Prob Asgn 5

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1. 40 blue marbles and 60 red marbles

We choose 10 marbles at random.

$X =$ number of blue marbles

$Y =$ number of red marbles

Since marbles are chosen without replacement, it is hypergeometric

$X + Y = 10$

$R_{XY} = \{ (x,y) \mid x+y = 10, x,y \geq 0, x,y \in \mathbb{Z} \}$

$= \{ (0,10), (1,9) \ldots\ldots (9,1), (10,0) \}$

For a hypergeometric distribution

$$P_X(k) = \frac{{}^{K}C_{k} \, {}^{N-K}C_{n-k}}{{}^{N}C_{n}}$$

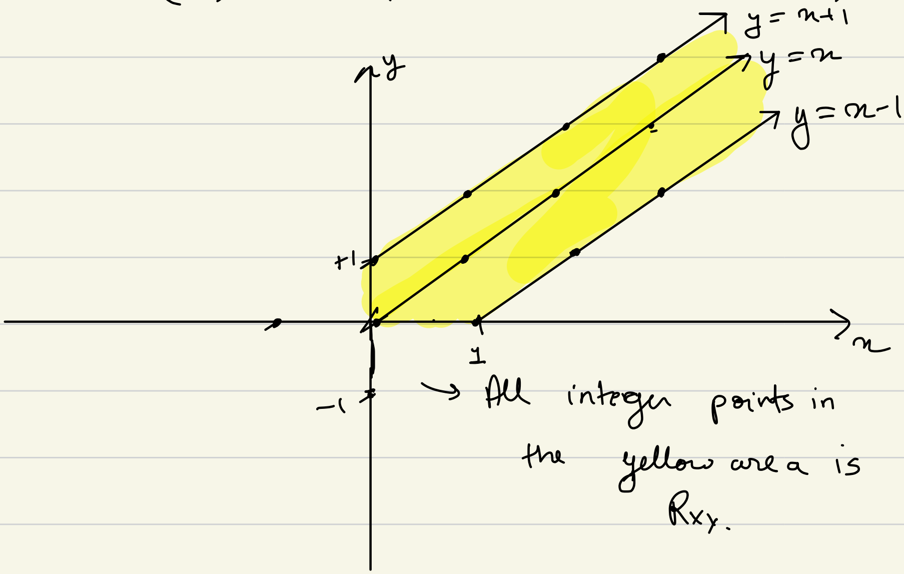
where $N =$ population size $\quad k =$ number of successes

$n =$ number of draws $\quad K =$ number of success states

$$\Rightarrow P_{X,Y}(x,y) = \begin{cases} \dfrac{{}^{40}C_{x} \cdot {}^{60}C_{y}}{{}^{100}C_{10}}, & x+y = 10, x,y \geq 0, x,y \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

Q2 $\quad R_{XY} = \{(i,j) \in \mathbb{Z}^2 \mid i,j \geq 0 ,\ |i-j| \leq 1\}$

1.



2. $$P_{XY}(i,j) = \frac{1}{6 \cdot 2^{\min(i,j)}}$$

$$P_X(i) = \sum_{y} \frac{1}{6 \cdot 2^{\min(i,y)}}$$

for $x = i \Rightarrow y \in \{i, i-1, i+1\}$ $(i-1 \geq 0)$

So $$P_X(i) = \frac{1}{6}\left(\frac{1}{2^{\min(i,i-1)}} + \frac{1}{2^{\min(i,i)}} + \frac{1}{2^{\min(i,i+1)}}\right)$$

$$= \frac{1}{6}\left(\frac{1}{2^{i-1}} + \frac{2}{2^{i}}\right)$$

$$= \frac{1}{6 \cdot 2^{i-2}} \quad \text{for } \quad i \geq 1$$

for $x = 0$, $y \in \{0, 1\}$

$$\Rightarrow \quad P_X(0) = \frac{1}{6 \cdot 2^{\min(0,0)}} + \frac{1}{6 \cdot 2^{\min(0,1)}}$$

