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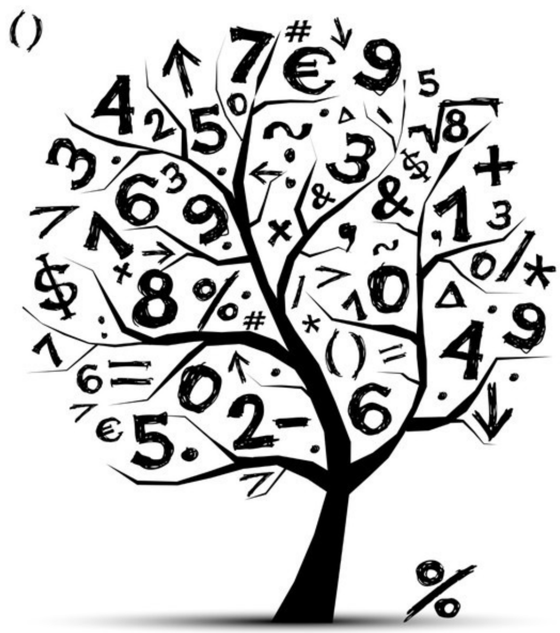
System of linear equations



- **Author**: Fabio Furini

Contents

1 Cramer's rule

Cramer's rule is an explicit formula for the solution of a system of m linear equations with m variables (valid whenever the system has a unique solution).

- Given a column vector $\boldsymbol{b} \in \mathbb{R}^{m\times 1}$ of m rows, a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ of m rows and m columns and a column vector $\boldsymbol{x} \in \mathbb{R}^{m}$ of m rows containing the m variables:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

if $\det(\boldsymbol{A}) \neq 0$, the solution of the system of m linear equations:

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$$

is given by the formula:

$$x_j = \frac{\det(\boldsymbol{A}_j)}{\det(\boldsymbol{A})}, \qquad \forall j \in \{1, 2, \dots, m\} \tag{1}$$

where $\boldsymbol{A}_j$ is the matrix formed by replacing the j-th column of $\boldsymbol{A}$ by the column vector $\boldsymbol{b}$.

1.1 Systems of two equations and two variables

With $m = 2$, we have:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \boldsymbol{A}_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix} \qquad \boldsymbol{A}_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$

and

$$\begin{aligned} \det(\boldsymbol{A}) &= a_{11}\,a_{22} - a_{12}\,a_{21} \\ \det(\boldsymbol{A}_1) &= b_1\,a_{22} - a_{12}\,b_2 \\ \det(\boldsymbol{A}_2) &= a_{11}\,b_2 - b_1\,a_{21} \end{aligned}$$

If $\det(\boldsymbol{A}) \neq 0$, we then have:

$$\begin{cases} a_{11}\,x_1 + a_{12}\,x_2 &= b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 &= b_2 \end{cases} \implies (x_1, x_2) = \left(\frac{\det(\boldsymbol{A}_1)}{\det(\boldsymbol{A})}, \frac{\det(\boldsymbol{A}_2)}{\det(\boldsymbol{A})} \right)$$

Example 1: solution of a systems of two equations and two variables

- Given

$$A = \begin{bmatrix} -1 & 1 \\ 8 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 19 \end{bmatrix}$$

we have

$$A_1 = \begin{bmatrix} 2 & 1 \\ 19 & 2 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 2 \\ 8 & 19 \end{bmatrix}$$

and

$$\det(A) = (-1)\cdot 2 - 1\cdot 8 = -10$$
$$\det(A_1) = 2\cdot 2 - 1\cdot 19 = -15$$
$$\det(A_2) = (-1)\cdot 19 - 2\cdot 8 = -35$$

Since $\det(A) \neq 0$, we then have:

$$\begin{cases} -x_1 & + & x_2 & = & 2 \\ 8\,x_1 & + & 2\,x_2 & = & 19 \end{cases} \implies (x_1, x_2) = \left(\frac{-15}{-10}, \frac{-35}{-10}\right) = \left(\frac{3}{2}, \frac{7}{2}\right)$$

- Given

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

we have

$$A_1 = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}$$

and

$$\det(A) = (-1)\cdot(-1) - (-1)\cdot 1 = 2$$
$$\det(A_1) = (-2)\cdot(-1) - (-1)\cdot 0 = 2$$
$$\det(A_2) = (-1)\cdot 0 - (-2)\cdot 1 = 2$$

Since $\det(A) \neq 0$, we then have:

$$\begin{cases} -x_1 & - & x_2 & = & -2 \\ x_1 & - & x_2 & = & 0 \end{cases} \implies (x_1, x_2) = \left(\frac{2}{2}, \frac{2}{2}\right) = (1, 1)$$

1.2 Systems of three equations and three variables

With $m = 3$, we have:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\boldsymbol{A}_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} \qquad \boldsymbol{A}_2 = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} \qquad \boldsymbol{A}_3 = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

and

$$\det(\boldsymbol{A}) = a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{11}\, a_{23}\, a_{32} - a_{12}\, a_{21}\, a_{33} - a_{13}\, a_{22}\, a_{31}$$

$$\begin{aligned}
\det(\boldsymbol{A}_1) &= b_1\, a_{22}\, a_{33} + a_{12}\, a_{23}\, b_3 + a_{13}\, b_2\, a_{32} - b_1\, a_{23}\, a_{32} + a_{12}\, b_2\, a_{33} + a_{13}\, a_{22}\, b_3 \\
\det(\boldsymbol{A}_2) &= a_{11}\, b_2\, a_{33} + b_1\, a_{23}\, a_{31} + a_{13}\, a_{21}\, b_3 - a_{11}\, a_{23}\, b_3 - b_1\, a_{21}\, a_{33} - a_{13}\, b_2\, a_{31} \\
\det(\boldsymbol{A}_3) &= a_{11}\, a_{22}\, b_3 + a_{12}\, b_2\, a_{31} + b_1\, a_{21}\, a_{32} - a_{11}\, b_2\, a_{32} - a_{12}\, a_{21}\, b_3 - b_1\, a_{22}\, a_{31}
\end{aligned}$$

If $\det(\boldsymbol{A}) \neq 0$, we then have:

$$\left\{ \begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 &= b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 &= b_2 \\ a_{31}\, x_1 + a_{32}\, x_2 + a_{33}\, x_3 &= b_3 \end{aligned} \right. \implies (x_1, x_2, x_3) = \left(\frac{\det(\boldsymbol{A}_1)}{\det(\boldsymbol{A})}, \frac{\det(\boldsymbol{A}_2)}{\det(\boldsymbol{A})}, \frac{\det(\boldsymbol{A}_3)}{\det(\boldsymbol{A})} \right)$$