STATISTICAL ALGORITHMS
FOR IMAGE RESTORATION FROM PROJECTIONS
or
Explorations of the heart

alternative title for
those close call
competition with afternoon
soapies on TV

Contributors:

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TITLE:

When a picture is represented by pixels, ie ~~an activity~~ picture, 2-d histogram of the activity being generated, we naturally have a choice ~~to~~ as to which level of resolution or scale to represent it.

~~Analogy of~~ may help Keep in mind the analogy Correspondence to density estimation via histograms is close.

$\boxed{1.1}$ Some ways in which we can make use of multiple scales
2 & 3 are ~~related to~~ methods from multigrid solution of PDE's, elegant review is reference

OHP 2.1 Application ~~is SPECT we have~~ one group at Macquarie is studying is SPECT, a medical imaging technique.

OHP 2.2 5x5 prototype, design probabilities p, ~~the~~ μ & λ notation
activity in patient circle

$\boxed{2.3}$ Restoration method studied here is EM

$\boxed{2.4}$ May assess a restoration by CSQ or MSE.

~~We assess use of multiple scales~~

$\boxed{3.1}$ We evaluate the methods above in simulations using a myocardium - lung phantom. plot activity traverse $\boxed{\text{Cross sections}}$

λ's translated into grey levels.

64 x 64 image, 32 proj~~n~~ ~~4 64~~ matching res. about 10,000 scintillation counts per projection

$\boxed{4.1}$

$\boxed{4.2-4.4}$ EM restoration, 6 iterations

$\boxed{5.1}$ Nested iteration. Persistence of grid squares.

$\boxed{5.2-5.4}$

$\boxed{6.1}$ Introduce smoothing.

7. Smoothness trade off against fit to data.
Why does smoothing improve the fit to the data here? CSQ
Persistence.

SOCIAL WELFARE

8. Coarse grid correction

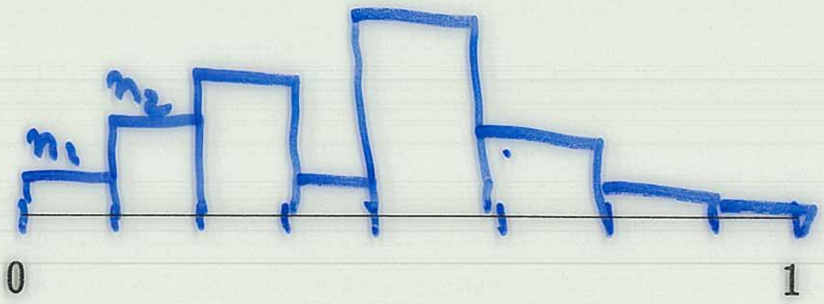
OHP 8.1 Correction principle
OHP 8.2 Two grid method
OHP 8.3 5x5 Prototype two grid experiment.
OHP 8.4 Full multi grid

$\boxed{8.1}$ Full multi grid ~~solution~~ restoration.

$\boxed{8.2-8.4}$ ~~Note: resolution of data bins was also~~

9. Review.

1 DIMENSIONAL ANALOGUE



A large number N of realisations of X provide binned counts $n_1,\ldots,n_I$ (a histogram)

When X has convolution density $a*f$ (with known error distribution a), we wish to recover the underlying density f. "Deconvolution"

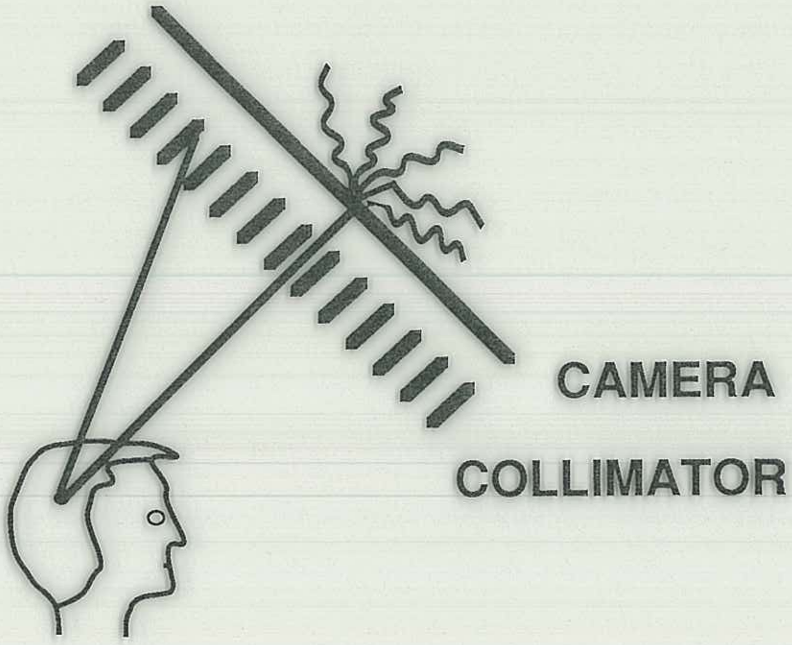
Example: deblur the Hubble telescope photos

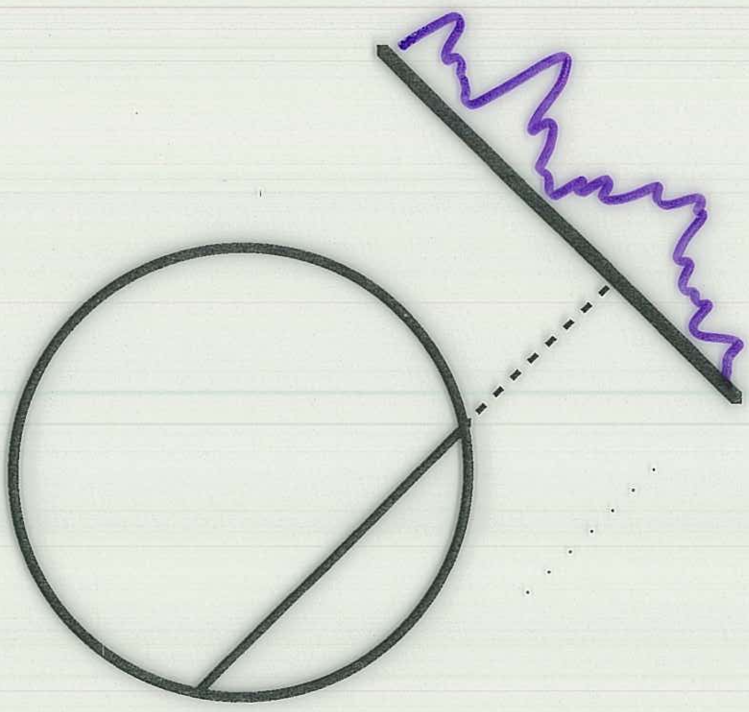
One approach is to write $E(n_i) = \Sigma\, a_{ij}\, \lambda_j$

Estimating $\{\lambda_j\}$ will then provide a discretised estimate of f.

OUTLINE

- SPECT background,
  sinograms and restorations
- EM algorithm as an iterative reallocation
  ("backprojection") of counts
- multi-scale versions and simulation results
- the 2-stage EM algorithm
- some other algorithms
  - MFS
  - dynamic EM


CAMERA
COLLIMATOR



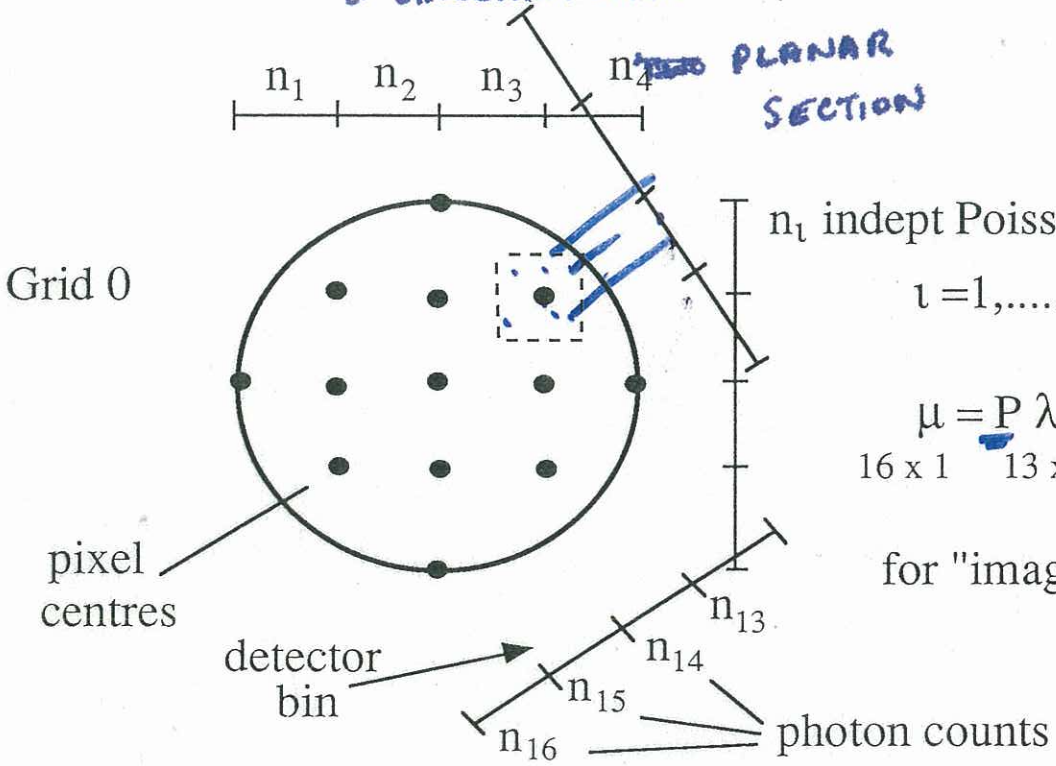
Ideal sensing

- would locate the source (e.g. pixel of o
  of each emission sensed
  level 1 data
- approximated by PET with time
  of flight recording

- observable data is a "histogram"
  emission counts

- different pixels within a region of
  uniform activity may record different
  counts (Poisson variability).

RELATIONSHIP BETWEEN ACTIVITY INSIDE BODY & SINOGRAM DATA

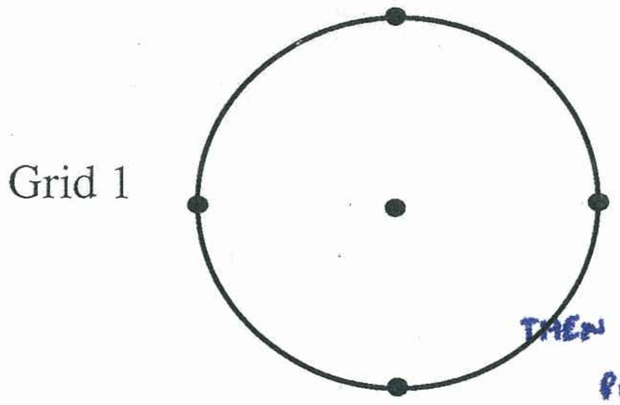


n_ι indept Poisson (μ_ι)

$\iota = 1, \ldots\ldots, 16$

$$\underset{16 \times 1}{\mu} = \underline{P}\, \underset{13 \times 1}{\lambda}$$

for "image" λ



THEN VARY ANGLE OF PROJECTION

CONVERT TO GRAY LEVELS IN SINOGRAM

OVERSIMPLIFICATION

Key: Data is Radon transform of image

Counts on projections must be realistic

THIS TALK :

EXPLORING POTENTIAL

IN ~~EMISSION T~~ MEDICAL IMAGING APPLICATION.

APPROACHES WITH MULTIPLE SCALES

1. NESTED ITERATION
2. COARSE GRID CORRECTION
3. FULL MULTI GRID

EM ALGORITHM IN SPECT

Principle: Redistribute detector bin counts of photons to pixels from which the photons may have originated . . . "back-projection"

Mechanism: Bayes rule

Let $\tilde{x}$ be the (discretised) pixel of origin and $\tilde{y}$ the bin location of a sensed emission

Then,

$$P(\tilde{x}=j \mid \tilde{y}=i) = \frac{\lambda_j \, p_{ij}}{\mu_i}$$

EM iteratively "improves" any image, creating a new image more consistent with the data. ← KEY TODAY

EM algorithm may be applied on any grid scale

CRITERIA ASSESSING QUALITY OF RECONSTRUCTION

1. **Agreement of projections with observed projection counts**

e.g. $$\mathrm{CSQ} = \sum \frac{(n_i - \mu_i)^2}{\mu_i}$$

2. **Agreement on pixels (given the true picture)**

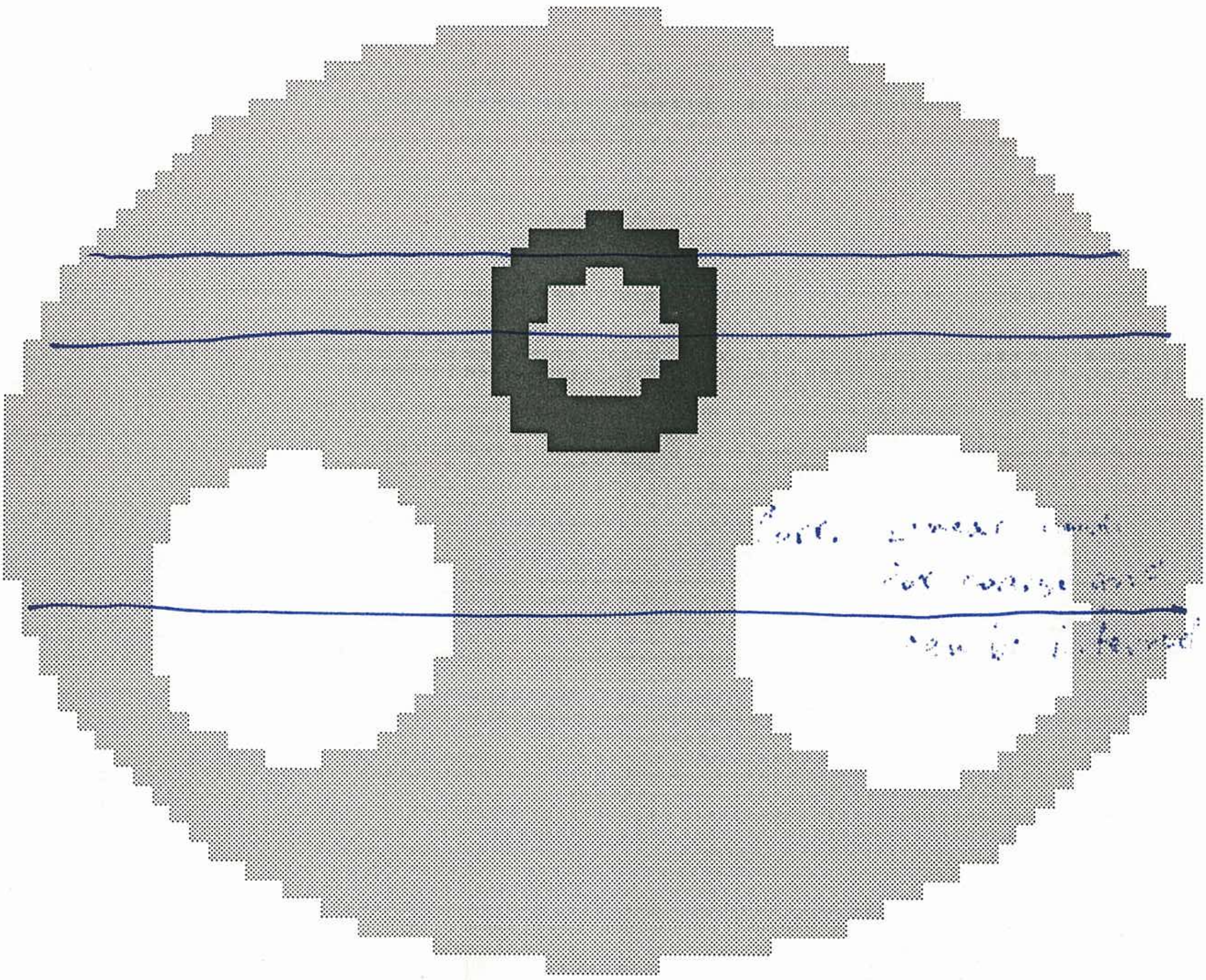
e.g. $$\mathrm{MSE} = \sum (\hat{\lambda}_j - \lambda_j)^2 / \sum \lambda_j$$

Our goal . . .

CONDUCTED A ~~FULL~~ LARGE SCALE SIMULATION

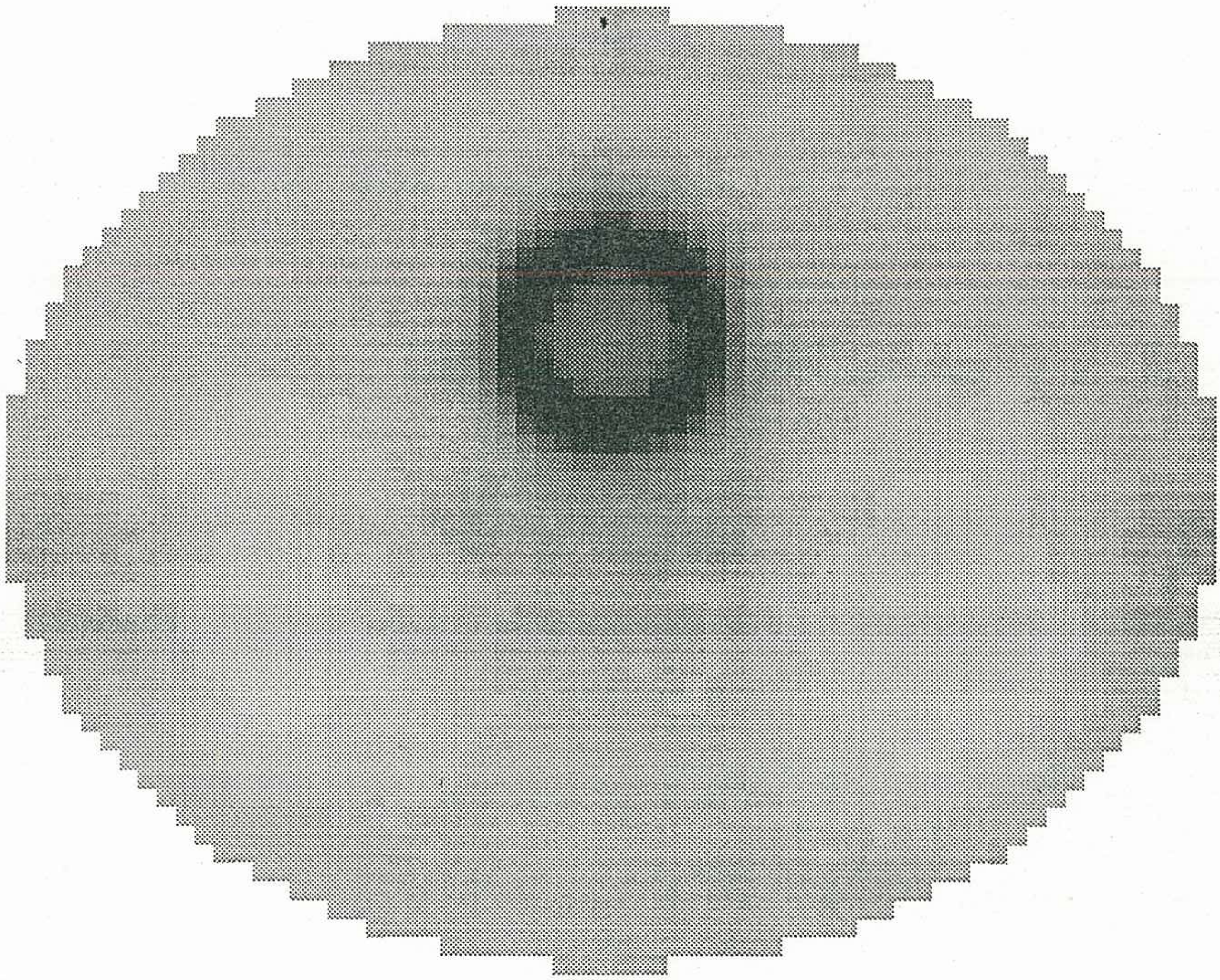
REPRESENTS

RD1.P - Original picture



64 x 64
draw the lines

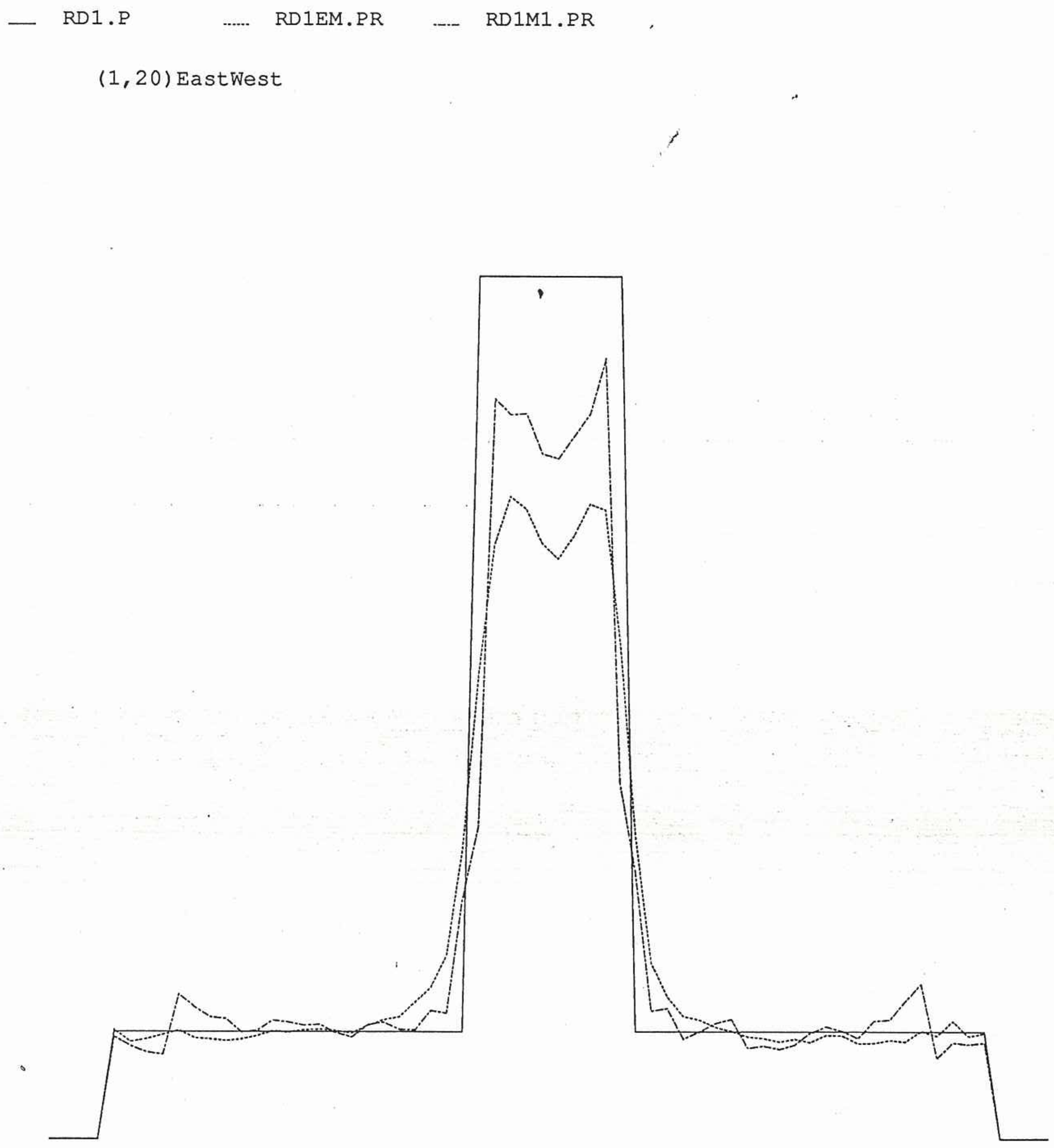
RD1EM.PR - 6] 98s CSQ: 8244.019 MSE: 86.443



No cross sections yet

Next slide RD1 M1

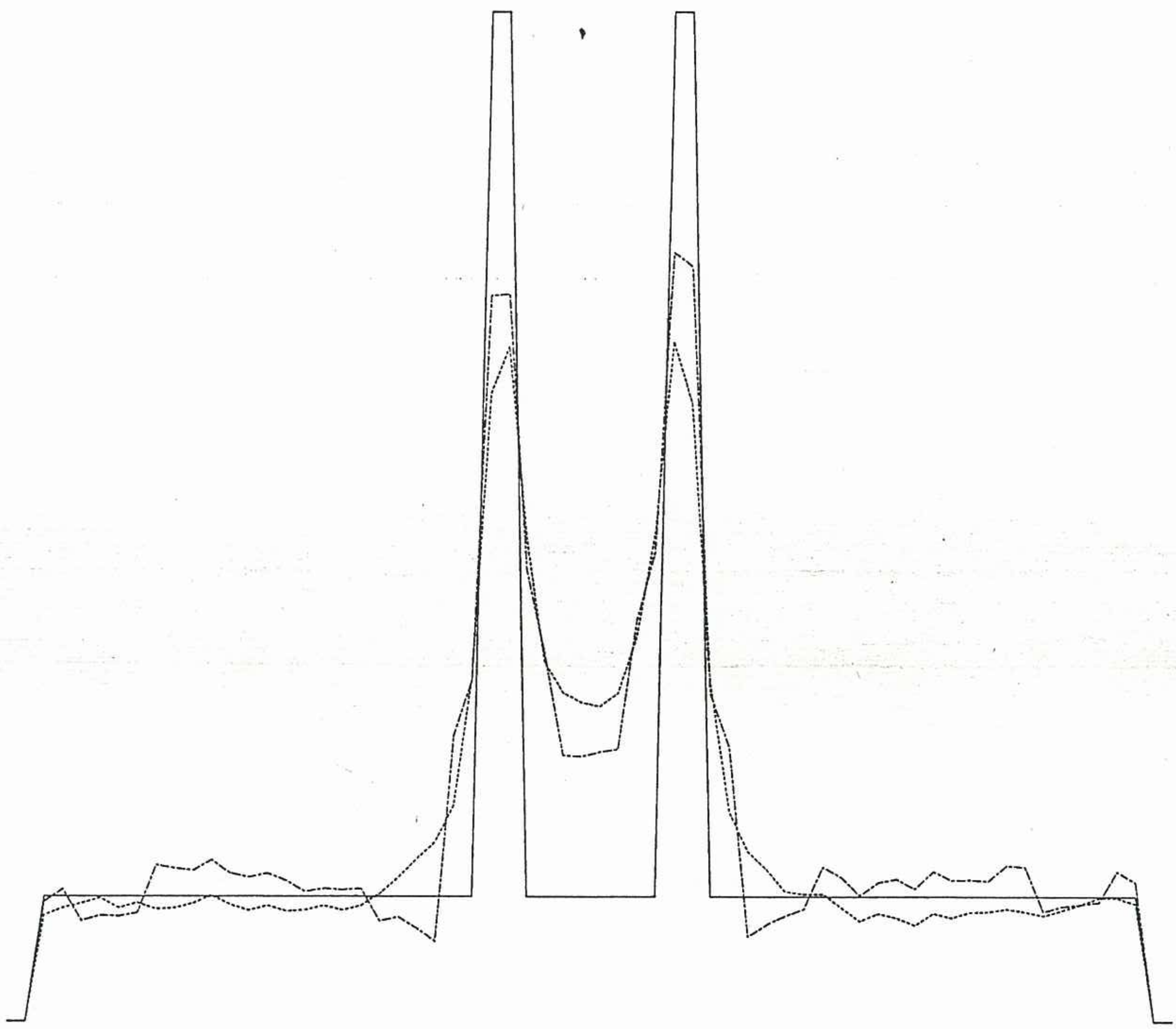
nested iteration
(no smoothing)
sq grid artifacts



RD1.P
RD1EM.PR
RD1M1.PR
(1,20)EastWest

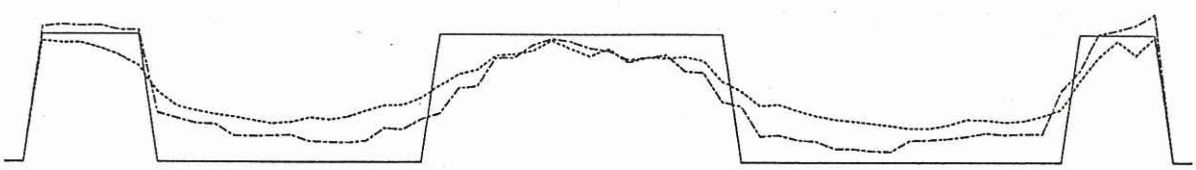
— RD1.P ...... RD1EM.PR ---- RD1M1.PR

(1,24)EastWest



— RD1.P ...... RD1EM.PR -.- RD1M1.PR

(1,42)EastWest



CORRECTION PRINCIPLE

SUPPOSE x AND y ARE IMAGES, y OBSERVABLE, RELATED BY

$$y = A x$$

RELAXATION METHODS PROVIDE SUCCESSI IMPROVEMENTS ON AN ESTIMATED SOLUTION

THE CORRECTION ξ SATISFIES THE <u>DEFECT</u> <u>EQUATION</u>

$$A \xi = r \qquad \text{WHERE } r = y -$$

SOLVING THE DEFECT EQUATION PROVIDES AN IMPROVED SOLUTION

$$x = x_0 + \xi$$

TO SOLVE ON A COARSE GRID WE DO BETT

TWO GRID METHOD

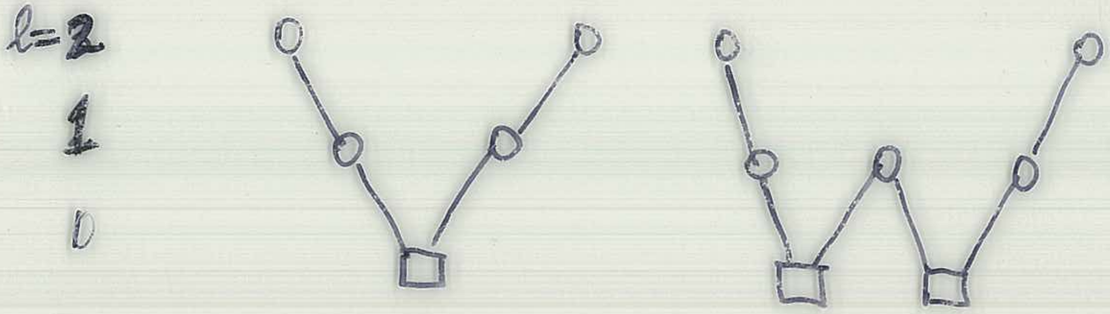
O = ν RELAXATI
STEPS

□ = SOLVE EXA

\ = FINE TO COAR
RESTRI

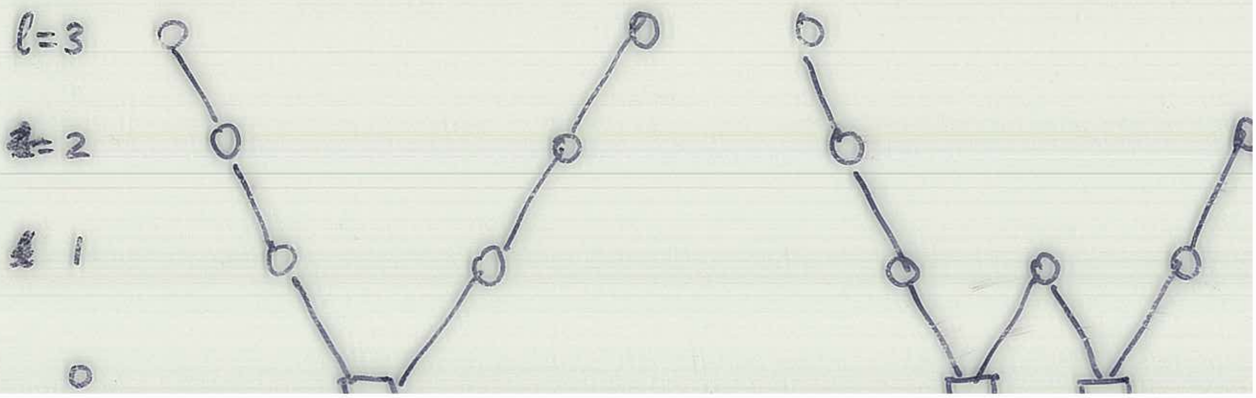
/ = COARSE TO FIN
INTERPOLAT

THREE GRID



$\underline{\gamma=1}$ $\underline{\gamma=2}$

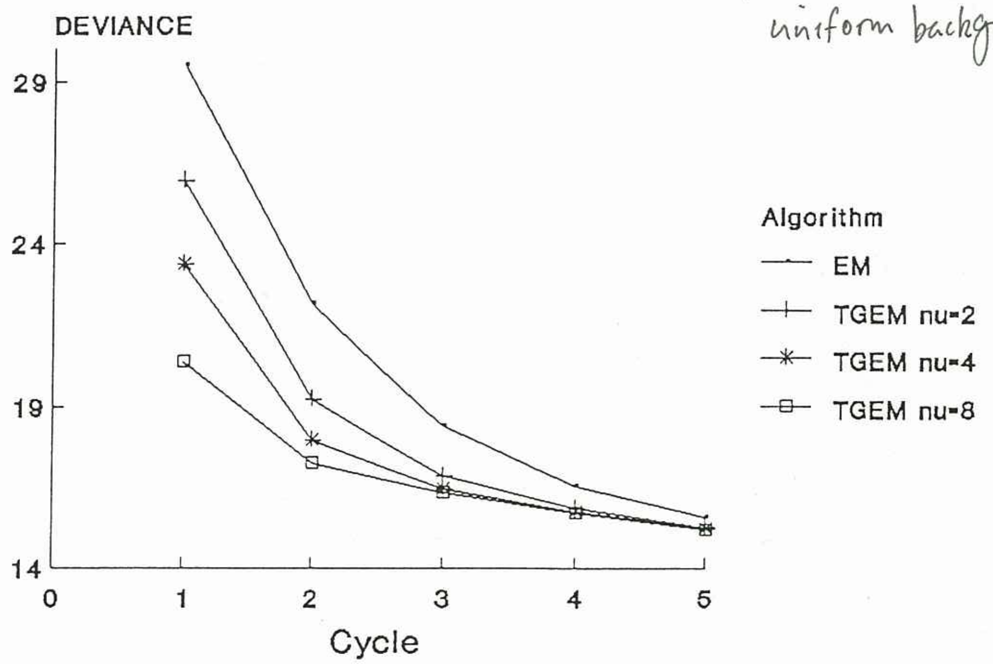
FOUR GRID



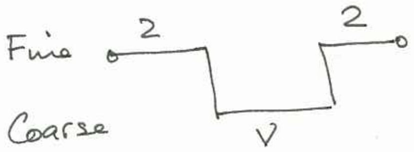
Results – two grid experiment on ~~the~~ 5×5 prototype

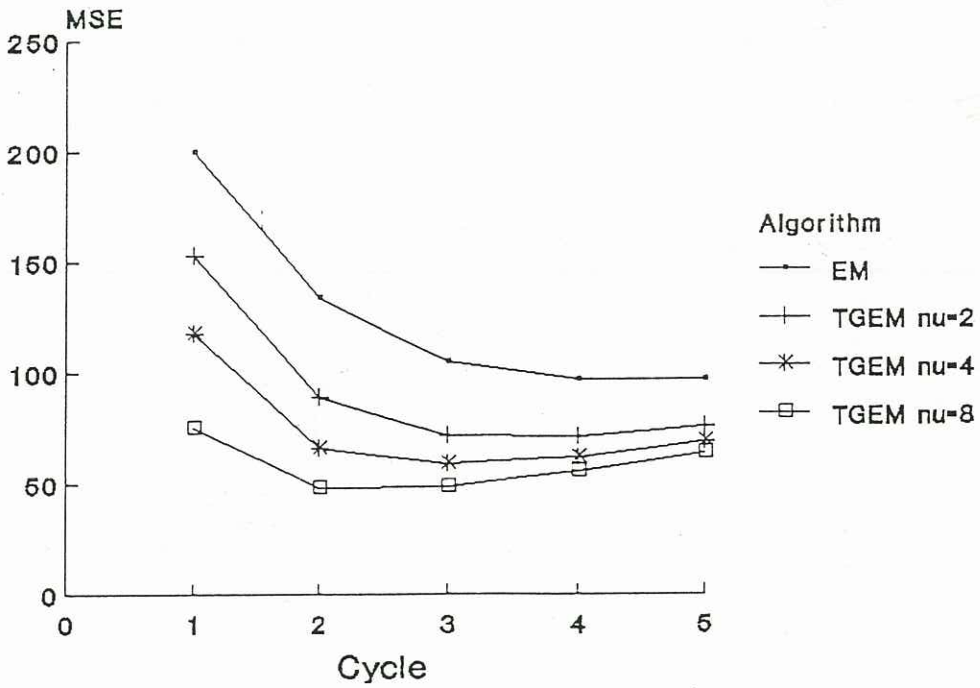
image here was point source on uniform background.

