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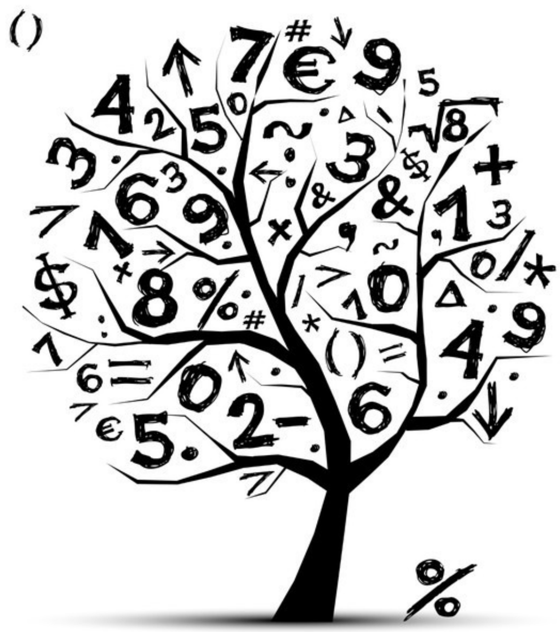
Induction principle



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1 The induction principle

- The induction principle is a technique to prove theorems with the following structure:

  " for every $n \in \mathbb{N}$, $n \geq n_0$, the property $p(n)$ holds"

- The value n_0 is the smallest integer for which the property is true; if $n_0 = 0$ the theorem simply states that the property is true for every $n \in \mathbb{N}$.

The proof by induction consists of the following two steps:

1. Show that $p(n)$ is true for $n = n_0$ (**first step of induction**)
2. Show that, for a generic natural number n greater than or equal to n_0, from the fact that $p(n)$ is true it follows that $p(n+1)$ is true (**inductive step**).

We can then conclude that for any $n \geq n_0$, $p(n)$ is true.

- The validity of this demonstration method is based on this fact:

  1. By point 1, we know that $p(n_0)$ is true. Without loss of generality, suppose $n_0 = 1$: therefore we know that $p(1)$ is true (this must be proved explicitly).
  2. For the point 2, since $p(1)$ is true, $p(2)$ is true (we have proved that whatever n is, if $p(n)$ is true $p(n+1)$ is also true). But then, since $p(2)$ is true, then $p(3)$ is true. Since $p(3)$ is true then $p(4)$ is true and so on ... so $p(n)$ is true for each $n \geq 1$.

- Summarizing, the proofs by induction requires these two phases:

  1. Prove directly $p(n_0)$;
  2. Take as hypothesis $p(n)$ (**inductive hypothesis**) and prove that $p(n+1)$ is true.

The logical structure of the proofs by induction is as follows:

" for every $n \in \mathbb{N}$, $n \geq n_0$, if $p(n)$ is true then also $p(n+1)$ is true"

2 Examples of proofs by induction

2.1 Summations of the terms in geometric progressions

Observation 1: sum of the first n terms of the geometric progression with $a = 1$

Given a common ratio $r \in \mathbb{R}$ and $n \in \mathbb{N}_+$ we have:

$$\sum_{j=0}^{n-1} r^j = \begin{cases} \frac{r^n-1}{r-1} & \text{if} \quad r \neq 1 \\ n & \text{otherwise} \end{cases} \tag{1}$$

Proof. By induction on n with $r \neq 1$.

- **First step of the induction**

  Let $n = 1$. Then the statement becomes:

  $$\sum_{j=0}^{0} r^j = \frac{r^1 - 1}{r - 1} \quad \text{that is} \quad 1 = 1$$

  which is evidently true.

- **Inductive step**

  Suppose it is true for n, and we prove it for $n+1$. By inductive hypothesis, we have:

  $$\sum_{j=0}^{n-1} r^j = \frac{r^n - 1}{r - 1}$$

  Then we can write:

  $$\sum_{j=0}^{n} r^j = \sum_{j=0}^{n-1} r^j + r^n = \frac{r^n - 1}{r - 1} + r^n$$

  $$= \frac{r^n - 1 + r^{n+1} - r^n}{r - 1} = \frac{r^{n+1} - 1}{r - 1}$$

  which is exactly the statement for $n+1$.

If $r = 1$, we have:

$$\sum_{j=0}^{n-1} r^j = \sum_{j=0}^{n-1} 1^j = \sum_{j=0}^{n-1} 1 = n$$

□

- If $a \in \mathbb{R}$ and $r \neq 1$, we have:

$$\sum_{j=0}^{n-1} a\, r^j = a \sum_{j=0}^{n-1} r^j = a\left(\frac{r^n - 1}{r-1}\right) = a\left(\frac{1-r^n}{1-r}\right) \tag{2}$$

2.2 Some important summations

Observation 2: sum of the first n natural numbers (without zero)

For every natural numbers $n \geq 1$, it holds:

$$\sum_{j=1}^{n} j = \frac{n^2+n}{2}$$

Proof. By induction on n.

