U. V. Patel College of Engineering

B. Tech. Semester - I (All)

Subject: 2BS101 Mathematics-I

Unit: 3 Integral Calculus

Chapter: 2 & 3 Gamma function and Beta function

(1) Gamma Function

The definite integral

$$\int_0^\infty e^{-x}\, x^{n-1}\, dx, \quad ; \quad n > 0$$

is a function of n and is called gamma function. It is denoted by Γ (read as gamma) thus

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx \quad ; \quad n > 0$$

This definition is also written as

$$\Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\, dt \quad ; \quad n > 0 \qquad \text{OR} \qquad \Gamma(n) = \int_0^\infty e^{-u}\, u^{n-1}\, du \quad ; \quad n > 0$$

Properties of Gamma function:

(1) $\boldsymbol{\Gamma(n+1) = n\,\Gamma(n)}$

We know that $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx$

Putting $n = n+1$ in above definition

$$\Gamma(n+1) = \int_0^\infty e^{-x}\, x^{n}\, dx$$

$$= \left[-e^{-x}\, x^n\right]_0^\infty - \int_0^\infty \left[n\, x^{n-1}(-e^{-x})\right]\, dx \qquad \left(u = x^n \;\; v = e^{-x}\right)$$

$$= 0 + n \int_0^\infty e^{-x}\, x^{n-1}\, dx \qquad \left(\because \lim_{n\to\infty} e^{-x}\, x^n = 0\right)$$

$$= n\,\Gamma(n)$$

Particular case:

When n is a positive integer then $\Gamma(n+1) = n!$

We know that

$$\begin{aligned}
\Gamma(n+1) &= n\,\Gamma(n) \\
&= n\,\Gamma\left((n-1)+1\right) \\
&= n\,(n-1)\,\Gamma(n-1) \\
&= n\,(n-1)\,\Gamma\left((n-2)+1\right) \\
&= n\,(n-1)\,(n-2)\,\Gamma(n-2) \qquad \text{Repeated application of this formula} \\
&= n\,(n-1)\,(n-2)\,(n-3)\,\ldots\ldots\ldots\, 2\cdot 1\cdot \Gamma(1)
\end{aligned}$$

But

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = -\left[e^{-x}\right]_0^\infty = -\left[e^{-\infty} - e^0\right] = 1$$

Therefore

$$\begin{aligned}
\Gamma(n+1) &= n\,(n-1)\,(n-2)\,(n-3)\,\ldots\ldots\ldots\, 2\cdot 1\cdot \Gamma(1) \\
&= n\,(n-1)\,(n-2)\,(n-3)\,\ldots\ldots\ldots\, 2\cdot 1\cdot 1 \\
&= n!
\end{aligned}$$

(2) $\boldsymbol{\int_0^\infty e^{-ax}\,x^{n-1}\,dx = \dfrac{\Gamma(n)}{a^n}}$ (Without Proof)

(3) $\boldsymbol{\Gamma(n) = 2\int_0^\infty e^{-u^2}\,u^{2n-1}\,du}$

We know that $\Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1}\,dx$

Taking $x = t^2 \quad \therefore \quad dx = 2t\,dt$

When $x = 0 \implies t = 0$ and when $x = \infty \implies t = \infty$

$$\begin{aligned}
\Gamma(n) &= \int_0^\infty e^{-x}\,x^{n-1}\,dx \\
&= \int_0^\infty e^{-t^2}\,(t^2)^{n-1}\,2t\,dt \\
&= 2\int_0^\infty e^{-t^2}\,t^{2n-2}\,t\,dt = 2\int_0^\infty e^{-t^2}\,t^{2n-1}\,dt = 2\int_0^\infty e^{-u^2}\,u^{2n-1}\,du
\end{aligned}$$

(4) $\boldsymbol{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$ (We will prove this later on)

Evaluate following integrals in terms of Gamma function.

