Lectures 18-19

In this chapter, we introduce expected value or the mean of a random variable. First we define expectation for discrete random variables and then for general random variable.

First we give a useful representation of discrete random variables.

Theorem 0.1 *Let X be a discrete random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Then there exists a partition $\{A_n \mid n = 1, , \ldots, N\} \subseteq \mathcal{F}$ of Ω and $\{a_n \mid n = 1, 2, \ldots, N\} \subseteq \mathbb{R}$ with a_n's distinct, so that*

$$X = \sum_{n=1}^{N} a_n I_{A_n} \text{ a.s.},$$

where N may be ∞.

Proof. Let F be the distribution function of X. Let $\{a_2, \ldots, a_N\}$ be the set of all discontinuities of F. Here N may be ∞. Since X is discrete, we have

$$\sum_{n=2}^{N} F(a_n) - F(a_n-) = 1 .$$

Set

$$A_n = \{X = a_n\} .$$

Then $\{A_n\}$ is pairwise disjoint. Let $\Omega' = \cup_{n=1}^{N} A_n$, a_1 be such that $a_1 \neq a_n$ for all $n \geq 2$ and $A_1 = \Omega \setminus \Omega'$. Then $\{A_n : n \geq 1\}$, is a partition of Ω, $P(A_1) = 0$ and

$$\left\{X \neq \sum_{n=1}^{N} a_n I_{A_n}\right\} \subseteq A_1.$$

Hence $P\Big(X \neq \sum_{n=1}^{N} a_n I_{A_n}\Big) = 0$, i.e.

$$X = \sum_{n=1}^{N} a_n I_{A_n} \text{ a.s.},$$

Remark 0.1 *(i) If X is a discrete random variable on a probability space $(\Omega, \mathcal{F}, P)$, then the 'effective' range of X is at the most countable. Here 'effective' range means those values taken by X which has positive probability. This leads to the name 'discrete' random variable.*

(ii) If two random vraibles are such that $X = Y$ a.s., then $F_X(x) = F_Y(x)$ for all x (exercise).

Remark 0.2 *If X is a discrete random variable given by $(A_n, a_n)|n = 1, 2, \cdots, N\}$ then one can assume without the loss of generality that*

$$X = \sum_{n=1}^{\infty} a_n I_{A_n} .$$

Since if $N < \infty$, then set $A_n = \emptyset$ for $n \geq N+1$ and $a_n, n \geq N+1$ are chosen so that they are distinct.

Theorem 0.2 *Let $\{A_n\}$ and $\{B_n\}$ be countable partitions of Ω from $\mathcal{F}$ and $\{a_n\}$, $\{b_n\}$ be a sequence of real numbers such that a_n's are distinct and b_n's are not necessarily distinct. Then if*

$$\sum_{n=1}^{\infty} a_n I_{A_n} = \sum_{n=1}^{\infty} b_n I_{B_n}$$

then

$$\sum_{n=1}^{\infty} a_n P(A_n) = \sum_{n=1}^{\infty} b_n P(B_n)$$

provided l.h.s. series converges absolutely.

Proof. Using the hypothesis given it follows that

- Each B_m intersect atmost one A_n. To see this, firstly if B_m intersect say A_n and A_k, then choose $\omega_1 \in B_m \cap A_n$ and $\omega_2 \in B_m \cap A_k$.

  $$\sum_{n=1}^{\infty} a_n I_{A_n}(\omega_1) = a_n = b_m = \sum_{n=1}^{\infty} b_n I_{B_n}(\omega_1)$$

  and

  $$a_k = \sum_{n=1}^{\infty} a_n I_{A_n}(\omega_2) = \sum_{n=1}^{\infty} b_n I_{B_n}(\omega_2) = b_m.$$

  i.e., $b_m = a_n = a_k$, a contradiction to the fact that $a_n \neq a_k$ for $n \neq k$.

- Each non empty B_m intersect atleast one A_n. If not, there exists a $B_k \neq \emptyset$ such that $B_k \cap A_n = \emptyset$ for all n. This implies $B_k \cap \Omega = \emptyset$. This contradicts $B_k \neq \emptyset$.

