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Induction (3)

For any natural number $n \geq 1$, we have:

$$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n} \tag{1}$$

Questions

1. Prove formula (1) using the induction principle.
2. Compute the value of the summation with $n = 10$.

Solution

By induction on n.

- **First step of the induction**

  Let $n = 1$. Then the statement becomes:

  $$\sum_{k=1}^{1} \frac{k}{2^k} = 2 - \frac{1+2}{2^1} \quad \text{that is} \quad \frac{1}{2} = \frac{1}{2}$$

  which is evidently true.

- **Inductive step**

  Suppose it is true for n, and we prove it for $(n+1)$. By inductive hypothesis, we have:

  $$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

  Then we can write:

  $$\begin{aligned}\sum_{k=1}^{n+1} \frac{k}{2^k} &= \sum_{k=1}^{n} \frac{k}{2^k} + \frac{n+1}{2^{n+1}} \\ &= 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}} = 2 - \frac{2\,(n+2)}{2^{n+1}} + \frac{n+1}{2^{n+1}} \\ &= 2 - \left(\frac{2\,(n+2) - n - 1}{2^{n+1}}\right) = 2 - \left(\frac{n+3}{2^{n+1}}\right) \\ &= 2 - \frac{(n+1)+2}{2^{n+1}}\end{aligned}$$

  which is exactly the statement for $n+1$.

The value of the summation with $n = 10$ is:

$$\sum_{k=1}^{10} \frac{k}{2^k} = \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \frac{5}{2^5} + \frac{6}{2^6} + \frac{7}{2^7} + \frac{8}{2^8} + \frac{9}{2^9} + \frac{10}{2^{10}} = 2 - \frac{12}{2^{10}} = \frac{509}{256}$$