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SE $A = A^T$ (MATRICI QUADRATE) ALLORA A E' DIAGONALIZZABILE

$A = V \Lambda V^{-1}$ PER UNA CERTA V MATRICE INVERTIBILE $\in \mathbb{R}^{n \times n}$

DIAGONALIZZABILITA' DI A $\quad$ Λ MATRICE DIAGONALE

MATRICI SIMMETRICHE $(A = A^T)$

TEOREMA SPETTRALE $\rightsquigarrow$ LA DIAGONALIZZAZIONE E' DATA DA:

$$A = V \Lambda V^{-1}$$ V PUO' ESSERE SCELTA ORTOGONALE

SE SI SCEGLIE UNA V ORTOGONALE CHIAMIAMO X QUESTA PARTICOLARE V COME $A = X \Lambda X^T$

ESEMPIO

$$A \in \mathbb{R}^{3 \times 3}$$

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$A = A^T$. E' SIMMETRICA

$$\det(A - \lambda I) = \det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(2-\lambda)^2 - 1\right] =$$

$$= (2-\lambda) \cdot \left[\lambda^2 - 4\lambda + 3\right] = (2-\lambda)(\lambda - 1)(\lambda - 3)$$

$$\lambda = 1 \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \\ x_3 = 0 \end{cases} =$$

$$= \begin{cases} x_2 = -x_1 \\ x_3 = 0 \end{cases} = \underline{v}_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 2 \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{cases} x_2 = 0 \\ x_1 = 0 \\ 0 = 0 \end{cases} \sim \underline{v}_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 3 \quad \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{cases} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \\ -x_3 = 0 \end{cases} \sim \underline{v}_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Lambda = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}} \qquad V = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

NON ABBIAMO CRITERIO, POSSIAMO ORDINARLI COME VOGLIO

$\boxed{\text{ESSENDO } A \text{ SIMMETRICA NON DEVO FARE L'INVERSA}}$

ATTENZIONE PERÒ: O FACCIO L'INVERSA O IL METODO CHE VEDREMO DOPO

$$A = V \Lambda V^{-1} \quad \tilde{\Lambda} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \tilde{V} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \tilde{V} \tilde{\Lambda} \tilde{V}^{-1}$$

A SIMMETRICA $\Rightarrow \exists$ $A = X D X^T$ CON X ORTOGONALE

$$X = \left(\underline{x_1} \,\middle|\, \underline{x_2} \,\middle|\, \underline{x_3} \right)$$

$\underline{x_1} \perp \underline{x_2} \perp \underline{x_3}$ $\qquad \|x_1\|_2 = \|x_2\|_2 = \|x_3\|_2 = 1$

SE NON VOGLIAMO FARE L'INVERSA DOBBIAMO FARE:

$$V = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \left.\begin{array}{l} \|\underline{v_1}\| = \sqrt{2} \\ \|\underline{v_2}\| = 1 \\ \|\underline{v_3}\| = \sqrt{2} \end{array}\right\} \quad X = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$\underline{v_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \underline{v_2} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \qquad \begin{array}{l} \underline{v_1}^T \underline{v_2} = 0 \\ \underline{v_1}^T \underline{v_3} = 0 \\ \underline{v_2}^T \underline{v_3} = 0 \end{array} \qquad \boxed{X^{-1} = X^T}$$

$$\frac{\underline{v_1}}{\|v_1\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \qquad \left\| \frac{\underline{v_1}}{\|v_1\|} \right\|_2 = \left\| \frac{1}{\|v_2\|_2} \underline{v_2} \right\|_2 = \frac{\|\underline{v_1}\|_2}{\|\underline{v_1}\|_2} = 1$$

ALTRO ESEMPIO:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad A = A^T \text{ SIMMETRICA} \qquad \text{HA RANGO } 1$$

$$\det(A - \lambda I) = \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)^2 - 1\right] - \left[1-\lambda-1\right] + \left[1-(1-\lambda)\right] =$$

$$= (1-\lambda)\left[\lambda^2 - 2\lambda - 1\right] - \left[-\lambda\right] + \left[1-1-\lambda\right] =$$

$$= \lambda\left[(1-\lambda)(\lambda-2) + 2\right] = \lambda\left[\lambda - 2 - \lambda^2 + 2\lambda + 2\right] =$$

$$= \lambda^2(3 - 2\lambda)$$

$\lambda = 0$ MOLTEPLICITÀ 2 $(\lambda^2 = 0)$

$\lambda = 3$ // 1

$$\lambda = 3 \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$= \begin{cases} -2x_1 + x_2 + x_3 = 0 \\ x_1 - 2x_2 + x_3 = 0 \\ x_1 + x_2 - 2x_3 = 0 \end{cases}$$

