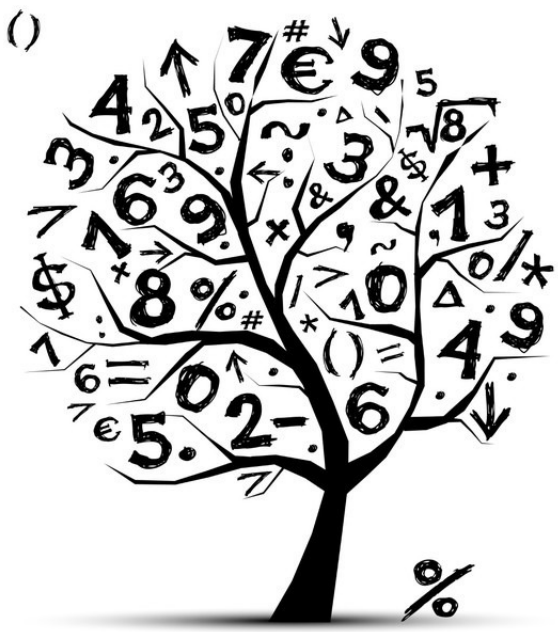
Last update: March 1, 2024

Absolute values and Euclidean norms



- **Author**: Fabio Furini

Contents

1 Absolute values

Definition 1: of absolute value

*The **absolute value** of element $a \in \mathbb{R}$ is the non-negative number defined as follows:*

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} \tag{1}$$

From the definition of absolute value it immediately follows that:

$$\forall \varepsilon \geq 0, a \in \mathbb{R}, \qquad |a| \leq \varepsilon \iff -\varepsilon \leq a \leq \varepsilon \tag{2}$$

1.1 Triangle inequality in $\mathbb{R}$

Observation 1: triangle inequality in $\mathbb{R}$

$$|b+c| \leq |b| + |c| \qquad \forall b, c \in \mathbb{R} \tag{3}$$

Proof. We write the two relations:

$$-|b| \leq b \leq |b|, \qquad -|c| \leq c \leq |c|$$

and add member to member:

$$-(|b|+|c|) \leq b + c \leq |b| + |c|$$

So for (2), with $\varepsilon = |b| + |c| \geq 0$ and $a = b + c$, it follows (3). □

- The triangle inequality is also used in the following form:

$$|d-e| \leq |d-f| + |e-f| \qquad \forall d, e, f \in \mathbb{R} \tag{4}$$

To obtain it, it suffices to set (3):

$$b = d - f, \quad c = f - e$$

and we get:

$$|d-f+f-e| = |d-e| \leq |d-f| + |f-e| = |d-f| + |e-f|$$

since:

$$|f-e| = |e-f| \qquad \forall e, f \in \mathbb{R}$$

- Finally, another form of the triangle inequality is:

$$|g| \leq |g-h| + |h| \quad \text{that is} \quad |g| - |h| \leq |g-h| \qquad \forall g, h \in \mathbb{R} \tag{5}$$

To obtain it, it suffices to set in (3):

$$b = g - h, \quad c = h$$

Observation 2: reverse triangle inequality in $\mathbb{R}$

$$\big||g| - |h|\big| \leq |g-h|, \quad \forall g, h \in \mathbb{R} \tag{6}$$

Proof. From (5), we have:

$$|g| - |h| \leq |g-h| \qquad \forall g, h \in \mathbb{R}$$

Similarly exchanging g for h in (5) we get:

$$|h| - |g| \leq |h-g| = |g-h| \quad \text{that is} \quad |g| - |h| \geq -|g-h|$$

then we have

$$-(|g-h|) \leq |g| - |h| \leq |g-h| \qquad \forall g, h \in \mathbb{R}$$

Accordingly from (2), with $\varepsilon = |g-h| \geq 0$ and $a = |g| - |h|$, we get the reverse triangle inequality.

□

- The triangle inequality (3) can easily be extended to the case of k addends:

$$\left|\sum_{i=1}^{k} b_i\right| \leq \sum_{i=1}^{k} |b_i|. \tag{7}$$

- The following immediate properties also hold:

$$|b\,c| = |b|\,|c|, \qquad \left|\frac{b}{c}\right| = \frac{|b|}{|c|}, \qquad |-b| = |b| \qquad \forall b, c \in \mathbb{R}. \tag{8}$$