- $B_m \cap A_n \neq \emptyset$, then $B_m \subseteq A_n$. To see this, if $A_n \cap B_m \neq \emptyset$, then $B_m \cap A_k = \emptyset$ for all $k \neq n$. Hence $B_m \cap \cup_{k \neq k} A_k = \emptyset$, i.e. $B_m \subseteq A_n$.

  $$\sum_{n=1}^{\infty} a_n I_{A_n} = \sum_{n=1}^{\infty} b_n I_{B_n}$$

Thus we conclude that for each $n \geq 1$, either $B_m \subseteq A_n$ or $B_m \cap A_n = \emptyset$ for all $m \geq 1$. Hence, for each $n \geq 1$, set

$$I_n = \{m \geq 1 | A_n \cap B_m \neq \emptyset\}.$$

Then clearly

$$A_n = \cup_{m \in I_n} B_m, \ n \geq 1.$$

Also if $m \in I_n$ then $a_n = b_m$. Therefore (using absolute summability of the series)

$$\begin{aligned}\sum_{m=1}^{\infty} b_m P(B_m) &= \sum_{n=1}^{\infty} \sum_{m \in I_n} b_m P(B_m) \\ &= \sum_{n=1}^{\infty} \sum_{m \in I_n} a_n P(B_m) \\ &= \sum_{n=1}^{\infty} a_n P(A_n).\end{aligned}$$

This completes the proof.

Definition 0.1 *Let X be a discrete random variable represented by $\{(A_n, a_n) \,|\, n \geq 1\}$. Then expectation of X denoted by EX is defined as*

$$EX = \sum_{n=1}^{\infty} a_n P(A_n),$$

provided the right hand side series converges absolutely.

Remark 0.3 *In view of Remark 0.1., if X has range $a_1, a_2, \ldots, a_N$, then*

$$EX = \sum_{n=1}^{N} a_n P\{X = a_n\}.$$

Example 0.1 *Let X be a Bernoulli(p) random variable. Then*

$$X = I_A, \text{ where } A = \{X = 1\}.$$

Hence

$$EX = P(A) = p.$$

Example 0.2 *Let X be a Binomial(n, p) random variable. Then*

$$X = \sum_{k=0}^{n} k\, I_{A_k}, \text{ where } A_k = \{X = k\}.$$

Hence

$$\begin{aligned} EX &= \sum_{k=0}^{n} kP(A_k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k (1-p)^{n-k} = np. \end{aligned}$$

Here we used the identity

$$k \binom{n}{k} = n \binom{n-1}{k-1}.$$

Example 0.3 *Let X be a Poisson (λ) random variable. Then*

$$X = \sum_{n=0}^{\infty} n\, I_{A_n}, \text{ where } A_n = \{X = n\}.$$

Hence

$$EX = \sum_{n=0}^{\infty} n\, e^{-\lambda} \frac{\lambda^n}{n!} = \lambda.$$

Example 0.4 *Let X be a Geometric (p) random variable. Then*

$$X = \sum_{n=1}^{\infty} n\, I_{A_n}, \text{ where } A_n = \{X = n\}$$

Hence

$$EX = \sum_{n=1}^{\infty} n\, p\, (1-p)^{n-1} = \frac{p}{(1-(1-p))^2} = \frac{1}{p}.$$

Theorem 0.3 *(Properties of expectation) Let X and Y be discrete random variables with finite means. Then*

(i) If $X \geq 0$, then $EX \geq 0$.

(ii) For $a \in \mathbb{R}$

$$E(aX + Y) = aEX + EY.$$

Proof. (i) Let $\{(A_n,\ a_n)|n \geq 1\}$ be a representation of X. Then $X \geq 0$ implies $a_n \geq 0$ for all $n \geq 1$. Hence

$$EX \ = \ \sum_{n=1}^{\infty} a_n P(A_n) \ \geq \ 0 \, .$$

(ii) Let Y has a representation $\{(B_n, b_n) \,|\, n \geq 1\}$. Now set

$$C_{nm} \ = \ A_n \cap B_m,\ n, m \geq 1, a_{nm} \ = \ a_n, m \geq 1, b_{nm} \ = \ b_m, n \geq 1$$

