NOTATION

Stüben &
Trottenberg

For given L, f find the solution u
of

$$L u = f$$

On the grid Ω_h :

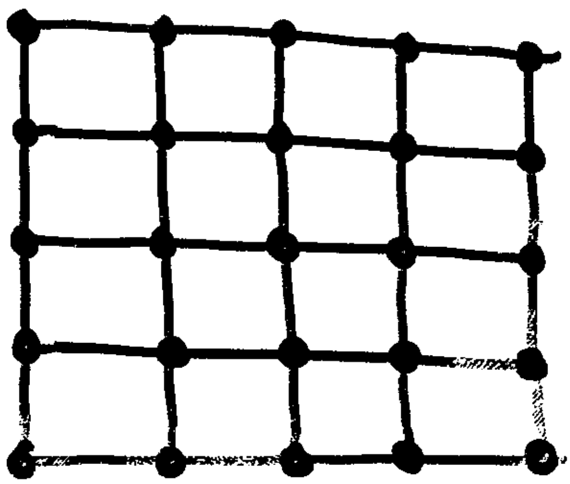
$$L_h u_h = f_h$$

where L_h is a known matrix
f_h is a known vector

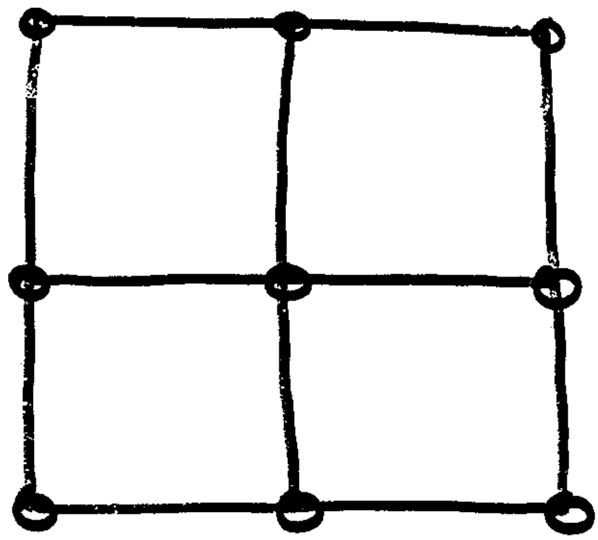
u_h and f_h are vectors with components for each grid point (pixel).

Ω_h :

FINE GRID



COARSE GRID Ω_H:



- any solution u_h can be restricted to an approximate solution u_H

$$u_H := I_h^H u_h$$

- any coarse grid solution u_H when interpolated (prolonged) provides an approximate fine grid solution

$$u_h := I_H^h u_H$$

restriction – local averaging
interpolation – bilinear interpolation

APPROACH

- ITERATIVE $\quad \dots\ u_h^j \longrightarrow u_h^{j+1} \dots$
- SOLVE A DUAL PROBLEM (THE DEFECT EQUATION)
- COMBINE A RELAXATION ALGORITHM WITH A COARSE GRID ALGORITHM

<u>Defect equation</u>:

$$\left.\begin{array}{l}\text{error}\\ \text{correction}\end{array}\right\} v_h^j := u_h - u_h^j$$

$$\left.\begin{array}{l}\text{defect}\\ \text{residual}\end{array}\right\} d_h^j := f_h - L_h u_h^j$$

Trivially $\quad L_h v_h^j = d_h^j$

Given estimate u_h^j solve this equation for v_h^j

ITERATIVE METHODS FOR SOLVING THE DEFECT EQUATION $L_h v_h^j = d_h^j$

1. RELAXATION METHODS

Replace L_h by a simpler operator $\hat{L}_h$.

