# CSE107 Exercise Set

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注意：Assessment 2 只考 week6—week12 内容

Attention!

Assessment 2 only contains what we learnt from week 6 to week 12

# CONTENTS

# Week 2 Number systems and Proof Techniques

第一部分主要包括 2 个内容

1. 各个数系（包括自然数，整数，有理数，素数，实数）
2. 几种常用的证明方法

（1）.找反例

（2）反证法

（3）数学归纳法

Tutorial1 部分

1. Give examples of natural numbers $x$ and $y$ such that $x - y$ is not a natural number.

Such an example shows that the natural numbers are not “closed under subtraction”.

Sol:

题目要求举例出自然数 x, y 使得 x-y 不是自然数。由自然数定义为大于等于 0 的整数得知秩序满足 y>x 即可。

$x = 4$, $y = 5$.

2. Give examples of integers $x$ and $y$ such that $x/y$ is not an integer. Such an example

shows that the integers are not “closed under division”.

Sol:只需举出一个反例证明两整数相除不是整数即可。

Let x=2， y=3

3. Consider an operation which takes numbers $x$ and $y$ and returns $x^2 - y$. Which of the following number systems are closed under this operation?
   - The natural numbers?
   - The positive integers?
   - The integers?
   - The rationals?

Sol：此题就是简单的找符合题给 operation 的数系。

1 No， Take x=2 ， y=5

2 No， Take x=2, y=5

3 Yes

4 Yes

4. Show that the rational numbers are closed under subtraction. That is, show that if $p$ and $q$ are rational numbers, then $q - p$ is a rational number.

Sol：根据题给条件，q 和 p 均为有理数，即可将其表示为分数形式，再相减通分即可得出证明。

Since $q$ and $p$ are rational, then can be written as $q = \frac{x}{y}$ and $p = \frac{w}{z}$ where $w, x, y, z$ are integers and $y \neq 0$ and $z \neq 0$. Then

$$q - p = \frac{x}{y} - \frac{w}{z} = \frac{xz}{yz} - \frac{wy}{yz} = \frac{xz - wy}{yz}.$$

Now note that $xz - wy$ is an integer (since the integers are closed under multiplication and subtraction) and $yz$ is an integer which is not 0 since $y$ and $z$ are not 0. Thus, $q - p$ is rational.

5. In the notes, we defined a positive integer to be "even" if it has 2 as a factor, and "odd" otherwise. Write down a list of all prime numbers that are even.

Sol:根据条件已知所有以 2 为因子的正整数称为偶数，其余正整数为奇数，而所有素数中有且仅有 2 为偶数。

The only prime number that is even is "2"

6. Show that if positive integers $x$ and $y$ are even, then so is $x + y$.
Sol：证明偶数相加依然为偶数，抓住偶数是基于 2 为因子该条件证明即可。
Since $x$ and $y$ are even positive integers, we can write $x = 2w$ and $y = 2z$ where $w$ and $z$ are positive integers. Thus $x + y = 2w + 2z = 2(w + x)$. So $x + y$ has 2 as a factor, so it is even.

7. Show that if a positive integer $x > 2$ is even, then $x - 2$ is also an even positive integer.

Sol：证明一个大于 2 的偶数减去 2 之后仍为偶数，属简单证明。

We can write $x = 2w$ where $w$ is a positive integer. Since $x > 2$, we know $w > 1$. Then $x - 2 = 2w - 2 = 2(w - 1)$, which is a positive integer with 2 as a factor, so it is even.

8. Prove by induction that, for all $n \in Z_+$, $\sum_{i=1}^{n}(-1)^i i^2 = (-1)^n n(n+1)/2$ .

$$\sum_{i=1}^{n}(-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$$

Sol:此题目的证明要用到数学归纳法，过程并不复杂，步骤格式要熟悉。尤其抓住利用 n=k 时等式成立证明 n=k+1 时也成立该条思路。

Proof:

(*) $$\sum_{i=1}^{n}(-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$$

Base case: When $n = 1$, the left side of (*) is $(-1)1^2 = -1$, and the right side is $\frac{(-1)1(1+1)}{2} = -1$, so both sides are equal and (*) is true for $n = 1$.

Induction step: Let $k \in \mathbb{Z}_+$ be given and suppose (*) is true for $n = k$. Then

$$\sum_{i=1}^{k+1}(-1)^i i^2 = \sum_{i=1}^{k}(-1)^i i^2 + (-1)^{k+1}(k+1)^2$$

$$= \frac{(-1)^k k(k+1)}{2} + (-1)^{k+1}(k+1)^2$$

$$= \frac{(-1)^k (k+1)}{2}\left(k - 2(k+1)\right)$$

$$= \frac{(-1)^k (k+1)}{2}(-k-2)$$

$$= \frac{(-1)^{k+1}(k+2)}{2}$$

Thus, (*) holds for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: By the principle of induction, (*) is true for all $n \in \mathbb{Z}_+$

连等式中，第一行到第二行运用的是 n=k 时成立的条件，将一个求和用已知式子表示，后几行的目的都是为了将式子的形式通过运算凑配为相同格式。

9. Prove that for any real number $x > -1$ and any positive integer $n$, $(1+x)^n \geq 1 + nx$.

Sol：使用数学归纳法。

Let $x$ be a real number in the range given, namely $x > -1$. We will prove by induction that for any positive integer $n$,

(*) $$(1+x)^n \geq 1+nx.$$

holds for any $n \in \mathbb{Z}_+$.

Base case: For $n = 1$, the left and right sides of (*) are both $1 + x$, so (*) holds.

Induction step: Let $k \in \mathbb{Z}_+$. be given and suppose (*) is true for $n = k$. We have

$$(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x) = 1+(k+1)x+kx^2$$
$$\geq 1+(k+1)x$$

Hence (*) holds for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: By the principle of induction, it follows that (*) holds for all $n \in \mathbb{Z}_+$.

10. Please use the Proof by Contradiction to show that the square root of 3 is irrational.

Sol:根据题目要求要使用反证法，因而假设$\sqrt{3}$是有理数，通过找出矛盾推翻这一观念从而证明。

Proof:

Suppose $\sqrt{3}$ is rational, then $\sqrt{3} = a/b$ for some (a,b) suppose we have a/b in simplest form.

$$\sqrt{3} = a/b$$
$$a^2 = 3b^2$$

If b is even, then a is also even in which case a/b is not simplest form.

If b is odd then a is also odd. Therefore:

$$a = 2n + 1$$
$$b = 2m + 1$$

$$(2n+1)^2 = 3(2m+1)^2$$

$$4n^2 + 4n + 1 = 12m^2 + 12m + 3$$

$$2n^2 + 2n = 6m^2 + 6m + 1$$

$$2(n^2 + n) = 2(3m^2 + 3m) + 1$$

Since $(n^2 + n)$ is an integer, the left hand side is even. Since $(3m^2 + 3m)$ is an integer, the right hand side is odd and we have found a contradiction. Therefore our hypothesis is false.

**有理数表示为分数的形式是常用手段。**

## Week 3　Set Theory

# Contents

- Notation for *sets*.
- Important sets.
- What is a *subset* of a set?
- When are two sets *equal*?
- *Operations* on sets.
- *Algebra* of sets.
- *Cardinality* of sets.
- The *cartesian product* of sets.
- Bit strings.

第二部分主要介绍了各种集合的定义概念以及如何运算。

1. Notation（集合）

## Notation

For a large set, especially an infinite set, we cannot write down all the elements. We use a predicate $P$ instead.

$$S = \{x \mid P(x)\}$$

denotes the set of objects $x$ for which the predicate $P(x)$ is true.

**Examples**: Let $S = \{1, 3, 5, 7, \ldots\}$. Then

$$S = \{x \mid \ x \text{ is an odd positive integer}\}$$

and

$$S = \{2n - 1 \mid \ n \text{ is a positive integer }\}.$$

2. Important Set（几个重要集合）

The empty set has no elements. It is written as $\emptyset$ or as $\{\}$.

We have seen some other examples of sets in Part 1.

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (the natural numbers)
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (the integers)
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ (the positive integers)
- $\mathbb{Q} = \{x/y \mid x \in \mathbb{Z}, y \in \mathbb{Z}, y \neq 0\}$ (the rationals)
- $\mathbb{R}$: (real numbers)

3. Subsets（子集）

**Definition** A set $B$ is called a *subset* of a set $A$ if every element of $B$ is an element of $A$. This is denoted by $B \subseteq A$.

4. Equality(相等集合)

**Definition** A set $A$ is called *equal* to a set $B$ if $A \subseteq B$ and $B \subseteq A$. This is denoted by $A = B$.

**Examples**:

$$\{1\} = \{1, 1, 1\},$$

$$\{1, 2\} = \{2, 1\},$$

$$\{5, 4, 4, 3, 5\} = \{3, 4, 5\}.$$

5. The union of two sets（两集合的并集）

**Definition** The union of two sets $A$ and $B$ is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

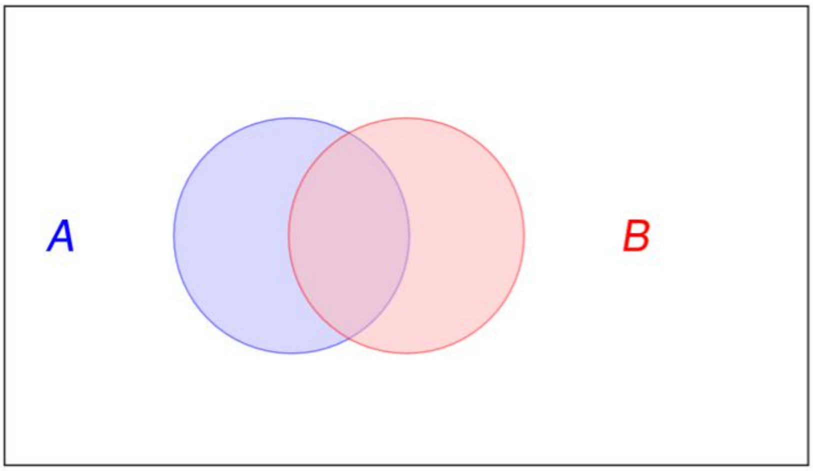



Figure: Venn diagram of $A \cup B$.

