Assignment 2

October 15, 2020

1 Section 1

1.1 Answer 1

From the definition of moment generating function we can construct a random variable X which fits the given moment generating function.

$$P_X(x) = \left\{ \begin{array}{cc} \frac{1}{10}, & \text{for } x = -20 \\ \frac{1}{5}, & \text{for } x = -3 \\ \frac{3}{10}, & \text{for } x = 4 \\ \frac{2}{5}, & \text{for } x = 5 \end{array} \right\} \tag{1}$$

$$mgf_X(t) = \sum_x e^{tx} P_X(x) \tag{2}$$

(2) is equal to the given moment generating function.
From uniqueness theorem of moment generating function the PDF of the function must be (1).

$$P(|X| \leq 2) = P(-2 \leq X \leq 2) = 0 \tag{3}$$

1.2 Answer 2

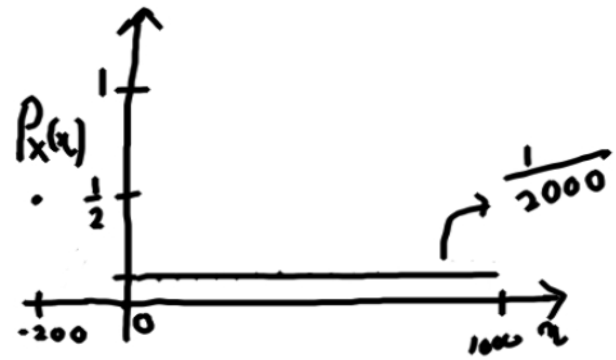
Let X be a random variable for the profit earned.
Since guessing odd or even number has $\frac{1}{2}$ probability

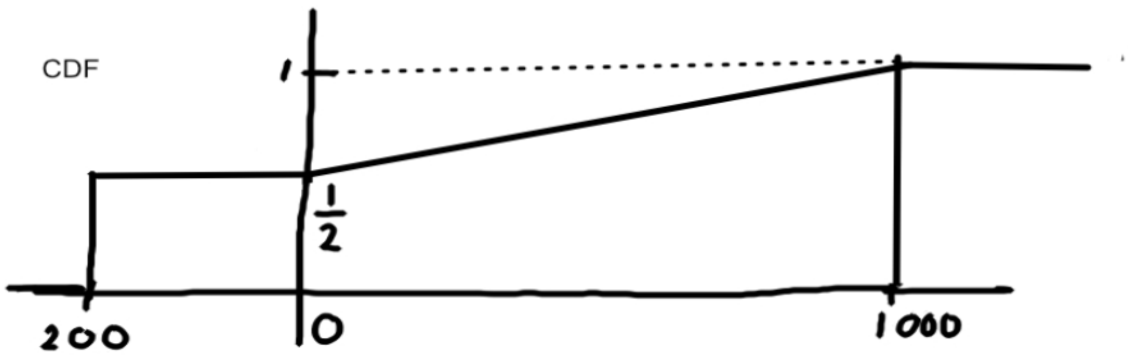
$$P(X = -200) = \frac{1}{2} \tag{1}$$

Assumption: Uniform continuous distribution For $0 \leq X \leq 1000$ first we must guess correctly which has probability $1/2$ and then there is uniform distribution between 0 to 1000.
Thus for $P(0 \leq X \leq 1000) = \frac{1}{2*(1000-0)}$
Since the PDF is constant for $0 \leq x \leq 1000$ the CDF in the range would be a linear function g(x) under the constraint CDF(0) = $1/2$ and CDF(1000) = 1

$$g(x) = \frac{1}{2} + \frac{x}{2000} \tag{2}$$

$$F_X(x) = \begin{cases} 0 & x < -200 \\ \frac{1}{2} & -200 \leq x < 0 \\ \frac{1}{2} + \frac{x}{2000} & 0 \leq x \leq 1000 \\ 1 & 1000 < x \end{cases} \tag{3}$$





1.3 Answer 3

Since P is uniform $P_X(x) = 1/3$ for $x \in [0, 3]$.
Also expected value is $(3 + 0)/2 = 1.5$.
By Law of the unconscious statistician

$$E[Q(x)] = \int_0^3 f(x)q(x)dx \tag{1}$$

where f(x) is the PDF of P, and q(x) is PDF of Q in terms of x

$$\begin{aligned} E[Q(x)] &= \int_0^3 f(x)q(x)dx \\ &= \frac{3}{2}\int_0^3 \log\frac{x}{3-x}dx \\ &= \frac{3}{2} * 0 \\ &= 0 \end{aligned}$$

1.4 Answer 4

Let X be a random variable which denotes the waiting time for Ashish.
Let event A = Taxi is waiting and he boards it i.e $X = 0$

$$P(A) = \frac{2}{3}$$

Let event B = He boards the next taxi before 5 min i.e $0 < X \le 5$
In this case the event A does not happen and then a taxi has to arrive before 5 min.
Since the arrival of Taxi is a uniform random variable between 0 - 10 min the probability of taxi coming before in x min is $x/10$
Lets say the taxi arrives in x min

$$P(B_x) = P(\bar{A})P(\text{Taxi arrives in x min}) = \frac{1}{3}.\frac{x}{10} = \frac{x}{30} \tag{1}$$

In this case the $E[B_5]$ is $(0+5)/2 = 5/2$ since this is uniform distribution
Let event C = He boards Bus at exactly 5 min i.e $X = 5$
In this case A and B must fail. This has unit probability since the bus always arrives at 5 min.

