Geometric Modeling

Notes based on a course given by Boris Thibert for M1 Applied Mathematics, Université Grenoble Alpes.

Malik Hacini

Last updated: **October 16, 2025**

Contents

Chapter 1: Interpolation

Lagrange interpolation

Theorem 1.1

Let $a = t_0 < \ldots < t_n = b, y_0, \ldots, y_n \in \mathbb{R}$

There exists a unique polynomial L_n of degree $\leq n$ such that:

$$\forall i \in \{0, n\}, L_n(t_i) = y_i \qquad [1]$$

These polynomials are called Lagrange interpolation polynomials A simple expression is :

$$L_n(t) = \sum_{i=0}^{n} y_i P_i(t), \qquad \text{where } P_i(t) = \prod_{i \neq j} \frac{t - t_j}{t_i - t_j} \qquad [2]$$

Proof: We denote $E = \mathbb{R}_n[X]$ the space of polynomials of degree $\leq n$.

- E is a $\mathbb{R}$-vector space

- Let

$$\begin{aligned} \varphi : E &\longrightarrow \mathbb{R}^{n+1} \\ P &\longmapsto (P(t_0), \ldots, P(t_n)) \end{aligned} \quad [3]$$

φ is a linear map between vector spaces of some dimension. To prove the result, we need φ to be one-to-one map.

Let $P \in \ker \varphi$

Then :

- P is of degree $\leq n$
- P vanishes at n+1 points

then by the fundamental theorem of algrebra, $P \equiv 0$

Runge Phenomenon

We observe in practice the Runge Phenomenon Let

$$\begin{aligned} f : [-1, 1] &\to \mathbb{R} \\ x &\mapsto \frac{1}{1 + 25x^2} \end{aligned} \quad [4]$$

ti uniformly distributed in $[-1, 1]$

We observe $\|L_n - f\|_\infty \underset{n \to +\infty}{\longrightarrow} +\infty$

We have an approximation result :

Theorem 1.2

:

Let $f : [a, b] \to \mathbb{R} \in \mathcal{C}^{n+1}$

L_n the Lagrange polynoomial associated to $a = t_0 < \ldots < t_n = b$

Then $\|f - L_n\| \leq \frac{1}{(n+1)!} \|q_{n+1}\|_\infty \left\|f^{n+1}\right\|_\infty$

when $q_{n+1}(t) = \Pi_{i=0}^n (t - t_i)$

Runge phenomenon : $\left\|f^{n+1}\right\|_\infty$ goes to infinity quickly

Proof:

We introduce the error $g = f - L_n$

Let $t \in [0, b]\{t_i\}$

$$\begin{aligned} k(u) &:= g(u) - q_{n+1}(u) \frac{g(t)}{q_{n+1}(t)} \\ k(t_i) &= 0 \text{ for } i = 0, \ldots, n \\ k(t) &= 0 \end{aligned} \quad [5]$$

$k(t)$ vanishes at $(n + 2)$ points

so $k'(t)$ vanishes at $(n + 1)$ points (by Rolle)

so k^{n+1} vanishes at 1 point ξ

=> $0 = k^{n+1}(\xi) = g^{n+1}(\xi) - q_{n+1}^{n+1}(\xi) \frac{g(t)}{q_{n+1}(t)}$

$g^{n+1} = f(n + 1)(\xi)$ and $q^{n+1} = (n + 1)!$

=> $|g(t)| = \frac{1}{(n+1)!} q_{n+1}(t) f^{n+1}(\xi) \leq \|\ldots\|_\infty$

Interpolation with splines

Theorem 1.3

Let $y_0, ..., y_n \in \mathbb{R}$

$a = t_0, ..., t_n = b$ and $\alpha, \beta \in \mathbb{R}$

Then there exists a unique function S

such that

- $S_{|[t_i, t_{i+1}]}$ is a polynomial of degree ≤ 3
- S is $\mathcal{C}^2$
- $S(t_i) = y_i$
- $S'(a) = \alpha$ and $S'(b) = \beta$

order of continuity is one less than the order of polynomial (see order 0 case)

This function S is called cubic splines

Minimization result

Theorem 1.4

Let $y_0, ..., y_n \in \mathbb{R}$ ...

