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Induction (4)

For any natural number $n \geq 1$, we have:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1} \tag{1}$$

Questions

1. Prove formula (1) using the induction principle.
2. Compute the value of the summation with $n = 10$.

Solution

1. By induction on n.

   - **First step of the induction**
     Let $n = 1$. Then the statement becomes:

     $$\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{2} \quad \text{that is} \quad \frac{1}{2} = \frac{1}{2}$$

     which is true.

   - **Inductive step**
     Suppose it is true for n, and we prove it for $(n+1)$. By inductive hypothesis, we have:

     $$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

     Then we can write:

     $$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n(n+2)+1}{(n+1)(n+2)} \\ &= \frac{n^2+2n+1}{(n+1)(n+2)} \\ &= \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{(n+1)+1}\end{aligned}$$

     which is exactly the statement for $n+1$.

2. The value of the summation with $n = 10$ is:

   $$\sum_{k=1}^{10} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \frac{1}{90} + \frac{1}{110} = \frac{10}{11}$$