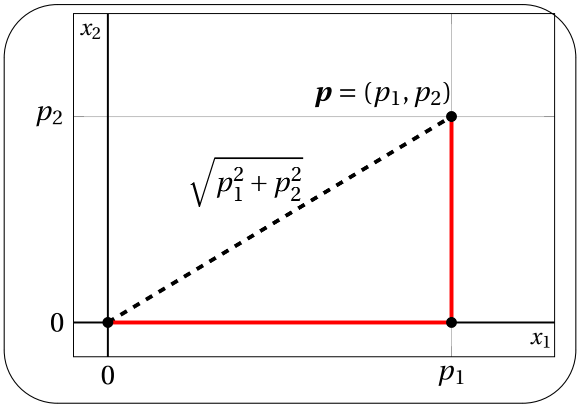
2 Euclidean norms

Definition 2: Euclidean norm

The ***Euclidean norm*** *of a point* $\boldsymbol{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ *is the non-negative number defined as follows:*

$$||\boldsymbol{p}|| = \sqrt{\sum_{j=1}^{n} p_j^2} \tag{9}$$

The Euclidean norm of a point in $\mathbb{R}^n$ is its **Euclidean distance** from the origin. This is due to the Pythagoras's theorem as illustrated in $\mathbb{R}^2$ by the picture.



Given two points $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ in $\mathbb{R}^n$ we have:

$$||\boldsymbol{p} - \boldsymbol{w}|| = \sqrt{\sum_{j=1}^{n} (p_j - w_j)^2}$$

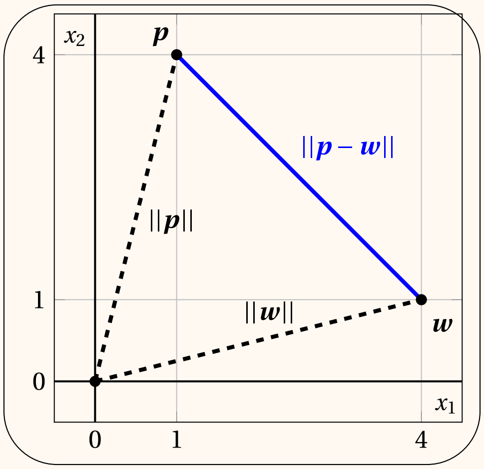
The value $||\boldsymbol{p} - \boldsymbol{w}||$ is the **Euclidean distance** between the points $\boldsymbol{p}$ and $\boldsymbol{w}$.

Example 1: Euclidean norms and Euclidean distances

Consider the points $\boldsymbol{p} = (1, 4) \in \mathbb{R}^2$ and $\boldsymbol{w} = (4, 1) \in \mathbb{R}^2$, we have:

$$||\boldsymbol{p}|| = \sqrt{1^2 + 4^2}, \quad ||\boldsymbol{w}|| = \sqrt{4^2 + 1^2}$$

$$||\boldsymbol{p} - \boldsymbol{w}|| = \sqrt{(1-4)^2 + (4-1)^2}$$



Observation 3: absolute homogeneity

$$||\lambda\, \boldsymbol{p}|| = |\lambda|\; ||\boldsymbol{p}||, \quad \forall \lambda \in \mathbb{R}, \boldsymbol{p} \in \mathbb{R}^n \tag{10}$$

Proof. For any $\lambda \in \mathbb{R}$ and $\boldsymbol{p} \in \mathbb{R}^n$, we have:

$$||\lambda\, \boldsymbol{p}|| = \sqrt{\sum_{j=1}^{n} (\lambda\, p_j)^2} = \sqrt{\sum_{j=1}^{n} \lambda^2\, p_j^2} = |\lambda| \sqrt{\sum_{j=1}^{n} p_j^2} = |\lambda|\; ||\boldsymbol{p}||$$

□

Given two points $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ in $\mathbb{R}^n$, the value :

$$\boldsymbol{p} \cdot \boldsymbol{w} = \sum_{j=1}^{n} \left(p_j \cdot w_j\right)$$

is called **scalar product** of $\boldsymbol{p}$ and $\boldsymbol{w}$. Its properties (derived from the properties of the sum and the product operations in $\mathbb{R}$) are:

$$\boldsymbol{p} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{p}, \quad \forall \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n \tag{11}$$

$$\boldsymbol{p} \cdot (\boldsymbol{w} + \boldsymbol{u}) = \boldsymbol{p} \cdot \boldsymbol{w} + \boldsymbol{p} \cdot \boldsymbol{u}, \quad \forall \boldsymbol{p}, \boldsymbol{w}, \boldsymbol{u} \in \mathbb{R}^n \tag{12}$$

$$\lambda\, (\boldsymbol{p} \cdot \boldsymbol{w}) = (\lambda\, \boldsymbol{p}) \cdot \boldsymbol{w}, \quad \forall \lambda \in \mathbb{R},\;\; \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n \tag{13}$$

$$\boldsymbol{p} \cdot \boldsymbol{p} \geq 0, \quad \forall \boldsymbol{p} \in \mathbb{R}^n \quad \text{and} \quad \boldsymbol{p} \cdot \boldsymbol{p} = 0 \iff \boldsymbol{p} = \boldsymbol{0}, \quad \forall \boldsymbol{p} \in \mathbb{R}^n \tag{14}$$

where $\boldsymbol{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$ (the origin of the Cartesian axes).

