# Assignment 1

September 27, 2020

# 1 Section 1

## 1.1 Answer 1

1. Since X can be modelled as binomial variable

$$P\{X = i\} = \binom{10}{i}(0.2)^i(0.8)^{10-i} \ \forall \ i \ \in \{0, 1, ..10\}$$

2. Expected Value

$$E[X] = np = 10 * 0.2 = 2$$

Variance

$$\mathsf{Var}(X) = np(1-p) = 10 * (0.2)(0.8) = 1.6$$

3. Y can be described as a linear function of X

$$Y = 2X - 3$$

Expected Value

$$E[Y] = 2 * E[X] - 3 = 1$$

Variance

$$\mathsf{Var}(Y) = 2^2 * \mathsf{Var}(X) = 6.4$$

4. Let money earned be a random variable P

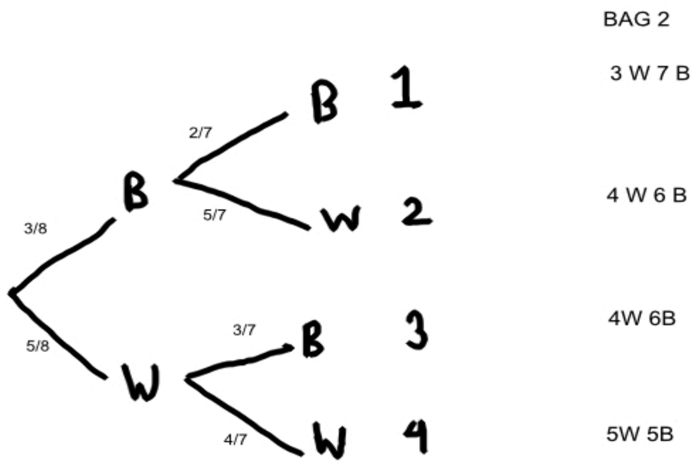
$$P = X^2$$

Since

$$E[X^2] = Var(X) + E[X]^2$$
$$E[P] = 1.6 + 4 = 5.6$$

## 1.2 Answer 2

The process of taking out the balls from the bag1 can be represented by the following tree




Let $B_i$ represent the ith Branch E = Probability of drawing 1 white and 1 black ball from bag 2
E = (BW OR WB)

$$P(E|B_1) = \frac{3}{10}.\frac{7}{9} + \frac{7}{10}.\frac{3}{9} = \frac{42}{90} \tag{1}$$

$$P(E|B_2) = \frac{4}{10}.\frac{6}{9} + \frac{6}{10}.\frac{4}{9} = \frac{48}{90} \tag{2}$$

$$P(E|B_3) = \frac{4}{10}.\frac{6}{9} + \frac{6}{10}.\frac{4}{9} = \frac{48}{90} \tag{3}$$

$$P(E|B_4) = \frac{5}{10}.\frac{5}{9} + \frac{5}{10}.\frac{5}{9} = \frac{50}{90} \tag{4}$$

$$\tag{5}$$

$$P(E) = P(E|B_1)P(B_1) + P(E|B_2)P(B_2) + P(E|B_3)P(B_3) + P(E|B_4)P(B_4) \tag{6}$$

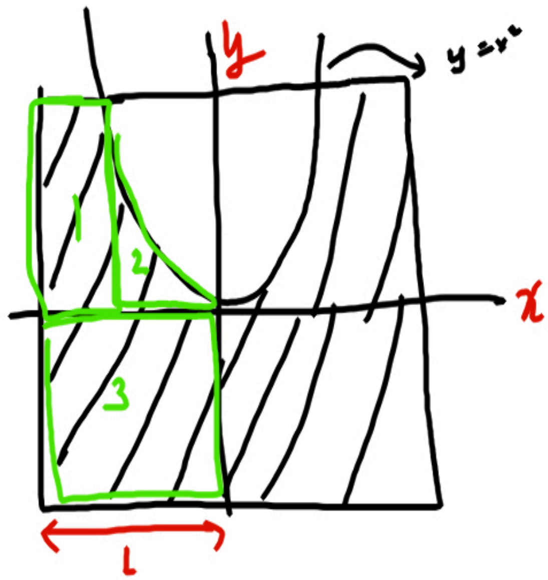
$$P(E) = \frac{42}{90} * \frac{6}{56} + \frac{48}{90} * \frac{15}{56} + \frac{48}{90} * \frac{15}{56} + \frac{50}{90} * \frac{20}{56} = 0.534126984 \tag{7}$$

## 1.3 Answer 3

For the equation to have at least one root $D >= 0$

$$\Rightarrow (2k)^2 >= 4m \Rightarrow k^2 >= m$$

Assuming $y = m$ and $x = k$, we get $x^2 > y$




For $-L <= x, y <= L$ we get the following graph
Area under the graph $= 2 * (Area_1 + Area_2 + Area_3) = A$
Area of the total domain $= (L + L)^2 = 4L^2 = D$
Probabilty required is

$$
\begin{aligned}
&= \lim_{l \to \infty} (A/D) \\
&= \lim_{l \to \infty} \frac{2 * (Area_1 + Area_2 + Area_3)}{4L^2} \\
&= \lim_{l \to \infty} \frac{(L - \sqrt{L})L + \int_0^{\sqrt{L}} x^2 dx + L^2}{2L^2} \\
&= \lim_{l \to \infty} \frac{L^2 - L^{3/2} + L^{3/2}/3 + L^2}{2L^2} \\
&= \lim_{l \to \infty} \frac{2L^2 - 2/3 * L^{3/2}}{2L^2} \\
&= \lim_{l \to \infty} 1 - \frac{1}{3\sqrt{L}} \\
&= 1
\end{aligned}
$$

## 1.4 Answer 4

The following cases are possible of the two children are possible
$\{(B, B), (B, G), (G, B), (G, G)\}$
where B refers to Boy and G refers to Girl

