Series Expansions & Indeterminate forms

* Maclaurin's series :-

Condition ⇒ $(x-a)=0$ where, $a=0 \longrightarrow (x=0)$

$$f(x) = f(0) + \frac{(x)f'(0)}{1!} + \frac{(x)^2 f''(0)}{2!} + \frac{(x)^3 f'''(0)}{3!} + \cdots$$

① $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

② $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

③ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

④ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

⑤ $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$

⑥ $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots$

⑦ $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \cdots$

{ standard function by maclaurin's series

→ Expansion of inverse function $\{\tan^{-1}x, \cot^{-1}x, \sin^{-1}x, \cos^{-1}x\}$ by using maclaurin's series is carried out by substituting these values;

$$(1+x)^m = 1 + \frac{mx}{1!} + \frac{m(m-1)(x)^2}{2!} + \frac{m(m-1)(m-2)(x)^3}{3!} \cdots$$

$$(1-x)^m = 1 - \frac{mx}{1!} + \frac{m(m-1)(x)^2}{2!} - \frac{m(m-1)(m-2)(x)^3}{3!} \cdots$$

* **Taylor's Series :-**

→ Condition = $(x-a)=0 \Rightarrow (x=a)$

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

* **Taylor's Series :-**

→ Condition ⇒ if $f(a+h)$ is given.

$$f(a+h) = f(a) + \frac{f'(a)\cdot h}{1!} + \frac{f''(a)h^2}{2!} + \frac{f'''(a)h^3}{3!} + \cdots$$

* **Imp formula :-**

① $\tanh^{-1}x = \frac{1}{2}\log\left[\frac{(1+x)}{(1-x)}\right]$

② $\coth^{-1}x = \frac{1}{2}\log\left[\frac{(1+\frac{1}{x})}{(1-\frac{1}{x})}\right]$

③ $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

④ $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

⑤ $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

⑥ $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

⑦ $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

⑧ $\frac{d}{dx}(\sec^2x) = 2\sec^2x\cdot\tan x$

* Indeterminate form :- $\left\{ \frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0.\infty, 0^0, \infty^0, 1^\infty \right\}$

① $\left(\frac{0}{0}\right)$ = • Solve by using L'Hospital Rule.

② $\left(\frac{\infty}{\infty}\right)$ = • Solve by using L'Hospital Rule.

③ $(\infty - \infty)$ = • first of all take LCM
• check it will be converted by $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form.
• Now apply L-H Rule.

④ $(0.\infty)$ = • take inverse of any one function.
$\{f(x) \cdot g(x)\}$

$$f(x) \cdot g(x) = \frac{g(x)}{1/f(x)} = \frac{g(x)}{f^{-1}(x)}$$

Or

$$f(x) \cdot g(x) = \frac{f(x)}{1/g(x)} = \frac{f(x)}{g^{-1}(x)}$$

• Check it will be converted to $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form
• Now apply L-H Rule.
• {log / ln} function cannot be inverse.

⑤ $(0^0, \infty^0, 1^\infty)$ = • first of all assume $y = \lim_{x \to a} \{f(x)\}^{g(x)}$

• Now take log to the both side of Equation.

$$\log y = \lim_{x \to a} \log \{f(x)\}^{g(x)}$$

$$\log y = \lim_{x \to a} g(x) \cdot \log \{f(x)\}$$

• Check it will be converted to $\left(\frac{0}{0}\right)$, $\left(\frac{\infty}{\infty}\right)$ or $(0.\infty)$
• apply respective step to respective function/form.

* Important terms :-

$$\left[\lim_{x\to 0} \frac{\sin(X)}{X} = 1\right]$$

X = any no/function

① $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ or $\lim\limits_{x\to 0} \frac{x}{\sin x} = 1$

② $\lim\limits_{x\to 0} \frac{\tan x}{x} = 1$ or $\lim\limits_{x\to 0} \frac{x}{\tan x} = 1$

③ $\lim\limits_{x\to \infty} \left(1+\frac{1}{x}\right)^x = e$

④ $\lim\limits_{x\to 0} \frac{\sin^{-1}x}{x} = 1$

⑤ $\lim\limits_{x\to 0} \frac{\tan^{-1}x}{x} = 1$

{Practice Question}

(1) $\lim_{x \to 0} \dfrac{\sinh x - x}{\sin x - x\cos x}$ Ans $= \frac{1}{2}$

(2) $\lim_{x \to 0} \dfrac{e^{ax} - e^{-ax}}{\log(1+bx)}$ Ans $= \frac{2a}{b}$

(3) $\lim_{x \to 0} \dfrac{1+\sin x - \cos x + \log(1-x)}{x \tan^2 x}$ Ans $= -\frac{1}{2}$

(4) $\lim_{x \to 0} \dfrac{1+\sin x - \cos x + \log(1-x)}{x \tan x (e^x - 1)}$ Ans $= \frac{-1}{2}$