6. The intersection of two sets（两集合的交集）

**Definition** The intersection of two sets $A$ and $B$ is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

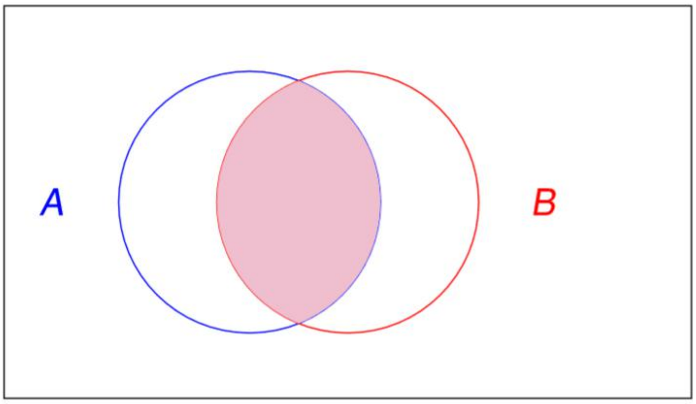



Figure: Venn diagram of $A \cap B$.

7. The relative complement（相对补）

**Definition** The relative complement of a set $B$ relative to a set $A$ is the set

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

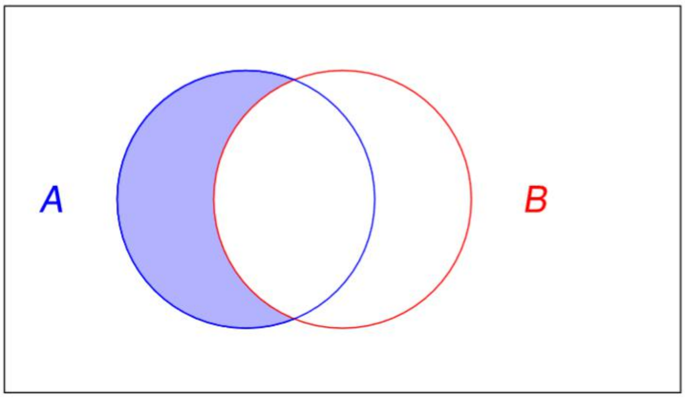



Figure: Venn diagram of $A - B$.

8. The complement（补集）

When we are dealing with subsets of some large set $U$, then we call $U$ the *universal set* for the problem in question.

**Definition** The complement of a set $A$ is the set

$$\sim A = \{x \mid x \notin A\} = U - A.$$

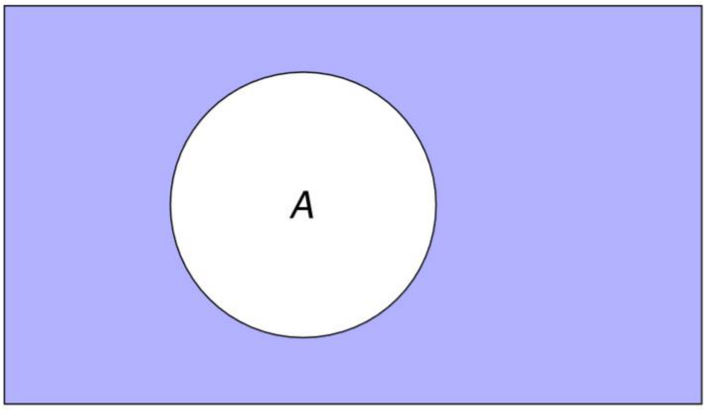



Figure: Venn diagram of $\sim A$.

9. The symmetric difference（对称差异，即两集合的并集除去两集合的交集）

**Definition** The symmetric difference of two sets $A$ and $B$ is the set

$$A \Delta B = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B)\}.$$

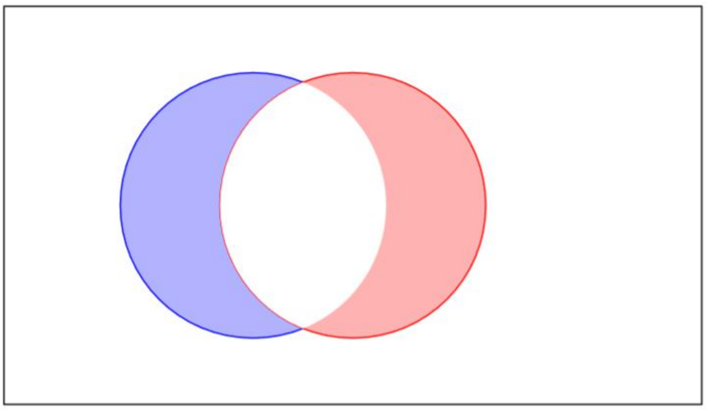


Figure: Venn diagram of $A \Delta B$.

10.The algebra of sets （集合的代数运算）

11. The Power Set（所有子集组成的集合）

**Definition** The power set $Pow(A)$ of a set $A$ is the set of all subsets of $A$. In other words,

$$Pow(A) = \{C \mid C \subseteq A\}.$$

**Example:**

Let $A = \{1, 2, 3\}$. Then

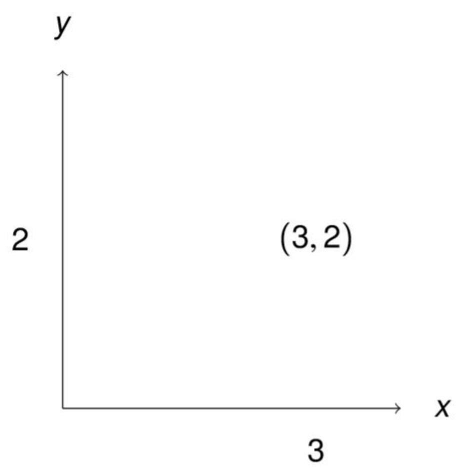
$$Pow(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}.$$

12．Cardinality of sets（集合的基数）

**Definition** The cardinality of a *finite* set $S$ is the number of elements in $S$, and is denoted by $|S|$.

13.Cartesian Plane

The set $\mathbb{R} \times \mathbb{R}$, or $\mathbb{R}^2$ as it is often written, consists of all pairs of real numbers $(x, y)$. $\mathbb{R}^2$ is called the *Cartesian plane*.




14.Bit strings of length n

Bit strings can be used to represent the subsets of a set.

Suppose we have a set $S = \{s_1, \ldots, s_n\}$.

(I have given the $n$ elements of the set names. $s_1$ is the name of the first element. $s_2$ is the name of the second element (which is different from the first element) and so on.)

If we have a subset $A \subseteq S$, the characteristic vector of $A$ is the $n$-bit string $(b_1, \ldots, b_n) \in B^n$ where

$$b_i = \begin{cases} 1 & \text{if} \quad s_i \in A \\ 0 & \text{if} \quad s_i \notin A \end{cases}$$

下面给出一个典例

Let $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$ and $B = \{3, 4\}$.

- The characteristic vector of $A$ is $(1, 0, 1, 0, 1)$.
- The characteristic vector of $B$ is $(0, 0, 1, 1, 0)$.
- The characteristic vector of $A \cap B$ is $(0, 0, 1, 0, 0)$.
- The characteristic vector of $A \cup B$ is $(1, 0, 1, 1, 1)$.

Tutorial 2

1. If $X$ and $Y$ are empty sets, then $X = Y$. If you think that it is true, prove it. If not, explain why.

   Sol：利用反证法找矛盾，易得。

   This is true. Assume that X and Y are not equal, then we should be able to find an element from set X which does not belong to Y, but X contains no element. Similarly, we cannot find an element in Y that does not belong to X (again Y contains no elements at all).

2. Let $X$,$Y$ and Z be sets. If $X \subset Y$ and $Y \subset Z$, then $X \subset Z$. If you think that it is true, prove it. If not, explain why.

   Sol： This is true. Let x ∈ X. Since X ⊂ Y and x ∈ X, we can conclude that x ∈ Y. Since Y ⊂ Z and x ∈ Y, we can conclude that x ∈ Z and therefore X ⊂ Z.

3. Let $X,Y$ and $Z$ be sets. Is this expression $(X - Y) \cup (Y - Z) = X - Z$ true? If your answer is yes, then prove it. Otherwise, give a counterexample.

Sol: No. A counterexample can be X={1,2,4,5}, Y={2,3,5,6}, Z={4,5,6,7}

4. Prove that the following sets are equal:

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
3. $A \times (B - C) = (A \times B) - (A \times C)$

Sol:此题运用集合的代数运算即可得出证明。

- Let (a,x) ∈ A x (B ∪ C). This means that a ∈ A and x ∈ B ∪ C. It logically follows by the distributive laws that a ∈ A and x ∈ B, or a ∈ A and x ∈ C. This means that (a,x) ∈ (AxB) ∪ (AxC). Thus every element of A x (B ∪ C) is an element of (AxB) ∪ (AxC). Therefore, A x (B ∪ C) ⊆ (A x B) ∪ (A x C).
- For the reverse containment, let (a,x) ∈ (A x B) ∪ (A x C). This means that either a ∈ A and x ∈ B, or a ∈ A and x ∈ C. It follows that a ∈ A, and either x ∈ B or x ∈ C. Thus, (a,x) ∈ A x (B ∪ C). So (A x B) ∪ (A x C) ⊆ A x (B ∪ C). Since both containments hold it follows that the sets in question are equal.
- The proof of the other ones are similar.

5.Let A and B be two sets. Prove the following statements: A ⊆ B if and only if ~ B ⊆ ~A.

