Assignment 4

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1 Section 1

1.1 Answer 1

1.

$$\begin{aligned} E[X] &= \int_{x=0}^{\infty} x.4xe^{-2x}dx \\ &= 4\int_{x=0}^{\infty} x^2e^{-2x}dx \end{aligned}$$

Substitute $2x = y, 2dx = dy$

$$\begin{aligned} E[X] &= 4\int_{x=0}^{\infty} x^2e^{-2x}dx \\ &= \frac{1}{2}\int_{y=0}^{\infty} y^2e^{-y}dy \\ &= \frac{1}{2}\int_{y=0}^{\infty} y^{3-1}e^{-y}dy \\ &= \frac{1}{2}\Gamma(3) \\ &= \frac{1}{2}2! \\ &= 1 \end{aligned}$$

$$\begin{aligned} E[X^2] &= \int_{x=0}^{\infty} x^2.4xe^{-2x}dx \\ &= 4\int_{x=0}^{\infty} x^3e^{-2x}dx \end{aligned}$$

Substitute $2x = y, 2dx = dy$

$$\begin{aligned} E[X^2] &= 4\int_{x=0}^{\infty} x^3e^{-2x}dx \\ &= \frac{1}{4}\int_{y=0}^{\infty} y^3e^{-y}dy \\ &= \frac{1}{4}\int_{y=0}^{\infty} y^{4-1}e^{-y}dy \\ &= \frac{1}{4}\Gamma(4) \\ &= \frac{1}{4}3! \\ &= 1.5 \end{aligned}$$

$$\begin{aligned}
E[X^3] &= \int_{x=0}^{\infty} x^3.4xe^{-2x}dx \\
&= 4\int_{x=0}^{\infty} x^4e^{-2x}dx
\end{aligned}$$

Substitute $2x = y, 2dx = dy$

$$\begin{aligned}
E[X^3] &= 4\int_{x=0}^{\infty} x^4e^{-2x}dx \\
&= \frac{1}{8}\int_{y=0}^{\infty} y^4e^{-y}dy \\
&= \frac{1}{8}\int_{y=0}^{\infty} y^{5-1}e^{-y}dy \\
&= \frac{1}{8}\Gamma(5) \\
&= \frac{1}{8}4! \\
&= 3
\end{aligned}$$

2.

$$\begin{aligned}
E[X^n] &= \int_{x=0}^{\infty} x^n.4xe^{-2x}dx \\
&= 4\int_{x=0}^{\infty} x^{n+1}e^{-2x}dx
\end{aligned}$$

Substitute $2x = y, 2dx = dy$

$$\begin{aligned}
E[X^n] &= 4\int_{x=0}^{\infty} x^{n+1}e^{-2x}dx \\
&= \frac{1}{2^n}\int_{y=0}^{\infty} y^{n+1}e^{-y}dy \\
&= \frac{1}{2^n}\int_{y=0}^{\infty} y^{n+2-1}e^{-y}dy \\
&= \frac{1}{2^n}\Gamma(n+2) \\
&= \frac{1}{2^n}(n+1)!
\end{aligned}$$

1.2 Answer 2

1. $\Gamma(7/2)$

$$\begin{aligned}
\Gamma(7/2) &= \frac{5}{2}.\Gamma(\frac{5}{2}) && (1) \\
&= \frac{5}{2}.\frac{3}{2}.\Gamma\left(\frac{3}{2}\right) && (2) \\
&= \frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right) && (3)
\end{aligned}$$

Finding $\Gamma(\frac{1}{2})$

$$\Gamma(\frac{1}{2}) = \int_0^{\infty} x^{-\frac{1}{2}}e^{-x}dx \quad (4)$$

Substitute $x = t^2, dx = 2tdt$

$$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-t^2} dt \tag{5}$$

Consider a Normal random variable with mean = 0, $\sigma^2 = \frac{1}{2}$

$$f_X(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} dx \tag{6}$$

Area under random variable PMF is 1

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2} dx = 1 \tag{7}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \tag{8}$$

Since the function is even, from (8) we can conclude

$$\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2 \tag{9}$$

From (5) and (9)

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{10}$$

From (3)

$$\Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi} \tag{11}$$

2.

$$I = \int_0^\infty x^6 e^{-5x} dx \tag{12}$$

Substitute $5x = t, 5dx = dt$

$$I = \frac{1}{5^7}\int_0^\infty t^6 e^{-t} dt \tag{13}$$

$$= \frac{1}{5^7}\int_0^\infty t^{7-1} e^{-t} dt \tag{14}$$

$$= \frac{1}{5^7}\Gamma(7) \tag{15}$$

$$= \frac{1}{5^7}6! \tag{16}$$

$$= \frac{360}{5^7} \tag{17}$$

$$= 0.009216 \tag{18}$$

1.3 Answer 3

(a)

$$\sum_x \sum_y f_{X,Y}(x,y) = 1 \tag{1}$$

$$c(2.(1^2 + 2^2 + 4^2) + 3(1^2 + 3^2)) = 1 \tag{2}$$

$$c = \frac{1}{72} \tag{3}$$

(b)

$$P(Y < X) = \sum \sum_{y<x} f_{X,Y}(x,y) \tag{4}$$

$$= \frac{1}{72}((1^2 + 2^2) + (1^2 + 4^2) + (3^2 + 4^2)) \tag{5}$$

$$= \frac{47}{72} \tag{6}$$

(c)

$$P(Y > X) = \sum\sum_{y>x} f_{X,Y}(x, y) \tag{7}$$
$$= \frac{1}{72}((3^2 + 2^2) + (3^2 + 1^2)) \tag{8}$$
$$= \frac{23}{72} \tag{9}$$

(d)

$$P(Y = X) = \sum\sum_{y=x} f_{X,Y}(x, y) \tag{10}$$
$$= \frac{1}{72}((1^2 + 1^2)) \tag{11}$$
$$= \frac{2}{72} \tag{12}$$

(e)

$$P(Y = 3) = \sum_{y=3}\sum_{x} f_{X,Y}(x, y) \tag{13}$$
$$= \frac{1}{72}((1^2 + 3^2) + (2^2 + 3^2) + (4^2 + 3^2)) \tag{14}$$
$$= \frac{48}{72} \tag{15}$$

(f)

$$f_X(x) = \sum_{y_i} f_{X,Y}(x, y_i) \tag{16}$$
$$= \frac{1}{72}((x^2 + 1^2) + (x^2 + 3^2)) \tag{17}$$
$$= \frac{2x^2 + 10}{72} \tag{18}$$
$$= \frac{x^2 + 5}{36}, x \in (1, 2, 4) \tag{19}$$

$$f_Y(y) = \sum_{x_i} f_{X,Y}(x_i, y) \tag{20}$$
$$= \frac{1}{72}((1^2 + y^2) + (2^2 + y^2) + (4^2 + y^2)) \tag{21}$$
$$= \frac{3y^2 + 21}{72} \tag{22}$$
$$= \frac{y^2 + 7}{24}, y \in (1, 3) \tag{23}$$

(g)

$$E[X] = \sum x_i f_X(x_i) \tag{24}$$
$$= \frac{1^2 + 5}{36} + 2\frac{2^2 + 5}{36} + 4\frac{4^2 + 5}{36} \tag{25}$$
$$= \frac{6 + 2.(9) + 4.(21)}{36} \tag{26}$$
$$= \frac{108}{36} \tag{27}$$
$$= 3 \tag{28}$$

$$\begin{aligned} E[Y] &= \sum y_i f_Y(y_i) && (29) \\ &= \frac{(1^2+7)+3(3^2+7)}{24} && (30) \\ &= \frac{56}{24} && (31) \\ &= \frac{7}{3} && (32) \end{aligned}$$

$$\begin{aligned} E[XY] &= \sum_{y_i}\sum_{x_i} xy f_{XY}(x,y) && (33) \\ &= \frac{(1^2+1^2)+3.(1^2+3^2)+2(2^2+1^2)+6(2^2+3^2)+4(4^2+1^2)+12(4^2+3^2)}{72} && (34) \\ &= \frac{2+3.(10)+2(5)+6(13)+4(17)+12(25)}{72} && (35) \\ &= \frac{488}{72} && (36) \\ &= 6.777777778 && (37) \end{aligned}$$

