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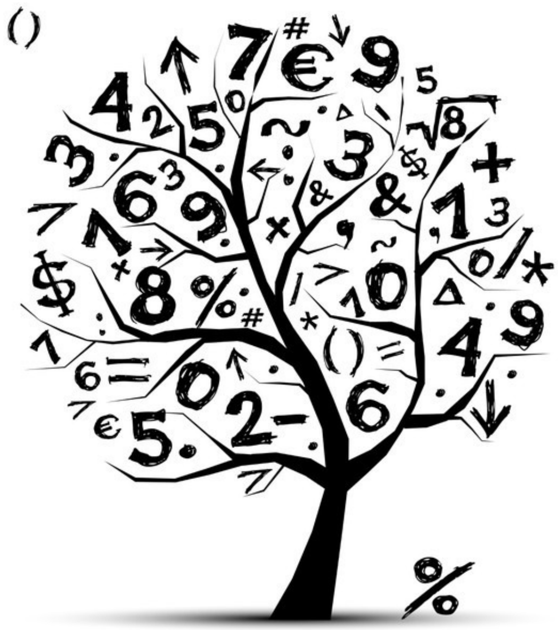
Scalar linear functions, half-spaces and polyhedra



- **Author**: Fabio Furini

Contents

1 Scalar linear functions

Definition 1: scalar linear function

Given n values $c_j \in \mathbb{R}$ with $j \in \{1,2,\ldots,n\}$, the associated ***scalar linear function*** *is:*

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f : \underbrace{(x_1, x_2, \ldots, x_n)}_{=\boldsymbol{x}} \mapsto \sum_{j=1}^{n} c_j \, x_j$$

- The n values c_j with $j \in \{1,2\ldots,n\}$ can be represented by a column vector $\boldsymbol{c} \in \mathbb{R}^{n\times 1}$ of n rows. The n **variables** x_j with $j \in \{1,2\ldots,n\}$ can also be represented by a column vector $\boldsymbol{x}$ of n rows. We can then use the following **matrix notation**:

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad\qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

  Accordingly any scalar linear function f can be equivalently rewritten in **matrix notation** as follows:

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f : \boldsymbol{x} \mapsto \boldsymbol{c}' \, \boldsymbol{x}$$

Observation 1

Given a column vector $\boldsymbol{c} \in \mathbb{R}^{n\times 1}$, the associated scalar linear function $f : \mathbb{R}^n \to \mathbb{R}$, $f : \boldsymbol{x} \mapsto \boldsymbol{c}' \, \boldsymbol{x}$ is convex and concave.

Proof. For any two points $\boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n$ and any $\lambda \in [0,1]$ we have:

$$\boldsymbol{c}' \left(\lambda \, \boldsymbol{p} + (1-\lambda) \, \boldsymbol{w}\right) \; = \; \lambda \, \boldsymbol{c}' \, \boldsymbol{p} \; + \; (1-\lambda) \, \boldsymbol{c}' \, \boldsymbol{w}$$

So we have $f\left(\lambda \, \boldsymbol{p} + (1-\lambda) \, \boldsymbol{w}\right) = \lambda \, f(\boldsymbol{p}) + (1-\lambda) \, f(\boldsymbol{w})$ for all points $\boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n$ and the domain $\mathbb{R}^n$ is convex, then the function is convex and concave. □

The **gradient** (vector of the partial derivatives) of a scalar linear function f is:

$$\nabla f(\boldsymbol{x}) = \left[\frac{\partial f(\boldsymbol{x})}{\partial x_1} \;\; \frac{\partial f(\boldsymbol{x})}{\partial x_2} \;\; \ldots \;\; \frac{\partial f(\boldsymbol{x})}{\partial x_n} \right] = \boldsymbol{c}'$$

The **contours lines** of the scalar linear function are the hyperplanes perpendicular to the gradient (the sets of points in which the function is **constant**):

