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Convex function (1)

Given m convex functions:

$$f_i : F_i \subseteq \mathbb{R}^n \to \mathbb{R}, \quad f_i : \underbrace{(x_1, x_2, \ldots, x_n)}_{=\boldsymbol{x}} \mapsto f_i(\boldsymbol{x}) \quad \text{with} \quad i \in \{1, 2, \ldots, m\}$$

where F_1, F_2, $\ldots$, F_m are the domains of the m functions, consider the function:

$$g : \bigcap_{i=1}^{m} F_i \subseteq \mathbb{R}^n \to \mathbb{R}, \quad g : \underbrace{(x_1, x_2, \ldots, x_n)}_{=\boldsymbol{x}} \mapsto \underbrace{\max \left\{ f_1(\boldsymbol{x}),\, f_2(\boldsymbol{x}),\, \ldots,\, f_m(\boldsymbol{x}) \right\}}_{=g(\boldsymbol{x})}$$

Questions

1. Prove that the function g is convex using the definition of convex functions.
2. Consider the following two convex functions ($m = 2$):

$$f_1 : [0, 10] \to \mathbb{R}, \;\; f_1 : x \mapsto \frac{1}{10}(x - 10)^2 \qquad \text{and} \qquad f_2 : [0, 10] \to \mathbb{R}, \;\; f_2 : x \mapsto \frac{1}{10} x^2$$

   Plot the graphic of the associated function g.

Solution

A Function $f : F \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex if:

$\forall\, \boldsymbol{p}, \boldsymbol{w} \in F$ and $\forall \lambda \in [0,1]$ we have $f\big(\lambda\, \boldsymbol{p} + (1-\lambda)\, \boldsymbol{w}\big) \le \lambda\, f(\boldsymbol{p}) + (1-\lambda)\, f(\boldsymbol{w})$

and the domain F of the function is a convex set.

1. First, we note that the domain G of the function g, i.e., the set:

$$G = \bigcap_{i=1}^{m} F_i$$

is convex since is given by the intersection of the m convex sets: $F_1, F_2, \ldots, F_m$. Each set F_i, with $i \in \{1, 2, \ldots, m\}$, is convex since the associated function f_i is convex.

Then, $\forall\, \boldsymbol{p}, \boldsymbol{w} \in G$ and $\forall \lambda \in [0,1]$, we have:

$$\begin{aligned} g(\lambda \boldsymbol{p} + (1-\lambda)\boldsymbol{w}) &= f_i(\lambda \boldsymbol{p} + (1-\lambda)\boldsymbol{w}) && \text{(for some } i \in \{1,2,\ldots,m\}) \\ &\le \lambda f_i(\boldsymbol{p}) + (1-\lambda) f_i(\boldsymbol{w}) && \text{(since } f_i \text{ is convex } \forall i \in \{1,2,\ldots,m\}) \\ &\le \lambda \max\left\{f_1(\boldsymbol{p}), f_2(\boldsymbol{p}), \ldots, f_m(\boldsymbol{w})\right\} + (1-\lambda) \max\left\{f_1(\boldsymbol{w}), f_2(\boldsymbol{w}), \ldots, f_m(\boldsymbol{w})\right\} \\ &= \lambda g(\boldsymbol{p}) + (1-\lambda) g(\boldsymbol{w}). \end{aligned}$$

Accordingly, the function g is convex.

2. The graphic of the function g: