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Polyhedra (2)

Consider the following half-spaces in $\mathbb{R}^2$:

$$\begin{array}{rrrrrrr} - & x_1 & + & 3 & x_2 & \leq & 12 \\ & x_1 & + & 2 & x_2 & \leq & 12 \\ 3 & x_1 & - & 2 & x_2 & \leq & 8 \\ & x_1 & & & & \geq & 0 \\ & & & & x_2 & \geq & 0 \end{array}$$

Questions

1. Draw a plot of their supporting hyper-planes.
2. Draw the polyhedron given by the intersection of the half-spaces. Is it a polytope?
3. What happens if we add constraint $x_1 \geq 5$ to the polyhedron? What if we add constraint $x_1 \leq 5$?

Solution

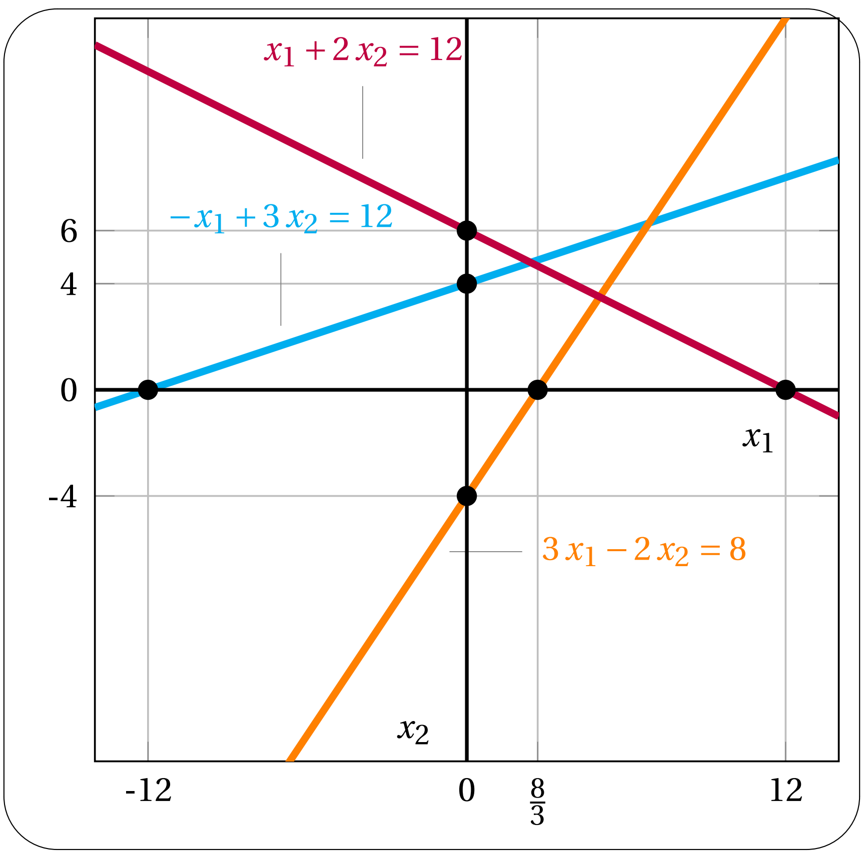
Given $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 \in \mathbb{R}$, if $(a_{11}\,a_{22} - a_{12}\,a_{21}) \neq 0$, we have:

$$\begin{cases} a_{11}\,x_1 + a_{12}\,x_2 &= b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 &= b_2 \end{cases} \implies (x_1, x_2) = \left(\frac{b_1\,a_{22} - a_{12}\,b_2}{a_{11}\,a_{22} - a_{12}\,a_{21}} \;,\; \frac{a_{11}\,b_2 - b_1\,a_{21}}{a_{11}\,a_{22} - a_{12}\,a_{21}} \right)$$

1. The supporting hyper-planes associated with the given half-spaces are:

$$\begin{array}{rcrcl} -\,x_1 & + & 3\,x_2 & = & 12 \\ x_1 & + & 2\,x_2 & = & 12 \\ 3\,x_1 & - & 2\,x_2 & = & 8 \\ x_1 & & & = & 0 \\ & & x_2 & = & 0 \end{array}$$

Their plot is shown below.