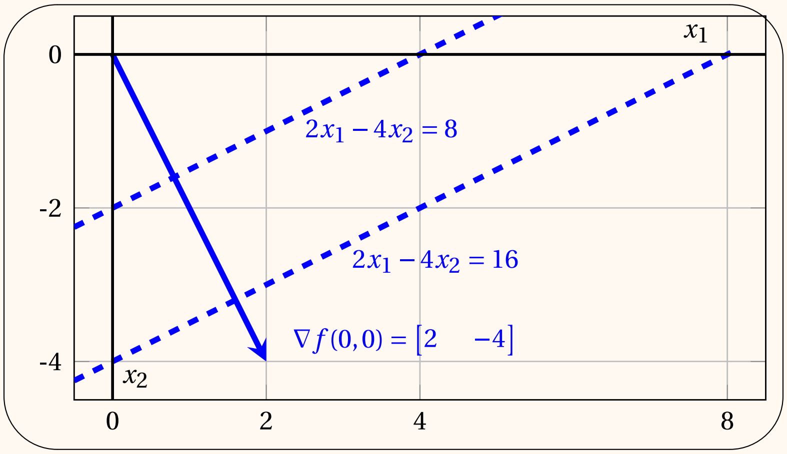
$$\boldsymbol{c}' \, \boldsymbol{x} = k, \quad k \in \mathbb{R}$$

Example 1: gradient and contour lines of scalar linear functions

Consider for example the vector $\boldsymbol{c}' = \underbrace{\begin{bmatrix} 2 & -4 \end{bmatrix}}_{c_1 \quad c_2}$ with $n = 2$. The associated scalar linear function is:

$$f(x_1, x_2) = \begin{bmatrix} 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1 - 4x_2, \text{ its gradient is } \nabla f(x_1, x_2) = \begin{bmatrix} \dfrac{\partial f(x_1, x_2)}{\partial x_1} & \dfrac{\partial f(x_1, x_2)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2 & -4 \end{bmatrix}$$

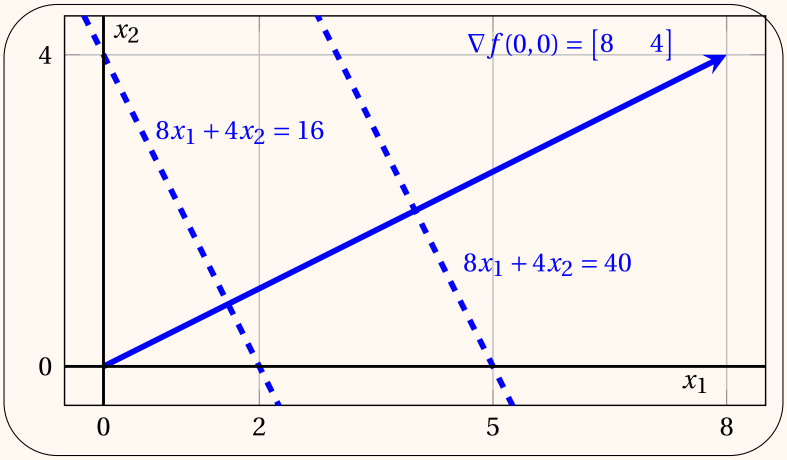
The gradient in $(0, 0)$ and two contour lines of the scalar linear function (dashed blue lines):



Consider for example the vector $\boldsymbol{c}' = \underbrace{\begin{bmatrix} 8 & 4 \end{bmatrix}}_{c_1 \quad c_2}$ with $n = 2$. The associated scalar linear function is:

$$f(x_1, x_2) = \begin{bmatrix} 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8\,x_1 + 4\,x_2 \text{ its gradient is } \nabla f(x_1, x_2) = \begin{bmatrix} \dfrac{\partial f(x_1, x_2)}{\partial x_1} & \dfrac{\partial f(x_1, x_2)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 8 & 4 \end{bmatrix}$$

The gradient in $(0, 0)$ and two contour lines of the scalar linear function (dashed blue lines):



2 Half-spaces and supporting hyperplanes

Definition 2: half-space and supporting hyperplane

Given n values $a_j \in \mathbb{R}$ with $j \in \{1,2,\dots,n\}$ and a value $b \in \mathbb{R}$,

$$\text{the set } \left\{ \underbrace{(x_1, x_2, \dots, x_n)}_{=\boldsymbol{x}} \in \mathbb{R}^n : \ \sum_{j=1}^{n} a_j\, x_j \le b \right\} \text{ is called an } \textbf{\textit{half-space}}$$

$$\text{and the set } \left\{ \underbrace{(x_1, x_2, \dots, x_n)}_{=\boldsymbol{x}} \in \mathbb{R}^n : \ \sum_{j=1}^{n} a_j\, x_j = b \right\} \text{ is its } \textbf{\textit{supporting hyperplane}}.$$

- The n **coefficients** a_j with $j \in \{1,2,\dots,n\}$ can be represented by a row vector $\boldsymbol{a} \in \mathbb{R}^{1\times n}$ of n columns:

$$\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$$

  Accordingly an half-space and its supporting hyperplane can be equivalently rewritten in **matrix notation** as follows:

$$\left\{\boldsymbol{x} \in \mathbb{R}^n : \ \boldsymbol{a}\,\boldsymbol{x} \le b\right\} \quad \text{and} \quad \left\{\boldsymbol{x} \in \mathbb{R}^n : \ \boldsymbol{a}\,\boldsymbol{x} = b\right\}$$

Observation 2

Given a row vector $\boldsymbol{a} \in \mathbb{R}^{1\times n}$ and $b \in \mathbb{R}$, the associated half-space $\left\{\boldsymbol{x} \in \mathbb{R}^n : \ \boldsymbol{a}\,\boldsymbol{x} \le b\right\}$ is a convex set.

Proof. For any two points $\boldsymbol{p}$ and $\boldsymbol{w}$ in a half-space, we have:

$$\boldsymbol{a}\,\boldsymbol{p} \le b \quad \text{and} \quad \boldsymbol{a}\,\boldsymbol{w} \le b$$

Moreover, for any $\lambda \in [0,1]$, we have:

$$\boldsymbol{a}\left(\lambda\,\boldsymbol{p} + (1-\lambda)\,\boldsymbol{w}\right) \ = \ \lambda\, \underbrace{\boldsymbol{a}\,\boldsymbol{p}}_{\le b} + (1-\lambda)\, \underbrace{\boldsymbol{a}\,\boldsymbol{w}}_{\le b} \ \le \ \lambda\, b + (1-\lambda)\, b = b$$

Then the point $\left(\lambda\,\boldsymbol{p} + (1-\lambda)\,\boldsymbol{w}\right)$ belongs to the half-space and, accordingly, the half-space is convex. □

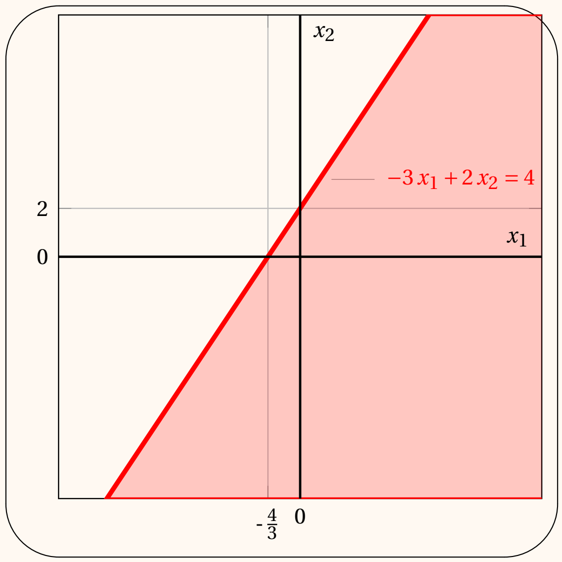
- Any supporting hyperplane is clearly a convex set.

