A LINEAR ALGEBRAIC APPROACH TO *LIGHTS OUT!* ON $G \square P_T$ [*]

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Abstract. The game *LIGHTS OUT!* is played on a 5×5 square grid of buttons; each button may be on or off (lit or unlit). Pressing a button changes the on/off state of the light of the button pressed and of all its vertical and horizontal neighbors. Given an initial configuration of buttons that are lit, the object of the game is to turn all the lights out. The game can be generalized to arbitrary graphs. We investigate graphs of the form $G \square P_t$, where G is an arbitrary graph and P_t is a path of length t. In particular, we provide conditions for which $G \square P_t$ is universally solvable (every initial configuration of lights can be turned out by a finite sequence of button presses). To do so, we extend properties of Fibonacci polynomials from $GF(2)$, the finite field of order 2, to $M_k(GF(2))$, the set of all $k \times k$ matrices over $GF(2)$.

Key words. matrix, determinant, graph, Fibonacci polynomials, Lights Out

AMS subject classifications. 05C50, 15A15, 15A03, 15B33

1. Introduction. The popular electronic game *LIGHTS OUT!* was released by Tiger Electronics in 1995. The game is played on a 5×5 square grid of buttons, where each button is either on or off (lit or unlit). When you start the game, it generates a random puzzle or configuration of lit and unlit buttons. The object of the game is simple - turn the lights out. However, every press of a button has an effect on the puzzle. If you press a button that is lit, it will shut off. Conversely, pressing an unlit button causes it to light up. When you press a button, not only does it change that light, but it also changes the adjacent lights (those that are directly above, below, or next to the pressed button). The object of the game is to solve each puzzle with the fewest number of button presses.

The traditional game is played on a square grid but can be generalized to arbitrary graphs. In this paper, we consider the game applied to graphs of the form $G \square P_t$, where G is an arbitrary graph and P_t is a path of length t. In particular, we address the question of whether or not $G \square P_t$ is *universally solvable*, i. e. whether or not every initial configuration on $G \square P_t$ is *solvable* (all of the lights can be turned off by a finite sequence of button presses). Recall that the *Cartesian Product* of G with H, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if (1) $u = u'$ and $v \sim v'$ in H, or (2) $v = v'$ and $u \sim u'$ in G.

For an extensive survey of the work that has been done on the game, see [5]. Sutner [8, 9] was the first to study the game, and he did so in the context of cellular automata. He showed that for any graph, it is possible to turn all the lights off if initially all the lights are on. Anderson and Feil [4] obtained a complete strategy for the traditional game on the 5×5 grid. Amin et al. [1, 2, 3] and Goldwasser et al. [6] concentrated on universally solvable graphs.

We represent the state of each light by an element of $GF(2)$, the field of integers modulo 2; 1 means on, 0 means off. All calculations are done modulo 2. Let $G = (V, E)$ be an undirected

graph of order n. For each $v \in V$, the *open neighborhood* $N(v)$ of v is the set of vertices adjacent to v, $N(v) = \{u \in V : (u, v) \in E\}$. The *closed neighborhood* $N[v]$ of v is the open neighborhood along with v itself, $N[v] = N(v) \cup \{v\}$. Throughout the paper, we denote by A_G the adjacency matrix of G where all entries on the main diagonal are 1. Often, A_G is referred to as the closed neighborhood matrix of G.

THEOREM 1.1. [6, 8, 9] *A graph G is universally solvable if and only if its adjacency matrix A_G is invertible over $GF(2)$.*

Since A_G is invertible if and only if $\det(A_G) \neq 0$, we approach the problem of determining whether a graph G is universally solvable by studying the determinant of the adjacency matrix A_G. Amin et al. [1] called the nullity of A_G the *parity dimension* of G. In much of their early work [1, 2], they investigated parity dimension for graphs using graph-theoretic techniques.

In Section 2, we investigate universal solvability of the Cartesian Product $G \square P_2$, where G is a path, cycle, complete graph, or complete bipartite graph. We do so using a linear algebraic technique (i. e. we compute the determinant of the adjacency matrix). In Section 3, we investigate universal solvability of $G \square P_t$, where G is an arbitrary graph and P_t is a path of length t. In particular, we extend properties of Fibonacci polynomials (used by Goldwasser et al. [6] to determine universal solvability of grid graphs) from $GF(2)$ to $M_k(GF(2))$, the set of all $k \times k$ matrices over

