U. V. Patel College of Engineering

B. Tech. Semester - I (All Branches)

Subject: 2BS101 Mathematics-I

Unit: 3 Integral Calculus

Chapter: 1 Reduction Formula

$$(1)\ \int_0^{\frac{\pi}{2}} \sin^n x\, dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} & :\rightarrow \text{If } n \text{ is even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1 & :\rightarrow \text{If } n \text{ is odd} \end{cases}$$

For $n \geq 2$ and $n \in \mathbb{N}$

$$(2)\ \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} & :\rightarrow \text{If } n \text{ is even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1 & :\rightarrow \text{If } n \text{ is odd} \end{cases}$$

For $n \geq 2$ and $n \in \mathbb{N}$

Evaluate following integrals in terms of reduction formula.

(1) $\boldsymbol{\int_0^{\frac{\pi}{2}} \sin^6 x\, dx}$

Here $n = 6$ is an even number. Applying above reduction formula we get,

$$\int_0^{\frac{\pi}{2}} \sin^6 x\, dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}$$

(2) $\boldsymbol{\int_0^{\frac{\pi}{2}} \sin^7 x\, dx}$

Here $n = 7$ is an odd number. Applying above reduction formula we get,

$$\int_0^{\frac{\pi}{2}} \sin^7 x\, dx = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{16}{35}$$

(3) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^8 x \, dx$

Here $n = 8$ is an even number. Applying above reduction formula we get,

$$\int_0^{\frac{\pi}{2}} \cos^8 x \, dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{35\pi}{256}$$

(4) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^9 x \, dx$

Here $n = 9$ is an odd number. Applying above reduction formula we get,

$$\int_0^{\frac{\pi}{2}} \cos^9 x \, dx = \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{128}{315}$$

Homework-1

Evaluate following integrals in terms of reduction formula.

(1) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$ (2) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{10} x \, dx$ (3) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{11} x \, dx$ (4) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{12} x \, dx$

Evaluate following integrals in terms of reduction formula.

(1) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^7 2\theta \, d\theta$

We are taking

$$\begin{aligned} 2\theta &= x \\ \therefore \; 2\, d\theta &= dx \\ \therefore \; d\theta &= \frac{dx}{2} \end{aligned}$$

When $\theta = 0 \implies x = 0$ and when $\theta = \frac{\pi}{4} \implies x = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{4}} \sin^7 2\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^7 x \, \frac{dx}{2} = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^7 x \, dx = \frac{1}{2}\left[\frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1\right] = \frac{8}{35}$$

(2) $\displaystyle\int_0^{\frac{\pi}{8}} \cos^3 4\theta \, d\theta$

We are taking

$$4\theta = x$$
$$\therefore \quad 4\, d\theta = dx$$
$$\therefore \quad d\theta = \frac{dx}{4}$$

When $\theta = 0 \implies x = 0$ and when $\theta = \frac{\pi}{8} \implies x = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{8}} \cos^3 4\theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^3 x \, \frac{dx}{4} = \frac{1}{4}\int_0^{\frac{\pi}{2}} \cos^3 x \, dx = \frac{1}{4}\left[\frac{2}{3}\cdot 1\right] = \frac{1}{6}$$

(3) $\displaystyle\int_0^{\pi} \sin^7 \frac{\theta}{2} \, d\theta$

We are taking

$$\frac{\theta}{2} = x$$
$$\therefore \quad \theta = 2x$$
$$\therefore \quad d\theta = 2\, dx$$

When $\theta = 0 \implies x = 0$ and when $\theta = \pi \implies x = \frac{\pi}{2}$

$$\int_0^{\pi} \sin^7 \frac{\theta}{2} \, d\theta = \int_0^{\frac{\pi}{2}} \sin^7 x \; 2\, dx = 2\int_0^{\frac{\pi}{2}} \sin^7 x \, dx = 2\left[\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1\right] = \frac{32}{35}$$

Homework-2

Evaluate following integrals in terms of reduction formula.