Then $\{C_{nm} : n, m \geq 1\}$ is a partion of Ω and

$$X \ = \ \sum_{n,m=1}^{\infty} a_{nm} I_{C_{nm}},\ Y \ = \ \sum_{n,m=1}^{\infty} b_{nm} I_{C_{nm}},$$

i.e., X, Y are represented using a common partition of Ω. Therefore

$$aX + Y \ = \ \sum_{n,m=1}^{\infty} (a a_{nm} \ + \ b_{nm})\, I_{C_{nm}} \, .$$

Hence

$$\begin{aligned}
a\,EX \ + \ EY \ &= \ a \sum_{n=1}^{\infty} a_n \, P(A_n) \ + \ \sum_{m=1}^{\infty} b_m \, P(B_m) \\
&= \ a \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \, P(A_n \cap B_m) \ + \ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_m \, P(A_n \cap B_m) \\
&= \ a \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \, P(C_{nm}) \ + \ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{nm} \, P(C_{nm}) \\
&= \ \sum_{n,m=1}^{\infty} (a a_{nm} \ + \ b_{nm}) \, P(C_{nm}) \\
&= \ E(aX + Y) \, .
\end{aligned}$$

Definition 0.2 *(Simple random variable) A random variable is said to be simple if it is discrete and the distribution function has only finitely many discontinuities.*

Theorem 0.4 *Let X be random variable in $(\Omega, \mathcal{F}, P)$ such that $X \geq 0$, then there exists a sequence of simple random variables $\{X_n\}$satisfying*

(i) For each $n \geq 1,\ X_n \geq 0,\ X_n \leq X_{n+1} \leq X$.

(ii) For each $\omega \in \Omega,\ X_n(\omega) \to X(\omega)$ as $n \to \infty$.

Proof. For $n \geq 1$, define simple random variable X_n as follows:

$$X_n(\omega) \;=\; \begin{cases} \dfrac{k}{2^n} & \text{if} \quad \dfrac{k}{2^n} \leq X(\omega) < \dfrac{k+1}{2^n} \;, k = 0, \cdots, n2^n - 1 \\ 0 & \text{if} \quad X(\omega) \geq n \,. \end{cases}$$

Then X_n's satisfies the following:

- if $X_n(\omega) = \frac{k}{2^n}$ (i.e. $X(\omega) \in [\frac{k}{2^n}, \ \frac{k+1}{2^n})$), then $X(\omega) \in [\frac{2k}{2^{2n}}, \ \frac{2k+1}{2^{2n}}) \cup [\frac{2k+1}{2^{2n}}, \ \frac{2k+2}{2^{2n}})$. Hence $X_{n+1}(\omega) = \frac{k}{2^n}$ or $= \frac{2k+1}{2^{2n}}$.
- For $X(\omega) < n$,
$$|X_n(\omega) - X(\omega)| \leq \frac{1}{2^n}.$$

The above implies that

$$X_n \;\leq X_{n+1}, \; n \geq 1$$
$$\lim_{n\to\infty} X_n(\omega) \;=\; X(\omega), \; \omega \in \Omega \,.$$

Lemma 0.1 *Let X be a non negative random variable and $\{X_n\}$ be a sequence of simple random variables satisfying (i) and (ii) of Theorem 0.4. Then $\lim_{n\to\infty} EX_n$ exists and is given by*

$$\lim_{n\to\infty} EX_n \;=\; \sup\{EY \,|\, Y \text{ is simple and } Y \leq X\} \,.$$

Proof. (Reading exercise) Since $X_n \leq X_{n+1}$, we have $EX_n \leq EX_{n+1}, n \geq 1$ (see exercise). Hence $\lim_{n\to\infty} EX_n$ exists. Also since X_n's are simple, clearly,

$$EX_n \;\leq\; \sup\{EY \,|\, Y \text{ is simple and } Y \leq X\}, \; n \geq 1.$$

Therefore

$$\lim_{n\to\infty} EX_n \;\leq\; \sup\{EY \,|\, Y \text{ is simple and } Y \leq X\} \,.$$

Hence to complete the proof, it suffices to show that for Y simple and $Y \leq X$,

$$EY \;\leq\; \lim_{n\to\infty} EX_n \,.$$

Let

$$Y \;=\; \sum_{k=1}^{m} a_k \, I_{A_k} \,,$$

where $\{A_k \,|\, k = 1, \ldots, m\}$ is a partition of Ω. Fix $\epsilon > 0$, set for $k \geq 1$ and $n \geq 1$,

$$A_{kn} \;=\; \{\omega \in A_k \,|\, X_n(\omega) \;\geq\; a_k \,-\, \epsilon\}\,.$$

