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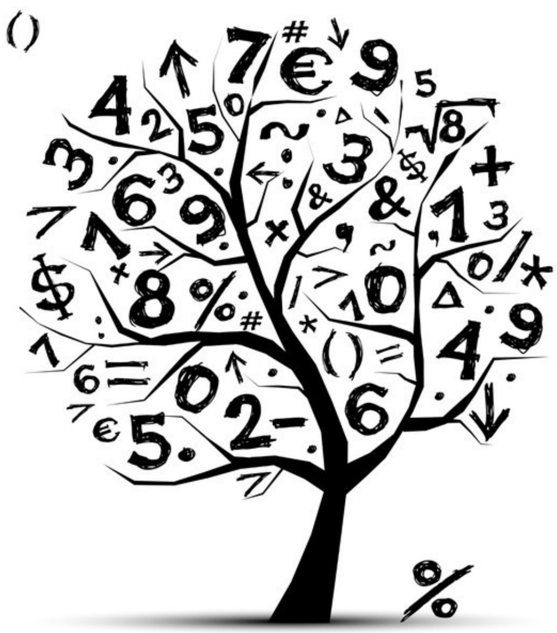
Summations and geometric progressions/series



- **Author**: Fabio Furini

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1 Summations

Definition 1: summation

Given n values $a_j \in \mathbb{R}$ with $j \in \{1, 2, \ldots, n\}$, the ***summation***

$$a_1 + a_2 + \ldots + a_n$$

can be indicated in compact form with the summation symbol:

$$\sum_{j=1}^{n} a_j$$

which reads: "summation for j from 1 to n of a_j". The symbol j is called ***summation index****.*

- The summation symbol is therefore very useful when the terms a_j are explicitly defined as a function of the summation index j.

Example 1: summations

$$\sum_{j=1}^{10} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$$

$$\sum_{j=3}^{n} j^2 = 3^2 + 4^2 + 5^2 + \ldots + n^2$$

- The summation index is a **dummy index**. This means that if j is replaced with i, k or any other index in all its occurrences the value of the summation does not change.

Example 2: dummy index

We have:

$$\sum_{j=1}^{n} j^2 = \sum_{i=1}^{n} i^2 = \sum_{k=1}^{n} k^2$$

as the two symbols indicate the summation of the squares of the first n the natural numbers (without zero). On the contrary, we have:

$$\sum_{j=1}^{n} j^2 \neq \sum_{j=1}^{m} j^2$$

as the two symbols indicate the summation, respectively, of the first n and the first m squares of the natural numbers (without zero). Clearly, if $n \neq m$, the result is different.

1.1 Summation properties

Observation 1

Given $c \in \mathbb{R}$, we have:

$$\sum_{j=1}^{n} (c \cdot a_j) = c \sum_{j=1}^{n} a_j \qquad \text{(product by a constant)} \tag{1}$$

$$\sum_{j=1}^{n} c = n \cdot c \qquad \text{(summation with constant terms)} \tag{2}$$

Proof. First property (1). From the distributive property we have:

$$\underbrace{c\,a_1 + c\,a_2 + \ldots + c\,a_n}_{=\sum_{j=1}^{n}(c \cdot a_j)} = \underbrace{c\,(a_1 + a_2 + \ldots + a_n)}_{=c\,\sum_{j=1}^{n} a_j}$$

Second property (2):

$$\underbrace{c + c + \ldots + c}_{=\sum_{j=1}^{n} c,\quad c \text{ summed up } n \text{ times}} = n\,c$$

□

Observation 2

Given two summations $\sum_{j=1}^{n} a_j$ e $\sum_{j=1}^{n} b_j$, we have:

$$\sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j = \sum_{j=1}^{n} (a_j + b_j) \tag{3}$$

Proof. We have:

$$\underbrace{a_1 + a_2 + \ldots + a_n + b_1 + b_2 + \ldots + b_n}_{=\sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j} = \underbrace{a_1 + b_1 + a_2 + b_2 + \ldots + a_n + b_n}_{=\sum_{j=1}^{n}(a_j + b_j)}$$

□

Observation 3

Given two natural numbers $n, m \in \mathbb{N}$, we have:

$$\sum_{j=1}^{n+m} a_j = \sum_{j=1}^{n} a_j + \sum_{j=n+1}^{n+m} a_j \qquad \text{(scomposition)} \tag{4}$$

$$\sum_{j=1}^{n} a_j = \sum_{j=1+m}^{n+m} a_{j-m} = \sum_{j=1-m}^{n-m} a_{j+m} \qquad \text{(index translation)} \tag{5}$$

$$\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} a_{n-j+1} = \sum_{j=0}^{n-1} a_{n-j} \qquad \text{(index reflexion)} \tag{6}$$