$$P(C) = P(\bar{A})P(\bar{B})P(\text{Boards bus}) = \frac{1}{3}.\frac{1}{2}.\frac{1}{1} = \frac{1}{6} \tag{2}$$

Now pmf of X

$$P_X(x) = \begin{cases} 2/3 & x = 0 \\ 1/30 & 0 < x < 5 \\ 1/6 & x = 5 \end{cases} \tag{3}$$

Since $CDF_X(k) = P_X(x \leq k) = \int_{-\infty}^{\infty} P_X(x)$

$$CDF_X(x) = \begin{cases} 0 & x < 0 \\ 2/3 & x = 0 \\ 2/3 + x/30 & 0 < x < 5 \\ 1 & x \geq 5 \end{cases} \tag{4}$$

From total expectation theorem

$$\begin{aligned} E[X] &= E[X|A]P(A) + E[X|B_5]P(B_5) + E[X|C]P(C) \\ &= 0 * 2/3 + 5/2 * 1/6 + 5 * 1/6 \\ &= 5/12 + 5/6 \\ &= 15/12 \end{aligned}$$

1.5 Answer 5

The pmf can be simplified as

$$p_G(g) = \begin{cases} 1/3 & loss \\ 2/3 & gain \end{cases} \tag{1}$$

1. Let the total number of rounds played be tracked random variable X
   Assume they end in round k
   This means both of them had $k-1$ gains followed by their first loss on kth turn
   This can be modelled as product two independent geometric random variables where p = $\frac{2}{3}$ and the case of k - 1 success followed by a failure i.e $p^{k-1}(1-p)$
   $P(X = k) = p^{k-1}(1-p) * p^{k-1}(1-p)$
   $P(X = k) = (p^{k-1}(1-p))^2$
   $P(X = k) = (\frac{2}{3}^{k-1}.\frac{1}{3})^2$

2. Let random variable Z be the number of rounds before M gets his third loss.
   Here the result of V does not matter.
   The third loss has to be on the z^{th} round. Before this there would be $z-1$th rounds in which two result in loss.
   This can be modelled as binomial random variable.

$$P(\text{two losses in z-1 rounds}) = \binom{z-1}{2}\frac{2}{3}^{z-3}\frac{1}{3}^2 \tag{2}$$

   The final loss has $\frac{1}{3}$ chance

$$P_Z(z) = \binom{z-1}{2}\frac{2}{3}^{z-3}\frac{1}{3}^3 \tag{3}$$

   Also if we consider T as the random variable where T is the number of total gambles before M wins. Each round has two gambles

$$P_T(t) = \binom{t/2-1}{2}\frac{2}{3}^{t/2-3}\frac{1}{3}^3 \tag{4}$$

3. Lets say they both succeed by round k
   One of them won on last round, and the other wins in round before least once, or both of them might win in last round.
   P(Winning first time on kth round) $= \frac{1}{3}^{k-1}\frac{2}{3}$
   P(Winning atleast once in k - 1 rounds) $= 1 - \frac{1}{3}^{k-1}$
   P(Both win first time in kth round) $= (\frac{1}{3})^{2(k-1)})\frac{4}{9}$
   $P(N = k) = \binom{2}{1}P$(Winning first time on kth round)P(Winning atleast once in k - 1 rounds)$+P$(Both win first time in kth round)

$$\begin{aligned} P(N = k) &= \binom{2}{1}\frac{1}{3}^{k-1}\frac{2}{3} * (1 - \frac{1}{3}^{k-1}) + \frac{1}{3}^{2(k-1)}\frac{4}{9} \\ &= \frac{4}{3}.\frac{1}{3}^{k-1} - \frac{8}{9}.\frac{1}{3}^{2(k-1)} \end{aligned}$$

$$\begin{aligned} E[N] &= \sum_{k=1}^{\infty}(k * P(N = k)) \\ E[N] &= \sum_{k=1}^{\infty} k * (\frac{4}{3}\frac{1}{3}^{k-1} - \frac{8}{9}\frac{1}{3}^{2(k-1)}) \\ E[N] &= \frac{4}{3}.\frac{9}{4} - \frac{8}{9}.\frac{81}{64} \\ E[N] &= \frac{15}{8} = 1.875 \end{aligned}$$

1.6 Answer 6

Since 500 pages contains 500 errors on average the rate is 1 error per page
Let X be the random variable for number of errors on one page.
X can be modelled as a poisson random variable.

$$P(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!} \tag{1}$$

Probability of having k errors on one page, $\lambda = 1$

$$P(X = k) = \frac{e^{-1}(1)^k}{k!} \tag{2}$$

$$\begin{aligned} P(X \geq 3) &= 1 - (P(X = 0) + P(X = 1) + P(X = 2)) \\ &= 1 - \left(\frac{e^{-1}(1)^0}{0!} + \frac{e^{-1}(1)^1}{1!} + \frac{e^{-1}(1)^2}{2!}\right) \\ &= 1 - \frac{5e^{-1}}{2} \\ &= 0.080301397 \end{aligned}$$

1.7 Answer 7

$$M_X(t) = \sum_x e^{tx}P_X(x) = E[e^{tX}] \tag{1}$$

Let $N(t)$ be the new mgf
1. kX

$$\begin{aligned} N(t) &= E[e^{tkX}] \\ &= E[e^{(tk)X}] \\ &= M_X(kt) \end{aligned}$$

