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Norms

Given a point $\boldsymbol{p} \in \mathbb{R}^n$ and a positive semidefinite matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$

$$\boldsymbol{Q} = \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1n} \\ q_{21} & q_{22} & \dots & q_{2n} \\ \vdots & \vdots & \dots & \vdots \\ q_{n1} & q_{n2} & \dots & q_{nn} \end{bmatrix}$$

we consider the **extended norm** which is defined as follows:

$$||\boldsymbol{p}||_{\boldsymbol{Q}} = \sqrt{\boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{p}} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}\, p_i\, p_j}$$

If $\boldsymbol{Q}$ is the identity matrix $\boldsymbol{I}$, the extended norm corresponds to the euclidean norm. Since the matrix $\boldsymbol{Q}$ is positive semidefinite, we have:

$$\boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{p} \geq 0, \quad \forall \boldsymbol{p} \in \mathbb{R}^n \tag{1}$$

Questions

1. Prove the following properties

Observation 1: absolute homogeneity

$$||\lambda\,\boldsymbol{p}||_{\boldsymbol{Q}} = |\lambda|\,||\boldsymbol{p}||_{\boldsymbol{Q}}, \quad \forall \lambda \in \mathbb{R}, \boldsymbol{p} \in \mathbb{R}^n \tag{2}$$

Observation 2: Cauchy-Schwarz inequality

$$|\boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{w}| \leq ||\boldsymbol{p}||_{\boldsymbol{Q}}\,||\boldsymbol{w}||_{\boldsymbol{Q}}, \quad \forall \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n \tag{3}$$

Observation 3: triangle inequality in $\mathbb{R}^n$

$$||\boldsymbol{p} + \boldsymbol{w}|| \leq ||\boldsymbol{p}|| + ||\boldsymbol{w}||, \quad \forall \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n \tag{4}$$

Solution

The following properties (derived from the properties of the sum and the product operations in $\mathbb{R}$) hold:

$$\boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{w} = \boldsymbol{w}'\,\boldsymbol{Q}\,\boldsymbol{p}, \quad \forall\,\boldsymbol{p},\boldsymbol{w} \in \mathbb{R}^n \tag{5}$$

$$\boldsymbol{p}'\,\boldsymbol{Q}\,(\boldsymbol{w}+\boldsymbol{u}) = \boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{w} + \boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{w}, \quad \forall\,\boldsymbol{p},\boldsymbol{w},\boldsymbol{u} \in \mathbb{R}^n \tag{6}$$

$$\lambda\,(\boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{w}) = (\lambda\,\boldsymbol{p})'\,\boldsymbol{Q}\,\boldsymbol{w}, \quad \forall\,\lambda \in \mathbb{R},\;\; \boldsymbol{p},\boldsymbol{w} \in \mathbb{R}^n \tag{7}$$

1. For all $\lambda \in \mathbb{R}$ and $\boldsymbol{p} \in \mathbb{R}^n$, we have:

$$||\lambda\,\boldsymbol{p}|| = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}\lambda\,p_i\,\lambda\,p_j} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda^2\,q_{ij}\,p_i\,p_j} = |\lambda|\,\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}\,p_i\,p_j} = |\lambda|\,||\boldsymbol{p}||_{\boldsymbol{Q}}$$

2. If $\boldsymbol{p} = \boldsymbol{0}$ or $\boldsymbol{w} = \boldsymbol{0}$ (or both), it clearly holds. If $\boldsymbol{p} \neq \boldsymbol{0}$ and $\boldsymbol{w} \neq \boldsymbol{0}$, we define the set of points:

$$\boldsymbol{u} = \alpha\,\boldsymbol{p} + \beta\,\boldsymbol{w} \quad \text{with} \quad \alpha,\beta \in \mathbb{R}$$