$$= \frac{1}{3}$$

$$\Rightarrow \quad P_X(x) = \begin{cases} \dfrac{1}{3 \cdot 2^{x-1}} & x \geq 1 \\ \dfrac{1}{3} & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since the domain and function is symmetric in x, y

$$P_Y(y) = \begin{cases} \dfrac{1}{3 \cdot 2^{y-1}} & y \geq 1 \\ \dfrac{1}{3} & y = 0 \\ 0 & \text{otherwise} \end{cases}$$

3. $P(X=Y \mid X<2)$

$$= P(X=Y \mid X=0 \cup X=1)$$

$$= \frac{P(X=Y \cap (X=0 \cup X=1))}{P(X=0 \cup X=1)}$$

$$= \frac{P_{XY}(0,0) + P_{XY}(1,1)}{P_X(0) + P_X(1)}$$

$$= \frac{\frac{1}{6} + \frac{1}{12}}{\frac{1}{3} + \frac{1}{3}} = \frac{\frac{1}{6} \cdot \frac{3}{2}}{\frac{2}{3}} = \frac{1}{6} \times \frac{9}{4} = \boxed{\frac{3}{8}}$$

4. $P(1 \leq X^2 + Y^2 \leq 5)$

$T = X, Y \in \{(0,1), (2,1), (1,0), (1,1), (1,2)\}$

$\min T = (0, 1, 0, 1, 1)$

$$P(1 \leq X^2 + Y^2 \leq 5) = \frac{1}{6}\left(2 \cdot \frac{1}{2^0} + 3 \cdot \frac{1}{2^1}\right)$$

$$= \frac{1}{6}\left(2 + \frac{3}{2}\right)$$

$$= \frac{1}{6} \frac{7}{2} = \boxed{\frac{7}{12}}$$

5. $$P(X=Y) = \sum_{x=0}^{\infty} \frac{1}{6\cdot 2^{\min(x,x)}} = \sum_{x=0}^{\infty} \frac{1}{6\cdot 2^{x}}$$

$$= \frac{1}{6} \underbrace{\sum_{x=0}^{\infty} \frac{1}{2^{x}}}_{2}$$

$$= \boxed{\frac{1}{3}}$$

6. $E[X \mid Y=2]$

if $y=2$, $x \in \{1, 2, 3\}$

$$E[X \mid Y=2] = \sum_{x} x \cdot P(X=x \mid Y=2)$$

$$= \sum_{x} x \cdot \frac{P_{XY}(x,2)}{P_Y(2)}$$

$$= \sum_{x} x \cdot \frac{1}{6\cdot 2^{\min(x,2)}} \times \frac{3\cdot 2^{2-1}}{1}$$

$$= \sum_{x} \frac{x}{2^{\min(x,2)}}$$

$$= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^2}$$

$$= \frac{2+2+3}{4} = \boxed{\frac{7}{4}}$$

7. $Var(X \mid Y=2)$

$$E[X^2 \mid Y=2] = \sum_x x^2 \cdot P(X=x \mid Y=2)$$

$$= \sum_x x^2 \cdot \frac{P_{XY}(x,2)}{P_Y(2)}$$

$$= \sum_x x^2 \cdot \frac{1}{6 \cdot 2^{\min(x,2)}} \times \frac{3 \cdot 2^{2-1}}{1}$$

$$= \sum_x \frac{x^2}{2^{\min(x,2)}}$$

$$= \frac{1}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^2}$$

$$= \frac{2+4+9}{4} = \frac{15}{4}$$

$$Var(X \mid Y=2) = E[X^2 \mid Y=2] - E[X \mid Y=2]^2$$

$$= \frac{15}{4} - \left(\frac{7}{4}\right)^2$$

$$= \frac{15}{4} - \frac{49}{16}$$

$$= \frac{60-49}{16} = \boxed{\frac{11}{16}}$$

Q3 For a bivariate normal distribution with parameters $\mu_x, \sigma_x^2, \mu_y, \sigma_y^2, \rho$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}$$