2. X + k
Using linearity of expectation

$$\begin{aligned} N(t) &= E[e^{t(X+k)}] \\ &= E[e^{tX}.e^{tk}] \\ &= e^{tk}E[e^{tX}] \\ &= e^{tk}M_X(t) \end{aligned}$$

3. $\sum_{i=0}^{N} X_i$

$$N(t) = E[e^{tX_1+X_2+...+X_n}]$$

Since they are independently sampled we can distribute expectation over multiplication, and also the mgf of all X_i is the same since they are from the same distribution.

$$\begin{aligned} N(t) &= E[e^{t(X_1+X_2+...+X_n)}] \\ &= \Pi_{i=1}^{n} E[e^{tX_i}] \\ &= \Pi_{i=1}^{n} M_x(t) \\ &= M_X(t)^n \end{aligned}$$

For continuous random variables $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$
4. PDF(y) = PDF(x + k)
I have two solutions for this as the question seemed unclear
Solution 1: Assumption this means : $f_Y(y) = f_X(y+k)$

$$N_Y(t) = \int_{-\infty}^{\infty} e^{ty} f_Y(y)dy$$
$$N_Y(t) = \int_{-\infty}^{\infty} e^{ty} f_X(y+k)dy$$

Substitute $x = y + k, dx = dy$

$$N_Y(t) = \int_{-\infty}^{\infty} e^{t(x-k)} f_X(x)dx$$
$$N_Y(t) = e^{-kt} \int_{-\infty}^{\infty} e^{tx)} f_X(x)dx$$
$$N_Y(t) = e^{-kt} M_X(t)$$

Solution 2: Assumption this means $f_Y(x) = f_{X+k}(x)$
or $Y = X + k$
From the result in 2nd part we can say that $M_Y(t) = e^{tk}M_X(t)$
5. PDF(y) = PDF(2x)
Again two solutions:
Solution 1:
Assuming it means $f_Y(y) = f_X(2y)$

$$\begin{aligned} N_Y(t) &= \int_{-\infty}^{\infty} e^{ty} f_Y(y)dy \\ &= \int_{-\infty}^{\infty} e^{ty} f_X(2y)dy \end{aligned}$$

Substitute $2y = x, dy = dx/2$

$$\begin{aligned}
N_Y(t) &= \int_{-\infty}^{\infty} e^{tx/2} f_Y(x) dx/2 \\
&= \frac{1}{2} \int_{-\infty}^{\infty} e^{(t/2)x} f_X(x) dx \\
&= \frac{1}{2} M_X(t/2)
\end{aligned}$$

Solution 2:
Assuming it means $f_Y(x) = f_{2X}(x) \Rightarrow Y = 2X$
From the 1st part
$M_Y(t) = M_X(2t)$

1.8 Answer 8

Let a X_i be an indicator random variable such that it takes the value 1 if the ith person selects his own hat, and takes the value 0 otherwise
Since $P(X_i = 1) = \frac{1}{n}$ and $P(X_i = 0) = 1 - \frac{1}{n}$
$E[X_i] = 1 * \frac{1}{n} + 0 * (1 - \frac{1}{n}) = \frac{1}{n}$
Total number of people with own hat
$X = X_1 + X_2 + ... + X_n$
Using linearity of expectation,

$$E[X] = E[X_1 + X_2 + ... + X_n] = E[X_1] + E[X_2]... + E[X_n] \tag{1}$$

From symmetry,

$$E[X] = E[X_1] + E[X_2]... + E[X_n] = nE[X_i] = n * \frac{1}{n} = 1 \tag{2}$$

1.9 Answer 9

Using method of transformation in this question.
Let $g(x) = 6x - 3$
$|\frac{d}{dy} g(x)| = 6$
Since g is strictly increasing inverse exists $x \in [0, 2]$
$g^{-1}(y) = \frac{y+3}{6}, -3 \le y \le 9$
From method of transform

$$\begin{aligned}
f_Y(y) &= \frac{f_X(g^{-1}(y))}{|g'(g^{-1}y)|} \\
&= \frac{y+3}{2*6} . \frac{1}{6} \\
&= \frac{y+3}{72}, -3 \le y \le 9
\end{aligned}$$

$$f_Y(y) = \begin{cases} 0 & y < -3 \\ \frac{y+3}{72}, & -3 \le y \le 9 \\ 0 & y > 9 \end{cases} \tag{1}$$

1.10 Answer 10

Let X_i be the interarrival time between X_{i-1} and X_i arrival.
$P(X_i > t) = P(0 \text{ arrival in t seconds}) = \frac{(\lambda)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$
$P(X_i < t) = 1 - P(X_i > t) = 1 - e^{-\lambda t}$
Let X be the random variable for the number of cars that appear before no cars for τ seconds

$$P_X(x = n) = P(X_1 < \tau)P(X_2 < \tau)..P(X_n < \tau)P(X_{n+1} > \tau) \tag{1}$$

$$P_X(x = n) = (1 - e^{-\lambda\tau})^n (e^{-\lambda\tau}) \tag{2}$$

$$
\begin{aligned}
E[X] &= \sum_{i=0}^{\infty} i P_X(i) \\
&= \sum_{i=0}^{\infty} i(1-e^{-\lambda\tau})^i (e^{-\lambda\tau}) \\
&= e^{-\lambda\tau} \sum_{i=0}^{\infty} i(1-e^{-\lambda\tau})^i \\
&= e^{-\lambda\tau} \frac{(1-e^{-\lambda\tau})}{(1-e^{-\lambda\tau}-1)^2} \\
&= \frac{(e^{-\lambda\tau})(1-e^{-\lambda\tau})}{(e^{-\lambda\tau})^2} \\
E[X] &= e^{\lambda\tau} - 1
\end{aligned}
$$

1.11 Answer 11

Let's say each shot takes 1 unit of time. So s shots take s units of time.