(1) $\displaystyle\int_0^{\pi} (1-\cos x)^3 \, dx$ (2) $\displaystyle\int_0^{\pi} (1+\cos x)^4 \, dx$ (3) $\displaystyle\int_0^{\frac{\pi}{8}} \cos^{10} 4\theta \, d\theta$

Remember following results:

(1) $\int\limits_{-a}^{a} f(x)\,dx = 2\int\limits_{0}^{a} f(x)\,dx$ when $f(x)$ is an even function

(2) $\int\limits_{0}^{2a} f(x)\,dx = \int\limits_{0}^{a} f(x)\,dx + \int\limits_{0}^{a} f(2a-x)\,dx$

Evaluate following integrals in terms of reduction formula.

(1) $\boldsymbol{\int\limits_{-\pi}^{\pi} \sin^4 x\,dx}$

The integrand $f(x) = \sin^4 x$ is an even function hence we can use above result (1)

$$\int\limits_{-\pi}^{\pi} \sin^4 x\,dx = 2\int\limits_{0}^{\pi} \sin^4 x\,dx$$

Using $\int\limits_{0}^{2a} f(x)\,dx = \int\limits_{0}^{a} f(x)\,dx + \int\limits_{0}^{a} f(2a-x)\,dx$ for which $2a = \pi$ therefore $a = \dfrac{\pi}{2}$

$$\begin{aligned}
\int\limits_{-\pi}^{\pi} \sin^4 x\,dx &= 2\left[\int\limits_{0}^{\pi/2} \sin^4 x\,dx + \int\limits_{0}^{\pi/2} \sin^4(\pi - x)\,dx\right] \\
&= 2\left[\int\limits_{0}^{\pi/2} \sin^4 x\,dx + \int\limits_{0}^{\pi/2} \sin^4 x\,dx\right] \\
&= 2\left[2\int\limits_{0}^{\pi/2} \sin^4 x\,dx\right] \\
&= 4\left[\int\limits_{0}^{\pi/2} \sin^4 x\,dx\right] \\
&= 4\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] \\
&= \frac{3\pi}{4}
\end{aligned}$$

(3) $$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)(m-5)\cdots][(n-1)(n-3)(n-5)\cdots]}{[(m+n)(m+n-2)(m+n-4)\cdots\cdots]} \cdot \frac{\pi}{2}$$

If m and n both are even numbers. For $m, n \geq 2$ and $n \in \mathbb{N}$

$$= \frac{[(m-1)(m-3)(m-5)\cdots][(n-1)(n-3)(n-5)\cdots]}{[(m+n)(m+n-2)(m+n-4)\cdots\cdots]} \cdot 1$$

For remaining choices of m and n. For $m, n \geq 2$ and $n \in \mathbb{N}$

Evaluate following integrals in terms of reduction formula.

(1) $$\int_0^{\frac{\pi}{2}} \sin^6 x \ \cos^8 x \, dx$$

Here $m = 6$ and $n = 8$. Applying above reduction formula we get,

$$\int_0^{\frac{\pi}{2}} \sin^6 x \ \cos^8 x \, dx = \frac{[(6-1)(6-3)(6-5)] \cdot [(8-1)(8-3)(8-5)(8-7)]}{[(14)(14-2)(14-4)(14-6)(14-8)(14-10)(14-12)]} \cdot \frac{\pi}{2}$$

$$= \frac{[5 \cdot 3 \cdot 1] \cdot [7 \cdot 5 \cdot 3 \cdot 1]}{[14 \cdot 12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2]} \cdot \frac{\pi}{2}$$

$$= \frac{5\pi}{4096}$$

(2) $$\int_0^{\frac{\pi}{2}} \sin^4 x \ \cos^6 x \, dx$$

Here $m = 4$ and $n = 6$. Applying above reduction formula we get,

$$\int_0^{\frac{\pi}{2}} \sin^4 x \ \cos^6 x \, dx = \frac{[3 \cdot 1] \cdot [5 \cdot 3 \cdot 1]}{[10 \cdot 8 \cdot 6 \cdot 4 \cdot 2]} \cdot \frac{\pi}{2} = \frac{3\pi}{512}$$

