The Azimi Attenuation Model
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Amplitude attenuation model: The amplitude $A(\omega,x)$ declines exponentially with distance x, according to an attenuation function $\alpha(\omega)$, or equivalently, the quality factor $Q(\omega)$ or tee-star $t^*(\omega)$:

$$A(\omega,x) = A_0(\omega)\ \exp\{-\alpha(\omega)\ x\} = A_0(\omega)\exp\left\{-\ \frac{\omega x}{2c(\omega)Q(\omega)}\right\} = A_0(\omega)\exp\left\{-\ \frac{\omega t^*(\omega):}{2}\right\}$$

Here ω is angular frequency, $A_0(\omega)$ is the initial amplitude at $x=0$, and $c(\omega)$ is phase velocity. In some seismological settings, the quality factor $Q(\omega)$ is only weakly frequency-dependent, and one can speak, approximately, of a constant-Q attenuation model (with quality factor Q_0). Similarly, while the phase velocity is dispersive, in some settings it is only weakly so, and one can speak of a non-dispersive model (with velocity c_0). When dispersion is negligible, the attenuation function and quality factor are related by:

$$\alpha(\omega) \approx \frac{\omega}{2c_0 Q(\omega)} \quad \text{and} \quad Q(\omega) \approx \frac{\omega}{2c_0\alpha(\omega)}$$

Propagation Model: A harmonic wave with angular frequency ω, initial amplitude $A_0(\omega)$ and time dependence $\exp\{-i\omega t\}$ is propagates to a position x via:

$$A_0(\omega)\ \exp\{ikx - i\omega t\} = A_0(\omega)\ \exp\{i\omega[s(\omega)x - t]\}$$

Here $s(\omega) = k/\omega = 1/c(\omega)$ is the phase slowness.

Causality requires that the attenuation function $\alpha(\omega)$ and phase slowness $s(\omega)$ be related through an integral equation called the Kramer-Kronig Relationship. It can be shown that no constant-Q model can satisfy this relationship. Azimi found an (α, s) pair that satisfies the relationship and is approximately constant-Q, at least for frequencies much less than some corner frequency ω_0:

$$a(\omega) = \frac{a_2\omega}{1+a_3\omega} \quad \text{and} \quad \Delta s(\omega) = s(\omega) - s_0 = -\frac{2a_2\ \ln a_3\omega}{\pi(1-a_3^2\omega^2)} \quad \text{with} \quad \omega \geq 0$$

Here a_2 and a_3 are constants. Note that when we set $a_2 = 1/(2c_0Q_0)$ and $a_3 = 1/\omega_0$, the attenuation function obeys:

$$a(\omega) \approx a_2\omega \ \ \text{and} \ \ Q(\omega) \approx \frac{\omega}{2c_0\alpha(\omega)} = \frac{1}{2c_0a_2} = Q_0 \quad \text{for} \quad \omega \ll \omega_0$$

That is, it is constant-Q for frequencies much less than the corner frequency.

A real displacement pulse $u_0(t) = u(x = 0, t)$ can be attenuated and propagated to the position x in the following steps"

Step 1: Fourier transform $u_0(t)$ to $\tilde{u}_0(\omega)$ and focus on the non-negative frequency values of $\tilde{u}_0(\omega)$ only.

Step 2: Multiply $\tilde{u}_0(\omega)$ by $\exp\{-\alpha(\omega)\, x\} \exp\{i\omega s(\omega) x\}$ to obtain $\tilde{u}(\omega)$.

Step 3: Set $\tilde{u}(\omega = 0)$ to unity.