1. ith color can be chosen in n ways.
   Lets say s^{th} shot is hits the an ith color balloon for the second time.
   This means in the rest of $s-1$ shots one of the shot hit ith color balloon. This shot can be selected in $\binom{s-1}{1}$
   Rest $s-2$ shots can have the other $n-1$ balloons without repetition. This has $\binom{n-1}{s-2}$ ways and arranged in $(s-2)!$ ways
   Let T denote the number of ways the event ends in s shots.

$$T(s) = n\binom{s-1}{1}\binom{n-1}{s-2}(s-2)! \tag{1}$$

   The total way of having s shots is n^s

$$P(X=s) = \begin{cases} 0 & s < 2 \\ n(s-1)\binom{n-1}{s-2}(s-2)!\ n^{-s} & s \leq n+1 \end{cases} \tag{2}$$

2. Let s^{th} shot be the last shot. This means i colors have already been shot.
   The way of choosing these i colors is $\binom{n}{i}$
   In the previous $s-1$ shots these i colors are shot
   Since all i colors have to be present this means that at least one shot of each color is there. There are $\binom{s-1}{i}i!$ ways of giving i different color shots
   The rest $s-i-1$ shots have i^{s-i-1} ways. The new color on s^{th} shot has $\binom{n-i}{1}$ ways. Let T denote the number of ways the event ends in s shots.

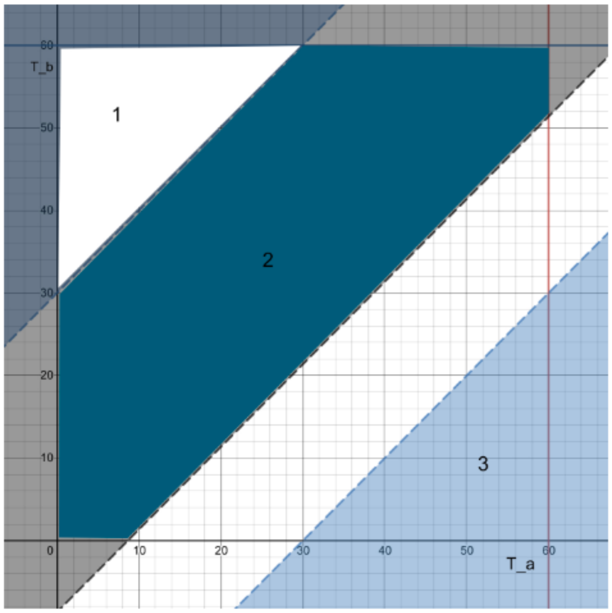
$$T(s) = \binom{n}{i}\ i!\ \binom{s-i}{i} i^{s-i-1} \binom{n-i}{1} \tag{3}$$

   Total ways of having s shots is n^s

$$P(Y_i = s) = \begin{cases} \binom{n}{i}\ i!\ \binom{s-i}{i} i^{s-i-1}\binom{n-i}{1} n^{-s} & s \geq i+1 \\ 0 & s < i+1 \end{cases} \tag{4}$$

1.12 Answer 12

Time is in minutes.
Speed is in km/min $= 5/6km/min$
Let us say they meet before $d$km distance from college.
Time of A $T_a < \frac{6d}{5}$ min
Time of B $T_b > \frac{6(25-d)}{5}$ min
Difference in starting time $= T_a - T_b < \frac{6d}{5} - \frac{6(25-d)}{5} = \frac{12d-150}{5}$
Now this can be plotted on a graph since this is uniform distribution
Also the edge cases when if $T_a - T_b > 30 \Rightarrow d = 25, T_a - T_b < -30, \Rightarrow d = 0$
Let x-axis denote T_a and T_b denote y-axis.



$Area_1$ denotes the area in which $d = 0$.
$Area_2$ denotes the area in which $0 < d < 25$, $Area_2$ is variable
$Area_3$ denotes the area in which $d > 25$
Let X be the meeting distance.

$$F_X(x = d) = \begin{cases} \frac{Area_1}{Total} & d = 0 \\ \frac{Area_1 + Area_2}{Total} & 0 < d \leq 25 \\ \frac{Area_1 + Area_2 + Area_3}{Total} & d > 25 \end{cases} \tag{1}$$

$Total = 60 * 60 = 3600$
$Area_1$ = 1/2*30*30 = 450
$Area_1 + Area_2 = \begin{cases} \frac{1}{2}(\frac{150+12d}{5})^2 & d <= 12.5 \\ 3600 - \frac{1}{2}(\frac{450-12d}{5})^2 & d > 12.5 \end{cases}$
$Area_3 = 1/2 * 30 * 30 = 450$
Putting these values in

$$F_X(x = d) = \begin{cases} 0.125 & d = 0 \\ \frac{1}{7200}(\frac{150+12d}{5})^2 & d <= 12.5 \\ 1 - \frac{1}{7200}(\frac{450-12d}{5})^2 & 12.5 < d <= 25 \\ 1 & d > 25 \end{cases}$$

1.13 Answer 13

1. 3 Bulb Replacements = ABAB
   Expected time = E[A] + E[B] + E[A] + E[B] = 0.25 + 0.5 + 0.25 + 0.5 = 1.5 year