(h)

$$\begin{aligned} Var(X) &= E[X^2]-E[X]^2 && (38) \\ E[X^2] &= \sum_x x^2 f_X(x) && (39) \\ &= \frac{1^2+5}{36} + 2^2\frac{2^2+5}{36} + 4^2\frac{4^2+5}{36} && (40) \\ &= \frac{1+5+16+20+256+80}{36} && (41) \\ &= \frac{378}{36} && (42) \\ &= 10.5 && (43) \\ Var(X) &= 10.5-(3)^2 && (44) \\ &= 1.5 && (45) \end{aligned}$$

$$\begin{aligned} Var(Y) &= E[Y^2]-E[Y]^2 && (46) \\ E[Y^2] &= \sum_y y^2 f_Y(y) && (47) \\ &= \frac{1^2+7}{24} + 3^2\frac{3^2+7}{24} && (48) \\ &= \frac{8+9.(16)}{24} && (49) \\ &= \frac{152}{24} && (50) \\ &= 6.333 && (51) \\ Var(Y) &= 6.333-5.444 && (52) \\ &= 0.889 && (53) \end{aligned}$$

$$\begin{aligned} Var(X+Y) &= E[(X+Y)^2]-E[X+Y]^2 && (54) \\ &= E[X^2]+E[Y^2]+2E[XY]-(E[X]^2+E[Y]^2+2E[X]E[Y]) && (55) \\ &= 10.5+6.333+2*6.777-(9+5.444+2*7) && (56) \\ &= 1.943 && (57) \end{aligned}$$

(i)

$$P(A) = P(X \geq Y) = P(Y < X) + P(Y = X) = \frac{49}{72} \tag{58}$$

$$\begin{aligned}
P(X = 1|A) &= \frac{P(X = 1 \cap A)}{P(A)} && (59)\\
&= \frac{P(1,1)}{P(A)} && (60)\\
&= \frac{72.(1^2 + 1^2)}{72.49} && (61)\\
&= \frac{2}{49} && (62)\\
P(X = 2|A) &= \frac{P(X = 2 \cap A)}{P(A)} && (63)\\
&= \frac{P(2,1)}{P(A)} && (64)\\
&= \frac{72.(2^2 + 1^2)}{72.49} && (65)\\
&= \frac{5}{49} && (66)\\
P(X = 4|A) &= \frac{P(X = 4 \cap A)}{P(A)} && (67)\\
&= \frac{P(4,1) + P(4,3)}{P(A)} && (68)\\
&= \frac{72.(4^2 + 1^2) + 72.(4^2 + 3^2)}{72.49} && (69)\\
&= \frac{42}{49} && (70)
\end{aligned}$$

$$\begin{aligned}
E[X|A] &= 1.\frac{2}{49} + 2.\frac{5}{49} + 4.\frac{42}{49} && (71)\\
&= \frac{180}{49} = 3.673 && (72)
\end{aligned}$$

$$\begin{aligned}
E[X^2|A] &= 1^2.\frac{2}{49} + 2^2.\frac{5}{49} + 4^2.\frac{42}{49} && (73)\\
&= \frac{694}{49} = 14.1632 && (74)
\end{aligned}$$

$$\begin{aligned}
Var(X|A) &= E[X^2|A] - (E[X|A])^2 && (75)\\
&= \frac{694}{49} - (\frac{180}{49})^2 && (76)\\
&= 0.66888 && (77)
\end{aligned}$$

1.4 Answer 4

Since X is bernoulli,

$$\begin{aligned}
P(X = 1|Q = q) &= q && (1)\\
P(X = 0|Q = q) &= 1 - q && (2)\\
f_{Q|X}(q|x) &= \frac{P(Q = q \cap X = x)}{P(X = x)}, x \in \{0, 1\} && (3)
\end{aligned}$$

$$f_{Q,X}(q,x) = P(Q = q \cap X = x) \tag{4}$$
$$= P(X = x|Q = q)P(Q = q) \tag{5}$$
$$f_{Q,X}(q,1) = P(X = 1|Q = q)P(Q = q) \tag{6}$$
$$f_X(1) = \int_0^1 P(X = 1|Q = q)P(Q = q)dq \tag{7}$$
$$= \int_0^1 q(6q(1-q))dq \tag{8}$$
$$= 6\int_0^1 q^2 - q^3 \; dq \tag{9}$$
$$= 6(\frac{q^3}{3} - \frac{q^4}{4})\Big|_0^1 = \frac{1}{2} \tag{10}$$

