1.3 SEQUENCES IN GENERAL – SERIES

♦ SEQUENCE

A **sequence** is just an ordered list of numbers (**terms** in a definite order). For example

$$\begin{array}{cccccc} 2, & 5, & 13, & 5, & -4, & \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ 1^{st} & 2^{nd} & 3^{rd} & 4^{th} & 5^{th} & \\ \text{term} & \text{term} & \text{term} & \text{term} & \text{term} & \end{array}$$

Usually, the terms of a sequence follow a specific pattern, for example

0,2,4,6,8,10,…	(even numbers)
1,3,5,7,9,11,…	(odd numbers)
5,10,15,20,25,…	(positive multiples of 5)
2,4,8,16,32,…	(powers of 2)

We use the notation u_n to describe the n-th term. Thus, the terms of the sequence are denoted by

$$u_1, u_2, u_3, u_4, u_5, \ldots$$

♦ SERIES

A **series** is just a sum of terms:

$S_n = u_1 + u_2 + u_3 + \cdots + u_n$ (the sum of the first n terms)

$S_\infty = u_1 + u_2 + u_3 + \cdots$ (the sum of all terms, ∞ terms)

We say that S_∞ is an infinite series, while the finite sums $S_1, S_2, S_3, \ldots$ are called **partial sums**.

EXAMPLE 1

Consider the sequence

1,3,5,7,9,11,... (odd numbers)

Some of the terms are the following

$u_1=1$, $u_2=3$, $u_3=5$, $u_6=11$, $u_{10}=19$

Also,

$S_1=1$,

$S_2=1+3=4$,

$S_3=1+3+5=9$,

$S_4=1+3+5+7=16$

Finally,

$S_\infty=1+3+5+7+\cdots$ (in this case the result is $+\infty$)

- SIGMA NOTATION ($\sum_{n=1}^{k}$)

Instead of writing

$$u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9$$

we may write

$$\sum_{n=1}^{9} u_n$$

It stands for the sum of all terms u_n, where n ranges from 1 to 9.

In general,

$$\boxed{\sum_{n=1}^{k} u_n}$$

expresses the sum of all terms u_n, where n ranges from 1 to k.

We may also start with another value for n, instead of 1, e.g. $\sum_{n=4}^{9} u_n$

EXAMPLE 2

- $\sum_{n=1}^{3} 2^n = 2^1 + 2^2 + 2^3 = 2+4+8 = 14$
- $\sum_{n=1}^{4} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{12+6+4+3}{12} = \frac{25}{12}$
- $\sum_{k=1}^{3} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4+2+1}{8} = \frac{7}{8}$
- $\sum_{n=3}^{6} (2n+1) = 7+9+11+13 = 40$
- $\sum_{x=3}^{20} \frac{x}{x+2} = \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \cdots + \frac{20}{22} =$ ... whatever that is, I don't mind!!!

We can also express an infinite sum as follows

- $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ (it never finishes!)

The result is 1. (I know it looks strange, but believe me, it is right!)

♦ NOTICE

There are two basic ways to describe a sequence

A) <u>by a GENERAL FORMULA</u>

We just describe the general term u_n in terms of n.

For example, $u_n = 2n$ (It gives $u_1 = 2$, $u_2 = 4$, $u_3 = 6$, ...)

It is the sequence 2,4,6,8,10,...

EXAMPLE 3

$u_n = n^2$ is the sequence $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

that is 1, 4, 9, 16, 25, ...

$u_n = 2^n$ is the sequence 2, 4, 8, 16, 32, ...

B) by a RECURSIVE RELATION (mainly for Math HL)

Given: u_1 , the first term

u_{n+1} in terms of u_n

For example,

$u_1 = 10$

$u_{n+1} = u_n + 2$

This says that the first term is 10 and then

$u_2 = u_1+2$

$u_3 = u_2+2$

$u_4 = u_3+2$ and so on.

In simple words, begin with 10 and keep adding 2 in order to find the following term.

It is the sequence 10, 12, 14, 16, 18, …

EXAMPLE 4

$u_1 = 3 \qquad u_{n+1} = 2u_n + 5$

It is the sequence 3, 11, 27, 59, …

EXAMPLE 5

Sometimes, we are given the first two terms u_1, u_2 and then a recursive formula for u_{n+1} in terms of u_n and u_{n-1}.
The most famous sequence of this form is the **Fibonacci sequence**

$u_1 = 1,\ u_2 = 1$

$u_{n+1} = u_n + u_{n-1}$

In other words,

we add u_1, u_2 in order to obtain u_3,

we add u_2, u_3 in order to obtain u_4, and so on.

It is the sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …
