An Agent-based Ramsey growth model with endogenous technical progress (ABRam-T)

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1 Overview

1.1 Purpose and Patterns

The purpose of the model is to analyze and test a decentralized economy composed of maximizing agents, with a particular focus on understanding the growth dynamics of the system. The model is built upon the well-known one-sector Ramsey growth model, with the introduction of endogenous technical progress through mechanisms of learning by doing and knowledge spillovers. In this setting, we aim to investigate farms that adopt different investment strategies based on different assumptions about the information available to them, in particular about the nature of technical progress.

The model is based on an analysis of three building blocks of Green Growth [3]; it is constructed directly from the first building block, which it extends from just two time steps, in order to explore long-term dynamics and growth of the economic system.

The model is abstract and does not aim to fit specific empirical data patterns for any given country or region. We evaluate it based on its ability to reproduce two well-known theoretical results in economics:

1. **Decentralized Economy vs. Benevolent Planner:** This pattern illustrates the dilemma that a decentralized economy is not able to reproduce the optimal results of a benevolent planner in the Ramsey growth model.

2. **Reproduction of Kaldor Facts:** Our model is capable of reproducing the Kaldor Facts, a set of stylized facts about economic growth.

By focusing on these patterns, we aim to provide a tool for understanding mechanisms in a decentralized economy of Ramsey-type agents.

1.2 Entities, State Variables, and Scales

The model's entities are:

1. **Agent Farm**: Agent that combines activities usually attributed to firms (like production) and households (like labour and consumption).

2. **Agent Statistician**: Agent that collects the system's information.

3. **Economy**: Entity that contains the agents and the simulation procedures.[1]

The model operates on a timeline of discrete time steps. As our model is abstract, these do not correspond to specific real-world time periods.

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[1]The model code uses the agent-based modelling framework agentPy [2], requiring the specification of a "model" entity. In other implementations, this might not be considered an entity in its own right.

1.2.1 Farms

Farms are agents that integrate the economic concepts of firms and households. They are characterized by their `capital` and `investment strategy`. These variables guide the farms in their `investment`, `production`, `consumption`, and their utility evaluation. We distinguish three types of farms depending on their investment strategy: *collaborative farms* (represented by value 0), *ignorant farms* (represented by value 1), and *witty farms*. Witty farms use expectations, they either follow an adaptive rule (represented by value 2), or a trend-follower rule (represented by value 3). When a witty farm follows an adaptive rule, the value of the adaptive constant can take two values. Similarly, the trend following rule can also take two values. The variables and parameters characterizing these agents are listed in the following tables.

Variable type	**Variable**	**Domain**	**Description**
State Variables	`K`	$\mathbb{R}^+$	Capital of the farm.
	`InvStrg`	$\{0, 1, 2, 3\}$	Investment strategy of the farm.
Supplementary Variables	`P`	$\mathbb{R}$	Production of the farm.
	`I`	$\mathbb{R}$	Investment of the farm.
	`expected_I_others`	$\mathbb{R}$	Farm's expectation on how much the other farms are going to invest.
	`C`	$\mathbb{R}^+$	Consumption of the farm.
	`u`	$\mathbb{R}^+$	Instantaneous utility of the farm
	`U`	$\mathbb{R}^+$	Cumulative utility of the farm
Information Variables	`past_I`	$\mathbb{R}$	Farm's investment from two time steps ago.
	`past_total_I`	$\mathbb{R}$	Statistician's variable `Total_I` from two time steps ago.
	`eta`	$\mathbb{R}$	Variable `real_tech_prg` from the Statistician

Parameter	**Values**	**Description**
`alpha`	0.33	Output elasticity of the capital
`delta`	0.05	The capital stock decay rate
`rho`	0.99	The factor to discount future felicity
`adapt_cns`	{0.65, 1.00}	Expectation weight factor for adaptive rule.
`trend_cns`	{0.40, 1.30}	Extrapolation coefficient for the trend following rule.

Table 1: Farms' variables and parameters.