1. If Atsi is a girl
   P(Sister | Atsi is girl) = P({(G,G)}) / P({(G,B), (B,G), P(G,G)}) = $\frac{1}{3}$
2. If Atsi is a boy
   P(Sister | Atsi is boy) = P({(B,G), (G,B)}) / P({(G,B), (B,G), P(B,B)}) = $\frac{2}{3}$

## 1.5 Answer 5

F, A, S are correct; L, H are skipped; T is extra;
P(FAST instead of FLASH) = P(correct(F)).P(correct(A)).P(correct(S)).P(skipped(L)).P(skipped(H)).P(extra(T))
$= 0.8^3 * 0.1^2 * 0.1$
$= 0.000512$

## 1.6 Answer 6

Probability that person$_1$ has a unique birthday $(p_1)$ = 365/ 365
Probability that person$_2$ has a unique birthday $(p_2)$ = 364/ 365

....
Probability that person$_{30}$ has a unique birthday ($p_{30}$)= 335/ 365
P(All people have unique birthday) = $p_1 p_2 .. p_{30} = \frac{365.364.363..335}{365^{30}} = 0.2695$
P(At least two people share birthday) = 1 - P(All people have unique birthday) = 0.7304

## 1.7 Answer 7

The number of total paths to reach (n,m) from (0, 0) = The number of ways arrange n right steps and m down steps in n + m spaces

$$P_{(0,0)\to(n,m)} = \binom{n+m}{n} \tag{1}$$

The number of paths to reach (n,m) via some (x,y) = (Number of paths to reach (x,y) from (0, 0)) $*$ (Number of paths to reach (n, m) from (x, y)) By following similar logic as stated before

$$P_{(0,0)\to(x,y)} * P_{(x,y)\to(n,m)} = \binom{x+y}{x} * \binom{(n-x)+(m-y)}{n-x} \tag{2}$$

Thus probability of going via (x,y) to (n, m) = (2)/(1)

$$= \frac{\binom{x+y}{x} * \binom{(n-x)+(m-y)}{n-x}}{\binom{n+m}{n}}$$

## 1.8 Answer 8

Since $P(E)$ and $P(F)$ are independent

$$P(E \cap F) = P(E)P(F) = 1/6 \tag{1}$$

Also applying Demorgan's Law on

$$P(\bar{E} \cap \bar{F}) = 1/3$$
$$\Rightarrow P(E \cup F) = 2/3 \tag{2}$$

Using $P(E \cap F) = 1/6, (2)$ and $P(E \cup F) + P(E \cap F) = P(E) + P(F)$

$$P(E) + P(F) = \frac{5}{6} \tag{3}$$

Substitute the value of $P(F)$ from (1) into (3), and solve to get P(E) = 1/2 or 1/3
From the question $(P(E) - P(F)(1 - P(F)) > 0 \Rightarrow P(E) > P(F)$ since $1 - P(F) >= 0$
Thus P(E) = 1/3 and P(F) = 1/2

## 1.9 Answer 9

Let Q be the event that at least one container has 3 balls
P(Every container has <= 2 balls) = 1 - P(Q)
This question can be formulated as sum i.e.

$$x_1 + x_2 + x_3 ... x_n = n$$

The total number of distribution possible is

$$= \binom{n+n-1}{n-1} = \binom{2n-1}{n-1} \tag{1}$$

Now for P(Q) we assume that 3 balls have already been given to $x_1$, so total number of balls is now n - 3.

$$x_1 + x_2 + x_3 ... x_n = n - 3$$

And the solution for this:

$$= \binom{n-3+n-1}{n-1} = \binom{2n-4}{n-1} \tag{2}$$

Now P(Q) = (2)/(1) since (2) is the required event and (1) is entire sample space

$$P(Q) = \frac{\binom{2n-4}{n-1}}{\binom{2n-1}{n-1}} \tag{3}$$
$$= \frac{n(n-1)(n-2)}{(2n-1)(2n-2)(2n-3)} \tag{4}$$
$$= \frac{n(n-2)}{2(2n-1)(2n-3)} \tag{5}$$

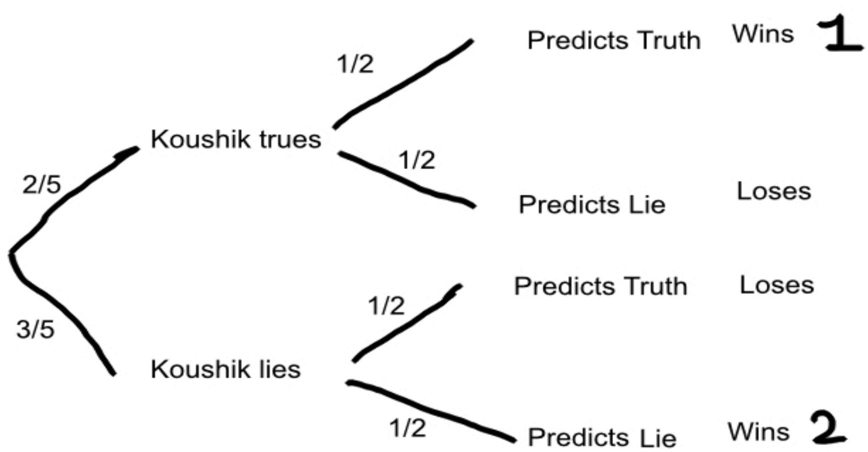
P(Every container has <= 2 balls)

$$= 1 - P(Q) \tag{6}$$
$$= 1 - \frac{n(n-2)}{2(2n-1)(2n-3)} \tag{7}$$
$$= \frac{7n^2 - 14n + 6}{(2n-1)(2n-3)} \tag{8}$$

# 2 Section 2

## 2.1 Answer 12

The situation can be modelled as probability tree.




$$P(Wins) = Branch_1 + Branch_2$$
$$P(Wins) = \frac{2}{5} * \frac{1}{2} + \frac{3}{5} * \frac{1}{2} = \frac{1}{2}$$

## 2.2 Answer 13

While doing a guess the probability of being correct is 1/4 and wrong is 3/4.
We can draw the following probability tree to represent this situation.