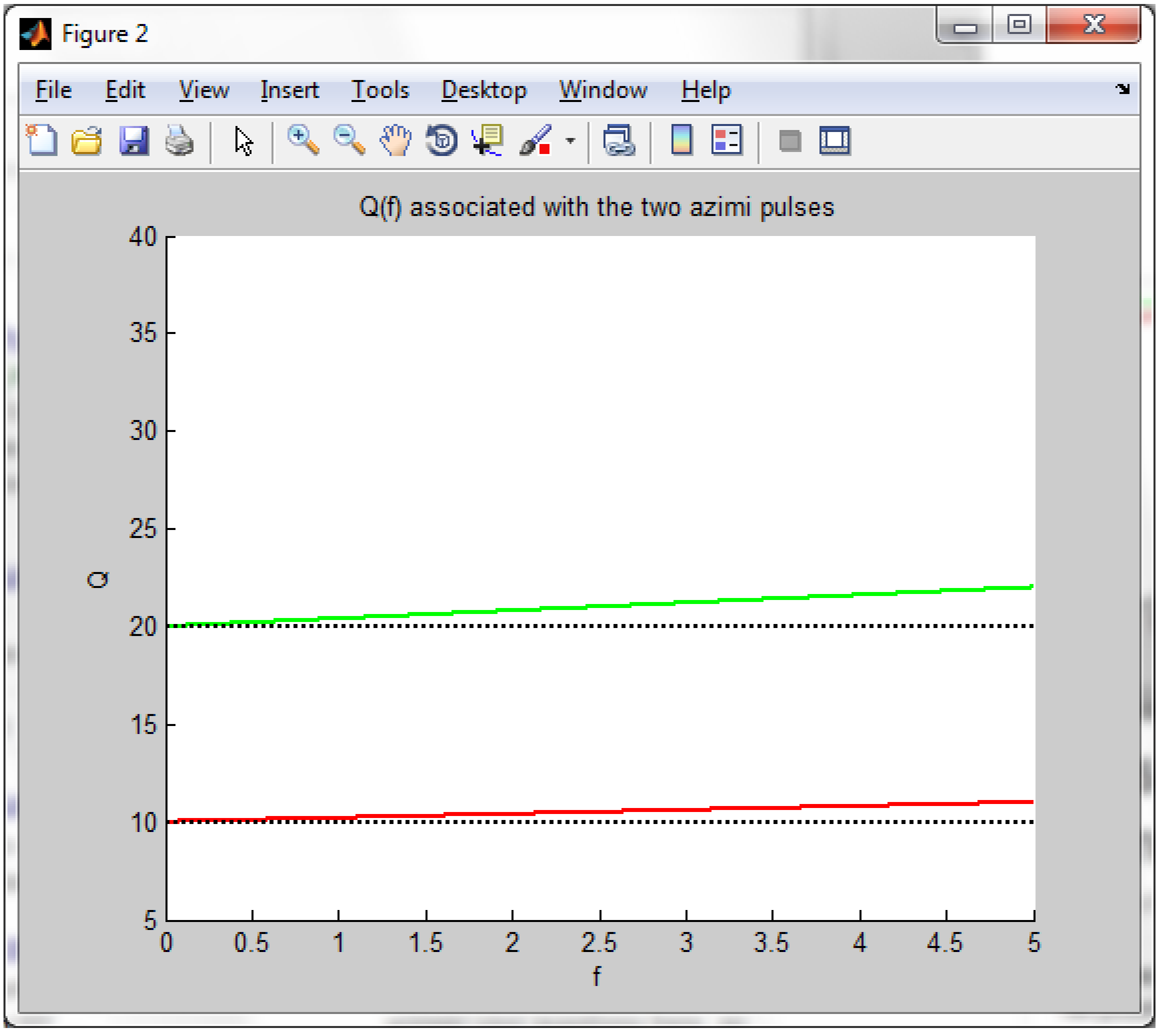
Step 4: Form the negative frequency values of $\tilde{u}(\omega)$ by taking the complex conjugate of the positive frequency values.

Step 5: Inverse Fourier transform $\tilde{u}(\omega)$ back to $u(t)$.

