2)(108)} + \frac{x^3}{(6)(648)} + - - - - - -$$

$$\sqrt{10} = \sqrt{9+1} = \sqrt{a+h} = \qquad h = x = 1.$$

$$\sqrt{10} = 3 + \frac{1}{6} - \frac{1}{216} + \frac{1}{3888} = 3 + (1 \lrcorner 6) - (1 \lrcorner 216) + (1 \lrcorner 3888)$$

$= (3 \lrcorner 631 \lrcorner 3888)$ press $a\,b/c$

$= 3.16229$ Ans

$\boxed{Q\rightarrow}$ $\sqrt{36.12}$ correct upto 4 decimals.

$f(a+h) = \sqrt{a+h} = \sqrt{36+0.12}$

$a = 36$ [take closest square value]

$h = x = 0.12$

$f(x) = \sqrt{x}$

$f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$f''(x) = \dfrac{-1}{4x.\sqrt{x}}$

$f'''(x) = \dfrac{3}{8x^2\sqrt{x}}$

$f(a) = 6$

$f'(a) = \dfrac{1}{2\times6} = \dfrac{1}{12}$

$f''(a) = -\dfrac{1}{4(36)(6)} = -\dfrac{1}{864}$

$f'''(a) = \dfrac{3}{8(36)^2(6)} = \dfrac{1}{20736}$

$$\sqrt{a+h} = 6 + \frac{(x)(\frac{1}{12})}{1!} + \frac{(x)^2(-\frac{1}{864})}{2!} + \frac{(x)^3(\frac{1}{20736})}{3!} + --$$

$$\sqrt{a+h} = 6 + \frac{x}{12} - \frac{x^2}{1728} + \frac{x^3}{124416} + ----$$

$$\sqrt{36.12} = 6 + \frac{(0.12)}{12} - \frac{(0.12)^2}{1728} + \frac{(0.12)^3}{124416} = 6.00999 \text{ Ans}$$

$\boxed{Q\rightarrow}$ $\sqrt[3]{25}$ correct upto 3 decimals.

$f(a+h) = \sqrt[3]{a+h} = \sqrt[3]{(27-2)}$

$a = 27$ [take closest cube value]

$h = x = -2$

$f(x) = \sqrt[3]{x}$

$f(x) = (x)^{1/3}$

$f'(x) = \dfrac{1}{3(x)^{+2/3}}$

$f''(x) = -\dfrac{2}{9}(x)^{-5/3}$

$f'''(x) = \dfrac{5}{27}(x)^{-8/3}$

$f(a) = \sqrt[3]{27} = 3$

$f'(a) = \dfrac{1}{3(x)^{2/3}} = \dfrac{1}{3(27)^{2/3}} = \dfrac{1}{3\times9} = \dfrac{1}{27}$

$f''(a) = \dfrac{-2}{9(x)^{5/3}} = \dfrac{-2}{9(3)^5} = \dfrac{-2}{2187}$

$f'''(a) = \dfrac{5}{27(x)^{8/3}} = \dfrac{5}{27(3)^8} = \dfrac{5}{177147}$

$$f(a+h) = f(a) + \frac{(h)f'(a)}{1!} + \frac{(h)^2 f''(a)}{2!} + \frac{(h)^3 f'''(a)}{3!} + \cdots$$

$$\sqrt[3]{a+h} = 3 + \frac{(x)\left\{\frac{1}{27}\right\}}{1!} + \frac{(x)^2\left\{\frac{-2}{2187}\right\}}{2!} + \frac{(x)^3\left\{\frac{5}{177147}\right\}}{3!} + \cdots$$

$$\sqrt[3]{25} = 3 + \frac{(-2)}{27} + \frac{(-2)^2(-2)}{2\times 2187} + \frac{(-2)^3(5)}{6\times 177147} + \cdots$$

$$\sqrt[3]{25} = 2.92402 \quad \text{Ans}$$

*** Miscellaneous type :-

Q→ obtain the series $\log_e(1+x)$ and find the series $\log_e\left(\frac{1+x}{1-x}\right)$ & hence find $\log_e\left(\frac{11}{9}\right)$.

$$f(x) \Rightarrow \log_e\left(\frac{1+x}{1-x}\right) = \log_e(1+x) - \log_e(1-x)$$

$$f(x) \Rightarrow \log_e\left(\frac{1+x}{1-x}\right) = \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots\right\} - \left\{-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots\right\}$$

$$\log_e\left(\frac{1+x}{1-x}\right) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$\log_e\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$$

(Put $x = \frac{1}{10}$)

$$\Rightarrow \log_e\left(\frac{1+\frac{1}{10}}{1-\frac{1}{10}}\right) = 2\left[\left(\frac{1}{10}\right) + \frac{1}{3}\left(\frac{1}{10}\right)^3 + \frac{1}{5}\left(\frac{1}{10}\right)^5 + \cdots\right]$$

$$= 0.20067 \quad \text{Ans}$$

note →

$$\boxed{\begin{array}{l} ① \ \tanh^{-1}x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) \\ ② \ \coth^{-1}x = \frac{1}{2}\log_e\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right) \end{array}}$$