Since X is bernoulli $f_X(0) = \frac{1}{2}$
From (5),

$$P(Q = q \cap X = 1) = q * (6q(1-q)) \tag{11}$$
$$P(Q = q \cap X = 0) = (1-q) * (6q(1-q)) \tag{12}$$

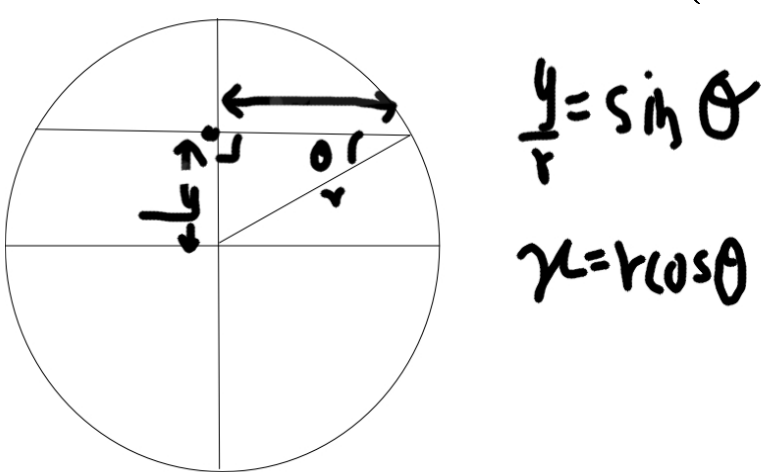
Substituting value from (12), (11), (10) in (3),

$$f_{Q|X}(q|x) = \begin{cases} 12q^2(1-q) & x = 1 \\ 12q(1-q)^2 & x = 0 \end{cases} q \in [0,1] \tag{13}$$

1.5 Answer 5

Area of the target $= \pi r^2$
Since by defn the probability is uniformly distributed

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$



For Y = y, the given range of x is $(r\cos\theta, -r\cos\theta)$ where $\sin\theta = \frac{y}{r}$
This means $f_{X|Y}(x|y)$ is of total length to $2r\cos\theta$ The probability is uniformly distributed along this length hence the probability of each point is just the reciprocal.

$$f_{X|Y}(x|y) = \frac{1}{2r\cos\theta} = \frac{1}{2r\sqrt{1 - \frac{y^2}{r^2}}}, x^2 + y^2 \leq r^2 \tag{2}$$

1.6 Answer 6

Total ways of distributing 4 balls in 4 different bins is 4^4 since each ball can go in one of the 4 bins.
If there are x balls in bin 3, then we have $\binom{4}{x}$ ways of choosing these balls
Rest $4 - x$ balls have $(3)^{4-x}$ ways of going into other 3 bins.

$$p_X(x) = \frac{\binom{4}{x}3^{(4-x)}}{4^4}, x \in (0,1,2,3,4) \tag{1}$$

Let us assume that there are n non empty bins.
To solve for this we will use principle of exclusion and inclusion.
Total ways of distribution of 4 balls in n bins = $t_1 = n^4$
But this also includes in which all objects just go to $n-1$ bins so we will subtract that
$t_2 = n^4 - \binom{n}{n-1}(n-1)^4$, but in this case we will over-subtract the cases in which all balls go to $n-2$ bins, so we add them.
$t_3 = n^4 - \binom{n}{n-1}(n-1)^4 + \binom{n}{n-2}(n-2)^4 = n^4 - \binom{n}{1}(n-1)^4 + \binom{n}{2}(n-2)^4$
While doing this we have over added the cases in which balls go to $n-3$ bins so we have to subtract them.
This pattern continues and can be generalized as

$$
\begin{aligned}
t &= n^4 - \binom{n}{1}(n-1)^4 + ... + (-1)^{n-1}\binom{n}{n-1}(1)^4 \\
&= \sum_{i=0}^{n-1}(-1)^i \binom{n}{i}(n-i)^4
\end{aligned}
$$

$$
f_N(n) = \frac{t}{4^4} = \sum_{i=0}^{n-1}(-1)^i \binom{n}{i}\left(\frac{n-i}{4}\right)^4, 1 <= n <= 4 \tag{2}
$$

Number of ways of having A = $4^3 = 64$ (ball 2, 3, 4 have 4 choices)
For the last part lets consider the cases one by one. Let n_i denote the ith non empty bin.

1. X = x, N = 2
   Choose x ball out of Ball2-4 and put it in 3rd bin and rest go to first (including 1st ball to satisfy A)

$$
p_{X,N|A}(x,2) = \frac{\binom{3}{x}}{64} \quad x \in (1,2,3) \tag{3}
$$

2. X = 1, N = 3
   Choose 1 ball out of Ball2-4 and put it in 3rd bin(n_1). Ball 1 goes to bin 1(n_2). Then then either bin 2 or bin 4 can be n_3 (2 ways). The remaining two balls can either just go in the n_3 (1 way) or 1 ball goes to n_2 and 1 goes goes to n_3 (2 ways)

$$
p_{X,N|A}(1,3) = \frac{\binom{3}{1}\binom{2}{1}(1+2)}{64} = \frac{9}{32} \tag{4}
$$

3. X = 2, N = 3
   Choose 2 ball out of Ball2-4 and put it in 3rd bin(n_1)(3 ways). Ball 1 goes to bin 1(n_2). Then then either bin 2 or bin 4 can be n_3 (2 ways). The remaining ball goes to n_3 (1 way)

$$
p_{X,N|A}(2,3) = \frac{\binom{3}{2}\binom{2}{1}(1)}{64} = \frac{3}{32} \tag{5}
$$

4. X = 3, N = 3
   This is not possible since Ball2-4 go to bin 3, and ball 1 goes to bin 1. Hence only two bins are non empty

$$
p_{X,N|A}(3,3) = 0 \tag{6}
$$

$$
p_{X,N|A}(x,n) = \begin{cases} \frac{\binom{3}{x}}{64} & n = 2, x = (1,2,3) \\ \frac{9}{32} & x = 1, n = 3 \\ \frac{3}{32} & x = 2, n = 3 \\ 0 & \text{otherwise} \end{cases} \tag{7}
$$

1.7 Answer 7

Let M be the number of steps in 1 unit of time,

$$
E[M] = \frac{1}{2} * 1 + \frac{1}{4} * 2 + \frac{1}{4} * 0 = 1 \tag{1}
$$

(a) Since he has $\frac{1}{3}$ prob of passing out at 1, 2 or 3 time each, let Z be total number of steps. From total expectation theorem,

$$\begin{aligned} E[Z] &= \sum_{i=1}^{3} P(i) * (i * E[M]) \\ &= \frac{1}{3}(1*1) + \frac{1}{3}(2*1) + \frac{1}{3}(3*1) \\ &= \frac{6}{3} = 2 \end{aligned}$$

Displacement is synonymous to the final position, the D_i be the displacement in ith unit of time.

$$E[D_i] = \frac{1}{2} * 1 + \frac{1}{4} * -2 + \frac{1}{4} * 0 = 0 \tag{2}$$

(b) Denote a step forward by F, backward by B and stay at place by N Define S = total Displacement The unordered set of steps to have the given displacements is
S = -6 = (BBB)
S = -4 = (BB, BBN)
S = -3 = (BBF)
S = -2 = (B, BN, BNN)
S = -1 = (BF, BFN)
S = 0 = (N, NN, NNN, FFB)
S = 1 = (F, FN, FNN)
S = 2 = (FF, FFN)
S = 3 = (FFF)

$$\begin{aligned} P(S=-6) &= \frac{1}{3}\frac{1}{4^3} = \frac{1}{192} = 0.0052 \\ P(S=-4) &= \frac{1}{3}(\frac{1}{4^2} + 3\frac{1}{4^3}) = \frac{7}{192} \\ P(S=-3) &= \frac{1}{3}3\frac{1}{32} = \frac{1}{32} \\ P(S=-2) &= \frac{1}{3}(\frac{1}{4} + 2\frac{1}{16} + 3\frac{1}{64}) = \frac{9}{64} \\ P(S=-1) &= \frac{1}{3}(2\frac{1}{8} + 6\frac{1}{32}) = \frac{7}{48} \\ P(S=0) &= \frac{1}{3}(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + 3\frac{1}{16}) = \frac{11}{64} \\ P(S=1) &= \frac{1}{3}(\frac{1}{2} + 2\frac{1}{8} + 3\frac{1}{32}) = \frac{9}{32} \\ P(S=2) &= \frac{1}{3}(\frac{1}{4} + 3\frac{1}{16}) = \frac{7}{48} \\ P(S=3) &= \frac{1}{3}(\frac{1}{8}) = \frac{1}{24} \end{aligned}$$