Solve $\hat{L}_h \hat{v}_h^j = d_h^j$,

this provides the new approximation

$$u_h^{j+1} = u_h^j + \hat{v}_h^j$$

Example: Jacobi method $\hat{L}_h := \mathrm{diag}(L)$

2. COARSE GRID APPROXIMATION

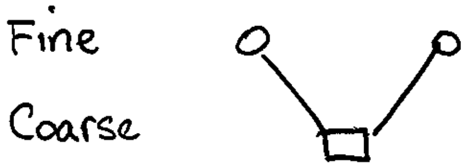
Approximate L_h by L_H

Solve $L_H \hat{v}_H^j = d_H^j$

<u>Thus</u>:

1. compute the defect
2. restrict the defect
3. solve on Ω_H
4. interpolate the correction
5. compute the new approximation

TWO GRID METHOD



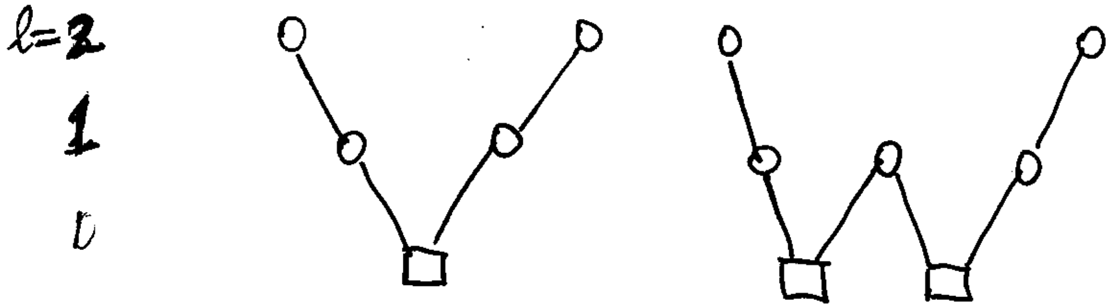
$\circ = \nu$ RELAXATION STEPS

$\square$ = SOLVE EXACTLY

\ = FINE TO COARSE RESTRICTION

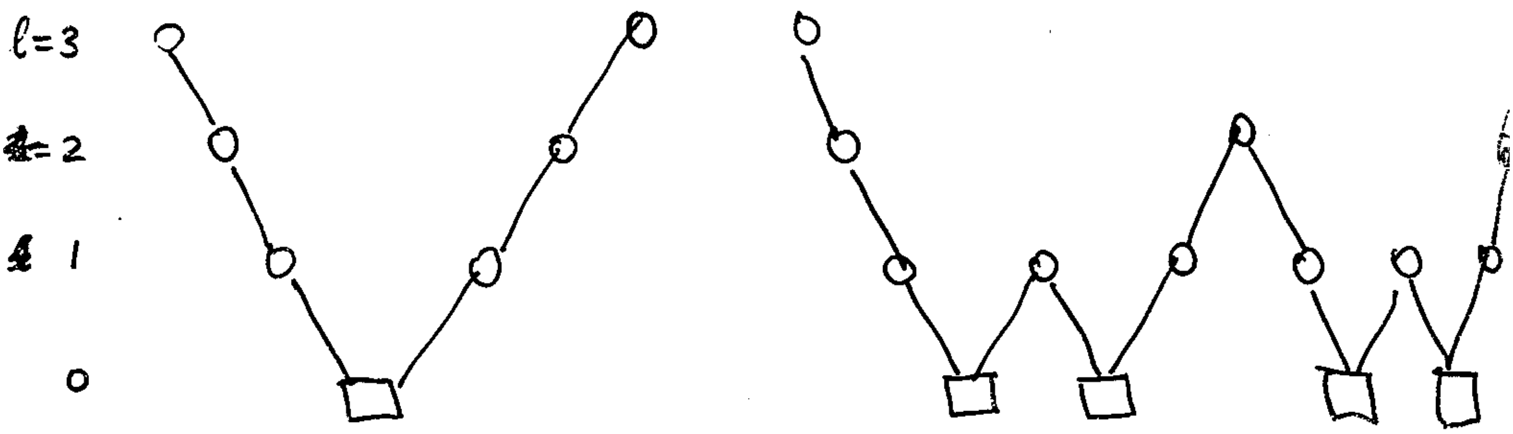
/ = COARSE TO FINE INTERPOLATION

THREE GRID



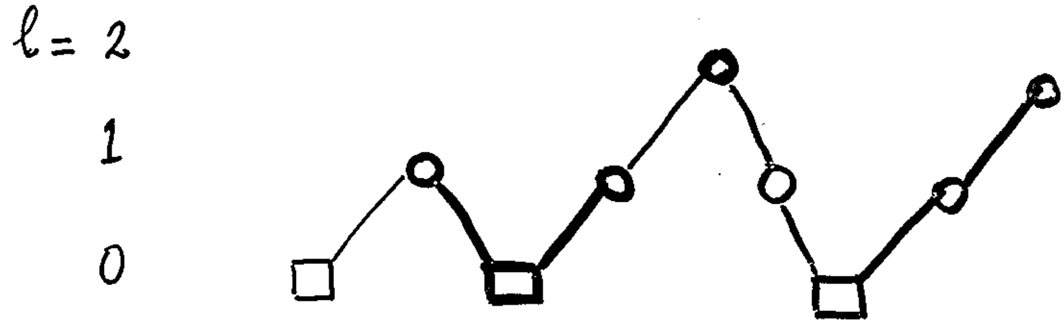
$\underline{\gamma=1}$ $\underline{\gamma=2}$

FOUR GRID



$\gamma=1$ $\gamma=2$

FULL MULTI GRID (nested iteration of Multigrid)



Properties: For large l

- the approximation $\tilde{u}_h$ of the discrete solution u_h is computed to an error $\|\tilde{u}_h - u_h\|$ which is smaller than the discretization error $\|u - u_h\|$

- the number of arithmetic operations needed is proportional to the number of grid points of Ω_h (with only a small constant of proportionality).

APPLICATION TO RECONSTRUCTION

ML AS ITERATIVE REWEIGHTED LS (FISHER SCORING)

To maximize likelihood, λ must satisfy the normal equations

$$\underset{J\times J}{F}\ \underset{J\times 1}{\lambda} = \underset{J\times I}{P'}\ \underset{I\times I}{D_\mu^{-1}}\ \underset{I\times 1}{n} \qquad \text{where } F = P' D_\mu^{-1} P$$

For a current estimate λ^j the <u>defect</u> is

$$g^j := P' D_\mu^{-1} n - F\lambda^j = P' D_\mu^{-1}(n-\mu)$$

$$= \underline{\mathbf{gradient}}$$

<u>Relaxation</u>

We may solve $\boxed{F\,\xi^j = g}$ defect equation where $\xi^j = \lambda - \lambda^j$

