

INDETERMINATE FORMS

- * Some limits of functions which have indeterminate forms, that have no meaning on direct substitution, but it may be a meaningful and have existence of limit after simplification or in other way.
- * We will use an important 'L' Hospital rule for finding such limits, which is an important application of Taylor's theorem.
- * We shall discuss the indeterminate forms;
 - a) $\frac{0}{0}$ form
 - b) $\frac{\infty}{\infty}$ form
 - c) $\infty - \infty$ form
 - d) $0 \times \infty$ form
 - e) $0^0, \infty^0, 1^\infty$ forms.

** L'Hospital's Rule for $\left(\frac{0}{0}\right)$ form:

Statement: If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}, \quad g'(a) \neq 0$$

→ Using Taylor's Series,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f(a) + \frac{(x-a)f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!} + \dots}{g(a) + \frac{(x-a)g'(a)}{1!} + \frac{(x-a)^2 g''(a)}{2!} + \frac{(x-a)^3 g'''(a)}{3!} + \dots}$$

$$\left. \begin{aligned} \lim_{x \rightarrow a} f(x) &= f(a) = 0 \\ \lim_{x \rightarrow a} g(x) &= g(a) = 0 \end{aligned} \right\}$$

$$\begin{aligned} \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{(x-a) \frac{f'(a)}{1!} + \frac{(x-a)^2 f''(a)}{2!} + \frac{(x-a)^3 f'''(a)}{3!} + \dots}{(x-a) \frac{g'(a)}{1!} + \frac{(x-a)^2 g''(a)}{2!} + \frac{(x-a)^3 g'''(a)}{3!} + \dots} \\ \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{(x-a) \left\{ \frac{f'(a)}{1!} + \frac{(x-a) \cdot f''(a)}{2!} + \frac{(x-a)^2 \cdot f'''(a)}{3!} + \dots \right\}}{(x-a) \left\{ \frac{g'(a)}{1!} + \frac{(x-a) \cdot g''(a)}{2!} + \frac{(x-a)^2 \cdot g'''(a)}{3!} + \dots \right\}} \\ \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{f'(a) + \frac{(x-a) \cdot f''(a)}{2!} + \frac{(x-a)^2 \cdot f'''(a)}{3!} + \dots}{g'(a) + \frac{(x-a) \cdot g''(a)}{2!} + \frac{(x-a)^2 \cdot g'''(a)}{3!} + \dots} \\ \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \frac{f'(a)}{g'(a)} \\ \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}, \text{ hence proved} \end{aligned}$$

*** Rule to Evaluate : { $\frac{0}{0}$ form }.

- 1) Check whether the limit is an indeterminate form.
- 2) Apply L'Hospital's Rule and Continue Differentiate of $f(x)$ & $g(x)$ till the required value is reached.

*** Shortcut formula:-

$$1) \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{or} \quad \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$$

$$2) \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$3) \lim_{x \rightarrow 0} \frac{\sin hx}{x} = 1$$

Remember

$$\left\{ \lim_{x \rightarrow 0} \frac{\sin kx}{x} = 1 \right.$$

$$4) \lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1$$

$$5) \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1$$

$$6) \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log a$$

$$7) \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

Q → 1) Evaluate, $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^2} \left(\frac{0}{0} \right)$

→ apply LH Rule, $\lim_{x \rightarrow 0} \frac{\sec^2 x - \cos x}{2x} \left(\frac{0}{0} \right)$

→ apply LH Rule, $\lim_{x \rightarrow 0} \frac{2 \cdot \sec^2 x \cdot \tan x + \sin x}{2}$

→ $\frac{2(1)(0) + (0)}{2} = 0$ Ans

2) Evaluate, $\lim_{x \rightarrow 0} \frac{\log(1-x^2)}{\log \cos x} \left(\frac{0}{0} \right)$

$$\begin{cases} \cos 0 = 1 \\ \log 1 = 0 \end{cases}$$

→ apply LH Rule; $\lim_{x \rightarrow 0} \frac{-2x}{\frac{-\sin x}{\cos x}}$

→ $\lim_{x \rightarrow 0} \frac{2x}{(1-x^2) \cdot \tan x}$

$$\left\{ \lim_{x \rightarrow 0} \frac{x}{\tan x} = 1 \right\}$$

$$\lim_{x \rightarrow 0} \frac{2}{1+x^2} = \frac{2}{1} = 2. \quad \text{Ans}$$

(Q-3) $\lim_{x \rightarrow 0} \frac{x - \tan x}{x^3} \quad \left(\frac{0}{0} \right)$

applying LH Rule

$$\lim_{x \rightarrow 0} \frac{1 - \frac{1}{1+x^2}}{3x^2} = \lim_{x \rightarrow 0} \frac{1+x^2 - 1}{3x^2(1+x^2)}$$