Proof. The three properties are simply different notations and/or arrangements of the terms in the summations. □

1.2 Some important summations

Observation 4: sum of the first n natural numbers (without zero)

For any natural number $n \geq 1$:

$$\sum_{j=1}^{n} j = \frac{n^2+n}{2} \tag{7}$$

Proof. We have:

$$\sum_{j=1}^{n} j = \frac{1}{2}\left(\sum_{j=1}^{n} j + \sum_{j=1}^{n} j\right) = \frac{1}{2}\left(\sum_{j=1}^{n} j + \sum_{j=1}^{n} (n-j+1)\right)$$

$$= \frac{1}{2}\sum_{j=1}^{n} (j+n-j+1) = \frac{1}{2}\sum_{j=1}^{n} (n+1) = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

□

Observation 5: sum of the first n odd numbers

For any natural number $n \geq 1$:

$$\sum_{j=1}^{n} (2j-1) = n^2 \quad \text{or equivalently} \quad \sum_{j=0}^{n-1} (2j+1) = n^2$$

Proof. We have:

$$\begin{aligned} \sum_{j=1}^{n} (2j-1) &= 2\sum_{j=1}^{n} j - \sum_{j=1}^{n} 1 \\ &= 2\left(\frac{n^2+n}{2}\right) - n = n^2 + n - n = n^2 \end{aligned}$$

$$\begin{aligned} \sum_{j=0}^{n-1} (2j+1) &= 2\sum_{j=0}^{n-1} j + \sum_{j=0}^{n-1} 1 = 2\sum_{j=1}^{n} (j-1) + \sum_{j=1}^{n} 1 \\ &= 2\left(\sum_{j=1}^{n} j - \sum_{j=1}^{n} 1\right) + n \\ &= 2\left(\frac{n^2+n}{2} - n\right) + n = n^2 + n - 2n + n = n^2 \end{aligned}$$

$\square$

Observation 6: sum of the first n even numbers (without zero)

For any natural number $n \geq 1$:

$$\sum_{j=1}^{n} 2j = n^2 + n \tag{8}$$

Proof. We have:

$$\sum_{j=1}^{n} 2j = 2\sum_{j=1}^{n} j = 2\left(\frac{n^2+n}{2}\right) = n^2 + n$$

$\square$

2 Geometric progressions

Definition 2: geometric progression

*A sequence of real numbers is in a **geometric progression** if the ratio between each term (starting from the second) and its predecessor is constant. This constant is called the **common ratio** of the progression.*

Given the first term $a \in \mathbb{R}$ and the common ratio $r \in \mathbb{R}$, the associated geometric progression is:

$$a, \; a\,r, \; a\,r^2, \; a\,r^3, \; a\,r^4, \; \ldots$$

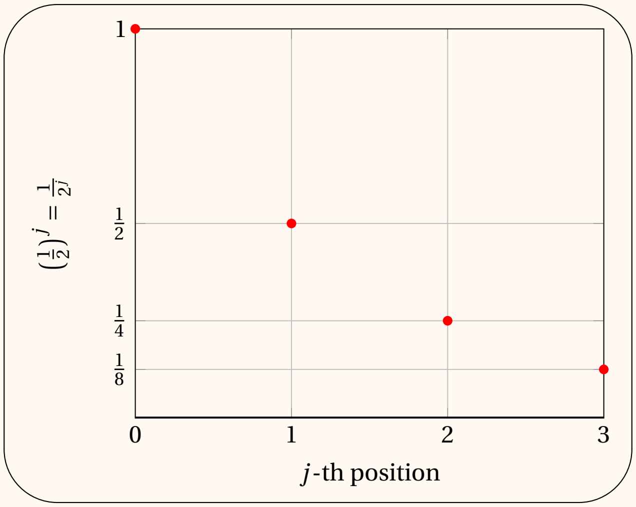
- Each term (starting from the second) is obtained by multiplying the previous term by b. The j-th term ($j \in \mathbb{N}$) can be expressed as $a\,b^j$ and we have:

$$\begin{aligned} &a\,r^0 = a && \text{first term, in position } j = 0 \\ &a\,r^1 = a\,b && \text{second term, in position } j = 1 \\ &a\,r^2 && \text{third term, in position } j = 2 \\ &a\,r^3 && \text{fourth term, in position } j = 3 \\ &\ldots && \ldots \end{aligned}$$