putting $\rho = 0$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left\{\frac{-1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}$$

$$= \underbrace{\frac{1}{2\pi\sigma_x}\exp\left\{\frac{-1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right\}} \times \underbrace{\frac{1}{2\pi\sigma_y}\exp\left\{\frac{-1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}}$$

$$= f_X(x) \cdot f_Y(y)$$

Since $f_{XY}(x,y) = f_X(x) f_Y(y)$

X, Y are independent.

Q4 $\quad C_x = E[(X - E[x])(X - E[x])^T] = \mathrm{Cov}(X, X)$

Since $y = AX + b$

$E[y] = E[AX + b]$

Using linearity of expectation

$E[y] = A E[x] + b$

$$C_y = E[(y - E[y])(y - E[y])^T]$$
$$= E[(AX + b - (AE[x] + b))(AX + b - (AE[x] + b))^T]$$
$$= E[(AX - AE[x])(AX - AE[x])^T]$$
$$= E[A(X - E[x])(A(X - E[x]))^T]$$

From properties of transpose $(AB)^T = B^T A^T$

$$\Rightarrow E[A(X - E[x])(x - E[x])^T A^T]$$

Since A, A^T are scalars we can use linearity of expectation

$$\Rightarrow A\, E[(X - E[x])(x - E[x])^T]\, A^T$$
$$= A C_x A^T$$

Hence proved $\quad \boxed{C_y = A C_x A^T}$

Q5

$X \sim N(\mu_x, \sigma_x)$

$Y \sim N(\mu_y, \sigma_y)$

For a general x with mean m and covariance matrix C

$$f_X(x) = \frac{1}{(2\pi)^{n/2}\sqrt{\det C}} \exp\left[\frac{-1}{2}(x-\mu_x)^T C^{-1}(x-\mu_x)\right]$$

We need to show that this formula is same as

$$f_{XY}(x,y) = \overbrace{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}}^{a} \exp\left[\overbrace{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]}^{b}\right]$$

for the PDF of $\begin{bmatrix} X \\ Y \end{bmatrix}$

Both formulas are of form $ae^{-\frac{1}{2}b}$ hence we need to show a & b are same for both formulas.

$$m = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$$

$$C = \begin{bmatrix} \text{Var}(X) & \text{Cov}(X,Y) \\ \text{Cov}(Y,X) & \text{Var}(Y) \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$$

$$\det C = \sigma_x^2\sigma_y^2 - \rho^2\sigma_x^2\sigma_y^2 = \sigma_x^2\sigma_y^2(1-\rho^2)$$

$$\therefore \frac{1}{(2\pi)^{n/2}\sqrt{\det C}} = \overbrace{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}}^{a}$$

Hence the value of a is the same in the formula.

$$C^{-1} = \frac{\operatorname{adj}(C)}{\det C} = \frac{1}{\sigma_x^2\sigma_y^2(1-\rho)^2}\begin{bmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix}$$

Now by matrix multiplication

$$(x-m)^T C^{-1}(x-m)$$

$$= \frac{1}{\sigma_x^2\sigma_y^2(1-\rho^2)}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}^T \begin{bmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}$$

$$= \frac{1}{\sigma_x^2\sigma_y^2(1-\rho^2)}\begin{bmatrix} x-\mu_x & y-\mu_y \end{bmatrix}\Big[(x-\mu_x)^2\sigma_y^2 - \rho\sigma_x\sigma_y(x-\mu_x)(y-\mu_y) - \rho\sigma_x\sigma_y(x-\mu_x)(y-\mu_y) + \sigma_x^2(y-\mu_y)^2\Big]$$

$$= \frac{1}{(1-\rho)^2}\underbrace{\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]}_{b}$$

Hence the value of b in both formula is the same.