Since the matrix $\boldsymbol{Q}$ is positive semidefinite and for the properties (5),(6),(7), for all $\alpha,\beta \in \mathbb{R}$ we have:

$$\begin{aligned}
\underbrace{\boldsymbol{u}'\,\boldsymbol{Q}\,\boldsymbol{u}}_{\geq 0} &= (\alpha\,\boldsymbol{p} + \beta\,\boldsymbol{w})'\,\boldsymbol{Q}\,(\alpha\,\boldsymbol{p} + \beta\,\boldsymbol{w}) \\
&= \alpha^2\,\boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{p} + 2\,\alpha\,\beta\,\boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{w} + \beta^2\,\boldsymbol{w}'\,\boldsymbol{Q}\,\boldsymbol{w} \\
&= \alpha^2\,||\boldsymbol{p}||_{\boldsymbol{Q}}^2 + 2\,\alpha\,\beta\,\boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{w} + \beta^2\,||\boldsymbol{w}||_{\boldsymbol{Q}}^2 \geq 0, \quad \forall\,\alpha,\beta \in \mathbb{R}
\end{aligned}$$

For $\alpha = ||\boldsymbol{w}||_{\boldsymbol{Q}}^2$ and $\beta = -\boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{w}$, we then have:

$$||\boldsymbol{w}||_{\boldsymbol{Q}}^4\,||\boldsymbol{p}||_{\boldsymbol{Q}}^2 - 2\,||\boldsymbol{w}||_{\boldsymbol{Q}}^2\,(\boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{w})^2 + (\boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{w})^2\,||\boldsymbol{w}||_{\boldsymbol{Q}}^2 =$$

$$= \underbrace{||\boldsymbol{w}||_{\boldsymbol{Q}}^4\,||\boldsymbol{p}||_{\boldsymbol{Q}}^2 - ||\boldsymbol{w}||_{\boldsymbol{Q}}^2\,(\boldsymbol{p}'\,\boldsymbol{Q}\,\boldsymbol{w})^2}_{\geq 0}$$

It follows that:

$$||\boldsymbol{w}||_{\boldsymbol{Q}}^2 \, (\boldsymbol{p}' \, \boldsymbol{Q} \, \boldsymbol{w})^2 \leq ||\boldsymbol{w}||_{\boldsymbol{Q}}^4 \, ||\boldsymbol{p}||_{\boldsymbol{Q}}^2 \quad \Longrightarrow \quad (\boldsymbol{p}' \, \boldsymbol{Q} \, \boldsymbol{w})^2 \leq ||\boldsymbol{w}||_{\boldsymbol{Q}}^2 \, ||\boldsymbol{p}||_{\boldsymbol{Q}}^2$$

obtained by diving by $||\boldsymbol{w}||_{\boldsymbol{Q}}^2 > 0$. This implies $|\boldsymbol{p}' \, \boldsymbol{Q} \, \boldsymbol{w}| \leq ||\boldsymbol{p}||_{\boldsymbol{Q}} \, ||\boldsymbol{w}||_{\boldsymbol{Q}}$.

3. The extension of the triangle inequality is equivalent to the inequality:

$$||\boldsymbol{p} + \boldsymbol{w}||_{\boldsymbol{Q}}^2 \leq \left(||\boldsymbol{p}||_{\boldsymbol{Q}} + ||\boldsymbol{w}||_{\boldsymbol{Q}}\right)^2, \quad \forall \, \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n$$

Directly from the definitions and the extension of the Cauchy-Schwarz inequality we have:

$$\begin{aligned} ||\boldsymbol{p} + \boldsymbol{w}||_{\boldsymbol{Q}}^2 &= (\boldsymbol{p} + \boldsymbol{w})' \, \boldsymbol{Q} \, (\boldsymbol{p} + \boldsymbol{w}) \\ &= \boldsymbol{p}' \, \boldsymbol{Q} \, \boldsymbol{p} + \boldsymbol{w}' \, \boldsymbol{Q} \, \boldsymbol{w} + 2 \;\, \boldsymbol{p}' \, \boldsymbol{Q} \, \boldsymbol{w} \\ &= ||\boldsymbol{p}||_{\boldsymbol{Q}}^2 + ||\boldsymbol{w}||_{\boldsymbol{Q}}^2 + 2 \;\, \boldsymbol{p}' \, \boldsymbol{Q} \, \boldsymbol{w} \\ &\leq ||\boldsymbol{p}||_{\boldsymbol{Q}}^2 + ||\boldsymbol{w}||_{\boldsymbol{Q}}^2 + 2 \, ||\boldsymbol{p}||_{\boldsymbol{Q}} \, ||\boldsymbol{w}||_{\boldsymbol{Q}} = \left(||\boldsymbol{p}||_{\boldsymbol{Q}} + ||\boldsymbol{w}||_{\boldsymbol{Q}}\right)^2 \end{aligned}$$