Figure 8: Lack of fit of EM and TGEM restorations in small scale simulations



each Cycle

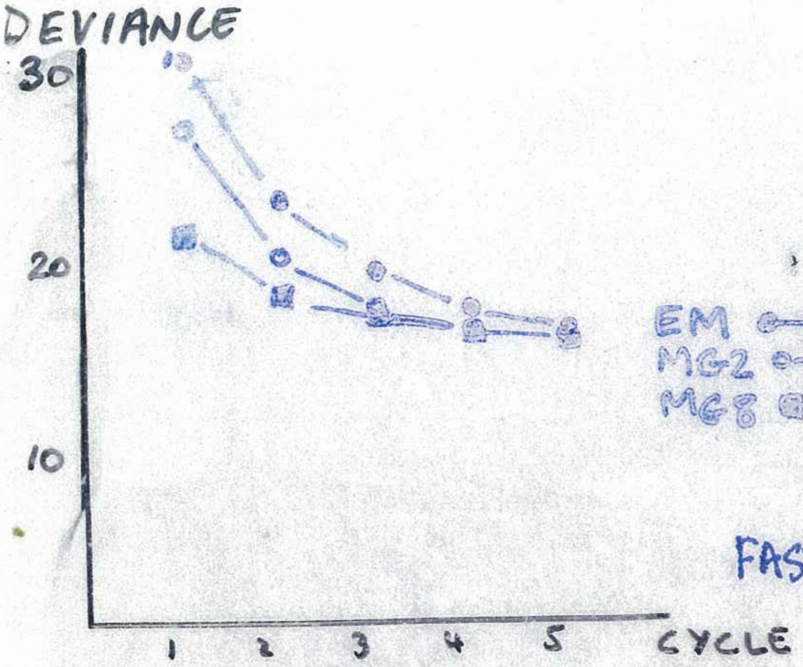


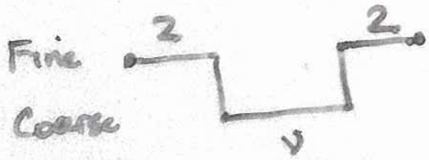


TMG can reach a better solution

RESULTS - PROTOTYPE TWO GRID EXPT

Image: point source on uniform background

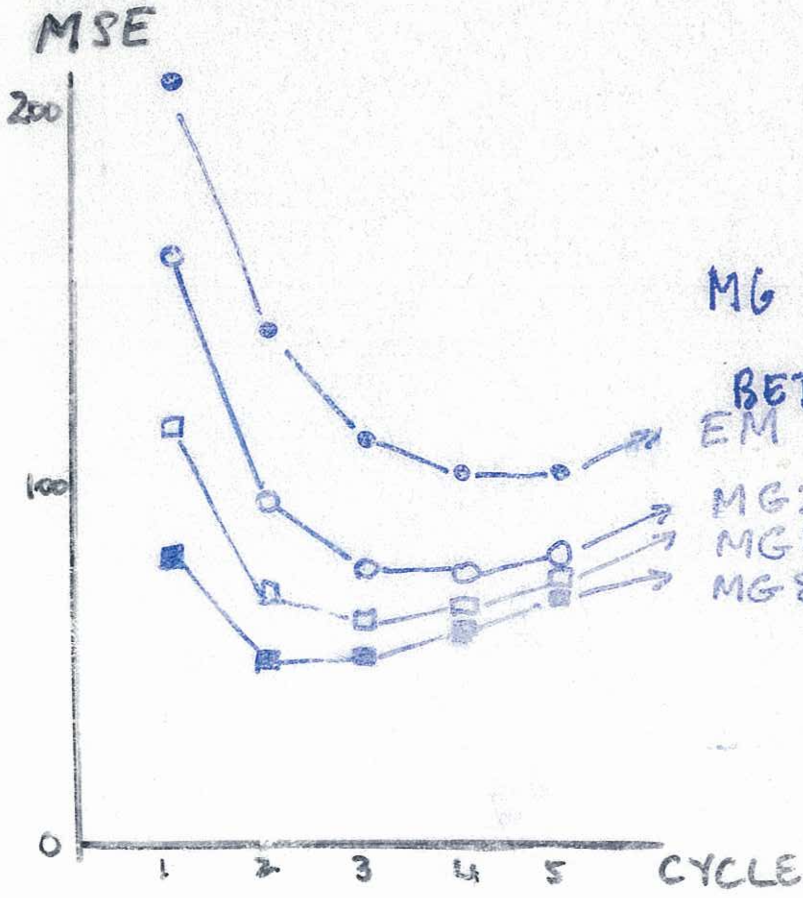




= 1 CYCLE

EM v=0

FASTER



MG CAN REACH A BETTER SOLUTION

SOME ISSUES RESOLVABLE BY PROTOTYPE INVESTIGATIONS

- pixel grid choice:
  relative efficiency of pixel grids (e.g. ring, square);
  inestimable contrasts & model singularity.
  (Margaret Donald)

  non uniqueness of circular grid reconstruction

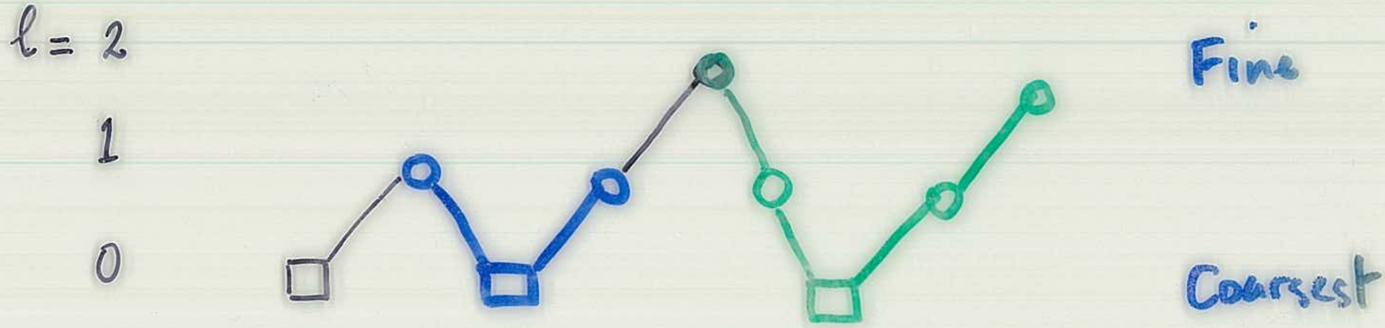
- projection data: how much information has been lost?
  Variance of reconstruction can be compared with
  Poisson variation of the "ideal" tomograph.
  (c.f. Silverman & Johnstone)

- penalised likelihood benefits & methods for
  hyperparameter choice
  (Pui Lam Leung & Khairil Notodiputro)

  regularisation

- Evaluating and refining new procedures.
  Three examples:
  Modified Fisher Scoring algorithm (Notodiputro)
  EMS (Jun Ma)
  Twogrid & multigrid procedures (Hudson & Gidas)

FULL MULTI GRID (nested iteration of Multigrid)



Properties: For large l

- the approximation $\tilde{u}_h$ of the discrete solution u_h is computed to an error $\|\tilde{u}_h - u_h\|$ which is smaller than the discretization error $\|u - u_h\|$

- the number of arithmetic operations needed is proportional to the number of grid point of Ω_h (with only a small constant of proportionality).

REVIEW

o EM WIDELY APPLICABLE

o Stochastic relaxation GIBBS PRIORS & POSTERIORS

~~REQUIRE ITERATIVE~~

1. Multi grid algorithms have widespread application in image restoration.

2. In SPECT:

   - o EM is too slow to restore high frequency elements;

   - o EM will not alter noisy restorations with good projections;

   - o smoothing the algorithm may retain the fit to projections, while improving the restoration visually;

   - o nested iterations and full multi grid do improve the image provided smoothing is incorporated.

EM algorithm - Statistical formulation.

x - complete, but unobserved data
y - incomplete, but observed

distributions depending on ϕ

Dempster, Laird and Rubin formulation

$y = y(x)$, a many to one mapping ; deterministic for y, given x. y INCOMPLETE FORM OF x.

