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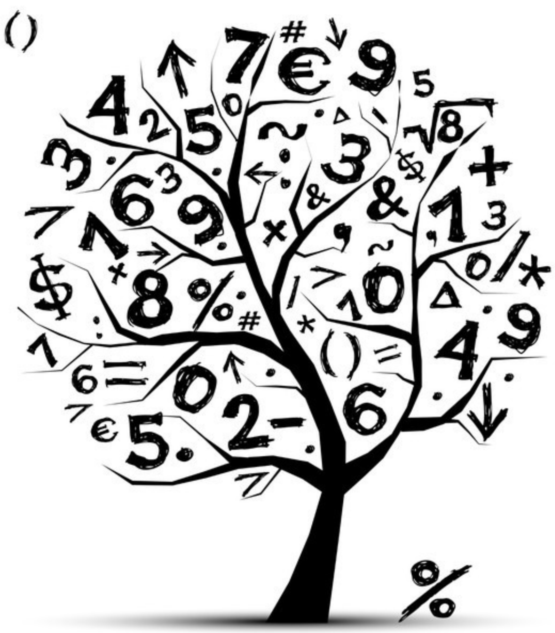
# Binomial coefficients

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## Contents

# 1 Binomial coefficients

**Definition 1: binomial coefficient**

*The **binomial coefficient** is the value:*

$$c_{n,k} = \frac{n!}{k!\,(n-k)!} \qquad \text{with } 0 \le k \le n, k \in \mathbb{N}. \tag{1}$$

- The binomial coefficient $c_{n,k}$ is usually indicated by the symbol: $\binom{n}{k}$ which is read "$n$ choose $k$".
- Some properties of the factorial, of immediate verification, are:

$$n! = n \cdot (n-1)!$$

$$\frac{n!}{(n-k)!} = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1), \ \text{ con } \ k \ge 1 \tag{2}$$

With $k \ge 1$, it boils down to the product of $k$ factors, starting from $n$ and decreasing by one unit at a time.

- Given (1) and (2), we have:

$$c_{n,k} = \binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{k!} \tag{3}$$

with $k \ge 1$, expression that is more manageable for calculating the binomial coefficient.

**Observation 1**

*For any $n \ge 0$ and $0 \le k \le n$, it holds:*

$$\binom{n}{n-k} = \binom{n}{k}$$

*Proof.* We have:

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

□

**Observation 2**

*For any $n \geq 1$ and $1 \leq k \leq n$, it holds:*

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k} \tag{4}$$

*Proof.* We have:

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)! \underbrace{(n-1-(k-1))!}_{=(n-k)!=(n-k)(n-k-1)!}} + \frac{(n-1)!}{k!\,(n-k-1)!} \\
&= \frac{(n-1)!}{(k-1)!\,(n-k)\,(n-k-1)!} + \frac{(n-1)!}{k\,(k-1)!\,(n-k-1)!} \\
&= \frac{k\,(n-1)! + (n-k)\,(n-1)!}{k\,(k-1)!\,(n-k)\,(n-k-1)!} = \frac{\overbrace{(n-1)!\,n}^{=n!}}{\underbrace{k\,(k-1)!}_{k!}\,\underbrace{(n-k)\,(n-k-1)!}_{(n-k)!}} \\
&= \frac{n!}{k!\,(n-k)!} = \binom{n}{k}
\end{aligned}$$

□

The relation (4) allows to calculate the binomial coefficients $\binom{n}{k}$ in a recursive manner by means of the so-called **Pascal's triangle**

- The rules for creating the Pascal's triangle are:

1. At the top of the triangle there is the number $\binom{0}{0} = 1$ (by definition).
2. On the sides there are the numbers $\binom{n}{0} = \binom{n}{n} = 1$ for any $n \geq 1$.
3. For $0 < k < n$, the value $\binom{n}{k}$ is written at the intersection of the $n$-th row and $k$-th column.
4. The value $\binom{n}{k}$ results from the sum of the two numbers that are in the previous row, the one on the same column and the one on the previous column:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for $n \geq 2$ and $1 \leq k < n$.

