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# Convex set (1)

Given a positive semidefinite matrix $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$

$$\boldsymbol{Q} = \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1n} \\ q_{21} & q_{22} & \dots & q_{2n} \\ \vdots & \vdots & \dots & \vdots \\ q_{n1} & q_{n2} & \dots & q_{nn} \end{bmatrix}$$

and a column vector $\boldsymbol{c} \in \mathbb{R}^{n\times 1}$, consider the set:

$$F = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \quad \underbrace{||\boldsymbol{x}-\boldsymbol{c}||_{\boldsymbol{Q}}}_{=\sqrt{(\boldsymbol{x}-\boldsymbol{c})'\,\boldsymbol{Q}\,(\boldsymbol{x}-\boldsymbol{c})}} \leq 1 \right\} = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \quad \underbrace{||\boldsymbol{x}-\boldsymbol{c}||^2_{\boldsymbol{Q}}}_{=(\boldsymbol{x}-\boldsymbol{c})'\,\boldsymbol{Q}\,(\boldsymbol{x}-\boldsymbol{c})} \leq 1 \right\}$$

Since the matrix $\boldsymbol{Q}$ is positive semidefinite, we have:

$$(\boldsymbol{x}-\boldsymbol{c})'\,\boldsymbol{Q}\,(\boldsymbol{x}-\boldsymbol{c}) \geq 0, \quad \forall \boldsymbol{x} \in \mathbb{R}^n$$

## Questions

1. Prove that $F$ is a convex set.

2. With $n = 2$, consider:

$$\boldsymbol{Q} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{c} = (0,0)$$

first prove that the matrix $\boldsymbol{Q}$ is positive semidefinite then plot the set:

$$F = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \ \underbrace{||\boldsymbol{x}||^2_{\boldsymbol{Q}}}_{=\boldsymbol{x}'\,\boldsymbol{Q}\,\boldsymbol{x}} \leq 1 \right\}$$

## Solution

A set $F \subseteq \mathbb{R}^n$ (subset of the $n$-dimensional space) is **convex** if

$$\forall \boldsymbol{p}, \boldsymbol{w} \in F \text{ and } \forall \lambda \in [0,1] \quad \text{we have} \quad \underbrace{\lambda\, \boldsymbol{p} + (1-\lambda)\, \boldsymbol{w}}_{\text{convex combination}} \in F$$

The following three properties hold:

1. Absolute homogeneity:

$$||\lambda\, \boldsymbol{p}||_{\boldsymbol{Q}} = |\lambda|\, ||\boldsymbol{p}||_{\boldsymbol{Q}}, \quad \forall \lambda \in \mathbb{R}, \boldsymbol{p} \in \mathbb{R}^n$$

2. Cauchy-Schwarz inequality:

$$|\boldsymbol{p}'\, \boldsymbol{Q}\, \boldsymbol{w}| \le ||\boldsymbol{p}||_{\boldsymbol{Q}}\, ||\boldsymbol{w}||_{\boldsymbol{Q}}, \quad \forall \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n$$

3. Triangular inequality:

$$||\boldsymbol{p} + \boldsymbol{w}||_{\boldsymbol{Q}} \le ||\boldsymbol{p}||_{\boldsymbol{Q}} + ||\boldsymbol{w}||_{\boldsymbol{Q}}, \quad \forall \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n$$

1. For any two points $\boldsymbol{p}$ and $\boldsymbol{w}$ in the set, we have:

$$||\boldsymbol{p} - \boldsymbol{c}||_{\boldsymbol{Q}} \le 1 \quad \text{and} \quad ||\boldsymbol{w} - \boldsymbol{c}||_{\boldsymbol{Q}} \le 1$$