1.2.2 Statistician

The **statistician** computes all the aggregate variables of the model and communicates to the farms those values that they need to know: aggregate capital, "Gross Domestic Product", and the technical progress of the economy at each time step. We introduce the statistician agent given the aggregate nature of the technological progress and the limited information the farms have about the economy. Table 2 list the variables of the statistician.

Variable	Domain	Description
Total_K	$\mathbb{R}$	Total capital of the system given by the sum of all farms' capital
real_tech_prg	$\mathbb{R}$	Technical progress of the economy.
GDP	$\mathbb{R}$	Gross domestic product of the system given by the sum of all farms' production.
Total_I	$\mathbb{R}$	Gross investment of the system given by the sum of all farms' investment.
Total_C	$\mathbb{R}^+$	Total consumption of the system given by the sum of all farms' consumption.
g_rate	$\mathbb{R}$	GDP growth rate
K_output_rate	$\mathbb{R}$	Capital to output ratio

Table 2: State variables of the Statistician agent.

1.2.3 Economy

The entity **Economy** represents the global environment. It contains the farms, the statistician, and the simulation procedures. It is in charge of initializing the agents, managing the model's schedule, and recording the variables associated with the agents. The attribute `parameters` contains the simulation parameters of the economy, i.e. the parameters that determine a certain simulation scenario. The parameters encompass the agents' parameters (Table 1) and the simulation's parameters (see Table 3)

The total number of farms in the simulation is given by the attribute `agents` and the attribute `steps` defines the simulation's length.

Name	Domain	Parameter description
agents	$\mathbb{N}$	Number of farms in the simulation
steps	$\mathbb{N}$	Length of the simulation
eta0	$\mathbb{R}^+$	Initial technical progress
InvStrg	$\mathbb{N}^{\text{agents}}$	List with each farm initial investment strategy
Tot_K0	$\mathbb{R}^+$	The system's initial capital
K0	$\mathbb{R}^{\text{agents}}$	List with each farm initial capital

Table 3: Simulation parameters

1.3 Process overview and scheduling

1.3.1 Simulation setup

Upon initialization of the model, the *economy* is created. During this initialization the farms and the statistician are created, for more information see section 3.1.

1.3.2 Proceeding in time

In the model, time proceeds in discrete time steps. There is not an explicit real time equivalent of a time step. In each time step, beginning with time $t = 1$ the **model's step function** is called. This function does the following.

1. For each farm, the **farm's capital function** is called, which updates and computes the farm's capital $K_t^{(i)}$ for the current time step (for more information see Section 3.3.1).
2. The statistician computes the technological progress `real_tech_prg` for the current time step using its function **technical_progress** (for more detail see Section 3.3.2).
3. The statistician computes the system's total capital `Tot_K` by adding the farms' capital.
4. For each farm, the **farm's production function** is called (for more information see 3.3.3)
5. The farms' **investment** process starts.

5.1. For each farm, the **farm's expectations formulation function** is called, which updates the farm's expectations about the other farms' investment (for more information see 3.3.4).

5.2. For each farm the **farm's investment function** is called as described in section 3.3.5.

6. The statistician computes the gross investment of the system by adding the farms' investment.
7. The statistician computes the GDP growth rate of the system (for more details see Section 3.3.6).
8. The statistician computes the GDP of the system by adding the farms' production.
9. For each farm the **farm's consumption function** is called as described in Section 3.3.7.
10. For each farm the **farm's utility function** is called (for more details see 3.3.8).
11. The statistician computes the system's gross consumption by adding the farms' consumption.
12. The statistician computes the system's capital-to-output ratio by calculating the ratio of the total capital to the GDP.

After each time step the **model's update function** (including at $t = 0$) is called. This function records the dynamic variables of the system.

2 Design Concepts

2.1 Basic Principles

The model addresses the well known **one sector Ramsey growth model** with endogenous technical progress. Endogenous technical progress is introduced via learning by doing and knowledge spillovers. Each agent **maximizes its intertemporal utility** function for its present and next time step, to decide how much to invest in its present time.

