2/12/24

AUTOVALORI E AUTOVETTORI

$$\mathbb{R}^{n\times m} \ni A = \begin{bmatrix} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \cdots \end{bmatrix}$$

$$A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

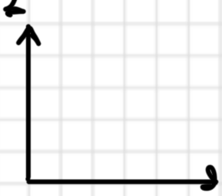
$$\underline{v} = \sum_{i=1}^{m} v_i \underline{e}_i \longrightarrow \underline{w} = \sum_{i=1}^{m} v_i (\underline{a}_i) \in \mathbb{R}^n$$

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \quad \underline{e}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$A \underline{e}_i = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow i\text{-esima} \quad \begin{array}{c} 0 \cdots 0\ 1\ 0 \cdots 0 \\ \begin{bmatrix} & \boxed{A} & \end{bmatrix} \end{array} = \underline{a}_i$$

NOME COLONNA i-ESIMA DI A

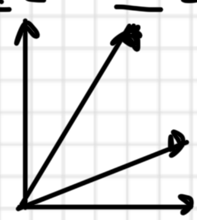
$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{e}_2$



$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\xrightarrow{A}$

$\underline{e}_2$ $\quad \underline{a}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



$\underline{a}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{e}_1$

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\begin{pmatrix} e_1 \mid & \mid e_m \end{pmatrix} = I_m$

BASE DI $\mathbb{R}^m$

$$\mathbb{R}^m \rightarrow \overbrace{\begin{pmatrix} \underline{e}_1 \end{pmatrix} \dots \begin{pmatrix} \underline{e}_m \end{pmatrix}}^{\text{IDENTITA'}} \text{ su } \mathbb{R}^m$$

$$4\begin{array}{c} 10 \\ \begin{pmatrix} a_1 & \Big| & & \Big| & a_{10} \end{pmatrix} \end{array} \begin{pmatrix} v_1 \\ \vdots \\ v_{10} \end{pmatrix} = v_1 \begin{pmatrix} \vdots \\ a_1 \\ \vdots \end{pmatrix} + \dots + v_{10} \begin{pmatrix} \underline{a}_{10} \end{pmatrix} \in \mathbb{R}^4$$

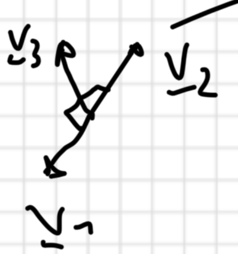
$a_i \in \mathbb{R}^4$

$$\underline{v} = \sum_{i=1}^{10^m} v_i \underline{e}_i$$

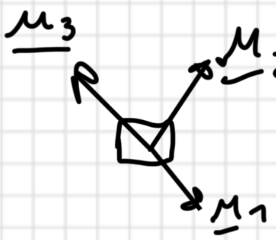
$$A = I_n A I_m = \begin{array}{c} I_n \\ \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} \end{array} \qquad 4\begin{array}{c} 10 \\ \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \end{array} \begin{pmatrix} A & & \\ & I_m & \\ & & \ddots \\ & & & 1 \end{pmatrix}$$

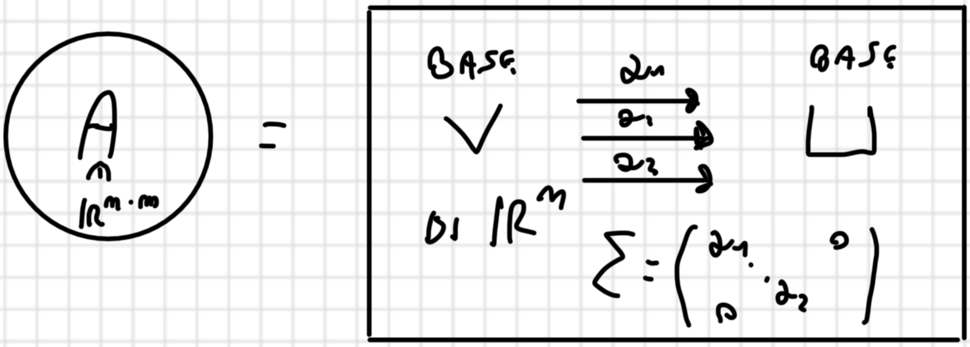
$$= U \Sigma V^T \qquad \Sigma = \left(\begin{array}{ccc|c} a_1 & & 0 & \\ & a_2 \ddots & & 0 \\ 0 & & a_m & \end{array}\right)$$

$$V \xrightarrow{\Sigma} U$$



$$\begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ell_2 & 0 \\ 0 & 0 & \ell_2 \end{pmatrix}$$



