Geometric Modeling

Notes based on a course given by Boris Thibert for M1 Applied Mathematics, Université Grenoble Alpes.

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Contents

Chapter 1: Interpolation

Lagrange interpolation

Theorem 1.1

Let $a = t_0 < \ldots < t_n = b, y_0, \ldots, y_n \in \mathbb{R}$

There exists a unique polynomial L_n of degree $\leq n$ such that:

$$\forall i \in \{0, n\}, L_n(t_i) = y_i \quad [1]$$

These polynomials are called Lagrange interpolation polynomials A simple expression is :

$$L_n(t) = \sum_{i=0}^{n} y_i P_i(t), \quad \text{where } P_i(t) = \prod_{i \neq j} \frac{t - t_j}{t_i - t_j} \quad [2]$$

Proof: We denote $E = \mathbb{R}_n[X]$ the space of polynomials of degree $\leq n$.

- E is a $\mathbb{R}$-vector space
- Let

$$\begin{aligned} \varphi : E &\longrightarrow \mathbb{R}^{n+1} \\ P &\longmapsto (P(t_0), \ldots, P(t_n)) \end{aligned} \quad [3]$$

φ is a linear map between vector spaces of some dimension. To prove the result, we need φ to be one-to-one map.

Let $P \in \ker \varphi$

Then :

- P is of degree $\leq n$
- P vanishes at n+1 points

then by the fundamental theorem of algrebra, $P \equiv 0$

Runge Phenomenon

We observe in practice the Runge Phenomenon Let

$$\begin{aligned} f : [-1, 1] &\to \mathbb{R} \\ x &\mapsto \frac{1}{1 + 25x^2} \end{aligned} \qquad [4]$$

ti uniformly distributed in $[-1, 1]$

We observe $\|L_n - f\|_\infty \underset{n \to +\infty}{\longrightarrow} +\infty$

We have an approximation result :

Theorem 1.2

:

Let $f : [a, b] \to \mathbb{R} \in \mathcal{C}^{n+1}$

L_n the Lagrange polynoomial associated to $a = t_0 < \ldots < t_n = b$

Then $\|f - L_n\| \leq \frac{1}{(n+1)!} \|q_{n+1}\|_\infty \left\| f^{n+1} \right\|_\infty$

when $q_{n+1}(t) = \Pi_{i=0}^{n} (t - t_i)$

Runge phenomenon : $\left\| f^{n+1} \right\|_\infty$ goes to infinity quickly

Proof:

We introduce the error $g = f - L_n$

Let $t \in [0, b]\{t_i\}$

$$\begin{aligned} k(u) &:= g(u) - q_{n+1}(u) \frac{g(t)}{q_{n+1}(t)} \\ k(t_i) &= 0 \text{ for } i = 0, \ldots, n \\ k(t) &= 0 \end{aligned} \qquad [5]$$

$k(t)$ vanishes at $(n + 2)$ points

so $k'(t)$ vanishes at $(n + 1)$ points (by Rolle)

so k^{n+1} vanishes at 1 point ξ

=> $0 = k^{n+1}(\xi) = g^{n+1}(\xi) - q_{n+1}^{n+1}(\xi) \frac{g(t)}{q_{n+1}(t)}$

$g^{n+1} = f(n + 1)(\xi)$ and $q^{n+1} = (n + 1)!$

=> $|g(t)| = \frac{1}{(n+1)!} q_{n+1}(t) f^{n+1}(\xi) \leq \|\ldots\|_\infty$

Interpolation with splines

Theorem 1.3

Let $y_0, \ldots, y_n \in \mathbb{R}$

$a = t_0, \ldots, t_n = b$ and $\alpha, \beta \in \mathbb{R}$

Then there exists a unique function S

such that

- $S_{|[t_i, t_{i+1}]}$ is a polynomial of degree ≤ 3
- S is $\mathcal{C}^2$
- $S(t_i) = y_i$
- $S'(a) = \alpha$ and $S'(b) = \beta$

order of continuity is one less than the order of polynomial (see order 0 case)

This function S is called cubic splines

Minimization result

Theorem 1.4

Let $y_0, \ldots, y_n \in \mathbb{R}$ ...

Then the associated spline S satisfies :

$$S = \arg\min \int_a^b f''(t)^2 dt \qquad [6]$$

So splines are the solution to the minimization problem of reducing the energy (L^2 norm) of the second derivative.

Skecth of proof : Let $g \in E$ and $e = f - S$ the error with S spline

Step 1 :

Show

$\forall h : [a, b] \to \mathcal{C}^0$, linear on $[t_i, t_i + 1]$,

$$\int_a^b e''(x)h(x)dx = 0 \qquad [7]$$

Step 2:

$$\begin{aligned} \int_a^b f''^2 &= \int (e + S)^2 \\ &= \int e''^2 + 2\int e''S'' + \int S''^2 \\ &= \int e''^2 + \int S''^2 \end{aligned} \qquad [8]$$

$$\Rightarrow \forall f \in E \int f''^2 \geq \int S''^2$$

We need to show uniqueness of the minimizer

If $\int f''^2 = \int S''^2$ then $\int e''^2 = 0$

Chapter 2: Differential geometry

Representation of curves

- Drawings
- Parametrized curves :

$$\gamma : [a, b] \to \mathbb{R}^d \tag{9}$$

- Implicit representation:

$$\mathcal{C} = \{(x, y), x^2 + y^2 = R^2\} = f^{-1}(0) \text{ where } f : t \to x^2 + y^2 - R^2 \tag{10}$$

Generalities on parametrized curves

Reminder :

Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}^2$ of class $\mathcal{C}^n$, and $t_0 \in]a, b[$. Performing a Taylor expansion around t_0 yields :

$$f(t) = f(t_0) + (t - t_0)f'(t_0) + \frac{(t - t_0)^2}{2!}f''(t_0) + \dots + \frac{(t - t_0)^n}{n!}f^n(t_0) + o((t - t_0)^n) \tag{11}$$

Geometrically, we have $\overrightarrow{f(t_0)f(t)} = f(t) - f(t_0) = ((t - t_0))f'(t_0)$. Thus, when it does not vanish, $f'(t_0)$ is tangent to $\mathcal{C} = f([a, b])$ at $f(t_0)$.

Now, let p be the smallest $k \geq 1$ such that $f^k(t_0) \neq 0$ and q the smallest $k > p$ such that $(f^p(t_0), f^q(t_0))$ are linearly independant.

Then, in the reference frame centered around $f(t_0)$ and with basis vectors $(f^p(t_0), f^q(t_0))$, we have

$$f(t) \approx \begin{pmatrix} \lambda(t) \\ \mu(t) \end{pmatrix} \text{ where } \lambda(t) = \frac{(t - t_0)^p}{p!} \text{ and } \mu(t) = \frac{(t - t_0)^q}{q!}. \tag{12}$$

TODO : Add figure

Remark : In practice, $\lambda(t) >> \mu(t)$, thus the influence of f^q is vastly exaggerated in the figure.

Metric properties of curves

Length of a rectifiable curve

In this section $\|.\|$ is implicitely assumed to be the euclidean norm $\|.\|_2$.

Let's first define the length of a curve in the most geometrical way possible.

Length of a rectifiable curve Definition 2.1

Let $\gamma : [a, b] \to \mathbb{R}^d$ be $\mathcal{C}^k$ with $k \geq 1$, and $\mathcal{S}$ be the set of uniform subdivisions of $[a, b]$. For $s = (t_0, t_1, \dots, t_n) \in \mathcal{S}$, we define

$$l(s) = \sum_{i=0}^{n-1} \|\gamma(t_{i+1}) - \gamma(t_i)\|. \tag{13}$$

If $\{l(s), s \in \mathcal{S}\}$ is bounded, we say γ is **rectifiable** and define the length $L(\gamma)$ of γ as

$$L(\gamma) = \sup_{s \in \mathcal{S}} l(s) \tag{14}$$

This leads to the natural integral caracterisation that follows.