Sol: 此题关键词为 if and only if，这就等价于是证明充分必要条件。因此在证明的时候既要考虑充分性也要考虑必要性。两个都需证明。

<u>If A ⊆ B then ~ B ⊆ ~A:</u>

Assume A ⊆ B . This means that if x ∈ A, then x ∈ B. Let x∈ ~B. This means that x ∉ B, Then, it follows that x ∉ A, which means that x∈ ~A. Hence, we have shown ~ B ⊆ ~A.

<u>If ~B ⊆ ~A then A ⊆ B:</u>

Apply the forward direction with ~B and ~A in place of A and B, respectively, and use the fact that ~ (~C ) = C.

# Week 4 Relations

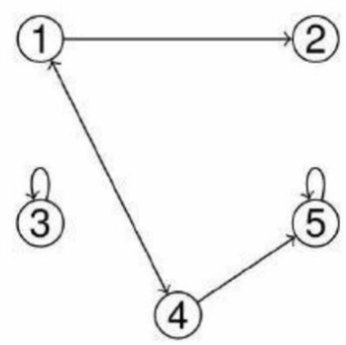
## Contents:

Representation of binary relations by directed graphs
Representation of binary relations by matrices
Unary relations
Properties of binary relations
Transitive closure
Equivalence relations and partitions
Partial orders and total orders

## In Class Exercises:

### (1)

(a) Which relation is represented by the following digraph?




**Analysis:**

Refer to *Digraphs of binary relations on a single set* In PPT(Relations)
It is straightforward

**Answer:**

{(1,2),(3,3),(5,5,),(1,4),(4,1),(4,5)}

### (2)

(b) Let $A = \{a, b, c, d\}$. What is the relation $R$ on $A$ represented by the below matrix?

$$M = \begin{bmatrix} F & F & T & T \\ T & F & T & T \\ T & F & T & F \\ T & F & T & F \end{bmatrix}$$

**Analysis:**

Refer to *Representation of binary relations: matrices* in PPT(Relations)

$$M(i,j) = \begin{cases} T & \text{if} \quad (a_i, b_j) \in R \\ F & \text{if} \quad (a_i, b_j) \notin R \end{cases}$$

**Answer:**

{( a,c), (a,d), (b,a), (b,c), (b,d), (c,a), (c,c), (d,a), (d,c) }

## (3)

*Let $R$ be the relation on the set $\mathbb{R}$ real numbers defined by $xRy$ iff $x - y$ is an integer. Prove that $R$ is an equivalence relation on $\mathbb{R}$.*

**Analysis:**

Refer to *Properties of binary relations* in PPT(relations)

To prove an equivalence relation, we need to prove it is reflexive, transitive and symmetric.

**Answer:**

PROOF.

I. Reflexive: Suppose $x \in \mathbb{R}$. Then $x - x = 0$, which is an integer. Thus, $xRx$.

II. Symmetric: Suppose $x, y \in \mathbb{R}$ and $xRy$. Then $x - y$ is an integer. Since $y - x = -(x - y)$, $y - x$ is also an integer. Thus, $yRx$.

III. Suppose $x, y \in \mathbb{R}$, $xRy$ and $yRz$. Then $x - y$ and $y - z$ are integers. Thus, the sum $(x - y) + (y - z) = x - z$ is also an integer, and so $xRz$.

Thus, $R$ is an equivalence relation on $\mathbb{R}$. □

# Tutorial:

## (4)

List the set of ordered pairs and draw the graphical representation of the relation $R$ between $\{1,2,3,4\}$ and $\{a, b, c\}$ with the matrix:

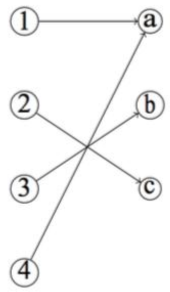
$$M = \begin{bmatrix} T & F & F \\ F & F & T \\ F & T & F \\ T & F & F \end{bmatrix}$$

**Analysis:**

Refer to *Representation of binary relations: matrices And Representation of binary* *relations: directed graphs* in PPT(Relations)

**Answer:**

- The list of ordered pairs is: {(1, a), (2, c), (3, b), (4, a)}.
- The digraph representation is:



## (5)

Let $R$ be the relation on $\{1,2,3,4\}$ given by $xRy$ if and only if $\mathrm{x}-\mathrm{y}=0$. Represent $R$ in the following ways:

- as a set of ordered pairs;
- in graphical form;
- in matrix form.

**Analysis:**

Refer to *Representation of binary relations: matrices And Digraphs of binary relations on a single set* in PPT(Relations)

**Answer:**

As a set of ordered pairs: {(1, 1), (2, 2), (3, 3), (4, 4)};

As a digraph:

As a matrix:

$$M=\begin{bmatrix} T & F & F & F \\ F & T & F & F \\ F & F & T & F \\ F & F & F & T \end{bmatrix}$$

## (6)

Determine which of the following relations on the set of people is reflexive, symmetric, or transitive:

(a) 'has the same parents as'
(b) 'is a brother of'
(c) 'is at least as clever as'.

**Analysis:**

Refer to *Properties of binary relations* in PPT(Relations)

**Answer:**

(a) reflexive, symmetric and transitive.
(b) transitive, not reflexive, not symmetric.
(c) reflexive and transitive but not symmetric.

## (7)

For each of the following relations on

$$A = \{a \mid a \in \mathbb{Z} \text{ and } 1 \leq a \leq 12\}$$

list the ordered pairs belonging to:

- $S_1 = \{(a, b) \in A \times A \mid a \cdot b = 9\}$
- $S_2 = \{(a, b) \in A \times A \mid 2a = 3b\}$

**Analysis:**

Refer to *binary relation* in PPT(Relations)

**Answer:**

S1 = {(1, 9), (3, 3), (9, 1)};
S2 = {(3, 2), (6, 4), (9, 6), (12, 8)};

## (8)

Is there a mistake in the following proof that any transitive and symmetric relation $R$ is reflexive? If so, what is it?
Let $aRb$. By symmetry, $bRa$. By transitivity, if $aRb$ and $bRa$, then $aRa$. This proves reflexivity.

**Analysis:**

Refer to *Properties of binary relations* in PPT(Relations)

Use a counterexample to address this problem

**Answer:**

Yes, there is a mistake. The following is a counterexample to the above proof: Let A = {1, 2, 3} and let R = {(1, 1), (1, 2), (2, 1), (2, 2)}. Clearly, R in the counterexample is symmetric and transitive but it is not reflexive ((3, 3) ∉R).

## (9)

For each of the following equivalence relations $R$ on a given set $A$, describe the equivalence classes $E_x$ into which the relation partitions the set $A$:

(a) $A$ is the set of books in a library; $R$ is given by $xRy$ if and only if the colour of $x's$ cover is the same as the colour of $y's$ cover.
(b) $A = \mathbb{Z}$; $R$ is given by $xRy$ if and only if $x - y$ is even.
(c) $A$ is the set of people; $R$ is given by $xRy$ if and only if $x$ has the same sex as $y$.

**Analysis:**

Refer to *Properties of binary relations* in PPT(Relations)

**Answer:**

(a) Each equivalence class consists of all those books of a fixed colour.
(b) There are two equivalence classes, the set of odd integers and the set of even integers.
(c) There are two equivalence classes, the set of females and the set of males.

## (10)

Define an equivalence relation $R$ on $\mathbb{N}$ as follows: $xRy$ if and only if 3 is a divisor of $x - y$.
Determine the equivalence classes

- $E_0$
- $E_1$
- $E_3$

**Analysis:**

Refer to *Equivalence Relations* in PPT(Relations)

**Definition** The *equivalence class* $E_x$ of any $x \in A$ is defined by

$$E_x = \{y \mid yRx\}.$$

**Answer:**

The equivalence classes are:
(a) E0 = {0, 3, 6, 9, 12, . . .};
(b) E1 = {1, 4, 7, 10, 13, . . .};
(c) E3 = E0.

## (11)

Let $A = \{a \in \mathbb{Z} \mid a \neq 0\}$ and $R = \{(a, b) \mid$ a and b have the same parity$\}$. Show that $R$ over $A$ is an equivalence notion.

**Analysis:**

Refer to *Properties of binary relations* in PPT(Relations)

Prove it is reflexivity, symmetry and transitivity

**Answer:**

The parity relation is an equivalence relation because it preserves the properties of Reflexivity, Symmetry and Transitivity.

- Reflexivity: For any $x \in A$, $x$ has the same parity as itself, so $(x, x) \in R$.
- Symmetry: If $(x, y) \in R$, $x$ and $y$ have the same parity, so $(y, x) \in R$.
- Transitivity: If $(x, y) \in R$, and $(y, z) \in R$, then $x$ and $z$ have the same parity as $y$, so they have the same parity as each other (if $y$ is odd, both $x$ and $z$ are odd; if $y$ is even, both $x$ and $z$ are even), thus $(x, z) \in R$.

## (12)

A relation $R$ is defined on the integers by $aRb\ if\ a^2 - b^2 \leq 3$. Show that $R$ is reflexive, but not symmetric and not transitive.

**Analysis:**

Refer to *Properties of binary relations* in PPT(Relations)

Use a counterexample to address this problem

**Answer:**

It is reflexive since $a^2 - a^2 = 0 \leq 3$. Through counter-examples, it is not hard to see that: it is not symmetric since $0^2 - 10^2 \leq 3$ but $10^2 - 0^2 > 3$. Also, it is not transitive since $2^2 - 1^2 \leq 3$ and $1^2 - 0^2 \leq 3$ but $2^2 - 0^2 > 3$.

## (13)

Let $R$ be a binary relation from $A$ to $B$ and $C, D \subseteq A$. If $C \subseteq D$, then $R(C) \subseteq R(D)$. If you think that it is true, prove it. If not, give a counter-example.

