Assignment 3

October 22, 2020

1 Section 1

1.1 Answer 1

1.

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(aX + b \le y) \\ &= P(X \le \frac{y-b}{a}) \\ &= F_X(\frac{y-b}{a}) \end{aligned}$$

We can get PDF by differentiating the CDF

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} F_X(\frac{y-b}{a}) \\ &= \frac{1}{a} f_X(\frac{y-b}{a}) \end{aligned}$$

2.

$$\begin{aligned} f_Y(y) &= \frac{1}{a} f_X(\frac{y-b}{a}) \\ &= \frac{\lambda}{a} e^{-\lambda \frac{y-b}{a}} \end{aligned}$$

Let $\frac{\lambda}{a} = \lambda'$

$$f_Y(y) = \lambda' e^{-\lambda'(y-b)}$$

Clearly for $b = 0, a > 0$ it is exponential

$$f_Y(y) = \lambda' e^{-\lambda' y}$$

3. $X = N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1}$$

$Y = aX + b$
From the first part in this question:

$$f_Y(y) = \frac{1}{a} f_X(\frac{y-b}{a}) \tag{2}$$

Substituting from equation (1) into (2)

$$f_Y(y) = \frac{1}{a\sqrt{2\pi}\sigma} e^{-\frac{(y-(b+a\mu))^2}{2\sigma^2 a^2}} \tag{3}$$

$a \neq 0$
Thus $aX + b = N(a\mu + b, a^2\sigma^2)$
Thus $aX + b$ is normal $\forall a, b \in R^2 - (a = 0, b)$

1.2 Answer 2

1. From definition of CDF,

$$F_Y(y) = P(Y \leq y) \tag{1}$$
$$= P(e^X \leq y) \tag{2}$$
$$= P(x \leq \ln y) \tag{3}$$
$$= F_X(\ln y) \tag{4}$$
$$= \phi(\frac{\ln y - \mu}{\sigma}) \tag{5}$$

2. Solving it two ways

   (a) differentiating CDF

$$f_Y(y) = F_Y'(y) = \frac{1}{y\sigma}\phi'(\frac{\ln y - \mu}{\sigma}) \tag{6}$$
$$= \frac{1}{y\sigma}\frac{1}{\sqrt{2\pi}}e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}} \tag{7}$$

   (b) method of transform

$$g(x) = e^x \tag{8}$$
$$g'(x) = e^x \tag{9}$$
$$g^{-1}(x) = ln(x) \tag{10}$$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y)|} \tag{11}$$
$$= \frac{\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}}}{e^{\ln y}} \tag{12}$$
$$= \frac{1}{y}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}} \tag{13}$$

1.3 Answer 3

Define X_i to be an indicator random variable which is 1 if the die rolled 2 otherwise 0

$$E[X_i] = P(X_i = 1) = \frac{1}{6}$$
$$Var(X_i) = \frac{1}{6}(1 - \frac{1}{6})^2 + \frac{5}{6}(0 - \frac{1}{6})^2 = \frac{5}{36}$$
$$\sigma = 0.372677996$$

Define $S_n = X_1 + X_2 + ... + X_n$
$n = 12000$
Required $P(1900 < S_n < 2150) = P(S_n < 2150) - P(S_n <= 1900)$
From the central limit theorem

$$Z_n = \frac{S_n - n\mu}{\sqrt{n}\sigma} = \frac{S_n - 2000}{40.824829046} \tag{1}$$

Z_n can be approximated as normal random variable
Using binomial approximation of random variable $P(S_n < 2150) \sim P(S_n < 2149.5)$

$$P(S_n < 2149.5) = P(S_n - 2000 < 149.5)$$
$$= P(\frac{S_n - 2000}{40.824829046} < 3.6619871654)$$
$$= P(Z_n < 3.6619871654)$$
$$= \phi(3.6619871654)$$

$P(S_n <= 1900) \sim P(S_n < 1900.5)$

$$\begin{aligned}
P(S_n > 1900) &= P(S_n - 2000 < -100.5) \\
&= P(\frac{S_n - 2000}{40.824829046} < -2.4617371915) \\
&= P(Z_n < -2.4617371915) \\
&= \phi(-2.4617371915)
\end{aligned}$$

$$\begin{aligned}
P(1900 < S_n < 2150) &= \phi(3.6619871654) - \phi(-2.4617371915) \\
&= 0.99987486 - 0.00691401 \\
&= 0.99296085 \\
&= 0.993
\end{aligned}$$