Then

$$X_n \geq \sum_{k=1}^{m} (a_k - \varepsilon)\, I_{A_{kn}}.$$

Also, since $X_n \nearrow X$, it follows that [1]

$$A_{kn} \nearrow A_k,\ k = 1, 2, \cdots, m.$$

Hence

$$EX_n \;\geq\; \sum_{k=1}^{m} (a_k - \epsilon) P(A_{kn})\,. \tag{0.1}$$

Using continuity property of probability, we have

$$\lim_{n\to\infty} P(A_{kn}) \;=\; P(A_k),\ 1 \leq k \leq m\,.$$

Now let, $n \to \infty$ in (0.1), we get

$$\lim_{n\to\infty} EX_n \geq \sum_{k=1}^{m} (a_k - \epsilon) P(A_k) \;=\; EY - \epsilon\,.$$

Since, $\epsilon > 0$ is arbitrary, we get

$$\lim_{n\to\infty} EX_n \geq\; EY\,.$$

This completes the proof.

Definition 0.3 *The expectation of a non negative random variable X is defined as*

$$EX \;=\; \sup\{EY \,|\, Y \text{ is simple and } Y \leq X\}\,. \tag{0.2}$$

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[1]Since $X_n \leq X_{n+1},\ n \geq 1$, we have for each $k \geq 1,\ A_{k\,n} \subseteq A_{k\,n+1}, n \geq 1\,.$

$$\begin{aligned} \omega \in A_k \quad &\Longrightarrow \quad X(\omega) \geq a_k \quad \Longrightarrow \quad \lim_{n\to\infty} X_n(\omega) \geq a_k \\ &\Longrightarrow \quad X_{n_0}(\omega) \geq a_k - \epsilon \text{ for some } n_0 \\ &\Longrightarrow \quad \omega \in A_{kn_0} \subseteq \cup_{n=1}^{\infty} A_{kn}\,. \end{aligned}$$

Therefore $\cup_{n=1}^{\infty} A_{kn} \;=\; A_k,\ 1 \leq k \leq m\,.$

Remark 0.4 *From Lemma 0.1, we have*

$$EX = \lim_{n\to\infty} EX_n \,,$$

where X_n is a sequence of simple random variables given by

$$X_n(\omega) = \begin{cases} \dfrac{k}{2^n} & \text{if } \dfrac{k}{2^n} \le X(\omega) < \dfrac{k+1}{2^n} \,, k = 0, \cdots, n2^n - 1 \\ 0 & \text{if } X(\omega) \ge n \,. \end{cases}$$

Theorem 0.5 *Let X be a continuous non negative random variable with pdf f. Then*

$$EX = \int_0^\infty x f(x)\, dx$$

provided the rhs integral exists.

Proof. By using the simple random variables X_n given in Remark 0.1 above, we get

$$\begin{aligned} EX &= \lim_{n\to\infty} EX_n \\ &= \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \frac{k}{2^n} P\Big(\frac{k}{2^n} \le X(\omega) < \frac{k+1}{2^n}\Big) \\ &= \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \Big[F((k+1)/2^n) - F(k/2^n)\Big] \\ &= \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} f(t)dt \\ &= \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} tf(t)dt + \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (k/2^n - t)f(t)dt \\ &= \lim_{n\to\infty} \int_0^n tf(t)dt + \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (k/2^n - t)f(t)dt. \end{aligned} \tag{0.3}$$

Consider

$$\begin{aligned} \Big| \sum_{k=0}^{n2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (k/2^n - t)f(t)dt \Big| &\le \frac{1}{2^n} \sum_{k=0}^{n2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} f(t)dt \\ &= \frac{1}{2^n} \int_0^n f(t)dt \le \frac{1}{2^n}. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (k/2^n - t) f(t) dt \;=\; 0.$$

Hence by substituting the above in (0.3), we get

$$EX \;=\; \int_0^\infty x f(x)\, dx\,.$$

Definition 0.4 *Let X be a random variable on $(\Omega, \mathcal{F}, P)$. The mean or expectation of X is said to exists if either EX^+ or EX^- is finite. In this case EX is defined as*

$$EX \;=\; EX^+ - EX^-\,,$$

where

$$X^+ \;=\; \max\{X, 0\},\;\; X^- \;=\; \max\{-X, 0\}\,.$$

Note that X^+ is the positive part and X^- is the negative part of X.