Example 2: half-space and supporting hyper-plane

With $n = 2$, $a_1 = -3$, $a_2 = 2$ and $b = 4$, let us consider the following half-space in $\mathbb{R}^2$:

$$-3\,x_1 + 2\,x_2 \leq 4 \quad \text{where} \quad \boldsymbol{a} = \begin{bmatrix} -3 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

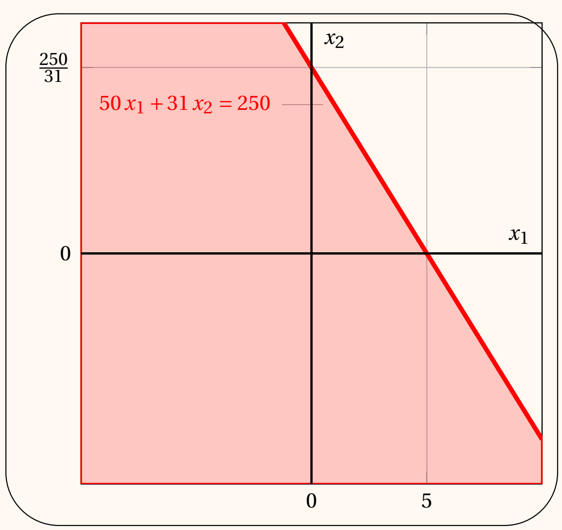
Its supporting hyper-plane is $-3\,x_1 + 2\,x_2 = 4$. The shaded part of the figure is the half-space and the red line is its supporting hyperplane:



With $n = 2$, $a_1 = 50$, $a_2 = 31$ and $b = 250$, let us consider the following half-space in $\mathbb{R}^2$:

$$50\,x_1 + 31\,x_2 \leq 250 \quad \text{where} \quad \boldsymbol{a} = \begin{bmatrix} 50 & 31 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Its supporting hyper-plane is $50\,x_1 + 31\,x_2 = 250$. The shaded part of the figure is the half-space and the red line is its supporting hyperplane:



3 Polyhedra

Definition 3: of polyhedron

Given $m \cdot n$ values $a_{ij} \in \mathbb{R}$ with $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$ and m values $b_i \in \mathbb{R}$ with $i \in \{1, 2, \ldots, m\}$, the set:

$$\left\{ \underbrace{(x_1, x_2, \ldots, x_n)}_{= \boldsymbol{x}} \in \mathbb{R}^n : \quad \sum_{j=1}^{n} a_{ij}\, x_j \le b_i, \quad \forall i \in \{1, 2, \ldots, m\} \right\}$$

*is called a **polyhedron**.*

A polyhedron is given by the intersection of half-spaces. If a polyhedron is bounded then it is also called a **polytope**.

- For a given $i \in \{1, 2, \ldots, m\}$, the n **coefficients** a_{ij} with $j \in \{1, 2, \ldots, n\}$ can be represented by a row vector $\boldsymbol{a}_i \in \mathbb{R}^{1 \times n}$ of n columns:

  $$\boldsymbol{a}_i = \begin{bmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{bmatrix}$$

  Accordingly a polyhedron can be equivalently rewritten in **matrix notation** as follows:

  $$\left\{ \ \boldsymbol{x} \in \mathbb{R}^n : \quad \boldsymbol{a}_i\, \boldsymbol{x} \le b_i, \quad \forall i \in \{1, 2, \ldots, m\} \ \right\}$$

- The $m \cdot n$ **coefficients** a_{ij} with $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$ can be represented by a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ of m rows and n columns and the m **coefficients** b_i with $i \in \{1, 2, \ldots, m\}$ can be represented by a column vector $\boldsymbol{b} \in \mathbb{R}^{m \times 1}$ of m rows:

  $$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}$$

  Accordingly a polyhedron can be equivalently rewritten in matrix notation as follows:

  $$\left\{ \boldsymbol{x} \in \mathbb{R}^n : \ \boldsymbol{A}\, \boldsymbol{x} \le b \right\}$$

Observation 3

Given a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, the associated polyhedron $\left\{ \boldsymbol{x} \in \mathbb{R}^n : \ \boldsymbol{A}\, \boldsymbol{x} \le b \right\}$ is a convex set.

