Last update: March 4, 2024

Polyhedra (3)

Consider the following half-spaces in $\mathbb{R}^2$:

$$
\begin{array}{rrcrcr}
- & x_1 & - & x_2 & \leq & -2 \\
4 & x_1 & - & 2\ x_2 & \leq & 7 \\
 & x_1 & + & 5\ x_2 & \leq & 21 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

Questions

1. Draw a plot of their supporting hyper-planes.
2. Draw the polyhedron given by the intersection of the half-spaces. Is it a polytope?

Solution

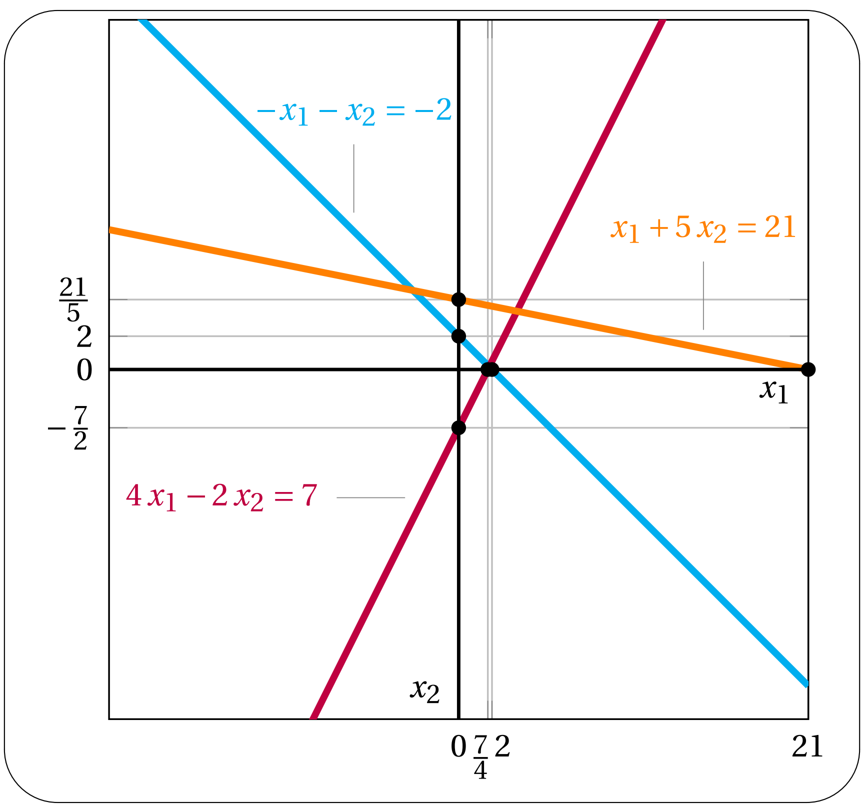
Given $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 \in \mathbb{R}$, if $(a_{11}\, a_{22} - a_{12}\, a_{21}) \neq 0$, we have:

$$\begin{cases} a_{11}\, x_1 + a_{12}\, x_2 & = b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 & = b_2 \end{cases} \implies (x_1, x_2) = \left(\frac{b_1\, a_{22} - a_{12}\, b_2}{a_{11}\, a_{22} - a_{12}\, a_{21}} \;,\; \frac{a_{11}\, b_2 - b_1\, a_{21}}{a_{11}\, a_{22} - a_{12}\, a_{21}} \right)$$

1. The supporting hyper-planes associated with the given half-spaces are:

$$\begin{array}{rrrrrrr} - & x_1 & - & & x_2 & = & -2 \\ 4 & x_1 & - & 2 & x_2 & = & 7 \\ & x_1 & + & 5 & x_2 & = & 21 \\ & x_1 & & & & = & 0 \\ & & & & x_2 & = & 0 \end{array}$$

Their plot is shown below.

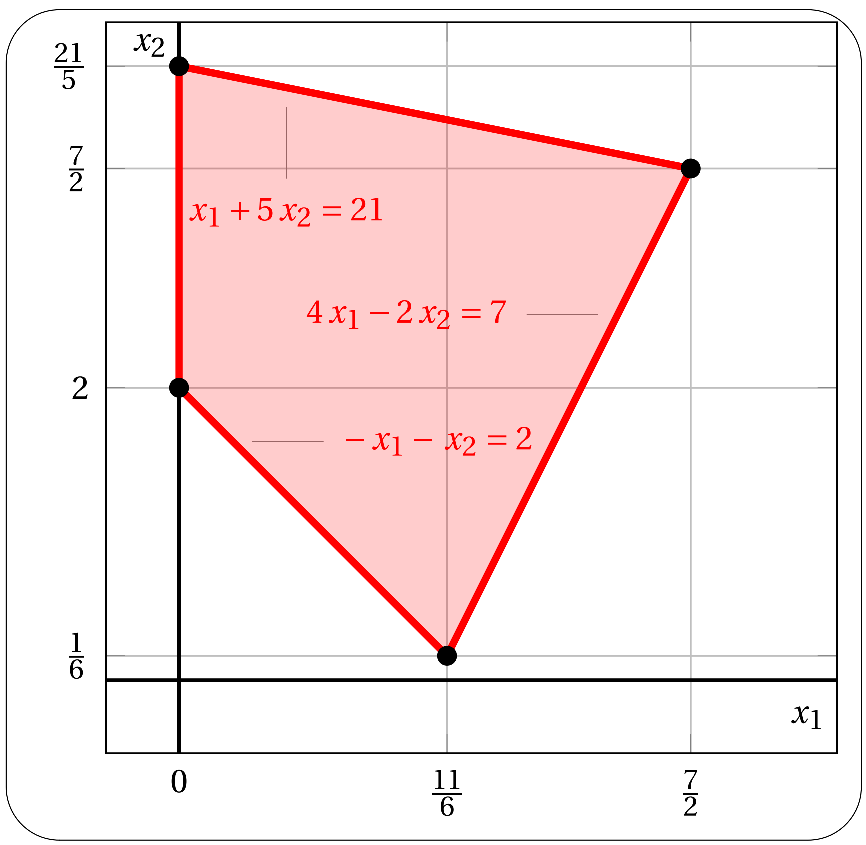


We compute the intersection points:

$$\begin{cases} x_1 + 5\,x_2 &= 21 \\ 4\,x_1 - 2\,x_2 &= 7 \end{cases} \implies (x_1, x_2) = \left(\frac{21\cdot(-2) - 5\cdot 7}{1\cdot(-2) - 5\cdot 4} \;,\; \frac{1\cdot 7 - 21\cdot 4}{1\cdot(-2) - 5\cdot 4} \right) = \left(\frac{7}{2} \;,\; \frac{7}{2} \right)$$

$$\begin{cases} -x_1 - x_2 &= -2 \\ 4\,x_1 - 2\,x_2 &= 7 \end{cases} \implies (x_1, x_2) = \left(\frac{(-2)\cdot(-2) - (-1)\cdot 7}{(-1)\cdot(-2) - (-1)\cdot 4} \;,\; \frac{(-1)\cdot 7 - (-2)\cdot 4}{(-1)\cdot(-2) - (-1)\cdot 4} \right) = \left(\frac{11}{6} \;,\; \frac{1}{6} \right)$$

2. The polyhedron, given by the intersection of the given half-spaces, is:



Since the polyhedron is bounded, it is also a polytope.