$$= \lim_{x \rightarrow 0} \frac{x^2}{3x^2(1+x^2)} = \lim_{x \rightarrow 0} \frac{1}{3(1+x^2)}$$

$$= \frac{1}{3} \quad \text{Ans}$$

(Q-4) $\lim_{x \rightarrow \pi/2} \frac{\log \sin x}{(\pi - 2x)^2} \quad \left(\frac{0}{0} \right)$

applying LH Rule; $\lim_{x \rightarrow \pi/2} \frac{\frac{\cos x}{\sin x}}{2(\pi - 2x) \cdot (-2)}$

$$\Rightarrow \lim_{x \rightarrow \pi/2} \frac{\cot x}{-4(\pi - 2x)} \quad \left(\frac{0}{0} \right)$$

applying LH Rule; $\lim_{x \rightarrow \pi/2} -\frac{\operatorname{cosec}^2 x}{8}$

$$= -\frac{1}{8} \quad \text{Ans}$$

$$Q \rightarrow 5 \quad \lim_{x \rightarrow y} \frac{x^y - y^x}{x^x - y^y}$$

$\left(\frac{0}{0} \right)$

→ applying LH Rule; $\lim_{x \rightarrow y} \frac{y^x - y \cdot \log y}{x^x (1 + \log x)}$

$$\frac{d}{dx} \{x^x\} \Rightarrow \text{let } y = x^x \\ \log y = \log(x)^x$$

$$\log y = x \cdot \log x$$

$$\frac{1}{y} \frac{dy}{dx} = x \left\{ \frac{1}{x} \right\} + \log x \left\{ 1 \right\}$$

$$\frac{dy}{dx} = y \{ 1 + \log x \}$$

$$\frac{dy}{dx} = x^x \{ 1 + \log x \}$$

$$\rightarrow \frac{y \cdot y^{(y)}}{y^y (1 + \log y)} - \frac{y^y \cdot \log y}{y^y (1 + \log y)} = \frac{y^y - y \cdot \log y}{y^y (1 + \log y)}$$

$$\Rightarrow \frac{y^y \{ 1 - \log y \}}{y^y \{ 1 + \log y \}} = \frac{1 - \log y}{1 + \log y}$$

$$Q \rightarrow 6 \quad \lim_{x \rightarrow 0} \frac{\cosh x - \cos x}{x \cdot \sin x}$$

if denominator differentiated
then terms will be increasing
and it will tough to calculate.

$$= \lim_{x \rightarrow 0} \frac{\cosh x - \cos x}{x \cdot \sin x} \cdot \frac{x}{x} \quad \left(\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1 \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\cosh x - \cos x}{x^2} \quad \left(\frac{0}{0} \right) \quad (\cosh 0 = 1) \\ (\cos 0 = 1)$$

applying LH Rule, $\lim_{x \rightarrow 0} \frac{\sinhx + \sinx}{2x}$

$$= \lim_{x \rightarrow 0} \frac{1}{2} \left[\frac{\sinhx}{x} + \frac{\sinx}{x} \right]$$

$$= \frac{1}{2} [1+1] = \frac{2}{2} = 1$$

Q7 $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{\sin^2 x - x^2} \quad \left(\frac{0}{0} \right)$

\Rightarrow Applying LH Rule, $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{2 \cdot \sin x \cdot \cos x - 2x}$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{\sin 2x - 2x} \quad \left(\frac{0}{0} \right)$$

\Rightarrow applying LH Rule, $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{2 \cos 2x - 2} \quad \left(\frac{0}{0} \right)$

\Rightarrow applying LH Rule, $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{-4 \sin 2x} \quad \left(\frac{0}{0} \right)$

\Rightarrow applying LH Rule, $\lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{-8 \cos 2x}$

$$\Rightarrow \frac{1+1}{-8(1)} = -\frac{1}{4}$$

$$\text{Q8} \lim_{x \rightarrow 0} \frac{\tan^2 x - x^2}{x^2 \cdot \tan^2 x} \quad (\text{need to remove } \tan^2 x)$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\tan^2 x - x^2}{x^2 / \tan^2 x} \cdot x^2$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\tan^2 x - x^2}{x^4} \left(\frac{0}{0} \right)$$

$$\text{applying LH Rule; } \lim_{x \rightarrow 0} \frac{2 \cdot \tan x \cdot \sec^2 x - 2x}{4x^3} \left(\frac{0}{0} \right)$$

$$\text{applying LH Rule; } \lim_{x \rightarrow 0} \frac{2 \sec^4 x + 4 \tan^2 x \cdot \sec^2 x - 2}{12x^2} \left(\frac{0}{0} \right)$$

$$\text{applying LH Rule; } \lim_{x \rightarrow 0} \frac{8 \sec^4 x \cdot \tan x + 4 \left[\tan^3 x \cdot 2 \sec^2 x + 2 \tan x \cdot \sec^4 x \right]}{24x}$$

