28/10

METODO DI GAUSS - JORDAN ($A \mapsto A^{-1}$)

ESEMPIO

$$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\downarrow \; r_2 \leftarrow r_2 + \frac{1}{2} r_1$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 9/2 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}$$

$$\downarrow \; r_1 \leftarrow \frac{1}{2} r_1, \quad r_2 \leftarrow \frac{2}{9} r_2$$

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1/2 & 0 \\ 1/9 & 2/9 \end{pmatrix}$$

$$\downarrow \; r_1 \leftarrow r_1 - \frac{1}{2} R_2$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 4/9 & -1/9 \\ 1/9 & 2/9 \end{pmatrix}$$

$$\underbrace{E_{12}\left(\frac{2}{9}\right) E_1\left(\frac{1}{2}\right) E_{21}\left(\frac{1}{2}\right)}_{A^{-1}} A = I$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4/9 & -1/9 \\ 1/9 & 2/9 \end{pmatrix} = A^{-1}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} = \frac{1}{\underbrace{8+1}_{9}} \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -4 \\ 0 & 0 & 2 \end{pmatrix} \longrightarrow A^{-1}\,?$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{3} & -4 & 0 & 1 & 0 \\ 0 & 0 & \boxed{2} & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_3 \leftarrow \frac{1}{2} r_3]{r_2 \leftarrow \frac{1}{3} r_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -4/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$$

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FAI APPARIRE / $\boxed{0}$ SOPRA I PIVOT

$$\xrightarrow{r_2 \leftarrow r_2 + \frac{4}{3} r_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{3}{3} \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) \xrightarrow{r_1 \leftarrow r_1 - 3r_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -3/2 \\ 0 & 1 & 0 & 0 & 1/3 & 3/3 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) \xrightarrow{r_1 \leftarrow r_1 + 2r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{6} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$$

A^{-1}

$A \in \mathbb{R}^{n \times n} \longrightarrow$ COSTO $= n^3$

$$\boxed{\begin{array}{c}\text{SIST. LIN (GAUSS)} \\ n^3/3\end{array}}$$

DETERMINANTI

GAUSS:

$$\dots E_{21}(\lambda) E_{12} \dots A = A' = A^{(n)} = \begin{pmatrix} \bullet & \cdots & \bullet \\ & \ddots & \vdots \\ 0 & & \bullet \end{pmatrix}$$

REGOLA DI BINET: $\det(E \cdot A) = \det(E) \det(A)$

1) $R_i \leftrightarrow R_j \mid E_{ij} = \begin{pmatrix} 1 & \cdots & 1 \\ \boxed{1} & \ddots & 1 \end{pmatrix} \begin{matrix} \leftarrow i \\ \leftarrow j \end{matrix} \Rightarrow \det E_{ij} = -1$

2) $R_i \leftarrow \lambda R_i \mid E_i(\lambda) = \begin{pmatrix} 1 & & \\ & \ddots \lambda & \\ & & \ddots 1 \end{pmatrix} \Rightarrow \det E(\lambda) = \lambda$

3) $R_i \leftarrow R_i + \lambda R_j \mid E_{ij}(\lambda) = \begin{pmatrix} 1 & \ddots & \\ \lambda & & 1 \end{pmatrix} \Rightarrow \det E_{ij}(\lambda) = 1$

j

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1) $R_i \leftrightarrow R_j \Rightarrow \det A \mapsto -\det A$

2) $R_i \leftarrow \lambda R_i \Rightarrow \det A \mapsto \lambda \det A$

3) $R_i \leftarrow R_i + \lambda R_j \Rightarrow \det A$ <u>NON CAMBIA</u>

$A \xrightarrow{\text{GAUSS}} A' = \begin{pmatrix} a'_{11} & \cdots & \cdots \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{pmatrix} \Rightarrow \det A' = \pm \det A$

$\pm \to (-1)^{\#\text{SCAMBI}}$

$A^{(1)} \to A^{(2)} \to \ldots \to A^{(n)} = U$

$m_{ik} = a_{ik}^{(k)} / a_{kk}^{(k)}$

SE NON SI FANNO SCAMBI

FATTORIZZAZIONE "LU":

$$L = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ m_{ik} & & 1 \end{pmatrix} \leftarrow i \quad , \quad U = A^{(n)} \Rightarrow A = L \cdot U$$

$\uparrow$
k

SE INVECE SI FANNO SCAMBI:

$P \cdot A = LU$

$\hookrightarrow$ PRODOTTO DELLE E_{ij}

DETERMINANTI

$$A = \begin{pmatrix} 0 & 1 & 2 & 4 \\ -1 & 1 & 2 & 0 \\ 3 & 1 & 0 & 4 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 4 \\ 3 & 1 & 0 & 4 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{r_3 + 3r_1} \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 4 & 6 & 4 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{r_3 - 4r_2} \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -2 & -12 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4 - r_2} \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -2 & -12 \\ 0 & 0 & -1 & -3 \end{pmatrix} \xrightarrow{r_4 - \frac{1}{2} r_3} \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -2 & -12 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$= A^{(4)} \Rightarrow \det A^{(4)} = -1 \cdot 1 \cdot (-2) \cdot 3 = 6$$

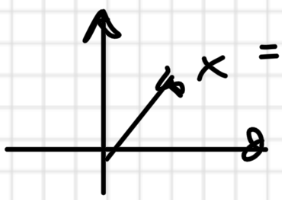
$$\det A = -6$$

CONDIZIONAMENTO DEL PROBLEMA $Ax = b$ (ERR. INERENTE)

x → OUTPUT

A, b → DATI

NORME VETTORIALI



$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad |x| = \sqrt{x_1^2 + x_2^2}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots x_m^2} = \sqrt{x^t \cdot x}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$\|\cdot\|_2 : \mathbb{R}^m \to \mathbb{R}$ PROPRIETÀ

1) $\|x\|_2 \geq 0 \quad \forall x \in \mathbb{R}^m$, INOLTRE $\|x_2\| = 0 \iff x = 0$

2) OMOGENEITÀ: $\|\alpha \cdot x_2\| = |\alpha| \cdot \|x\|_2 \quad \forall x \in \mathbb{R}^m$
$\forall \alpha \in \mathbb{R}$

3) DISUGUAGLIANZA TRIANGOLARE: $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$

DEFINIZIONE

NORMA VETTORIALE $\|\cdot\| : \mathbb{R}^m \to \mathbb{R}$ TALE CHE

1) $\|x\| \geq 0 \quad \forall x \in \mathbb{R}$ E $\|x\| = 0 \iff x = 0$

2) $\|\alpha x\| = |\alpha| \cdot \|x\| \quad \forall \alpha \in \mathbb{R} \ \forall x \in \mathbb{R}^m$

3) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^m$

Esempi

1) $\|x\|_\infty = \max\limits_{i=1\dots n} |x_i|$

2) $\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$

TEOREMA

$$\forall \|\cdot\|', \|\cdot\|'' \ \exists\, \alpha, \beta > 0 : \alpha \|x\|'' \le \|x\|' \le \beta \|x\|'' \quad \forall x \in \mathbb{R}^n$$

(EQUIVALENZA METRICA DELLE NORME)

$\|\cdot\|'$	$\|\cdot\|''$	α	β
1	∞	1	n
1	2	1	$\sqrt{n}$
2	∞	1	$\sqrt{n}$

$\rightarrow \|x\|_\infty \le \|x\|_1 \le n \|x\|_\infty$

SFERA UNITARIA $= \{ x \in \mathbb{R}^n : \|x\| = 1 \}$