1.4 Answer 4

1. In this case we know that at least one of the child is the boy. $P(B, B) = P(B, G) = P(G, B) = P(G, G) = \frac{1}{4}$

$$\begin{aligned}
P(\text{Other is female}|\text{At least one is boy}) &= \frac{P((B, G) + (G, B))}{P((B, G) + (G, B) + (B, B))} \\
&= \frac{1/4 + 1/4}{1/4 + 1/4 + 1/4} \\
&= \frac{2}{3}
\end{aligned}$$

2. A parent who has 2 boys is twice as likely to have a boy named Atsi than a parent who has a boy and a girl
Thus this skews the probability of (B, B) case to $\frac{1}{2}$.

$$\begin{aligned}
P(\text{Other is female}|\text{A boy named Atsi}) &= \frac{P((B, G) + (G, B))}{P((B, G) + (G, B) + (B, B))} \\
&= \frac{1/4 + 1/4}{1/4 + 1/4 + 1/2} \\
&= \frac{1}{2}
\end{aligned}$$

3. A parent who has 2 boys is twice as likely to have a boy named Atsi than a parent who has a boy and a girl
Thus this skews the probability of (B, B) case case to $\frac{1}{2}$.
Naming the boy provides more information as we can assume that a specific name is given to all boys with the same probability so a parent with two boys is 2 times more likely have a boy with a specific name. This will make them more likely to report this fact to us when we meet them. Hence this extra information changes the probability.

4. From part (a) we can say that

$$P(\text{One is female}|\text{At least one is boy}) = \frac{2}{3} \tag{1}$$

$$P(\text{One child is female}) = \frac{1}{2} \tag{2}$$

We can use bayes theorem to take into account the new information.
Let $P(\text{Name of a Boy is Atsi}) = x = P(I)$
If it is given that other is female this means there is only 1 boy $P(\text{Boy is Atsi}|\text{Other is female}) = x = P(I)$
Using the above expressions and applying bayes theorem,

$$\begin{aligned}
P(\text{Other is female}|\text{Boy named Atsi}) &= \frac{P(\text{Boy named Atsi}|\text{Other is female})P(\text{Other is female})}{P(\text{Boy named Atsi})} \\
&= \frac{x * 1/2}{P(I|BB)P(BB) + P(I|BG)P(BG) + P(I|GB)P(GB) + P(I|GG)P(GG)} \\
&= \frac{x * 1/2}{(2x + x + x + 0) * 1/4} \\
&= \frac{1}{2}
\end{aligned}$$

1.5 Answer 5

1. Since for exponential random variable PDF is non zero for $z > 0$

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\ &= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx \\ &= \int_0^z \lambda^2 e^{-\lambda z} dx \\ &= \lambda^2 e^{-\lambda z} \int_0^z dx \\ &= \lambda^2 e^{-\lambda z} z \end{aligned}$$

For $z < 0, f_Y(z-x) = 0$ or $f_X(x) = 0$ so $f_Z(z) = 0$

$$f_Z(z) = \begin{cases} \lambda^2 e^{-\lambda z} z & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{1}$$

2.

$$f_X(x) = f_Y(x) = \begin{cases} 0 & x > 1 \\ 1 & 0 \leq x \leq 1 \\ 0 & x < 0 \end{cases} \tag{2}$$

Since $Z = X + Y, f_Z(z) = 0$ for $z < 0$ and $z \geq 2$
From convolution of two independent random variables,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx$$

Take two cases
For $0 < z \leq 1 \Rightarrow x \geq 0, z - x \geq 0$

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\ &= \int_0^z 1.1.dx \\ &= z \end{aligned}$$

For $1 < z < 2$
$z - x \leq 1 \Rightarrow x \geq z - 1$
$0 \leq x \leq 1$

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\ &= \int_{z-1}^1 1.1.dx \\ &= 2 - z \end{aligned}$$

Combining the two cases

$$f_Z(z) = \begin{cases} 0 & z \leq 0 \\ z & 0 < z \leq 1 \\ 2 - z & 1 < z < 2 \\ 0 & z \geq 2 \end{cases} \tag{3}$$

2 Section 2

2.1 Answer 6

1. Consider case $t \geq 0$
   Let $T_1 = X, T_2 = Y$, $X - Y = T$

$$\begin{aligned}
F_T(t) &= P(X - Y \leq t) \\
&= 1 - P(X - Y > t) \\
&= 1 - \int_0^\infty \left(\int_{t+y}^\infty f_{X,Y}(x,y)dx \right) dy \\
&= 1 - \int_0^\infty \lambda e^{-\lambda y} \left(\int_{t+y}^\infty \lambda e^{-\lambda x} dx \right) dy \\
&= 1 - \int_0^\infty \lambda e^{-\lambda y} e^{-\lambda(t+y)} dy \\
&= 1 - e^{-\lambda t} \int_0^\infty \lambda e^{-2\lambda y} dy \\
&= 1 - \frac{1}{2} e^{-\lambda t}
\end{aligned}$$