(1) $\displaystyle\boldsymbol{\int_0^\infty e^{-x^4}\, x^2\, dx}$

We are taking

$$x^4 = t$$
$$\therefore \quad x = t^{1/4}$$
$$\therefore \quad dx = \frac{1}{4}\, t^{\frac{1}{4}-1}\, dt$$
$$\therefore \quad dx = \frac{1}{4}\, t^{-3/4}\, dt$$

When $x = 0 \implies t = 0$ and when $x = \infty \implies t = \infty$

$$\int_0^\infty e^{-x^4}\, x^2\, dx = \int_0^\infty e^{-t}\ t^{1/2}\ \frac{1}{4}\ t^{-3/4}\, dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t}\ t^{1/2}\ t^{-3/4}\, dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t}\ t^{\frac{1}{2}-\frac{3}{4}}\, dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t}\ t^{\frac{-1}{4}}\, dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t}\ t^{\left(\frac{-1}{4}+1\right)-1}\, dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t}\ t^{\left(\frac{3}{4}\right)-1}\, dt$$

$$= \frac{1}{4}\,\Gamma\left(\frac{3}{4}\right)$$

(2) $\displaystyle\int_{-\infty}^{\infty} e^{-k^2x^2}\, dx$

Here $f(x) = k^2x^2$ is an even function.

$$\int_{-\infty}^{\infty} e^{-k^2x^2}\, dx = 2\int_{0}^{\infty} e^{-k^2x^2}\, dx$$

Now, we are taking

$$\begin{aligned} k^2x^2 &= t \\ \therefore \quad kx &= t^{1/2} \\ \therefore \quad k\, dx &= \frac{1}{2}\, t^{-1/2}\, dt \\ \therefore \quad dx &= \frac{1}{2k}\, t^{-1/2}\, dt \end{aligned}$$

When $x = 0 \implies t = 0$ and when $x = \infty \implies t = \infty$

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-k^2x^2}\, dx &= 2\int_{0}^{\infty} e^{-k^2x^2}\, dx \\ &= 2\int_{0}^{\infty} e^{-t}\, \frac{1}{2k}\, t^{-1/2}\, dt \\ &= \frac{1}{k}\int_{0}^{\infty} e^{-t}\, t^{-1/2}\, dt \\ &= \frac{1}{k}\int_{0}^{\infty} e^{-t}\, t^{(-1/2+1)-1}\, dt \\ &= \frac{1}{k}\int_{0}^{\infty} e^{-t}\, t^{(1/2)-1}\, dt \\ &= \frac{1}{k}\,\Gamma\left(\frac{1}{2}\right) \\ &= \frac{1}{k}\,\sqrt{\pi} \end{aligned}$$

(3) $\boldsymbol{\int\limits_0^\infty \frac{x^a}{a^x}\,dx \qquad (a > 1)}$

Now, we are taking

$$a^x = e^t$$

$$\therefore \;\; x\;\log a = t \qquad \therefore \;\; x = \frac{t}{\log a}$$

$$\therefore \;\; dx\;\log a = dt$$

$$\therefore \;\; dx = \frac{dt}{\log a}$$

When $x = 0 \implies t = 0$ and when $x = \infty \implies t = \infty$

$$\int\limits_0^\infty \frac{x^a}{a^x}\,dx = \int\limits_0^\infty \frac{\left(\frac{t}{\log a}\right)^a}{e^t}\,\frac{dt}{\log a}$$

$$= \int\limits_0^\infty \frac{t^a}{e^t\;(\log a)^a \cdot \log a}\,dt$$

$$= \frac{1}{(\log a)^{a+1}} \int\limits_0^\infty \frac{t^a}{e^t}\,dt$$

$$= \frac{1}{(\log a)^{a+1}} \int\limits_0^\infty e^{-t}\;t^a\;dt$$

$$= \frac{1}{(\log a)^{a+1}} \int\limits_0^\infty e^{-t}\;t^{(a+1)-1}\;dt$$

$$= \frac{1}{(\log a)^{a+1}}\,\Gamma\,(a+1)$$

$$= \frac{a!}{(\log a)^{a+1}}$$

Homework

Evaluate following integrals in terms of Gamma function.

(1) $\displaystyle\int_0^\infty \frac{x^4}{4^x}\, dx$

(2) $\displaystyle\int_0^\infty e^{-x^2}\, x^6\, dx$

(2) Beta Function

The definite integral

$$\int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx, \quad (m>0,\ n>0)$$

is a function of m and n is called beta function. It is denoted by $B(m,n)$ (read as beta m , n) thus

$$B(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$$

Properties of Beta function:

(1) $\boldsymbol{B(m\,,\,n) = B(n\,,\,m)}$

We know that $B(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx$

Now, we are taking

$$1-x = y$$
$$\therefore \quad x = 1-y$$
$$\therefore \quad dx = -dy$$

When $x=0 \implies y=1$ and when $x=1 \implies y=0$

$$\begin{aligned}
B(m,n) &= \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx \\
&= \int_1^0 (1-y)^{m-1}\,y^{n-1}\,(-dy) \\
&= -\int_1^0 y^{n-1}\,(1-y)^{m-1}\,dy \\
&= \int_0^1 y^{n-1}\,(1-y)^{m-1}\,dy \\
&= B(n,m)
\end{aligned}$$

(2) $\boldsymbol{B\left(m\,,\,n\right) = 2\int\limits_{0}^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta}$

We know that $B\left(m,n\right) = \int\limits_{0}^{1} x^{m-1}\ \left(1-x\right)^{n-1}\ dx$

Now, we are taking

$$x = \sin^2\theta$$

$$\therefore\ \ dx = 2\sin\theta\cos\theta\ d\theta$$

When $x = 0 \implies \theta = 0$ and when $x = 1 \implies \theta = \pi/2$

$$B\left(m,n\right) = \int\limits_{0}^{1} x^{m-1}\ \left(1-x\right)^{n-1}\ dx$$

$$= \int\limits_{0}^{\pi/2} \left(\sin^2\theta\right)^{m-1}\ \left(1-\sin^2\theta\right)^{n-1}\ \left(2\sin\theta\cos\theta\ d\theta\right)$$

$$= 2\int\limits_{0}^{\pi/2} \sin^{2m-2}\theta\ \cos^{2n-2}\theta\ \sin\theta\cos\theta\ d\theta$$

$$= 2\int\limits_{0}^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ \ d\theta$$

Evaluate following integrals in terms of Beta function.

(1) $\boldsymbol{\int\limits_{0}^{1} \frac{dx}{\sqrt{1-x^4}}}$

We are taking

$$x^4 = t$$

$$\therefore\ \ x = t^{1/4}$$

$$\therefore\ \ dx = \frac{1}{4}\,t^{\frac{1}{4}-1}\ dt$$

$$\therefore\ \ dx = \frac{1}{4}\,t^{\frac{-3}{4}}\ dt$$

When $x = 0 \implies t = 0$ and when $x = 1 \implies t = 1$

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 \frac{1}{\sqrt{1-t}} \ \frac{1}{4} \ t^{\frac{-3}{4}} \, dt$$

$$= \frac{1}{4} \int_0^1 t^{\frac{-3}{4}} \ (1-t)^{-1/2} \ dt$$

$$= \frac{1}{4} \int_0^1 t^{\left(\frac{-3}{4}+1\right)-1} \ (1-t)^{(-1/2+1)-1} \ dt$$

$$= \frac{1}{4} \int_0^1 t^{\left(\frac{1}{4}\right)-1} \ (1-t)^{\left(\frac{1}{2}\right)-1} \ dt$$

$$= \frac{1}{4} \, B\left(\frac{1}{4}, \frac{1}{2}\right)$$

(2) $\displaystyle \int_0^1 \frac{x\,dx}{\sqrt{1-x^5}}$

We are taking

$$x^5 = t$$

$$\therefore \quad x = t^{1/5}$$

$$\therefore \quad dx = \frac{1}{5} \, t^{\frac{1}{5}-1} \, dt$$

$$\therefore \quad dx = \frac{1}{5} \, t^{\frac{-4}{5}} \, dt$$

When $x = 0 \implies t = 0$ and when $x = 1 \implies t = 1$

$$\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}} = \int_0^1 \frac{t^{1/5}}{\sqrt{1-t}} \ \frac{1}{5} \ t^{\frac{-4}{5}} \, dt$$

$$= \frac{1}{5} \int_0^1 t^{\frac{1}{5}} \ (1-t)^{-1/2} \ \ t^{\frac{-4}{5}} \ dt$$

$$= \frac{1}{5} \int_0^1 t^{\frac{-3}{5}} \ (1-t)^{-1/2} \ dt$$

$$= \frac{1}{5} \int_0^1 t^{\left(\frac{-3}{5}+1\right)-1} \ (1-t)^{\left(\frac{-1}{2}+1\right)-1} \ dt$$

$$= \frac{1}{5} \int_0^1 t^{\left(\frac{2}{5}\right)-1} \ (1-t)^{\left(\frac{1}{2}\right)-1} \ dt$$

$$= \frac{1}{5} \ B\left(\frac{2}{5}, \frac{1}{2}\right)$$

Homework

Evaluate following integrals in terms of Beta function.