Theorem 0.6 *(Reading exercise) Let X be a continuous random variable with finite mean and pdf f. Then*

$$EX \;=\; \int_{-\infty}^\infty x f(x)\, dx\,.$$

Proof. Set

$$Y_n(\omega) \;=\; \begin{cases} \dfrac{k}{2^n} & \text{if} \;\; \dfrac{k}{2^n} \le X^+(\omega) < \dfrac{k+1}{2^n}\;, k = 0, \cdots, n2^n - 1 \\ 0 & \text{if} \;\; X^+(\omega) \ge n\,. \end{cases}$$

Hence

$$Y_n(\omega) \;=\; \begin{cases} \dfrac{k}{2^n} & \text{if} \;\; \dfrac{k}{2^n} \le X(\omega) < \dfrac{k+1}{2^n}\;, k = 0, \cdots, n2^n - 1 \\ 0 & \text{if} \;\; X(\omega) \ge n \text{ or } X(\omega) \le 0\,. \end{cases}$$

Then Y_n is a sequence of simple random variables such that

$$EX^+ \;=\; \lim_{n\to\infty} EY_n\,.$$

Similarly, set

$$Z_n(\omega) \;=\; \begin{cases} \dfrac{k}{2^n} & \text{if} \;\; \dfrac{k}{2^n} \le X^-(\omega) < \dfrac{k+1}{2^n}\;, k = 0, \cdots, n2^n - 1 \\ 0 & \text{if} \;\; X^-(\omega) \ge n\,. \end{cases}$$

Hence

$$Z_n(\omega) \;=\; \begin{cases} \dfrac{k}{2^n} & \text{if} \quad \dfrac{k}{2^n} \le -X(\omega) < \dfrac{k+1}{2^n}\ , k = 0, \cdots, n2^n - 1 \\ 0 & \text{if} \quad X(\omega) \le -n \text{ or } X(\omega) \ge 0\,. \end{cases}$$

Then

$$EX^- \;=\; \lim_{n\to\infty} EZ_n\,.$$

Now

$$\lim_{n\to\infty} EY_n \;=\; \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \frac{k}{2^n} P(\{\frac{k}{2^n} \le X(\omega) < \frac{k+1}{2^n}\}) \tag{0.4}$$

and

$$\begin{aligned} -\lim\nolimits_{n\to\infty} EZ_n \;&=\; -\lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \frac{k}{2^n} P\Big(\frac{k}{2^n} \le -X(\omega) < \frac{k+1}{2^n}\Big) \\ &=\; \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \Big(-\frac{k}{2^n}\Big) P\Big(\frac{-k-1}{2^n} < X(\omega) \le \frac{-k}{2^n}\Big) \\ &=\; \lim_{n\to\infty} \sum_{k=0}^{-n2^n+1} \frac{k}{2^n} P\Big(\frac{k-1}{2^n} < X(\omega) \le \frac{k}{2^n}\Big) \\ &=\; \sum_{k=0}^{-n2^n+1} \frac{k-1}{2^n} P\Big(\frac{k-1}{2^n} < X(\omega) \le \frac{k}{2^n}\Big)\,. \end{aligned} \tag{0.5}$$

The last equality follows by the arguments from the proof of 0.5. Combining (0.4) and (0.4), we get

$$EX \;=\; \lim_{n\to\infty} \sum_{k=-n2^n}^{n2^n-1} \frac{k}{2^n} P\Big(\frac{k}{2^n} \le X(\omega) < \frac{k+1}{2^n}\Big)\,.$$

Now as in the proof of Theorem 0.5, we complete the proof.

We state the following useful properties of expectation.

Theorem 0.7 *Let X, Y be random variables with finite mean. Then*

(i) If $X \ge 0$, then $EX \ge 0$.

(ii) For $a \in \mathbb{R}$,

$$E(aX + Y) \;=\; aEX \,+\, EY\,.$$

(iii) Let $Z \ge 0$ be a random variable such that $Z \le X$. Then Z has finite mean and $EZ \,\le\, EX$.

The proof of the above theorem is as follows.

(1) We have already seen the proofs of (i) and (ii) when X, Y are simple random variables. Proof of (iii) for simple random variables X and Z is an easy exercise.