Moreover, we have:

$$||\boldsymbol{p}|| = \sqrt{\boldsymbol{p} \cdot \boldsymbol{p}} \quad \text{and} \quad ||\boldsymbol{p}||^2 = \boldsymbol{p} \cdot \boldsymbol{p}, \quad \forall \boldsymbol{p} \in \mathbb{R}^n \tag{15}$$

Observation 4: Cauchy-Schwarz inequality

$$|\boldsymbol{p} \cdot \boldsymbol{w}| \leq ||\boldsymbol{p}||\; ||\boldsymbol{w}||, \quad \forall \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n \tag{16}$$

Proof. If $\boldsymbol{p} = \mathbf{0}$ or $\boldsymbol{w} = \mathbf{0}$ (or both), it clearly holds. If $\boldsymbol{p} \neq \mathbf{0}$ and $\boldsymbol{w} \neq \mathbf{0}$, we define the set of points:

$$\boldsymbol{u} = \alpha\, \boldsymbol{p} + \beta\, \boldsymbol{w} \quad \text{with} \quad \alpha, \beta \in \mathbb{R}$$

For the properties (11),(12),(13),(14), for all $\alpha, \beta \in \mathbb{R}$ we have:

$$\underbrace{\boldsymbol{u} \cdot \boldsymbol{u}}_{\geq 0} = (\alpha\, \boldsymbol{p} + \beta\, \boldsymbol{w}) \cdot (\alpha\, \boldsymbol{p} + \beta\, \boldsymbol{w})$$

$$= \alpha^2\, ||\boldsymbol{p}||^2 + 2\, \alpha\, \beta\, \boldsymbol{p} \cdot \boldsymbol{w} + \beta^2\, ||\boldsymbol{w}||^2 \geq 0$$

For $\alpha = ||\boldsymbol{w}||^2$ and $\beta = -\boldsymbol{p} \cdot \boldsymbol{w}$, substituting, we have:

$$||\boldsymbol{w}||^4\, ||\boldsymbol{p}||^2 - 2\, ||\boldsymbol{w}||^2\, (\boldsymbol{p} \cdot \boldsymbol{w})^2 + (\boldsymbol{p} \cdot \boldsymbol{w})^2\, ||\boldsymbol{w}||^2 = \underbrace{||\boldsymbol{w}||^4\, ||\boldsymbol{p}||^2 - ||\boldsymbol{w}||^2\, (\boldsymbol{p} \cdot \boldsymbol{w})^2}_{\geq 0}$$

Then we have:

$$||\boldsymbol{w}||^2\, (\boldsymbol{p} \cdot \boldsymbol{w})^2 \leq ||\boldsymbol{w}||^4\, ||\boldsymbol{p}||^2 \Longrightarrow (\boldsymbol{p} \cdot \boldsymbol{w})^2 \leq ||\boldsymbol{w}||^2\, ||\boldsymbol{p}||^2$$

obtained by diving by $||\boldsymbol{w}||^2 > 0$. This implies $|\boldsymbol{p} \cdot \boldsymbol{w}| \leq ||\boldsymbol{p}||\, ||\boldsymbol{w}||$. □

2.1 Triangle inequality in $\mathbb{R}^n$

Observation 5: triangle inequality in $\mathbb{R}^n$

$$||\boldsymbol{p} + \boldsymbol{w}|| \leq ||\boldsymbol{p}|| + ||\boldsymbol{w}||, \quad \forall\, \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n \tag{17}$$

- In $\mathbb{R}^2$, the triangle inequality states that **for any triangle the sum of the lengths of any two sides must be greater than or equal to the length of the remaining side**.

Proof. The triangle inequality is equivalent to the inequality:

$$||\boldsymbol{p} + \boldsymbol{w}||^2 \leq \left(||\boldsymbol{p}|| + ||\boldsymbol{w}||\right)^2, \quad \forall\, \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n$$

Directly from the definitions and the Cauchy-Schwarz inequality, we have:

$$||\boldsymbol{p} + \boldsymbol{w}||^2 = (\boldsymbol{p} + \boldsymbol{w}) \cdot (\boldsymbol{p} + \boldsymbol{w}) = ||\boldsymbol{p}||^2 + ||\boldsymbol{w}||^2 + 2\;\, \boldsymbol{p} \cdot \boldsymbol{w}$$

$$\leq ||\boldsymbol{p}||^2 + ||\boldsymbol{w}||^2 + 2\, ||\boldsymbol{p}||\, ||\boldsymbol{w}|| = \left(||\boldsymbol{p}|| + ||\boldsymbol{w}||\right)^2$$