**Example 1: Pascal's triangle for $n \leq 10$ e $k \leq 10$**

| | $k=0$ | $k=1$ | $k=2$ | $k=3$ | $k=4$ | $k=5$ | $k=6$ | $k=7$ | $k=8$ | $k=9$ | $k=10$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n=0$ | 1 | | | | | | | | | | |
| $n=1$ | 1 | 1 | | | | | | | | | |
| $n=2$ | 1 | 2 | 1 | | | | | | | | |
| $n=3$ | 1 | 3 | 3 | 1 | | | | | | | |
| $n=4$ | 1 | 4 | 6 | 4 | 1 | | | | | | |
| $n=5$ | 1 | 5 | 10 | 10 | 5 | 1 | | | | | |
| $n=6$ | 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | | |
| $n=7$ | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | |
| $n=8$ | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | |
| $n=9$ | 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | |
| $n=10$ | 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 |

To compute the binomial coefficient $\binom{5}{3}$, corresponding to the blue cell, we can use the (4) relation and add the two binomial coefficients: $\binom{4}{2}$ and $\binom{4}{3}$, corresponding to the red cells:

$$\binom{5}{3} = \binom{4}{2} + \binom{4}{3} = 6 + 4 = 10$$

Given a set of $n$ elements, there are $\binom{n}{k}$ subsets of $k \leq n$ elements.

## 2 Newton's binomial

- The $n$-th power of a binomial $(a+b)$ can be calculated with the following formula (hence the name of binomial coefficient):

**Theorem 1: Newton's binomial**

*For any integer $n \geq 0$ and $a, b \in \mathbb{R}$, it holds:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} \, a^{n-k} \, b^k \tag{5}$$

*Proof.* By induction on $n$.

- **First step of the induction**

  Let $n = 0$. Then the statement becomes: $(a+b)^0 = \binom{0}{0}\ a^0\ b^0$ that is $1 = 1$ which is evidently true.

- **Inductive step**

  Suppose it is true for $n$, and we prove it for $n+1$. By inductive hypothesis, we have: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k}\ a^{n-k}\ b^k$. Accordingly we have:

$$(a+b)^{n+1} = (a+b)\cdot(a+b)^n = (a+b)\sum_{k=0}^{n}\binom{n}{k}\ a^{n-k}\ b^k$$

$$= \sum_{k=0}^{n}\binom{n}{k}\ a^{n-k+1}\ b^k + \underbrace{\sum_{k=0}^{n}\binom{n}{k}\ a^{n-k}\ b^{k+1}}_{=\sum_{k=1}^{n+1}\binom{n}{k-1}\ a^{n-k+1}\ b^k}$$

$$= a^{n+1} + \sum_{k=1}^{n}\binom{n}{k}\ a^{n-k+1}\ b^k + b^{n+1} + \sum_{k=1}^{n}\binom{n}{k-1}\ a^{n-k+1}\ b^k$$

$$= a^{n+1} + b^{n+1} + \sum_{k=1}^{n}\underbrace{\left(\binom{n}{k}+\binom{n}{k-1}\right)}_{=\binom{n+1}{k}}\ a^{n-k+1}\ b^k$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}\ a^{n+1-k}\ b^k$$

  which is exactly the intended statement, for $n+1$.

$\square$

**Observation 3**

*For any $n \geq 0$ and $0 \leq k \leq n$, it holds:*

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n \tag{6}$$

*Proof.* We have:

$$2^n = (1+1)^n$$

then by applying the Newton's binomial we get:

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \, 1^{n-k} \, 1^k = \sum_{k=0}^{n} \binom{n}{k}$$

$\square$

Given a set of $n$ elements, the number of all possible subsets is $2^n$.