TABLE 1.1
Summary of solvable graph families

result #	Graph	Universally Solvable?
[2]	Path P_n, $n \geq 1$	Y if $n \equiv 0, 1 \pmod 3$ N if $n \equiv 2 \pmod 3$
[1]	Cycle C_k, $k \geq 3$	Y if $k \equiv 1, 2 \pmod 3$ N if $k \equiv 0 \pmod 3$
[1]	Complete Graph K_n, $n \geq 2$	N
[1]	Complete Bipartite Graph $K_{k,i}$, $k, i \geq 1$	Y if k even N if k, i odd
[6]	$P_2 \square P_n$	Y if n even N if n odd
Sect. 2	$C_k \square P_2$	N
Sect. 2	$K_n \square P_2$	Y if n even N if n odd
Sect. 2	$K_{k,i} \square P_2$	N
[3], 2.5	Hypercube Q_n, $n \geq 1$	Y if n even N if n odd
3.3	$G \square P_t$, $t = 2^i - 1$ $(i \geq 1)$	Y iff G is universally solvable
3.3	$G \square P_t$, $t \geq 1$ odd, $t + 1 = 2^i \cdot j$ $(i \geq 1$ and $j > 1$ is the largest odd divisor of $t + 1)$	Y iff G and $G \square P_{j-1}$ are universally solvable
3.4	$G \square P_t$, $t \equiv 1 \pmod 4$	Y iff G and $G \square P_{\frac{t-1}{2}}$ are universally solvable

$GF(2)$. Table 1.1 provides a summary of universal solvability for the graph families considered in Sections 2 and 3.

2. $G \Box P_2$. The following result is extremely useful in computing the determinant of a 2×2 block matrix and will be used throughout the section.

THEOREM 2.1. [7] *If*

$$M = \left[\begin{array}{cc} A & B \\ C & D \end{array} \right],$$

where $A, B, C, D \in M_n(F)$ for a field F and $CD = DC$, then $\det(M) = \det(AD - BC)$.

Amin et al. [1, 2] used graph-theoretic techniques to compute the parity dimension of many graphs, including paths, cycles, complete graphs, and complete bipartite graphs. Recall that a graph is universally solvable if its parity dimension is 0. For a summary of these results, see Table 1.1. These results can also be established by computing the determinant of the adjacency matrix, in keeping with the theme of this paper.

We now investigate the Cartesian Product $G \Box P_2$ where G is a path, cycle, complete graph, or complete bipartite graph. The proof technique is similar for each of these, so we do not include each proof here. Rather, we illustrate the technique and the usefulness of Theorem 2.1 by supplying a proof for the hypercube. For a summary of the results for the other Cartesian Products, see Table 1.1.

The *n-cube Q_n*, $n \geq 1$, is defined as the repeated Cartesian product of n paths of length two. Specifically, $Q_1 = P_2$ and $Q_n = Q_{n-1} \Box P_2$ for $n \geq 2$. The n-cube is often referred to as the *nth hypercube*. If $V(P_2) = \{0, 1\}$, then the vertex set of Q_n can be viewed as the set of n-tuples $(v_1, v_2, ..., v_n)$, where $v_i \in \{0, 1\}$. Moreover, two n-tuples share an edge if they differ in exactly one coordinate. The hypercubes Q_3 and Q_4 are shown in Figure 2.1.
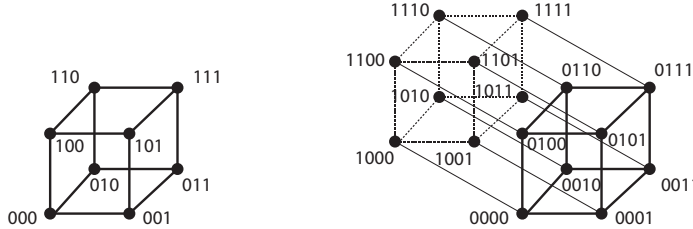

FIG. 2.1. *The hypercubes Q_3 and Q_4*

Amin et al. [3] computed the parity dimension of the hypercube by observing that the hypercube is a connected, orthogonal, bipartite graph, thereby establishing which hypercubes are universally solvable. We establish the result by computing the determinant of the adjacency matrix. Again, all calculations are done modulo 2.

DEFINITION 2.2. *Let $C_1 = [1]$. For each positive integer k, define C_{2^k} recursively by*

$$C_{2^k} = \left[\begin{array}{cc} C_{2^{k-1}} & I_{2^{k-1}} \\ I_{2^{k-1}} & C_{2^{k-1}} \end{array} \right],$$

where $I_{2^{k-1}}$ is the $(2^{k-1}) \times (2^{k-1})$ identity matrix.