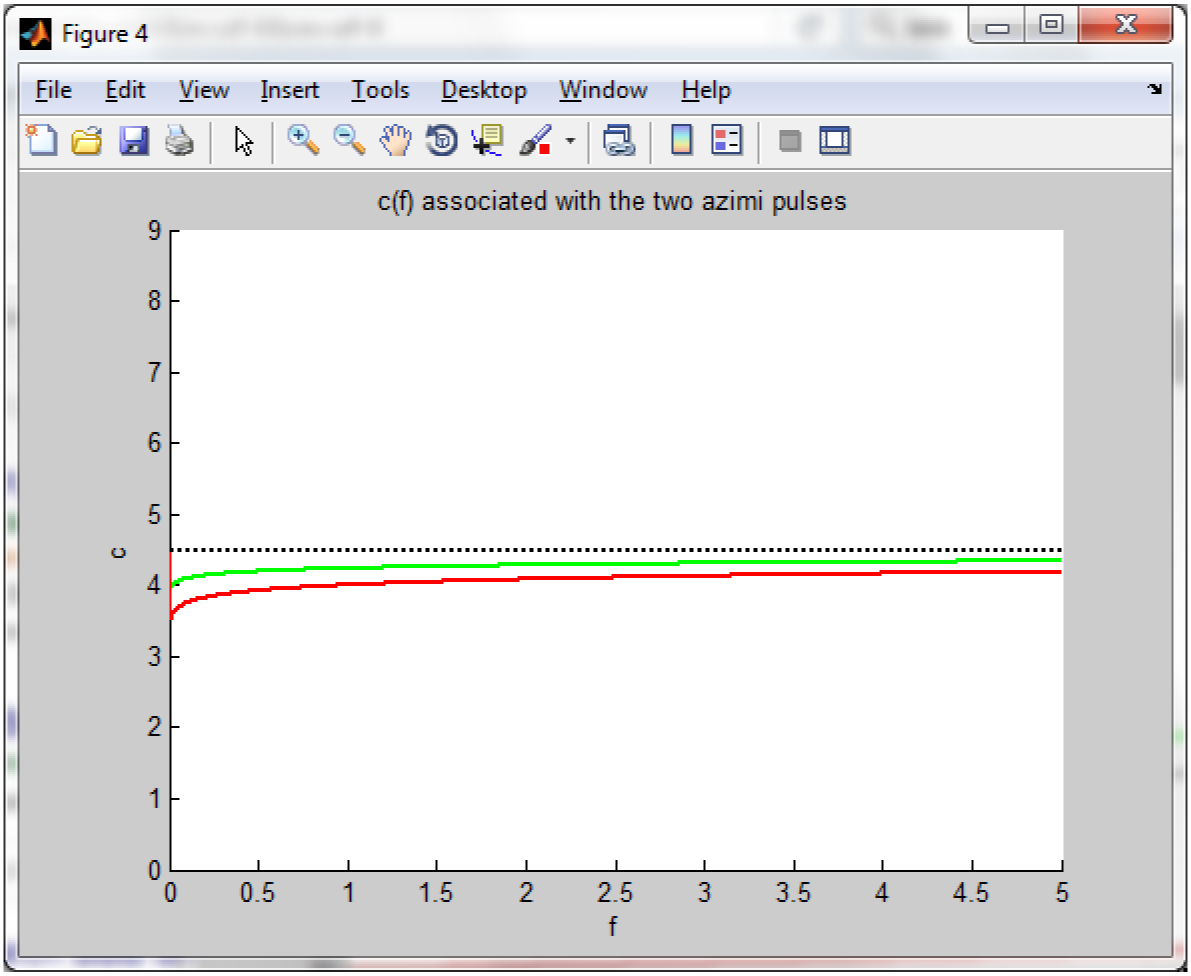
Sometimes, it may be convenient to replace $s(\omega)$ with $\Delta s(\omega)$ in Step 2, so that the pulse is only delayed by the deviation in phase velocity. In this way, several pulses can be aligned on the same plot.

Note that c_0 and Q_0 appear only in the constant $a_2 \propto 1/(c_0 Q_0)$, and not in a_3 and that a_2 appears in $a(\omega)$ and $\Delta s(\omega)$ only as a leading multiplicative factor. Thus, both decay rate $a(\omega)x$ and phase delay $\Delta s(\omega)x$ are proportional to $x/(c_0 Q_0) = t_0^*$. Therefore, the pulse shape contains only enough information to determine t_0^* and not enough to determine x and Q_0 individually.

