Last update: March 1, 2024

Polyhedra (1)

Consider the following half-spaces in $\mathbb{R}^2$:

$$\begin{aligned} -4\ x_1 + 10\ x_2 &\leq 35 \\ 3\ x_1 + x_2 &\leq 12 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Questions

1. Draw a plot of their supporting hyper-planes.
2. Draw the polyhedron given by the intersection of the half-spaces. Is it a polytope?

Solution

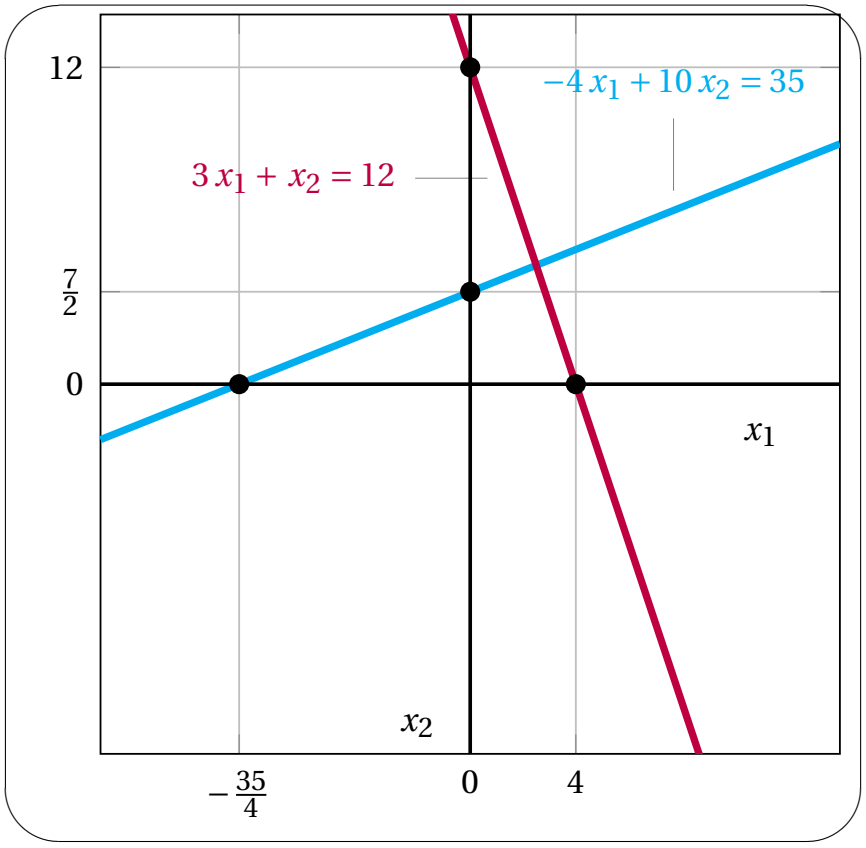
Given $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 \in \mathbb{R}$, if $(a_{11}\ a_{22} - a_{12}\ a_{21}) \neq 0$, we have:

$$\begin{cases} a_{11}\ x_1 + a_{12}\ x_2 &= b_1 \\ a_{21}\ x_1 + a_{22}\ x_2 &= b_2 \end{cases} \implies (x_1, x_2) = \left(\frac{b_1\ a_{22} - a_{12}\ b_2}{a_{11}\ a_{22} - a_{12}\ a_{21}} \ , \ \frac{a_{11}\ b_2 - b_1\ a_{21}}{a_{11}\ a_{22} - a_{12}\ a_{21}} \right)$$

1. The supporting hyper-planes associated with the given half-spaces are:

$$\begin{aligned} -4\ x_1 \ + \ 10\ x_2 &= 35 \\ 3\ x_1 \ + \ x_2 &= 12 \\ x_1 &= 0 \\ x_2 &= 0 \end{aligned}$$

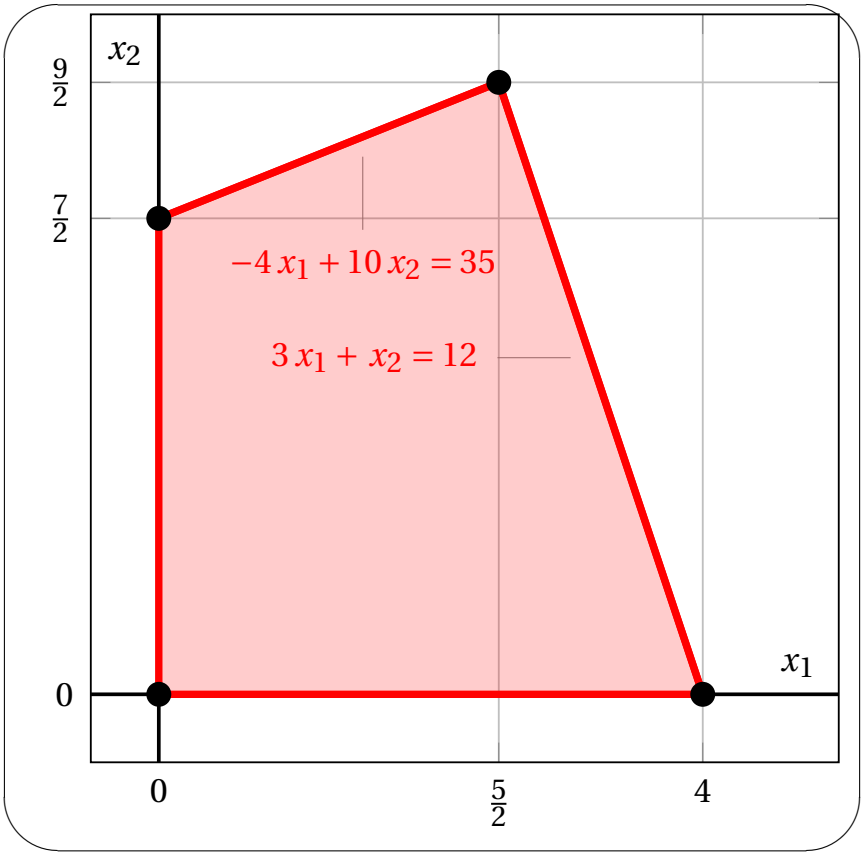
Their plot is shown below.



We compute the intersection point:

$$\begin{cases} -4\,x_1 + 10\,x_2 = 35 \\ 3\,x_1 + x_2 = 12 \end{cases} \implies (x_1, x_2) = \left(\frac{35\cdot 1 - 10\cdot 12}{(-4)\cdot 1 - 10\cdot 3}\ ,\ \frac{(-4)\cdot 12 - 35\cdot 3}{(-4)\cdot 1 - 10\cdot 3} \right) = \left(\frac{5}{2}\ ,\ \frac{9}{2} \right)$$

2. The polyhedron, given by the intersection of the given half-spaces, is:



Since the polyhedron is bounded, it is also a polytope.