**Analysis:**

Refer to *Binary Relation* in PPT(Relations)

**Answer:**

This is true. For any $y \in R(C)$, there exists an $a \in C$ such that $(a, y) \in R$. Clearly, $a \in D$ and then $y \in R(D)$.

# Week 5 Relations(bis)

## Content:

Properties of relations

Relational closures

Symmetric closure

Partial orderings

Hasse Diagram

Totally ordered

## In Class Exercises:

### (1)

Let $A = \{1,2,3,4\}$. What is the transitive closure of the relation: $\{(1, 2), (2, 3), (3, 1), (3, 4)\}$ on $A$?

**Analysis:**

Refer to *Transitive Closure* in PPT(Relations(bis))

<u>Example of Transitive Closure:</u>

Let $S = \{1, 2, 3\}$.
$R = \{(1,1), (1,2), (1,3), (2,3), (3,1)\}$.

$(2,3) \in R \wedge (3,1) \in R \rightarrow (2,1) \in R^t$
$(3,1) \in R \wedge (1,2) \in R \rightarrow (3,2) \in R^t$
$(3,1) \in R \wedge (1,3) \in R \rightarrow (3,3) \in R^t$
$(2,1) \in R^t \wedge (1,2) \in R \rightarrow (2,2) \in R^t$ (*Must be done iteratively)

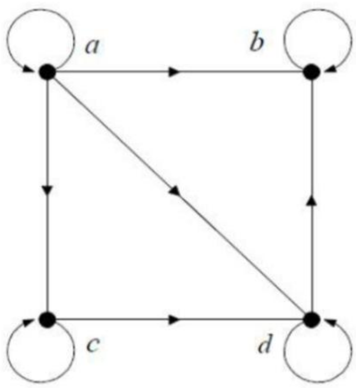
So, $R^t = R \cup \{(2,1), (3,2), (3,3), (2,2)\}$

**Answer:**

{(1,2),(2,3),(3,1),(3,4),(1,1),(2,2),(3,3),(1,3),(1,4),(2,4),(2,1),(3,2)}

## (2)

Prove that the relation with the directed graph below is a partial order.




**Analysis:**

Refer to *Partial orderings* in PPT(Relations(bis)

**Answer:**

Since the relation is not transitive, it Is neither partial order. An counterexample can be c→d and d→b, but not c→b.

# Tutorial:

## (1)

Show that the relation "is a divisor of" on the set $\mathbb{Z}^+$ of positive integers is a partial order.

**Analysis:**

Refer to *Partial orderings* in PPT(Relations(bis)

We need to prove the relation is reflexive, transitive and antisymmetric.

**Answer:**

For $x, y, z \in \mathbb{Z}$

- The relation is reflexive: x is a divisor of x.
- The relation is transitive: If x is a divisor of y and y is a divisor of z then x is a divisor of z.
- The relation is antisymmetric: If x is a divisor of y and y is a divisor of x then x = y.

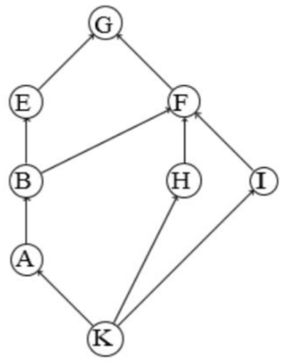
## (2)

**2:** Suppose that A, B, E, F, G, H, I and K represent people on twitter. We assume that everybody follows himself on twitter. Also, everybody follows G. Everybody follows E except H, I, F and G. Everybody follows F except E and G. Also, A follows B and K follows everybody. It turns out that the relation "follows" is a partial order in this case. Draw the Hasse Diagram of this partial order. Construct a pair of people, such that, if the first of them started following the second, then "follows" would no longer be a partial order.

**Analysis:**

Refer to *Hasse Diagram* in PPT(Relations(bis))

**Answer:**




## (3)

**3**: Let $M = \{a, b, c\}$. What is the transitive closure of the relation $\{(a, a), (a, b), (a, c), (c, a), (b, c)\}$ on M?

**Analysis:**

Refer to *Transitive Closure* in PPT(Relations(bis))

**Answer:**

{(a,a),(a,b),(a,c),(c,a),(b,c),(c,b),(c,c),(b,a),(b,b)}

## (4)

**4:** Define the relation on ℝ x ℝ by (a,b) R (x,y) iff a ≤ x and b ≤ y. Prove that R is a partial order on ℝ x ℝ.

**Analysis:**

Refer to *Partial orderings* in PPT(Relations(bis)

A partial ordering (or partial order) is a relation that is reflexive, antisymmetric, and transitive. Thus, we need to prove it is reflexive, antisymmetric, and transitive.

**Answer:**

**Solution:** For all (a,b) ∈ ℝ x ℝ, we have (a,b) R (a,b) since a ≤ a and b ≤ b. So, R is reflexive. If (a,b) R (x,y) and (x,y) R (a,b) then a ≤ x and b ≤ y, as well as, x ≤ a and y ≤ b. Since, a ≤ x and x ≤ a we have a = x. Similarly, b = y and we have (a,b) = (x,y). Thus, R is antisymmetric. If (a,b) R (x,y) and (x,y) R (w,z), then a ≤ x and x ≤ w, so a ≤ w. Similarly, b ≤ y and y ≤ z, so b ≤ z. Thus, (a,b) R (w,z) and R is transitive. Therefore, R is a partial order.

## (5)

**5:** Given a partially ordered set, is a pair $\mathbf{P} = (X, \leq)$, where $X$ is a nonempty set and $\leq$ is a partial order on $X$; that is, for $x, y$, and $z \in X$

(1) $x \leq x$ (reflexivity)

(2) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry)

(3) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity)

Prove that: If $(X, \leq)$ is a finite partially ordered set, then $X$ has a maximal and a minimal element.

**Analysis:**

Refer to *Partial orderings* in PPT(Relations(bis)

**Answer:**

**Solution:** For $x, y \in X$, $x \in X$ is maximal if $x \leq y$ implies $x = y$. Pick $x_1 \in X$. If $x_1$ is not maximal, there is another element $x_2 \in X$ with $x_1 < x_2$. Continuing in this way for $r$ steps, we get $x_1 < x_2 < \cdots < x_r$. Since $<$ is transitive, $x_i < x_j$ whenever $i < j$. In particular they are distinct. Since $X$ is finite this process must stop and the last element is a maximal element. The proof to show that $X$ has a minimal element is similar.

## (6)

**6:** Prove that the divides relation on $A = \{1,2,4,8,\ldots,2^n\}$ is a total order; where n is a nonnegative integer.

**Analysis:**

Refer to *Totally ordered* in PPT(bis)

We need to prove the relation is reflexivity, transitivity, antisymmetry, aRb and bRa.

**Answer:**

It is not hard to see that "divides relation on A" is a partial order.

**Reflexivity:** Trivial.

**Transitivity:** If $(a, b)$ is an element of the divides relation then we know that $\boldsymbol{a}$ divides $\boldsymbol{b}$; similarly if $(b, c)$ is and element of the divides relation, then we know that $\boldsymbol{b}$ divides $\boldsymbol{c}$. Transitivity holds because for elements $(a, b), (b, c)$ ; $(a, c)$ will be also part of the relation and the divides property will still hold.

**Antisymmetry:** If $(a, b)$ is a member of the divides relation, then $(b, a)$ would be a member only if $a = b$.

**Then we have to show that $\boldsymbol{aRb}$ or $\boldsymbol{bRa}$:**
Let $a$ and $b$, be particular but arbitrarily chosen elements of A. By definition of A, there are nonnegative integer $r$ and $s$ such that $a = 2^r$ and $b = 2^s$ (Since $r, s$ are the exponents, nonnegative integers). Now either $r < s$ or $s < r$;
If $r < s$, then $b = 2^s = 2^r 2^{s-r} = a2^{s-r}$ where $s - r > 0$. It follows, by definition of divisibility, that $a$ divides $b$. For $s < r$, the proof is similar.

# Week 6 Functions

## Content:

Inverse relations
Compositions of relations
Functions: definitions and examples
Domain, codomain, and range
Injective, surjective, and bijective functions
Invertible functions
Compositions of functions
Pigeonhole principle

## In Class Exercises:

### (1)

(a) Let $A = \{7,2,1,9\}$ and $R \subseteq A \times A$ be given by $R = \{(7,7), (2,1), (1,9)\}$. Determine the inverse relation $R^{-1}$ of $R$.

**Analysis:**

Refer to *Inverse functions* in PPT(Functions)

**Answer:**

{(7,7),(1,2),(9,1)}

### (2)

(b) $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \frac{5}{7+x}$. Find the inverse of $f(x)$.

**Analysis:**

Refer to *Inverse functions* in PPT(Functions)

**Answer:**

The inverse of *f(x)* does not exist as *f(x)* breaks at x=-7 (i.e. 5/-7+7 = 5/0)

**(3)**

(c) $f: \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = 4x^2 - 1$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ is given by $g(x) = x + 3$.

**Analysis:**

Refer to *Composition of Relations* in PPT(Functions)

**Answer:**

$$(f \circ g)(x) = 4x^2 + 24x + 35$$
$$(g \circ f)(x) = 4x^2 + 2$$
$$(f \circ f)(x) = 64x^4 - 32x^2 + 3$$
$$(g \circ g)(x) = x + 6$$

## Tutorial:

**(1)**

Q1. The relation R on A = {1,2,3, 4, 5} is represented by the matrix

$$M = \begin{bmatrix} F & F & T & T & F \\ F & T & F & F & T \\ F & F & F & T & F \\ F & F & T & F & F \\ F & T & F & F & F \end{bmatrix}$$

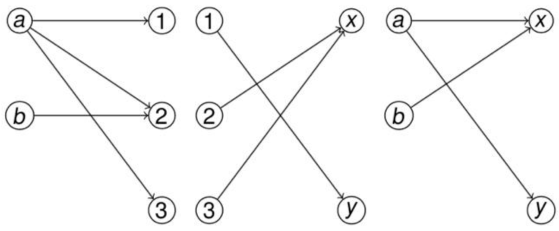
Determine the matrix representing R o R.