LEMMA 2.3. *For each positive integer k, $\det(C_{2^k}) = 0$ if k is odd and $\det(C_{2^k}) = 1$ if k is even.*

Proof. The proof is by induction. We prove the case for k odd. The proof of the case for k even is similar. Note that $\det(C_2) = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 0$. Suppose $\det(C_{2^k}) = 0$ for some odd integer $k > 1$. Then

$$
\begin{aligned}
\det(C_{2^{k+2}}) &= \det\left(\begin{bmatrix} C_{2^{k+1}} & I_{2^{k+1}} \\ I_{2^{k+1}} & C_{2^{k+1}} \end{bmatrix}\right) \\
&= \det\left((C_{2^{k+1}})^2 - (I_{2^{k+1}})^2\right) \qquad\qquad \text{(by Theorem 2.1)} \\
&= \det\left((C_{2^{k+1}} - I_{2^{k+1}})(C_{2^{k+1}} + I_{2^{k+1}})\right) \\
&= \left[\det(C_{2^{k+1}} + I_{2^{k+1}})\right]^2 \\
&= \left[\det\left(\begin{bmatrix} C_{2^k} + I_{2^k} & I_{2^k} \\ I_{2^k} & C_{2^k} + I_{2^k} \end{bmatrix}\right)\right]^2 \\
&= \left[\det\left((C_{2^k} + I_{2^k})^2 - (I_{2^k})^2\right)\right]^2 \\
&= \left[\det(C_{2^k})^2\right]^2 \\
&= \left[\det(C_{2^k})\right]^4 \\
&= 0
\end{aligned}
$$

by the induction hypothesis. $\square$

OBSERVATION 2.4. *For each positive integer k, the adjacency matrix for the k-cube Q_k is*

$$
A_k = \begin{bmatrix} C_{2^{k-1}} & I_{2^{k-1}} \\ I_{2^{k-1}} & C_{2^{k-1}} \end{bmatrix},
$$

where $C_{2^{k-1}}$ is defined as in Definition 2.2 and $I_{2^{k-1}}$ is the $(2^{k-1}) \times (2^{k-1})$ identity matrix.

THEOREM 2.5. *For each positive integer k, $\det(A_k) = 0$ if k is odd and $\det(A_k) = 1$ if k is even. In other words, not every puzzle on the k-cube Q_k is solvable if k is odd, whereas every puzzle on the k-cube Q_k is solvable if k is even.*

Proof. Observe that $\det(A_1) = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 0$ and $\det(A_2) = \det\left(\begin{bmatrix} C_2 & I_2 \\ I_2 & C_2 \end{bmatrix}\right) =$

$$\left[\det(C_2 + I_2)\right]^2 = \left[\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\right]^2 = 1. \text{ For } k \geq 3,$$

$$\begin{aligned}
\det(A_k) &= \det\left(\begin{bmatrix} C_{2^{k-1}} & I_{2^{k-1}} \\ I_{2^{k-1}} & C_{2^{k-1}} \end{bmatrix}\right) \\
&= \det\left((C_{2^{k-1}})^2 - (I_{2^{k-1}})^2\right) \\
&= \det\left((C_{2^{k-1}} - I_{2^{k-1}})(C_{2^{k-1}} + I_{2^{k-1}})\right) \\
&= \left[\det(C_{2^{k-1}} + I_{2^{k-1}})\right]^2 \\
&= \left[\det\left(\begin{bmatrix} C_{2^{k-2}} + I_{2^{k-2}} & I_{2^{k-2}} \\ I_{2^{k-2}} & C_{2^{k-2}} + I_{2^{k-2}} \end{bmatrix}\right)\right]^2 \\
&= \left[\det\left((C_{2^{k-2}} + I_{2^{k-2}})^2 - (I_{2^{k-2}})^2\right)\right]^2 \\
&= \left[\det(C_{2^{k-2}})^2\right]^2 \\
&= \left[\det(C_{2^{k-2}})\right]^4.
\end{aligned}$$

By Lemma 2.3, $\det(A_k) = 0$ if k is odd and $\det(A_k) = 1$ if k is even. $\square$

3. $G\square P_t$. Let G be a finite graph of order n and let P_t be a path of length t. The adjacency matrix of $G\square P_t$ where all entries on the main diagonal are equal to 1 is obtained from A_{P_t} by replacing each main diagonal entry by A_G, each off-diagonal 1 by I_n, and each off-diagonal 0 by O_n, where I_n is the $n \times n$ identity matrix and O_n is the $n \times n$ matrix of zeros. We define the sequence $\{A_k\}$ of G as follows: $A_0 = I_n$, $A_1 = A_G$, and $A_k = A_G \cdot A_{k-1} + A_{k-2}$. The sequence $\{A_k\}$ satisfies the Fibonacci recurrence with initial conditions shifted by one, so $A_k = f_{k+1}(A_G)$ $(k = 0, 1, 2, ...)$ with $f_0(A_G) = O_n$ and $f_1(A_G) = I_n$.