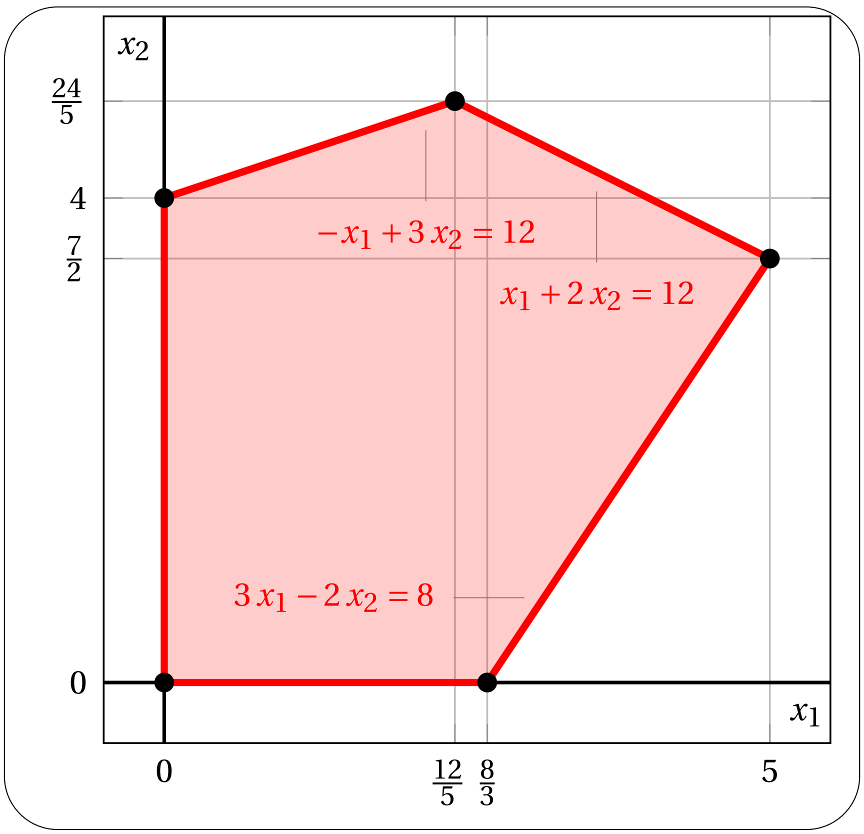


We compute the intersection points:

$$\begin{cases} -x_1 + 3\,x_2 &= 12 \\ x_1 + 2\,x_2 &= 12 \end{cases} \implies (x_1, x_2) = \left(\frac{12 \cdot 2 - 3 \cdot 12}{(-1) \cdot 2 - 3 \cdot 1} \;,\; \frac{(-1) \cdot 12 - 12 \cdot 1}{(-1) \cdot 2 - 3 \cdot 1} \right) = \left(\frac{12}{5} \;,\; \frac{24}{5} \right)$$

$$\begin{cases} x_1 + 2\,x_2 &= 12 \\ 3\,x_1 - 2\,x_2 &= 8 \end{cases} \implies (x_1, x_2) = \left(\frac{12 \cdot (-2) - 2 \cdot 8}{1 \cdot (-2) - 2 \cdot 3} \;,\; \frac{1 \cdot 8 - 12 \cdot 3}{1 \cdot (-2) - 2 \cdot 3} \right) = \left(5 \;,\; \frac{7}{2} \right)$$

2. The polyhedron, given by the intersection of the given half-spaces, is:



Since the polyhedron is bounded, it is also a polytope.

3. If we add constraint $x_1 \geq 5$ the polyhedron reduces to the point $\tilde{\boldsymbol{x}} = (\,\tilde{x}_1,\ \tilde{x}_2\,) = \left(5,\ \frac{7}{2}\right)$.
   If we add constraint $x_1 \leq 5$ the polyhedron does not change since for any $\tilde{\boldsymbol{x}} \in P$ we have that $\tilde{x}_1 = \leq 5$.