Since knowing, guessing and copying are disjoint conditions we can add the branches to get total probability.

$$\begin{aligned} P(\text{Knowing Answer}|\text{Correct}) &= \frac{Branch_1}{Branch_1 + Branch_3 + Branch_5} \\ &= \frac{\frac{1}{2}.\frac{1}{1}}{\frac{1}{2}.\frac{1}{1} + \frac{1}{3}.\frac{1}{4} + \frac{1}{6}.\frac{1}{8}} \\ &= \frac{24}{29} \end{aligned}$$

## 2.3 Answer 14

Since rolling of dice is independent events
P(Atleast one Pawn)

$$\begin{aligned} &= 1 - \text{P(No pawn on Dice1} \cap \text{No pawn on Dice2} \cap \text{No pawn on Dice3)} \\ &= 1 - \text{P(No pawn on Dice1).P(No pawn Dice2).P(No pawn on Dice3)} \\ &= 1 - \frac{5}{6}\frac{5}{6}\frac{5}{6} \\ &= \frac{91}{216} \end{aligned}$$

## 2.4 Answer 15

In this problem we can divide the required Event E into a set of disjoint events $\{S_2, S_4, S_6 ...\}$ where $S_i$ denotes that the event ends on the ith toss. From total probability theorem we can say

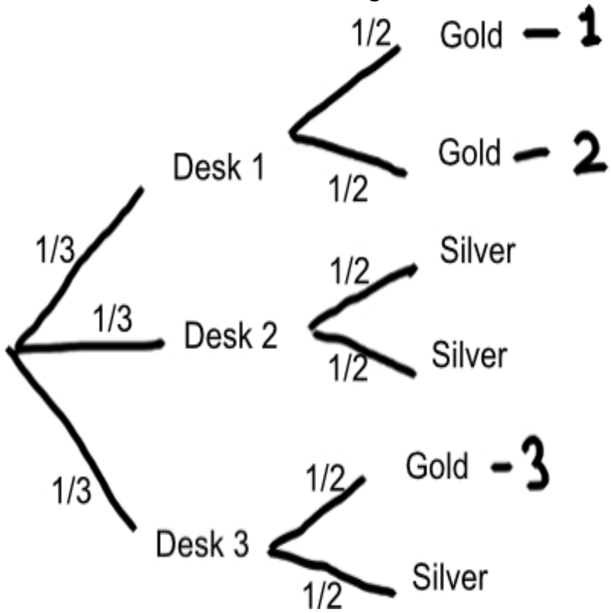
$$P(E) = P(S_2) + P(S_4) + P(S_6)...$$

where $P(S_i) = (1-p)^{i-1}p$ (tail in i - 1 tosses followed by a head)

$$\begin{aligned} P(E) &= P(S_2) + P(S_4) + P(S_6)... \\ &= p(1-p) + p(1-p)^3 + p(1-p)^5... \\ &= p(1-p)(1 + (1-p)^2 + (1-p)^4...) \\ &= p(1-p)\frac{1}{1-(1-p)^2} \\ &= \frac{p(1-p)}{p(2-p)} \\ &= \frac{1-p}{2-p} \end{aligned}$$

## 2.5 Answer 17

Let the Desk with two gold medals be Desk1, the desk with two silver medals be Desk2 and the desk with 1 one gold and 1 silver medal be Desk3. Let's label the gold medals in the drawers as $G_1, G_2, G_3$, where $G_1$ and $G_2$ belong to Desk1 and $G_3$ to Desk3.




The probability of initially selecting a desk and gold medal is independent of which medal it is and is 1/6 from above diagram. When selecting $G_1$ and $G_2$ the other drawer contain a gold medal.
P(Other drawer has gold medal | Gold medal was selected) =

$$\begin{aligned} &= \frac{P(G_1 \cup G_2)}{P(G_1 \cup G_2 \cup G_3)} \\ &= \frac{\frac{1}{6} + \frac{1}{6}}{\frac{1}{6} + \frac{1}{6} + \frac{1}{6}} \\ &= 2/3 \end{aligned} \tag{1}$$

## 2.6 Answer 18

To be confident she needs to have $\alpha$ hits before $\beta$ misses.
Let $P_{i,j}$ denote the probability that there are exactly j misses before i hits. To have exactly j failure before ith success this means there were i - 1 hits and j misses before the last hit.

$$\begin{aligned} P_{i,j} &= p * (P(\text{having a combination of i - 1 hits and j misses})) \\ &= p * \binom{i-1+j}{j} p^{i-1}(1-p)^j \\ &= \binom{i-1+j}{j} p^{i}(1-p)^j \end{aligned}$$

Now having $\alpha$ hits before $\beta$ misses, means we can have at most $\beta - 1$ misses before $\alpha$ hits

$$\begin{aligned} P(\alpha \text{ hits before } \beta \text{ misses}) &= \sum_{j=0}^{\beta-1} P_{\alpha,j} \\ &= \sum_{j=0}^{\beta-1} \binom{\alpha-1+j}{j} p^{\alpha}(1-p)^j \\ &= p^{\alpha} \sum_{j=0}^{\beta-1} \binom{\alpha-1+j}{j} (1-p)^j \end{aligned}$$