Then the associated spline S satisfies :

$$S = \arg\min \int_a^b f''(t)^2 dt \quad [6]$$

So splines are the solution to the minimization problem of reducing the energy (L^2 norm) of the second derivative.

Skecth of proof : Let $g \in E$ and $e = f - S$ the error with S spline

Step 1 :

Show

$\forall h : [a, b] \to \mathcal{C}^0$, linear on $[t_i, t_i + 1]$,

$$\int_a^b e''(x) h(x) dx = 0 \quad [7]$$

Step 2:

$$\begin{aligned} \int_a^b f''^2 &= \int (e + S)^2 \\ &= \int e''^2 + 2 \int e'' S'' + \int S''^2 \\ &= \int e''^2 + \int S''^2 \end{aligned} \quad [8]$$

$$\Rightarrow \forall f \in E \int f''^2 \geq \int S''^2$$

We need to show uniqueness of the minimizer

If $\int f''^2 = \int S''^2$ then $\int e''^2 = 0$

Chapter 2: Differential geometry of curves

Representation of curves

- Drawings
- Parametrized curves :

$$\gamma : [a, b] \to \mathbb{R}^d \tag{9}$$

- Implicit representation:

$$\mathcal{C} = \left\{(x, y), x^2 + y^2 = R^2\right\} = f^{-1}(0) \text{ where } f : t \to x^2 + y^2 - R^2 \tag{10}$$

Generalities on parametrized curves

Reminder :

Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}^2$ of class $\mathcal{C}^n$, and $t_0 \in]a, b[$. Performing a Taylor expansion around t_0 yields :

$$f(t) = f(t_0) + (t - t_0)f'(t_0) + \frac{(t - t_0)^2}{2!} f''(t_0) + \ldots + \frac{(t - t_0)^n}{n!} f^n(t_0) + o\left((t - t_0)^n\right) \tag{11}$$

Geometrically, we have $\overrightarrow{f(t_0)f(t)} = f(t) - f(t_0) = ((t - t_0))f'(t_0)$. Thus, when it does not vanish, $f'(t_0)$ is tangent to $\mathcal{C} = f([a, b])$ at $f(t_0)$.

Now, let p be the smallest $k \geq 1$ such that $f^k(t_0) \neq 0$ and q the smallest $k > p$ such that $(f^p(t_0), f^q(t_0))$ are linearly independant.

Then, in the reference frame centered around $f(t_0)$ and with basis vectors $(f^p(t_0), f^q(t_0))$, we have

$$f(t) \approx \begin{pmatrix} \lambda(t) \\ \mu(t) \end{pmatrix} \text{ where } \lambda(t) = \frac{(t - t_0)^p}{p!} \text{ and } \mu(t) = \frac{(t - t_0)^q}{q!}. \tag{12}$$

TODO : Add figure

Remark : In practice, $\lambda(t) >> \mu(t)$, thus the influence of f^q is vastly exaggerated in the figure.

Metric properties of curves

Length of a rectifiable curve

In this section $\|.\|$ is implicitely assumed to be the euclidean norm $\|.\|_2$.

Let's first define the length of a curve in the most geometrical way possible.

Length of a rectifiable curve Definition 2.1

Let $\gamma : [a, b] \to \mathbb{R}^d$ be $\mathcal{C}^k$ with $k \geq 1$, and $\mathcal{S}$ be the set of uniform subdivisions of $[a, b]$. For $s = (t_0, t_1, \ldots, t_n) \in \mathcal{S}$, we define

$$l(s) = \sum_{i=0}^{n-1} \|\gamma(t_{i+1}) - \gamma(t_i)\|. \tag{13}$$

If $\{l(s), s \in \mathcal{S}\}$ is bounded, we say γ is **rectifiable** and define the length $L(\gamma)$ of γ as

$$L(\gamma) = \sup_{s \in \mathcal{S}} l(s) \tag{14}$$

This leads to the natural integral caracterisation that follows.

Integral expression of the length of a rectifiable curve Theorem 2.1

Let $\gamma : [a, b] \to \mathbb{R}^d$ be a rectifiable curve. We have :

$$L(\gamma) = \int_a^b \|\gamma'(t)\| dt \tag{15}$$

Arc-length

Arc-length of a rectifiable curve — Definition 2.2

Let

Notion of regular curve

Regular Curve — Definition 2.3

A curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^d$ of class $\mathcal{C}^k$ is said to be regular at t if $\gamma'(t) \neq 0$.

