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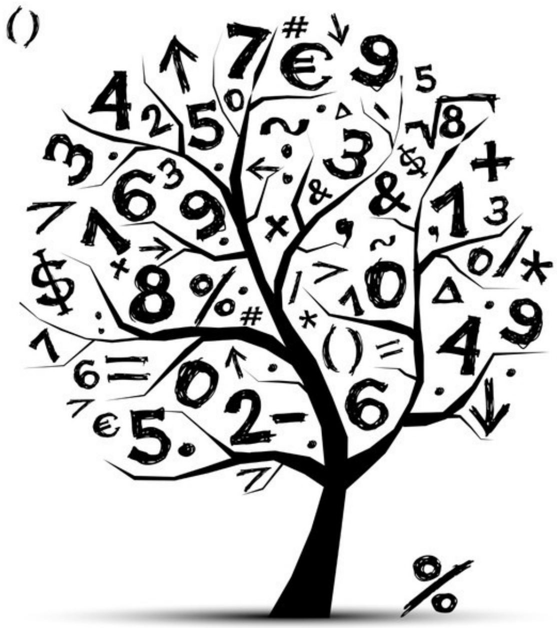
Scalar quadratic functions



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1 Scalar quadratic functions

Definition 1: of scalar quadratic function

*Given $n \cdot n$ values $q_{ij} \in \mathbb{R}$ with $i, j \in \{1, 2, \dots, n\}$, the associated **scalar quadratic function** is:*

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f : \underbrace{(x_1, x_2, \dots, x_n)}_{= \boldsymbol{x}} \mapsto \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}\, x_i\, x_j$$

- The $n \cdot n$ values q_{ij} with $i, j \in \{1, 2, \dots, n\}$ can be represented by a square matrix $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ of n rows and n columns. The n **variables** x_j with $j \in \{1, 2 \dots, n\}$ can be represented by a column vector $\boldsymbol{x}$ of n rows. We can use the following **matrix notation**:

$$\boldsymbol{Q} = \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1n} \\ q_{21} & q_{22} & \dots & q_{2n} \\ \vdots & \vdots & \dots & \vdots \\ q_{n1} & q_{n2} & \dots & q_{nn} \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

  Accordingly any scalar quadratic function f can be equivalently rewritten in matrix notation as follows:

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f : \boldsymbol{x} \mapsto \boldsymbol{x}' \boldsymbol{Q}\, \boldsymbol{x}$$

- If for some $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$ we have $q_{ij} \neq q_{ji}$, we can set both values as follows:

$$q_{ij} = q_{ji} = \frac{q_{ij} + q_{ji}}{2}$$

  and obtain an equivalent square symmetric matrix $\boldsymbol{Q}$. This can be done since $x_i\, x_j = x_j\, x_i$. For this reason, without loss of generality, **we can consider only square symmetric matrices $\boldsymbol{Q}$**.

Example 1: square symmetric Q matrices

Consider for example the following square non-symmetric matrix $\boldsymbol{Q}$ with 3 row and 3 column:

$$\begin{bmatrix} 2 & -4 & 0 \\ 2 & 2 & 6 \\ 0 & -8 & 2 \end{bmatrix}$$

an equivalent square symmetric matrix $\boldsymbol{Q}$ can be obtained as follows:

$$\begin{bmatrix} 2 & \frac{2+(-4)}{2} & 0 \\ \frac{2+(-4)}{2} & 2 & \frac{-8+6}{2} \\ 0 & \frac{-8+6}{2} & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Definition 2: positive semidefinite matrix

A square matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ *is* ***positive semidefinite*** *if:*

$$\boldsymbol{x}'\boldsymbol{Q}\,\boldsymbol{x} \geq 0, \quad \forall \boldsymbol{x} \in \mathbb{R}^n$$

Example 2: positive semidefinite matrix

The identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is positive semidefinite since we have:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 \geq 0, \qquad \forall (x_1, x_2) \in \mathbb{R}^2$$