Integral expression of the length of a rectifiable curve Theorem 2.1

Let $\gamma : [a, b] \to \mathbb{R}^d$ be a rectifiable curve. We have :

$$L(\gamma) = \int_a^b \|\gamma'(t)\| dt \tag{15}$$

Arc-length

Arc-length of a rectifiable curve — Definition 2.2

Let

Notion of regular curve

Regular Curve — Definition 2.3

A curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^d$ of class $\mathcal{C}^k$ is said to be regular at t if $\gamma'(t) \neq 0$.

Remarks:
- A $\mathcal{C}^1$ regular curve admits a tangent at every point
- We sometimes call the set $\tilde{C} = \gamma(I) \subset \mathbb{R}^d$ the "geometric curve" or the support of the curve.
- For $\mathcal{C}^1$ regular curves, I is connected/compact $\Rightarrow \tilde{C}$ is connected/compact

A non regular smooth curve — Example 2.1

Consider the following $\mathcal{C}^0$ curve :

$$t \to \begin{cases} \left(t, t^{\frac{3}{2}}\right) \text{ if } t \geq 0 \\ \left(|t| - |t|^{\frac{3}{2}}\right) \text{ else} \end{cases} \qquad [16]$$

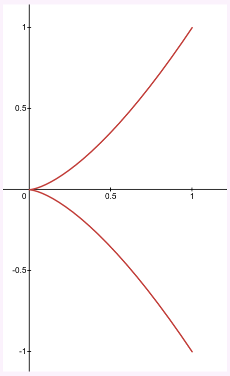

Figure 1: The geometric curve

This geometric curve can be represented by the map $g : t \to \left(t^2, t^3\right)$ which is $\mathcal{C}^\infty$. The curve is not regular in 0 because $g'(0) = 0$.

Reparametrization — Proposition 2.1

Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^d$ and $\varphi : I \to J$ a $\mathcal{C}^k$ diffeomorphism (i.e φ bijective, φ of class $\mathcal{C}^k$, $\varphi'(t) \neq 0$, thus φ^{-1} is of class $\mathcal{C}^k$).

Then $\gamma \circ \varphi^{-1} : J \to \mathbb{R}^d$ is called an admissible reparametrization of γ and φ or φ^{-1} is called an admissible change of variable.

Remark: $\varphi(t) = t^3$ is not an admissible change of variable, because $\varphi'(0) = 0$, even tho φ is bijective and $\mathcal{C}^k$.

We sometimes also see the following alternative definition of geometric curve, which states that a geometric curve does not depend on the parametrization.

Geometric Curve Definition 2.4

A "geometric curve" is an equivalence class for the relation

$$\gamma : I \to \mathbb{R}^d \sim g : J \to \mathbb{R}^d \text{ if } \exists \varphi : J \to I \text{ a } \mathcal{C}^k\text{-diffeomorphism s.t } \gamma = g \circ \varphi^{-1}. \qquad [17]$$

Proposition 2.2

If $\gamma : I \to \mathbb{R}^d$ is $\mathcal{C}^k$ regular and $\varphi : J \to I$ is an admissible change of variable, then $\gamma \circ \varphi$ is $\mathcal{C}^k$ regular.

Proof: $(\gamma \circ \varphi)'(t) = \gamma'(\varphi(t)) \cdot \varphi'(t) \neq 0$

Remark : Non-admissible reparametrization exist and can change the regularity of the curve, but not the admissible ones.

Plane curves

We will now study the curvature of plane and space curves separately, starting with plane curves.

Serret-Fresnet frame Definition 2.5

Let $\gamma : I \to \mathbb{R}^2$ a $\mathcal{C}^1$ regular curve. We have

$$T(t) = \gamma'(t) \qquad [18]$$