Multidisciplinary Optimization on Water Kettle

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Abstract: The main object of this paper is performing a multidisciplinary optimization on a water kettle. Different parameters of a water kettle has been taken into concern for the optimization system.

1. Introduction

Generally, a water kettle is a kind of pot that dedicated for boiling water, usually with a lid, spout and handle. The main concern to a kettle is the speed of boiling water (Hoad, 1993). It is known that there are different factors that will affect a kettle's performance on heating water. In this project, electric kettle will be taking concern on analysis the boiling speed.

2. Object Function (Target of optimization)

The objective function is to find the minimum time to heat up a volume of water to the temperature of 100°C (volume of water must be greater than the minimum one).

3. Design Variable

Table 1 List of Design Variables

Symbol	Unit	Description
h	[m]	Height of the kettle
D	[m]	Diameter of base of the kettle
α	[deg]	Angle between the Base and side wall of kettle
w	[m]	Thickness of the wall

4. Constraints

Table 2 List of Constraints

Symbol	Unit	Description
$\boldsymbol{h > 0}$ $\boldsymbol{D > 0}$ $\boldsymbol{H > 0}$ $\boldsymbol{d > 0}$	[m]	All dimension of kettle should be greater than zero
$\boldsymbol{0 < \alpha < 90}$	[°]	Angle between the base and side wall of kettle between 0 and 90 deg.
$\boldsymbol{0 < w < 0{,}005}$	[m]	Thickness of the wall between 0 and 5 mm
$\boldsymbol{t \leq 500}$	[s]	Time needed to boil the water shorter than 500 seconds

5. Geometrical Dependencies

A basic model of a kettle and its geometrical parameters are shown in the below diagram.

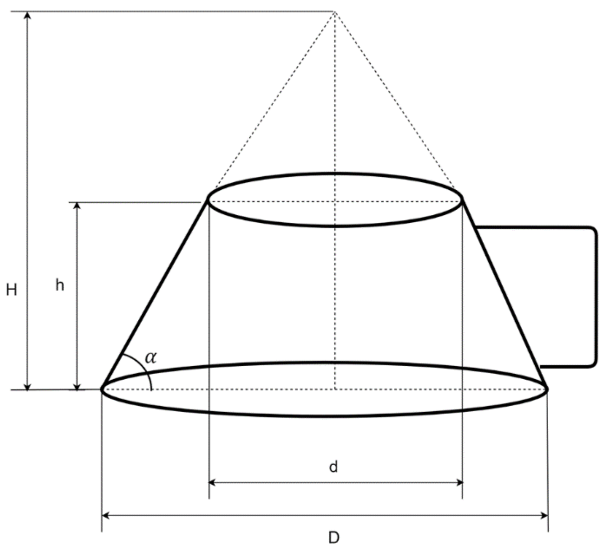

Figure 1 Geometrical Parameter of Kettle

And related parameter can be calculated by following equations.

$$\mathrm{H} = \frac{D}{2}\tan\alpha \quad \text{(Eq 1)}$$

$$\frac{d}{2} = \frac{(H-h)}{\tan\alpha} \quad \text{(Eq 2)}$$

Volume of the kettle can be calculated by:

$$V_c = \frac{1}{3}\pi\left(\frac{D}{2}\right)^2 \mathrm{H}$$

Then,

$$V_{c1} = \frac{1}{3}\pi\left(\frac{d}{2}\right)^2 (\mathrm{H}-\mathrm{h})$$

Hence, (Eq 3)

$$\mathrm{V} = V_c - V_{c1} = \frac{1}{3}\pi\left(\frac{D}{2}\right)^2 \mathrm{H} - \frac{1}{3}\pi\left(\frac{d}{2}\right)^2 \mathrm{H} + \frac{1}{3}\pi\left(\frac{d}{2}\right)^2 \mathrm{h}$$

Length of the side wall can be calculated by:

$$\mathrm{L} = \frac{\left(\frac{D}{2}\right)}{\cos\alpha} \quad \text{(Eq 4)}$$

$$\mathrm{l} = \frac{\left(\frac{d}{2}\right)}{\cos\alpha} \quad \text{(Eq 5)}$$

Total surface area can be calculated by:

$$\mathrm{S} = \pi R^2 + \pi\left(\frac{D}{2}\right)\mathrm{L} - \pi\left(\frac{d}{2}\right)\mathrm{l} + \pi\left(\frac{d}{2}\right)^2 \quad \text{(Eq 6)}$$