$$E[S^5] = \sum_s s^5.P(S=s) \tag{3}$$

Substituting values we get,

$$\begin{aligned} E[S^5] = &(-6)^5/192 + (-4)^5 * 7/192 + (-3)^5/32 + (-2)^5 * 9/64 + \\ &(-1)^5 * 7/48 + 9/32 + 2^5 * 7/48 + 3^5 * 1/24 \\ &= -75.0 \end{aligned}$$

1.8 Answer 8

[REMOVED]

1.9 Answer 9

$$f_{X,Y}(x,y) = P(X = x \text{ and } Y = y) \tag{1}$$

Let say there are total n tosses. Total outcomes $= 6^n$
To get X = x and Y = y, first we need to select $x+y$ tosses $= \binom{n}{x+y}$
Out of these $x+y$ we need to select x tosses which will have 1 $= \binom{x+y}{x}$
Remaining y tosses will have 2.
The unselected tosses can have anything else that is 4 choice for each $4^{n-(x+y)}$ Total favorable outcomes =

$$= \binom{n}{x+y}\binom{x+y}{x}4^{n-(x+y)} \tag{2}$$

$$\tag{3}$$

$$\begin{aligned} f_{X,Y}(x,y) &= P(X = x \text{ and } Y = y) \\ &= \frac{\binom{n}{x+y}\binom{x+y}{x}4^{n-(x+y)}}{6^n} \\ &= \binom{n}{x+y}\binom{x+y}{x}\left(\frac{2}{3}\right)^{n-(x+y)}\left(\frac{1}{6}\right)^{x+y} \end{aligned}$$

Note : $x+y <= n, n > 0, x, y >= 0$
Given n = 4

$$\begin{aligned} f_{X,Y}(x,y) &= P(X = x \text{ and } Y = y) \\ &= \frac{\binom{4}{x+y}\binom{x+y}{x}4^{4-(x+y)}}{6^4} \\ &= \binom{4}{x+y}\binom{x+y}{x}\left(\frac{2}{3}\right)^{4-(x+y)}\left(\frac{1}{6}\right)^{x+y} \end{aligned}$$

Note : $x+y <= 4, x, y >= 0$

1.10 Answer 10

$$Y = \sum_{i}^{n} X_i \tag{1}$$

If exactly n customers visit the shop, for a given number of customers n

$$E[Y|N = n] = E[\sum_{i=0}^{n} X_i] \tag{2}$$

From linearity of expectation,

$$\begin{aligned} E[Y|N = n] &= \sum_{i=0}^{n} E[X_i] && (3) \\ &= \sum_{i=0}^{n} k && (4) \\ &= nk && (5) \\ E[Y|N] &= Nk && (6) \end{aligned}$$

From law of iterated expectations,

$$\begin{aligned} E[Y] &= E[E[Y|N]] && (7) \\ E[Y] &= E[Nk] && (8) \\ E[Y] &= kE[N] && (9) \\ E[Y] &= ke && (10) \end{aligned}$$

To calculate variance lets consider the conditional case first

$$Var(Y|N=n) = Var(\sum_{i=0}^{n} X_i) \tag{11}$$

Since X_i are independent of each other we can apply linearity of variance,

$$\begin{aligned} Var(Y|N=n) &= \sum_{i=0}^{n} Var(X_i) && (12)\\ Var(Y|N=n) &= nm && (13)\\ Var(Y|N) &= Nm && (14) \end{aligned}$$

Using result from (14),

$$\begin{aligned} E[Var(Y|N)] &= E[Nm] && (15)\\ &= mE[N] && (16)\\ &= me && (17)\\ & && (18) \end{aligned}$$

Using result from (6),

$$\begin{aligned} Var(E[Y|N]) &= Var(Nk) && (19)\\ &= k^2 Var(N) && (20)\\ &= k^2 v && (21) \end{aligned}$$

$$\begin{aligned} Var(Y) &= E[Var(Y|N)] + Var(E[Y|N])\\ &= me + k^2 v \end{aligned}$$