**Analysis:**

Refer to *Matrix representation of compositions* in PPT(Functions)

Example:

Now let's go back and see how this works for matrices representing relations




$R : \begin{bmatrix} T & T & T \\ F & T & F \end{bmatrix}$ $S : \begin{bmatrix} F & T \\ T & F \\ T & F \end{bmatrix}$ $S \circ R : \begin{bmatrix} T & T \\ T & F \end{bmatrix}$

**Answer:**

$R \circ R$ is represented by

$$M = \begin{bmatrix} F & F & T & T & F \\ F & T & F & F & T \\ F & F & T & F & F \\ F & F & F & T & F \\ F & T & F & F & T \end{bmatrix}$$

## (2)

Let A = {0, 2, 4,6} and B = {1, 3, 5, 7}. Determine which of the following relations between A and B forms a function with domain A and codomain B:

(a) {(6, 3), (2, 1), (0, 3), (4, 5)},

(b) {(2, 3), (4, 7), (0, 1), (6, 5)},

(c) {(2, 1), (4, 5), (6, 3)},

(d) {(6, 1), (0, 3), (4, 1), (0, 7), (2, 5)}.

For those which are functions, which are injective and which are surjective?

**Analysis:**

Refer to *Injective (one to one) functions and Surjective (or onto) functions* in PPT(Functions)

**Definition** $f : A \to B$ is *surjective* (or onto) if the range of $f$ coincides with the codomain of $f$. This means that for every $b \in B$ there exists $a \in A$ with $b = f(a)$.

**Definition** Let $f : A \to B$ be a function. We call $f$ an *injective* (or one-to-one) function if

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \text{ for all } a_1, a_2 \in A.$$

**Answer:**

(a) a function, but neither surjective nor injective

(b) a function, surjective and injective

(c) not a function since no element of B is associated with 0 in A,

(d) not a function since 3 and 7 are associated with 0.

## (3)

Which of the following functions are injective? Which are surjective?

*a)* $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2 + 1$.

*b)* $g : \mathbb{N} \to \mathbb{N}$ given by $g(x) = 2^x$.

*c)* $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = 5x - 1$

**Analysis:**

Refer to *Injective (one to one) functions and Surjective (or onto) functions* in PPT(Functions)

**Answer:**

(a) $f(-1) = f(1)$, therefore $f$ is not injective. $f(x) = 3$ has no solution in $\mathbf{Z}$. Therefore $f$ is not surjective.

(b) If $g(a) = g(b)$, then $2^a = 2^b$ and $a = b$. Hence, $g$ is injective. $g(x) = 3$ has no solution in $\mathbf{N}$. Hence, $g$ is not surjective.

(c) $h$ is surjective. $h(b) = h(a)$ implies $a = b$. Hence, $h$ is injective.

## (4)

The function $f : A \to B$ is given by $f(x) = 1 + \frac{2}{x}$ where $A$ denotes the set of real numbers excluding 0 and $B$ denotes the set of real numbers excluding 1. Show that $f$ is bijective and determine the inverse function.

**Analysis:**

Refer to *Injective (one to one) functions and Surjective (or onto) functions and Inverse* *functions* in PPT(Functions)

**Answer:**

If $f(a) = f(b)$ then $1+\frac{2}{a} = 1+\frac{2}{b}$ and so $a = b$. Hence, $f$ is injective. If $f(x) = y$ then $1 + \frac{2}{x} = y$ and so $x = \frac{2}{(y-1)}$. Hence, $f$ is surjective. Therefore, $f$ is bijective. The inverse function is given by $f^{-1} : B \to A$ where $f^{-1}(x) = \frac{2}{(x-1)}$.

## (5)

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x$ and the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x + 9$.

Calculate $g \circ f$, $f \circ g$, $f \circ f$ and $g \circ g$.

**Analysis:**

Refer to *Composition of Relations* in PPT(Functions)

**Answer:**

$$g \circ f(x) = g(f(x)) = g(3x) = 3x + 9$$
$$f \circ g(x) = f(g(x)) = f(x + 9) = 3x + 27$$
$$f \circ f(x) = f(f(x)) = f(3x) = 9x$$
$$g \circ g(x) = g(g(x)) = g(x + 9) = x + 18$$

## (6)

Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$. Show that:

a) if $f$ and $g$ are surjective, then $fg$ are surjective.

b) if $f$ and $g$ are injective, then $f+g$ are injective.

**Analysis:**

Refer to *Injective (one to one) functions and Surjective (or onto) functions* in PPT(Functions)

Use a counterexample

**Answer:**

Counter example:

$f(x) = x$ which is surjective and injective

$g(x) = -x$ which is surjective and injective

$fg(x) = -x^2$ *which is not surjective.*

$(f + g)(x) = 0$ which is not injective.

## (7)

For some domains and codomains, show the composition:

a) is commutative (e.g. $(g \circ f) = (f \circ g)$);

b) is associative (e.g. $(h \circ g) \circ f = h \circ (g \circ f)$).

**Analysis:**

Refer to *Composition of Relations* in PPT(Functions)

Use a counterexample for a

**Answer:**

**Q7.a**

Counter example : $f(x) = x^2$ and $g(x) = x + 1$.

$$g(f(x)) = x^2 + 1$$

$$f(g(x)) = (x + 1)^2 = x^2 + 2x + 1$$

**Q7.b**

Let a from A.

$$\begin{aligned}[(h \circ g) \circ f](a) &= (h \circ g)\big(f(a)\big) \\ &= h\Big(g\big(f(a)\big)\Big) \\ &= h(g \circ f(a)) \\ &= [h \circ (g \circ f)](a)\end{aligned}$$

## (8)

Let $A$, $B$ and $C$ be sets and let $f : A \to B$ and $g : B \to C$ be functions. Prove that:

a) if $f$ and $g$ are injective, then $g \circ f$ is also injective.

b) if $f$ and $g$ are surjective, then $g \circ f$ is also surjective.

c) if $g \circ f$ is injective, then $f$ is injective.

d) if $g \circ f$ is surjective, then $g$ is surjective.

**Analysis:**

Refer to *Composition of Relations and Injective (one to one) functions and Surjective (or onto) functions* in PPT(Functions)

**Answer:**

a) if $f$ and $g$ are injective, then $g \circ f$ is also injective.

We must show that for all $a_1, a_2 \in A$, if $g(f(a_1))= g(f(a_2))$, then $a_1 = a_2$.

Assume we have $g(f(a_1))= g(f(a_2))$, Let $f(a_1) = b_1$ and $f(a_2) = b_2$. So we have $g(b_1)= g(b_2)$.

Because $g$ is injective, this implies $f(a_2) = b_1 = b_2 = f(a_2)$.

Because $f$ is also injective, this implies $a_1 = a_2$.

b) if $f$ and $g$ are surjective, then $g \circ f$ is also surjective.

We must show that for all $c \in C$, these exists at least one $a$ in $A$ such that $g\big(f(a)\big) = c$.

Since $g : B \to C$ is surjective, there exists $b \in B$ such that $g(b) = c$.

Since $f : A \to B$ is surjective, there exists $a \in A$ such that $f(a) = b$.

Then $g\big(f(a)\big) = g(b) = c$.

# Week 8 Logic

**Outline of knowledge (本节知识点大纲)**

1. Propositional Logic 命题逻辑
2. Formulas of Propositional Logic
3. Truth Value & Truth Table
4. Tautology， contradiction and semantic consequence
5. Local Equivalence

**In - class exercises**

(a) Construct a proposition/statement in that is true precisely when:
1. p is true and q is false
2. p is false and q is false or when p is false and q is true
3. p is false or q is true, and r is false

**Analysis：**
**The point of this question is the Syntax of Propositional logic.**
**You can find details in Week 8's lecture P4-8**

Answer:
1. $p \land \neg q$
2. $(\neg p \land \neg q) \lor (\neg p \land q)$
3. $(\neg p \lor q) \land \neg r$

(b) Show that the statements below are tautologies and/or contradictions:
1. $((p \to q) \land (q \to r)) \to (p \to r)$
2. $\neg (p \land \neg p) \land (r \land \neg r)$

**Analysis：**
**The point of this question is the logical equivalence/truth table.**
**You can find details in Week 8's lecture P13-18 P29-32**
**Answer:**
**Use logical equivalence/truth table, one can show that:**
**1. is tautology**
**2. is contradiction**

(c) Use the laws of equivalence to show that the below statements are logically equivalent:

1. $(p \wedge \neg q) \vee q \equiv p \vee q$
2. $\neg p \vee p \equiv p \wedge \neg p$

Answer:

Using commutative, distributive, or tautology, identity and commutative laws, one can show:

1. $(p \wedge \neg q) \vee q \equiv q \vee (p \wedge \neg q) \equiv (q \vee p) \wedge (q \vee \neg q) \equiv (q \vee p) \wedge T \equiv q \vee p \equiv p \vee q$
2. Note that $\neg p \vee p \not\equiv p \wedge \neg p$ since one is Or Tautology and another one is And Contradiction

**Analysis：**

**The point of this question is the logical equivalence.**

**You can find details in Week 8's lecture P29-32**

**Tutorial**

The relation $R$ on $A = \{1, 2, 3, 4, 5\}$ is represented by the matrix

$$M = \begin{bmatrix} F & F & T & T & F \\ F & T & F & F & T \\ F & F & F & T & F \\ F & F & T & F & F \\ F & T & F & F & F \end{bmatrix}$$

Determine the matrix representing $R \circ R$.