2.2 Emergence

In the economy of n farms, where $n \in \mathbb{N}$, the key outcome is it's growth pattern. The economy can grow, stagnate, or contract, depending on the initial capital and parameters of the farms. The growth of the economy throughout a simulation is given by the Gross Domestic Product (GDP) and its growth rate.

2.3 Adaptation

The model includes no adaptation.

2.4 Objectives

Each farm aims to maximize its intertemporal utility over two time steps. The utility of a farm depends on its current and future consumption. By solving this maximization problem, farms can determine the optimal investment for the present time period that will maximize their utility. How the farms solve their maximization problem depends on various assumptions about the information available to them. These assumptions form the basis of the farms' investment strategy.

2.5 Learning

The model includes no learning.

2.6 Prediction

To formulate the **witty farm's** investment, the farm needs to predict how much the other farms are going to invest. To do this, they formulate expectations. The witty farm uses one of two rules to make these predictions. They either use an **adaptive rule**, or a **trend following rule** for more detail see Section 3.3.4.

2.7 Sensing

The **farms** are aware of the system's technological progress to produce. Consequently, the farms are assumed to sense the present time step technological progress from the statistician. In addition, the **witty farms** need the system's gross investment and the total capital to formulate their expectations. Hence, they sense the system's total investment from the statistician.

2.8 Interaction

There are two types of interactions in the model: the interaction between the farms and the statistician, and the interaction among the farms themselves. The farms directly interact with the statistician when the statistician gathers their data and shares information about the current technological progress (and total investment).

The farms do not interact with one another directly, but only through the technological progress resulting from all farms' investment. This can be seen as a mediated interaction.

2.9 Stochasticity

In the model, stochasticity is used to initialize the system by providing different capitals to the farms. The total initial capital of the system is distributed among the farms using a Pareto distribution. By providing different capitals to the farms, we create a heterogeneous economy to study effects of technological progress in the system. Other than at initialization, the model is deterministic.

2.10 Collectives

The model does not include collectives.

2.11 Observation

In the model, we collect all the agents' variables at each time step. The statistics of the system are computed by the statistician.

3 Details

3.1 Initialization

The initialization of the model is done when the model is created (see Section 1.3.1 for scheduling). A `parameters` dictionary is set up to run scenarios with differing initial conditions. The simulation `parameters` that can be changed upon initialization are listed in Table 3. We describe the setup of the agents in the following.

3.1.1 Initialization of the farms

The simulation starts with the initialization of the farms and their state variables. The attribute `InvStrg` of the `parameters` dictionary determine how many farms of each type are created. The `InvStrg` is a list that contains the investment strategies of the farms. In simulations that include witty farms, both the expectation weight factor for the adaptive rule and the extrapolation coefficient for the trend follower rule are restricted to a single value.

The initial capital variable is either the same constant $K_0^{(i)}$ for the farms or is set randomly using a Pareto distribution. The farms' initial capital is saved to the list `k0`. After the capital initialization, the state variable for production (`f`) is initialized by computing the farms' output using the farm's initial capital and the production function (for details see Section 3.3.3). For the farm's state variable for investment (`I`) we use as well the initial capital and the farm's investment function given by their investment strategy (for details see Section 3.3.5). All the farms' expectations on how much the others are going to invest are set to zero, i.e. `expected_I_others = 0`, as well the variables that indicate past investment and gross past investment (`past_I` and `past_gross_I`) since we assume no prior history or information before time zero. The farms' initial consumption (`c`) is obtained by the farm's consumption function (for more detail see 3.3.7) and the farm's utilities are computed using the farm's utility function (for details see Section 3.3.8).

3.1.2 Initialization of the statistician agent.

Then one agent statistician is created for the simulation. Since the statistician agent computes aggregate variables, it can access the values of all farms. Then the statistician computes `Total_K` which is the sum of the agent's respective capitals. Technical progress is given by the parameter `eta0`. The state variables `GDP`, `Total_I` and `Total_C` (see Table 2) are initialized by summing the corresponding farm's variables. We initialize the growth rate variable `g_rate` by setting it equal to zero (since we assume no prior history or information before time zero). The variable for the capital-to-output ratio is initialized by the ratio between the total capital and the GDP.