Since a, b are the Hence Proved that both the given formulas are the same i.e. $ae^{-\frac{1}{2}b}$

Q 6

Jensens Inequality

for convex functions $g(x)$

$$E[g(X)] \geq g(E[X])$$

Since $\ln\sqrt{x}$ is concave we take $-\ln\sqrt{x}$ as $g(x)$

$$g(x) = -\ln\sqrt{x}$$

$$g'(x) = -\frac{1}{\sqrt{x}} \cdot \frac{+1}{2\sqrt{x}} = \frac{-1}{2x}$$

$$g''(x) = \frac{1}{2x^2}$$

since $x > 0 \Rightarrow g''(x) > 0$

the function is convex

$E[g(x)] \geq g(E[X])$ when g is convex

$$\Rightarrow E[-\ln\sqrt{X}] \geq -\ln\sqrt{E[X]}$$

$= -E[\ln\sqrt{X}] \geq -\ln\sqrt{E[X]}$ (from linearity of $E[\cdot]$)

$$= E[\ln\sqrt{X}] \leq \ln\sqrt{E[X]}$$

$$\Rightarrow \boxed{E[\ln\sqrt{X}] \leq \ln\sqrt{10} \leq 1.513}$$

Q7 Generalized Union Bounds

The union bound inequality is

For any events $A_1, A_2, \ldots, A_n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Proof by induction

Base case: $n=1$

$$P(A_1) \leq P(A_1)$$

Induction

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

Using $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and the facts that the union operation is associative

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) - P\left(\bigcup_{i=1}^{k} A_i \cap A_{k+1}\right)$$

Since $P\left(\bigcup_{i=1}^{k} A_i \cap A_{k+1}\right) \geq 0$

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) \leq P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) = P\left(\bigcup_{i=1}^{k+1} A_i\right)$$

Hence proved by induction.

Now to gennalize this we use $\boxed{\text{inclusion-exclusion}}$ principle

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \cdots\cdots + (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

Stopping at the first term, the remaining terms add up to a nonpositive number, hence we obtain an upper bound

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Stopping at the second term, the remaining terms add up to a non negative number, hence we obtain an lower - bound

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

Stopping at the third term, the remaining terms add up to a nonpositive number, hence we obtain an upper bound

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

and this pattern continues.

This can be gennalized to odd terms giving an upperbound and even terms giving a lower bound.

Define

$$S_1 = \sum_{i=1}^{n} P(A_i)$$

$$S_2 = \sum_{1 \le i_1 \le i_2 \le n} P(A_{i_1} \cap A_{i_2})$$

$$\vdots$$

$$S_k = \sum_{1 \le i_1 \le i_2 \dots \le i_k \le n} P(A_{i_1} \cap A_{i_2} \cap \dots A_{i_k})$$

for $k = 3, 4, \dots, n$

For odd k

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{j=1}^{k} (-1)^{j-1} S_j$$

For even k

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{j=1}^{k} (-1)^{j-1} S_j$$

This is generalization of the union bound

GRAPH QUESTION

Considering isolated node not a general node

Let A_i be the event that the ith node is isolated

Hence B_n, $\bigcup_{i=1}^{n} A_i$

From the previous result

$$P(B_n) = P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

By Symmetry $\sum_{i=1}^{n} P(A_i) = nP(A_1)$

$$\sum_{i<j} P(A_i \cap A_j) = {}^nC_2\, P(A_1 \cap A_2)$$

Hence

$$P(B_n) \geq nP(A_1) - {}^nC_2\, P(A_1 \cap A_2)$$

The event A_1 occurs if node 1 is not connected to any of the other $(n-1)$ nodes

$\therefore$ Since the connections are independent

$P(A_1) = (1-p)^{n-1}$

The event $A_1 \cap A_2$ occurs when nodes 1 and 2 are isolated

There are $(n-2)$ potential edges between node 1 and the rest of graph (excluding node 2)

There are $(n-2)$ potential edges between node 2 and the rest of graph (excluding node 1)

There is one potential edge b/w node 1 and node 2.

