Mathematical Underpinnings Lab 1 28.02.2024

Task 1 (logistic regression - generative vs discriminative approach)

a) **Generative approach**

- Generate a sample from $\mathcal{N}(m_1, \Sigma)$ $(Y = 1)$ and $\mathcal{N}(m_2, \Sigma)$ $(Y = -1)$[1]. Does the distribution of $P(Y = 1|X = x)$ correspond to a logistic model?
- If the distribution of $P(Y = 1|X = x)$ corresponds to a logistic model, find the formulas for the parameters of the logistic model (the coefficients and the intercept). Compare obtained parameters β with the estimated $\hat{\beta}$.

Simulations details: each sample from a normal distribution consists of $n = 500$ observations (in total we have 1000 observations), $m_1 = (1, 1)$, $m_2 = (0, 0)$, and

$$\Sigma = \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix}.$$

Assume that $\pi = P(Y = 1) = 0.5$.

b) **Discriminative approach**

- Generate a sample from a gaussian mixture

$$\frac{1}{2}\mathcal{N}(m_1, \Sigma) + \frac{1}{2}\mathcal{N}(m_2, \Sigma).$$

  Next, generate y_i for $i = 1, 2, \ldots, n$ from Bernoulli distribution with probability of success $1/(1 + \exp(-\beta_0 - \beta x_i)$, where $\beta_0 = -2$ and $\beta = (2, 2)$. Fit a logistic model and compare β with the estimated $\hat{\beta}$.
- Draw a scatterplot of data generated in a) and in b). Compare the distributions. What distinguishes the generative approach from the discriminative approach?

c) Generate n_1 observations from the distribution $\mathrm{Exp}(1)$[2] (corresponding to $Y = 1$) and n_2 observations from a certain distribution such that $P(Y = 1|X = x)$ corresponds to a logistic model (you may fix $n_1 = 1000$, $n_2 = 2000$ and $\beta_1 = 1$). Fit the model.
Q1. What is the distribution of $f_{X|Y=-1}$?
Q2. Once π is fixed, do we have freedom to choose an intercept in the logistic model thus constructed?
Q3. If π is not given, is the intercept in the logistic model uniquely determined?
If that is doable, given β_1 and π compute β_0.

[1] Recall: $\phi_{\mu,\Sigma}(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp(-\frac{1}{2}(x - \mu)'\Sigma^{-1}(x - \mu))$

[2] Recall: $\phi_\lambda(x) = \lambda \exp(-\lambda x)\mathbb{I}(x \geq 0)$