Remarks:
- A $\mathcal{C}^1$ regular curve admits a tangent at every point
- We sometimes call the set $\tilde{C} = \gamma(I) \subset \mathbb{R}^d$ the "geometric curve" or the support of the curve.
- For $\mathcal{C}^1$ regular curves, I is connected/compact $\Rightarrow \tilde{C}$ is connected/compact

A non regular smooth curve — Example 2.1

Consider the following $\mathcal{C}^0$ curve :

$$t \to \begin{cases} \left(t, t^{\frac{3}{2}}\right) \text{ if } t \geq 0 \\ \left(|t| - |t|^{\frac{3}{2}}\right) \text{ else} \end{cases} \qquad [16]$$

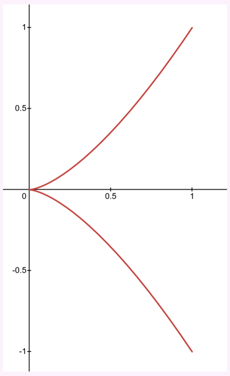

Figure 1: The geometric curve

This geometric curve can be represented by the map $g : t \to (t^2, t^3)$ which is $\mathcal{C}^\infty$. The curve is not regular in 0 because $g'(0) = 0$.

Reparametrization — Proposition 2.1

Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^d$ and $\varphi : I \to J$ a $\mathcal{C}^k$ diffeomorphism (i.e φ bijective, φ of class $\mathcal{C}^k$, $\varphi'(t) \neq 0$, thus φ^{-1} is of class $\mathcal{C}^k$).

Then $\gamma \circ \varphi^{-1} : J \to \mathbb{R}^d$ is called an admissible reparametrization of γ and φ or φ^{-1} is called an admissible change of variable.

Remark: $\varphi(t) = t^3$ is not an admissible change of variable, because $\varphi'(0) = 0$, even tho φ is bijective and $\mathcal{C}^k$.

We sometimes also see the following alternative definition of geometric curve, which states that a geometric curve does not depend on the parametrization.

Geometric Curve — Definition 2.4

A "geometric curve" is an equivalence class for the relation

$$\gamma : I \to \mathbb{R}^d \sim g : J \to \mathbb{R}^d \text{ if } \exists \varphi : J \to I \text{ a } \mathcal{C}^k\text{-diffeomorphism s.t } \gamma = g \circ \varphi^{-1}. \quad [17]$$

Proposition 2.2

If $\gamma : I \to \mathbb{R}^d$ is $\mathcal{C}^k$ regular and $\varphi : J \to I$ is an admissible change of variable, then $\gamma \circ \varphi$ is $\mathcal{C}^k$ regular.

Proof: $(\gamma \circ \varphi)'(t) = \gamma'(\varphi(t)) \cdot \varphi'(t) \neq 0$

Remark : Non-admissible reparametrization exist and can change the regularity of the curve, but not the admissible ones.

Plane curves

We will now study the curvature of plane and space curves separately, starting with plane curves.

Serret-Fresnet frame — Definition 2.5

Let $\gamma : I \to \mathbb{R}^2$ a $\mathcal{C}^1$ regular curve. The frame $(f(s), T(s), N(s))$ is called the Serret-Fresnet frame at t.

Remark : When the curve is parametrized with arc-length, $T(s) = \gamma'(s)$, thus $N(s) = \mathrm{Rot}_{\frac{\pi}{2}}(\gamma'(s))$.

Curvature

Curvature for plane curves — Definition 2.6

Let $\gamma : I \to \mathbb{R}^2$ a $\mathcal{C}^2$ regular curve parametrized by arc length. The curvature of γ at $\gamma(s)$ is defined by

$$\kappa_s(s) = \langle \gamma''(s), N(s) \rangle = \pm \|\gamma''(s)\| \quad [18]$$

Remark :

- We define curvature to be positive when $\gamma''(s) = N(s)$ and negative when $\gamma''(s) = -N(s)$. This can be interpreted by convexity $\Leftrightarrow$ positive curvature for graphs of $\mathbb{R} \to \mathbb{R}$ functions.
- For any non arc-length parametrization, $\gamma''(t)$ is not (in general) orthogonal to $\gamma'(t)$ and the above formula doesn't hold. It still holds if $\|\gamma'(t)\|$ is constant as $T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$. We can however derive a general formula for any parametrization.