Example 3: geometric progression

- With $a = 1$ and $r = \frac{1}{2}$, the first 4 terms are:

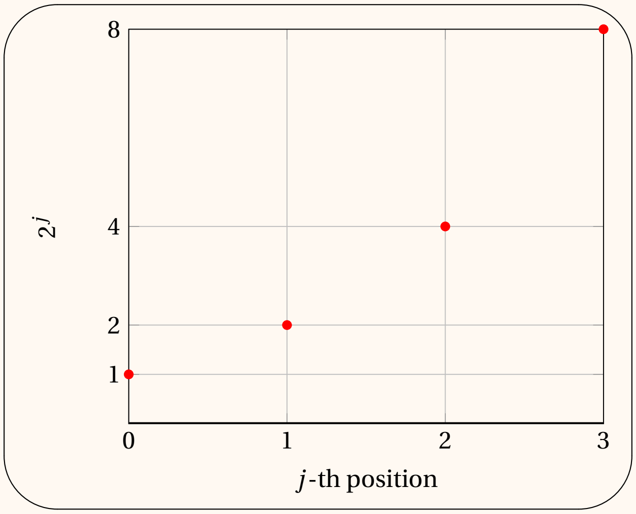
$$1, \; \frac{1}{2}, \; \frac{1}{4}, \; \frac{1}{8}$$



Example 4: geometric progression

- With $a = 1$ and $r = 2$, the first 4 terms are:

$$1,\ 2,\ 4,\ 8$$



2.1 Summations of terms in geometric progression

Observation 7: sum of the first n terms of the geometric progression with $a = 1$

Given a common ratio $r \in \mathbb{R}$ and $n \in \mathbb{N}_+$ we have:

$$\sum_{j=0}^{n-1} r^j = \begin{cases} \frac{r^n-1}{r-1} & \text{if} \quad r \neq 1 \\ n & \text{otherwise} \end{cases} \tag{9}$$

Proof. If $r \neq 1$, applying the properties of summations, we have:

$$\begin{aligned}(r-1)\sum_{j=0}^{n-1} r^j &= r\sum_{j=0}^{n-1} r^j - \sum_{j=0}^{n-1} r^j \\ &= \sum_{j=0}^{n-1} r^{j+1} - \sum_{j=0}^{n-1} r^j = \sum_{j=1}^{n} r^j - \sum_{j=0}^{n-1} r^j \\ &= \sum_{j=1}^{n-1} r^j + r^n - \left(1 + \sum_{j=1}^{n-1} r^j\right) = r^n - 1\end{aligned}$$

□

Proof. If $r = 1$, we have:

$$\sum_{j=0}^{n-1} r^j = \sum_{j=0}^{n-1} 1^j = \sum_{j=0}^{n-1} 1 = n$$

□

- Given $r \in \mathbb{R}, n \in \mathbb{N}_+$ and $a \in \mathbb{R}$, the formula (9) can be extended as follows:

$$\sum_{j=0}^{n-1} a\, r^j = \begin{cases} a\left(\frac{r^n-1}{r-1}\right) = a\left(\frac{1-r^n}{1-r}\right) & \text{if} \quad r \neq 1 \\ a\, n & \text{otherwise} \end{cases} \qquad \text{since} \qquad \sum_{j=0}^{n-1} a\, r^j = a \sum_{j=0}^{n-1} r^j \tag{10}$$

Example 5: sum of the first n terms of geometric progressions

- With $a = 1$ and $r = \frac{1}{2}$, the first 4 terms are:

$$1, \ \frac{1}{2}, \ \frac{1}{4}, \ \frac{1}{8} \qquad \text{and} \qquad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{8+4+2+1}{8} = \frac{15}{8}$$

The sum of the first $n = 4$ terms is given by the formula:

$$\sum_{j=0}^{3} \left(\frac{1}{2}\right)^j = \sum_{j=1}^{4} \frac{1}{2^j} = \frac{1 - \frac{1}{2^4}}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{16}}{\frac{1}{2}} = \frac{15}{16} 2 = \frac{15}{8}$$