Example 3: positive semidefinite matrix

The following matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

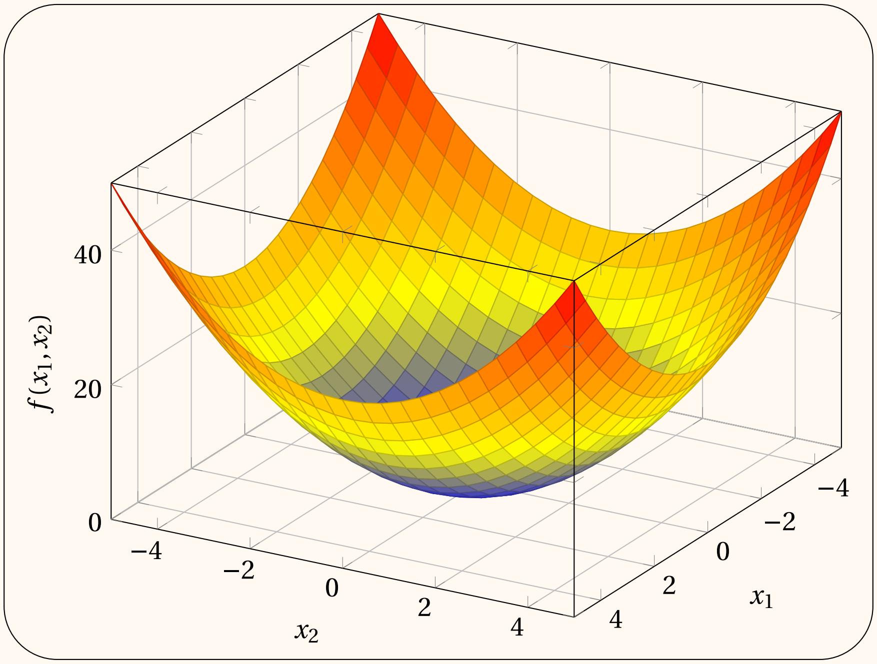
is positive semidefinite since we have:

$$\begin{aligned}
\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} (2\,x_1 - x_2) & (-x_1 + 2\,x_2 - x_3) & (-x_2 + 2\,x_3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= (2\,x_1 - x_2)\,x_1 + (-x_1 + 2\,x_2 - x_3)\,x_2 + (-x_2 + 2\,x_3)x_3 \\
&= 2\,x_1^2 - x_2\,x_1 - x_1\,x_2 + 2\,x_2^2 - x_3\,x_2 - x_2\,x_3 + 2\,x_3^2 \\
&= 2\,x_1^2 - 2\,x_1\,x_2 + 2\,x_2^2 - 2\,x_2\,x_3 + 2\,x_3^2 \\
&= x_1^2 + x_1^2 - 2\,x_1\,x_2 + x_2^2 + x_2^2 - 2\,x_2\,x_3 + x_3^2 + x_3^2 \\
&= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 \geq 0, \qquad \forall (x_1, x_2, x_3) \in \mathbb{R}^3
\end{aligned}$$

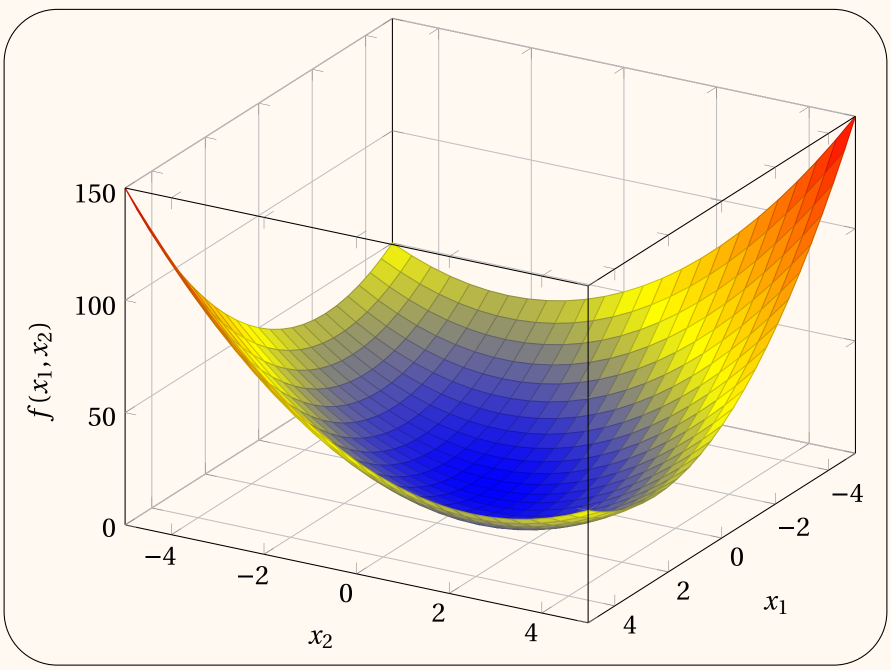
A square symmetric matrix $\boldsymbol{Q}$ is positive semidefinite if all its eigenvalues are real and non-negative. It is an equivalent definition.