Planar curvature for general parametrizations — Proposition 2.3

Let $\gamma : I \to \mathbb{R}^2$ a $\mathcal{C}^2$ regular curve. We have

$$\kappa_t(t) = \frac{\det(\gamma'(t), \gamma''(t))}{\|\gamma'(t)\|^3} \quad [19]$$

Proof: Let $\tilde{\gamma} = \gamma \circ \varphi^{-1}$ be the arc-length reparametrization of γ. We denote $t = \varphi^{-1}(s)$. Then,

$$\tilde{\gamma}'(s) = \left(\gamma \circ \varphi^{-1}\right)'(s) = \gamma'\left(\varphi^{-1}(s)\right) \cdot \left(\varphi^{-1}\right)'(s) = \frac{\gamma'(t)}{\|\gamma'(t)\|}. \quad [20]$$

Hence

$$\begin{aligned} \tilde{\gamma}''(s) &= \frac{\gamma''(t)}{\|\gamma'(t)\|^2} + \gamma'(t)\frac{d}{dt}\left(\frac{1}{\|\gamma'(t)\|}\right) \\ \Rightarrow \det(\tilde{\gamma}'(s), \tilde{\gamma}''(s)) &= \frac{\det(\gamma'(t), \gamma''(t))}{\|\gamma'(s)\|^3}. \end{aligned} \quad [21]$$

Indeed, $\gamma'(t)\frac{d}{dt}\left(\frac{1}{\|\gamma'(t)\|}\right)$ is colinear to $\gamma'(t)$ and thus vanishes in the det. Furthermore, since $\kappa(s) = \langle\tilde{\gamma}''(s), N(s)\rangle$ and $\tilde{\gamma}''(s)$ is colinear to $N(s)$, we have $\det(\tilde{\gamma}'(s), \tilde{\gamma}''(s)) = \det(T(s), \kappa_s(s)N(s))$. Thus, since $(T(s), N(s))$ is a direct basis, $\det(T(s), N(s)) = 1$, hence $\det(T(s), k(s)N(s)) = \kappa_s(s)$. Finally,

$$\kappa_s(s) = \kappa_t(t) = \det(\tilde{\gamma}'(s), \tilde{\gamma}''(s)) = \frac{\det(\gamma'(t), \gamma''(t))}{\|\gamma'(s)\|^3}. \qquad [22]$$

Serret-Fresnet formula — Proposition 2.4

Let $\gamma : I \to \mathbb{R}^2$ a $\mathcal{C}^2$ regular curve parametrized by arc length. We have

$$\begin{cases} T'(s) = \kappa(s)N(s) \\ N'(s) = -\kappa(s)T(s) \end{cases} \qquad [23]$$

Proof:

- TODO
- We have $\langle N(s), T(s)\rangle = 0$. Differentating this equation yields

$$\langle N'(s), T(s)\rangle + \underbrace{(\langle N(s), T'(s)\rangle)}_{\kappa(s)} = 0. \qquad [24]$$

  Moreover,

$$\langle N(s), N(s)\rangle = 1 \Rightarrow \langle N'(s), N(s)\rangle = 0. \qquad [25]$$

  Thus, in the direct basis $(T(s), N(s))$, we have $N'(s) = \begin{pmatrix} -\kappa(s) \\ 0 \end{pmatrix}$, i.e $N'(s) = -\kappa(s)T(s)$.

Osculating circle and center of curvature

Center and radius of curvature — Definition 2.7

Let $\gamma : I \to \mathbb{R}^2$ a $\mathcal{C}^2$ regular curve. For all t where $\kappa(t) \neq 0$, we define the center of curvature $c(t)$ as

$$c(t) = f(t) + \frac{1}{\kappa(t)}N(t). \qquad [26]$$

The quantity $R(t) = \frac{1}{\|\kappa(t)\|}$ is called the radius of curvature.