- With $a = 1$ and $r = 2$, the first four terms are:

$$1, \ 2, \ 4, \ 8 \qquad \text{and} \qquad 1 + 2 + 4 + 8 = 15$$

The sum of the first $n = 4$ terms is given by the formula:

$$\sum_{j=1}^{3} 2^j = \frac{2^4 - 1}{2 - 1} = 16 - 1 = 15$$

- With $a = 1$ and $r = -2$, the first four terms are:

$$1, \ -2, \ 4, \ -8 \qquad \text{and} \qquad 1 - 2 + 4 - 8 = 5$$

The sum of the first $n = 4$ terms is given by the formula:

$$\sum_{j=1}^{3} -2^j = \frac{-2^4 - 1}{-2 - 1} = \frac{-15}{-3} = 5$$

2.2 Geometric series

Definition 3: geometric series

Given a common ratio $r \in \mathbb{R}$, the limit

$$\lim_{n\to+\infty} \sum_{j=0}^{n} r^j$$

*is called the **geometric series** of common ratio r and it is denoted by $\sum_{j=0}^{\infty} r^j$.*

Observation 8: convergence of the geometric series

Given a common ration $r \in \mathbb{R}$,

$$\text{the geometric series } \sum_{j=0}^{\infty} r^j \quad \text{is} \quad \begin{cases} \text{convergent} & \text{if } |r| < 1 \\ \text{divergent at } +\infty & \text{if } r \geq 1 \\ \text{irregular} & \text{if } r \leq -1 \end{cases}$$

If the geometric series is convergent, we have $\sum_{j=0}^{\infty} .r^j = \frac{1}{1-r}$

Proof. Given $r \in \mathbb{R}$, we have

$$\lim_{n\to+\infty} r^n = \begin{cases} +\infty & \text{if } r > 1 \\ 1 & \text{if } r = 1 \\ 0 & \text{if } |r| < 1 \\ \text{does not exist} & \text{if } r \leq -1 \end{cases}$$

If $r \neq 1$ we have $\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}$ and:

$$\lim_{n\to+\infty} \sum_{j=0}^{n} r^j = \lim_{n\to+\infty} \frac{r^{n+1}-1}{r-1} = \frac{1}{r-1} \lim_{n\to+\infty} \left(r^{n+1}-1\right) = \begin{cases} \dfrac{1}{1-r} & \text{if } |r| < 1 \\ +\infty & \text{if } r > 1 \\ \text{does not exist} & \text{if } r \leq -1 \end{cases}$$

If $r = 1$, we have $\sum_{j=0}^{n} r^j = n+1$ and accordingly :

$$\lim_{n\to+\infty} \sum_{j=0}^{n} r^j = \lim_{n\to+\infty} (n+1) = +\infty$$

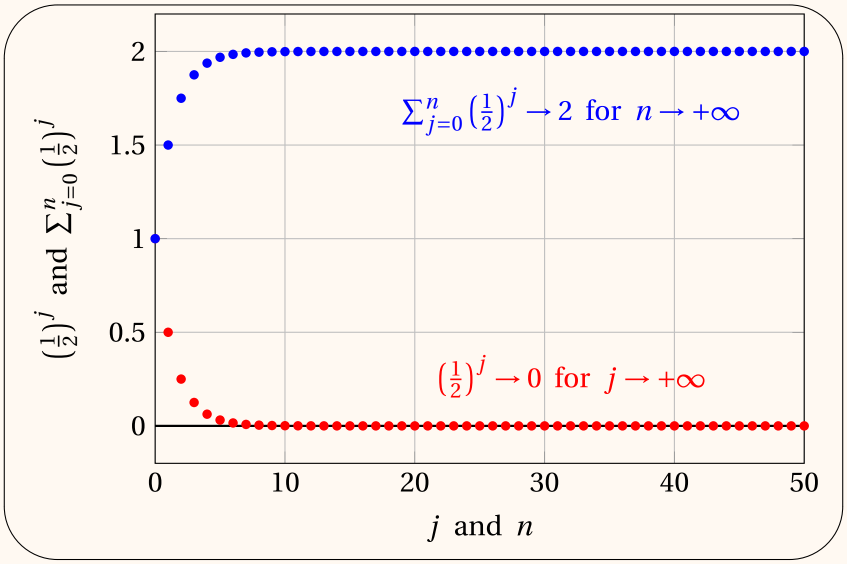
$\square$

Example 6: geometric series

For the following geometric series we have:

$$\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^j = \frac{1}{1-\frac{1}{2}} = 2$$

since the common ration r is $1/2$ and $|1/2| < 1$.



For the following geometric series we have:

$$\sum_{k=0}^{\infty}\left(\frac{13}{12}\right)^k = +\infty$$

since the common ration r is $13/12$ and $13/12 > 1$.