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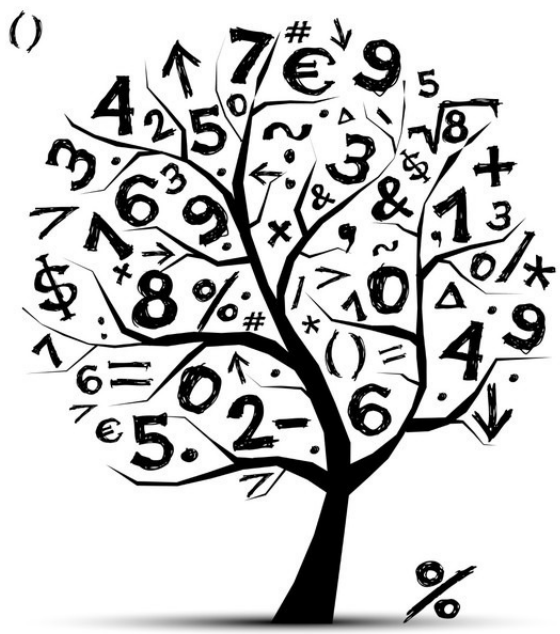
Convex sets and convex functions



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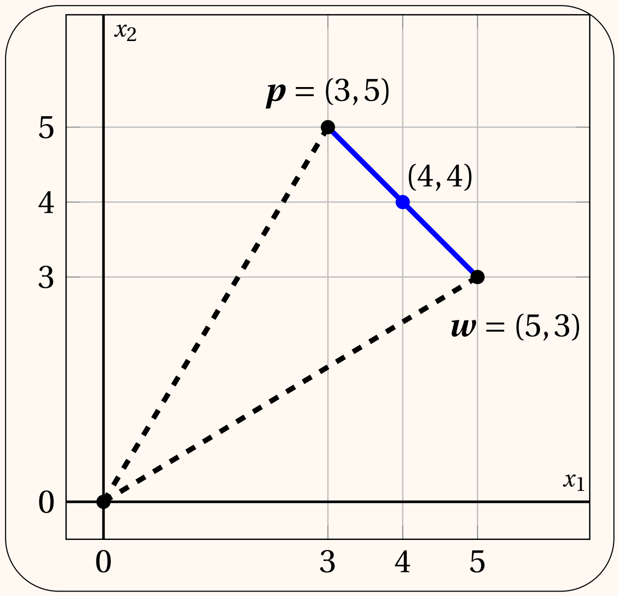
1 Convex sets

Example 1: convex combinations

Let us consider for example two points: $\boldsymbol{p} = (3,5)$ and $\boldsymbol{w} = (5,3)$ in $\mathbb{R}^2$, for any $\lambda \in [0,1]$ their convex combination is:

$$\big(\lambda \cdot 3 + (1-\lambda)\cdot 5\ ,\ \lambda \cdot 5 + (1-\lambda)\cdot 3\big)$$

which are the points on the blue segment of the following figure.



If we consider for example $\lambda = \frac{1}{2}$, we have the point shown in blue in the figure:

$$\left(\frac{1}{2}\,3 + \left(1-\frac{1}{2}\right)5\ ,\ \frac{1}{2}\,5 + \left(1-\frac{1}{2}\right)3\right) = (4,4)$$

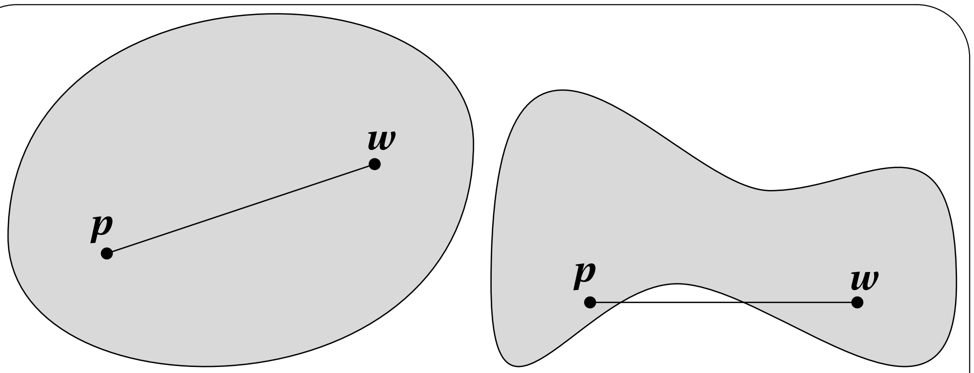
Definition 1: of convex set

*A set $F \subseteq \mathbb{R}^n$ (subset of the n-dimensional space) is **convex** if*

$$\forall \boldsymbol{p}, \boldsymbol{w} \in F \ \text{ and } \ \forall \lambda \in [0,1] \quad \text{we have} \quad \underbrace{\lambda\, \boldsymbol{p} + (1-\lambda)\, \boldsymbol{w}}_{\text{convex combination}} \in F$$

A set is convex if for any of its two points, any convex combination of the two points belongs to the set. In other words, if all points on the segment that joins any two points of the set belong to the set.

