Last update: March 1, 2024

Summations (2)

For any natural number $n \geq 1$, we have:

$$\sum_{k=1}^{n} k^3 = \frac{n^4 + 2n^3 + n^2}{4} \tag{1}$$

Sum of the cubes of the first n natural numbers (without zero)

Questions

1. Prove formula (1) using the properties of the summations.
2. Compute the sum of the squares of the first 10 natural numbers (without zero).
3. Prove that sum of the cubes of the first n natural numbers (without zero) equals the square of the sum of the first n natural numbers:

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

Solution

1. We start by writing $\sum_{k=0}^{n}(k+1)^4$ in two different ways:

$$1)\quad \sum_{k=0}^{n}(k+1)^4 = \sum_{k=0}^{n}(k^4+4\,k^3+6\,k^2+4\,k+1)$$

$$= \left(\sum_{k=1}^{n}k^4+4\sum_{k=1}^{n}k^3+6\sum_{k=1}^{n}k^2+4\sum_{k=1}^{n}k+\sum_{k=1}^{n}1\right)+1$$

$$= \sum_{k=1}^{n}k^4+4\sum_{k=1}^{n}k^3+2n^3+3n^2+n+2\,n^2+2\,n+n+1$$

$$= \sum_{k=1}^{n}k^4+4\sum_{k=1}^{n}k^3+2n^3+5n^2+4n+1$$

$$2)\quad \sum_{k=0}^{n}(k+1)^4 = \sum_{k=1}^{n+1}k^4 = \sum_{k=1}^{n}k^4+(n+1)^4$$

Equating the expressions and canceling the sum with k^4 we get:

$$4\sum_{k=1}^{n}k^3+2n^3+5n^2+4n+1=(n+1)^4$$

Isolating the sum with k^3 we get:

$$4\sum_{k=1}^{n}k^3=n^4+4n^3+6n^2+4n+1-2n^3-5n^2-4n-1=n^4+2n^3+n^2$$

Then we have:

$$\sum_{k=1}^{n}k^3=\frac{n^4+2n^3+n^2}{4}$$

2. The sum of cubes of the first 10 natural numbers (without zero) is:

$$\sum_{k=1}^{10}k^2=1^3+2^3+3^3+4^3+5^3+6^3+7^3+8^3+9^3+10^3=\frac{10^4+2\cdot 10^3+10^2}{4}=3025$$

3. The sum of the cubes of the first n natural numbers equals the square of the sum of the first n natural numbers since we have:

$$\sum_{k=1}^{n}k^3=\frac{n^4+2n^3+n^2}{4}=\frac{n^2\,(n^2+2n+1)}{4}=\frac{n^2\,(n+1)^2}{4}=\left(\frac{n\,(n+1)}{2}\right)^2=\left(\sum_{k=1}^{n}k\right)^2$$