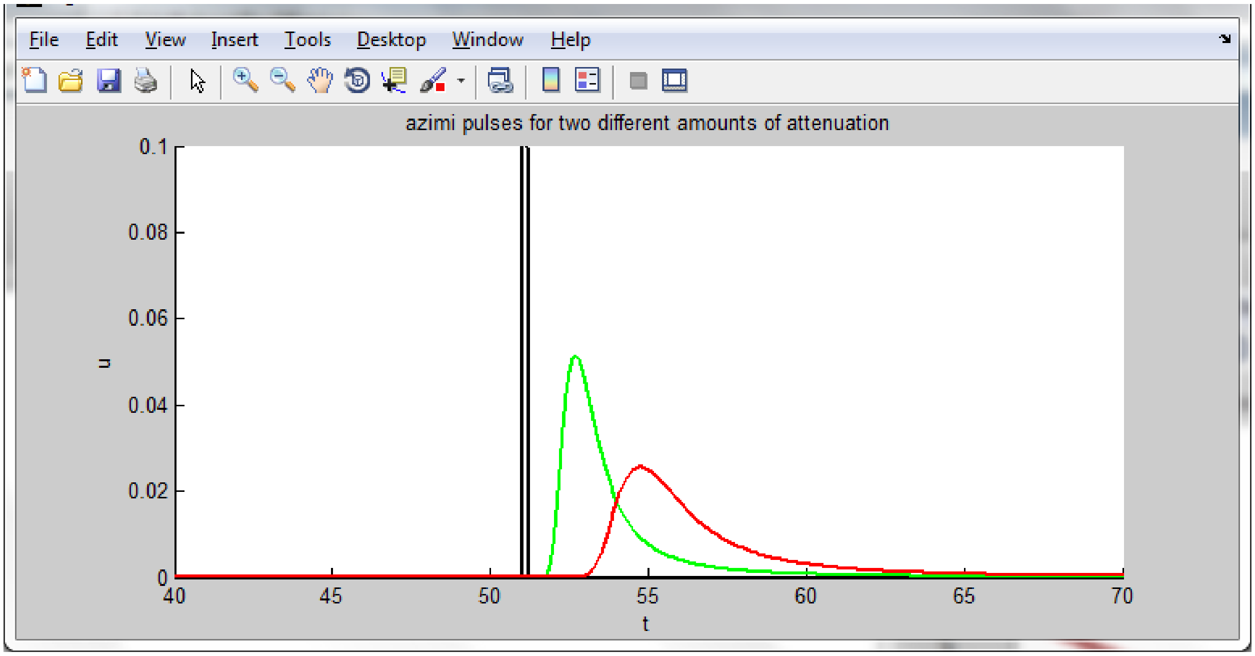
Sample $Q(f)$'s for $f_0 = 2\pi\omega_0 = 50$ Hz, $Q_0 = 10$ (red) and 20 (green) and $c_0 = 4.5\ \mathrm{km/s}$.