by using $\hat{F} = \mathrm{diag}(F)$ ... MFS

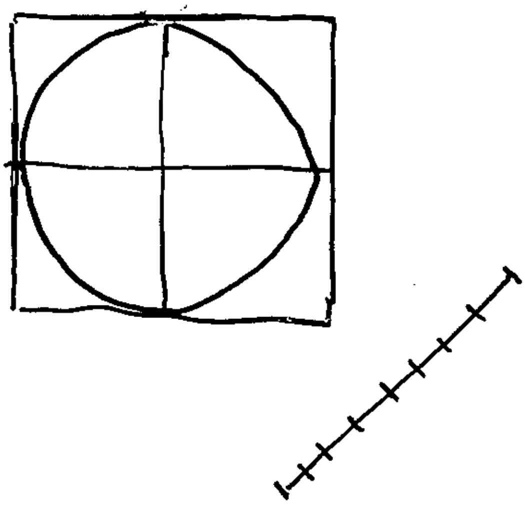
<u>Coarse grid approximation</u>

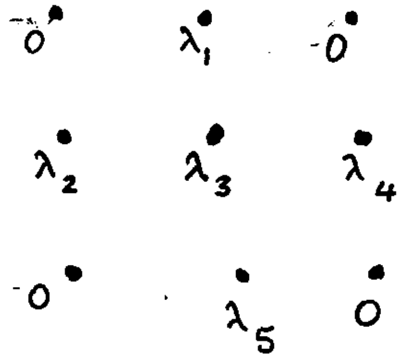
- restrict g to grid H
- solve $F_H\,\xi_H^j = g_H$
- interpolate ξ_H^j to ξ_h^j
- $\lambda_h^{j+1} := \lambda_h^j + \xi_h^j$

EXAMPLE

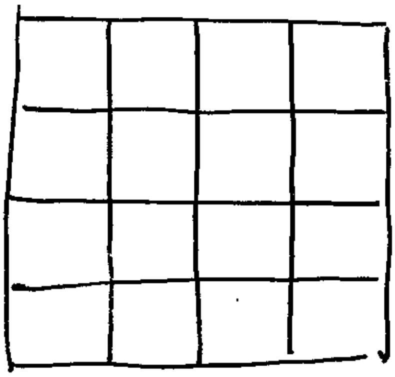
FOUR GRID ALGORITHM – 8 DETECTORS

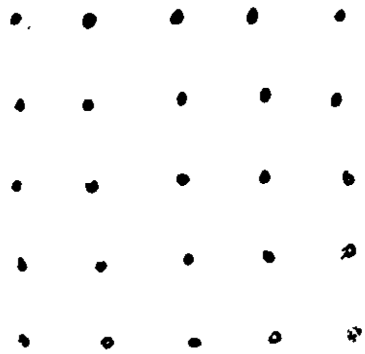
Level 1



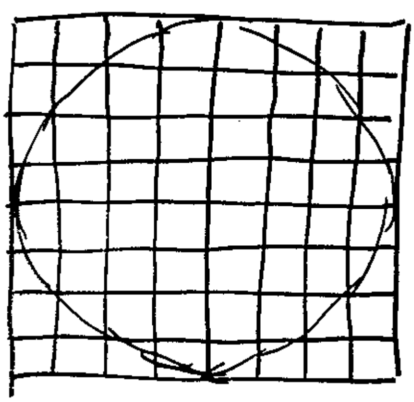


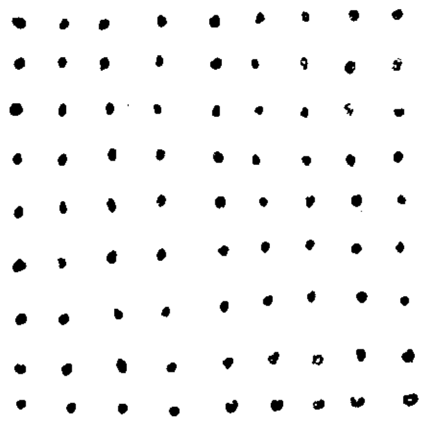
Level 2

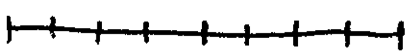




Level 3







Restriction I_h^H

Full weighting $\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ local average on fine grid

boundary points exceptional

Interpolation

$\frac{1}{4} \left] \begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{matrix} \right[$ allocation from course to fine grid

H-grid equation

1. course grid gradient and ~~the~~ Fisher Info calculated on a fine grid interpolation

<u>Ex</u> <u>Grid 0</u> $\boxed{\lambda}$ <u>Grid 1</u>

$$\begin{matrix} & \lambda_\bullet & \\ \lambda^\bullet & \lambda^\bullet & \lambda^\bullet \\ & \lambda^\bullet & \end{matrix}$$

+ normalize

2. gradient from local average of fine grid values

3. equation from MFS on the coarse grid with new P matrix

4. equation from MFS on the coarse grid with counts grouped.