2. Let expected life time of A = t_A = 0.25
   Let expected life time of B = t_B = 0.5
   The first bulb is A which has t_A expected time, and the rest n can be modelled as binomial
   Let k out of n replacements be bulb A.
   Expected lifetime $E[k] = t_A + kt_A + (n-k)t_B$
   P(k out of n replacements be bulb A) = P(R = k) = $\binom{n}{k}p^k(1-p)^{n-k}$

$$
\begin{aligned}
\text{Total Expectation} &= \sum_{k=0}^{t} \binom{n}{k} P(R=k) * E[k] \\
&= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} (t_A + kt_A + (n-k)t_B) \\
&= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} (1/4 + k/4 + (n-k)/2) \\
&= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \left(\frac{2n+1-k}{4}\right) \\
&= 0.25 \left(\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} + 2n \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} - \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}(k) \right) \\
&= 0.25 \left(1 + 2n - \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}(k) \right)
\end{aligned}
$$

Solving for $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}(k)$
This is same as expectation of binomial random variable $= np$

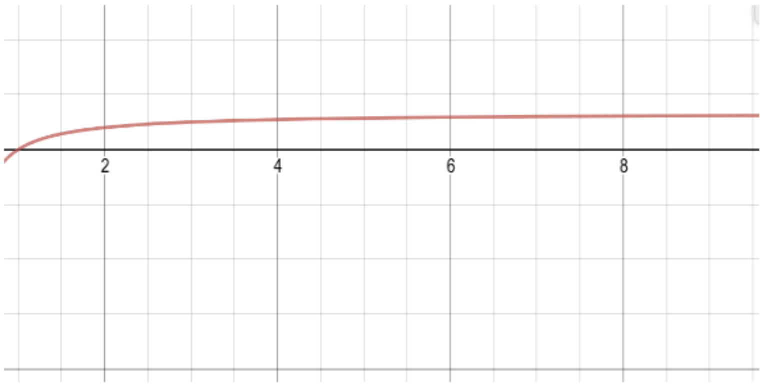
$$\text{Total Expectation} = 0.25\,(1 + 2n - np)$$

1.14 Answer 14

(a) Let (a, b, c) be three vertices in the graph such that $(a, b), (b, c), (c, a) \in E$
X_{ab}, X_{bc}, X_{ca} represent the monochromaticity of three edges connecting the three vertices
$P(X_{ab} = 1 \cap X_{bc} = 1 \cap X_{ca} = 1) = \frac{3}{3^3} = \frac{1}{9}$ (3 favorable outcomes out of 3^3 in total since each vertex can have 3 ways of coloring)
$P(X_{ab} = 1)P(X_{bc} = 1)P(X_{ca} = 1) = (\frac{3}{3^2})^3 = \frac{1}{27}$
(each edge has 3 favorable outcomes out of 3^2 in total since each vertex can have 3 ways of coloring)
Since $P(X_{ab} = 1 \cap X_{bc} = 1 \cap X_{ca} = 1) \neq P(X_{ab} = 1)P(X_{bc} = 1)P(X_{ca} = 1)$
Hence X_e are dependent

(b) Let the total number of edges be n
Let T_i be a an indicator random variable which is one such that i^{th} edge is non monochromatic and 0 otherwise
$P(T_i = 0) = \frac{3}{3^2} = \frac{1}{3} = E[T_i]$ (3 favorable outcomes out of 3^2 in total since each vertex can have 3 ways of coloring)
$P(T_i = 1) = 1 - P(T_i = 0) = \frac{2}{3}$
$Y = T_1 + T_2 + T_3 + ... + T_n$
$E[Y] = E[T_1 + T_2 + T_3 + ... + T_n]$
From linearity of expectation
$E[Y] = E[T_1] + E[T_2] + ... + E[T_n]$
From symmetry
$E[Y] = \sum_{i=1}^{n} E[T_i] = n * E[T_i] = \frac{2n}{3}$