2. Using symmetry $Z = X - Y$ and $-Z = Y - X$ have the same distribution. For case $t < 0$

$$F_T(t) = P(T \leq t) = P(-T \geq -t) = P(T \geq -t) = 1 - F_T(-t) \tag{1}$$

   With $t < 0$ and we have $-t > 0$

$$F_T(t) = 1 - F_T(-t) = 1 - (1 - \frac{1}{2} e^{-\lambda(-t)}) = \frac{1}{2} e^{\lambda t} \tag{2}$$

   Combining result from part 1 and this part

$$F_T(t) = \begin{cases} 1 - \frac{\lambda}{2} e^{-\lambda t} & t \geq 0 \\ \frac{1}{2} e^{\lambda t} & t < 0 \end{cases} \tag{3}$$

3.

$$\begin{aligned}
f_T(t) &= \frac{d}{dt} F_T(t) = \begin{cases} \frac{\lambda}{2} e^{-\lambda t} & t \geq 0 \\ \frac{\lambda}{2} e^{\lambda t} & t < 0 \end{cases} \\
f_T(t) &= \frac{\lambda}{2} e^{-\lambda |t|}
\end{aligned}$$

4. It is called Laplace PDF.

2.2 Answer 7

1. Since the arrival time of Atsi is uniform, if we condition it on the 10 min interval in which the next train comes we get a uniform random variable in 10 min interval. Hence, waiting time to board train is uniformly distributed in the 10 min interval before the next train arrives

$$E[\text{waiting time}] = (10 - 0)/2 = 5\text{min} \tag{1}$$

2. Waiting 3 more minutes if waiting for 6 minutes already = Waiting for 9 minutes in total

$$\begin{aligned}
P(X > 9 | X > 6) &= P(\text{arrives in 1st minute after previous train}) / P(\text{arrives in first 4 minutes after previous train}) \\
&= 1/4 = 0.25
\end{aligned}$$

3. Train arrival is an exponential distribution.
   Exponential distribution exhibits memorylessness probability.

Let T be the event of an arrival.
Let the previous arrival be at time t and the waiting time is time s.

$$\begin{aligned}
P(T > s + t \mid T > s) &= \frac{P(T > s + t \cap T > s)}{P(T > s)} && (2)\\
&= \frac{P(T > s + t)}{P(T > s)} && (3)\\
&= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} && (4)\\
&= e^{-\lambda t} && (5)\\
&= P(T > t). && (6)
\end{aligned}$$

The probability of next event is independent of when previous event happened. Thus the waiting time is always the same = mean = 10 minutes.

2.3 Answer 8

1. Let X_1 be the first arrival. The waiting time for first arrival

$$\begin{aligned}
P(X_1 > t) &= P(\text{no arrival in (0,t]})\\
&= e^{-\lambda t}
\end{aligned}$$

This can be generalized to any event in poisson. Since the individual events in disjoint intervals are independent in poisson distribution the inter-arrival / waiting times for all events have exponential distribution.

$$\begin{aligned}
P(X_n > t | X_{n-1} = s) &= P(\text{no arrival in (s,s + t)} | X_{n-1} = s)\\
&= P(\text{no arrival in (s,s + t)})\\
&= e^{-\lambda t}
\end{aligned}$$

$$P(X_n \leq t | X_{n-1} = s) = 1 - P(X_n > t | X_{n-1} = s) \tag{1}$$

$$F_{X_i}(t) = \begin{cases} 1 - e^{-\lambda t} & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{2}$$

differentiating to get PDF

$$f_{X_i}(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{3}$$

is exponential with λ. Hence Proved.

2. Waiting time for bernoulli
Waiting till first success of a series of bernoulli trial is geometric random variable by definition.
This can be extended to any success of bernoulli trials since these trials are independent.
Let us consider there has been n bernoulli trials and no success recorded and T is the time until the first success.
Hence T is the waiting time for success of a bernoulli variable.

$$P(T - n = t | T > n) = (1 - p)^{t-1} p = P(T = t) \tag{4}$$

This is the geometric PMF. This shows that the waiting time of bernoulli is a Geometric Random variable.

