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Convex set (2)

Given n values $a_j \in \mathbb{R}_+$, with $j \in \{1, 2, \ldots, n\}$, and a point $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$, consider the set:

$$F = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} \frac{(x_j - c_j)^2}{a_j^2} \leq 1 \right\}$$

It corresponds to the set of points inside an ellipsoid centered in $(c_1, c_2, \ldots, c_n)$ with lengths of the semi-axes given by the values a_j, with $j \in \{1, 2, \ldots, n\}$.

Questions

1. Provide the condition for the values $a_j \in \mathbb{R}_+$, with $j \in \{1, 2, \ldots, n\}$, under which F is the set of points within a ball of radius $r \in \mathbb{R}_+$ centered in $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$.
2. Prove that F is a convex set.
3. Consider $n = 2$, $a_1 = 3$, $a_2 = 2$ and the center $(c_1, c_2) = (4, 3)$, draw the set F in $\mathbb{R}^2$.

Solution

A set $F \subseteq \mathbb{R}^n$ (subset of the n-dimensional space) is **convex** if

$$\forall \boldsymbol{p}, \boldsymbol{w} \in F \text{ and } \forall \lambda \in [0,1] \text{ we have } \underbrace{\lambda\, \boldsymbol{p} + (1-\lambda)\, \boldsymbol{w}}_{\text{convex combination}} \in F$$

The Cauchy-Schwarz inequality:

$$|\boldsymbol{p} \cdot \boldsymbol{w}| \leq ||\boldsymbol{p}||\, ||\boldsymbol{w}||, \quad \forall \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n$$

1. Given a center $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, the set of points inside a ball of radius r centered in $(c_1, c_2, \ldots, c_n)$ is given by:

$$\left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} (x_j - c_j)^2 \leq r^2 \right\} = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} \frac{(x_j - c_j)^2}{r^2} \leq 1 \right\}$$

Therefore, F is a ball of radius r centered in $(c_1, c_2, \ldots, c_n)$ iff $a_j = r$ for all $j \in \{1, 2, \ldots, n\}$.

2. For any two points $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ in the set F, we have:

$$\sum_{j=1}^{n} \frac{(p_j - c_j)^2}{a_j^2} \leq 1 \quad \text{and} \quad \sum_{j=1}^{n} \frac{(w_j - c_j)^2}{a_j^2} \leq 1$$

Moreover, for any $\lambda \in [0,1]$, we have:

$$\begin{aligned}
\sum_{j=1}^{n} \frac{\left(\lambda\, p_j + (1-\lambda)\, w_j - c_j\right)^2}{a_j^2} &= \sum_{j=1}^{n} \frac{\left(\lambda\,(p_j - c_j) + (1-\lambda)\,(w_j - c_j)\right)^2}{a_j^2} \\
&= \sum_{j=1}^{n} \frac{\lambda^2\,(p_j - c_j)^2 + (1-\lambda)^2\,(w_j - c_j)^2 + 2\,\lambda\,(1-\lambda)\,(p_j - c_j)\,(w_j - c_j)}{a_j^2} \\
&= \lambda^2 \underbrace{\sum_{j=1}^{n} \frac{(p_j - c_j)^2}{a_j^2}}_{\leq 1} + (1-\lambda)^2 \underbrace{\sum_{j=1}^{n} \frac{(w_j - c_j)^2}{a_j^2}}_{\leq 1} + 2\,\lambda\,(1-\lambda) \sum_{j=1}^{n} \frac{(p_j - c_j)\,(w_j - c_j)}{a_j^2} \\
&\leq \lambda^2 + (1-\lambda)^2 + 2\,\lambda\,(1-\lambda) \sum_{j=1}^{n} \frac{(p_j - c_j)\,(w_j - c_j)}{a_j^2}
\end{aligned}$$

By the Cauchy-Schwarz inequality we have:

$$\left|\sum_{j=1}^{n} \frac{p_j - c_j}{a_j} \frac{w_j - c_j}{a_j}\right| \leq \sqrt{\underbrace{\sum_{j=1}^{n} \frac{(p_j - c_j)^2}{a_j^2}}_{\leq 1}} \sqrt{\underbrace{\sum_{j=1}^{n} \frac{(w_j - c_j)^2}{a_j^2}}_{\leq 1}} \leq 1$$

Accordingly, we can conclude:

$$\sum_{j=1}^{n} \frac{\left(\lambda\, p_j + (1-\lambda)\, w_j - c_j\right)^2}{a_j^2} \leq \lambda^2 + (1-\lambda)^2 + 2\,\lambda\,(1-\lambda) = 1$$

Then the point $\lambda\, \boldsymbol{p} + (1-\lambda)\, \boldsymbol{w}$ belong to the set and, accordingly, the set F is convex.

3. For the specific values, we have the following convex set:

$$F = \left\{ \ (x_1, x_2) \in \mathbb{R}^2 : \ \frac{(x_1 - 4)^2}{3^2} + \frac{(x_2 - 3)^2}{2^2} \ \leq \ 1 \ \right\}$$

The set F corresponds to the set of points inside an ellipse surrounding two focal points with orthogonal semi-axes of length $a_1 = 3$ and $a_2 = 2$ and centered in $\bar{\boldsymbol{x}} = (4, 3)$. They are the points shown by the red-shaded portion of the space in the following figure: