NEW APPLICATIONS OF ITERATIVE SCALING IN TRANSMISSION TOMOGRAPHY

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- TRANSMISSION: $E(y) = \mu^{0} \exp(-\underset{J\times 1}{Ax})$

  Objective - reconstruct x by ML.

- GIS = SMART APPLIED TO A DUAL-SYSTEM.

  A key multiplicative form of solution.
  what about x?

- ACCELERATION VIA OS

  Subset balance

- OTHER PRACTICAL ISSUES

  Avoiding exponentiation

SMART in multiplicative form:

$$x_j^{k+1} = x_j^k \prod_i (\frac{y_i}{\hat{y}_i^k})^{a_{ij}}$$

In Transmission : we could apply ~~this~~ SMART to $\tilde{y} = -\log(y/\mu^0)$

(Method A)

This is NOT a ML-solution

Equations are INCONSISTENT

Noise amplified at low counts.

The solution retains a $\boxed{\text{MULTIPLICATIVE FORM}}$

$$\hat{x}_j = x_j^0 \prod_i \xi_i^{a_{ij}}$$

METHOD 1 :

Apply SMART to linear system

for $\tilde{y} = -\log(y/\mu^0)$

$\llcorner$ blank scan vals.

- NOT a ML-solution
- The linear system is INCONSISTENT
- This method amplifies noise at low counts.

METHOD 2 : Generalized Iterative Scaling

Apply SMART to a DUAL linear system

for $g = A^T y$

- Start point must be μ^0
- The linear system is CONSISTENT
- The method provides a ML solution and thus is efficient.

KNOWN RESULTS - Problem T

Only previous work providing ~~&~~ algebraic solution of ML

1. *Lange, 1984*

   An EM algorithm for *transmission* requires approximate solution (M-step).

2. *Darroch & Ratcliffe, 1972*

   Consider solutions

$$A^T \mu = \overbrace{A^T y}^{\text{new data}}, \tag{5}$$

Dual Equations

J equations in I unknowns

consistent

considered equations in unknowns μ which are restricted to the multiplicative form

$$\mu_i = \mu_i^0 \prod \eta_j^{a_{ij}}. \tag{6}$$

NOT system from - log(survival fraction)

A unique solution of system (5) of the multiplicative (log-linear) form (6) exists and may be recovered by generalized iterative scaling.

NB We are solving ~~for unknowns~~ linear equns in unknowns μ — the ray ~~sums~~ atten sums. ~~or projections~~

GENERALIZED ITERATIVE SCALING

1. Initial computations : $g = A^T y$ (backproject)

   $\eta^0 \leftarrow 1$, $\mu^0 \leftarrow$ expected blank scan

2. Iterations : $k = 0, 1, 2, \ldots$

$$\hat{g}^k = A^T \mu^k$$

$$\mu_i^{k+1} = \mu_i^k \prod_j \left(\frac{g_j}{\hat{g}_j^k} \right)^{a_{ij}} \qquad \text{(SMART)}$$

$$\eta_j^{k+1} = \eta_j^k \left(\frac{g_j}{\hat{g}_j^k} \right)$$

NB ROLE REVERSALS in DUAL system :

$i \leftarrow j$, $j \leftarrow i$ voxels / rays

Unknowns in linear system $x_j \leftarrow \mu_i$

GIS for TRANSMISSION

The GIS/SMART solution from start-point μ^0 always retains the multiplicative form

$$\mu_i = \mu_i^0 \prod \eta_j^{a_{ij}}.$$

which respects the tomographic model

$$\mu = \mu^0 \exp\left(-Ax\right),$$

with $\eta_j = \exp(-x_j)$.

Thus we advocate seeking a solution of the dual system ~~of equation (5)~~ employing GIS, or SV-EM, algorithms to provide the solution μ for transmission tomography!

We want x, not μ, though. See Arghami and Hudson.

A further issue, acceleration . . . by ordered subsets!

What of acceleration by subset methods.

THEOREM

For projection matrix A with row sums equal to 1, accelerated GIS $\{\mu^m\}$ converges to the unique solution of the Normal equations for which $KL(\mu, \mu^0)$ is minimized. The solution may be expressed in the product form required for Transmission.

The product form of solution is retained, ... a key point for solution in Transmission.

Which subsets of pixels?

... 'subset balance'.

FURTHER CONSIDERATIONS

SMART/GIS requires exponentiation on every $a_{ij} > 0$.
SV-EM does NOT, AND provides a similar solution.

$\Rightarrow$ Solve DUAL system from start μ^0 using SV-EM (or accelerate by OS-EM) to provide solutions $\hat{\mu}$.

Since the solution may not have PRODUCT FORM search for the solution x of

$$\hat{\mu} = \mu^0 \exp(-Ax)$$

e.g. $\tilde{y} = -\log \frac{\hat{\mu}}{\mu^0} = Ax$ should have much reduced noise

If ~~it~~ SMART has a night sky property IN CONSISTENT SYSTEMS

and if we solve ~~it~~ DUAL equations J equs in I unknowns ($J < I$)

then we expect ~~it~~ at most ~~[illegible]~~ $J-1$ non-zero fitted μ_i.

The remaining rays are estimated to have $\underline{\underline{\infty}}$ ~~attenuat~~ cumulative hazard.

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avoiding exponentiation:

- SMART/GIS requires exponentiation on every non-zero a_{ij} (~~problem~~ considered comput. problem by Fessler)

- SV-EM does <u>NOT</u>, ~~but~~ AND provides a similar solution to SMART

⇒ 1. SOLVE DUAL SYSTEM FROM START μ^0
USING SV-EM / OS-EM (FAST!!) $\Rightarrow \hat{\mu}$

2. If the solution is not ~~quite~~ (quite) of PRODUCT FORM

$$\cancel{1 = \sum_j a_{ij} \frac{(A^T y)_j}{(A^T \hat{\mu})_j}} \qquad 0 = \sum_j a_{ij} \log \frac{(A^T y)_j}{(A^T \hat{\mu})_j} \quad \underline{\forall i}$$

Now search for solution x of $\hat{\mu} = \mu^0 \exp(-Ax)$:

a. FBP

b. OS-EM on ~~log~~ $\tilde{y} = -\log \frac{\hat{\mu}}{\mu^0} = Ax$ (much reduced noise ~~[illegible]~~ in system)

~~c. SMART~~ (requires logs <u>just once</u>).

CONCLUSION

GIS algorithm in Transmission
applies a very similar approach to DUAL equations
that SV-EM algorithm applies
to forward equations in Emission.

Acceleration of both algorithms is
effected by Ordered Subsets.
with GIS provable convergence

Each iteration reduces a KL-distance ... safe cgence.

with effective subsets {work to be done...}
GIS would provide a general purpose algorithm
perhaps as effective as SV-EM.

PLAN

1. Linear systems
2. Two algorithms for their solution

   Symmetry EM / SMART

   OSEM / MART
3. Models for Medical Imaging
4. Known results for consistent and inconsistent linear systems
5. A new approach in Transmission Tomography

   MPL / Bayesian