E step: Estimate complete data sufficient statistics

$$t^{(m)} = E_m\{ t(x) | y \}$$

M step: Obtain $\phi^{(m+1)}$ from ML equns with data x

$$E_{m+1}(t(x)) = t^{(m)}$$

Stochastic variant

TRIVIAL BUT CONCEPTUALLY INTERESTING VARIANT

y STATISTICALLY DEGRADED FORM OF x.

(x, y) jointly distributed. Conditional distribution $y|x$ specifies the dependence of y on x.

E, M steps as above. Same result.

Proof: $X = (x, y)$ $Y = y$

$(x,y) \longmapsto y$ is the map from X to Y.

Error in camera recordings

$$i \xrightarrow{p_{ij}} j \xrightarrow{q_{jk}} k$$

Level 1 Level 2 Level 3

Complete

unobserved <u>unobserved</u> <u>observe</u>

q_{jk}'s determined by the p.s.f. of the camera.

corresponding to discretization of

x - location of photon emissi

y - actual location of camera sci

z - recorded location of camer scintillation

Approaches

1. EM with appropriate transition pro ($r = p * q$)

2. 2-stage EM

Two stage EM

1. estimate the level 2 data

$$\hat{n}_j^{(2)} = E\{ n_j^{(2)} \mid n^{(3)}, \phi^{(m)} \}$$

$$= \mu_j^{(2)} \sum_k \frac{q_{jk}\, n_k^{(3)}}{\mu_k^{(3)}} \qquad \text{'de b}$$

2. update $\phi^{(m)}$ to $\phi^{(m+1)}$ using the standard algorithm with estimated proje data

$$\phi_i^{(m+1)} = \phi_i^{(m)} \sum_j \frac{p_{ij}\, \hat{n}_j^{(2)}}{\mu_j^{(2)}}$$

3. revise $\mu^{(2)}$ and $\mu^{(3)}$ in light of the change to ϕ and continue to c through 1 – 3.

In Hudson & Gidas we generalize 2-stage and prove it proceeds uphill at ev step.

Alternatives to EM algorithm

Iterative reweighted least squares is the usual a

to optimization in exponential families

EM step : $\lambda_j^{(m+1)} = \lambda_j^{(m)} + \lambda_j^{(m)} g_j$

$\underset{\sim}{\lambda}^{(m+1)} = \underset{\sim}{\lambda}^{(m)} + J \underset{\sim}{g}$ — Poisson natural varia   J - diag

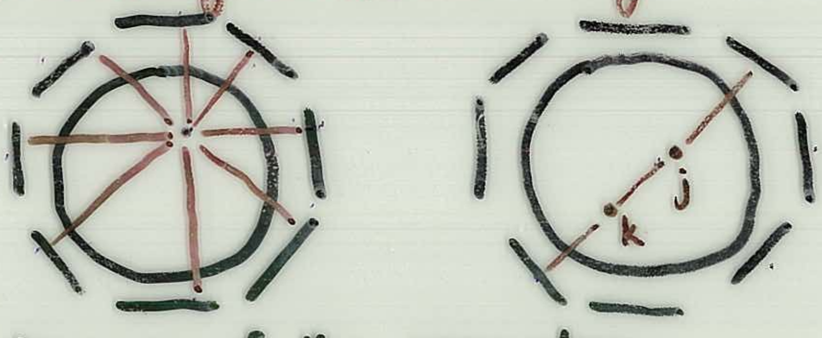
IRLS step : $\underset{\sim}{\lambda}^{(m+1)} = \underset{\sim}{\lambda}^{(m)} + M^{-1} \underset{\sim}{g}$ — Reconstruction varian   M - Fishe