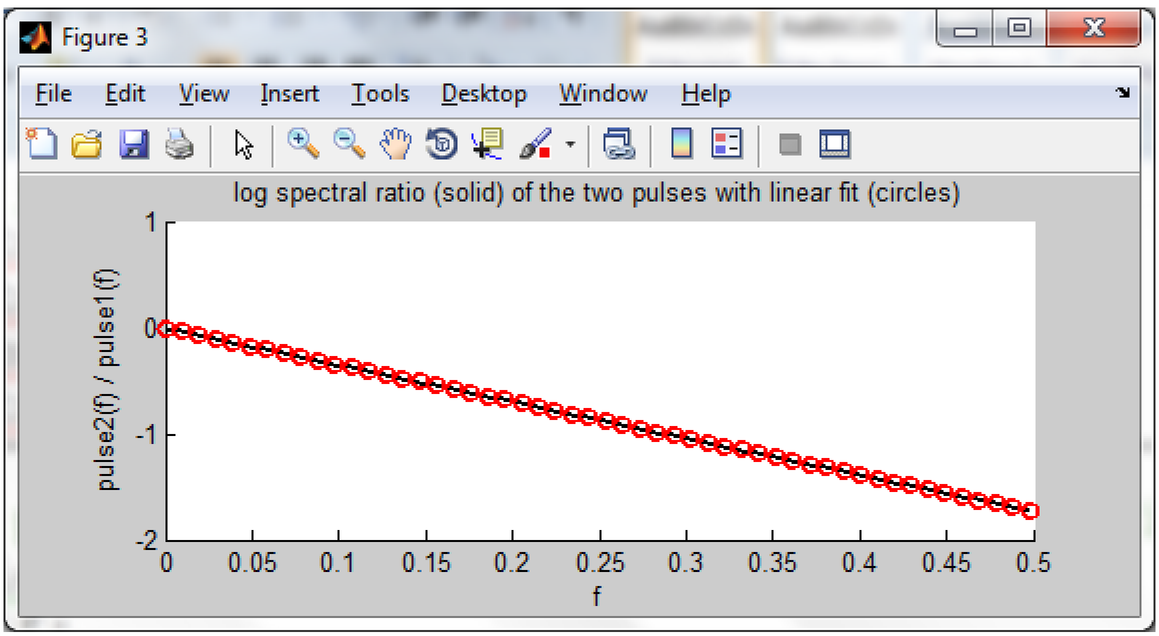
Sample $c(f)$'s for $f_0 = 2\pi\omega_0 = 50$ Hz, $Q_0 = 10$ (red) and 20 (green) and $c_0 = 4.5$ km/s.



Sample $u(x,t)$ for and $x = 100$ km and $u_0(t)$ a length $N = 1024$ time-series with a sampling interval of 0.1 s and a unit spike at position $N/2$:



The differential attenuation between the two Azimi pulses (black) and the best-fitting log-linear model (red).



The true differential t^* and the one estimated via the linear fit:

