

The Poynting vector and power in a simple circuit

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(Received 12 July 1999; accepted 13 January 2000)

This paper outlines a simple technique for visualizing the flow of energy from a power supply to elements in a circuit as a flow through the electric and magnetic fields—or the Poynting vector—surrounding the circuit. In addition to providing the reader experience with the Poynting vector and its relation to energy flow and power, we also present a quick method for solving Laplace's equation in two dimensions. © 2000 American Association of Physics Teachers.

I. INTRODUCTION

The role of surface charges on wires connecting elements in an electric circuit has been discussed by Jackson¹ and Heald.² While these papers also discuss the Poynting vector and its relationship to energy flow, they require the reader to have experience with higher level mathematics (e.g., Fourier–Bessel series in Jackson's paper and cylindrical harmonics in Heald's). Other work has described the relationship of the Poynting vector and energy flow in transformers³ and in time-changing systems with electromagnetic induction.⁴ This paper analyzes a simple circuit in two dimensions. We first solve Laplace's equation with the relaxation method using a spreadsheet. From the resulting potential we calculate the electric field. With the two-dimensional circuit we then have a simple, constant magnetic field. Thus all the calculations can be performed numerically in a single spreadsheet with no need for introducing special mathematical functions. We feel this approach is useful in teaching introductory electricity and magnetism.

Figure 1 illustrates the circuit under consideration. Throughout the paper we'll assume that the values of the resistors are $\frac{1}{2}\Omega$ and the power source is a 1-V battery. Hence the power generated by the battery is 1 W while each resistor dissipates $\frac{1}{2}$ W. That is the simple way to look at things. But let's take this simple example with solutions we know and look at it with more sophisticated tools; i.e., let's make the simple complicated.

II. THE ELECTRIC POTENTIAL AND LAPLACE'S EQUATION

The electric field is obtained by solving Laplace's equation in two dimensions. Note that this effectively turns the circuit into a long square channel with long ribbon-like resistors and battery. For this paper we assume the channel is sufficiently long that “end effects” are negligible. This idealization, in addition to making the electric field simple to solve (via Laplace's equation in two dimensions), makes the magnetic field trivial as we now have a solenoid of square cross section. Also, qualitative features of the electric and Poynting vector fields of an actual circuit, in addition to quantitative results for power, are preserved.

To solve Laplace's equation we use the “relaxation method,”⁵ which is an iterative method that simply replaces the potential at a point by the average of the points surrounding it. After a number of iterations, the potential “relaxes” to the solution. The Laplacian can be thought of as an averaging operator. We define the boundary conditions in a spreadsheet on a square 100×100 grid. For our particular

circuit, shown in Fig. 1, the boundary of the square grid consists of the wires connecting the circuit elements. This means the outer boundary of the first quadrant is at 0.5 V, the second at 1.0 V, and the third and fourth at 0.0 V (assuming the origin is the center in Fig. 1). Note that we are assuming that the resistors and battery are point-like (in the plane of the figure) objects. We then set the interior cells to be the average of the four cells (nearest neighbors) around them. In EXCEL this produces a “circular references” error, which can be overcome by setting the spreadsheet up to resolve circular references.⁶ Thus, within minutes, the spreadsheet has performed the “relaxation method” and has solved Laplace's equation in two dimensions.

III. THE ELECTRIC FIELD

From here one could proceed in the spreadsheet by implementing numerical derivatives⁷ on additional second sheets. Alternatively, one could output the array of numbers to a file for processing in another mathematical program. We have used MATHCAD, MATLAB, and MATHEMATICA; all work very well. MATHEMATICA has strong graphical capabilities (all the figures in this paper were generated with MATHEMATICA) and built-in functions for handling the gradient of a discrete array of numbers. From the potential the gradient is taken and the electric field is obtained. Figure 2 shows a contour plot of the potential with the direction of the electric field (unit vectors) overlaid.