3.1.3 Finalize Initialisation

In the last part of the initialization, the **model's update function** is called, this function records the variables of interest of the system.

3.2 Input data

The model does not use input data to represent time-varying processes.

3.3 Submodels

In this section we describe the submodels, these are methods either used by the **farms** or by the **statistician**.

3.3.1 Capital (Farm)

The farm's capital at the current time step $K_t^{(i)}$ is computed using the following recursive function:

$$K_t^{(i)} = (1-\delta)K_{t-1}^{(i)} + I_{t-1}, \tag{1}$$

where $K_{t-1}^{(i)}$ is the capital from the previous time step, δ is the depreciation rate (see Table 1), and $I_{t-1}^{(i)}$ is the investment from the previous time step. The new capital `K` is computed from these previous variables.

3.3.2 Technical Progress (Statistician)

Technical progress η grows proportionally to aggregate capital. We assume learning-by-doing which goes hand in hand with capital accumulation. The state variable `real_tech_prg` is calculated as a recursive function. It is the product of the previous technical progress η_{t-1} and the ratio of current total capital to previous total capital:

$$\eta_t = \frac{K_t}{K_{t-1}}\eta_{t-1}. \tag{2}$$

3.3.3 Production (Farm)

The farms produce an output using the Cobb-Douglas production function:

$$f\left(K_t^{(i)}\eta_t\right) = \left(K_t^{(i)}\right)^{\alpha}(\eta_t)^{1-\alpha}, \tag{3}$$

where $K_t^{(i)}$ is their current capital given by `K`, η_t is the system's current technical progress given by the statistician's variable `real_tech_prg`, and α is the output elasticity of the capital (see Table 1). Note that in this model we consider that farms always employ their constant labor amount of $L = 1$.

3.3.4 Formulation of expectations (Farm)

The farms state variable `expected_I_others` is updated depending on the farms' investment strategy. Farms that are *collaborative* or *ignorant* do not need expectations, hence the value of their is zero and needs no updating. When the farms are *witty*, they generate their expectations about how much the other farms invest following either an **adaptive rule** or a **trend follower rule**.

If the witty farm follows an **adaptive rule**, to compute her expectations on how much the other farms invest $I_t^{(\sim i),e(i)}$, she uses:

$$I_t^{(\sim i),e(i)} = (1-\lambda_A) I_{t-1}^{(\sim i),e(i)} + \lambda_A (I_{t-1} - I_{t-1}^{(i)}), \tag{4}$$

where $I_{t-1}^{(\sim i),e(i)}$ is the agent's last forecast, $I_{t-1}^{(i)}$ is the agent's last time investment, I_{t-1} is the system's past gross investment, and λ_A is the expectations weight factor (see Table 1).

If the witty farm uses a **trend following rule**, to compute her expectations on how much the other farms invest $I_t^{(\sim i),e(i)}$, the farm uses:

$$I_t^{(\sim i),e(i)} = I_{t-1} - I_{t-1}^{(i)} + \lambda_T (I_{t-1} - I_{t-2} - (I_{t-1}^{(i)} - I_{t-2}^{(i)})). \tag{5}$$

where $I_{t-1}^{(i)}, I_{t-2}^{(i)}$ correspond to the farm's previous investment and the farm's investment from two-time steps ago, I_{t-1}, I_{t-2} correspond to the system's previous gross investment and the gross investment from two-time steps ago, and λ_T is the extrapolation coefficient for the trend following rule.