Q→ Expand $\tanh^{-1}x$ in the power of x.

$$f(x) = \tanh^{-1}x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left[\log_e(1+x) - \log_e(1-x)\right]$$

$$= \frac{1}{2}\left[\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots\right\} - \left\{-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots\right\}\right]$$

$$= \frac{1}{2}\left[2x + \frac{2x^3}{3} + \frac{2x^5}{5} \cdots\right].$$

$$\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\cdots$$ Ans

Q→ $\coth^{-1}x$ in the power of x by using maclusion series.

$$f(x) = \coth^{-1}x = \frac{1}{2}\log_e\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right) = \frac{1}{2}\left[\log_e\left(1+\frac{1}{x}\right) - \log_e\left(1-\frac{1}{x}\right)\right]$$

$$= \frac{1}{2}\left[\left\{\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} \cdots\right\} - \left\{-\frac{1}{x} - \frac{1}{2x^2} \ \frac{1}{3x^3} - \frac{1}{4x^4} \cdots\right\}\right]$$

$$= \frac{1}{2}\left[\frac{2}{x} + \frac{2}{3x^3} + \frac{2}{5x^5} \cdots\right]$$

$$\coth^{-1}x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots$$ Ans

note →

- $\cos 2x = 2\cos^2 x - 1$
- $\cos 2x = 1 - 2\sin^2 x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$
- $\cos 3x = 4\cos^3 x - 3\cos x$

- $\sin 2x = 2\sin x \cos x$
- $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$
- $\sin 3x = 3\sin x - 4\sin^3 x$

- $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

Q → Prove that $\sin^{-1}\left[\dfrac{2x}{1+x^2}\right] = 2x - \dfrac{2x^3}{3} + \dfrac{2x^5}{5} - \dfrac{2x^7}{7} \ldots$

$$f(x) = \sin^{-1}\left[\frac{2x}{1+x^2}\right]$$

Let $x = \tan\theta$
or $\theta = \tan^{-1}x$

$$f(x) = \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$f(x) = \sin^{-1}(\sin 2\theta)$$

$$f(x) = 2\theta = 2\tan^{-1}x$$

$$f'(x) = \frac{2}{1+x^2} = 2(1+x^2)^{-1} \approx (1+X)^m$$

$X = x^2$
$m = -1$

$$f'(x) = 2\left[1 + \frac{(x^2)(-1)}{1!} + \frac{(x^2)^2(-1)(-2)}{2!} + \frac{(x^2)^3(-1)(-2)(-3)}{3!} + \ldots\right]$$

$$f'(x) = 2\left[1 - x^2 + x^4 - x^6 \ldots\right]$$

integrate w.r.t x both side.

$$\int f'(x)\,dx = \int (2 - 2x^2 + 2x^4 - 2x^6 \ldots)\,dx$$

$$f(x) = 2x - \frac{2x^3}{3} + \frac{2x^5}{5} - \frac{2x^7}{7} \ldots$$

$$\sin^{-1}\left[\frac{2x}{1+x^2}\right] = 2x - \frac{2x^3}{3} + \frac{2x^5}{5} - \frac{2x^7}{7} \ldots$$

proved.

Q→ Expand $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ in the power of x.

Ans $f(x) = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ Let $x = \tan\theta$

$\theta = \tan^{-1}x$

$$f(x) = \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right] = \tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right] = \tan^{-1}\left[\frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\right] = \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right] = \tan^{-1}\left[\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right] = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$f(x) = \frac{\theta}{2} = \frac{\tan^{-1}x}{2} = \frac{1}{2}\left\{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - - \right\}$$

$$f(x) = \frac{x}{2} - \frac{x^3}{6} + \frac{x^5}{10} - \frac{x^7}{14} - - - -$$ Ans

Q→ Prove that $\sin^{-1}[3x-4x^3] = 3\left\{x + \frac{x^3}{6} + \frac{3}{40}x^5 - - -\right\}$

Ans $f(x) = \sin^{-1}[3x-4x^3]$ Let $x = \sin\theta$

$\theta = \sin^{-1}x$

$$= \sin^{-1}[3\sin\theta - 4\sin^3\theta]$$

$$= \sin^{-1}[\sin 3\theta]$$

$$f(x) = 3\theta = 3\sin^{-1}x$$

$$f'(x) = \frac{3}{\sqrt{1-x^2}} = 3(1-x^2)^{-1/2} \approx (1-X)^m \qquad X = x^2, \quad m = -1/2$$

$$f'(x) = 3\left[1 - \frac{(x^2)(-\frac{1}{2})}{1!} + \frac{(x^2)^2(-\frac{1}{2})(-\frac{3}{2})}{2!} - \frac{(x^2)^3(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} - -\right]$$

$$f'(x) = 3\left[1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} \cdots\right]$$

$$\int f'(x)\,dx = 3\int \left[1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} \cdots\right] dx$$

$$f(x) = 3\left[x + \frac{x^3}{6} + \frac{3x^5}{40} \cdots\right] \quad \text{As}$$