(2) approximate non negative random variables using simple random variables and then using a generalization of Lemma 0.1 namely monotone convergence theorem.

(3) For general random variable one use the decomposition $X = X^+ - X^-, Y = Y^+ - Y^-, Z = Z^+ - Z^-$.

Many results in probability theory uses the above technique, i.e. to prove a statement (involving expectation), first prove it for a class of random variables (which generally comes out to be easy) and then use convergence theorem involving expection, i.e.,

$$\lim_{n\to\infty} EX_n \;=\; E[\lim_n X_n].$$

For example, we already know that the above is true when $X \geq 0$, X_n's are simple and non negative and $X_n(\omega) \to X(\omega)$ for all $\omega \in \Omega$. In the context of Riemann integration, one can recall the following convergence theorem.

" If $g_n, n \geq 1$ is a sequence of continuous functions defined on the $[a, b]$ such that $g_n \to g$ uniformly in $[a,\ b]$, then

$$\lim_{n\to\infty} \int_a^b g_n(x)\, dx \;=\; \int_a^b g(x) dx \,."$$

i.e., to take limit inside the integral, one need uniform convergence of functions. In many situations, it is highly unlikely to get uniform convergence.

In fact uniform convergence is not required to take limit inside an integral. This is illustrated in the following theorems. The proof of the following theorem is same as the proof of Lemma 0.1. We leave it as an exercise for interested students.

Theorem 0.8 *(Monotone convergence theorem) Let X_n be a nondecreasing sequence of nonnegative random variables such that* $\lim_{n\to\infty} X_n \;=\; X$. *Then*

$$\lim_{n\to\infty} EX_n \;=\; EX\,.$$

[Here $\lim_{n\to\infty} X_n \;=\; X$ means $\lim_{n\to\infty} X_n(\omega) \;=\; X(\omega)$, $\omega \in \Omega$.]

Theorem 0.9 *(Fatou's lemma) Let $\{X_n\}$ be a sequence of non negative random variables. Then*

$$E\Big[\liminf_{n\to\infty} X_n\Big] \leq \liminf_{n\to\infty} EX_n.$$

Proof: Recall that

$$\liminf_n X_n \;=\; \lim_{n\to\infty} Y_n,\; Y_n = \inf\{X_n, X_{n+1}, \cdots\}.$$

Then $Y_n \nearrow X := \liminf_n X_n,\; Y_n \leq X_n$. Hence using monotone convergence theorem, we have

$$\begin{aligned} E\Big[\liminf_n X_n\Big] &= \lim_{n\to\infty} EY_n \\ &= \liminf_{n\to\infty} EY_n \\ &\leq \liminf_{n\to\infty} EX_n. \end{aligned}$$

□

Theorem 0.10 *(Dominated Convergence Theorem) Let X_n, X, Y be random variables such that*

(i) *Y has finite mean.*

(ii) $|X_n| \leq Y$.

(iii) $\lim_{n\to\infty} X_n \;=\; X$.

Then

$$\lim_{n\to\infty} EX_n \;=\; EX\,.$$

Proof: Given

$$-Y \leq X_n \leq Y, \text{i.e. } Y - X_n \geq 0,\;\; X_n + Y \geq 0.$$

Hence using Fatou's lemma twice (by noting that $\liminf_{n\to\infty} X_n = \lim_n X_n$)we have

$$E[Y - X] \leq EY + \liminf_{n\to\infty}(-EX_n) = EY - \limsup_{n\to\infty} EX_n$$

and

$$E[Y + X] \leq EY + \liminf_{n\to\infty} EX_n.$$

The above implies that

$$\limsup_{n\to\infty} EX_n \leq EX \leq \liminf_{n\to\infty} EX_n.$$

This completes the proof (why?) □

Example 0.5 *Let U be uniform $(0,\ 1)$ random variable (defined on $(\Omega, \mathcal{F}, P)$). Define*

$$X_n(\omega) \ = \ nI_{\{U(\omega) < \frac{1}{n}\}},\ n \geq 1.$$

Then clearly $X_n(\omega) \to 0$ as $n \to \infty$ for all $\omega \in \Omega$ and $EX_n = 1$. Hence

$$\lim_{n\to\infty} EX_n \neq EX = 0.$$

Here note that there exists no random variable Y with finite mean such that

$$|X_n| \leq Y.$$