Last update: March 1, 2024

Summations (1)

For any natural number $n \geq 1$, we have:

$$\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6} \tag{1}$$

Sum of the squares of the first n natural numbers (without zero)

Questions

1. Prove formula (1) using the properties of the summations.
2. Compute the sum of the squares of the first 10 natural numbers (without zero).

Solution

1. We start by writing $\sum_{k=0}^{n}(k+1)^3$ in two different ways:

$$1)\quad \sum_{k=0}^{n}(k+1)^3 = \sum_{k=0}^{n}(k^3+3\,k^2+3\,k+1)$$

$$= \left(\sum_{k=1}^{n}k^3+3\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+\sum_{k=1}^{n}1\right)+1$$

$$= \sum_{k=1}^{n}k^3+3\sum_{k=1}^{n}k^2+\frac{3\,(n^2+n)}{2}+n+1$$

$$2)\quad \sum_{k=0}^{n}(k+1)^3 = \sum_{k=1}^{n+1}k^3 = \sum_{k=1}^{n}k^3+(n+1)^3$$

Equating the expressions and canceling the sum with k^3 we get:

$$3\sum_{k=1}^{n}k^2+\frac{3\,(n^2+n)}{2}+n+1=(n+1)^3$$

Isolating the sum with k^2 we get:

$$3\sum_{k=1}^{n}k^2=(n+1)^3-\frac{3\,(n^2+n)}{2}-n-1$$

$$= n^3+3n^2+3n+1-\frac{3n^2+3n}{2}-n-1$$

$$= \frac{2n^3+3n^2+n}{2}$$

Then we have:

$$\sum_{k=1}^{n}k^2=\frac{2n^3+3n^2+n}{6}$$

2. The sum of squares of the first 10 natural numbers (without zero) is:

$$\sum_{k=1}^{10}k^2=1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2=\frac{2\cdot10^3+3\cdot10^2+10}{6}=385$$