13/12

DATA $A \in \mathbb{R}^{m \times n}$ $\exists\, U$ ORTOGONALE $(U^T U = I) \in \mathbb{R}^{m \times m}$

V " $(V^T V = I) \in \mathbb{R}^{n \times n}$

$$A = U \Sigma V^T \qquad \Sigma = \text{DIAG}(\sigma_1, \dots, \sigma_p) \quad p = \min\{m, n\}$$

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p \geq 0$$

$$\Sigma \in \mathbb{R}^{m \times n}$$

SE A E' DIAGONALIZZABILE QUALE RELAZIONE C'E' FRA DIAGONALIZZAZIONE E SVD

SUPPONIAMO $A = A^T$ SIMMETRICA. QUINDI AMMETTE DIAGONALIZZAZIONE

$A = X \Lambda X^T$ X ORTOGONALE $A \in \mathbb{R}^{n \times n}$

$\Lambda = \text{DIAG}(\lambda_1, \dots, \lambda_n)$

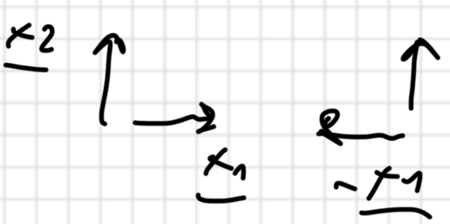
<u>ESEMPIO</u>

SE $X = \dfrac{1}{9}\begin{pmatrix} -4 & 7 & 4 \\ 8 & 4 & 1 \\ 1 & -4 & 8 \end{pmatrix}$ Λ DIAG$(-3, -1, 2)$

SVD = SINGULAR VALUE DECOMPOSITION

$A = X \Lambda X^T = U \Sigma V^T$ U, V ORTOGONALI

X E' ORTOGONALE $\longrightarrow$ E' DETERMINATA A MENO DI



i) PERMUTAZIONI DI COLONNE
ii) CAMBIO DI SEGNO

$\Big\}$ ALMENO QUESTE 2 CONDIZIONI

$\{\underline{x}_1, \underline{x}_2\}$ $\quad$ $\{-\underline{x}_1, \underline{x}_2\}$ ENTRAMBE SONO BASI ORTOGONALI DI $\mathbb{R}^2$

$$A \longrightarrow \boxed{\begin{array}{c} A^T A \\ \hline A A^T \end{array}} \longrightarrow \lambda_i(A^T A) \qquad \sigma_i = \sqrt{\lambda_i(A^T A)}$$

$i = 1, 2, 3$

$$\lambda_i(A^T A) = \left[\lambda_i(A)\right]^2 = \lambda_i^2$$

$\sigma_1 \geq \sigma_2 \geq \sigma_3$ SONO DA "PESCARE" TRA LE RADICI POSITIVE DI $(-3)^2, (-1)^2, 2^2$

$$X = \frac{1}{9}\begin{pmatrix} -4 & 7 & 4 \\ 8 & 4 & 1 \\ 1 & -4 & 8 \end{pmatrix} \qquad \Lambda = \mathrm{DIAG}\left(\overbrace{-3, -1, 2}\right)$$

$\sigma_i = |\lambda_j|$ PER UNA CERTA PERMUTAZIONE $i \longleftrightarrow j$

i		j
1	$\longleftrightarrow$	1
2	$\longleftrightarrow$	3
3	$\longleftrightarrow$	2

$$\boxed{\sigma_1 \geq \sigma_2 \geq \sigma_3} \qquad \underset{|\lambda_1|}{\overset{\sigma_1 = 3}{\parallel}} \qquad \underset{|\lambda_2|}{\overset{\sigma_2 = 2}{\parallel}} \qquad \underset{|\lambda_3|}{\overset{\sigma_3 = 1}{\parallel}}$$

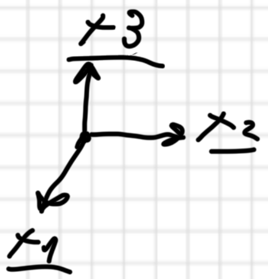
$$X = \frac{1}{9}\begin{array}{c} \begin{array}{ccc} \underline{x}_1 & \underline{x}_2 & \underline{x}_3 \end{array} \\ \begin{pmatrix} -4 & 7 & 4 \\ 8 & 4 & 1 \\ 1 & -4 & 8 \end{pmatrix} \end{array}$$