Example 4: scalar quadratic function

With $\boldsymbol{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we have $f(x_1, x_2) = x_1^2 + x_2^2$. Its 3D graphic is:

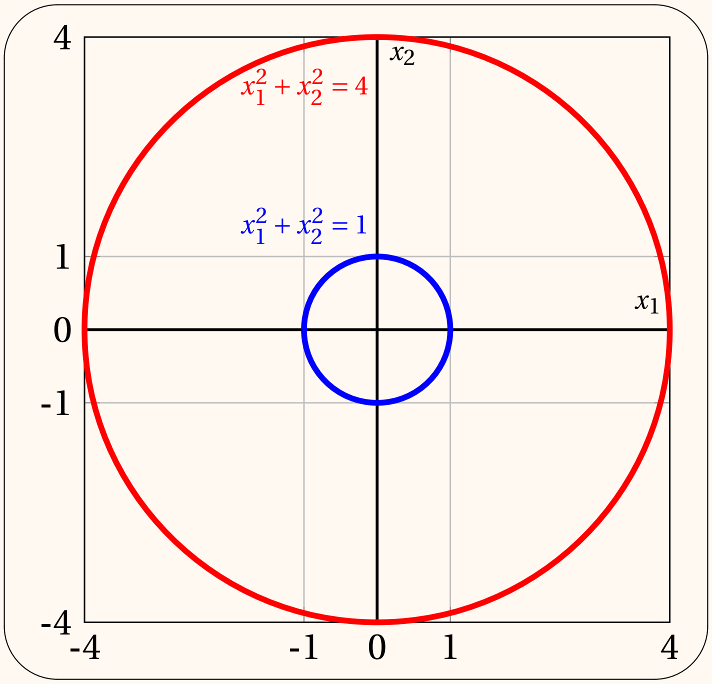


With $\boldsymbol{Q} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we have $f(x_1, x_2) = 2\,x_1^2 - 2\,x_1\,x_2 + 2\,x_2^2$. Its 3D graphic is:

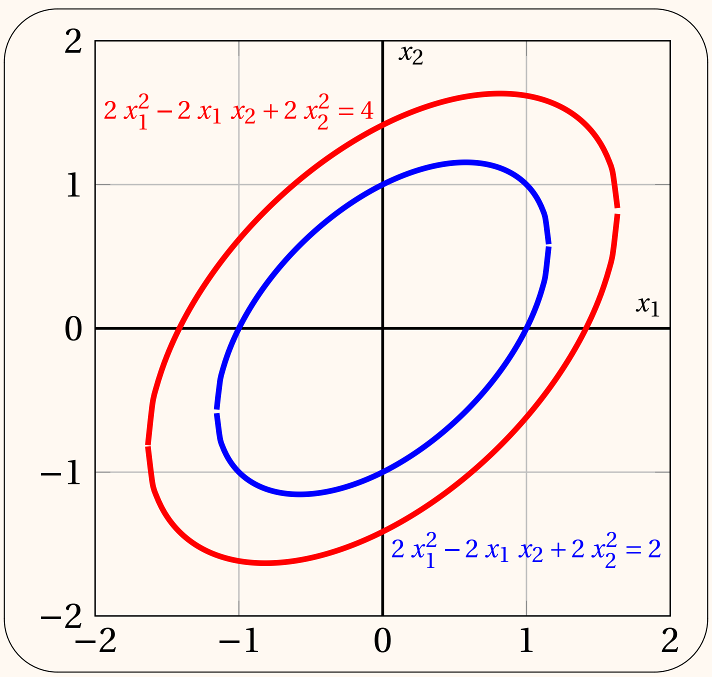


Example 5: contour lines of quadratic functions

Examples of contours lines of the function $f(x_1, x_2) = x_1^2 + x_2^2$ are:



Examples of contours lines of the functions $f(x_1, x_2) = 2\,x_1^2 - 2\,x_1\,x_2 + 2\,x_2^2$ and are:

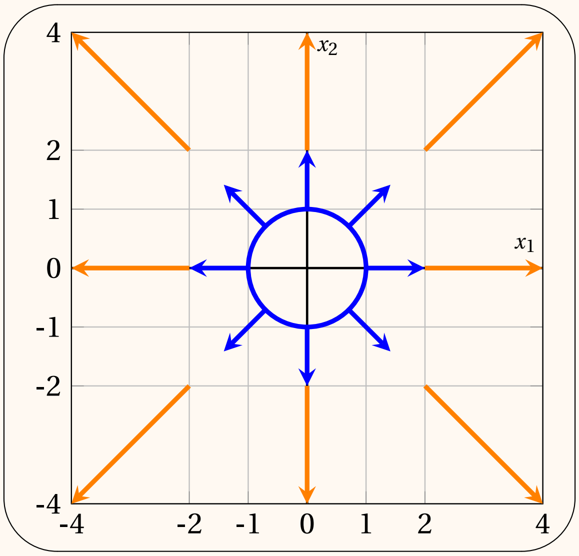


Example 6: gradients of quadratic functions

The gradient of the function $f(x_1, x_2) = x_1^2 + x_2^2$ is:

$$\nabla f(x_1, x_2) = \left[\frac{\partial f(x_1, x_2)}{\partial x_1} \quad \frac{\partial f(x_1, x_2)}{\partial x_2} \right] = [2\,x_1 \quad 2\,x_2]$$

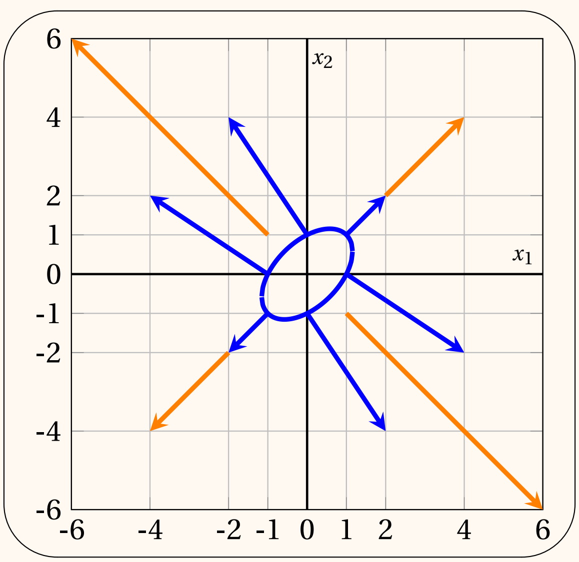
Example of gradient vectors on the contour line $x_1^2 + x_2^2 = 1$ (blue vectors) and in some other points (orange vectors):



The gradient of the function $f(x_1, x_2) = 2\,x_1^2 - 2\,x_1\,x_2 + 2\,x_2^2$ is:

$$\nabla f(x_1, x_2) = \left[\frac{\partial f(x_1, x_2)}{\partial x_1} \quad \frac{\partial f(x_1, x_2)}{\partial x_2} \right] = [4\,x_1 - 2\,x_2 \quad -2\,x_1 + 4\,x_2]$$

Example of gradient vectors on the contour line $2\,x_1^2 - 2\,x_1\,x_2 + 2\,x_2^2 = 2$ (blue vectors) and in some other points (orange vectors):



2 Necessary and sufficient conditions for convexity

Observation 1

Given a square matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$, if $\boldsymbol{Q}$ is positive semidefinite then the associated scalar quadratic function $f : \mathbb{R}^n \to \mathbb{R},\ f : \boldsymbol{x} \mapsto \boldsymbol{x}'\boldsymbol{Q}\,\boldsymbol{x}$ is convex.

Proof. If the matrix $\boldsymbol{Q}$ is positive semidefinite, for any two points $\boldsymbol{p}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ we have:

$$\underbrace{(\boldsymbol{p}-\boldsymbol{w})'\boldsymbol{Q}\,(\boldsymbol{p}-\boldsymbol{w})}_{=\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{p}+\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{w}-\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{w}-\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{p}} \geq 0$$