**Analysis：The point is calculation with matrix in relation**

**Answer：**

$R \circ R$ is represented by

$$M = \begin{bmatrix} F & F & T & T & F \\ F & T & F & F & T \\ F & F & T & F & F \\ F & F & F & T & F \\ F & T & F & F & T \end{bmatrix}$$

Let $A = \{0,2,4,6\}$ and $B = \{1,3,5,7\}$. Determine which of the following relations between A and B forms a function with domain A and codomain B:

a) $\{(6,3),(2,1),(0,3),(4,5)\}$
b) $\{(2,3),(4,7),(0,1),(6,5)\}$
c) $\{(2,1),(4,5),(6,3)\}$
d) $\{(6,1),(0,3),(4,1),(0,7),(2,5)\}$

For those which are functions, which are injective and which are surjective?

**Answer:**

**(a) a function, but neither surjective nor injective**

**(b) a function, surjective and injective**

**(c) not a function since no element of B is associated with 0 in A,**

**(d) not a function since 3 and 7 are associated with 0.**

Which of the following functions are injective? Which are surjective?

a) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2 + 1$.
b) $g : \mathbb{N} \to \mathbb{N}$ given by $g(x) = 2^x$.
c) $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = 5x - 1$

**Answer:**

(a) $f(-1) = f(1)$, therefore $f$ is not injective. $f(x) = 3$ has no solution in $\mathbf{Z}$. Therefore $f$ is not surjective.

(b) If $g(a) = g(b)$, then $2^a = 2^b$ and $a = b$. Hence, $g$ is injective. $g(x) = 3$ has no solution in $\mathbf{N}$. Hence, $g$ is not surjective.

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The function $f : A \to B$ is given by $f(x) = 1 + \frac{2}{x}$ where $A$ denotes the set of real numbers excluding 0 and $B$ denotes the set of real numbers excluding 1. Show that $f$ is bijective and determine the inverse function.

**Answer:**

If $f(a) = f(b)$ then $1+\frac{2}{a} = 1+\frac{2}{b}$ and so $a = b$. Hence, $f$ is injective. If $f(x) = y$ then $1 + \frac{2}{x} = y$ and so $x = \frac{2}{(y-1)}$. Hence, $f$ is surjective. Therefore, $f$ is bijective. The inverse function is given by $f^{-1} : B \to A$ where $f^{-1}(x) = \frac{2}{(x-1)}$.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x$ and the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x + 9$. Calculate $g \circ f$, $f \circ g$, $f \circ f$ and $g \circ g$.

**Answer:**

1. $g \circ f(x) = g(f(x)) = g(3x) = 3x + 9$
2. $f \circ g(x) = f(g(x)) = f(x + 9) = 3x + 27$
3. $f \circ f(x) = f(f(x)) = f(3x) = 9x$
4. $g \circ g(x) = g(g(x)) = g(x + 9) = x + 18$

Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$. Show that:

a) If $f$ and $g$ are surjective, then $fg$ are surjective.
b) If $f$ and $g$ are injective, then $f + g$ are injective.

**Answer:**

Counter example:

$f(x) = x$ which is surjective and injective

$g(x) = -x$ which is surjective and injective

$fg(x) = -x^2$ *which is not surjective.*

$(f + g)(x) = 0$ which is not injective.

For some domains and codomains, show the composition:

a) is commutative (e.g. $(g \circ f) = (f \circ g)$);
b) is associative (e.g. $(h \circ g) \circ f = h \circ (g \circ f)$).

**Answer:**

$$\begin{aligned}[(h \circ \mathrm{g}) \circ f](a) &= (h \circ g)(f(a)) \\ &= h\Big(g\big(f(a)\big)\Big) \\ &= h(g \circ f(a)) \\ &= [h \circ (g \circ f)](a)\end{aligned}$$

Let $A$, $B$ and $C$ be sets and let $f: A \to B$ and $g: B \to C$ be functions. Prove that:

a) if $f$ and $g$ are injective, then $g \circ f$ is also injective.
b) if $f$ and $g$ are surjective, then $g \circ f$ is also surjective.
c) if $g \circ f$ is injective, then $f$ is injective.
d) if $g \circ f$ is surjective, then $g$ is surjective.

**Answer:**

a) if $f$ and $g$ are injective, then $g \circ f$ is also injective.

We must show that for all $a_1, a_2 \in A$, if $g(f(a_1))= g(f(a_2))$, then $a_1 = a_2$.

Assume we have $g(f(a_1))= g(f(a_2))$, Let $f(a_1) = b_1$ and $f(a_2) = b_2$. So we have $g(b_1)= g(b_2)$.

Because $g$ is injective, this implies $f(a_2) = b_1 = b_2 = f(a_2)$.

Because $f$ is also injective, this implies $a_1 = a_2$.

b) if $f$ and $g$ are surjective, then $g \circ f$ is also surjective.

We must show that for all $c \in C$, these exists at least one $a$ in $A$ such that $g\big(f(a)\big) = c$.

Since $g : B \to C$ is surjective, there exists $b \in B$ such that $g(b) = c$.

Since $f : A \to B$ is surjective, there exists $a \in A$ such that $f(a) = b$.

Then $g\big(f(a)\big) = g(b) = c$.

c) if $g \circ f$ is injective, then $f$ is injective.

We have to prove when $f(a_1) = f(a_2)$, $a_1 = a_2$.

Let$a_1, a_2 \in A$. Then

$$f(a_1) = f(a_2) \Longrightarrow g\big(f(a_1)\big) = g\big(f(a_2)\big)$$
$$\Leftrightarrow g \circ f(a_1) = g \circ f(a_2) \quad \Leftrightarrow a1 = a2.$$

d) if $g \circ f$ is surjective, then $g$ is surjective.

Let $c \in C$. As $g \circ f$ is surjective there is an $a \in A$ with $c = g \circ f(a)$.

Then b $\stackrel{\text{def}}{=}$ f(a) ∈ B and $g(b) = c$.

# Week 10 Introduction to Predicate logic

**Outline of knowledge (本节知识点大纲)**

1. The definition of predicate logic
2. The statement of predicate logic
3. The translation of predicate logic
4. The elements of predicate logic
5. Signatures

**In class exercises**

Problem 1

Consider the signature S = {Friend, Female, alice, bob} consisting of a binary predicate symbol Friend, a unary predicate symbol Female, and two constant symbols alice and bob. Assume that these symbols have the following meaning:

• Friend means "is a friend of" (i.e., Friend(x, y) states: x is a friend of y).
• Female means "is female" (i.e., Female(x) states: x is female).
• alice and bob refer to "Alice" and "Bob", respectively.

Translate the following sentences into S-formulae, that is, for each of the following sentences provide an S-formula that expresses the sentence:

1. Alice is a friend of Bob.
2. Bob has a girlfriend.
3. All friends of Bob are also friends of Alice.
4. Alice has at least two friends.
5. Everybody has a friend.
6. Nobody is a friend of everybody else.

**Analysis:**
**The point of this question is the translation of predicate logic.**
**You can find details in Week 10's lecture P7-16**

**Answer:**
**1. Friend(alice, bob)**
**2. $\exists x$ Friend(bob, x) $\wedge$ Female(x)**
**3. $\forall x$ Friend(x, bob) $\rightarrow$ Friend(x, alice)**
**4. $\exists x \exists y$ Friend(alice, x) $\wedge$ Friend(alice, y) $\wedge$ $\neg x = y$**
**5. $\forall x \exists y$Friend(x, y)**
**6. $\neg \exists x \forall y$ $\neg x = y \rightarrow$ Friend(x, y)**

Problem 2

Translate the following statements into S-formulae for an appropriate signature S. Before

translating the sentences, describe the signature S and the meaning of the symbols in it.
1. Liverpool is a city in the UK.
2. There is a city that is larger than Liverpool.
3. London is larger than all other cities in the UK.
4. Some students at University of Liverpool do not live in Liverpool.
5. Not everyone living in Liverpool studies at University of Liverpool.

**Analysis:**
**The point of this question is the translation of predicate logic and signatures. You can find details in Week 10's lecture P7-16 & P18-20**

**Answer:**
**We choose a signature S = {City, LocatedIn, Larger, UoLStudent, LivesIn, liverpool, london, uk}, where City and UoLStudent are unary predicate symbols, LocatedIn, Larger, LivesIn are binary predicate symbols, and liverpool, london, UK are constant symbols. We assume that the symbols have the following meaning:**
**• City(x) means "x is a city";**
**• LocatedIn(x, y) means "x is located in y";**
**• Larger(x, y) means "x is larger than y";**
**• UoLStudent(x) means "x is a student at University of Liverpool";**
**• LivesIn(x, y) means "x lives in y";**
**• liverpool, london, and uk stand for "Liverpool", "London", and "UK", respectively.**
**We are now able to express the five statements:**
**1. City(liverpool) ∧ LocatedIn(liverpool, UK)**
**2. ∃x City(x) ∧ Larger(x, liverpool)**
**3. ∀x (City(x) ∧ LocatedIn(x, uk) ∧ ¬x = london) → Larger(london, x)**
**4. ∃x UoLStudent(x) ∧ ¬LivesIn(x, liverpool)**
**5. ¬∀x LivesIn(x, liverpool) → UoLStudent(x)**

**Tutorial**

1. Suppose P1 represents 'This car is red', P2 represents 'Your house is blue' and P3 represents 'You are tall'.
What does (P1 ∧ P2 ) represent?
What does ¬P3 represent?
What does ¬(P2 ∧ P3 )represent?

Represent in propositional logic:
You are not tall.
If this car is red, then your house is blue.
This car is not red or you are tall.
It is not the case that this car is not red.