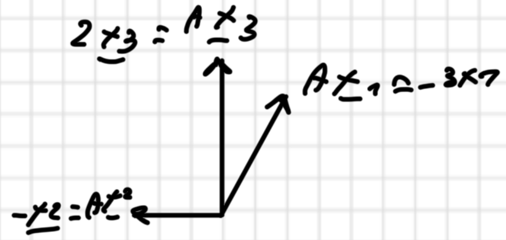
$$\Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$U \Sigma V^T = \underbrace{\left(-\underline{x}_1 \;\vdots\; \underline{x}_3 \;\vdots\; -\underline{x}_2 \right)}_{U} \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \hline \underline{x}_3 \\ \hline -\underline{x}_2 \end{pmatrix}$$

$$V^T = X^T$$

$$\Lambda \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$





$$A\underline{w} = U \Sigma \left(V^T \underline{w} \right)$$

COMPONENTI DI $\underline{w}$ SULLA BASE V

$$A = U \Sigma V^T \iff AV = U\Sigma$$

$$n \le m$$

$$A\underbrace{\left(\underline{v}_1 \,\vdots\, \underline{v}_2 \,\vdots\, \underline{v}_m \right)}_{m \times m} = \underbrace{\left(\underline{u}_1 \,\vdots\, \dots \,\middle|\, \underline{u} \right)}_{n \times n} \left(\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_m & \end{array} \right)$$

$$A\underbrace{\left(\underline{v}_1 \,\middle|\, \dots \,\middle|\, \underline{v}_m \right)}_{m \times m} = \underbrace{\left(u_1 \,\vdots\, \dots \,\middle|\, u_m \right)}_{n \times n} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{pmatrix} =$$

$$= \left(\sigma_1 u_1 \,\middle|\, \dots \,\middle|\, \sigma_m u_m \right)$$

$m = 10$, $n = 6$ $\qquad A : \mathbb{R}^6 \longrightarrow \mathbb{R}^6$

$rk(A)$

$$\Sigma = \left(\underbrace{\begin{array}{c|c|c|c|c|c} 100 & 0 & 0 & 0 & & \\ 0 & 50 & 0 & 0 & & \\ 0 & 0 & 10 & 0 & & \\ 0 & 0 & 0 & \frac{1}{100} & 0 & \\ 0 & & & & & 0 \\ 0 & & & & & \end{array}}_{6} \underbrace{\begin{array}{|c|c|c|c} 0 & 0 & 0 & 0 \end{array}}_{4} \right)$$

$$Im(A) = \{ \underline{u}_1, \underline{u}_2, u_3, u_4 \}$$

$\sigma_1 = 100$

$\sigma_2 = 50$

$\sigma_3 = 10$

$\sigma_4 = \frac{1}{100}$

$\sigma_5 = 0$

$\sigma_6 = 0$

$$AV = \left(100\,\underline{u}_1 \,\middle|\, 50\,u_2 \,\middle|\, 10\,u_3 \,\middle|\, \frac{1}{100}u_4 \,\middle|\, \right)$$

$$ker(A) = \{ \underline{v}_5, \underline{v}_6, \underline{v}_7, \underline{v}_8, \underline{v}_9, \underline{v}_{10} \}$$

$m=2$ $\quad n=4$ $\quad$ 2VERSION V

$\sigma_1 = 10$ $\qquad A: \mathbb{R}^2 \longrightarrow \mathbb{R}^4$

$\sigma_2 = 1$

$$\underset{4}{\left(\ \overset{2}{\ }\ \right)} \left(\underline{V}_1 \mid \underline{V}_2 \right) = \left(v_1 \mid v_2 \mid v_3 \mid v_4 \right) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ \hline 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \left(10\, \underline{v}_1 \mid 1\, \underline{v}_2 \right)$$

$$\mathrm{Im}(A) = \langle \underline{v}_1, \underline{v}_1 \rangle \qquad \ker(A^T) = \langle \underline{u}_3, \underline{u}_4 \rangle$$

$$A^T U = V \Sigma^T$$

$$A \left(\underline{v}_1 \mid \underline{v}_2 \mid \underline{v}_3 \mid \underline{v}_4 \right) = \left(\underline{V}_1 \mid \underline{V}_2 \right) \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) =$$

$$\left(10\, \underline{v}_1 \mid 1\, \underline{v}_2 \mid \underline{0} \mid \underline{0} \right)$$