- In $\mathbb{R}^2$ (the 2-dimensional space), an example of a convex set and an example of a non-convex set are given by the following picture:



Convex set   Non-convex set

- The set of points in $\mathbb{R}^n$ inside a ball (a circle in $\mathbb{R}^2$ and a sphere in $\mathbb{R}^3$) of radius $r \in \mathbb{R}_+$ centered in $\boldsymbol{c} \in \mathbb{R}^n$ is a convex set as stated by the following observation:

Observation 1

Given $\boldsymbol{c} \in \mathbb{R}^n$ *and* $r \in \mathbb{R}_+$, *the set* $\left\{ \boldsymbol{x} \in \mathbb{R}^n : \underbrace{||\boldsymbol{x} - \boldsymbol{c}||}_{=\sqrt{\sum_{j=1}^n (x_j - c_j)^2}} \leq r \right\}$ *is convex.*

Proof. For any two points $\boldsymbol{p}$ and $\boldsymbol{w}$ in the set, we have:

$$||\boldsymbol{p} - \boldsymbol{c}|| \leq r \quad \text{and} \quad ||\boldsymbol{w} - \boldsymbol{c}|| \leq r$$

Moreover, for any $\lambda \in [0, 1]$, we have:

$$\begin{aligned}
||\lambda\, \boldsymbol{p} + (1-\lambda)\, \boldsymbol{w} - \boldsymbol{c}|| &= ||\lambda\, (\boldsymbol{p} - \boldsymbol{c}) + (1-\lambda)\, (\boldsymbol{w} - \boldsymbol{c})|| \\
&\leq ||\lambda\, (\boldsymbol{p} - \boldsymbol{c})|| + ||(1-\lambda)\, (\boldsymbol{w} - \boldsymbol{c})|| \quad \text{(by the traingle ineq.)} \\
&= |\lambda|\; ||\boldsymbol{p} - \boldsymbol{c}|| + |(1-\lambda)|\; ||\boldsymbol{w} - \boldsymbol{c}|| \quad \text{(by the absolute homog.)} \\
&= \lambda\, \underbrace{||\boldsymbol{p} - \boldsymbol{c}||}_{\leq r} + (1-\lambda)\, \underbrace{||\boldsymbol{w} - \boldsymbol{c}||}_{\leq r} \quad (\text{since } \lambda \geq 0, (1-\lambda) \geq 0) \\
&\leq \lambda\, r + (1-\lambda)\, r = r
\end{aligned}$$

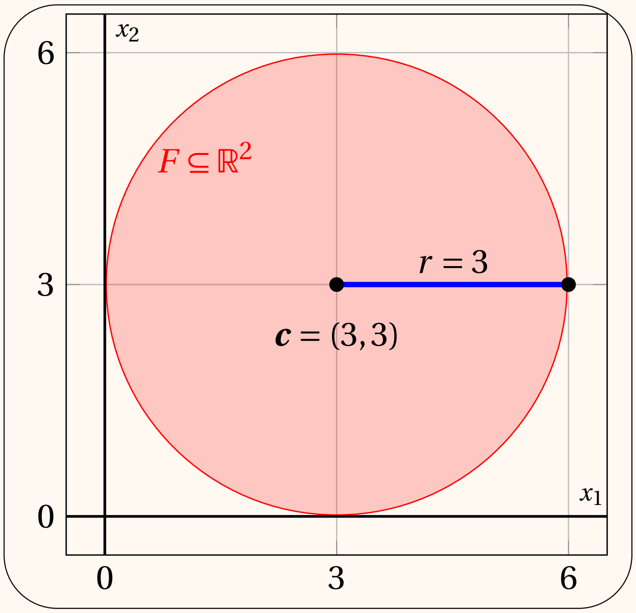
Then $\left(\lambda\, \boldsymbol{p} + (1-\lambda)\, \boldsymbol{w}\right)$ belongs to the set and, accordingly, the set is convex. □

Example 2: ball in $\mathbb{R}^2$

For example in $\mathbb{R}^2$, considering a radius $r = 3$ and the center $\boldsymbol{c} = (3,3)$, we have the following convex set:

$$F = \left\{ \; (x_1, x_2) \in \mathbb{R}^2 : \; \underbrace{||(x_1, x_2) - (3,3)||}_{=\sqrt{(x_1-3)^2+(x_2-3)^2}} \; \leq \; 3 \; \right\}$$

The points of this set (a ball in $\mathbb{R}^2$) are the points inside a circle of radius 3 centered in $(3,3)$. They are the points shown by the red shaded portion of the space in the following figure:



Observation 2

Given m convex sets F_i with $i \in \{1, 2 \dots, m\}$, their intersection $F = \bigcap_{i=1}^{m} F_i$ is a convex set.