$\therefore$ the potential edges $= 2(n-2)+1 = 2n-3$

These edges must be present in the graph

$\because$ connections are independent

$P(A_1 \cap A_2) = (1-p)^{2n-3}$

Hence substituting these expressions, we get

$P(B_n) \geqslant n\,P(A_1) - {}^nC_2\,P(A_1 \cap A_2)$

$\Rightarrow \quad P(B_n) \geqslant n(1-p)^{n-1} - {}^nC_2\,(1-p)^{2n-3}$

Hence Proved

Q8 Chebyshev inequality

$$P(|X - E[X]| \geq b) \leq \frac{Var(X)}{b^2}$$

$E[X] = np$, $Var(X) = np(1-p)$

$$P(X \geq \alpha n) = P(X - np \geq n\alpha - np)$$

$$\leq P(|X - np| \geq n\alpha - np)$$

$$\leq P(|X - E[X]| \geq \underbrace{n\alpha - np}_{b})$$

$$\leq \frac{Var(X)}{(n\alpha - np)^2} = \frac{np(1-p)}{n^2(\alpha - p)^2}$$ [Using chebyshev]

$$\leq \frac{p(1-p)}{n(\alpha - p)^2}$$

$$\therefore P(X \geq \alpha n) \leq \frac{p(1-p)}{n(\alpha - p)^2}$$

for $p = \frac{1}{2}$, $\alpha = \frac{3}{4}$

$$P\left(X \geq \frac{3n}{4}\right) \leq \frac{\frac{1}{2} \cdot \frac{1}{2}}{n\left(\frac{1}{2} - \frac{3}{4}\right)^2}$$

$$= \frac{1}{n\,4\,\frac{1}{16}} = \frac{4}{n}$$

$$P\left(X \geq \frac{3n}{4}\right) \leq \frac{4}{n}$$

Q9. In Q8 we got upper bound from Chebyshev as

$$= \boxed{\frac{p(1-p)}{n(\alpha-p)^2}}$$

$E[x] = np$, $Var(x) = np(1-p)$

Markov $\Rightarrow$ $P(x \geq a) \leq E[x]/a$

Applying this $\Rightarrow$ $P(X \geq \alpha n) \leq \dfrac{E[x]}{\alpha n}$

$$\leq \frac{np}{\alpha n}$$

$$P(X \geq \alpha n) \leq \boxed{\frac{p}{\alpha}}$$

Chernoff Bounds $\quad P(X \geq a) \leq \min\limits_{s>0} e^{-sa} M_x(s)$

Applying this $\quad P(X \geq \alpha n) \leq \min\limits_{s>0} e^{-s\alpha n} M_x(s)$

For Binomial, $M_x(s) = (pe^s + q)^n$ where $q = 1-p$

$$P(X \geq \alpha n) \leq \min_{s>0} \underbrace{e^{-s\alpha n}(pe^s+q)^n}_{f(s)}$$

To find minima fist we find the critical points

$$\Rightarrow \quad \frac{d f(s)}{ds} = 0$$

$$\Rightarrow \quad \frac{d\, e^{-s\alpha n}(pe^s+q)^n}{ds} = 0$$

$$\Rightarrow \quad e^{-s\alpha n}\left(n(pe^s+q)^{n-1} pe^s\right) + (pe^s+q)^n e^{-s\alpha n}(-\alpha n) = 0$$

$$\Rightarrow \quad \underbrace{e^{-s\alpha n}(pe^s+q)^{n-1}}_{\text{always } >0}\left[npe^s + (pe^s+q)(-\alpha n)\right] = 0$$