$$= \lim_{x \rightarrow 0} \frac{16 \sec^4 x \cdot \tan x + 8 \tan^3 x \cdot \sec^2 x}{24x} \left(\frac{0}{0} \right)$$

instead of applying LH Rule, I will separate denominator;

$$= \lim_{x \rightarrow 0} \left\{ \frac{16 \sec^4 x \cdot \tan x}{24x} + \frac{8 \tan^3 x \cdot \sec^2 x}{24x} \right\}$$

$$= \lim_{x \rightarrow 0} \left\{ \frac{16 \sec^4 x}{24} + \frac{8 \tan^3 x \cdot \sec^2 x}{24} \right\}$$

$$= \frac{16}{24} + 0 = \frac{2}{3} \quad \text{Ans}$$

* L'Hospital's Rule for the $(\frac{\infty}{\infty})$ Form;

If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \infty$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

→ hence, we can extend the rule for the $(\frac{\infty}{\infty})$ form by diff. the numerator and denominator separately as many times as would be necessary.

$$\Rightarrow \frac{\infty}{\infty} = \frac{1/0}{1/0} = \frac{0}{0}$$

$$(Q \rightarrow 1) \lim_{x \rightarrow 0} \frac{\log x}{\cot x} \quad \left(\frac{\infty}{\infty} \right) \quad (\log 0 = \infty)$$

$$\rightarrow \text{applying LH Rule; } \lim_{x \rightarrow 0} \frac{-\frac{1}{x}}{-\csc^2 x}$$

$$= \lim_{x \rightarrow 0} -\frac{\sin^2 x}{x} = \lim_{x \rightarrow 0} -\sin x \quad \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right)$$

$$= 0 \quad \text{Ans}$$

$$(Q \rightarrow 2) \lim_{x \rightarrow 0} \frac{\log(\sin x)}{\log(\tan x)} \quad \left(\frac{\infty}{\infty} \right)$$

$$\text{applying LH Rule; } \lim_{x \rightarrow 0} \frac{\frac{\cos x}{\sin x}}{\frac{\sec^2 x}{\tan x}}$$

$$= \lim_{x \rightarrow 0} \frac{\cos x \cdot \tan x}{\sin x \cdot \sec^2 x} = \lim_{x \rightarrow 0} \frac{\tan x}{\tan x \cdot \sec^2 x}$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{\sec^2 x} \right) = \lim_{x \rightarrow 0} (\cos^2 x) = 1$$

(Q-3) $\lim_{x \rightarrow \infty} \frac{x^n}{e^x}$

$$\left(\frac{\infty}{\infty} \right)$$

applying LH Rule; $\lim_{x \rightarrow \infty} \frac{n(x)}{e^x} \quad \left(\frac{\infty}{\infty} \right)$

applying LH Rule; $\lim_{x \rightarrow \infty} \frac{(n)(n-1)(x)^{n-2}}{e^x} \quad \left(\frac{\infty}{\infty} \right)$

applying n^{th} times LH Rule;

$\lim_{x \rightarrow \infty} \frac{(n)(n-1)(n-2) \dots (3)(2)(1)}{e^x}$

$$= \lim_{x \rightarrow \infty} \frac{n!}{e^x} = \frac{n!}{\infty} = 0$$

(Q-4) $\lim_{x \rightarrow \infty} \frac{x(\log x)^3}{1+x+x^2}$

Applying LH Rule; $\lim_{x \rightarrow \infty} \frac{(\log x)^3 \{1\} + (x) \{3(\log x)^2 \cdot \left(\frac{1}{x}\right)\}}{1+2x}$

$$= \lim_{x \rightarrow \infty} \frac{(\log x)^3 + 3(\log x)^2}{1+2x} \quad \left(\frac{\infty}{\infty} \right)$$

applying LH Rule; $\lim_{x \rightarrow \infty} \frac{3[\log x] \left(\frac{1}{x}\right) + 6(\log x) \left(\frac{1}{x}\right)}{2}$

$$= \lim_{x \rightarrow \infty} \frac{3(\log x)^2 + 6(\log x)}{2x} \quad \left(\frac{\infty}{\infty} \right)$$

applying LH Rule; $\lim_{x \rightarrow \infty} \frac{6(\log x)(\frac{1}{x}) + 6(\frac{1}{x})}{2}$

$$= \lim_{x \rightarrow \infty} \frac{6 \log x + 6}{2x} \quad \left(\frac{\infty}{\infty} \right)$$

applying LH Rule; $\lim_{x \rightarrow \infty} \frac{\frac{6}{x}}{2}$

$$= \lim_{x \rightarrow \infty} \frac{3}{x} = 0 \quad \cancel{\text{Ans}}$$