**Analysis:**
**The point of this question is the translation of predicate logic.**
**You can find details in Week 10's lecture P7-16**

**Sol:**

- **This car is red and your house is blue.**
- **You are not tall.**
- **Your house is not blue or you are not tall.**

- **¬p3**
- **(¬p1 v p2 )**
- **(¬p1 v p3 )**
- **¬¬p1**

2. Define P → Q.

**Sol: P → Q = (¬P ∨ Q )**

3. Let P = (p 1 ∨ ¬p 2 ) and Q = ¬P. Let I be an interpretation such that I(p 1 ) = 0 and I(p 2 ) = 0. Determine I(P) and I(Q) using a truth table.

**Analysis: The point is how to write a truth table and use it to judge interpretation.**

Sol:

| $p_1$ | $p_2$ | $\neg p_2$ | $(p_1 \vee \neg p_2)$ | $Q$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 |

Therefore $I(P) = 1$ and $I(Q) = 0$.

4. Let Q = ((p1 ∧ ¬p2 ) ∧ p3 ). Write down the truth table for Q. Determine the interpretations under which Q is true.

**Analysis: The point is how to write a truth table and use it to judge**

**interpretation.**

Sol:

| $p_1$ | $p_2$ | $p_3$ | $\neg p_2$ | $(p_1 \wedge \neg p_2)$ | $((p_1 \wedge \neg p_2) \wedge p_3)$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 |

P is true under the interpretation $I$ with $I(p_1) = 1, I(p_2) = 0$, and $I(p_3) = 1$. No other interpretation makes $P$ true.

5. What is a tautology?

**Sol: A formula P is a tautology if and only if ??(P) = 1 for every interpretation I.**

6. Which of the following are tautologies? Check using truth tables.
(a) (p1 ∨ p2 );
(b) (¬p1 ∨ (p2 ∧ p1 ));
(c) (¬¬p1 ↔ p1 );
(d) (¬p1 → ¬p1 ).

**Analysis: The point is the understand of truth table.**

Sol:

*a)* $(p_1 \vee p_2)$ is not a tautology. Take $I$ with $I(p_1) = 0, I(p_2) = 0$.

*b)* $(\neg p_1 \vee (p_2 \wedge p_1))$ is not a tautology. Take $I$ with $I(p_1) = 1, I(p_2) = 0$.

*c)* $(\neg\neg p_1 \leftrightarrow p_1)$ is a tautology.

d) $(\neg p_1 \rightarrow \neg p_1)$ is a tautology.

7. What is a contradiction?

**Sol: A formula P is a contradiction if and only if ??(P) = 0 for every interpretation I.**

8. Which of the following are contradictions? Check using truth tables.
(a) (¬p 1 ∧ p 2 );
(b) (p 1 → ¬p 1 ).
(c) (p1 ↔ ¬p 1 );

(d) (p1 ∧ (¬p2 ∨ ¬p1 ));

**Analysis: The point is the understand of truth table.**

Sol:

a) $(\neg p_1 \wedge p_2)$ is not a contradiction. Take $I$ with $I(p_1) = 0, I(p_2) = 1$.

b) $(p_1 \rightarrow \neg p_1)$ is not a contradiction. Take $I$ with $I(p_1) = 0$.

*c)* $(p_1 \leftrightarrow \neg p_1)$ is a contradiction.

*d)* $(p_1 \wedge (\neg p_2 \vee \neg p_1))$ is not a contradiction. Take $I$ with $I(p_1) = 1$, $I(p_2) = 0$.

9. Define the meaning of Γ⊨P.

Sol:

Γ⊨P if and only if, for every interpretation $I$, the following holds: if $I(Q)$=1 for all $Q \in \Gamma$, then $I(P)$=1.

10. Which of the following are true? Check using truth tables.

(a) {p 1 ,(p 1 → p 2 )} ⊨ p 2 ;

(b) {(p 1 → p 2 )} ⊨ (p 2 → p 1 );

(c) {(p 1 ∨ ¬p 2 )} ⊨ p 1

**Analysis: The point is the understand of truth table.**

Sol:

*a)* $\{p_1, (p_1 \rightarrow p_2)\} \vDash p_2$ is true.

*b)* $\{(p_1 \rightarrow p_2)\} \vDash (p_2 \rightarrow p_1)$ is not true. Take $I$ with $I(p_1) = 0, I(p_2) = 1$. Then $I(p_1 \rightarrow p_2)$=1, but $I(p_2 \rightarrow p_1)$=0.

*c)* $\{(p_1 \vee \neg p_2)\} \vDash p_1$ is not true. Take $I$ with $I(p_1) = 0, I(p_2) = 0$. Then $I((p_1 \vee \neg p_2)$=1, but $I(p_1)$=0.

11. Define the relation ≡ of logical equivalence.

**Sol: Q ≡ R if and only if for all interpretations I :I(Q) = I(R)**.

12. Show using truth tables

(P ∨ (Q ∧ R)) ≡ ((P ∨ Q) ∧ (P ∨ R)).

**Analysis: The point is the understand of truth table.**

Sol:

| P | Q | R | $(Q \wedge R)$ | $P \vee (Q \wedge R))$ | $(P \vee Q)$ | $(P \vee R)$ | $((P \vee Q) \wedge (P \vee R))$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

13. Is ¬(P ∧ Q) logically equivalent to (¬P ∧ ¬Q)? Discuss the relationship to De Morgan's Law.

**Analysis: De Morgan's Law**

Sol:

No, $\neg(P\wedge Q)$ is not logically equivalent to $(\neg P\wedge \neg Q)$?. Take an interpretation $I$ such that $I(P) = 1$ and $I(Q) = 0$. Then $I\big(\neg(P\wedge Q)\big) = 1$ but $I((\neg P\wedge \neg Q\ ) = 0$ .

One of De Morgan's Laws states that

$$\neg(P\wedge Q) \equiv (\neg P \vee \neg Q).$$

This is different to the incorrect equivalence in the question because $\wedge$ is replaced by $\vee$.

# Week 11 Combinatorics and Probability

**Outline of knowledge (本节知识点大纲)**

1. Notation for sums and products
2. The factorial function.
3. Counting outcomes of unions of disjoint events
4. Counting outcomes of sequences of events
5. Counting permutations and k-permutations.
6. Binomial coefficients and counting size-k subsets
7. Counting functions from one set to another.
8. Discrete probability: sample spaces and events
9. Conditional probability and independence
10. Random variables, expectation and linearity of expectation

**In class exercises**

(a) An e-banking account password consists of 8 characters. Characters can be both in upper and lower case letters (26 characters) as well as in digits (10 characters). However, the first two characters must be in upper case letters; the remaining six can be either in digits or in lower case letters; and the last two characters cannot be "5" or "x". How many different passwords are possible?

**Analysis:**
**The point of this question is the "sum" rule and the "product" rule.**
**You can find details in Week 11's lecture P9-13**
**Answer: There are 26 × 26 × 36 × 36 × 36 × 36× 34 × 34 passwords possible.**

(b) There are 50 kids in a class, 39 girls and 11 boys. How many ways are there to choose a group of 10 containing at least one boy and one girl?

**Analysis:**
**The point of this question is Conditional probability and independence**
**You can find details in Week 11's lecture P36-42**
**Answer: (50, 10) – (39, 10) – (11, 10)**

1. Let S = {-1, 1, 4}. What are the values of the following expressions?
   - $\sum_{x \in S}(x+1)$
   - $\sum_{i=3}^{6}(i^2+6)$
   - $\prod_{j=1}^{5}(\frac{j}{2})$
   - 4!
   - 0!

**Analysis:**

**Just use mathematical formulas in Week 11's Lecture to solve problems**

**Answer:**

1.
   - $\sum_{x \in S}(x+1) = (-1+1) + (1+1) + (4+1) = 7$
   - $\sum_{i=3}^{6}(i^2+6) = (3^2+6) + (4^2+6) + (5^2+6) + (6^2+6) = 110$
   - $\prod_{j=1}^{5}({}^{j}/_{2}) = \left({}^{5!}/_{2^5}\right) = 15/4$
   - $4! = 24$
   - $0! = 1$

2. A woman has six dresses, five pairs of trousers and three shirts. How many different outfits does she have?

**Answer:**

**The woman has 5 X 3 different outfits consisting of trousers and a shirt. (This is a simple application of the "product rule".) As an alternative she can wear six dresses. By the "sum rule", she has 15 + 6 = 21 possible outfits.**

3. A computer password consists of six characters. The first two must be lower case letters and the remaining four can be either digits or lower case letters. How many different passwords are possible?

**Answer:**

**There are 26 lower case letters and 36 digits and lower case letters. By the "product rule", there are 26 x 26 x 36 x 36 x 36 x 36 = 1 135 420 416 passwords possible.**

4. In a small start-up company, one person will be CEO, one person will be "scientific advisor", one person will be programmer, and one person will do all other jobs. There are 5 people available. In how many ways can the roles be assigned?

**Answer:**

**There are four designated jobs, so we want a 4-permutation from a set of size 5. There are 5!/(5-4)! = 5! = 120 choices.**

5. A hockey squad has 18 players; 11 players make a team. How many different teams are possible?

**Analysis:**

**The point of this question is Binomial coefficients**

**You can find details in Week 11's lecture P21-25**

$\binom{18}{11} = \frac{18!}{(18-11)!11!} = 31824$ possible team selections.

6. A phone number is allowed to start with any digit, including 0. How many 6-digit phone numbers have distinct digits? How many 6-digit phone numbers have distinct digits, and

don't start with 0.

**Answer:**

**The number of phone numbers with distinct digits is equal to the number of 6-permutations of the 10 digits, so it is**

**10!/(10 -6)! = 151 200.**

**For the second part of the question, there are 9 possibilities for the first digit, and then 9,8, 7, 6, 5 for the remaining digits, in turn, so the number of possibilities is 9 x 9 x 8 x 7 x 6 x 5 = 136080.**

7. How many 6-digit phone numbers are there? How many of these start with 0 or 1?

**Answer:**

**A phone number is just a function from the 6 positions in the phone number to the 10 digits, so there are 10^6 , phone numbers in all. The number that start with a0 or a1 is 2 x 10^5**

8. Suppose we want to pick 3 students for the SSLC. There are 10 volunteers from year one, 11 volunteers from year two, and 9 volunteers from year three. There are two volunteers from the MSc. How many ways can the committee be formed, assuming that at least one MSc student must be chosen?