(3) $$\int_0^{\frac{\pi}{2}} \sin^6 x \ \cos^2 x \, dx$$

Here $m = 6$ and $n = 2$. Applying above reduction formula we get,

$$\int_0^{\frac{\pi}{2}} \sin^6 x \ \cos^2 x \, dx = \frac{[5 \cdot 3 \cdot 1] \cdot [1]}{[8 \cdot 6 \cdot 4 \cdot 2]} \cdot \frac{\pi}{2} = \frac{5\pi}{256}$$

(4) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x \ \cos^7 x \, dx$

Here $m = 5$ and $n = 7$. Applying above reduction formula we get,

$$\int_0^{\frac{\pi}{2}} \sin^5 x \ \cos^7 x \, dx = \frac{[4 \cdot 2] \cdot [6 \cdot 4 \cdot 2]}{[12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2]} \cdot 1 = \frac{1}{120}$$

(5) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^6 x \ \cos^7 x \, dx$

Here $m = 6$ and $n = 7$. Applying above reduction formula we get,

$$\int_0^{\frac{\pi}{2}} \sin^6 x \ \cos^7 x \, dx = \frac{[5 \cdot 3 \cdot 1] \cdot [6 \cdot 4 \cdot 2]}{[13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1]} \cdot 1 = \frac{16}{3003}$$

(6) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x \ \cos^6 x \, dx$

Here $m = 5$ and $n = 6$. Applying above reduction formula we get,

$$\int_0^{\frac{\pi}{2}} \sin^5 x \ \cos^6 x \, dx = \frac{[4 \cdot 2] \cdot [5 \cdot 3 \cdot 1]}{[11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1]} \cdot 1 = \frac{11}{693}$$

Homework-3

Evaluate following integrals in terms of reduction formula.

(1) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x \ \cos^6 x \, dx$ (2) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^7 x \ \cos^9 x \, dx$ (3) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^6 x \ \cos^9 x \, dx$

(4) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4 x \ \cos^7 x \, dx$ (5) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x \ \cos^2 x \, dx$ (6) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 x \ \cos^8 x \, dx$

Evaluate the given integrals.

(1) $\displaystyle \int_0^{\frac{\pi}{6}} \sin^3 6\theta \ \cos^6 3\theta \ d\theta$

Here we can write,

$$\int_0^{\frac{\pi}{6}} \sin^3 6\theta \ \cos^6 3\theta \ d\theta = \int_0^{\frac{\pi}{6}} \sin^3 2(3\theta) \ \cos^6 3\theta \ d\theta$$

We are taking $3\theta = x \quad \therefore \quad 3d\theta = dx \quad \therefore \quad d\theta = \dfrac{dx}{3}$

When $\theta = 0 \implies x = 0$ and when $\theta = \dfrac{\pi}{6} \implies x = \dfrac{\pi}{2}$

$$\begin{aligned}
\int_0^{\frac{\pi}{6}} \sin^3 6\theta \ \cos^6 3\theta \ d\theta &= \int_0^{\frac{\pi}{6}} \sin^3 2(3\theta) \ \cos^6 3\theta \ d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin^3 2x \ \cos^6 x \ \frac{dx}{3} \\
&= \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^3 2x \ \cos^6 x \ dx \\
&= \frac{1}{3}\int_0^{\frac{\pi}{2}} (2\sin x\cos x)^3 \ \cos^6 x \ dx \\
&= \frac{1}{3}\int_0^{\frac{\pi}{2}} 8\sin^3 x\cos^3 x \ \cos^6 x \ dx \\
&= \frac{8}{3}\int_0^{\frac{\pi}{2}} \sin^3 x \ \cos^9 x \ dx \\
&= \frac{8}{3}\left[\frac{(2)\cdot(8\cdot 6\cdot 4\cdot 2)}{(12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2)}\right] \\
&= \frac{2}{45}
\end{aligned}$$