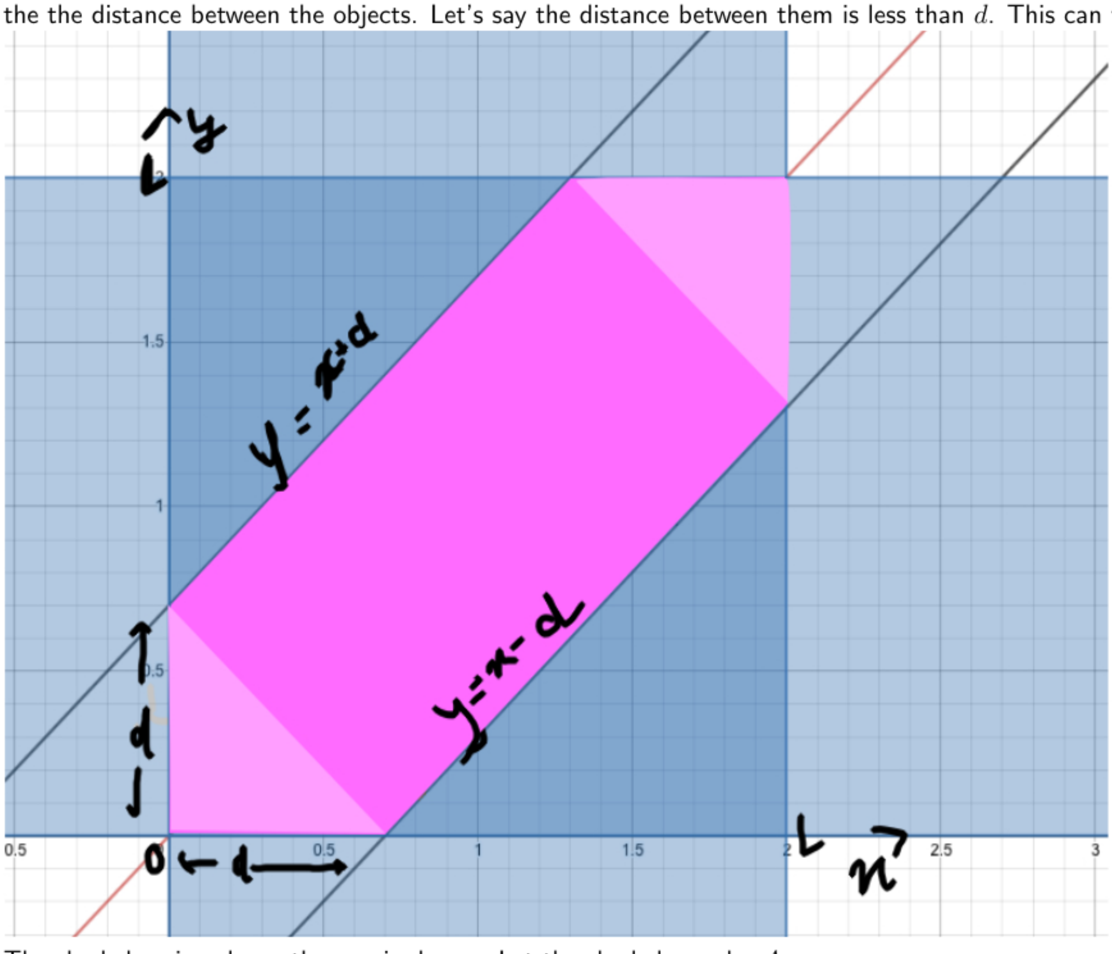
(c) The more connected is the graph the harder it is to not get monochromatic segments
For a complete graph of n vertex K_n there are n vertices and $\binom{n}{2}$ edges where each edge is connected to $n-1$ edges.
Since this is a symmetric color a third of vertex with each color.
For simplicity of calculation we are assuming total vertex $= 3n$ but this can be generalized by splitting remainder equally among colors
Total monochromatic edges $= \binom{n}{2} + \binom{n}{2} + \binom{n}{2}$
Total non monochromatic edges $= \binom{n}{2}$
Ratio $= \frac{3\binom{n}{2}}{\binom{3n}{2}} = \frac{n-1}{3n-1}$
This function is monotonic increasing for $n \geq 1$ and as n tends to ∞ it tends to $\frac{1}{3}$



Thus monochromatic edges are always less than $\leq \frac{1}{3}|E|$
This means non monochromatic edges are always $\geq \frac{2}{3}|E|$
Any non complete graph can be formed by removing edges from complete graph and for them remove the monochromatic edges first thus this holds for every graph
Hence proved.

1.15 Answer 15

Let x axis denote the position of ambulance and y axis denote position of the accident. Time T required is directly dependent on the the distance between the objects. Let's say the distance between them is less than d. This can visualized as follows



The shaded region shows the required area. Let the shaded area be A_s
Area of triangle be A_t(light pink) and area of rectangle be A_r (dark pink)

$$
\begin{aligned}
A_s &= 2 * A_t + A_r \\
&= 2 * \frac{d * d}{2} + L - d\sqrt{2} * d\sqrt{2}) \\
&= d^2 + 2(L-d)d \\
&= d^2 + 2Ld - 2d^2 \\
&= 2Ld - d^2
\end{aligned}
$$

Total Area is L^2
$P(\text{distance less than d}) = \frac{2Ld-d^2}{L^2}$
Let the speed of ambulance be v.
$d = v * t$
Let T be the random variable for time in which ambulance reaches. Here $t \geq 0$

$$
F_T(t) = \frac{2Ld - d^2}{L^2} = \frac{2Ltv - v^2t^2}{L^2} \tag{1}
$$

$$
\begin{aligned}
P_T(t) &= \frac{d}{dt} F_T(t) \\
&= \frac{d}{dt} \frac{2Ltv - v^2t^2}{L^2} \\
&= \frac{2Lv - 2v^2t}{L^2}
\end{aligned}
$$

1.16 Answer 16

(a) P(Z tails) $= \binom{z}{z}(1/2)^0(1/2)^z = (1/2)^z$
Since each trial is independent P(next two trials same) $= (1/2)^z * (1/2)^z = (1/2)^{2z}$

(b) i.
P(All same side) = P(Z tails) + P(Z heads) = 2P(Z tails) $= \frac{1}{2}^{z-1}$
P(Not all same side) = 1 - P(All same side) $= 1 - \frac{1}{2}^{z-1}$
Out of k, one of the trial is all same side and rest k - 1 are not all same side and the k + 1th trial has to be same side

$$
\begin{aligned}
P_X(k) &= \binom{k}{1} \frac{1}{2}^{z-1} \left(1 - \frac{1}{2}^{z-1}\right)^{k-1} \frac{1}{2}^{z-1} \\
&= k \frac{1}{2}^{z} \left(1 - \frac{1}{2}^{z-1}\right)^{k-1}
\end{aligned}
$$

ii.
Let X_i be the number of tails in an unsuccessful trial before the first successful trial
Since Z = 3,

$$
\begin{aligned}
P(X_i = 1) &= \frac{\binom{3}{1}}{6} = \frac{1}{2} \\
P(X_i = 2) &= \frac{\binom{3}{2}}{6} = \frac{1}{2} \\
E[X_i] &= \frac{1}{2} + 2\frac{1}{2} = \frac{3}{2} \\
Var(X_i) &= \frac{1}{2}(1 - \frac{3}{2})^2 + \frac{1}{2}(2 - \frac{3}{2})^2 = \frac{3}{12} = \frac{1}{4}
\end{aligned}
$$