$x_1 - 2x_2 + x_3 = x_1 + x_2 - 2x_3$

$3x_3 = 3x_2$

$x_3 = x_3$

$$-2x_1 + x_2 + \underset{x_2}{\overset{x_3}{\sim}} = -2x_1 = -2x_1 = \boxed{x_1 = x_2}$$

$$V_3 = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;\middle|\; t \neq 0 \right\} \qquad A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 0$ MOLTEPLICITÀ DI 2

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A} - 0 \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A} \rightsquigarrow \begin{cases} x_1 + x_2 + x_3 = 0 \\ \ldots = 0 \\ \ldots = 0 \end{cases}$$

$$A \to \ker(A) = \{ x \in \mathbb{R}^n : A \underline{x} = \underline{0} \}$$

$$\boxed{A \underline{x} = \lambda \underline{x}} \qquad (\lambda, \underline{x}) \qquad \lambda = 0 \qquad \boxed{A \underline{x} = 0 \quad \underline{x} = \underline{0}}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad V_{(\lambda = 0)} = \left\{ \underline{0} \neq \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 + x_2 + x_3 \overset{b}{=} 0 \right\}$$

$$\underbrace{x_1, \; x_2}_{\text{PARAMETRI LIBERI}} \qquad \underbrace{x_3 \overset{\text{VINCOLO LINEARE}}{=} -x_1 - x_2}_{\text{PARAMETRO VINCOLATO}}$$

SCELGO x_1, x_2

$x_1 = 1, x_2 = 0 \Rightarrow x_3 = -1$

$x_1 = 0, \; x_2 = 1 \Rightarrow x_3 = -1$

$$\boxed{V_{(\lambda=3)} = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} : t \neq 0 \right\} \quad \underline{v_1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \qquad \underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \underline{v_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\underline{V}_{(\lambda=0)} = \left\{ t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \;\middle|\; (t, s) \neq (0,0) \right\} \quad t, s \in \mathbb{R}$$

$$A \underline{v}_2 = \underline{0} \qquad A \underline{v_3} = \underline{0} \qquad A(t \underline{v}_3 + s \underline{v_3}) = \underline{0}$$

$\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\underline{v_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\lambda = 3$ $\quad \lambda = 0 \quad \lambda = 0$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = V \Lambda V^{-1} \quad \text{DOVE} \quad \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

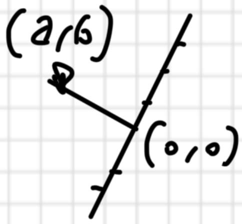
$$V = \left(v_2 \middle| v_3 \middle| \underline{v_1} \right) \Longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

NON $\perp$ $\qquad = \underline{v_3}$

ORTOGONALE

$\left.\begin{array}{l} \underline{v_3} \perp \underline{v_2} \\ \underline{v_3} \perp \underline{v_1} \end{array}\right\}$ CALCOLIAMO V^{-1}

$$\left.\begin{array}{l} \boxed{x_1 + x_2 + x_3 = 0} \\ (1, 0, -1) \end{array}\right\}$$



$\boxed{ax + by = 0}$

$(x,y) \in \mathbb{R}^2$

$\perp (a,b)$

$x_1 - x_2 = 0 \qquad \boxed{x_1 = x_2}$

$$(1, 0, -1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{cases} x_1 - x_3 = 0 \ \perp v_1 \\ x_1 + x_2 + x_3 = 0 \end{cases}$$

$2x_1 + x_2 = 0$

$x_2 = -2x_1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$V = \begin{array}{ccc} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{pmatrix} \\ \underline{v_1} \quad \underline{v_2} \quad \underline{v_3} \end{array}$$

$\underline{v_3} \perp \underline{v_2}$

$\underline{v_3} \perp \underline{v_1}$

$\underline{v_1} \perp \underline{v_2}$

$$\begin{cases} \underline{x_1} = \dfrac{v_1}{\|v_1\|_2} \\ x_2 = \dfrac{v_2}{\|v_2\|_2} \\ \underline{x_3} = \dfrac{v_3}{\|v_3\|_2} \end{cases}$$

$$X = \begin{array}{c} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\ x_1 \quad x_2 \quad x_3 \end{array}$$