THEOREM 3.1. *Let G be a finite graph of order n. Then $\det(A_{G\square P_t}) = \det(A_t)$.*

Proof. We show that

$$A_{G\square P_t} = \begin{bmatrix} A_G & I_n & O_n & O_n & \cdots & O_n \\ I_n & A_G & I_n & O_n & \cdots & O_n \\ O_n & I_n & A_G & I_n & \cdots & O_n \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & O_n & I_n & A_G & I_n \\ O_n & O_n & \cdots & O_n & I_n & A_G \end{bmatrix}$$

is row equivalent to

$$\begin{bmatrix} O_n & O_n & O_n & \cdots & O_n & A_t \\ I_n & O_n & O_n & \cdots & O_n & A_{t-1} \\ O_n & I_n & O_n & \cdots & O_n & A_{t-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & O_n & I_n & O_n & A_2 \\ O_n & O_n & \cdots & O_n & I_n & A_1 \end{bmatrix}.$$

For each $1 \leq i \leq t-1$, add $A_1 \cdot r_{i+1} + A_2 \cdot r_{i+2} + A_3 \cdot r_{i+3} + ... + A_{t-i} \cdot r_t$ to row r_i. $\square$

Goldwasser et al. [6] studied properties of the Fibonacci polynomials over $GF(2)$. We extend several of the properties to $M_k(GF(2))$.

THEOREM 3.2. *Let $A \in M_k(GF(2))$ and let $f_0, f_1, f_2, ...$ be the sequence of Fibonacci polynomials over $M_k(GF(2))$ with $f_0(A) = O_k$ and $f_1(A) = I_k$. Then*

    1. $Af_n(A) = f_n(A)A$ for $n \geq 2$.
    2. $f_{n-t}(A) + f_{n+t}(A) = Af_n(A)f_t(A)$ for $0 \leq t \leq n$.
    3. $f_{2n}(A) = Af_n^2(A)$ for $n \geq 0$.
    4. $f_{2^n}(A) = A^{(2^n-1)}$ for $n \geq 1$.

Proof. We use induction to prove (1). The base case ($n = 2$) is straightforward. Suppose $Af_n(A) = f_n(A)A$ for all $n \leq l$. Then

$$
\begin{aligned}
Af_{l+1}(A) &= A(Af_l(A) + f_{l-1}(A)) \\
&= AAf_l(A) + Af_{l-1}(A) \\
&= Af_l(A)A + f_{l-1}(A)A \qquad &\text{(by inductive hypothesis)} \\
&= (Af_l(A) + f_{l-1}(A))A \\
&= f_{l+1}(A)A.
\end{aligned}
$$

To prove (2), we fix n and use induction on t. The base case, $t = 0$ (or $t = 1$) is straightforward. For the inductive step we have that

$$
\begin{aligned}
f_{n-t}(A) + f_{n+t}(A) &= (Af_{n-t+1}(A) + f_{n-t+2}(A)) + (Af_{n+t-1}(A) + f_{n+t-2}(A)) \\
&= A(f_{n-(t-1)}(A) + f_{n+(t-1)}(A)) + (f_{n-(t-2)}(A) + f_{n+(t-2)}(A)) \\
&= A(Af_n(A)f_{t-1}(A)) + (Af_n(A)f_{t-2}(A)) \\
&= A[(Af_n(A)f_{t-1}(A)) + (f_n(A)f_{t-2}(A))] \\
&= A[f_n(A)Af_{t-1}(A) + f_n(A)f_{t-2}(A)] \\
&= Af_n(A)(Af_{t-1}(A) + f_{t-2}(A)) \\
&= Af_n(A)f_t(A),
\end{aligned}
$$

where the third equality follows from the inductive hypothesis, the first and last equalities each follow from the Fibonacci recurrence relation, and the fifth equality follows from property (1). The special case of (2) where $n = t$ gives (3).