(1) $\displaystyle\int_0^1 \frac{x^7 \ dx}{\sqrt{1-x^4}}$

(1) $\displaystyle\int_0^m x^m \ (m-x)^n \ dx$

Prove that

$$\int_0^{\pi/2} \sin^p\theta \ \cos^q\theta \ d\theta = \frac{1}{2} \ B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

We have already proved in the property(2) for Beta function that

$$B\left(m\,,\,n\right) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \ \cos^{2n-1}\theta \, d\theta \quad \therefore \quad \int_0^{\pi/2} \sin^{2m-1}\theta \ \cos^{2n-1}\theta \, d\theta = \frac{1}{2} \ B\left(m\,,\,n\right)$$

Let's write $2m-1 = p \quad \therefore \quad m = \dfrac{p+1}{2}$ and $2n-1 = q \quad \therefore \quad n = \dfrac{q+1}{2}$

By putting these values in the RHS of above equation we get

$$\int_0^{\pi/2} \sin^{2m-1}\theta \ \cos^{2n-1}\theta \, d\theta = \frac{1}{2} \ B\left(m\,,\,n\right)$$

$$\therefore \quad \int_0^{\pi/2} \sin^p\theta \ \cos^q\theta \ d\theta = \frac{1}{2} \ B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Relationship between Beta and Gamma functions.

$$B\,(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

By Property (3) of Gamma function

$$\Gamma(m) = 2\int_0^\infty e^{-x^2}\,x^{2m-1}\,dx$$

$$\Gamma(n) = 2\int_0^\infty e^{-y^2}\,y^{2n-1}\,dy$$

Multiplying these two equations

$$\Gamma(m)\cdot\Gamma(n) = \left\{2\int_0^\infty e^{-x^2}\,x^{2m-1}\,dx\right\}\cdot\left\{2\int_0^\infty e^{-y^2}\,y^{2n-1}\,dy\right\}$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,x^{2m-1}\,y^{2n-1}\,dx\,dy \tag{1}$$

By taking $x = r\cos\theta$ and $y = r\sin\theta$ so that $x^2 + y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$

The region of integration in (1) is the entire positive quadrant of the $xy-$ plane. Hence the limit for r will be 0 to ∞ and those for θ will be 0 to $\frac{\pi}{2}$

Now

$$\Gamma(m)\cdot\Gamma(n) = 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\,(r\cos\theta)^{2m-1}\,(r\sin\theta)^{2n-1}\,r\,dr\,d\theta$$

$$= 4\left\{\int_0^{\frac{\pi}{2}}\sin^{2n-1}\theta\,\cos^{2m-1}\theta\,d\theta\right\}\cdot\left\{\int_0^\infty e^{-r^2}\,r^{(2m-1)+(2n-1)+1}\,dr\right\}$$

$$= \left\{2\int_0^{\frac{\pi}{2}}\sin^{2n-1}\theta\,\cos^{2m-1}\theta\,d\theta\right\}\cdot\left\{2\int_0^\infty e^{-r^2}\,r^{2(m+n)-1}\,dr\right\}$$

$$= B\,(n\,,\,m)\;\Gamma\,(m+n) \quad \text{(Properties of Beta and Gamma)}$$

$$= B\,(n\,,\,m)\;\Gamma\,(m+n) \quad \therefore\; B\,(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Prove that

(4) $\boldsymbol{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

We know that

$$B\,(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Taking $m = \dfrac{1}{2}$ and $n = \dfrac{1}{2}$ in above equation.

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\;\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$$

$$\therefore\;\; B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2}{\Gamma(1)}$$

$$\therefore\;\; B\left(\frac{1}{2}, \frac{1}{2}\right) = \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 \tag{1}$$

Now we know that $B\,(m\,,\,n) = 2\int\limits_0^{\pi/2} \sin^{2m-1}\theta\;\cos^{2n-1}\theta\;d\theta$

Taking $m = \dfrac{1}{2}$ and $n = \dfrac{1}{2}$ in above equation(1)

$$\therefore\;\; B\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int\limits_0^{\pi/2} d\theta = 2\,[\theta]_0^{\frac{\pi}{2}} = 2\left(\frac{\pi}{2} - 0\right) = \pi$$

Using these result in equation(1) we get

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2$$

$$\therefore\;\; \pi = \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2$$

$$\therefore\;\; \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Hence the result.