□

Given two points $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ in $\mathbb{R}^n$, we have:

$$||\boldsymbol{p}+\boldsymbol{w}|| = \sqrt{\sum_{j=1}^{n} \left(p_j + w_j\right)^2}$$

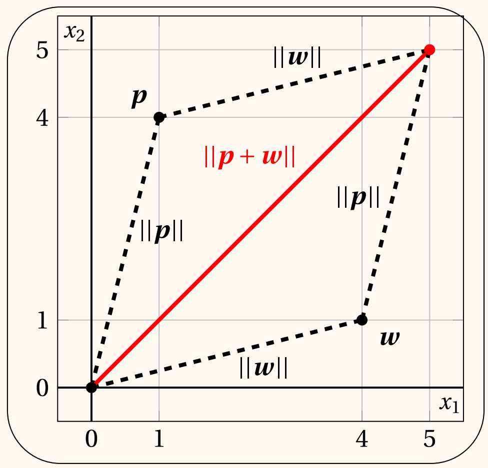
Example 2: triangle inequality in $\mathbb{R}^2$

Consider the points $\boldsymbol{p} = (1,4) \in \mathbb{R}^2$ and $\boldsymbol{w} = (4,1) \in \mathbb{R}^2$, we have:

$$||\boldsymbol{p}|| = \sqrt{1^2+4^2}, \quad ||\boldsymbol{w}|| = \sqrt{4^2+1^2}$$

$$||\boldsymbol{p}+\boldsymbol{w}|| = \sqrt{(1+4)^2+(4+1)^2}$$

$$\sqrt{(1+4)^2+(4+1)^2} \le \sqrt{1^2+4^2} + \sqrt{4^2+1^2}$$



- Another form of the triangle inequality is:

$$||\boldsymbol{r}|| \le ||\boldsymbol{r}-\boldsymbol{s}|| + ||\boldsymbol{s}|| \quad \text{that is} \quad ||\boldsymbol{r}|| - ||\boldsymbol{s}|| \le ||\boldsymbol{r}-\boldsymbol{s}||, \quad \forall \boldsymbol{r}, \boldsymbol{s} \in \mathbb{R}^n \tag{18}$$

  To obtain it, it suffices to set in (17):

$$\boldsymbol{p} = \boldsymbol{r} - \boldsymbol{s}, \quad \boldsymbol{w} = \boldsymbol{s}$$

Observation 6: reverse triangle inequality in $\mathbb{R}^n$

$$\Big| \, ||\boldsymbol{p}|| - ||\boldsymbol{w}|| \, \Big| \le ||\boldsymbol{p}-\boldsymbol{w}||, \quad \forall \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n \tag{19}$$

- In $\mathbb{R}^2$, the reverse triangle inequality states that **for any triangle the length of any side must be greater than or equal to the difference between the lengths of the other two sides**.

Proof. The reverse triangle inequality is equivalent to the inequality:

$$\underbrace{\left(||\boldsymbol{p}|| - ||\boldsymbol{w}||\right)^2}_{=\left|\,||\boldsymbol{p}||-||\boldsymbol{w}||\,\right|^2} \leq ||\boldsymbol{r} - \boldsymbol{w}||^2, \quad \forall \boldsymbol{p}, \boldsymbol{w} \in \mathbb{R}^n$$

Directly from the definitions and the Cauchy-Schwarz inequality, we have:

$$||\boldsymbol{p} - \boldsymbol{w}||^2 = (\boldsymbol{p} - \boldsymbol{w}) \cdot (\boldsymbol{p} - \boldsymbol{w}) = ||\boldsymbol{p}||^2 + ||\boldsymbol{w}||^2 - 2\ \boldsymbol{p} \cdot \boldsymbol{w}$$

$$\geq ||\boldsymbol{p}||^2 + ||\boldsymbol{w}||^2 - 2\,||\boldsymbol{p}||\,||\boldsymbol{w}|| = \left(||\boldsymbol{p}|| - ||\boldsymbol{w}||\right)^2$$

□

Example 3: reverse triangle inequality in $\mathbb{R}^2$

Consider the points $\boldsymbol{p} = (1,4) \in \mathbb{R}^2$ and $\boldsymbol{w} = (4,1) \in \mathbb{R}^2$, we have:

$$||\boldsymbol{p}|| = \sqrt{1^2 + 4^2}, \quad ||\boldsymbol{w}|| = \sqrt{4^2 + 1^2}$$

$$||\boldsymbol{p} - \boldsymbol{w}|| = \sqrt{(1-4)^2 + (4-1)^2}$$

$$\left|\sqrt{1^2 + 4^2} - \sqrt{4^2 + 1^2}\right| \leq \sqrt{(1-4)^2 + (4-1)^2}$$