Side & Upper surface area can be calculated by:

$$S_{side} = \pi\left(\frac{D}{2}\right)\mathrm{L} - \pi\left(\frac{d}{2}\right) \quad \text{(Eq 7)}$$

$$S_{upper} = \pi\left(\frac{d}{2}\right)^2 \quad \text{(Eq 8)}$$

Volume of water inside kettle can be calculated by:

$$\mathrm{V} = \mathrm{const} \quad \text{(Eq 9)}$$

Mass of water inside the kettle can be calculated by:

$$\mathrm{m} = \rho_{H2O} \cdot V \quad \text{(Eq 10)}$$

Density in relation to temperate can be calculated by:

$$\rho_{\mathrm{H2O}} = 1000\left[1 - \left[\frac{\mathrm{T_{int}} + 288.9414}{508929(\mathrm{T_{int}} + 68.129630)}\right](T_{int} - 3.9863)^2\right]$$

(Eq 11)

Where,

ρ_{H2O} – water density $\left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right]$

T_{int} – temperature of water [°C]

Input power of the heater can be calculated by:

$$\mathrm{P} = \mathrm{I} \cdot U = \frac{U}{R^2} \quad \text{(Eq 12)}$$

Where,

I – current [A]

U – voltage [V]

R – resistance of the coil [Ω]

Input energy can be calculated by:

$$\mathrm{E} = \eta P t \quad \text{(Eq 13)}$$

Where,

η – efficiency [%]

t – time [s]

Rate of change of heat content can be calculated by:

$$Q_i = c_v \mathrm{m}\frac{dT}{dt} \quad \text{(Eq 14)}$$

Where,

c_v – specific heat of water

m – mass of the liquid [kg[

T – temperature at certain time [K]

T_{boil} – boiling temperature

The heat flow through the wall can be illustrated by below diagram:

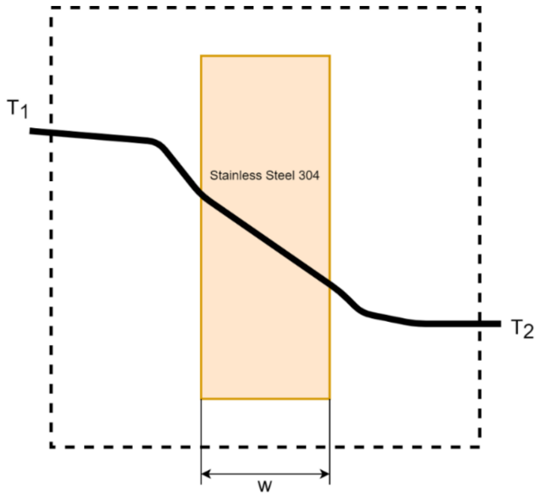

Figure 2 Heat exchange through wall

The heat exchange rate with surroundings can be calculated by:

$$\dot{Q} = \left(S_{side} + S_{upper}\right) \frac{T(t) - T_{ext}}{\frac{1}{\alpha_1} + \frac{w}{\lambda} + \frac{1}{\alpha_2}} \qquad \text{(Eq 15)}$$

Where,

T_{ext} – air temperature at normal conditions

α_1 – heat transfer coefficient between water and steel

α_2 – heat transfer coefficient between steel and air

λ – heat transfer coefficient for steel

w – thickness of the wall

The heat transfer coefficient can be calculated by:

$$\mathrm{k} = \frac{1}{\frac{1}{\alpha_1} + \frac{w}{\lambda} + \frac{1}{\alpha_2}} \qquad \text{(Eq 16)}$$

The energy balance equation can be calculated by:

$$\Sigma\, Q_i = \frac{\partial E}{\partial t} - \Sigma\, \dot{Q} \qquad \text{(Eq 17)}$$

Thus,

$$\sum c_p m \frac{dT}{dt} = \frac{d(\eta P t)}{dt} - \sum \left(S_{side} + S_{upper}\right) \frac{T(t) - T_{ext}}{\frac{1}{\alpha_1} + \frac{w}{\lambda} + \frac{1}{\alpha_2}}$$

(Eq 18)

6. Fixed Parameter

In order to simplify the model, thermal coefficients are assumed to be constant, although in reality their value may vary with changing temperature.

