Question # Convert from cylindrical to spherical

$x = r\cos\theta$; $x = R\sin\phi\cos\theta \Rightarrow r = R\sin\phi$

$y = r\sin\theta$; $y = R\sin\phi\sin\theta$

$z = z$; $z = R\cos\phi$

Now;
$$\frac{\partial u_r}{\partial R} = \frac{\partial u_r}{\partial r}\cdot\frac{\partial r}{\partial R} + \frac{\partial u_r}{\partial \theta}\cdot\frac{\partial \theta}{\partial R} + \cancelto{0}{\frac{\partial u_r}{\partial z}}\cdot\frac{\partial z}{\partial R}$$

$$= \frac{\partial}{\partial r}(u_x\cos\theta + u_y\sin\theta)\cdot\frac{\partial(R\sin\phi)}{\partial R} + \frac{\partial}{\partial\theta}(u_x\cos\theta + u_y\sin\theta)\frac{\partial\theta}{\partial R}$$

$$= \left[\frac{\partial u_x}{\partial r}\cos\theta + \frac{\partial u_y}{\partial r}\sin\theta\right]\cdot\sin\phi + \left[\frac{\partial u_x}{\partial\theta}\cos\theta + u_x(-\sin\theta) + \frac{\partial u_y}{\partial\theta}\sin\theta + u_y\cos\theta\right]\frac{\partial\theta}{\partial R}$$

$$\Rightarrow \frac{\partial u_r}{\partial R} = \left[\frac{\partial u_x}{\partial r}\cos\theta + \frac{\partial u_y}{\partial r}\sin\theta\right]\sin\phi + \left[\frac{\partial u_x}{\partial\theta}\cos\theta - u_x\sin\theta + \frac{\partial u_y}{\partial\theta}\sin\theta + u_y\cos\theta\right]\frac{\partial\theta}{\partial R}$$

$$\frac{\partial u_\theta}{\partial\theta} = \frac{\partial u_\theta}{\partial r}\cdot\frac{\partial r}{\partial\theta} + \frac{\partial u_\theta}{\partial\theta}\cdot\frac{\partial\theta}{\partial\theta} + \frac{\partial u_\theta}{\partial z}\cancelto{0}{\frac{\partial z}{\partial\theta}}$$

$$= \frac{\partial}{\partial r}(-ru_x\sin\theta + ru_y\cos\theta)\cdot\frac{\partial}{\partial\theta}(R\sin\phi) + \frac{\partial}{\partial\theta}(-ru_x\sin\theta + ru_y\cos\theta)(1)$$

$$= \left[-u_x\sin\theta\frac{\partial r}{\partial r} - r\sin\theta\frac{\partial u_x}{\partial r} - r u_x\cancelto{0}{\frac{\partial\sin\theta}{\partial r}} + u_y\cos\theta\frac{\partial r}{\partial r} + r\cos\theta\frac{\partial u_y}{\partial r} + r u_y\cancelto{0}{\frac{\partial\cos\theta}{\partial r}}\right]\cdot\sin\phi\frac{\partial R}{\partial\theta} + \left[-u_x\sin\theta\frac{\partial r}{\partial\theta} - r\sin\theta\frac{\partial u_x}{\partial\theta} - r u_x\frac{\partial\sin\theta}{\partial\theta} + u_y\cos\theta\frac{\partial r}{\partial\theta} + r\cos\theta\frac{\partial u_y}{\partial\theta} + r u_y\frac{\partial\cos\theta}{\partial\theta}\right].$$

$$= \left[-u_x\sin\theta - r\sin\theta\frac{\partial u_x}{\partial r} + u_y\cos\theta + r\cos\theta\frac{\partial u_y}{\partial r}\right]\sin\phi\frac{\partial R}{\partial\theta} + \left[(-u_x\sin\theta + u_y\cos\theta)\frac{\partial r}{\partial\theta} - r\sin\theta\frac{\partial u_x}{\partial\theta} + r\cos\theta\frac{\partial u_y}{\partial\theta} - r u_x\cos\theta - r u_y\sin\theta\right].$$

$$\frac{\partial u_z}{\partial \phi} = \frac{\partial u_z}{\partial r} \cdot \frac{\partial r}{\partial \phi} + \frac{\partial u_z}{\partial \theta} \cdot \frac{\partial \theta}{\partial \phi} + \frac{\partial u_z}{\partial \phi} \cdot \frac{\partial z}{\partial \phi}$$

$$= \frac{\partial u_z}{\partial r} \cdot \frac{\partial}{\partial \phi} (R \sin \phi) + \frac{\partial u_z}{\partial \theta} \cdot \frac{\partial \theta}{\partial \phi} + \frac{\partial u_z}{\partial \phi} \frac{\partial}{\partial \phi} (R \cos \phi)$$

$$= \frac{\partial u_z}{\partial r} \left(\frac{\partial R}{\partial \phi} \sin \phi + R \cos \phi \right) + \frac{\partial u_z}{\partial \theta} \cdot \frac{\partial \theta}{\partial \phi} + \frac{\partial u_z}{\partial \phi} \left(\frac{\partial R}{\partial \phi} \cos \phi - \sin \phi \cos \phi \right)$$

$$\Rightarrow \frac{\partial u_z}{\partial \phi} = \frac{\partial u_z}{\partial r} \left(\frac{\partial R}{\partial \phi} \sin \phi + R \cos \phi \right) + \frac{\partial u_z}{\partial \phi} \frac{\partial \theta}{\partial \phi} + \frac{\partial u_z}{\partial \phi} \left(\frac{\partial R}{\partial \phi} \cos \phi - \sin \phi \cos \phi \right).$$