(2) $\boldsymbol{\int_0^{\pi} \theta \ \sin^8\theta \ \cos^6\theta \ d\theta}$

Let's write

$$I = \int_0^{\pi} \theta \ \sin^8\theta \ \cos^6\theta \ d\theta \tag{1}$$

Using the result $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ we get

$$\begin{aligned}
I &= \int_0^{\pi} \theta \ \sin^8\theta \ \cos^6\theta \ d\theta \\
&= \int_0^{\pi} (\pi-\theta) \ \sin^8(\pi-\theta) \ \cos^6(\pi-\theta) \ d\theta \\
&= \int_0^{\pi} (\pi-\theta) \ \sin^8\theta \ \cos^6\theta \ d\theta \\
&= \pi \int_0^{\pi} \sin^8\theta \ \cos^6\theta \ d\theta - \int_0^{\pi} \theta \ \sin^8\theta \ \cos^6\theta \ d\theta \\
&= \pi \int_0^{\pi} \sin^8\theta \ \cos^6\theta \ d\theta - I \qquad (\because \ (1))
\end{aligned}$$

$$\therefore \ 2I = \pi \int_0^{\pi} \sin^8\theta \ \cos^6\theta \ d\theta$$

$$\therefore \ 2I = 2\pi \int_0^{\pi/2} \sin^8\theta \ \cos^6\theta \ d\theta \qquad \text{(even function)}$$

$$\therefore \ I = \pi \int_0^{\pi/2} \sin^8\theta \ \cos^6\theta \ d\theta$$

$$\therefore \ I = \pi \left[\frac{(7\cdot 5\cdot 3\cdot 1)\cdot(5\cdot 3\cdot 1)}{(14\cdot 12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2)} \right] \cdot \frac{\pi}{2} = \frac{5\pi^2}{4096}$$

(3) $\displaystyle \boldsymbol{\int_0^\infty \frac{x^2}{(1+x^2)^8}\, dx}$

Let's take $x = \tan\theta \quad \therefore \quad dx = \sec^2\theta\, d\theta$

When $x = 0 \implies \theta = 0$ and when $x = \infty \implies \theta = \dfrac{\pi}{2}$

$$\int_0^\infty \frac{x^2}{(1+x^2)^8}\, dx = \int_0^{\pi/2} \frac{\tan^2\theta}{(1+\tan^2\theta)^8}\, \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{\tan^2\theta}{(\sec^2\theta)^8}\, \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{\tan^2\theta}{\sec^{16}\theta}\, \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{\tan^2\theta}{\sec^{14}\theta}\, d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^{14}\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta \cos^{12}\theta\, d\theta$$

$$= \left[\frac{(1)\cdot(11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1)}{(14\cdot 12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2)}\right] \cdot \frac{\pi}{2}$$

$$= \frac{33\pi}{4096}$$

Homework-4

Evaluate following integrals in terms of reduction formula.

(1) $\displaystyle \int_0^{\frac{\pi}{6}} \cos^4 3\theta\ \sin^2 6\theta\, d\theta$ (2) $\displaystyle \int_0^{\pi} \theta\ \sin^6\theta\ \cos^4\theta\, d\theta$ (3) $\displaystyle \int_0^{a} x^{3/2}\sqrt{a-x}\, dx$

Reduction formula for $\int_0^{\pi/4} \tan^n \theta \, d\theta$; $n \geq 2$

$$I_n = \int_0^{\pi/4} \tan^n \theta \, d\theta$$

$$= \int_0^{\pi/4} \tan^{n-2} \theta \cdot \tan^2 \theta \, d\theta$$

$$= \int_0^{\pi/4} \tan^{n-2} \theta \cdot \left(\sec^2 \theta - 1\right) \, d\theta$$

$$= \int_0^{\pi/4} \tan^{n-2} \theta \cdot \sec^2 \theta \, d\theta - \int_0^{\pi/4} \tan^{n-2} \theta \, d\theta$$

$$= \left[\frac{\tan^{n-1} \theta}{n-1}\right]_0^{\pi/4} - I_{n-2} \quad \left(\because \quad \int [f(x)]^n \cdot f'(x) \, dx = \frac{f^{n+1}(x)}{n+1}\right)$$

$$= \frac{1}{n-1} \left[\tan^{n-1} \theta\right]_0^{\pi/4} - I_{n-2}$$

$$= \frac{1}{n-1} [1 - 0] - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2}$$