Dtstartrue 1.111111 Dtstarest 1.100142

MATLAB CODE

```
clear all;

% spectral ratio of two azimi pulses

% azimi attenuation model has 2 parameters
Q1 = 20;  % quality factor at low frequencies
Q2 = 10;  % quality factor at low frequencies
f0 = 50; % frequency below which Q(f) is approximately constant

N=1024; % number of samples
Dt=0.1; % sampling interval
c0=4.5; % low frequency velocity
x=100;  % propagation distance

[ t, pulse0, pulse1, f, Qf1, cw1 ] = azimi( N, Dt, x, c0, Q1, f0 );
[ t, pulse0, pulse2, f, Qf2, cw2 ] = azimi( N, Dt, x, c0, Q2, f0 );

% plot Azimi pulses
figure(1);
clf;
hold on;
axis( [40, 70, 0, 0.1] );
plot( t, pulse0, 'k-', 'LineWidth', 2 );
plot( t, pulse1, 'g-', 'LineWidth', 2 );
plot( t, pulse2, 'r-', 'LineWidth', 2 );
title('azimi pulses for two different amounts of attenuation');
xlabel('t');
ylabel('u');

% plot frequency-dependent quality factors
figure(2);
clf;
hold on;
axis( [f(1), f(end), 0.5*Q2, 2*Q1] );
plot( f, Qf1, 'g-', 'LineWidth', 2 );
plot( f, Qf2, 'r-', 'LineWidth', 2 );
plot( [f(1), f(end)], [Q1, Q1], 'k:', 'LineWidth', 2 );
plot( [f(1), f(end)], [Q2, Q2], 'k:', 'LineWidth', 2 );
title('Q(f) associated with the two azimi pulses');
xlabel('f');
ylabel('Q');
```

```
% plot frequency-dependent quality factors
figure(4);
clf;
hold on;
axis( [f(1), f(end), 0, 2*c0] );
plot( f, cw1, 'g-', 'LineWidth', 2 );
plot( f, cw2, 'r-', 'LineWidth', 2 );
plot( [f(1), f(end)], [c0, c0], 'k:', 'LineWidth', 2 );
title('c(f) associated with the two azimi pulses');
xlabel('f');
ylabel('c');

% standard fft setup
fny = f(end);
Df = f(2)-f(1);
N2 = N/2+1;

% compute spectral ratio
pulse1t = fft( pulse1 );
pulse1t = pulse1t(1:N2);
A1 = abs( pulse1t );
pulse2t = fft( pulse2 );
pulse2t = pulse2t(1:N2);
A2 = abs( pulse2t );
r = A2 ./ A1;
r(1)=1; % reset zero-frequency value

% confine analysis to f<fc band
fc = 0.5;
Nc = floor(fc/Df)+1;
f = f(1:Nc);
r = r(1:Nc);
logr = log(r);

% fit straight line to log spectral ratio
G = [ones(Nc,1), f];
mest = (G'*G)\(G'*logr);
b = mest(2);
logrpre = G*mest;
% A = A0 exp( -w x/2Qc ) = A0 exp( -f pi tstar )
% b = -pi tstar  so  tstar = -b/pi

% compare true and predicted tstar
Dtstarest = -b/pi;
Dtstartrue = x/(Q2*c0) - x/(Q1*c0);
```

```
fprintf('Dtstartrue %f Dtstarest %f\n', Dtstartrue, Dtstarest );

% plot spectral ratio and straight line fit
figure(3);
clf;
hold on;
axis( [0, fc, -2, 1] );
plot( f, logr, 'k-', 'LineWidth', 2 );
plot( f, logrpre, 'ro', 'LineWidth', 2 );
title('log spectral ratio (solid) of the two pulses with linear fit
(circles)');
xlabel('f');
ylabel('pulse2(f) / pulse1(f)');

function [ t, pulse0, pulse, f, Qw, cw ] = azimi( N, Dt, x, c0, Q, f0
)

% input parameters:
% f0 corner frequency of Azimi Q model, in hz (e.g. 50)
% c0 base velocity in km/s (e.g. 4.5);
% x  propagation distance in km (e.g. 100)
% Q  low frequency quality factor (e.g. 10)
% N  number of samples in pulse (e.g. 1024);
% Dt sampling interbal (e.g. 0.1)

% returned values
% t time array
% pulse0 input pulse, a unit spike at time N/2
% pulse attentated pulse
% f frequencies in Hz
% Qw frequency dependent quality factors
% cw frequency dependent phase velocities


% time series
t = Dt*[0:N-1]';
pulse0 = zeros(N,1);
pulse0(N/2)=1;

% standard fft setup
fny = 1/(2*Dt);
N2 = N/2+1;
df = fny / (N/2);
f = df*[0:N2-1]';
w = 2*pi*f;
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```
w0 = 2*pi*f0;

% attenuation factor
% exp( -a(w) x ) = exp( - wx / 2Qc )
%
% propagation law with velocity c=w/k and slowness s=1/c=k/w
% exp{ i(kx - wt) } = exp{ iw(sx - t) }
% propagation law
% exp( iwsx )

% Azimi's second law en.wikipedia.org/wiki/Azimi_Q_models
%
% a(w) = a2 |w| / [ 1 + a3 |w| ]
% note that for w<<w0 a(w) =
%
% s(w) = s0 + 2 a2 ln( a3 w ) / [ pi (1 - a3^2 w^2 ) ]

% now set a3 = 1/w0 where w0 is a reference frequency
% and set a2 = 1 / (2Qc0) where c0 is a reference velocity
% so that
% a(w) = (1/2Qc0) |w| / [ 1 +  |w/w0| ]
% so for w/w0 << 1
% a(w) = w/(2Qc0)  and Q(w) = w/(2 a c0)

a2 = 1 / (2*Q*c0);
a3 = 1 / w0;
a = a2*w ./ ( 1 + a3.*w );
Qw = w ./ (2.*a.*c0);
Qw(1) = Q;
ds = -2*a2*log(a3*w) ./ (pi*(1-(a3^2).*(w.^2 )));
ds(1)=0;

cw = 1./( (1/c0) + ds );

dt = fft(pulse0);
dp = dt(1:N2);
dp = dp .* exp(-a*x) .* exp(-complex(0,1)*w.*ds.*x);
dtnew = [dp(1:N2);conj(dp(N2-1:-1:2))]; % fold out negative
frequencies
pulse = ifft(dtnew);

end
```