3. PMF for Geometric random variable

$$f_X(x) = (1 - p)^{x-1} p \tag{5}$$

$$f_X(X > x) = (1 - p)^x \tag{6}$$

Let $p = \lambda\tau$ where λ is a positive constant and τ is the time increment. $x = \frac{t}{\tau}$ where t denotes the time passed $(t > 0)$.

$$\begin{aligned}
\lim_{\tau\to 0} f_X(X > x) &= \lim_{\tau\to 0}(1-p)^x \\
&= \lim_{\tau\to 0}(1-\lambda\tau)^{\frac{t}{\tau}} \\
\lim_{\tau\to 0} \ln f_X(X > x) &= \lim_{\tau\to 0} \ln(1-\lambda\tau)\frac{t}{\tau} \\
\lim_{\tau\to 0} \ln f_X(X > x) &= \lim_{\tau\to 0} \frac{t(-\lambda\tau - \lambda^2\tau^2/2 - \lambda^3\tau^3/3...)}{\tau} \\
\lim_{\tau\to 0} \ln f_X(X > x) &= \lim_{\tau\to 0} -t\lambda \\
\lim_{\tau\to 0} f_X(X > x) &= e^{-\lambda t}
\end{aligned}$$

This is exponential CDF and can be differentiated to get exponential PDF.

$$\begin{aligned}
F_X(x) = f_X(X \le x) = 1 - f_X(X > x) = 1 - e^{-\lambda t}, &\quad (t > 0) \\
f_X(x) = \lambda e^{-\lambda t} &\quad (t > 0)
\end{aligned}$$

Hence Proved.

2.4 Answer 9

1. Rayleigh distribution

(a) Mean
For $x > 0$,

$$R(x,\sigma) = \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} \tag{1}$$

$$E[X] = \int_0^\infty x\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx \tag{2}$$

$$\tag{3}$$

Note Γ refers to gamma distribution and Substitute $\frac{x^2}{2\sigma^2} = t$

$$\begin{aligned}
E(X) &= \int_0^\infty x\frac{x^2}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx \\
&= \int_0^\infty \sqrt{2}te^{-t}t^{-\frac{1}{2}}\sigma dt \\
&= \sqrt{2}\sigma\int_0^\infty t^{\frac{3}{2}-1}e^{-t}dt \\
&= \sqrt{2}\sigma\Gamma(\frac{3}{2}) \\
&= \frac{\sigma}{\sqrt{2}}\Gamma(\frac{1}{2}) \\
&= \frac{\sigma}{\sqrt{2}}\sqrt{\pi}
\end{aligned}$$

(b) Variance $= E[X^2] - E[X]^2$

$$E[X^2] = \int_{-\infty}^\infty x^2\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} \tag{4}$$

$$\tag{5}$$

Substitute $t = \frac{x^2}{2\sigma^2}$, $dt = \frac{2xdx}{\sigma^2}$

$$E[X^2] = \int_0^\infty 2\sigma^2 te^{-t}dt \tag{6}$$
$$= 2\sigma^2 \int_0^\infty t^{2-1}e^{-t}dt \tag{7}$$
$$= 2\sigma^2\Gamma(2) \tag{8}$$
$$= 2\sigma^2 \tag{9}$$

Variance $= E[X^2] - E[X]^2$

$$Var = 2\sigma^2 - \sigma^2\frac{\pi}{2} = \frac{\sigma^2}{2}(4-\pi) \tag{10}$$

(c) Mode

To find mode we need to find maxima of the PDF by differentiating

$$\frac{d}{dx}(\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}) = 0 \tag{11}$$
$$\frac{1}{\sigma^2}e^{\frac{-x^2}{2\sigma^2}}(\frac{-x^2}{\sigma^2}+1) = 0 \tag{12}$$
$$x = \sigma, -\sigma \tag{13}$$

Since $x > 0, x = \sigma$ is the mode.

2. Cauchy distribution $f(x;x_0,\gamma) = \frac{1}{\pi\gamma\left[1+\left(\frac{x-x_0}{\gamma}\right)^2\right]} = \frac{1}{\pi\gamma}\left[\frac{\gamma^2}{(x-x_0)^2+\gamma^2}\right]$

(a) Mean, the mean of distribution does not exist

$E[X] = \int_{-\infty}^\infty xf(x)\,dx$

$\int_a^\infty xf(x)\,dx + \int_{-\infty}^a xf(x)\,dx$

for an arbitrary real number a.

$$\int_a^\infty xf(x)dx = \int_a^\infty x\frac{1}{\pi\gamma}\frac{\gamma^2}{(x-x_0)^2+\gamma^2}dx \tag{14}$$
$$= \frac{\gamma}{\pi}\frac{1}{2}(\log(x_0^2 - 2x_0x + \gamma^2 + x^2) + \frac{x_0}{\gamma}\arctan\frac{x_0-x}{\gamma}\Big|_0^\infty \tag{15}$$
$$= \infty \tag{16}$$

Similarly this can be proved for $\int_{-\infty}^a xf(x)dx = -\infty$

For the integral $\int_{-\infty}^\infty xf(x)dx$ to exist (even as an infinite value), at least one of the terms in this sum should be finite, or both should be infinite and have the same sign. But in the case of the Cauchy distribution, both the terms in this sum are infinite and have opposite sign. $\infty - \infty$ is not defined and so is the mean.