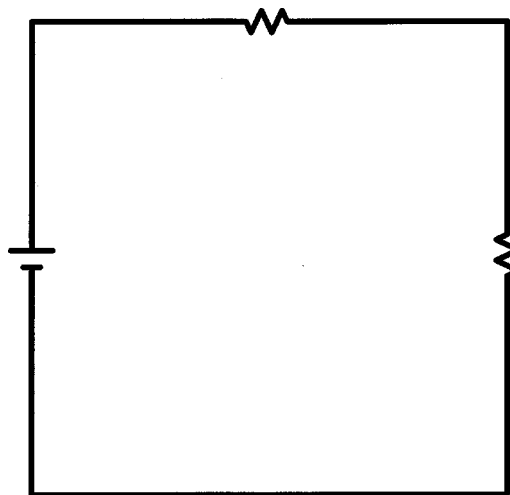


Fig. 1. The circuit under consideration. All elements are considered point-like. Battery and resistor values are noted in the text.

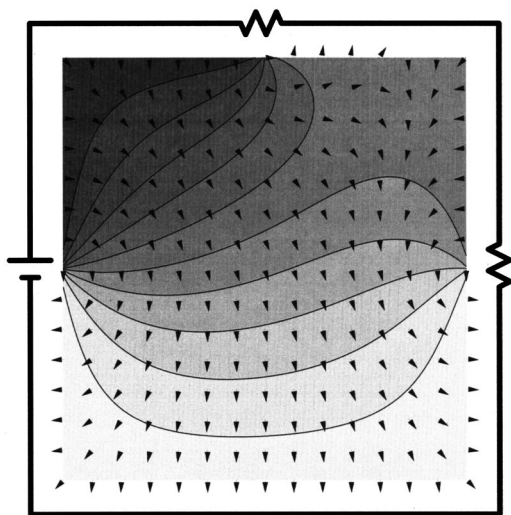


Fig. 2. The electric potential is shown in the contour plot with the electric field direction (unit vectors) overlaid.

IV. MAGNETIC FIELD

The magnetic field for a long square channel is that of a solenoid⁸ and is given by $B = \mu_0 I / L$ throughout the interior of the circuit and zero elsewhere. I is the current and L is the length of the solenoid (the solenoid is square in cross section). The idealization is for L to approach infinity and thus B approaches zero. However, when we obtain the total power we integrate over the length and thus the quantity $\int_{-L/2}^{L/2} B \, dl$ remains constant.

V. POYNTING VECTOR AND POWER

With the electric and magnetic fields the Poynting vector is computed from $\mathbf{S} = (1/\mu_0) \mathbf{E} \times \mathbf{B}$. It is shown in Fig. 3 (again with unit vectors overlaid). The outward power flow from each element, in our two-dimensional world, is given by the normal component of the Poynting vector integrated over each “surface,” such as those depicted by dashed lines in Fig. 4. We find that this integrated power is as expected (1

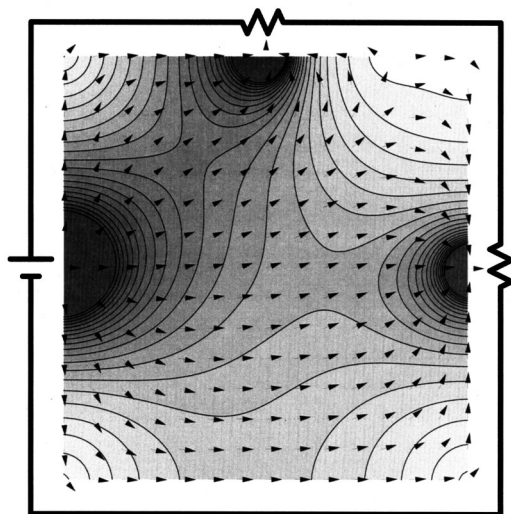


Fig. 3. The magnitude of the Poynting vector is shown in the contour plot with its direction (unit vectors) overlaid.