Proof. A polyhedron is a convex set since it given by the intersection of half-spaces which are convex sets. □

Example 3: polyhedron/polytope

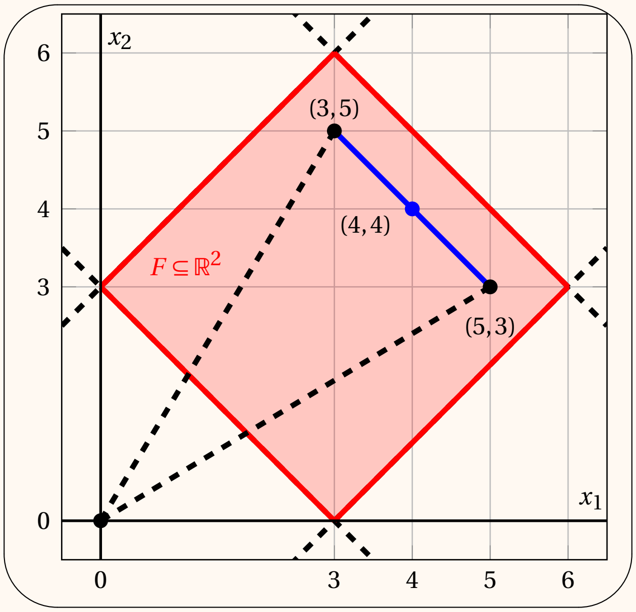
Consider the following polyhedron/polytope given by the intersection of four half-spaces:

$$F = \{ \ (x_1, x_2) \in \mathbb{R}^2 : \quad x_1 + x_2 \geq 3, \quad x_1 + x_2 \leq 9, \quad - x_1 + x_2 \leq 3, \quad - x_1 + x_2 \geq -3 \ \}$$

We consider, for example, two points $(3,5)$ and $(5,3)$ in the set F. Since the set is convex, all points on blue segment of the figure:

$$\big(\lambda\, 3 + (1-\lambda)\, 5 \ , \ \lambda\, 5 + (1-\lambda)\, 3\big) \quad \text{with} \quad \lambda \in [0,1]$$

belong to the set F. These points are all possible convex combinations of the two points..



For example with $\lambda = \frac{1}{2}$, the convex combination of the two points is the point $\big(\frac{1}{2}\, 3 + \big(1-\frac{1}{2}\big)\, 5 \ , \ \frac{1}{2}\, 5 + \big(1-\frac{1}{2}\big)\, 3\big) = (4,4)$ (shown in blue).

- It is worth noticing that if a polyhedron contains the following two half spaces:

$$\{\boldsymbol{x} \in \mathbb{R}^n : \ \boldsymbol{a}\, \boldsymbol{x} \leq b\} \quad \text{and} \quad \{\boldsymbol{x} \in \mathbb{R}^n : \ \boldsymbol{a}\, \boldsymbol{x} \geq b\}$$

it contains, *de facto*, the following hyperplanes:

$$\{\boldsymbol{x} \in \mathbb{R}^n : \ \boldsymbol{a}\, \boldsymbol{x} = b\}$$

Accordingly a polyhedron can be also given by the intersection of half-spaces and hyperplanes.

Example 4: polyhedron/polytope

Consider the following polyhedron/polytope given by the intersection of three half-spaces and one hyperplane:

$$F = \Big\{ \ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, \ x_2 \geq 0 \ x_3 \geq 0, \\ 2\,x_1 + 3\,x_2 + x_3 = 4 \ \Big\}$$

Its 3D picture is the hyperplane of the figure.