(b) Characteristic function

$\int_{-\infty}^\infty f(x;x_0,\gamma)e^{ixt}dx = e^{ix_0t-\gamma|t|}$

This function is not differentiable at $t = 0$ so the derivative does not exist. If a derivative would exist this means the mean would exist too.

(c) Higher order moments

The Odd Higher moments like mean are undefined. (They are not ∞, they are undefined like $\infty - \infty$ form)

The even powered higher moments have a value of ∞.

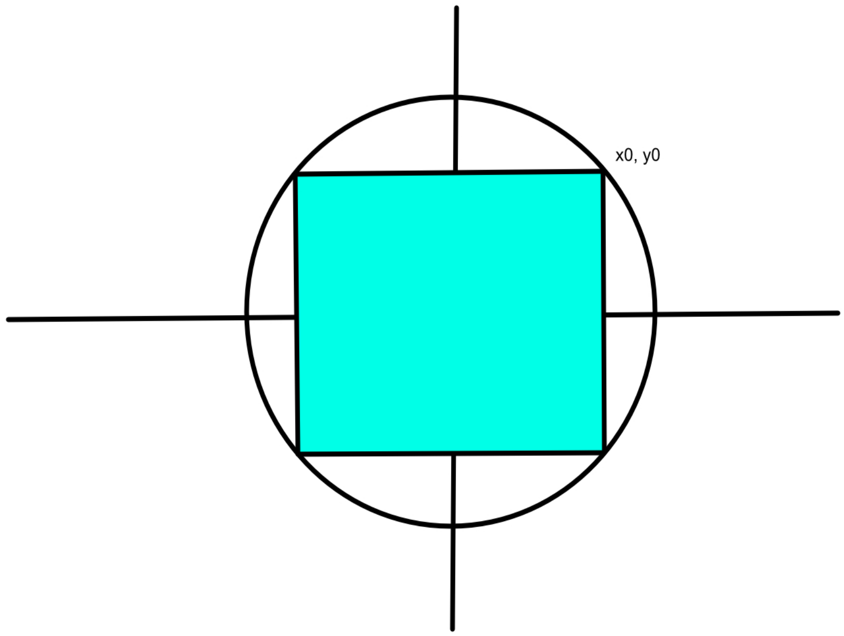
Variance is an even powered moment and evaluates to ∞, so it does not have a real value.

2.5 Answer 10

Let the first point be (x_0, y_0) and second point be (x_1, y_1) Now the other corners of the rectangle will be $(x_0, y_1), (x_1, y_0)$
For (x_1, y_0) to lie inside the triangle we get

$$x_1^2 + y_0^2 \leq 1$$
$$x_1^2 + 1 - x_0^2 \leq 1$$
$$x_1^2 \leq x_0^2$$
$$|x_1| \leq |x_0|$$

Similarly we also get $|y_0| \geq |y_1|$
These constraints can be graphed as follows



For any point (x_0, y_0) the point (x_1, y_1) must lie inside the cyan rectangle
Let the point (x_0, y_0) be $(\cos\theta, \sin\theta)$
The probability of this happening is

$$\begin{aligned} P(E) &= \frac{\text{Area of rectangle}}{\text{Area of circle}} \\ &= \frac{|4\cos\theta\sin\theta|}{\pi} \end{aligned}$$

For the entire circle we integrate this area for all values of theta and divide by the range of θ

$$P(X) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{|4\cos\theta\sin\theta|}{\pi} \tag{1}$$

From symmetry of the four quadrants we can say this is equivalent to

$$\begin{aligned} P(X) &= \frac{2}{\pi}\int_{\theta=0}^{\pi/2} \frac{|4\cos\theta\sin\theta|}{\pi} d\theta \\ &= \frac{8}{\pi^2}\int_{\theta=0}^{\pi/2} \cos\theta\sin\theta d\theta \\ &= \frac{8}{\pi^2}\frac{\sin^2\theta}{2}\Big|_0^{\pi/2} \\ &= \frac{8}{\pi^2}\left(\frac{1-0}{2}\right) \\ &= \frac{4}{\pi^2} \\ &= 0.405 \end{aligned}$$