$$\Rightarrow \quad \underbrace{n}_{\text{always } >0}[pe^s - \alpha pe^s - \alpha q] = 0$$

$$\Rightarrow \quad e^s = \frac{\alpha q}{p - \alpha p} = \frac{\alpha q}{p(1-\alpha)}$$

$$s = \ln\left[\frac{\alpha q}{p(1-\alpha)}\right]$$

Double derivative test

$$f''(s) > 0$$

on diffrentiating $f(s)$ twice and substituting $s = \ln\left[\frac{\alpha q}{p(1-\alpha)}\right]$ indeed $\boxed{f''(s) > 0}$

$$f(s) = e^{-s\alpha n}(pe^{s}+q)$$

$$= e^{\ln\left[\frac{\alpha q}{p(1-\alpha)}\right]^{(-\alpha n)}}\left[\frac{\alpha(1-p)}{(1-\alpha)} + (1-p)\right]^{n}$$

$$= \left(\frac{\alpha(1-p)}{p(1-\alpha)}\right)^{-\alpha n}\left[\frac{\alpha(1-p)+(1-p)-\alpha(1-p)}{(1-\alpha)}\right]^{n}$$

$$= \left(\frac{p}{\alpha}\right)^{\alpha n}\left(\frac{1-p}{1-\alpha}\right)^{-\alpha n}\cdot\left(\frac{1-p}{1-\alpha}\right)^{n}$$

$$= \left(\frac{p}{\alpha}\right)^{\alpha n}\left(\frac{1-p}{1-\alpha}\right)^{(1-\alpha)n}$$

$$\Rightarrow \boxed{P(X \geq \alpha n) \leq \left(\frac{p}{\alpha}\right)^{\alpha n}\left(\frac{1-p}{1-\alpha}\right)^{(1-\alpha)n}}$$

upper bound from chernoff

Q10 Since for each individual the coin toss is independent

Let X_i be an indicator random variable which is 1 if the ith person gets a present else it is zero. From the perspective of the ith person the result of coin toss of $(i+1)$th and $(i-1)$th person is independent from his.

All possible combinations are →

$$
\begin{array}{l}
H\ H\ H \\
\left.\begin{array}{l}
H\ H\ T \\
H\ T\ H\ ✓ \\
H\ T\ T \\
T\ H\ H \\
T\ H\ T\ ✓ \\
T\ T\ H
\end{array}\right\}\ \text{favourable outcomes} \\
T\ T\ T
\end{array}
$$

$$P(X_i = 0) = \frac{6}{8} = \frac{3}{4}$$

$$P(X_i = 1) = 1 - \frac{3}{4} = \frac{1}{4}$$

$$E[X_i] = \frac{1}{4}$$

Let X be the total no of people recieving presents

$X = X_1 + X_2 + X_3 + \ldots + X_n$

$E[X] = E[X_1 + X_2 + \ldots + X_n]$

Using linearity of expectation

$E[X] = E[X_1] + E[X_2] + \ldots + E[X_n]$

$= \sum_{i=1}^{n} E[X_i]$

$= n\, E[X_i] = \boxed{\frac{n}{4}}$

$$Var(X_i) = \frac{3}{4}\left(0 - \frac{1}{4}\right)^2 + \frac{1}{4}\left(1 - \frac{1}{4}\right)^2 = \frac{3}{4} \times \frac{1}{16} + \frac{1}{4}\frac{9}{16}$$

$$= \frac{12}{64} = \frac{3}{16}$$

$$Var(X) = \sum_{i=1}^{N} Var(X_i) + \sum_{i=1}^{N} \sum_{j \neq i} Cov(X_i, X_j)$$