Table 3 List of Fix Parameters

Symbol	Unit	Value	Description
$\boldsymbol{\alpha_1}$	$\left[\frac{\mathrm{W}}{\mathrm{m^2K}}\right]$	5000	Convective heat transfer coefficient for water
$\boldsymbol{\alpha_2}$	$\left[\frac{\mathrm{W}}{\mathrm{m^2K}}\right]$	50	Convective heat transfer coefficient for air
$\boldsymbol{\lambda}$	$\left[\frac{\mathrm{W}}{\mathrm{mK}}\right]$	16	Thermal conductivity coefficient
$\boldsymbol{c_p}$	$\left[\frac{\mathrm{kJ}}{\mathrm{kgK}}\right]$	4.22	Specific heat of water
$\boldsymbol{\rho_{H2O}}$	$\left[\frac{\mathrm{kg}}{\mathrm{m^3}}\right]$	958.38	Density of water
$\boldsymbol{T_{ext}}$	[°C]	25	Ambient temperature
$\boldsymbol{T_{init}}$	[°C]	20	Initial temperature of water
$\boldsymbol{U}$	[V]	230	Voltage
$\boldsymbol{I}$	[A]	22	Current
$\boldsymbol{\eta}$	[%]	85	Efficiency of heater

7. Formula used

Table 4 List of Formula Used

1.	$\mathbf{V} = \frac{1}{3}\pi\left(\frac{D}{2}\right)^{L}\mathbf{H} - \frac{1}{3}\pi\left(\frac{d}{2}\right)^{2}H + \frac{1}{3}\pi\left(\frac{d}{2}\right)^{2}h$
2.	$S_s + S_u = \pi\left(\frac{D}{2}\right)L - \pi\left(\frac{d}{2}\right)L + \pi\left(\frac{d}{2}\right)^2$
3.	$H = \frac{D}{2}\tan\alpha$
4.	$d = \frac{2(H-h)}{\tan\alpha}$
5.	$L = \frac{D}{2\cos\alpha}$

6.	$$l = \frac{d}{2\cos\alpha}$$
7.	$$m = \varrho\ H_2O(100deg) \times V$$
8.	$$c_p m \frac{dT}{dt} = \frac{\eta UIdt}{dt} - (S_s + S_u)\frac{T(t) - T_{ext}}{\frac{1}{\alpha_1} + \frac{w}{\lambda} + \frac{1}{\alpha_2}}$$

8. Partitioning of Equations

Objective function: Minimizing time needed to heat water to temperature of 100°C

Design variables: h , D , α , w

Constrains: V > 1 dm^3, S>0, d>0

Parameters: density, voltage, voltage, current, efficiency, cp, T_{ext} , T_{init}

	n, D ↓	D ↓	D, α ↓	h, α ↓	D, α ↓	α ↓	ϱ ↓	W ↓	
	1	2	3	4	5	6	7	8	
1	*						1		
2		*						1	
3	1		*	1					
4	1	1		*		1			
5		1			*				
6		1				*			
7							*	1	
8								*	t→

Output

Figure 3 Partition Graph 1

Output	3	5	1	2	4	6	7	8
3	*		1		1			
5		*		1				
1			*				1	
2				*				1
4			1	1	*	1		
6				1		*		
7							*	1
8								*

Input

Figure 4 Partition Graph 2

Output	3	5	4	7	1	2	6	8
3	*		1		1			
5		*				1		
4			*		1	1	1	
7				*				1
1				1	*			
2						*		1
6						1	*	
8								*

Input

Figure 5 Partition Graph 3

Output	3	5	4	1	7	2	6	8
3	*		1	1				
5		*				1		
4			*	1		1	1	
1				*	1			
7					*			1
2						*		1
6						1	*	
8								*

Input

Figure 6 Partition Graph 4

Output	3	5	4	1	7	6	2	8
3	*		1	1				
5		*					1	
4			*	1		1	1	
1				*	1			
7					*			1
6						*	1	
2							*	1
8								*

Input

Figure 7 Partition Graph 5

7. Critical Path

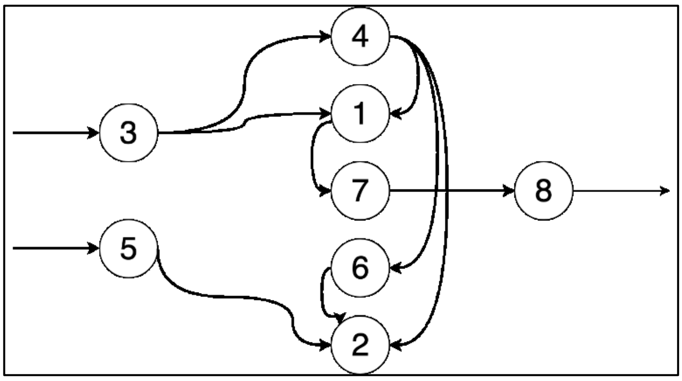

Figure 8 Order of calculation

8. Parameter study

There are 4 design variable, 4 factors and 4 levels per each. So the total number of experiment can be calculated by:

$$1 + n(l - 1) = 1 + 4(4 - 1) = 13 \text{ expts} \qquad \text{(Eq 19)}$$

Table 5 Experiment on 13 factors

Expt No.	Factor				Results	
	h [m]	D [m]	α [deg]	w [m]	V [m^3]	t [s]
1	0.1	0.2	70	0.001	0.0021	162.5
2	0.05	0.2	70	0.001	0.0013	100
3	0.15	0.2	70	0.001	0.0026	200
4	0.2	0.2	70	0.001	0.0028	225
5	0.1	0.15	70	0.001	0.0010	75
6	0.1	0.225	70	0.001	0.0028	225
7	0.1	0.25	70	0.001	0.0036	287.5
8	0.1	0.2	60	0.001	0.0017	125
9	0.1	0.2	65	0.001	0.0019	150
10	0.1	0.2	70	0.001	0.0021	162,5
11	0.1	0.2	70	0.0005	0.0021	162,5
12	0.1	0.2	70	0.002	0.0021	162,5
13	0.1	0.2	70	0.003	0.0021	162,5

And the best design variable are:

Table 6 Best design variables

Symbol	Value
h **[m]**	0.05
D **[m]**	0.15
α **[deg]**	60
w **[m]**	0.001
V **[m³]**	5.87e-4
t **[s]**	44.97

The obtained volume for this case is too small and the process should be repeated.

8. Optimization from MatLab Optimization Toolbox

5 MATLAB M file have been prepared from the optimization process.

Table 7 MatLab File & Description

File name	Description
energy_balance.m	Heat transfer efficiency formula, input function for ode45 solver
mass.m	Water mass equation
surface_area.m	Equation for total surface area of the kettle
volume.m	Equation for water volume in kettle
time_100C.m	Objective function calculating time
constraints.m	Function calculating constraints for Optimization Tool
test_parameters.m	Input of all used parameter, design variable data, which will be used for optimization process

8.1 Optimization Configuration & Result

Used Solver: fmincon – constrained nonlinear minimization

Algorithm: Interior Point

Start point: [0.1 0.15 60 0.0005]

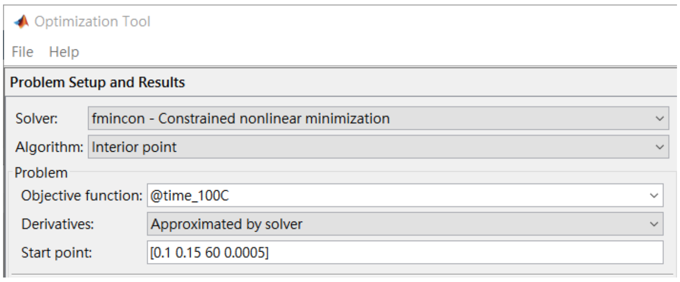

Figure 9 Optimization Toolbox Configuration

Table 8 Result of Optimization

Final point	**Value**	**Unit**	**Description**
x(1)	0.278	m	Kettle height
x(2)	0.144	m	Base diameter
x(3)	79.025	deg	Angle between base and side-wall
x(4)	0.004	m	Wall thickness

It will take 150second for the kettle to boil the water to 100℃ .

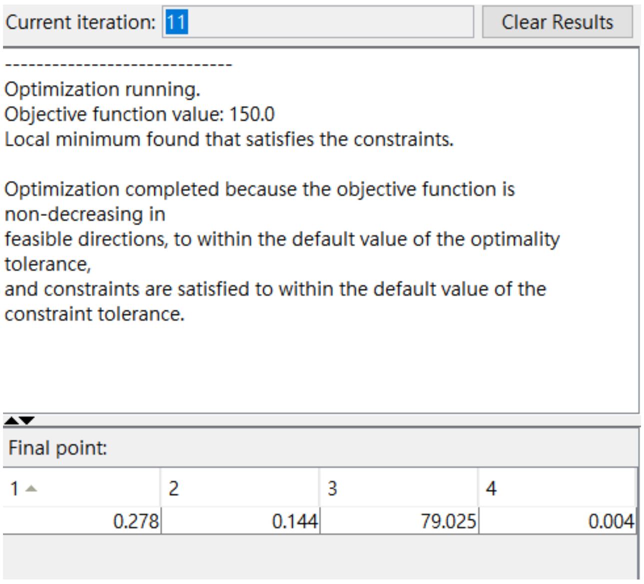

Figure 10 Optimization Result

9. Conclusion

The optimized result that returned from optimization toolbox is fruitful and reasonable. Value of kettle height, base diameter, angle between base & side wall and kettle thickness has got an achievable value. Therefore, it is believed that the multidisciplinary optimization for the kettle has succeeded.

10. Reference

[1] B. Barnes, G. R. Fulford: Mathematical Modelling with case studies. A Differential Equation Approach Using Maple, Taylor & Francis, London 2002

[2] Hoad, T. F. (1993). English Etymology. London: Oxford University Press.