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Induction (2)

For any natural number $n \geq 1$, we have:

$$\sum_{k=1}^{n} k^3 = \frac{n^4 + 2\,n^3 + n^2}{4} \tag{1}$$

Sum of the cubes of the first n natural numbers (without zero)

Questions

1. Prove formula (1) using the induction principle.
2. Compute the sum of the squares of the first 10 natural numbers (without zero).

Solution

1. By induction on n.

   - **First step of the induction**
     Let $n = 1$. Then the statement becomes:

     $$\sum_{k=1}^{1} k^2 = \frac{1+2+1}{4} \quad \text{that is} \quad 1 = 1$$

     which is evidently true.

   - **Inductive step**
     Suppose it is true for n, and we prove it for $(n+1)$. By inductive hypothesis, we have:

     $$\sum_{k=1}^{n} k^3 = \frac{n^4 + 2\,n^3 + n^2}{4}$$

     Then we can write:

     $$\begin{aligned}
     \sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 \\
     &= \frac{n^4 + 2\,n^3 + n^2}{4} + (n+1)^3 = \frac{n^4 + 2\,n^3 + n^2 + 4\,(n+1)^3}{4} \\
     &= \frac{n^4 + 2\,n^3 + n^2 + 4n^3 + 12n^2 + 12n + 4}{4} = \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4} \\
     &= \frac{(n^4 + 4n^3 + 6n^2 + 4n + 1) + 2(n^3 + 3n^2 + 3n + 1) + (n^2 + 2n + 1)}{4} \\
     &= \frac{(n+1)^4 + 2(n+1)^3 + (n+1)^2}{4}
     \end{aligned}$$

     which is exactly the statement for $n+1$.

2. The sum of the cubes of the first n natural numbers equals the square of the sum of the first n natural numbers since we have:

   $$\sum_{k=1}^{n} k^3 = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2\,(n^2 + 2n + 1)}{4} = \frac{n^2\,(n+1)^2}{4} = \left(\frac{n\,(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2$$