Proof. For any two points $\boldsymbol{p}$ and $\boldsymbol{w}$ in the set F, we have:

$$\boldsymbol{p} \in \bigcap_{i=1}^{m} F_i \text{ and } \boldsymbol{w} \in \bigcap_{i=1}^{m} F_i \implies \boldsymbol{p} \in F_i, \forall i \in \{1, 2, \dots, m\} \text{ and } \boldsymbol{w} \in F_i, \forall i \in \{1, 2, \dots, m\}$$

Then, since all sets F_i with $i \in \{1, 2, \dots, m\}$ are convex, for any $\lambda \in [0, 1]$ we have:

$$\lambda\, \boldsymbol{p} + (1 - \lambda)\, \boldsymbol{w} \in F_i, \quad \forall i \in \{1, 2, \dots, m\} \Longrightarrow \lambda\, \boldsymbol{p} + (1 - \lambda)\, \boldsymbol{w} \in \bigcap_{i=1}^{m} F_i$$

Accordingly $\lambda\, \boldsymbol{p} + (1 - \lambda)\, \boldsymbol{w} \in F$ and the set F is convex.

$\square$

2 Convex and concave functions

Definition 2: convex function

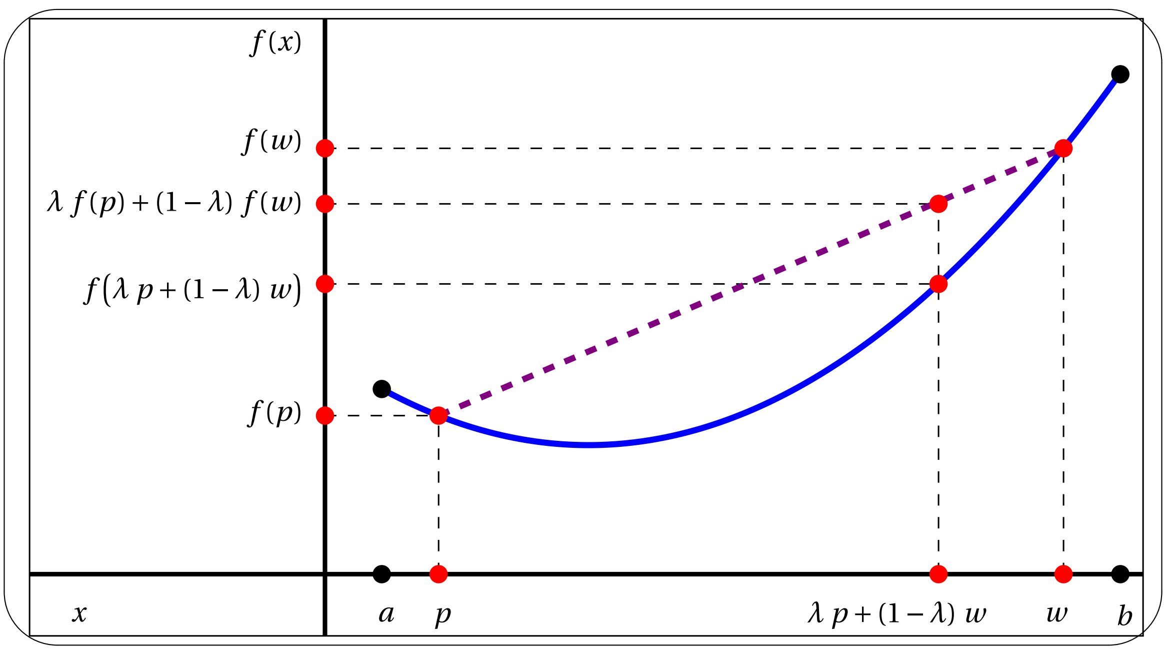
*A scalar function $f : F \subseteq \mathbb{R}^n \to \mathbb{R}$ is **convex** if:*

$$\forall \boldsymbol{p}, \boldsymbol{w} \in F \text{ and } \forall \lambda \in [0,1] \quad \text{we have} \quad f\big(\lambda\, \boldsymbol{p} + (1-\lambda)\, \boldsymbol{w}\big) \le \lambda\, f(\boldsymbol{p}) + (1-\lambda)\, f(\boldsymbol{w})$$

and the domain F of the function f is a convex set.

A scalar function is convex if for any two points in its convex domain the value computed in any convex combination of the two points is smaller or equal to the convex combination of the values computed in the two points.

- If $F = [a,b] \subseteq \mathbb{R}$ (an interval which is by nature a convex set), the graphic of a convex function is given by the following picture:



Definition 3: concave function

*A scalar function $f : F \subseteq \mathbb{R}^n \to \mathbb{R}$ is **concave** if:*

$$\forall \boldsymbol{p}, \boldsymbol{w} \in F \text{ and } \forall \lambda \in [0,1] \quad \text{we have} \quad f\big(\lambda\, \boldsymbol{p} + (1-\lambda)\, \boldsymbol{w}\big) \ge \lambda\, f(\boldsymbol{p}) + (1-\lambda)\, f(\boldsymbol{w})$$

and the domain F of the function f is a convex set.