$\boxed{Q \to}$ Expand $\cos^2 x$ in the power of x.

$$f(x) = \cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2}\left[1 + \cos 2x\right]$$

$$f(x) = \frac{1}{2}\left[1 + \left\{1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} \cdots\right\}\right]$$

$$= \frac{1}{2}\left[2 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} \cdots\right]$$

$$= 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} \cdots \quad \text{As}$$

$\boxed{Q \to}$ Expand $\sin^2 2x$ in the power of x.

$$f(x) = \sin^2 2x = \frac{1 - \cos 4x}{2} = \frac{1}{2}\left[1 - \cos 4x\right]$$

$$f(x) = \frac{1}{2}\left[1 - \left\{1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \frac{(4x)^6}{6!} \cdots\right\}\right]$$

$$= \frac{1}{2}\left[1 - 1 + 8x^2 - \frac{32}{3}x^4 + \frac{256}{45}x^6 \cdots\right]$$

$$= 4x^2 - \frac{16x^4}{3} + \frac{128}{45}x^6 \cdots \quad \text{As}$$

$\boxed{Q \to}$ $f(x) = \ln(\cos x)$ in the power of x.

$$f'(x) = \left(\frac{1}{\cos x}\right)\{-\sin x\} = -\tan x$$

$$f'(x) = -\left\{x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\right\}$$

$$f'(x) = -x - \frac{x^3}{3} - \frac{2}{15}x^5 - \ldots\ldots$$

$$\int f'(x)dx = \int \left(-x - \frac{x^3}{3} - \frac{2}{15}x^5 \ldots\right)dx$$

$$f(x) = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} \ldots\ldots \quad \underline{\underline{\text{Ans}}}$$

$\boxed{Q \to}$ $f(x) = e^{\sin x}$ in the power of x upto x^4 term.

$$f(x) = e^{\sin x} = e^{\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\right\}}$$

$$f(x) = 1 + \frac{1}{1!}\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\right\} + \frac{1}{2!}\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\right\}^2$$

$$= 1 + x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{1}{2}\left\{(x)^2 + 2(x)\left(-\frac{x^3}{3!}\right) \ldots\right\}$$

$$= 1 + x - \frac{x^3}{6} + \frac{x^5}{120} + \frac{1}{2}\left\{x^2 - \frac{x^4}{3} \ldots\right\}$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{6} \ldots \quad \underline{\underline{\text{Ans}}}$$

note $\to$ $\boxed{(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd}$

$\boxed{Q \rightarrow}$ $f(x) = e^{\cos x}$ upto x^4 terms.

$$f(x) = e^{\cos x} = e^{\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right\}} = e^1 . e^{\left\{-\frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right\}}$$

$$f(x) = e\left\{1 + \frac{1}{1!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right) + \frac{1}{2!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right)^2\right\}$$

$$f(x) = e\left\{1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{1}{2}\left(\left(-\frac{x^2}{2!}\right)^2 \cdots\cdots\right)\right\}$$

$$f(x) = e\left\{1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^4}{8} \cdots\right\}$$

$$f(x) = e\left(1 - \frac{x^2}{2} + \frac{x^4}{6} + \cdots\cdots\right)$$

Ans

OR

$f(x) = e^{\cos x}$ $\qquad a = 0 \rightarrow$ maclurin's series.

$y = f(x) = e^{\cos x}$ $\qquad f(0) = y(0) = e^{\cos 0} = e$

$y' = f'(x) = e^{\cos x}(-\sin x) = -y.\sin x$ $\qquad f'(0) = y'(0) = -(e)(0) = 0$

$y'' = (-y)\{\cos x\} + (\sin x)\{-y'\}$ $\qquad f''(0) = y''(0) = -(e)(1) - (0)(0) = -e$

$= -y\cos x - y'\sin x$ $\qquad f'''(0) = y'''(0) = (1)(-0) + (-e)(-0) + (0)(e) + (0)(1) = 0$

$y''' = ((\cos x)\{-y'\} + (-y)\{-\sin x\} + (\sin x)\{-y''\} + (-y')\{\cos x\}$

$$f(x) = f(0) + \frac{(x)f'(0)}{1!} + \frac{(x)^2 f''(0)}{2!} + \frac{(x)^3 f'''(0)}{3!} + \cdots\cdots$$

$$e^{\cos x} = e + \frac{(x)(0)}{1!} + \frac{(x)^2(-e)}{2!} + \frac{(x)^3(0)}{3!} + \cdots$$

$$e^{\cos x} = e - \frac{ex^2}{2} \cdots\cdots$$

$$e^{\cos x} = e\left\{1 - \frac{x^2}{2} \cdots\right\}$$ Ans