### 2.7 Answer 20

Let $T$ be the expected time to reach the city from the intersection.
The person on the intersection can take one of three different roads. Time to taken to reach city by $i$th road is $T_i$ hours

$$T_1 = 2 + T \tag{1}$$

$$T_2 = 4 + T \tag{2}$$

$$T_3 = 3 \tag{3}$$

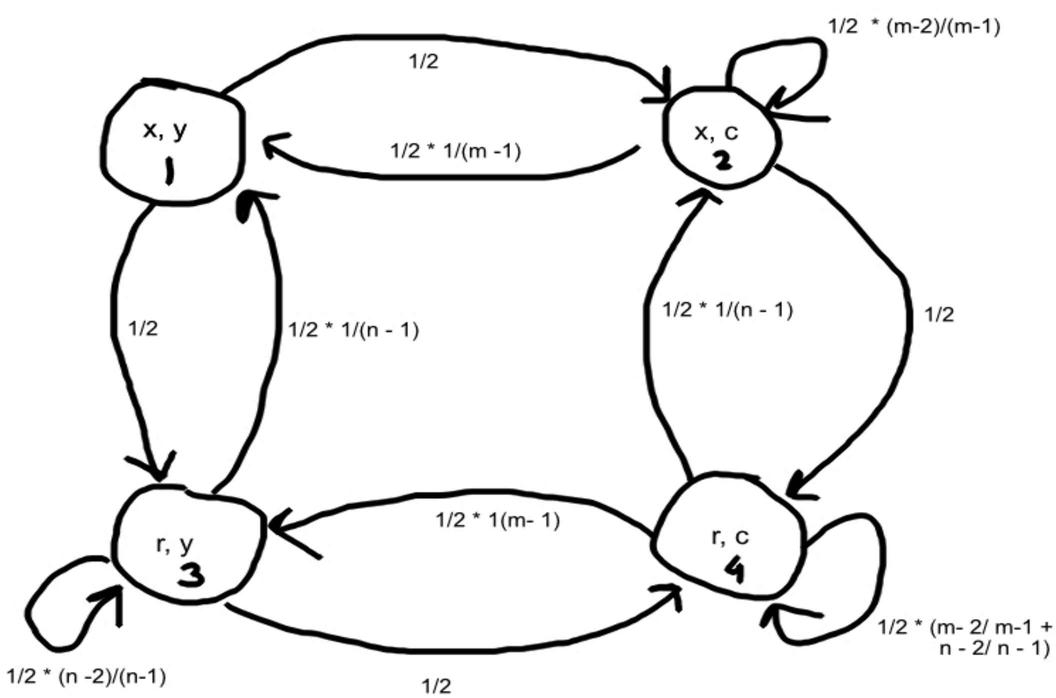
Atsi has $\frac{1}{3}$ probability of choosing a road.
Since choosing one of the 3 roads is disjoint and exhaustive we calculate $T$ by using the weighted sum of $T_i$ with their probabilities

$$\begin{aligned} T &= \frac{1}{3} * T_1 + \frac{1}{3} * T_2 + \frac{1}{3} * T_3 \\ &= \frac{1}{3}(T_1 + T_2 + T_3) \\ &= \frac{1}{3}(2 + T + 4 + T + 3) \\ &= \frac{1}{3}(9 + 2T) \\ \Rightarrow T &= 9 \text{ hours} \end{aligned}$$

## 3 Section 3

### 3.1 Answer 21

Note: By mistake i took n cells in a rows and m cells in a column while solving this questions
The following finite markov chain captures the process entierly




There are 4 states in the process above denoted as follows:

1. x, y (when we are on x,y cell)
2. x, c (same row as x, but c can be anything except y)
3. r, y (same column as y, but r can be anything except x)
4. r, c (any other cell not covered by previous three conditions)

Let $P_{ij}$ denote the probability of transition from state i to state j.
The probabilities are given in table below and can also be inferred from diagram above. (1, 2, 3, 4 denote the states and the $cell_{ij}$ denotes $P_{ij}$)

| $P_{ij}$ | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **1** | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| **2** | $\frac{1}{2(m-1)}$ | $\frac{m-2}{2(m-1)}$ | 0 | 1/2 |
| **3** | $\frac{1}{2(n-1)}$ | 0 | $\frac{n-2}{2(n-1)}$ | 1/2 |
| **4** | 0 | $\frac{1}{2(n-1)}$ | $\frac{1}{2(m-1)}$ | $\frac{m-2}{2(m-1)} + \frac{n-2}{2(n-1)}$ |

Recurrence Relations:
Let $r_{ij}(n) = P(X_n = j | X_0 = i)$ where $X_i$ denotes the ith state

$$r_{11}(k) = \sum_{s=1}^{4} r_{1s}(k-1)\ P_{s1} \tag{1}$$

In general

$$r_{ij}(k) = \sum_{s=1}^{4} r_{is}(k-1)\ P_{sj} \tag{2}$$

$$r_{ii}(0) = 1, r_{ij}(0) = 0\ (i \neq j) \tag{3}$$

Intuition behind recursion: For reaching any state, at the previous stage we will be at one of the 4 states. Hence we can partition the space into 4 disjoint events $r_{1i}(k-1), i = \{1, 2, 3, 4\}$.From each state the transition probability is known which is $P_{i1}$. By using total probability on these states we get the above recurrence.

## 3.2 Answer 22

Rename individuals to $N$ and $S$, where $N$ receives $n$ votes and $S$ receives $s$ votes and $n > s$
Let $n + s = t$ and $n - s = d$