**Answer:**

The number of possibilities with exactly one MSc student is

$$\binom{2}{1}\binom{10+11+9}{2} = 870$$

The number of possibilities with two MSc students is

$$\binom{2}{2}\binom{10+11+9}{1} = 30$$

Thus, the total number of allowable possibilities is 870 + 30 = 900. As a sanity check, note that the number of "unallowable" possibilities with no MSc students is $\binom{10+11+9}{3} = 4060$ and the total number of possibilities is $\binom{10+11+9+2}{3} = 4960$, which is equal to this plus the number of allowable possibilities.

9. How many injective functions are there from {1, 2, 3} to {1, 2, 3, 4, 5}?

**Answer:**

**The first element of A (e.g. 1) can take 5 values, the second element of A (e.g. 2) can only take 4 values and the third element (e.g. 3) of A can only take 3 values. Hence the total number of functions is 5x4x3 = 60.**

10. How many surjective functions are there from {1, 2, 3} to {1, 2, 3, 4, 5}?

**Answer:**

**No surjective functions, because |B|=5 is larger than |A|=3.**

11. There are 40 people in a room. They shake each other's hands once and only once. How many handshakes are there altogether?

**Answer:**

**We know that that 40 people shake hands, so each person shakes 39 hands. The first person shakes 39 hands, the second person shakes 38 hands, the third person shakes 37 hands, and so on. Hence, the total number of handshakes is 39+38 + 37 + 36 + ... + 3 + 2 + 1.**

12. How many ways are there to seat 10 people, consisting of 5 couples, in a row of seats (10 seats wide) if (a) the seats are assigned at random? and (b) all couples are to get adjacent seats?

**Answer:**

**a. 10!**

**b. 10x8x6x4x2: Fill 10 slots (seats) left to right keeping in mind the restriction that couples have to sit together. There are 10 choices for the first slot (no restriction), 1 choice for the second (since the first person's mate has to be seated there), 8 choices for the third slot (since 2 people have now been seated, leaving 8), 1 choice for the fourth slot (since this seat has to go to the mate of # 3), etc.**

# week 12 Tutorial 8

主体都是概率问题，知识点就是排列组合，不详述。

Tutorial8

1. Suppose that I have four fair coins with values 1p, 2p, 5p and 10p. I will call the 1p coin and the 2p coin "low-value" coins and the other two coins "high-value coins".

a. What is the probability, when I flip the coins, that the 1p and the 5p come up heads, and the other two coins come up tails?

b. What is the probability that at least one of the low-value coins comes up tails?

c. What is the probability that at least three of the coins come up tails?

d. What is the probability that at least three of the coins come up tails, conditioned on the fact that at least one of the low-value coins comes up tails?

e. Is the event that at least three of the coins come up tails independent of the event that at least one of the low-value coins comes up tails?

f. Suppose that, if a coin comes up heads, I am paid the value of that coin. What is the total value that I expect to receive if I flip all the coins?

Sol：简单题，看懂题目意思即可得出结果。简单说一下 A is independent of B 即加上 B 条件不影响 A 事件发生的概率。

1) Each outcome is a function from the 4 coins to the set {heads, tails}. There are $2^4 = 16$ functions, and they are all equally likely, so the probability of any one of them is 1/16. Thus, the probability that the 1p and the 5p come up heads and the other two coins come up tails is 1/16.
2) Let E1 be the event that the 1p coin comes up tails. Let E2 be the event that the 2p coin comes up tails.
$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$$
3) The event that at least three of the coins comes up tails is the union of five disjoint events: The event that all coins come up tails, and the four events in which a specified coin is the only heads. The probability of each of these five events is 1/16, so that the probability of their union is 5/16.
4) Let F be the event that at least three of the coins come up tails and G be the event that at least one of the low-value coins comes up tails. Then:
$$\Pr(F|G) = \frac{\Pr(F \cap G)}{\Pr(G)}.$$
But $\mathrm{F} \cap \mathrm{G} = \mathrm{F}$, So this is $\frac{P_r(F)}{P_r(G)} = \frac{\left(\frac{5}{16}\right)}{\left(\frac{3}{4}\right)} = \left(\frac{5}{12}\right)$
5) We have shown that $P_r(F \mid G) = \frac{5}{12}$ and $P_r(F) = \frac{5}{16}$. These are not the same. So events F and G are not independent.
6) Let $f_1$ be the amount of money that I get paid from the flip of the 1p coin, and $f_2$ be the amount of money that I get paid from the flip of the 2p coin, and so on.

   Then $E[f_1] = \frac{1}{2}$ and $E[f_2] = 2 \times \frac{1}{2} = 1$,

   $E[f_5] = 5 \times \frac{1}{2} = \frac{5}{2}$ ,

   $E[f_{10}] = 10 \times \frac{1}{2} = 5$,

   Thus: $E[f_1 + f_2 + f_5 + f_{10}] = \frac{1}{2} + 1 + \frac{5}{2} + 5 = 9$.

2. A card is taken at random from a standard 52-card pack of playing cards. What is the probability that the card is:
a. a five
b. a Diamond
c. not a Spade
d. a red Queen
e. a King, a Queen
f. a red jack
Sol: Diamond 表示方片 Spade 表示黑桃 Queen 是 Q King 是 K，jack 就是 J red Queen 包括方片以及红桃。

| | |
|---|---|
| a. a five | 1/13 |
| b. a Diamond | 1/4 |
| c. not a Spade | 3/4 |
| d. a red Queen | 1/26 |
| e. a King, a Queen | 3/13 |
| f. a red jack | 1/2 |

3. Tom rolls a fair dice 600 times. a. How many four would you expect him to obtain?
b. Should he be surprised if he obtained 110 fours?
Sol：expect to 在此表示平均，即表示投掷 600 次骰子平均可以有几次 4 出现。
a) 100
b) No

4. If Tom goes out with his friends, then the probability that Tom does his homework is 1/10. Otherwise (if he does not go out), the probability that Tom does his homework is 3/5. We know that the probability that he goes out with his friends is 3/4. What is the probability that Tom does his homework?
Sol: 这就是个组合题，Tom 做作业的可能性是由 2 个步骤组成的，第一步是是否出去与朋友玩，第二步才是是否做作业。因此只需要将 2 个步骤的概率分别相乘并组合一下即可得出答案。
p(Tom goes out and does homework) or p(Tom does not go out and does his homework) = p(Tom goes out) × p(does homework) + p(Tom does not go out) × p(does his homework) = (3/4) ×(1/10) + (1/4) ×(3/5)

5. How many people must there be in a room before the probability that someone has the same birthday as you do is at least 1/2?
Sol: 要使得一个房间里任意某个人跟你同一天生日的概率大于等于 1/2，即可从反方向考虑房间里任意一个人跟你不是同一天出生的概率。房间里一个人的时候就是 364/365，两个人的时候变成 364/365 x 364/365，即该 probability 的值为(364/365)$n$.，再用 1-该 probability 即可得出答案。(这道题本质上仍然是道组合题)

Probability that someone in the room has the same birthday as me, denoted by P(B) is 1 −

probability that no one in the room has the same birthday as me. $P(B)$ = 1 − (364/365)$n$.

We wish $P(B) \geqslant 1/2$, Taking logs, $n \geqslant 253$ is obtained.

6. If a gambler rolls two dice and gets a sum of 10, he wins 10 dollars and if he gets a sum of 3, he wins 20 dollars. The cost to play is 5 dollars. What is the expected value of the game?
Sol:
此题为简单题，分别计算获利的概率就可得出答案，唯一需要注意的是即便 sum 为 10 和 3 的时候你是赢家，但是你仍然需要花费 5 美元，所以你真正的获利分别是 5 和 15.

Sum of 10: (5,5), (6,4), (4,6): 3/36 (probability)
Sum of 3: (1,2), (2,1): 2/36 (probability)
Random number values (winner's gains): -5, 5, 15
Corresponding probabilities are: 31/36 , 3/36 , 2/36
E(x) = -5(31/36)+5(3/36)+15(2/36) = (-155+15+30)/36 = -110/36 = -$3.05

7. If the probability that it rains next Tuesday in Suzhou is twice the probability that it does not, what is the probability that it will rain next Tuesday in Suzhou?
Sol: 由下雨概率是不下雨概率的两倍易得出分别两事件发生的概率。

If Let X be the probability that it rains next Tuesday
Let Y be the probability that it does not next Tuesday
Given X = 2Y.
Then X + Y = 1 which means X + X/2 = 1 and X=2/3

8. In a popular lottery known as "Lotto 6/49", a player marks a card with six different numbers from the integers 1 -49 and wins if his or her numbers match six randomly selected such numbers:1. In how many ways can a player complete a game card? 2. How many cards should you complete in order to have at least one chance in a million of winning? 3. In how many ways complete a game card so that no number matches any of the six selected? 4. In how many ways can a player complete a game card so that at least one number matches one of the six selected?
Sol:此题是个简单的彩票机制，根据题目要求只要玩家选择的 6 张卡片与预先设定的 6 张一样，即可获得最终大奖。而卡片对应的其实就是 1-49 的数字。所以此题仍然为一道组合题。

(1) C(49,6) = 13,983,816
(2) n/C(49,6) = 1/1,000,000 . Therefore, n = 14
(3) C(49-6,6) = C(43,6) = 6,096,454
(4) C(49,6)- C(43,6) =13,983,816 -6,096,454 = 7,887,362