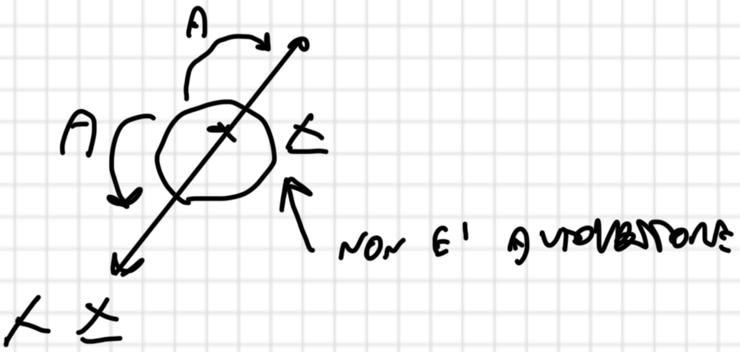


DECOMPOSIZIONE AI VALORI (Σ)
SINGOLARI

DEFINIZIONE

$A \in \mathbb{R}^{n\times m}$ SE $A\underline{x} = \lambda \underline{x}$ $\quad \lambda \in \mathbb{R}$ $\quad \underline{x} \in \mathbb{R}^n$

λ SI DICE AUTOVALORE E $\underline{x}$ AUTOVETTORE



$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \underline{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \underline{x}_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$A\underline{x}_2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= 1\, \underline{x}_2$

$\underline{x}_1$ VIENE MANDATO IN UN SUO MULTIPLO

$A\, \underline{x}_1 = \dots\dots = 3\underline{x}_1$

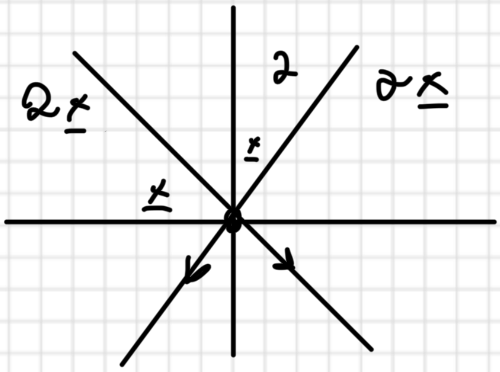
$A(2\underline{x}_1) = \dots\dots = 3(2\underline{x}_1)$

QUINDI $\underline{x}_1$ E' AUTOVETTORE DI A CON AUTOVALORE $\lambda_1 = 3$

$\underline{x}_2$ " " " " $\lambda_2 = 1$

OSSERVIAMO CHE TUTTI I MULTIPLI DI $\underline{x}_1$ (TRANNE $\underline{0}$) SONO AUTOVETTORI DI AUTOVALORE $\lambda_1 = 3$

$$A = 2I = \begin{pmatrix} 2 & & 0 \\ & 2 & \\ 0 & \ddots & 2 \end{pmatrix} \qquad A\underline{x} = 2I\underline{x} = \underset{\text{AUTOVALORE}}{2}\,\underline{x}$$



VALE $\forall \underline{x} \in \mathbb{R}^n$

TUTTI GLI $\underline{x} \in \mathbb{R}^n \setminus \{0\}$

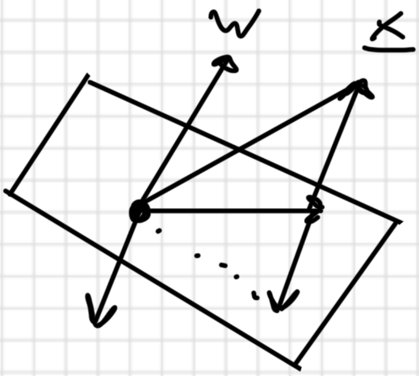
SONO AUTOVETTORI DI AUTOVALORE 2

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

IL VETTORE $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \underline{x}$ E' AUTOVETTORE DI AUTOVALORE $\lambda = 1$

A = P   RIFLESSIONE DI HOUSEHOLDER



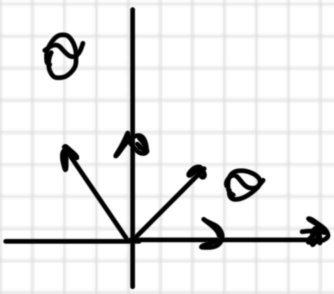
$$P = I - 2\,\underline{w}\,\underline{w}^T$$