$64^2 \times 64^2$ $m_{jk} = \sum_i \frac{p_i}{\ }$

also preserves total cell count

Options available

(i) use diagonal elements only



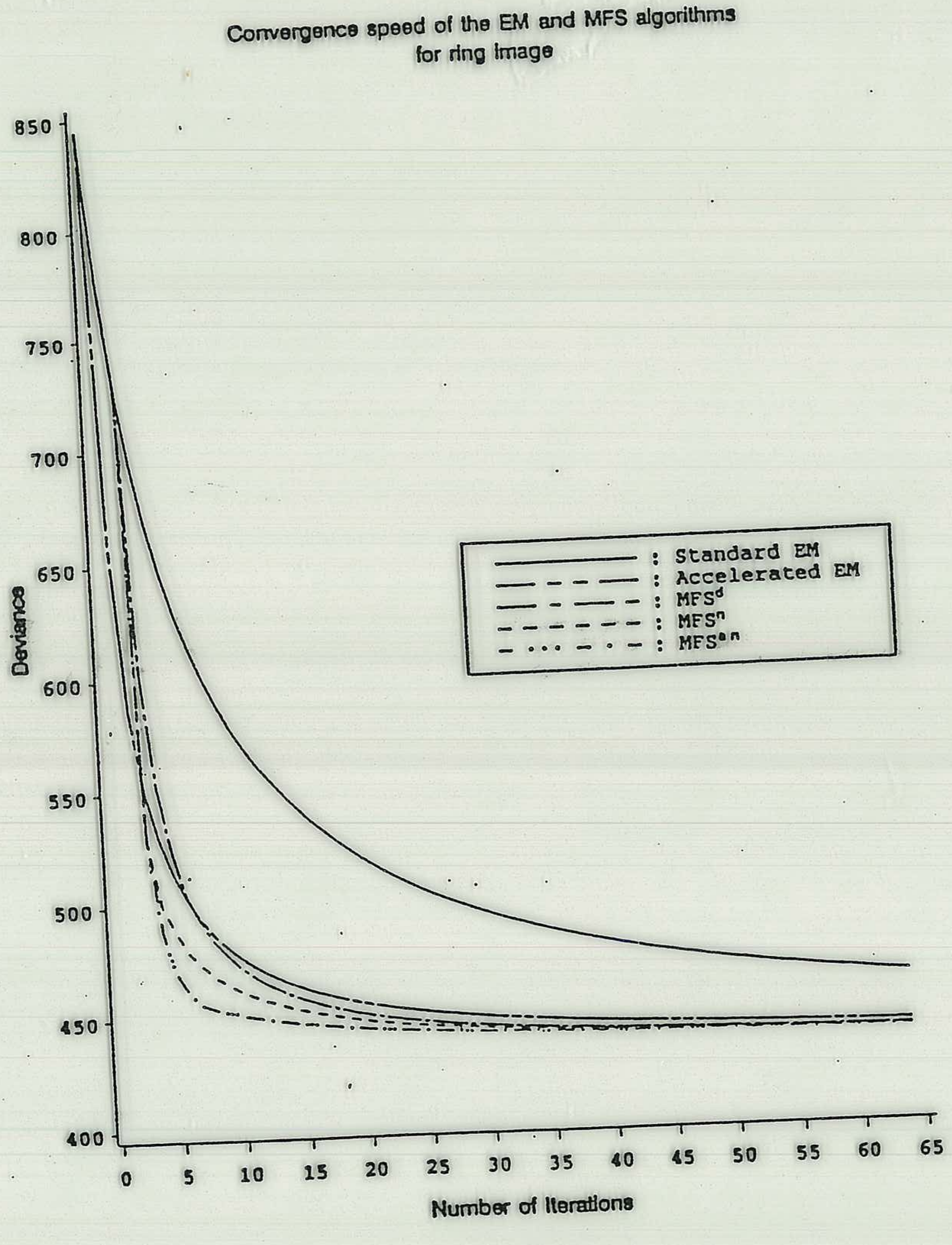
works because of the geometry

(ii) if necessary, because of limited detector resolutio

consider nearest neighbours also

$$M^{-1} = (M_d + M_{od})^{-1} \simeq M_d^{-1} - M_d^{-1} M_{od} M_d^{-1}$$

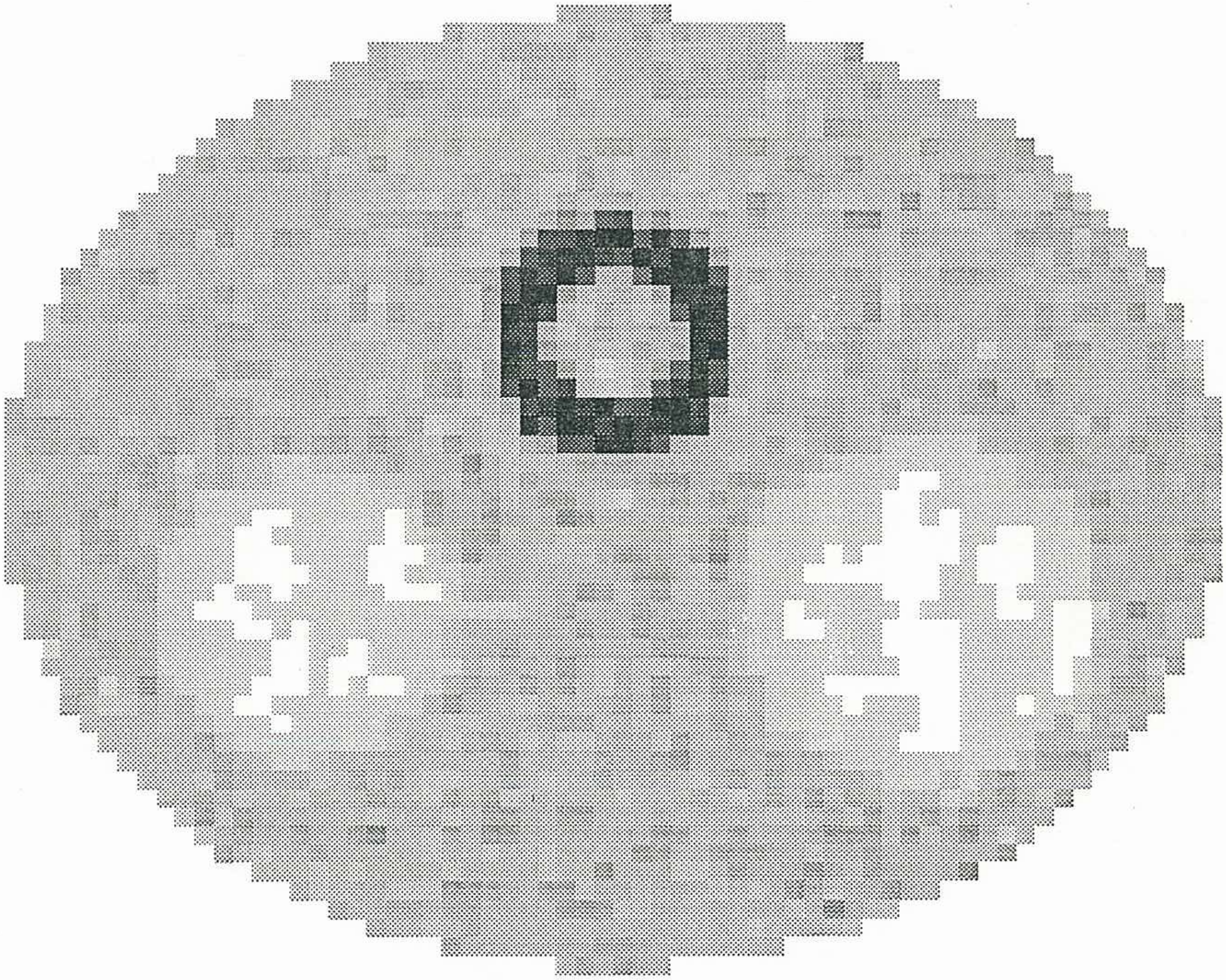
K. Notodiputro & M.H.



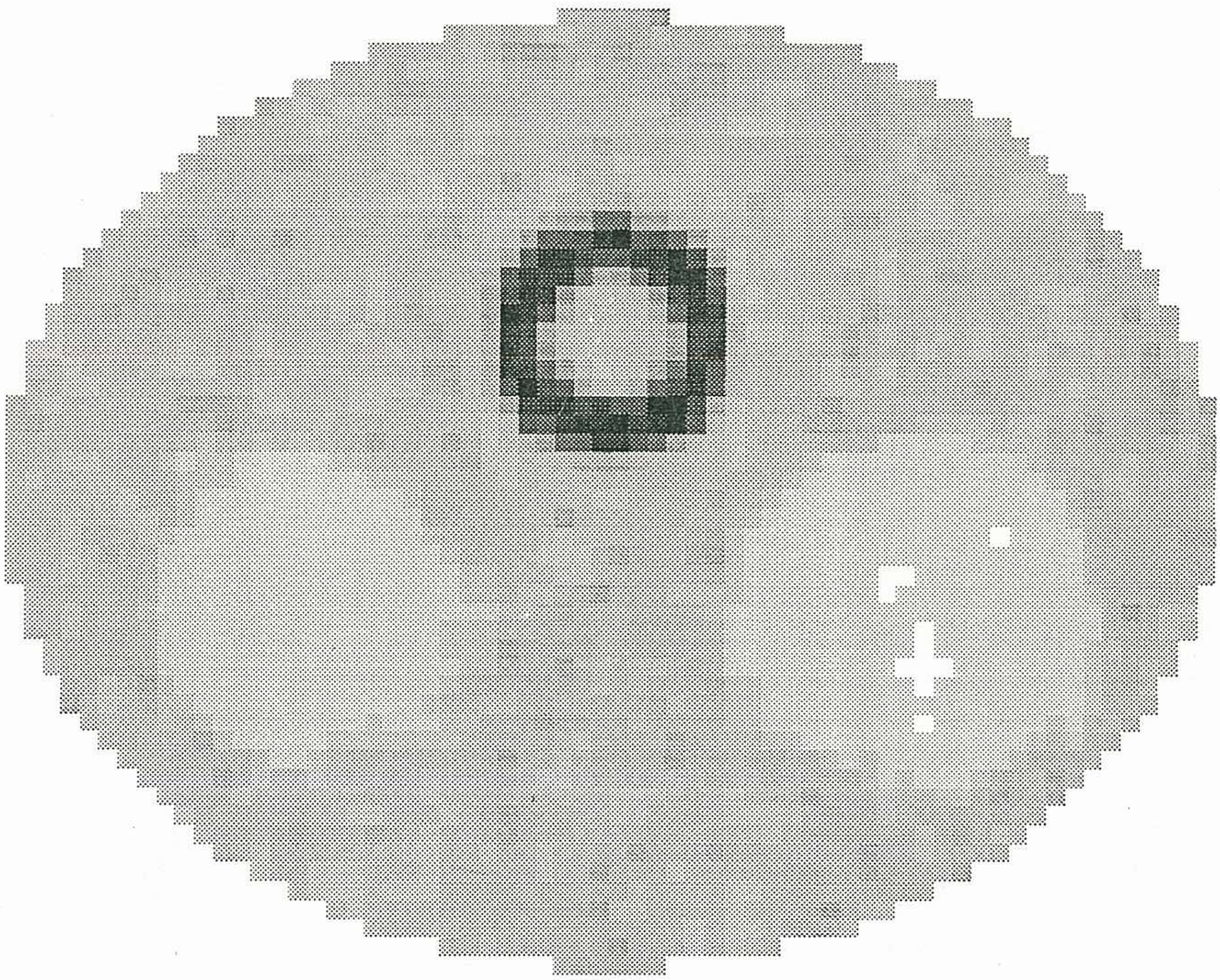
(b)

Fig 3.

RD1EM.PR - 128] 976s CSQ: 648.240 MSE: 31.164



RD1RF.PR - 1] 19s CSQ: 1601.954 MSE: 31.261



Review

Some themes in what we have done:

1. small prototypes distill the essence
   and isolate the issues;

2. smoothing, as can be
   imposed through a penalised likelihood criterion,
   can improve image quality;

3. there are a number of ways of
   greatly reducing computations;

4. multi-grid methods may be applied widely in
   image reconstruction.

These studies provide continuing feedback on the effectiveness of statistical strategies for high dimensional data.

Statistical algorithms for image restoration

by Malcolm Hudson

Seminar: Sydney University — 16 November

This talk will be a potpourri of results evaluating statistical algorithms for medical image restoration. In particular we will consider:

- the statistical model for single Photon emission tomography;
- the two-stage EM algorithm;
- multi-scale (multigrid) restorations;
- dynamic image restoration.

m. H.

30/10.

~~New~~ Statistical algorithms for image restoration.

~~Some topics~~ by Malcolm Hudson

Seminar: Sydney University — 16 November

This talk will be a potpourri of ~~recent developments in~~ statistical ~~methods~~ algorithms in ~~initiatives in topics image analysis~~ results evaluating statistical algorithms for medical image restoration. In particular we will consider:

~~the statistical~~

~~the physical~~

- the statistical model for single photon emission tomography;
- the ~~the~~ two-stage EM algorithm;
- multi-scale (multigrid)/~~reconstruction~~; restorations;
- ~~dynamic the~~ dynamic ~~EM~~ image restoration.

692.2222.

Sydney U. 16/11

- Outline – algorithms for medical image restoration.
- SPECT physics slide

Statistical description of sinogram data

n_i ~ Poisson (μ_i) indept $i = 1, \ldots,$

$\mu = A\lambda$, A known, ~~positive~~ transpose of a probability matrix

x is J×1 image of interest ⟺ activity distrib

~~(discretisation of a continuous analogue both in y & x)~~

~~$\mu(dy) = \int x \, a(y|x) \, dx$~~

Summary of ~~a~~ realizations of

x = pixel of origin

y = detector of scintillation

when $p(y|x)$ is known. $\boxed{\text{Count reallocation by probabilities}}$

- Small scale & ~~general~~ ~~phantom~~ heart lung phantom (myocardium)
- EM algorithm & ~~Bayesian~~ iterative reallocation interpretation
- Criteria
- ~~Multi Nested EM~~
- Comparisons of EM, nested EM, nested EMS and multigrid restorations.
- Comparisons with ~~non-statistical normality ff~~ filtered back projection
- -Role of smoothing projections.
- ~~Two grid~~ Two stage EM:

- Other algorithms: MFS

  dynamic EM

carousel

plastic sleeve & hang in filing cabinet (24 slides) } $1.50 ea.

$\boxed{\text{audio}}$

Kodak carousels. Harold

Quote 7573

Exploratory findings re MG in SPECT.