Hence

$$I_n = \int_0^{\pi/4} \tan^n \theta \, d\theta = \frac{1}{n-1} - I_{n-2}$$

Where

$$I_0 = \int_0^{\pi/4} \tan^0 \theta \, d\theta = \int_0^{\pi/4} 1 \, d\theta = [\theta]_0^{\pi/4} = \frac{\pi}{4}$$

$$I_1 = \int_0^{\pi/4} \tan^1 \theta \, d\theta = [\log \sec \theta]_0^{\pi/4} = [\log (\sec \pi/4) - \log (\sec 0)] = \frac{1}{2} \log 2$$

Homework-5

Derive given reduction formula

$$I_n = \int_{\pi/4}^{\pi/2} \cot^n \theta \, d\theta = \frac{1}{n-1} - I_{n-2} \; ; \quad n \geq 2$$

Evaluate following integrals

(1) $\displaystyle\int_0^{\pi/4} \tan^4 \theta \, d\theta$

We know that

$$I_n = \int_0^{\pi/4} \tan^n \theta \, d\theta = \frac{1}{n-1} - I_{n-2} \tag{1}$$

Putting $n = 4$ in equation (1)

$$\begin{aligned} I_4 = \int_0^{\pi/4} \tan^n \theta \, d\theta &= \frac{1}{4-1} - I_{4-2} \\ &= \frac{1}{3} - I_2 \\ &= \frac{1}{3} - \left[\frac{1}{1} - I_0\right] \qquad \text{(Putting } n = 2 \text{ in (1))} \\ &= \frac{1}{3} - 1 + I_0 \\ &= I_0 - \frac{2}{3} \\ &= \frac{\pi}{4} - \frac{2}{3} \end{aligned}$$

Because

$$I_0 = \int_0^{\pi/4} \tan^0 \theta \, d\theta = \int_0^{\pi/4} 1 \, d\theta = [\theta]_0^{\pi/4} = \frac{\pi}{4}$$

(2) $\displaystyle\int_{\pi/4}^{\pi/2} \cot^5 \theta \, d\theta$

We know that

$$I_n = \int_{\pi/4}^{\pi/2} \cot^n \theta \, d\theta = \frac{1}{n-1} - I_{n-2} \tag{1}$$

Putting $n = 5$ in equation (1)

$$\begin{aligned}
I_5 = \int_{\pi/4}^{\pi/2} \cot^n \theta \, d\theta &= \frac{1}{5-1} - I_{5-2} \\
&= \frac{1}{4} - I_3 \\
&= \frac{1}{4} - \left[\frac{1}{2} - I_1\right] \qquad (\text{Putting } \; n = 3 \; \text{ in } \; (1)) \\
&= \frac{1}{4} - \frac{1}{2} + I_1 \\
&= I_1 - \frac{1}{4} \\
&= \frac{1}{2}\log 2 - \frac{1}{4}
\end{aligned}$$

Because

$$I_1 = \int_0^{\pi/4} \tan^1 \theta \, d\theta = [\log \sec \theta]_0^{\pi/4} = [\log(\sec \pi/4) - \log(\sec 0)] = \frac{1}{2}\log 2$$

Finally

$$\int_{\pi/4}^{\pi/2} \cot^5 \theta \, d\theta = \frac{1}{2}\log 2 - \frac{1}{4} = \frac{1}{2}\left(\log 2 - \frac{1}{2}\right)$$

Homework-6

Evaluate following integrals

(1) $\displaystyle\int_0^{\pi/4} \tan^5 \theta \, d\theta$ (2) $\displaystyle\int_0^{1} \frac{x^6}{1+x^2} \, dx$

(3) $\displaystyle\int_{\pi/4}^{\pi/2} \cot^6 \theta \, d\theta$