Let N be the number of unsuccessful trials, it is a geometric random variable with $p = \frac{1}{4}$

$$
\begin{aligned}
P(N=i) &= (\frac{3}{4})^i \frac{1}{4} \\
E[N] &= \frac{1}{p} = 4 \\
Var[N] &= \frac{1-p}{p^2} = 12
\end{aligned}
$$

Let M be the number of tails before the first success

$$M = X_1 + X_2 + ... + X_n$$

Using linearity of expectation

$$E[M|N=n] = E[X_1 + X_2 + ... + X_n] = E[X_1] + E[X_2] + ... + E[X_n]$$

From symmetry

$$E[M|N=n] = n * E[X_i] = \frac{3n}{2}$$

From Law of total expectation,

$$
\begin{aligned}
E[M] &= \sum_{i=0}^{\infty} P(N=i)E[M|N=i] \\
&= \sum_{i=0}^{\infty} (\frac{3}{4})^i . \frac{1}{4} . \frac{3i}{2} \\
&= \frac{3}{8} * \sum_{i=0}^{\infty} \frac{3}{4}^i i \\
&= \frac{3}{8} * 12 \\
&= \frac{9}{2}
\end{aligned}
$$

Since X_is are independent,

$$
\begin{aligned}
Var(M|N=i) &= Var(X_1 + X_2 + X_3 + ... + X_i) \\
&= i * Var(X_t) \\
&= \frac{i}{4}
\end{aligned}
$$

From Law of conditional variance,

$$Var(M) = E(Var[M|N]) + Var(E[M|N]) \tag{1}$$

Using Law of Total Expectation

$$
\begin{aligned}
E[Var(M|N)] &= \sum_{i=0}^{\infty} P(N=i)Var(M|N=i) \\
&= \sum_{i=0}^{\infty} (\frac{3}{4})^i . \frac{1}{4} . \frac{i}{4} \\
&= \frac{1}{16} * \sum_{i=0}^{\infty} (\frac{3}{4}^i)i \\
&= \frac{1}{16} * 12 \\
&= \frac{3}{4}
\end{aligned}
$$

Using Law of Iterated Expectation,

$$\begin{aligned}
Var(E[M|N]) &= \sum_{i=0}^{\infty} P(N=i)(E[M|N=i] - E[E[M|N]])^2 \\
&= \sum_{i=0}^{\infty} P(N=i)(E[M|N=i] - E[M])^2 \\
&= \sum_{i=0}^{\infty} P(N=i)(E[M|N=i]^2 + E[M]^2 - 2E[M|N=i]E[M]) \\
&= \sum_{i=0}^{\infty} P(N=i)\left(\frac{9}{4}i^2 + \frac{81}{4} - \frac{27}{2}i\right) \\
&= \frac{1}{16}\sum_{i=0}^{\infty}(\frac{3}{4})^i(9i^2 + 81 - 54i) \\
&= \frac{432}{16}
\end{aligned}$$

$$\begin{aligned}
Var(M) &= E(Var[M|N]) + Var(E[M|N]) \\
&= \frac{3}{4} + \frac{432}{16} \\
&= 27.75
\end{aligned}$$

(c) Let X_i be the number of trials with i coins.
For all i there are only two cases which result in success while 2^i are the total cases.
$E[X_i] = \frac{1}{p}$ since this is geometric process.
$p = 2/2^i = 2^{1-i}$
$E[X_i] = 2^{i-1}$
Let X is the total number of trials before we have one coin where starting coins is n
$X = X_n + X_{n-1} + ... + X_2$
$E[X] = E[X_n] + E[X_{n-1}] + ... + E[X_2]$
$E[X] = \sum_{i=2}^{n} 2^{i-1}$
$E[X] = 2^n - 2$
According to question n is M
$E[X] = 2^M - 2$

1.17 Question 17

Let c_i denote the ith color.

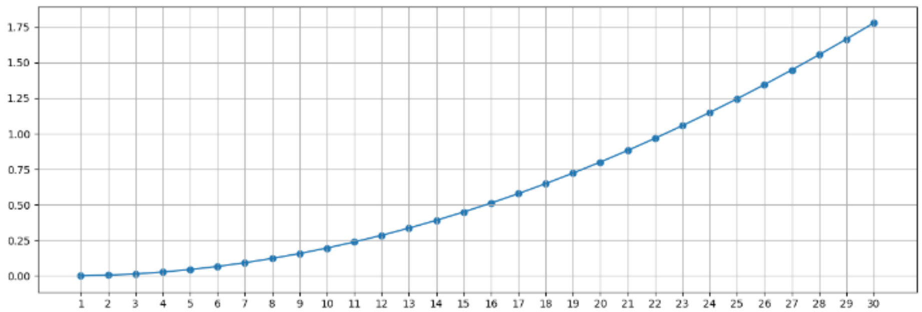
1. Lets say T_i is an indicator random variable which is 1 if the c_i has been assigned to exactly k balls else it is 0.
$P(T_i = 1) = \frac{\binom{n}{k}224^{n-k}}{225^n} = E[T_i]$ (choose k balls out of n and give them the ith color and give other $n-k$ balls from rest 224 colors)
$P(T_i = 0) = 1 - P(T_i = 1)$
Let T = number of colors which have exactly k balloons
$T = T_1 + T_2 + ... + T_{225}$
From linearity of expectation
$E[T] = E[T_1] + E[T_2] + ... + E[T_{225}]$
From symmetry
$$E[T] = 225 * E[T_i] = 225 * \frac{\binom{n}{k}(224)^{n-k}}{225^n} = \frac{\binom{n}{k}224^{n-k}}{225^{n-1}} \tag{1}$$