Moreover, for any $\lambda \in [0,1]$, we have:

$$\begin{aligned}
||\lambda\, \boldsymbol{p} + (1-\lambda)\, \boldsymbol{w} - \boldsymbol{c}||_{\boldsymbol{Q}} &= ||\lambda\, (\boldsymbol{p} - \boldsymbol{c}) + (1-\lambda)\, (\boldsymbol{w} - \boldsymbol{c})||_{\boldsymbol{Q}} \\
&\le ||\lambda\, (\boldsymbol{p} - \boldsymbol{c})||_{\boldsymbol{Q}} + ||(1-\lambda)\, (\boldsymbol{w} - \boldsymbol{c})||_{\boldsymbol{Q}} \quad \text{(by the traingle ineq.)} \\
&= |\lambda|\, ||\boldsymbol{p} - \boldsymbol{c}||_{\boldsymbol{Q}} + |(1-\lambda)|\, ||\boldsymbol{w} - \boldsymbol{c}||_{\boldsymbol{Q}} \quad \text{(by absolute homog.)} \\
&= \lambda\, \underbrace{||\boldsymbol{p} - \boldsymbol{c}||_{\boldsymbol{Q}}}_{\le 1} + (1-\lambda)\, \underbrace{||\boldsymbol{w} - \boldsymbol{c}||_{\boldsymbol{Q}}}_{\le 1} \quad \text{(since } \lambda \ge 0, (1-\lambda) \ge 0\text{)} \\
&\le \lambda + (1-\lambda) = 1
\end{aligned}$$

Then $\left(\lambda\, \boldsymbol{p} + (1-\lambda)\, \boldsymbol{w}\right)$ belong to the set and, accordingly, the set is convex.

2 The matrix $\boldsymbol{Q}$ is positive semidefinite since we have:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2\,x_1 - x_2 & -x_1 + 2\,x_2 & -x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= (2\,x_1 - x_2)\,x_1 + (-x_1 + 2\,x_2)\,x_2$$

$$= 2\,x_1^2 - x_2\,x_1 - x_1\,x_2 + 2\,x_2^2$$

$$= 2\,x_1^2 - 2\,x_1\,x_2 + 2\,x_2^2$$

$$= x_1^2 + x_1^2 - 2\,x_1\,x_2 + x_2^2 + x_2^2$$

$$= x_1^2 + (x_1 - x_2)^2 + x_2^2 \geq 0, \qquad \forall (x_1, x_2) \in \mathbb{R}^2$$

We have

$$\boldsymbol{x}'\,\boldsymbol{Q}\,\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\,x_1^2 - 2\,x_1\,x_2 + 2\,x_2^2$$

To plot

$$2\,x_1^2 - 2\,x_1\,x_2 + 2\,x_2^2 - 1 = 0$$

we need to express $x_2$ in function of $x_1$, so with $a = 2$, $b = -2\,x_1$ and $c = 2\,x_1^2 - 1$ we have:

$$c + b\,x_2 + a\,x_2^2 = 0 \quad \text{and} \quad x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

accordingly:

$$x_2 = \frac{-(-2\,x_1) \pm \sqrt{\left(-2\,x_1\right)^2 - 4 \cdot 2 \cdot \left(2\,x_1^2 - 1\right)}}{2 \cdot 2} = \frac{2\,x_1 \pm \sqrt{4\,x_1^2 - 16\,x_1^2 + 8}}{4}$$

$$= \frac{2\,x_1 \pm 2\sqrt{x_1^2 - 4\,x_1^2 + 2}}{4} = \frac{x_1 \pm \sqrt{2 - 3\,x_1^2}}{2}$$

Finally we need:

$$2 - 3\,x_1^2 \geq 0 \implies x_1 \in \left[-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right]$$

We can also compute some values as follows:

$$\text{when} \quad x_1 = \sqrt{\frac{2}{3}} \quad \text{then} \quad x_2 = \frac{\sqrt{\frac{2}{3}} + \sqrt{2 - 3 \cdot \frac{2}{3}}}{2} = \sqrt{\frac{2}{3}} \cdot \frac{1}{2} = \sqrt{\frac{1}{6}}$$

$$\text{when} \quad x_1 = -\sqrt{\frac{2}{3}} \quad \text{then} \quad x_2 = \frac{-\sqrt{\frac{2}{3}} + \sqrt{2 - 3 \cdot \left(-\frac{2}{3}\right)}}{2} = -\sqrt{\frac{2}{3}} \cdot \frac{1}{2} = -\sqrt{\frac{1}{6}}$$

$$\text{when} \quad x_1 = 0 \quad \text{then} \quad x_2 = \frac{0 + \sqrt{2 - 3 \cdot 0}}{2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \quad \text{or} \quad x_2 = \frac{0 - \sqrt{2 - 3 \cdot 0}}{2} = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}$$