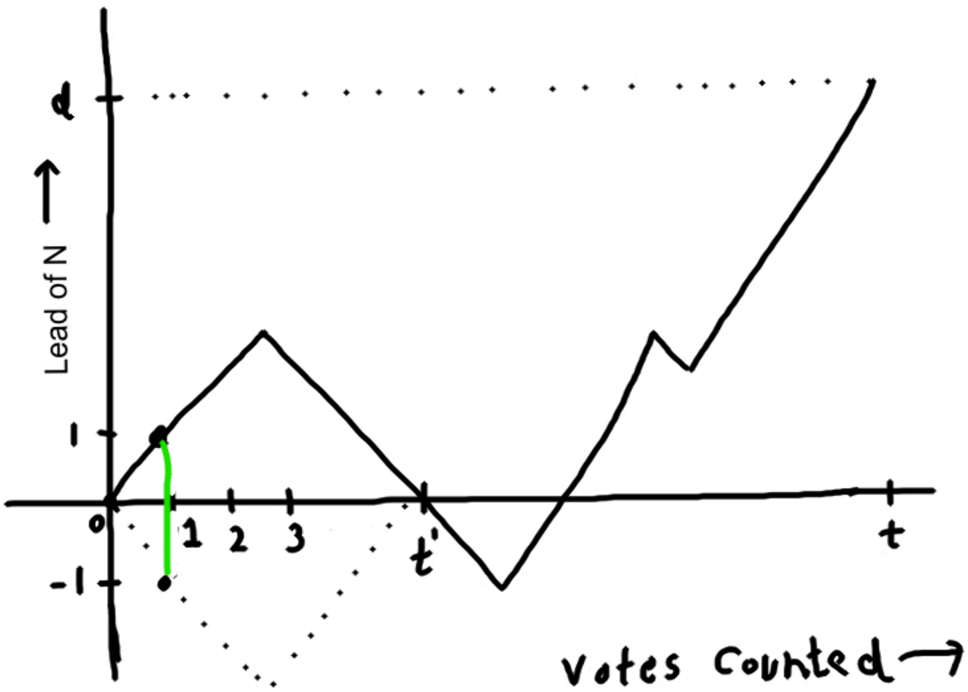
1. Let T denote total ways of counting $t$ votes. The total number of ways of counting $t$ votes is equivalent to choosing $n$ votes out of $t$

   $$T = \binom{t}{n} \tag{1}$$

   E = Set of all ways of counting votes such that $N$ is always ahead of $S$
   If the first vote is for $S$ then all those countings are not in E since $S$ is ahead of $N$ after the first vote.
   Thus if a counting that belongs to E, it must have the first vote for $N$
   Let G = Set of all ways of counting the votes such that first vote is for $N$
   Number of votes in G is equivalent of choosing $n-1$ votes for $N$ from the remaining of $t-1$ votes

   $$|G| = \binom{t-1}{n-1} \tag{2}$$

   We can represent a counting of votes as a simple walk (W)
   Each vote for $N$ represents a step up and each vote for $S$ represents a step down.
   The image below represents a general walk which passes belongs to G but does not belong to E




Plotting it on a graph we can make a few observations about the walk $W \in G - E$:

(a) The lead of N must be zero for atleast one point after the first vote is counted (tie), let us say the lead is zero on $t'$, $1 < t' < t$

(b) For each such walk we can take the reflection of the walk below the x axis from $[0, t']$ which can be joined with the previous from $(t', t]$ to give a new unique walk $W'$

(c) Let the set of all $W' = D$

(d) Since each $W$ gives a unique $W'$, $|G - E| = |D|$

(e) Since $W \in$ G, the lead of N for corresponding $W'$ must be -1 at t = 1, that means it passes through (1, -1)

(f) Consider a set $Z$ of all walks which start from (1, -1) and end at (t, d). Every walk in Z must cross the x axis at one point say $t'', 1 < t'' < t$. Let $Z'$ is the set of all walks in $Z$ reflected about x-axis from $[1, t'']$ and $(0, 0)$ point is added. $Z'$ is the same as $G - E$ where $t'' = t'$, $\Rightarrow |Z'| = |G - E|$

(g) Since $Z'$ is formed by reflection $Z$ and $D$ is formed by reflection of $G - E$, $|Z| = |Z'| = |G - E| = |D|$

$|Z|$ can be easily calculated from the definition in point (f), by using binomial coefficients (number of ways of choosing n up steps out of t - 1 moves)

$$|Z| = \binom{t-1}{n} = |G - E| \tag{3}$$

$$\begin{aligned} P(E) &= \frac{|E|}{|T|} \\ &= \frac{|G| - |G - E|}{|T|} \\ &= \frac{\binom{t-1}{n-1} - \binom{t-1}{n}}{\binom{t}{n}} \\ &= \frac{2n - t}{t} \\ &= \frac{2n - (n + s)}{n + s} \\ &= \frac{n - s}{n + s} \end{aligned}$$

F = A tie in counting votes
P(F) = 1 - P($N$ is always ahead ∪ $S$ is always ahead) = 1 - (P($N$ is always ahead) + P($S$ is always ahead))

$$P(F) = 1 - (\frac{n-s}{n+s} + 0) \tag{4}$$

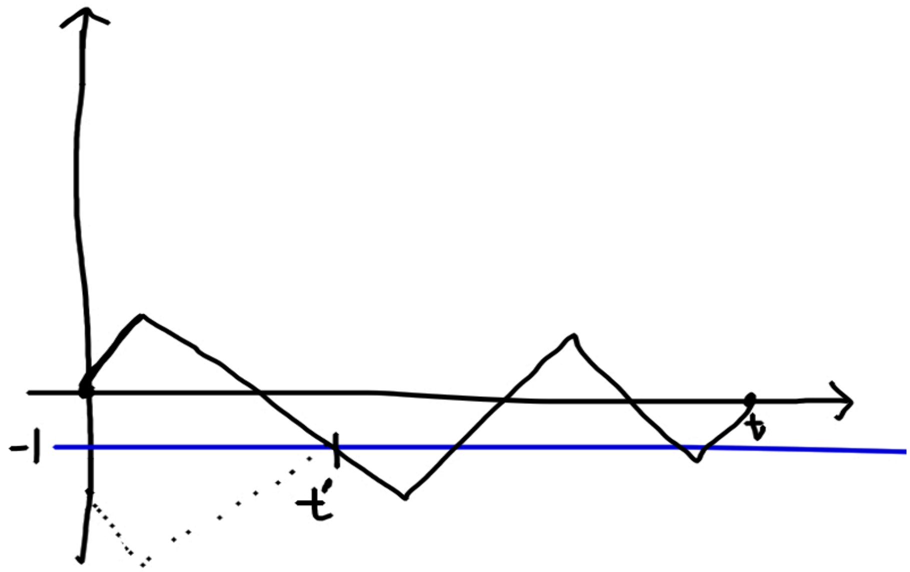
Since $n > s$, P($S$ is always ahead) has to be zero

$$P(F) = \frac{2s}{n+s} \tag{5}$$

2. For second part $n = s \Rightarrow d = 0, t = n + n = 2n$
   Total Ways T =

$$T = \binom{2n}{n} \tag{6}$$

Also since the $n = s$ this part is symmetric in $N$ and $S$
P(N never gets ahead) = P(S never gets ahead)