Remark :

- The circle of center $c(t)$ and radius $R(t)$ is called the osculating circle to γ at $\gamma(t)$.
- The osculating circle is placed towards the (local) interior of the curve : this justifies the sign convention in the curvature definition.

Osculating circle approximation — Proposition 2.5

The osculating circle to γ at $\gamma(t)$ is tangent to $\mathcal{C}(\gamma)$ at $\gamma(t)$ and approaches γ at order 2 around it.

Proof: Admitted.

Total curvature

Total curvature — Proposition 2.6

Let $\gamma : [a, b] \to \mathbb{R}^2$ a $\mathcal{C}^2$-regular curve parametrized by arc-length. We have

$$\int_a^b k(s)ds = \theta(b, a)(:= \text{total angle variation of } \gamma) \qquad [27]$$

This quantity is called total curvature.

Proof:

We denote $f(s) = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$ and $\theta(s) = (O_x, T(s))$ the angle between the tangent vector to f and the x-axis. We define θ to be differentiable thus continuous, thus defined up to $2\pi k$. Then,

$$T(s) = \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix} \quad \text{and} \quad N(s) = \begin{pmatrix} -\sin\theta(s) \\ \cos\theta(s) \end{pmatrix}. \tag{28}$$

Hence,

$$T'(s) = \theta'(s)\begin{pmatrix} -\sin\theta(s) \\ \cos\theta(s) \end{pmatrix} \quad \text{and} \quad T'(s) = k(s)N(s). \tag{29}$$

This yields:

$$\theta'(s) = k(s) \tag{30}$$

And the conclusion trivially follows.

Space curves

In this section, we let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be a $\mathcal{C}^k$ regular space curve (k will be specified).

Principal normal vector and curvature

Space curvature — Definition 2.8

If γ, $\mathcal{C}^2$ regular, is parametrized by arc-length, the curvature $\varkappa$ is defined by:

$$\varkappa(s) = \|\gamma''(s)\|. \tag{31}$$

Remark : Space curvature is unsigned, due to the fact there is no unique normal vector to the curve, but a plane.

Space curvature for general parametrizations — Proposition 2.7

For any regular parametrization of γ, we have

$$\varkappa(t) = \frac{\|\gamma'(t) \wedge \gamma''(t)\|}{\|\gamma'(t)\|^3}. \tag{32}$$

Proof: We denote $\tilde{\gamma} = \gamma \circ \varphi^{-1}$ the arc-length parametrization of γ, with $t = \varphi^{-1}(s)$. We have

$$\begin{aligned} \langle \tilde{\gamma}'(s), \tilde{\gamma}'(s) \rangle = 1 &\implies 2\langle \tilde{\gamma}''(s), \tilde{\gamma}'(s) \rangle = 0 \\ &\implies \tilde{\gamma}''(s) \perp \tilde{\gamma}'(s) \quad \text{and} \quad \|\tilde{\gamma}'(s) = 1\| \\ &\implies \varkappa(s) = \|\tilde{\gamma}''(s)\| = \|\tilde{\gamma}'''(s) \wedge \tilde{\gamma}'(s)\|. \end{aligned} \tag{33}$$

We already derived in 2D (same proof works) that

$$\tilde{\gamma}''(s) = \frac{\gamma''(t)}{\|\gamma''(t)\|} + \gamma'(t) \cdot \frac{d}{ds}\left(\frac{1}{\|\gamma'(t)\|}\right). \tag{34}$$

Hence,

$$\varkappa_s(s) = \left\| \frac{\gamma''(t)}{\|\gamma'(t)\|^2} + \gamma'(t) \cdot \lambda(t) \wedge \frac{\gamma'(t)}{\|\gamma'(t)\|} \right\| = \frac{\|\gamma''(t) \wedge \gamma'(t)\|}{\|\gamma'(t)\|^3} := k_t(t). \tag{35}$$

Principal normal vector — Definition 2.9

If γ is $\mathcal{C}^2$ regular and parametrized by arc-length, when $\varkappa(s) \neq 0$, i.e $\gamma''(s) \neq 0$, we define the principal normal as

$$N(s) = \frac{\gamma''(s)}{\|\gamma''(s)\|} = \left(\frac{1}{\varkappa(s)} \frac{d}{ds} T(s)\right) \tag{36}$$

Where:

- $T(s) = \gamma'(s)$ is a tangent vector
- γ is said to be biregular if $\forall s, \gamma''(s) \neq 0$.

Remark :

- The term **principal** comes from the fact that when you perform a Taylor expansion of γ around s, the curve lies (at order 2) in the $(T(s), N(s))$ plane, since $N(s)$ is defined through $\gamma''(s) \perp T(s) = \gamma'(s)$.

- We can also express the principal normal using a non-arc length parametrization.
- The principal normal and the planar normal vector are the same up to the sign. Thus, the principal normal is not continuous in general. Indeed, it can change orientation instantly on inflexion points.

Osculating plane/sphere, radius and center of curvature, evolute Definition 2.10

- The osculating plane at point $\gamma(t)$ is the plane spanned bt $T(t)$ and $N(t)$.
- The radius of curvature is $R(t) = \frac{1}{\kappa(t)}$.
- The center of curvature is $c(t) = f(t) + R(t)N(t)$.
- The evolute is the set of centers of curvatures.
- The osculating sphere at point $\gamma(t)$ is $\mathcal{S}(C(t), R(t))$.
- The osculating circle is the osculating sphere's cut through the osculating plane.

Binormal vector and Serret-Fresnet frame Definition 2.11

If γ is $\mathcal{C}^2$ regular and arc-length parametrized and s is a biregular point of γ, then we define

- $B(s) = T(s) \wedge N(s)$ the binormal vector at $\gamma(s)$.
- $\gamma(s), T(s), N(s), B(s)$ the Serret-Fresnet frame.

Another interesting property of γ is encoded by the variation of the osculating plane, that we call torsion. This can be defined using the derivative of the binormal vector, which is orthogonal to the osculating plane. This requires γ to be $\mathcal{C}^3$ regular.

Proposition 2.8

If γ is $\mathcal{C}^3$ regular and arc-length parametrized, then $B'(s)$ is colinear to $N(s)$.

Proof:

First method : $\|B(s)\|^2 = \langle B(s), B(s)\rangle = 1 \implies B'(s) \perp B(s)$. Hence, since $B(s) \perp N(s)$, $B'(s)$ is colinear to $N(s)$.
Second method : Differentiating $B(s)$ using the product rule yields:

$$\begin{aligned} B(s) = T(s) \wedge N(s) \implies B'(s) &= \underbrace{T'(s) \wedge N(s)}_{0} + T(s) \wedge N'(s) = T(s) \wedge N'(s) \\ &\implies B'(s) \perp T(s) \\ &\implies B'(s) \text{ is colinear to } N(s). \end{aligned} \tag{37}$$

Torsion Definition 2.12

For γ a $\mathcal{C}^3$ regular space curve parametrized by arc-length, we define torsion of γ in the point $\gamma(s)$ for all biregular points s as

$$\tau(s) = -\langle B'(s), N(s)\rangle = \pm\|B'(s)\| \tag{38}$$

Remark :

- Torsion measures the speed of rotation of the binormal vector at the given point. It's sign informs on the direction of rotation : the negative sign in the definition is a matter of convention. We thus define torsion to be negative when in the direction of the normal vector, and negative in the opposite case.
- As $N(s)$ is colinear to $B'(s)$, this is equivalent to $B'(s) = -\tau(s)N(s)$.
- As $N(s) \perp B(s)$, this is also equivalent to $\langle N'(s), B(s)\rangle$.

Torsion formula for arc-length parametrized curves — Proposition 2.9

For γ a $\mathcal{C}^3$ regular space curve parametrized by arc-length, we have

$$\tau(s) = \frac{\det(\gamma'(s), \gamma''(s), \gamma'''(s))}{\|\gamma''(s)\|^2}. \quad [39]$$

Proof:

$$\begin{aligned}\tau(s) &= -\langle N(s), B'(s)\rangle \\ &= \end{aligned} \quad [40]$$

Thus $\tau(s) = \det(T(s), N(s), N'(s))$. However,

$$\begin{aligned}&\cdot\ T(s) = \gamma'(s) \\ &\cdot\ N(s) = R(s)\gamma''(s) \\ &\cdot\ N'(s) = R'(s)\gamma''(s) + R(s)\gamma'''(s)\end{aligned} \quad [41]$$

Hence,

$$\begin{aligned}\tau(s) &= \det(\gamma'(s), R(s)\gamma''(s), R'(s)\gamma''(s) + R(s)\gamma'''(s)) \\ &= R(s)^2 \det(\gamma'(s), \gamma''(s), \gamma'''(s)).\end{aligned} \quad [42]$$

The conclusion trivially follows.