If X_i, X_j are more than two people apart then they are independent

if $2 < |i-j| < N-2$ then $Cov(X_i, X_j) = 0$ Since X_1 and X_2 are independent

From Symmetry

$Cov(X_1, X_2) = Cov(X_2, X_3) = Cov(X_3, X_4) = \dots = Cov(X_N, X_1)$

$Cov(X_1, X_3) = Cov(X_2, X_4) = Cov(X_{N-1}, X_1) = Cov(X_n, X_2)$

Thus we can write

$$Var(X) = N\,Var(X_1) + 2N\,Cov(X_1, X_2) + 2N\,Cov(X_1, X_3)$$

$$= \frac{3N}{16} + 2N\,Cov(X_1, X_2) + 2N\,Cov(X_1, X_3)$$

So we need to find $Cov(X_1, X_2)$ and $Cov(X_1, X_3)$

$$E[X_1 X_2] = P(X_1 = 1, X_2 = 1)$$

$$= P(H_N, T_1, H_2, T_3) + P(T_N, H_1, T_2, H_3)$$

$$= \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$

Thus $Cov(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2]$

$$= \frac{1}{8} - \frac{1}{16}$$

$$= \frac{1}{16}$$

$$E[X_1 X_3] = P(X_1 = 1, X_3 = 1)$$

$$= P(H_N, T_1, H_2, T_3, H_4) + P(T_N, H_1, T_2, H_3, T_4)$$

$$= \frac{1}{32} + \frac{1}{32} = \frac{1}{16}$$

Thus $\operatorname{cov}(X_1, X_3) = E[X_1 X_3] - E[X_1]E[X_3]$

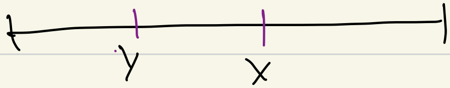
$$= \frac{1}{16} - \frac{1}{16} = 0$$

Therefore $\operatorname{Var}(X) = \frac{3N}{16} + 2N \operatorname{cov}(X_1, X_2) + 2N (X_1, X_3)$

$$= \frac{3N}{16} + \frac{2N}{16} + 0$$

$$= \frac{5N}{16}$$

Q11



$Y \leq X$

Let first break be at X and second at Y

X is uniformly distributed in $[0, l]$
and $Y|X$ is uniformly distributed in $[0, x]$

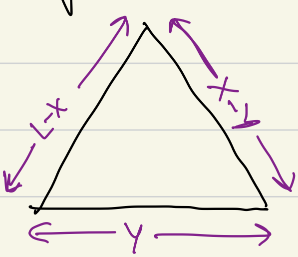
$$f_X(x) = \begin{cases} 1/l & 0 \leq x \leq l \\ 0 & \text{Otherwise} \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} 1/x & 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore f_{X,Y}(x,y) = f_{Y|X}(y,x)\, f_X(x)$$

$$\Rightarrow \begin{cases} \dfrac{1}{lx} & , 0 \leq y \leq x \leq l \\ 0 & , \text{otherwise} \end{cases}$$

To form a Δ sum of any 2 sides > 3rd side.



Using this property we get 3 conditions

1) $Y + X - Y > L - X$

$= 2X > L$

$= \boxed{X > \frac{L}{2}}$

2) $Y + L - X > X - Y$

$= 2Y - 2X > -L$

$= \boxed{Y > L - \frac{X}{2}}$

3) $L - X + X - Y > Y$

$= \boxed{Y < \frac{L}{2}} \Rightarrow \left\{ x > \frac{L}{2},\ y < \frac{L}{2},\ y > x - \frac{L}{2} \right\}$