(5) $\lim_{x \to a} \dfrac{\log(x-a)}{\log(e^x - e^a)}$ Ans $= 1$

(6) $\lim_{x \to 0} \left[\dfrac{1}{2x} - \dfrac{1}{x(e^{\pi x}+1)}\right]$ Ans $= \frac{\pi}{4}$

(7) $\lim_{x \to \infty} x\left[a^{\frac{1}{x}} - 1\right]$ Ans $= \log a$

(8) $\lim_{x \to 2} \left[\dfrac{1}{x-2} - \dfrac{1}{\log(x-1)}\right]$ Ans $= -\frac{1}{2}$

(9) $\lim_{x \to 0} (\operatorname{cosec} x)^{\tan^2 x}$ Ans $= \sqrt{e}$

(10) $\lim_{x \to 0} \left[\dfrac{1^x + 2^x + 3^x + 4^x}{4}\right]^{1/x}$ Ans $= (24)^{1/4}$

(11) $\lim_{x\to 0} (e^{3x} - 5x)^{\frac{1}{x}}$ Ans $= e^{-2}$

(12) $\lim_{x\to 0} \frac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x - x}$ Ans $= -15$

Solve this question → ① $\left(\frac{0}{0}\right)$ form
→ ② Expansion form.

Q→ $\lim_{x\to 0} \dfrac{\log_{\sec x} \cos\frac{x}{2}}{\log_{\sec\frac{x}{2}} \cos x}$

→ In logrethmic function; $\log_a b = \frac{\log b}{\log a}$ {change of base}

$\Rightarrow \lim_{x\to 0} \dfrac{\dfrac{\log\cos\frac{x}{2}}{\log\sec x}}{\dfrac{\log\cos x}{\log\sec\frac{x}{2}}} = \lim_{x\to 0} \dfrac{\log\cos\frac{x}{2} \cdot \log\sec\frac{x}{2}}{\log\sec x \cdot \log\cos x}$

$\Rightarrow \lim_{x\to 0} \dfrac{\log\cos\frac{x}{2} \cdot \log\left[\frac{1}{\cos\frac{x}{2}}\right]}{\log\cos x \cdot \log\left[\frac{1}{\cos x}\right]}$

$\Rightarrow \lim_{x\to 0} \dfrac{\log\cos\frac{x}{2} \cdot \log\left[\cos\frac{x}{2}\right]^{-1}}{\log\cos x \cdot \log\left[\cos x\right]^{-1}}$

$\Rightarrow \lim_{x\to 0} \dfrac{\log\cos\frac{x}{2} \cdot \left[-\log\left[\cos\frac{x}{2}\right]\right]}{\log\cos x \cdot \left[-\log\left[\cos x\right]\right]} = \lim_{x\to 0} \dfrac{-\left\{\log\cos\frac{x}{2}\right\}^2}{-\left\{\log\cos x\right\}^2}$

$\Rightarrow \lim_{x\to 0} \left[\dfrac{\log\cos\frac{x}{2}}{\log\cos x}\right]^2$

Let us assume $\dfrac{\log \cos\frac{x}{2}}{\log \cos x} = t$ --- (1)

$$\Rightarrow \lim_{x\to 0} \frac{\log \cos\frac{x}{2}}{\log \cos x} \quad \left[\frac{0}{0}\right]$$

$$\Rightarrow \lim_{x\to 0} \frac{\left[\frac{1}{\cos\frac{x}{2}}\right]\left[-\frac{\sin x}{2}\right]\left[\frac{1}{2}\right]}{\left[\frac{1}{\cos x}\right]\left[-\sin x\right]}$$

$$\Rightarrow \lim_{x\to 0} \frac{-\tan\frac{x}{2} \times \frac{1}{2}}{-\tan x}$$

$$\Rightarrow \lim_{x\to 0} \frac{\tan\left(\frac{x}{2}\right)}{2\tan x} \quad \left[\frac{0}{0}\right]$$

$$\Rightarrow \lim_{x\to 0} \frac{\sec^2\left(\frac{x}{2}\right)\left\{\frac{1}{2}\right\}}{2\cdot \sec^2(x)}$$

$$\Rightarrow \lim_{x\to 0} \frac{1}{4} \cdot \frac{\sec^2\left(\frac{x}{2}\right)}{\sec^2(x)}$$

putting x value $\Rightarrow \frac{1}{4}\left(\frac{1}{1}\right) = \boxed{\frac{1}{4}}$

$$\Rightarrow \lim_{x\to 0} [t]^2 = \lim_{x\to 0}\left[\frac{1}{4}\right]^2 = \boxed{\frac{1}{16}} \quad \text{Ans}$$

$\left\{ \left[\log \cos\frac{x}{2}\right]^2 \neq 2\log \cos\frac{x}{2} \right\}$

Hence, that way we have to assume $\boxed{\dfrac{\log \cos\frac{x}{2}}{\log \cos x}} = t$ because directly we cannot solve the limit.