1. Use of multiple grids in image reconstruction

PROCEDURES FOR MULTIPLE SCALES
1. NESTED ITERATION
2. CALCULATION OF CORRECTIONS USING COARSER GRIDS
3. FULL MULTI GRID

2 ~~3~~. Application : SPECT using EM algorithm

$\boxed{\text{SPECT ~~SLIDE~~ OHP}}$ ~~OH~~ 02.1

$\boxed{\text{EM algorithm ~~OH~~}}$ 2.3 new, backprojection by Bayes rule. iterative ~~proceeds from an init~~ improvements on previous reconstn.

~~Assessing~~ Quality of a ~~Aim of~~ reconstruction ~~– reconstruction with~~

• ~~Provide~~ Agreement of ~~counts observed and~~ projection counts observed & expected ~~projections of the reconstruction~~ $\sum(\frac{n_i - \mu_i)^2}{\mu_i}$ CHI[

2.4

• Agreement of pixel values MSE with ~~a~~ those generating the counts. $\sum(\hat{\lambda}_j - \lambda_j)^2 / \sum \lambda$

Ultimate goal to match ~~exp~~ CB on projection data by EM-style algorithm.

3 $\boxed{\text{Example 5×5 prototype}}$ ~~Slide~~ OHP 02.2

$\boxed{\text{Myocardium ~~The Heart~~ – lung phantom}}$ λ's translated into grey levels. [3.1]

~~4 NESTED ITERATION~~

~~Sinogram &~~

4. EM reconstruction after 6 iterations, myocardium 2-D [?] 4.1

& Cross section of RDI, RDI EM, RDI CB. [4.2], [4.3] [4.4]

5. ~~NESTED ITERATION~~ Nested iteration reference

Suppose we use the results of ~~lower~~ coarser grids as starting values for EM.

~~Cross~~ Picture & cross sections RDI, RDIEM, RDIMI [5.1]

Cross section [5.2]-[5.4]

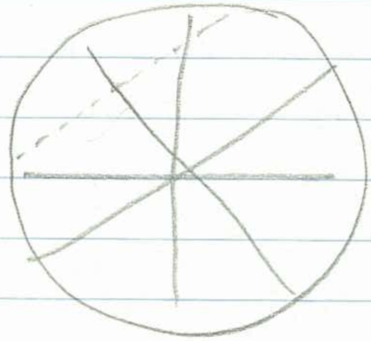
6. Incorporating smoothing RDIMIMDH.PR ? [6.1]

N.B. χ^2 & MSE both reduced.

the general rule
smoothing at the cost of fit to data
having no effect on

7. Why is the smoothing improving the fit to the data?

A. persistence of ~~squar~~ grid squares
angular Nyquist frequency — singularity



B. Grenander sieves.

8. Further ~~improving~~ improvements: coarse grid correction

$\boxed{\text{Correction principle}}$ OHP

$\boxed{\text{Two grid method}}$ ~~Two grid method~~ OHP

$\boxed{\text{Prototype two grid expt}}$ OHP

$\boxed{\text{Full multi grid}}$ OHP

$\boxed{\text{Myocardium lung phantom ~~with~~ M3MDH}}$ [8.1]
cross sections [8.2 – 8.4]

9. Review

PROCEDURES FOR MULTIPLE SC

1. NESTED ITERATION

THE USE OF COARSER GRIDS TO OB
GOOD INITIAL APPROXIMATIONS ON F
GRIDS

FOR PROJECTION DATA SEE
RANGANATH, DHAWAN & MULLANI IE

2. CALCULATION OF <u>CORRECTIONS</u> O
<u>COARSER</u> GRIDS

PARTICULARLY POWERFUL WHEN USED
CONJUNCTION WITH ITERATIVE (RELA
METHODS BECAUSE OF THEIR
<u>ERROR</u> <u>SMOOTHING</u> PROPERTY

3. COMBINE 1 & 2 IN FULL MULTI G
SEE STUBEN & TROTTENBERG (1981)

NESTED ITERATION

SOLVING $y = A_0 x$ for observable y

CAN BE APPROXIMATED BY INTERPOLATI

THE SOLUTION x (ON A COARSE GRID)

$$y' = A_1 x'$$

WHERE y' MAY BE A RESTRICTION (TO

COARSER GRID) OF y, AND WE M

CHOOSE $A_1 = RA_0$, WHERE R IS TH

<u>RESTRICTION</u> OPERATOR.

INTERPOLATION IS PROVIDED BY A

CORRESPONDING <u>PROLONGATION</u> OPE

ALGORITHMS FOR MULTIPLE GRID L

1. HUDSON & GIDAS — TWO GRID EM

$$\lambda = \lambda_0 + V(\lambda_0)g(\lambda_0)$$

$$\lambda' = \lambda'_0 + V(\lambda'_0)\,g(T\lambda'_0)$$

Note: 1. Restriction uses injection
2. Applies to EM algorithms only

2. REPARAMETERIZATION

Interpolation was used before projection to c
$E(n_i)$ for use in the gradient g, above.

$E(n) \approx \mu' = PT\lambda'$

is a model for data based on a coarse grid.
This is the standard model for data with P
~~and~~ there are corresponding ML equations

Incorporating the correction principle the cri

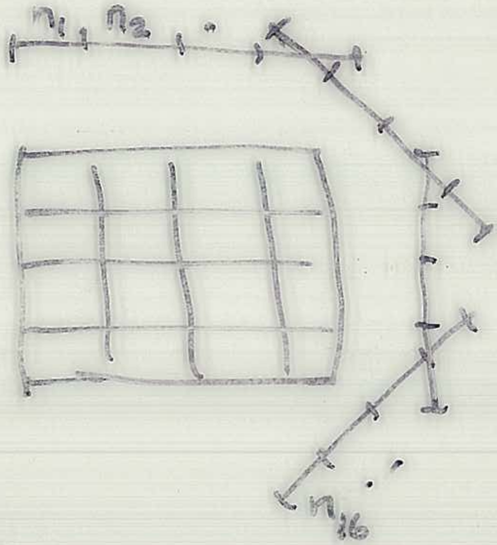
$$\begin{tikzcd}
\lambda_0 \arrow[d, "R"'] & \lambda_0 + T(\lambda'_m - \lambda'_0) \\
\lambda'_0 \arrow[r, "\text{any ML algorithm}"'] & \lambda'_m \arrow[u, "T"']
\end{tikzcd}$$

3. ALGORITHM RESCALING

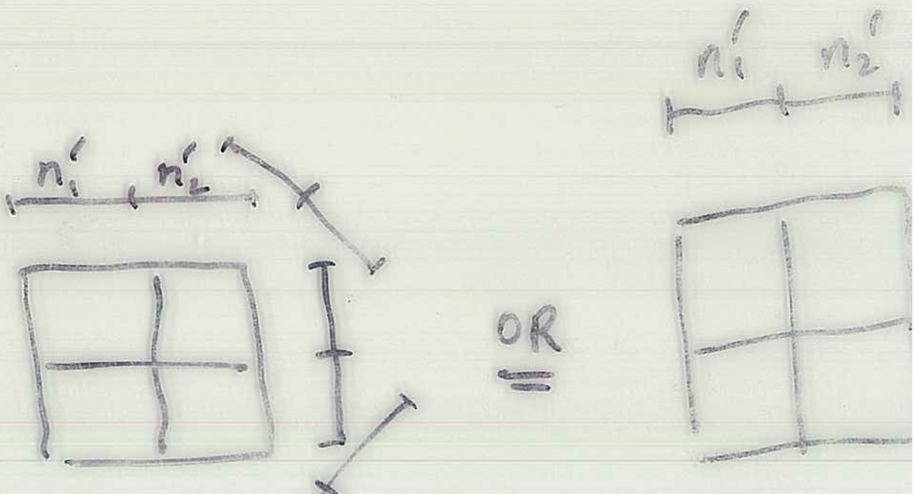
$$\begin{tikzcd}
(\lambda_0, n) \arrow[d, "R"'] & (\lambda_0 + T(\lambda_m' - \lambda_0'), n) \\
(\lambda_0', n') \arrow[r] & (\lambda_m', n') \arrow[u, "T"']
\end{tikzcd}$$

Example:

Grid 0



Grid 1



Multigrid algorithms (Uppsala)

Nested iteration in image reconstruction

~~Justification through Grenander's method of sieves ?~~

<u>Example</u> SPET ~~Myocardium - lung phantom~~ Up-down image

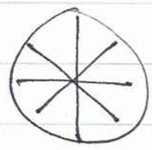
[1] EM reconstruction (4 iterations, lamda and trace)
demonstrates the failure to match high frequency components of the image

[2] EM from a median smoothed CBP start (4 iterations, lamda & trace)
demonstrates a better match to high frequency components after a better start for the low frequency component

The use of coarse grid solutions to provide good starting points
Justification through Grenander's method of sieves ?

uniform start
[3] Nested iterations, (lamda and trace after 4 surface iterations)
Failure due to persistence of the lower level grids ~~in the~~

Explanation of persistence through singularity of design
angular Nyquist frequency



Regularization via smoothing

[4] Nested iterations of EM<u>S</u>, uniform start (lambda & trace, 4 its)

Improving further an existing fine grid reconstruction
defect equation & problem of low frequency component in the <u>error</u>.

[5] ~~#~~MG iterations from [4] as starting point, descending^1 & reascending^3