We use induction to prove (4). The base case ($n = 1$) is obvious. Assume $f_{2^k}(A) = A^{(2^k-1)}$ for some $k > 1$. Then

$$
\begin{aligned}
f_{2^{k+1}}(A) &= f_{2 \cdot 2^k}(A) \\
&= Af_{2^k}^2(A) \qquad &\text{(by property (3))} \\
&= Af_{2^k}(A)f_{2^k}(A) \\
&= AA^{(2^k-1)}A^{(2^k-1)} \qquad &\text{(by inductive hypothesis)} \\
&= A^{(2^{k+1}-1)}.
\end{aligned}
$$

□

THEOREM 3.3. *Let t be an odd positive integer. Then $t + 1 = 2^i \cdot j$, where $i \geq 1$ and $j \geq 1$ is the largest odd divisor of $t + 1$.*

1. *If $j > 1$, then $\det(A_{G \square P_t}) = (\det A_G)^{(2^i - 1)} \cdot (\det A_{j-1})^{2^i}$. Hence, every puzzle on $G \square P_t$ is solvable if and only if every puzzle on G and every puzzle on $G \square P_{j-1}$ is solvable.*
2. *If $j = 1$, then $\det(A_{G \square P_t}) = (\det A_G)^{2^i - 1}$. Hence, every puzzle on $G \square P_t$ is solvable if and only if every puzzle on G is solvable for $t = 2^i - 1$.*

Proof. To prove (1), suppose $j > 1$. We apply property (3) of Theorem 3.2 i times. By Theorem 3.1,

$$
\begin{aligned}
\det(A_{G \square P_t}) &= \det(A_t) \\
&= \det(f_{t+1}(A_G)) \\
&= \det(f_{2 \cdot 2^{i-1} \cdot j}(A_G)) \\
&= \det(A_G f^2_{2^{i-1} \cdot j}(A_G)) \\
&= \det(A_G(A_G f^2_{2^{i-2} \cdot j}(A_G))^2) \\
&= \ldots = \det(A_G^{2^i - 1} f_j^{2^i}(A_G)) \\
&= (\det A_G)^{2^i - 1} \cdot \det(f_j(A_G))^{2^i} \\
&= (\det A_G)^{2^i - 1} \cdot (\det A_{j-1})^{2^i}.
\end{aligned}
$$

To prove (2), suppose $j = 1$. By Theorem 3.1,

$$
\begin{aligned}
\det(A_{G \square P_t}) &= \det(A_t) \\
&= \det(f_{t+1}(A_G)) \\
&= \det(f_{2^i}(A_G)) \\
&= \det(A_G^{(2^i - 1)}) \\
&= (\det A_G)^{2^i - 1},
\end{aligned}
$$

where the fourth equality follows from property (4) of Theorem 3.2. □

Theorem 3.3 is especially useful if $t = 2^i - 1$ for some $i \geq 1$. For example, the grid graph $P_k \square P_t$ is universally solvable if $k \equiv 0 \pmod 3$ or $k \equiv 1 \pmod 3$ as P_k is universally solvable for these values of k, and $P_k \square P_t$ is not universally solvable if $k \equiv 2 \pmod 3$ as P_k is not universally solvable for these values of k.

The following special case of Theorem 3.3 where t is odd and $t \equiv 1 \pmod 4$ provides a quick computation of $\det(A_{G \square P_t})$.

THEOREM 3.4. *If $t \equiv 1 \pmod 4$, then $\det(A_{G \square P_t}) = \det(A_G) \cdot (\det(A_{\frac{t-1}{2}}))^2$. Hence, every puzzle on $G \square P_t$ is solvable if and only if every puzzle on G and every puzzle on $G \square P_{\frac{t-1}{2}}$ is solvable.*

Proof. If $t \equiv 1 \pmod 4$, then $t + 1 \equiv 2 \pmod 4$. As such, $t + 1 = 2 \cdot \lceil \frac{(t+1)-1}{2} \rceil = 2 \cdot \lceil \frac{t}{2} \rceil$. By

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Theorem 3.1,

$$
\begin{aligned}
\det(A_{G \square P_t}) &= \det(A_t) \\
&= \det(f_{t+1}(A_G)) \\
&= \det(A_G \cdot f^2_{\lceil \frac{t}{2} \rceil}(A_G)) \\
&= \det(A_G) \cdot (\det(f_{\lceil \frac{t}{2} \rceil}(A_G)))^2 \\
&= \det(A_G) \cdot (\det(A_{\frac{t-1}{2}}))^2,
\end{aligned}
$$

where the third equality follows from property (3) of Theorem 3.2. □

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