2. Lets say T_i is an indicator random variable which is 1 if the c_i has been assigned to more than one ball else it is 0.
$P(T_i = 0) = \frac{224^n + n*224^{n-1}}{225^n}$ (if the c_i has been assigned to no balls + if c_i is assigned to one ball)
$P(T_i = 1) = 1 - P(T_i = 0) = E[T_i]$
Let T = number of colors which has been assigned to more than one ball
$T = T_1 + T_2 + ... + T_{225}$
From linearity of expectation

$E[T] = E[T_1] + E[T_2] + ... + E[T_{225}]$
From symmetry

$$E[T] = 225 * E[T_i] = 225 * \left(1 - \frac{224^n + n * 224^{n-1}}{225^n}\right) \tag{2}$$

3. For n = 23 the E[T] = 1.0567595402202672



1.18 Answer 18

X is a random variable for the number of tries it takes to succeed. This is $x - 1$ failure followed by one success.

$$P_X(x) = (1-p)^{x-1}p \text{ for } \{x \in 1, 2, 3...\} \tag{1}$$

Expectation is the average value

$$\begin{aligned} E[X] &= \sum_{x=1}^{\infty} x P_X(x) \\ &= \sum_{x=1}^{\infty} x(1-p)^{x-1}p \\ &= p\sum_{x=1}^{\infty} x(1-p)^{x-1} \\ &= p[\frac{d}{dp}(-\sum_{x=1}^{\infty}(1-p)^x)] \\ &= p[\frac{d}{dp}\frac{-1}{p}] \\ &= p(\frac{1}{p^2}) \\ &= \frac{1}{p} \end{aligned}$$

Variance is $E[X^2] - E[X]^2$ Let $q = 1 - p$

$$
\begin{aligned}
E[X^2] &= \sum_{x=1}^{\infty} x^2 q^{x-1} p \\
&= \sum_{x=1}^{\infty} (x-1+1)^2 q^{x-1} p \\
&= \sum_{x=1}^{\infty} (x-1)^2 q^{x-1} p + \sum_{x=1}^{\infty} 2(x-1)^2 q^{x-1} p + \sum_{x=1}^{\infty} q^{x-1} p \\
&= \sum_{y=1}^{\infty} y^2 q^y p + 2\sum_{y=1}^{\infty} y q^y p + 1 \\
&= qE[X^2] + 2qE[X] + 1 \\
&= \frac{q+1}{p^2} \\
&= \frac{2-p}{p^2}
\end{aligned}
$$

Variance is $\frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$

1.19 Answer 19

Let N_t^k be the number of students which spends time k in the system present at time t.
Thus at time t

$$N_t = \sum_{k=1}^{n} N_t^k \tag{1}$$

Since the total rate of arrival is λ we can represent the arrival as a sum of n independent poisson processes where kth subprocess is arrival of students which spend time k time in the system and has rate λp_k.
A student which spends k time is in the system at time t if and only if it arrives between time $t-k$ and t. Thus the mean for the kth subprocces is

$$E[N_t^k] = \lambda(t-(t-k))p_k = \lambda k p_k \tag{2}$$

$$E[N_t^k] = E[N_t^1 + N_t^2 + N_t^3 + ... + N_t^n]$$

From linearity of expectation:

$$
\begin{aligned}
E[N_t^k] &= E[N_t^1] + E[N_t^2] + E[N_t^3] + ... + E[N_t^n] \\
E[N_t^k] &= \lambda p_1 + \lambda 2 p_2 + ... + \lambda n p_n \\
E[N_t^k] &= \lambda \sum_{i=0}^{n} i * p_i \\
E[N_t^k] &= \lambda E[X]
\end{aligned}
$$

A combination of independent poisson processes is poisson process. N_t^k is a poisson process with mean $\lambda E[X]$
Since expectation is equal to rate for given time, $\lambda' = \lambda E[X]$ where λ' is rate for N_t^k

$$P(N_t^k) = \frac{e^{-\lambda E[X] t} (\lambda E[X] t)^k}{k!}$$

1.20 Answer 20

Using method of transform,
$g(x) = x^{\frac{1}{2}}$
$g'(x) = \frac{1}{2} x^{\frac{-1}{2}}$

g(x) is strictly increasing so inverse exists.
$g^{-1}(y) = y^2, y \geq 0$

$$\begin{aligned} f_Y(y) &= \frac{f_X(g^{-1}(y))}{|g'(g^{-1}y)|} \\ &= \frac{e^{-y^2}}{\frac{1}{2}y^{-1}} \\ &= 2ye^{-y^2}, y \geq 0 \end{aligned}$$

$$f_Y(y) = \begin{cases} 2ye^{-y^2} & y \geq 0 \\ 0 & y < 0 \end{cases} \tag{1}$$