$\underline{w}$ AUTOVETTORE DI AUTOVALORE -1

$$P\underline{w} = \underline{w} - 2\underline{w}(\underline{w}^T\underline{w}) = -\underline{w}$$

$Pv = v$ SE $v \perp w$ $v \in \mathbb{R}^{n-1}$ E' AUTOVETTORE DI AUTOVALORE 1

ROTAZIONI DI GIVENS



NON CI SONO AUTOVALORI, GLI AUTOVETTORI (REALI) NESSUN VETTORE VIENE TRASFORMATO IN UN SUO MULTIPLO

- RICAVARE L'AUTOVALORE AVENDO L'AUTOVETTORE

SIA $\underline{x}$ AUTOVETTORE $A\underline{x} = \lambda\underline{x}$ DOVE λ NON E' NOTO AL MOMENTO

$$A\underline{x} = \lambda\underline{x} \xrightarrow{(\underline{x} \neq 0)} \frac{\underline{x}^T A \underline{x}}{\underline{x}^T\underline{x}} = \frac{(x^T)(A)(x)}{(x^T)(x)} = \frac{x^T \lambda \underline{x}}{\underline{x}^T \underline{x}} = \frac{\lambda\, \cancel{\underline{x}^T\underline{x}}}{\cancel{\underline{x}^T\underline{x}}}$$

RICAVARE L'AUTOVETTORE A PARTIRE DA AUTOVALORE

CONOSCIAMO λ. SE λ È AUTOVALORE $\Rightarrow \exists \underline{x} : A\underline{x} = \lambda \underline{x}$

$$A\underline{x} - \lambda \underline{x} = \underline{0} \qquad A\underline{x} - \lambda I \underline{x} = 0 \qquad (A - \lambda I)\underline{x} = 0$$

TUTTI GLI $\underline{x} \neq 0$ TALI CHE $(A - \lambda I)\,\underline{x} = \underline{0}$ SONO AUTOVETTORI DI AUTOVALORE λ

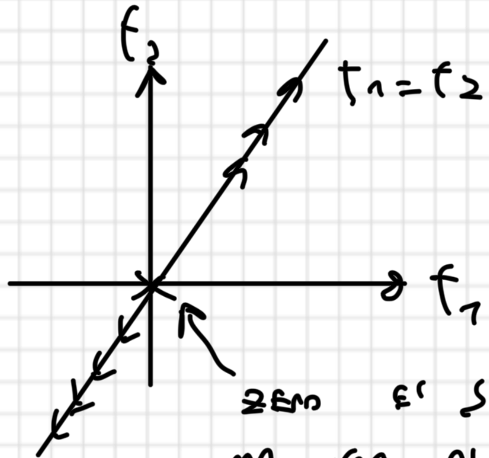
$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ AVEVAMO VISTO CHE $\underline{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ con $\lambda_1 = 3$

$\underline{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ con $\lambda_2 = -1$

$$(A - \lambda_1 I)\,\underline{x}_1 = \underline{0} \qquad \left[\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right]\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{x}_1 = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$\left.\begin{aligned} -t_1 + t_2 &= 0 \\ t_1 - t_2 &= 0 \end{aligned}\right\} \text{SONO DIPENDENTI}$$

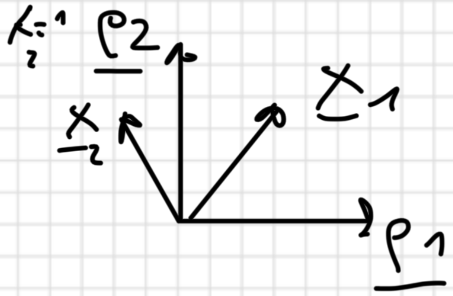


ZERO È SOLUZIONE MA NON È AUTOVETTORE

$$\det = 0$$

$$(A - \lambda_2 I)\,\underline{x}_2 = \underline{0} \qquad \left[\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$



OGNI VOLTA CHE ESISTE UN $\lambda \in \mathbb{R}$ E $\underline{x} \neq 0 \in \mathbb{R}^n$ TALE CHE $A\underline{x} = \lambda \underline{x}$ $\quad A\underline{x} - \lambda \underline{x} = 0$

$$\underbrace{(A - \lambda I)}_{\text{DEVE AVERE DET } = 0} \underline{x} = 0$$