Torsion formula for general biregular parametrization — Proposition 2.10

For γ a $\mathcal{C}^3$ regular space curve parametrized by t, we have

$$\tau(t) = \frac{\det(\gamma'(t), \gamma''(t), \gamma'''(t))}{\|\gamma'(t) \wedge \gamma''(t)\|^2} \quad [43]$$

Proof: Admitted.

Geometric interpretation of vanishing torsion — Proposition 2.11

Let γ be a $\mathcal{C}^3$ biregular space curve. We have :

$$\gamma \text{ is planar} \Leftrightarrow \tau \equiv 0. \quad [44]$$

Proof:

- $$\gamma \text{ is planar} \implies \gamma', \gamma'' \text{ and } \gamma''' \text{ are coplanar} \implies \tau \equiv 0. \quad [45]$$
- Suppose $\tau \equiv 0$, i.e $B(s) \equiv B_0$ (parametrizing γ by arc-length). Then $\forall s$,
  $$(\langle \gamma(s), B_0\rangle)' = \langle \gamma'(s), B_0\rangle = \langle T(s), B_0\rangle = 0. \quad [46]$$
  This tells us that $\langle \gamma(s), B_0\rangle$ is constant, thus $\gamma(s)$ lies in a plane orthogonal to B_0.

Serret-Fresnet formula — Proposition 2.12

Let γ be a arc-length $\mathcal{C}^3$ parametrized space curve. We have

- $T'(s) = \kappa(s)N(s)$
- $B'(s) = -\tau(s)N(s)$
- $N'(s) = -\kappa(s)T(s) + \tau(s)B(s)$.

Proof:

- Done
- Done
- $(\|N(s)\|) = 1 \Rightarrow N' \perp N$. Since $(T(s), B(s))$ is thus a direct basis, we have $N'(s) = \lambda(s)T(s) + \mu(s)B(s)$. We have:
  $$\lambda(s) = \langle N'(s), T(s)\rangle = -\langle N(s), T'(s)\rangle = -\langle N(s), \kappa(s)N(s)\rangle = -\kappa(s). \quad [47]$$

And

$$\mu(s) = \langle N'(s), B(s)\rangle = -\langle N(s), B'(s)\rangle = -\langle N(s), -\tau(s)N(s) = \tau(s). \quad [48]$$

Fundamental theorem of local theory of curves — Theorem 2.2

Let $\kappa : [a, b] \mapsto \mathbb{R}$ and $\tau : [a, b] \mapsto \mathbb{R}$ be $\mathcal{C}^1$ functions, and such that κ is always strictly positive.

Then, there exists a unique (up to a rigid motion) $\mathcal{C}^3$ arc-length parametrized curve $\gamma : [a, b] \mapsto \mathbb{R}^3$ whose curvature is κ and torsion τ.

Proof: Admitted. Relies on Cauchy-Lipschitz.

Chapter 3: Differential geometry of surfaces

Representations of surfaces include :

- Parametrization : $f : U \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$
  - Special case : the graph of a map $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$ is $f(x, y, z) = (x, y, \varphi(x, y))$.
- Implicit : An inverse image of a map $g : \mathbb{R}^3 \mapsto \mathbb{R}$.
  - Example : Let $g(x, y, z) = x^2 + y^2 + z^2 - R^2$. Then $g^{-1}(\{0\})$ is the sphere of center $0_{\mathbb{R}^3}$ and radius R.
- Drawings !

These representations are locally equivalent when surfaces are "regular" (existence of tangent spaces).

Generalities

Parametrized surface — Definition 3.1

A $\mathcal{C}^k$ parametrized surface is a $\mathcal{C}^k$ map $f : U \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ with U an open set of $\mathbb{R}^2$.