For any $\lambda \in [0,1]$, multiplying by $\lambda\,(1-\lambda) \geq 0$, we get:

$$\underbrace{\lambda\,(1-\lambda)\,\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{p}}_{=\lambda\,\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{p}-\lambda^2\,\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{p}} + \underbrace{\lambda\,(1-\lambda)\,\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{w}}_{=(1-\lambda)\,\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{w}-(1-\lambda)^2\,\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{w}} -\lambda\,(1-\lambda)\,\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{w} - \lambda\,(1-\lambda)\,\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{p} \;\geq\; 0$$

and rearranging:

$$\underbrace{\lambda^2\,\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{p} + (1-\lambda)^2\,\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{w} + \lambda\,(1-\lambda)\,\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{w} + \lambda\,(1-\lambda)\,\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{p}}_{=(\lambda\,\boldsymbol{p}+(1-\lambda)\,\boldsymbol{w})'\,\boldsymbol{Q}\,(\lambda\,\boldsymbol{p}+(1-\lambda)\,\boldsymbol{w})} \leq \lambda\,\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{p} + (1-\lambda)\,\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{w}$$

So we have $f\big(\lambda\,\boldsymbol{p} + (1-\lambda)\,\boldsymbol{w}\big) \leq \lambda\,f(\boldsymbol{p}) + (1-\lambda)\,f(\boldsymbol{w})$ for all points $\boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n$ and the domain $\mathbb{R}^n$ is convex, then the function is convex.

□

- We have shown that:

$$\boldsymbol{Q}\ \text{positive semidefinite} \implies f\ \text{convex}$$

so "$\boldsymbol{Q}$ positive semidefinite" is a sufficient condition to "f convex" and "f convex" is a necessary condition to "$\boldsymbol{Q}$ positive semidefinite". The implication works also in the other direction as stated by the following observation.

Observation 2

Given a square matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$, if the associated scalar quadratic function $f : \mathbb{R}^n \to \mathbb{R},\ f : \boldsymbol{x} \mapsto \boldsymbol{x}'\boldsymbol{Q}\,\boldsymbol{x}$ is convex then $\boldsymbol{Q}$ is positive semidefinite.

Proof. If the function f is convex, for any two points $\boldsymbol{p}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ and any $\lambda \in (0,1)$, we have:

$$\underbrace{(\lambda\,\boldsymbol{p}+(1-\lambda)\,\boldsymbol{w})'\,\boldsymbol{Q}\,(\lambda\,\boldsymbol{p}+(1-\lambda)\,\boldsymbol{w})}_{=\lambda^2\,\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{p}+(1-\lambda)^2\,\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{w}+\lambda\,(1-\lambda)\,\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{w}+\lambda\,(1-\lambda)\,\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{p}} \quad \le \lambda\,\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{p}+(1-\lambda)\,\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{w}$$

This implies, rearranging and simplifying, that:

$$\lambda\,(1-\lambda)\,\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{p}+\lambda\,(1-\lambda)\,\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{w}-\lambda\,(1-\lambda)\,\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{w}-\lambda\,(1-\lambda)\,\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{p}\ge 0$$

which, since $\lambda\,(1-\lambda)>0$, it implies:

$$\underbrace{\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{p}+\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{w}-\boldsymbol{p}'\boldsymbol{Q}\,\boldsymbol{w}-\boldsymbol{w}'\boldsymbol{Q}\,\boldsymbol{p}}_{=(\boldsymbol{p}-\boldsymbol{w})'\boldsymbol{Q}\,(\boldsymbol{p}-\boldsymbol{w})}\ge 0$$

Accordingly the matrix $\boldsymbol{Q}$ is positive semidefinite.

□

- We have shown that:

$$f \text{ convex} \implies \boldsymbol{Q} \text{ positive semidefinite}$$

so "f convex" is a sufficient condition to "$\boldsymbol{Q}$ positive semidefinite" and "$\boldsymbol{Q}$ positive semidefinite" is necessary condition to "f convex".

So we finally have:

$$\boldsymbol{Q} \text{ positive semidefinite} \iff f \text{ convex}$$

so "$\boldsymbol{Q}$ positive semidefinite" is a necessary and sufficient condition to "f convex" and vice versa.

Corollary 1

Given a square matrix $\boldsymbol{Q}\in\mathbb{R}^{n\times n}$, the associated scalar quadratic function $f:\mathbb{R}^n\to\mathbb{R},\ f:\boldsymbol{x}\mapsto\boldsymbol{x}'\boldsymbol{Q}\,\boldsymbol{x}$ is convex if and only if $\boldsymbol{Q}$ is positive semidefinite.