1.11 Answer 11

Let T denote the random variable for temperature
$T = N(10, 10^2)$
$59F = 15C$
Let X be a standard normal variable, for a shifted normal distribution $Y = N(\mu.\sigma)$:

$$\begin{aligned} F_Y(y) &= P(Y < y) && (1)\\ &= P(\sigma X + \mu < y) && (2)\\ &= P(X < \frac{y-\mu}{\sigma}) && (3)\\ &= \phi(\frac{y-\mu}{\sigma}) && (4) \end{aligned}$$

Using this result (4)

$$\begin{aligned} P(T \leq 59) &= F_T(15) && (5)\\ &= \phi(\frac{15-10}{10}) && (6)\\ &= \phi(0.5) && (7)\\ &= 0.691 && (8) \end{aligned}$$

1.12 Answer 12

Let rice be random variable X and curry be random variable Y
From the result proved in Question 13 we get PDF of Z = X + Y as

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma_Z} \exp[-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}] = N(\mu_X+\mu_Y, \sigma_X^2+\sigma_Y^2) \tag{1}$$

Substituting the values we get

$$f_Z(z) = N(540, 25^2) \tag{2}$$

$$\begin{aligned} P(Z \leq 575) &= F_Z(575) \tag{3}\\ & \tag{4} \end{aligned}$$

Let X be a standard normal variable, for a shifted normal distribution $Y = N(\mu.\sigma)$:

$$\begin{aligned} F_Y(y) &= P(Y < y) \tag{5}\\ &= P(\sigma X + \mu < y) \tag{6}\\ &= P(X < \frac{y-\mu}{\sigma}) \tag{7}\\ &= \phi(\frac{y-\mu}{\sigma}) \tag{8} \end{aligned}$$

Using this result

$$\begin{aligned} P(Z \leq 575) &= F_Z(575) \tag{9}\\ &= \phi(\frac{575-540}{25}) \tag{10}\\ &= \phi(1.4) \tag{11}\\ &= 0.919 \tag{12} \end{aligned}$$

1.13 Answer 13

$X = N(\mu_X, \sigma_X^2), Y = N(\mu_Y, \sigma_Y^2), Z = X + Y$

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{(z-x-\mu_Y)^2}{2\sigma_Y^2}\right] \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right] dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma_X\sigma_Y} \exp\left[-\frac{\sigma_X^2(z-x-\mu_Y)^2 + \sigma_Y^2(x-\mu_X)^2}{2\sigma_X^2\sigma_Y^2}\right] dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma_X\sigma_Y} \exp\left[-\frac{\sigma_X^2(z^2+x^2+\mu_Y^2-2xz-2z\mu_Y+2x\mu_Y) + \sigma_Y^2(x^2+\mu_X^2-2x\mu_X)}{2\sigma_Y^2\sigma_X^2}\right] dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma_X\sigma_Y} \exp\left[-\frac{x^2(\sigma_X^2+\sigma_Y^2) - 2x(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X) + \sigma_X^2(z^2+\mu_Y^2-2z\mu_Y)+\sigma_Y^2\mu_X^2}{2\sigma_Y^2\sigma_X^2}\right] dx \end{aligned}$$

Let $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Z} \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} \exp\left[-\frac{x^2 - 2x\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2} + \frac{\sigma_X^2(z^2+\mu_Y^2-2z\mu_Y)+\sigma_Y^2\mu_X^2}{\sigma_Z^2}}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right] dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Z} \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} \exp\left[-\frac{\left(x - \frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2 - \left(\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2 + \frac{\sigma_X^2(z-\mu_Y)^2+\sigma_Y^2\mu_X^2}{\sigma_Z^2}}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right] dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Z} \exp\left[-\frac{\sigma_Z^2\left(\sigma_X^2(z-\mu_Y)^2+\sigma_Y^2\mu_X^2\right) - \left(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X\right)^2}{2\sigma_Z^2\left(\sigma_X\sigma_Y\right)^2}\right] \\ &\quad \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} \exp\left[-\frac{\left(x - \frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right] dx \\ &= \frac{1}{\sqrt{2\pi}\sigma_Z} \exp\left[-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}\right] \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} \exp\left[-\frac{\left(x - \frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right] dx \end{aligned}$$

The integral evaluates to 1 since it is cdf of a normal random variable.