We are interested in the geometric surface $S = f(U)$. However, one such geometric surface can admit many different parametrizations. They are all the same up to a diffeomorphism. More precisely, with φ a C^k diffeomorphism, we have:

$$\begin{tikzcd}
U \subset \mathbb{R}^2 \arrow[r, "f"] \arrow[d, "\varphi"'] & S \\
V \subset \mathbb{R}^2 \arrow[ur, "g"] &
\end{tikzcd}$$

Admissible change of variable — Definition 3.2

We say $\varphi : U \mapsto V$ with V an open set in $\mathbb{R}^2$ is an admissible change of variable for a $\mathcal{C}^k$ parametrized surface if φ is a $\mathcal{C}^k$ diffeomorphism. The induced reparametrization is $g = f \circ \varphi^{-1}$. Indeed,

$$g(V) = f\big(\varphi^{-1}(V)\big) = f(U). \quad [49]$$

We are interested in **geometric** properties of curves, thus that do not depend on reparametrization by an admissible change of variable.

Tangent spaces

Curve on surface — Definition 3.3

Let $f : U \mapsto \mathbb{R}^3$ be a $\mathcal{C}^1$ surface and $\gamma : I \subset \mathbb{R} \mapsto U$ a $\mathcal{C}^1$ curve. Then, $f \circ \gamma$ is a curve on $S = f(U)$.

Examples: Let $I_1, I_2 \subset \mathbb{R}$ such that $I_1 \times I_2 \subset U$. For any point $(u_0, v_0) \in I_1 \times I_2$, we define the **coordinate curves** on S as $\gamma_{u_0} : I_1 \mapsto S$ and $\gamma_{v_0} : I_2 \mapsto S$:

$$\gamma_{u_0} : v \mapsto \gamma(u_0, v) \quad \text{and} \quad \gamma_{v_0} : u \mapsto \gamma(u, v_0). \qquad [50]$$

Remark that $\gamma'_{u_0}(v_0) = \frac{\partial f}{\partial v}(u_0, v_0)$ and $\gamma'_{v_0}(u_0) = \frac{\partial f}{\partial u}(u_0, v_0)$.

Tangent space — Definition 3.4

The tangent space to $S = f(U)$ as $m_0 = f(u_0, v_0)$ is the affine space passing through m_0 and spanned by $\left(\frac{\partial f}{\partial u}(u_0, v_0), \frac{\partial f}{\partial v}(u_0, v_0)\right)$. We denote it $T_{m_0}S$.

Remark : We will often abusively refer to $T_{m_0}S$ as a vector space, omitting the translation from 0 to m_0.

Proposition 3.1

$T_{m_0}S$ is (as a vector space) the set of derivatives of parametrized curves on S at 0 where $C(0) = m_0$.

Proof:

Let $C = f \circ \gamma$ be a curve on S. Then $C'(t) = D_f(\gamma(t)) \cdot \gamma'(t)$. We set $\gamma(0) = (u_0, v_0)$ and $\gamma'(0) = (\lambda, \mu)$. Then,

$$C'(0) = D_f(u_0, v_0) \cdot (\lambda, \mu) = \frac{\partial f}{\partial u}(u_0, v_0)\lambda + \frac{\partial f}{\partial v}(u_0, v_0)\mu \in T_{m_0}S. \qquad [51]$$

For the reciprocal inclusion, let $X \in T_{m_0}S$. We know

$$\exists(\lambda, \mu) \in \mathbb{R}^2,\ X = \frac{\partial f}{\partial u}(u_0, v_0)\lambda + \frac{\partial f}{\partial v}(u_0, v_0)\mu. \qquad [52]$$

We thus pick a curve $\gamma : I \subset R \mapsto U$ such that $\gamma(0) = (u_0, v_0)$ and $\gamma'(0) = (\lambda, \mu)$. We denote $C = f \circ \gamma$.

Then $X = D_f(u_0, v_0) \cdot (\lambda, \mu) = C'(0)$.

Remarks :

- We can express the tangent space (as a vector space) as $T_{m_0}S = \mathrm{Im}\big(D_f(u_0, v_0)\big)$.
- A parametrized surface f is regular at m_0 if $\mathrm{Rank}\big(D_f(u_0, v_0)\big) = 2$, i.e the tangent space is a **plane**.
- A parametrized surface f is said to be regular if it is regular at every point.
- A geometric surface S is said to be regular if there exists a regular paramerization of S.