3.3.5 Investment (Farm)

The investment functions are the farm's solution to the maximization problem of her intertemporal utility for two times steps, i.e.,

$$\max U = \max(\ln(C_t^{(i)}) + \rho \ln(C_{t+1}^{(i)}) \tag{6}$$

given:

$$C_t^{(i)} = P_t - I_t \tag{7}$$

$$K_{t+1}^{(i)} = (1-\delta) K_t + I_t \tag{8}$$

$$C_{t+1}^{(i)} = P_{t+1}. \tag{9}$$

Where ρ is the factor to discount future felicity, δ is the rate of capital depreciation, and the farm production P_t is given by a Cobb-Douglas production function see equation (3).

To solve this maximization problem, we consider different assumptions about the information available to the farms. This leads to different solutions and, consequently, different investment strategies. (for more detail see [1].

We say a farm is **collaborative** when the farm acts like a benevolent planner in a closed economy. Then, the investment of farm i in a decentralized economy, $I_t^{\mathrm{Co(i)}}$, at time t is

$$I_t^{\mathrm{Co(i)}} = \frac{1}{1+\rho} \left(\rho \left(K_t^{(i)} \right)^{\alpha} \eta_t^{1-\alpha} - (1-\delta) K_t^{(i)} \right). \tag{10}$$

When the farm is **ignorant**, she neglects her contribution to the aggregate capital stock, therefore she considers the technical progress of the future, η_{t+1}, as a given constant. Farm i's investment, $I_t^{\mathrm{Ig}(i)}$, is given by

$$I_t^{\mathrm{Ig}(i)} = \frac{1}{1+\alpha\rho} \left(\rho\alpha \left(K_t^{(i)} \right)^{\alpha} \eta_t^{1-\alpha} - (1-\delta) K_t^{(i)} \right). \tag{11}$$

The **witty** farm explicitly considers that the future technical progress η_{t+1} contains her investment. The solution to her maximization problem leads to a quadratic equation:

$$\begin{aligned} &-(1+\rho)\left(I_t^{(i)}\right)^2 + \left[\rho P_t^{(i)} - (2+\rho)\hat{K}_t^{(i)} - (1+\rho\alpha)\left(K_{t+1}^{(\sim i),e}\right)\right] I_t^{(i)} \\ &+ \rho P_t^{(i)} \hat{K}_t^{(i)} + \rho\alpha P_t^{(i)} K_{t+1}^{(\sim i),e} - \left(\hat{K}_t^{(i)}\right)^2 - \hat{K}_t^{(i)} K_{t+1}^{(\sim i),e} = 0 \end{aligned} \tag{12}$$

where $K_{t+1}^{(\sim i),e} = (1-\delta)\sum_{j\neq i} K_t^{(j)} + I_t^{(\sim i),e(i)}$ is the expected future capital of the others and $\hat{K}_t^{(i)}$ is the farm's depreciated capital. The solution of this quadratic equation provides two critical points. The farm chooses the solution that belongs to the feasible domain, i.e. $-(1-\delta)K_t^{(i)} < I_t^{\mathrm{Wi}(i)} < P_t^{(i)}$. Where $P_t^{(i)}$ corresponds to the farm's production (3).

3.3.6 GDP growth rate (Statistician)

The gross domestic product growth rate g_t is given by

$$g_t = \frac{GDP_t - GDP_{t-1}}{GDP_{t-1}}. \tag{13}$$

Where GDP_t is the sum of the farms' production and GDP_{t-1} is the system's previous time step gross domestic product.

3.3.7 Consumption (Farm)

Consumption $C_t^{(i)}$ of each farm is calculated as a difference between the farm's production and investment values

$$C_t^{(i)} = f(K_t^{(i)}, \eta_t L) - I_t^{(i)}$$

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3.3.8 Utility (Farm)

The farm computes her instantaneous utility $u(C_t^{(i)})$, given by

$$u(C_t^{(i)}) = \ln(C_t^{(i)}). \tag{14}$$

The farm's lifetime (or cumulative) utility $U(C_0^{(i)}, \dots, C_t^{(i)})$ is given by adding the farm's instantaneous utility to the previous value,

$$U(C_0^{(i)}, \dots, C_t^{(i)}) = \sum_{T=0}^{T=t-1} U(C_T^{(i)}) + u(C_t^{(i)}). \tag{15}$$

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