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma_Z} \exp[-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}] = N(\mu_X+\mu_Y, \sigma_X^2+\sigma_Y^2) \tag{1}$$

Hence proved

1.14 Answer 14

$$f_{X,Y}(x,y) = \begin{cases} 4y(x-y)e^{-(x+y)} & 0 < x < \infty, 0 \le y \le x \\ 0 & 0 \end{cases}$$

$0 < x < \infty, 0 \le y \le x$

$$\begin{aligned} f_Y(y) &= \int_{x=y}^{\infty} 4y(x-y)e^{-(x+y)}dx \\ &= 4ye^{-y}\int_{x=y}^{\infty}(x-y)e^{-x}dx,\ y \ge 0 \end{aligned}$$

After using integration by parts to solve this we get

$$\begin{aligned} f_Y(y) &= 4ye^{-y}(-x+y-1)\Big|_0^{\infty} \qquad (1) \\ &= 4e^{-2y}y,\ y \ge 0 \qquad (2) \end{aligned}$$

From definition, conditional prob $= \frac{f_{X,Y}(x,y)}{f_Y(y)}$

$$f_{X|Y}(x|y) = \begin{cases} (x-y)e^{-(x+y)}e^{2y} & x \ge y, \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$\begin{aligned} E[X|Y=y] &= \int_y^{\infty} x f_{X|Y}(x,y)dx \qquad (4) \\ &= \int_y^{\infty} x(x-y)e^{-(x+y)}e^{2y}dx \qquad (5) \end{aligned}$$

After using integration by parts and solving we get,

$$\begin{aligned} E[X|Y=y] &= e^{y-x}(-x^2+x(y-2)+y-2)\Big|_y^{\infty} \qquad (6) \\ &= y+2,\quad y \ge 0 \qquad (7) \end{aligned}$$

$$\begin{aligned} E[X^2|Y=y] &= \int_y^{\infty} x^2 f_{X|Y}(x,y)dx \qquad (8) \\ &= \int_y^{\infty} x^2(x-y)e^{-(x+y)}e^{2y}dx \qquad (9) \end{aligned}$$

After using integration by parts and solving we get,

$$\begin{aligned} E[X^2|Y=y] &= e^{y-x}(-x^3+x^2(y-3)+2x(y-3)+2(y-3))\Big|_y^{\infty} \qquad (10) \\ &= y^2+4y+6,\quad y \ge 0 \qquad (11) \end{aligned}$$

$$\begin{aligned} Var(X|Y=y) &= E[X^2|Y=y] - E[X|Y=y]^2 \qquad (12) \\ &= y^2+4y+6-(y+2)^2 \qquad (13) \\ &= 2 \qquad (14) \end{aligned}$$

1.15 Answer 15

Using properties of expectation and variance,

$$X = N(\mu_X, \sigma_X^2), Y = N(\mu_Y, \sigma_Y^2) \tag{1}$$

$$E[aX] = aE[X], \sigma_{aX}^2 = a^2\sigma_X^2, \Rightarrow aX = N(a\mu_X, a^2\sigma_X^2) \tag{2}$$

We can do similarly for bY. Using the result from 13 we get,

$$Z = (aX) + (bY) = N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2) \tag{3}$$

Using (3)

1.

$$\begin{aligned} Y &= 2X_1 + 3X_2 & (4)\\ &= N(2 * \mu_{X_1} + 3 * \mu_{X_2}, 2^2\sigma_{X_1}^2 + 3^2\sigma_{X_2}^2) & (5)\\ &= N(2 * 2 + 3 * 1, 2^2.3 + 3^2.4) & (6)\\ &= N(7, \sqrt{48}^2) & (7) \end{aligned}$$

2.

$$\begin{aligned} Y &= X_1 - X_2 & (8)\\ &= N(1 * \mu_{X_1} - 1 * \mu_{X_2}, 1^2\sigma_{X_1}^2 + 1^2\sigma_{X_2}^2) & (9)\\ &= N(1 * 2 - 1 * 1, 1^2.3 + 1^2.4) & (10)\\ &= N(1, \sqrt{7}^2) & (11) \end{aligned}$$