Let A, Band C be sets and let 𝑓∶𝐴→𝐵 and 𝑔∶𝐵→𝐶be functions. Prove that:

a) if f and g are injective, then𝑔∘𝑓is also injective.

b) if f and g are surjective, then 𝑔∘𝑓is also surjective.

c) if 𝑔∘𝑓is injective, then f is injective.

d) if 𝑔∘𝑓is surjective, then g is surjective.

A) if f and g are injective, then𝑔∘𝑓is also injective.

We must show that for all 𝑎1, 𝑎2∈𝐴, if 𝑔(𝑓(𝑎1))=𝑔(𝑓(𝑎2)), then 𝑎1=𝑎2.

Assume we have 𝑔(𝑓(𝑎1))=𝑔(𝑓(𝑎2)), Let 𝑓(𝑎1)=𝑏1and 𝑓(𝑎2)=𝑏2. So we have 𝑔(𝑏1)=𝑔(𝑏2).

Because 𝑔is injective, this implies 𝑓(𝑎2)=𝑏1=𝑏2=𝑓(𝑎2).

Because 𝑓is also injective, this implies 𝑎1=𝑎2.

b)if f and g are surjective, then 𝑔∘𝑓is also surjective.

We must show that for all 𝑐∈𝐶, these exists at least one 𝑎 in 𝐴 such that 𝑔𝑓𝑎=𝑐.

Since 𝑔∶𝐵→𝐶is surjective, there exists 𝑏∈𝐵such that 𝑔𝑏=𝑐.

Since 𝑓∶𝐴→𝐵is surjective, there exists 𝑎∈𝐴such that 𝑓𝑎=𝑏.

Then 𝑔𝑓𝑎=𝑔(𝑏)=𝑐.

c)if 𝑔∘𝑓is injective, then f is injective.

We have to prove when 𝑓𝑎1=𝑓𝑎2, 𝑎1=𝑎2.

Let 𝑎1, 𝑎2∈𝐴. Then

𝑓𝑎1=𝑓𝑎2⟹𝑔𝑓𝑎1=𝑔𝑓𝑎2⟺𝑔∘𝑓𝑎1=𝑔∘𝑓𝑎2⟺𝑎1=𝑎2.

d)if 𝑔∘𝑓is surjective, then g is surjective.

Let 𝑐∈𝐶. As 𝑔∘𝑓is surjective there is an 𝑎∈𝐴with 𝑐=𝑔∘𝑓𝑎.

Then b≝f(a)∈B and 𝑔𝑏=𝑐.