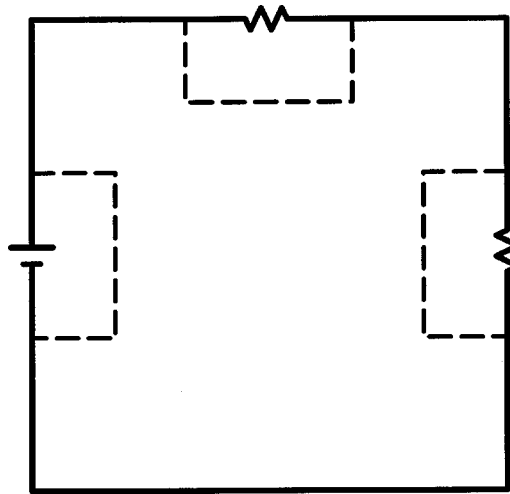


Fig. 4. The dashed lines indicate the surfaces used for the power calculations.

W for the battery and -0.5 W for each resistor) to within about 1%. (Square grids of 10, 30, and 100 were explored. We settled on 100 because the contour plot for the Poynting vector displayed slight “wiggles” near the circuit elements which vanished once the mesh was fine enough. The integrated powers for grid sizes of 30×30 , and even 10×10 , were still within a few percent.)

VI. CONCLUSION

We have presented a numerical technique to investigate the Poynting vector and its relationship to flow of power in a simple circuit. The entire investigation can be performed in a spreadsheet or in combination with other mathematical programs. One may also explore other circuits, treat circuits in full three dimensions, check the power flow through surfaces that do not include an element, and investigate the presence of surface charge.

ACKNOWLEDGMENTS

The authors would like to thank Dave Bell, Francis Chun, and Bill Pakula for many useful discussions.

¹J. D. Jackson, “Surface charges on circuit wires and resistors play three roles,” *Am. J. Phys.* **64**, 855–870 (1996).

²Mark A. Heald, “Electric fields and charges in elementary circuits,” *Am. J. Phys.* **52**, 522–526 (1984).

³F. Hermann and G. Bruno Schmid, “The Poynting vector field and the energy flow within a transformer,” *Am. J. Phys.* **54**, 528–531 (1986); N. Morton, “The Poynting vector distribution in a simple transformer,” *ibid.* **55**, 472–474 (1987).

⁴Mark A. Heald, “Energy flow in circuits with Faraday emf,” *Am. J. Phys.* **56**, 540–547 (1988); J. M. Aguirregabiria, A. Hernández, and M. Rivas, “Surface charges and energy flow in a ring rotating in a magnetic field,” *ibid.* **64**, 892–895 (1996).

⁵See, for example, E. M. Purcell, *Electricity and Magnetism, Berkeley Physics Course* (McGraw-Hill, New York, 1985), 2nd ed., Vol. 2, pp. 119–120 or David J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, Upper Saddle River, NJ, 1999), 2nd ed., pp. 113–114 or 3rd ed., pp. 112–114.

⁶In EXCEL, look under the “Tools, Options, Calculation” feature for settings to handle the circular references with an iterative procedure.

⁷Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical*

WHAT KEEPS AN AIRPLANE UP?

Everybody admits, I suppose, that the sciences other than mathematics are based on experiment. Things that can be checked by experiment are accepted: things that disagree with experiment are not. However, I am not aware of any physics or chemistry or biology course that repeats all the classical experiments, or even any of those that are particularly difficult or time-consuming. At least in the science courses I took (of course, this was a long time ago) we were told that certain things had been established experimentally, and maybe (not always) what the experiment was like. Inspection of some current textbooks suggests that things haven’t changed much in 40 years.

Now I do not know any experimental scientists who seem to feel uncomfortable about this state of affairs, although for all I know they may worry about it in secret. Nor do they seem to worry about the necessity of sometimes giving oversimplified or even mildly fallacious reasons why the experiment comes out as it does. For example, why does an airplane stay up? Elementary texts give theoretical reasons that do not seem very convincing; the real theoretical reasons are clearly too sophisticated for elementary courses; it presumably would be possible also to rely on experimental measurements of the flow around a wing, but few, if any, physics courses bring wind-tunnels into the classroom.

Ralph P. Boas, Jr., “Calculus as an Experimental Science,” in *Lion Hunting & Other Mathematical Pursuits*, edited by Gerald L. Alexanderson and Dale H. Mugler (Mathematical Association of America, Washington, 1995), p. 227.