INDIAN INSTITUTE OF TECHNOLOGY BOMBAY

Department of Mathematics

SI 427 (Probability Theory)

Tutorial Sheet-XI

1. Let $\{X_n | n \geq 1\}$ be a sequence of random variables and $a \in \mathbb{R}$. Then $X_n \to a$ in distribution iff $X_n \to a$ in probability.

2. Let $X_n \to X$ in probability. Show that $|X_n| \to |X|$ in probability.

3. Let X_n be a sequence of random variables and taking values in $[0,\ 1]$ and X is a random variable taking values in $[0,\ 1]$ such that $X_n \to X$ a.s.. Then show that
$$f(X_n) \to f(X) \text{ a.s}$$
for each continuous function $f : [0,\ 1] \to \mathbb{R}$.

4. Let $X_n \to X$ in probability and $X_n \to Y$ in probability. Show that $X = Y$ a.s.

5. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with mean zero, variance σ^2 and with finite third moment. Show that
$$\lim_{n\to\infty} E\Big(\frac{S_n}{\sigma\sqrt{n}}\Big)^3 \;=\; 0\,.$$

6. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with mean zero, variance σ^2 and with finite fourth moment. Show that
$$\lim_{n\to\infty} E\Big(\frac{S_n^4}{n^2}\Big) \;=\; 3\sigma^4\,.$$

7. Let $\{X_n | n \geq 1\}$ be a sequence of independent random variables such that
$$P\big(X_n = 3^n\big) = P\big(X_n = -3^n\big) = \frac{1}{2}\,.$$
Show that $\frac{X_1 + \cdots + X_n}{n}$ doesn't converge to 0 a.s.

8. Let $X_1, X_2, \ldots$ be independent and identically distributed Poisson (λ) random variables. Find
$$\lim_{n\to\infty} P\big(X_1 + \cdots + X_n \leq n\lambda + \sqrt{n\lambda}\big)\,.$$

9. (Normal approximation)Show that

$$\lim_{n\to\infty} \left| P\big(S_{2n} = n\big) - \frac{1}{\sqrt{n\pi}} \right| \;=\; 0$$

where $S_n \sim$ Binomial $(n, \frac{1}{2})$.

10. Using Central Limit Theorem prove that following.

$$\lim_{n\to\infty} e^{-n} \sum_{k=1}^{n} \frac{n^k}{k!} \;=\; \frac{1}{2}\,.$$