To calculate P(S never gets ahead)
The proof of this is similar to previous one. Here instead of x axis, line y = -1 is the line about which we can reflect the paths which touch the line y = -1 at some point $t'$. We find a one to one correspondence between these reflected paths and the set of ways of counting in which S gets ahead as we can reflect the reflected paths back about y = -1 to get the original path
To count these reflected part we make an observation that all of them must pass through the fixed point (0, -1) since all paths must begin with (0, 1). (In the previous part the fixed point point was (1, -1)).
Number of reflected paths is the ways of reaching (t, 0) from (0, -2) = K
Let the up and down steps required to reach (t, 0) from (0, -2) be $u$ and $b$

$$\begin{aligned} u + b &= t \\ u - b &= 2 \\ \Rightarrow b &= \frac{t}{2} - 1, u = \frac{t}{2} + 1 \end{aligned}$$

K = choosing $\frac{t}{2} - 1$ down steps out of $t$ steps

$$\begin{aligned} K &= \binom{t}{\frac{t}{2} - 1} \\ &= \binom{2n}{n-1} \end{aligned}$$

Number of countings in which S gets ahead = Number of reflected paths = K

$$
\begin{aligned}
P(\text{N never gets ahead}) &= P(\text{S never gets ahead}) \\
&= \frac{\text{Total - Number of Countings in which S gets ahead}}{\text{Total}} \\
&= \frac{\binom{2n}{n} - \binom{2n}{n-1}}{\binom{2n}{n}} \\
&= \frac{\frac{1}{n} - \frac{1}{n+1}}{\frac{1}{n}} \\
&= \frac{1}{n+1}
\end{aligned}
$$

## 3.3 Answer 24

Let $S_{\alpha H}$ be the event that $\alpha$ shots hit consecutively.
Let $S_{\beta M}$ be the event that $\beta$ shots miss consecutively.
$S_{\alpha H} < S_{\beta M}$ denotes the event that $\alpha$ shots hit consecutively before $\beta$ shots miss consecutively.
$P(Hit) = P(H) = p$
$P(Miss) = P(M) = 1 - p = q$
If the first shot is a hit either he gets $\alpha - 1$ hits and stops or in the next $\alpha - 1$ shot he misses, we can forget all the shots he hits before and assume the miss is the first miss.

$$
\begin{aligned}
P(S_{\alpha H} < S_{\beta M}|H) &= P(HHH..[\alpha - 1 \text{ times}])) + (1 - P(HHH..[\alpha - 1 \text{ times}]))P(S_{\alpha H} < S_{\beta M}|T) && (1) \\
&= p^{\alpha-1} + (1 - p^{\alpha-1})P(S_{\alpha H} < S_{\beta M}|M) && (2)
\end{aligned}
$$

In case the first shot is a miss if don't get $\beta - 1$ misses, we can treat the first hit as the first hit.

$$
\begin{aligned}
P(S_{\alpha H} < S_{\beta M}|M) &= (1 - P(MMM..[\beta - 1 \text{ times}]))(P(S_{\alpha H} < S_{\beta M}|H) && (3) \\
&= (1 - q^{\beta-1})(P(S_{\alpha H} < S_{\beta M}|H) && (4)
\end{aligned}
$$

Since equation (1) and (3) are linear in $P(S_{\alpha H} < S_{\beta M}|M)$ and $P(S_{\alpha H} < S_{\beta M}|H)$ we can solve them to get

$$
P(S_{\alpha H} < S_{\beta M}|H) = \frac{p^{\alpha-1}}{1 - (1 - p^{\alpha-1})(1 - q^{\beta-1})} \qquad (5)
$$

$$
P(S_{\alpha H} < S_{\beta M}|M) = \frac{(p^{\alpha-1})(1 - q^{\beta-1})}{1 - (1 - p^{\alpha-1})(1 - q^{\beta-1})} \qquad (6)
$$

From total probability theorem:

$$
P(S_{\alpha H} < S_{\beta M}) = P(S_{\alpha H} < S_{\beta M}|H)P(H) + P(S_{\alpha H} < S_{\beta M}|M)P(M) \qquad (7)
$$

Substituting from (5) and (6):

$$
\begin{aligned}
P(S_{\alpha H} < S_{\beta M}) &= \frac{p^{\alpha-1}}{1 - (1 - p^{\alpha-1})(1 - q^{\beta-1})}p + \frac{(p^{\alpha-1})(1 - q^{\beta-1})}{1 - (1 - p^{\alpha-1})(1 - q^{\beta-1})}q \\
&= \frac{p^{\alpha-1}(p + q - q^{\beta})}{1 - (1 - p^{\alpha-1})(1 - q^{\beta-1})} \\
&= \frac{p^{\alpha-1}(1 - q^{\beta})}{1 - (1 - p^{\alpha-1})(1 - q^{\beta-1})}
\end{aligned}
$$

where $q = 1 - p$

## 3.4 Answer 25

1. X = 2, Y = 3
   Let an ordered sequence of B,G denote the queue. Eg {BBBGG}
   Left side of the sequence denotes the most in front
   To satisfy the condition the first person must be a boy.
   This means the queue is of the form {B . . . .}.
   Also for the last girl in the queue there must be at least 2 boys in front of her.
   So we can have two cases when 2 boys are in front the last girl and when 3 boys are.

(a) 2 boys in front of last girl
{BBGGB}, {BGBGB}

(b) 3 boys in front of last girl
{BBBGG}, {BBGBG}, {BGBBG}

Thus there are 5 possible outcomes that satisfy the condition. Also in this question we can consider all boys as identical and all girls as identical since the permutation among them is constant.
Total possible queues = $\binom{5}{2} = 10$
P(E) = $\frac{5}{10} = \frac{1}{2}$

2. Let X = Y = n
Assume front of queue is at the left and end is at right.
As we go from left to right across the queue we make a few observations:

(a) If the last person was a boy then number of boys counted till now must be $>$ number of girls counted. If this is not true then the last girl does not have one more boy than girls in front of her.