Integrating the PDF with given constraints

$$\Rightarrow P(\text{valid}\,\Delta) = \int_{l/2}^{l} \int_{x - l/2}^{l/2} f_{XY}(x,y)\, dy\, dx$$

$$= \int_{l/2}^{l} \int_{x - l/2}^{l/2} \frac{dy}{lx}\, dx$$

$$= \int_{l/2}^{l} \frac{1}{lx}\, y \Big|_{x-\frac{l}{2}}^{l/2} \, dx$$

$$= \int_{l/2}^{l} \frac{1}{lx}\left[\frac{l}{2} - \left(x - \frac{l}{2}\right)\right] dx$$

$$= \frac{1}{l}\int_{l/2}^{l} \left[\frac{l - x}{x}\right] dx$$

$$= \frac{1}{l}\left[l \int_{l/2}^{l} \frac{1}{x}\, dx - \int_{l/2}^{l} 1\, dx \right]$$

$$= \frac{1}{l}\left[l \ln x \Big|_{l/2}^{l} - x \Big|_{l/2}^{l} \right]$$

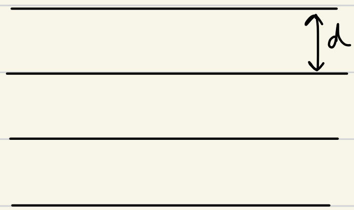
$$= \frac{1}{l}\left[l \ln\left(\frac{l}{l/2}\right) - \frac{l}{2} \right]$$

$$= \ln(2) - 1$$

$$= 0.1931$$

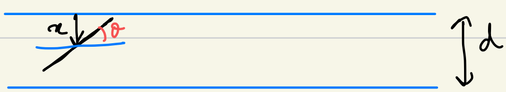
$$P(\text{valid } \Delta) = \boxed{0.1931}$$

Q12



Since we are throwing the needle randomly the center of needle and angle with the horizontal when it lands is uniformly distributed.

Let us say the center of needle is x distance away from the nearest line and at angle θ with the horizontal



if $\frac{l}{2}\sin\theta \geq x$ $\Rightarrow$ needle intersects

x can be maximum $\frac{d}{2}$ else it will be closer to other needle.

x varies uniformly from $\left(0, \frac{d}{2}\right)$ and θ varies uniformly from $\left(0, \frac{\pi}{2}\right)$

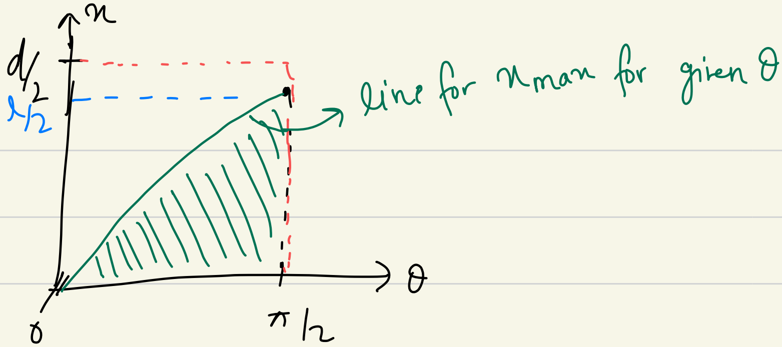
Note: θ from $\left(\frac{\pi}{2}, \pi\right)$ can be obtained by reflection if $\theta \in \left(0, \frac{\pi}{2}\right)$

For intersection:

$$\frac{l}{2}\sin\theta \geq x$$

$$x_{max} = \frac{l}{2}\sin\theta \quad (\text{for a given } \theta)$$

We can plot 2 uniform random variables on a 2D graph



if we look graphically the shaded region is the required area.

$$\text{Total area} = \frac{\pi}{2} \times \frac{d}{2} = \frac{\pi d}{4}$$

$$\text{Required} = \int_0^{\pi/2} \frac{l}{2}\sin\theta = \frac{l}{2}\int_0^{\pi/2} \sin\theta$$

$$= \frac{l}{2} \; -\cos\theta \Big|_0^{\pi/2}$$

$$= \frac{l}{2}$$

$$P(\text{Intersection}) = \frac{\text{Req. Area}}{\text{Total Area}} = \frac{l}{2} \times \frac{4}{\pi d} = \boxed{\frac{2l}{\pi d}}$$