- **First step of the induction**

  Let $n = 1$. Then the statement becomes:

  $$\sum_{j=1}^{1} j = \frac{1+1}{2} \quad \text{that is} \quad 1 = 1$$

  which is evidently true.

- **Inductive step**

  Suppose it is true for n, and we prove it for $(n+1)$. By inductive hypothesis, we have:

  $$\sum_{j=1}^{n} j = \frac{n^2+n}{2}$$

  Then we can write:

  $$\sum_{j=1}^{n+1} j = \sum_{j=1}^{n} j + n + 1 = \frac{n^2+n}{2} + n + 1$$

  $$= \frac{n^2+n+2\,(n+1)}{2} = \frac{n^2+2\,n+1+n+1}{2} = \frac{(n+1)^2+n+1}{2}$$

  which is exactly the statement for $n+1$.

$\square$

Observation 3: sum of the first n odd numbers

For every natural numbers $n \geq 1$, it holds:

$$\sum_{j=0}^{n-1} (2\,j+1) = n^2 \quad \text{or equivalently} \quad \sum_{j=1}^{n} (2\,j-1) = n^2$$

Proof. By induction on n.

- **First step of the induction**

  Let $n = 1$. Then the statement becomes:

  $$\sum_{k=0}^{0} (2\,j+1) = 1^2 \quad \text{that is} \quad 1 = 1$$

  which is evidently true.

- **Inductive step**

  Suppose it is true for n, and we prove it for $(n+1)$. By inductive hypothesis, we have:

  $$\sum_{k=0}^{n-1} (2\,j+1) = n^2$$

  Then we can write:

  $$\sum_{j=0}^{n-1+1} (2\,j+1) = \sum_{j=0}^{n-1} (2\,j+1) + 2\,n + 1 = n^2 + 2\,n + 1 = (n+1)^2$$

  which is exactly the statement for $n+1$.

To prove the second summation it is sufficient to observe, for example, that is it equivalent to the first one (obtained by an index translation):

$$\sum_{j=1}^{n} (2\,j-1) = \sum_{j=0}^{n-1} \big(2\,(j-1)-1\big) = \sum_{j=0}^{n-1} (2\,j+1)$$

$\square$

Observation 4: sum of the first n even numbers (without zero)

For every natural numbers $n \geq 1$, it holds:

$$\sum_{j=1}^{n} 2\,j = n^2 + n$$

Proof. By induction on n.

- **First step of the induction**

  Let $n = 1$. Then the statement becomes:

  $$\sum_{j=1}^{1} 2\,j = 1 + 1 \quad \text{that is} \quad 2 = 2$$

  which is evidently true.

- **Inductive step**

  Suppose it is true for n, and we prove it for $(n+1)$. By inductive hypothesis, we have:

  $$\sum_{j=1}^{n} 2\,j = n^2 + n$$

  Then we can write:

  $$\sum_{j=1}^{n+1} 2\,j = \sum_{j=1}^{n} 2\,j + 2\,(n+1) = 2\left(\sum_{j=1}^{n} j + n + 1\right)$$

  $$= 2\left(\frac{n^2+n}{2} + n + 1\right) = 2\left(\frac{n^2+n+2\,(n+1)}{2}\right)$$

  $$= n^2 + 2\,n + 1 + n + 1 = (n+1)^2 + n + 1$$

  which is exactly the statement for $n+1$.

□

2.3 Bernoulli's inequality

Observation 5: Bernoulli's inequality

For every natural numbers $n \geq 0$ and real numbers $x \in \mathbb{R}$, $x \geq -1$, it holds:

$$(1+x)^n \geq 1+n\,x \tag{3}$$

Proof. By induction on n.

- **First step of the induction**

  Let $n=0$. Then the statement becomes:

  $$(1+x)^0 \geq 1+0\,x \quad \text{that is} \quad 1 \geq 1$$

  which is evidently true.

- **Inductive step**

  Suppose it is true for n, and we prove it for $(n+1)$. By inductive hypothesis, we have:

  $$(1+x)^n \geq 1+n\,x$$

  and also we have

  $$(1+x) \geq 0 \quad \text{since} \quad x \geq -1.$$

  Then we can write:

  $$\begin{aligned}(1+x)^{n+1} &= (1+x)\cdot(1+x)^n \\ &\geq (1+x)\cdot(1+n\,x) \\ &= 1+(n+1)\,x+n\,x^2 \\ &\geq 1+(n+1)\,x\end{aligned}$$

  where in the last inequality we have exploited the fact that $n\,x^2 \geq 0$.

  The chain of inequalities show that, for $n+1$, we have

  $$(1+x)^{n+1} \geq 1+(n+1)\,x$$

  which is exactly the statement for $n+1$.

$\square$

Example 1: Bernoulli's inequality