(b) If the last person was a girl then number of boys counted till now must be $>=$ number of girls counted. If this is not true then the last girl does not have one more boy than girls in front of her.

(c) Combining these two we have that while counting from left to right, the number of boys must always be greater than or equal than the the number of girls counted at any time.

As we go from left to right assume whenever we encounter a boy it is vote for $S$ and whenever we encounter a girl it is a vote for $N$. Thus this problem can be reduced to the problem of when counting votes for $N$ and $S$ (with equal votes), $N$ never gets ahead of $S$. This is the same as Answer 24 second part. Thus the answer is

$$\frac{1}{n+1} \tag{1}$$

where n = number of boys = number of girls.

### 3.5 Answer 26

Let S be the set of all events when arrows hit are exactly 50
Lets analyse an event $E \in S$
The first two shots have constant probability of 1 as given
Rest 98 shots do not have constant probability and depend upon previous shots.
Since total 50 arrows hit and 50 miss, from $a_3$ to $a_{100}$ 49 arrows hit and 49 miss.
P(Hit) = Previous hits / Shots taken
P(Miss) = Previous miss / Shots taken
Let $a_i$ denote the ith arrow
P(E) has the following properties:

1. For any sequence of hits or misses to happen the probabilities have to be multiplied for each individual hit

2. We will calculate numerator and denominator independently for $P(E)$ and then divide

3. Since both P(Hit) and P(Miss) depend upon previous shots taken, the denominator of P(E) is constant. The denominator for $a_i$ is $i-1$. The denominator is $\prod_3^{100} i-1 = 99!$

4. Let say $a_k$ hits and before it $j$ arrows have already hit. The numerator of $P(a_k hit) = j$ (independent of k). Since total 49 arrows after $a_2$ hit and $a_1$ is guaranteed to hit Numerator contribution from hitting is $\prod_1^{49} i = 49!$

5. Let say $a_k$ misses and before it $j$ arrows have already missed. The numerator of $P(a_k miss) = j$ (independent of k). Since total 49 arrows after $a_2$ miss and $a_2$ is guaranteed to miss. Numerator contribution from missing is $\prod_1^{49} i = 49!$

6. Total numerator = $49! * 49!$

$$P(E) = \frac{\text{numerator}}{\text{denominator}} = \frac{49! * 49!}{99!} \tag{1}$$

This means that the all the events in S have the same probability.
The total number of events in S is equivalent to choosing 49 arrows (which hit) out of 98.

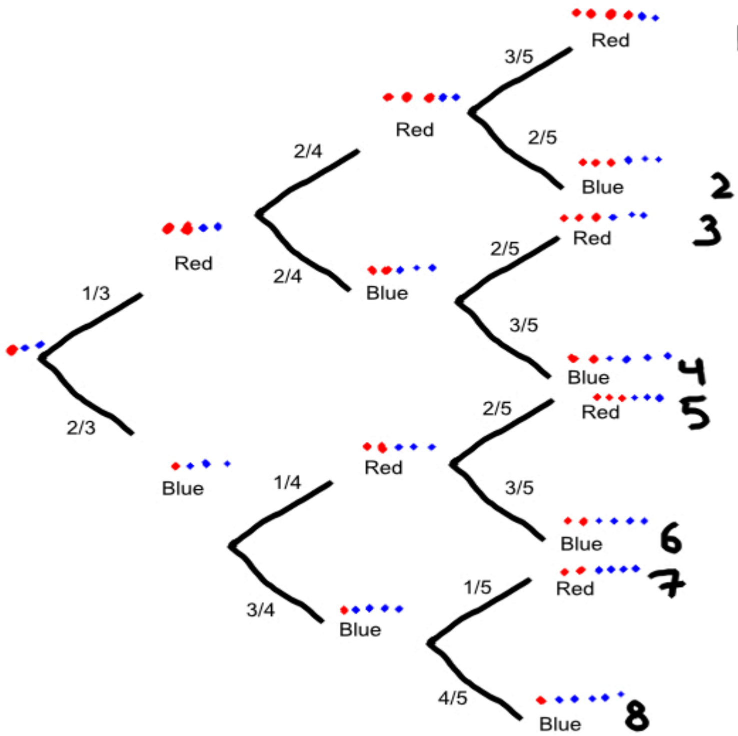
$$|S| = \binom{98}{49} \tag{2}$$

$$
\begin{aligned}
P(S) &= P(E)|S| \\
&= \frac{49! * 49!}{99!} * \binom{98}{49} \\
&= \frac{49! * 49!}{99!} * \frac{98!}{49! * 49!} \\
&= \frac{1}{99}
\end{aligned}
$$

Generally it can be proved that if P(X = k) denotes probability of exactly k shots hit then $P(X = k) = \frac{1}{99}, k \in \{2, 3, 4...99\}$.

### 3.6 Answer 28

The trials can be visualised as the following tree




1. E = Exactly one blue ball is drawn
   This means all the branches which have one Blue in them i.e $Branch_2 + Branch_3 + Branch_5$

$$
\begin{aligned}
P(E) &= \frac{1}{3}.\frac{2}{4}.\frac{2}{5} + \frac{1}{3}.\frac{2}{4}.\frac{2}{5} + \frac{2}{3}.\frac{1}{4}.\frac{2}{5} \\
&= \frac{12}{60} \\
&= \frac{1}{5}
\end{aligned}
$$

2. E = All drawn balls are red given that all the drawn balls are of the same color
   All balls are red corresponds to $Branch_1$
   All balls are of the same color corresponds to $Branch_1 + Branch_8$

$$
\begin{aligned}
P(E) &= \frac{Branch_1}{Branch_1 + Branch_8} \\
&= \frac{\frac{1}{3}.\frac{2}{4}.\frac{3}{5}}{\frac{1}{3}.\frac{2}{4}.\frac{3}{5} + \frac{2}{3}.\frac{3}{4}.\frac{4}{5}} \\
&= \frac{1}{5}
\end{aligned}
$$

3. E = At least one blue ball is drawn = Total - All Red
   This corresponds to $1 - Branch_1$

$$\begin{aligned} P(E) &= 1 - Branch_1 \\ &= 1 - \frac{1}{3}.\frac{2}{4}.\frac{3}{5} \\ &= \frac{9}{10} \end{aligned}$$

4. E = At least one red ball is drawn = Total - All Blue
   This corresponds to $1 - Branch_8$

$$\begin{aligned} P(E) &= 1 - Branch_8 \\ &= 1 - \frac{2}{3}.\frac{3}{4}.\frac{4}{5} \\ &= \frac{3}{5} \end{aligned}$$

## 3.7 Answer 29

Consider a random walk on a 1D plane which has probability p of going to a higher integer and (1 - p) of going to lesser integer.
Define $D_t$ be a random variable which is the integer at which we are on the random walk at a given time $t$
Define $P_n$ = P(Reaching i for the first time before reaching 0 | $D_0$ = n) where i > n

$$P_n = pP_{n+1} + (1-p)P_{n-1} \tag{1}$$

Base case:
$P_0 = 0$ since we can never reach i
$P_i = 1$ since we have reached i at time 0
The intuition behind the recursion is that since every move is independent relative to starting at (n + 1) we would need one more step forward with p probability and compared to starting at (n - 1) we need one more reverse step with (1 - p) probability.
Substitute p = 1/2 in the recursion (1)

$$2P_n = P_{n+1} + P_{n-1} \tag{2}$$

Solve the characteristic equation:

$$\begin{aligned} 2t^n &= t^{n+1} + t^{n-1} \\ 2t &= t^2 + 1 \\ 0 &= t^2 - 2t + 1 \\ t &= 1 \end{aligned}$$

Thus $P_n = An(1)^n + B(1)^n = An + B$ Using base cases we get

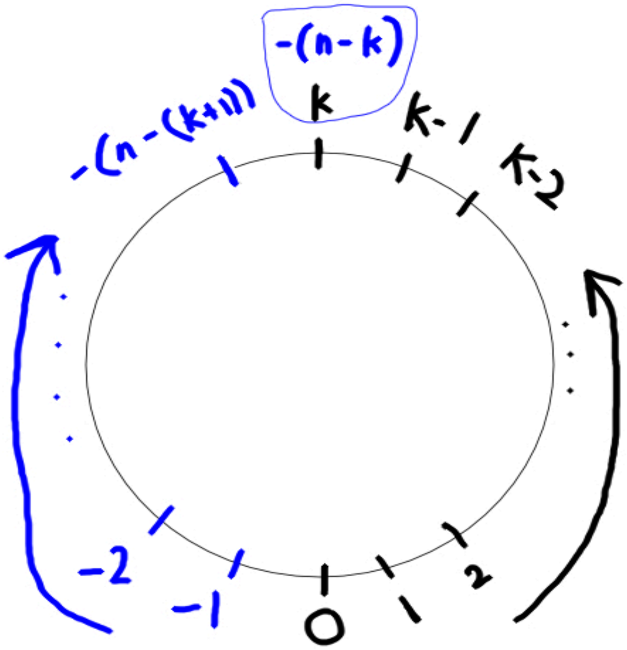
$$P_0 = 0 \Rightarrow B = 0, P_i = 1 \Rightarrow A = 1/i \tag{3}$$

Thus

$$P_n = \frac{n}{i} \tag{4}$$

$$P_n = \frac{\text{Distance b/w starting position (n) and point not visiting (0)}}{\text{Distance b/w ending position (i) to point not visiting (0)}} \tag{5}$$

If the robot reaches k for the first time when it has visited every other number before this means the robot has visited all other numbers at least once.
For the vertices assume that the vertices are numbered from $0$ to $k$ in anticlockwise direction and $0$ to $-(n-k)$ in clockwise direction.




E = Robot reaches $kth$ vertex last $(k \neq 0)$
Visiting a vertex last means that all other vertices have been visited atleast once before visiting the vertex for the first time.
To reach $kth$ vertex either robot reaches $k-1$ vertex first and then to $-(n-(k+1))$ through $0$ (without going to $k$) or robot reaches $-(n-(k+1))$ vertex first and then goes to $k-1$ vertex through $0$ (without going to $-(n-k)$)
This can modelled as two simple walks E1 and E2.
E1 = (0 → k - 1 before -(n - (k + 1))) ∩ (k - 1 → -(n - (k + 1)) before k)
E2 = (0 → -(n - (k + 1)) before k - 1) ∩ (-(n - (k + 1)) → k - 1 before -(n - k))
From the above result (5)

$$P(E1) = \frac{n-(k+1)}{n-2} * \frac{1}{n-1} \tag{6}$$

While calculating P(E2) the side clockwise from $0$ to $-(n-k)$ is treated as positive instead of negative since that direction is considered increasing and vice versa for anticlockwise.

$$P(E2) = \frac{k-1}{n-2} * \frac{1}{n-1} \tag{7}$$

$P(E) = P(E1) + P(E2)$ since E1 and E2 are disjoint

$$\begin{aligned} P(E) &= \frac{n-(k+1)}{n-2} * \frac{1}{n-1} + \frac{k-1}{n-2} * \frac{1}{n-1} \\ &= \frac{n-2}{(n-2)*(n-1)} \\ &= \frac{1}{n-1} \end{aligned}$$

Since the P(E) is independent of k

$$P(\text{Visit kth vertex last}) = \begin{cases} 0 & x = 0 \\ \frac{1}{n-1} & x \in \{1, 2, ...., n-1\} \end{cases}$$