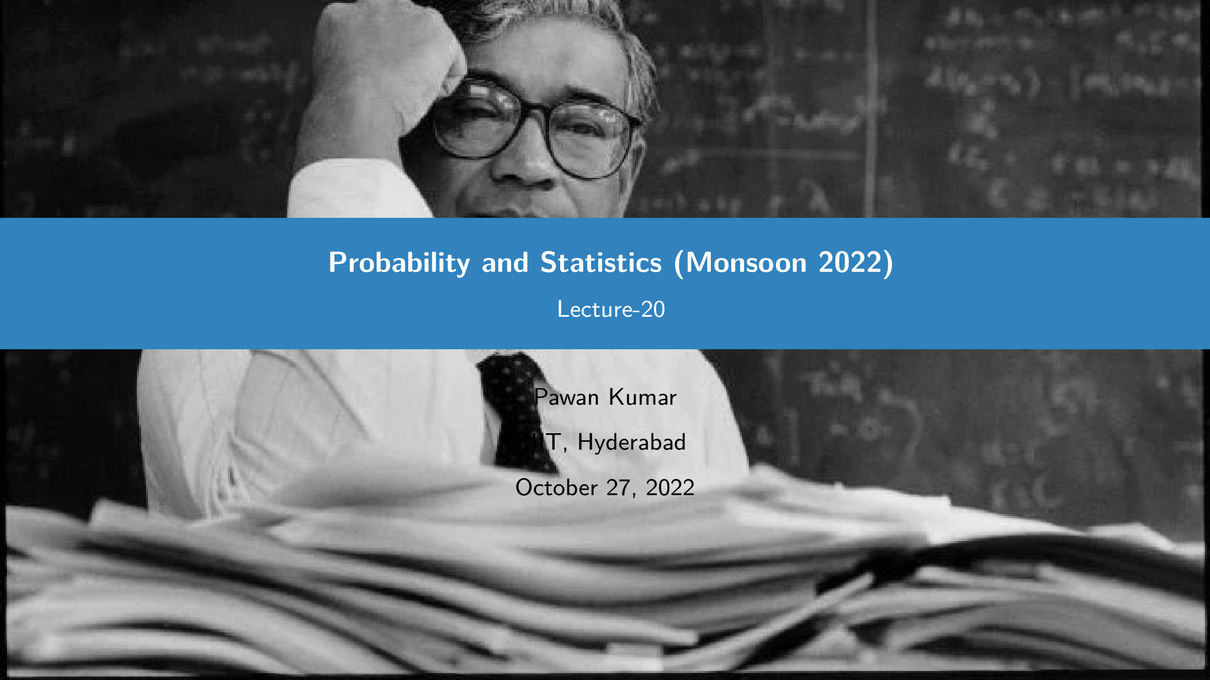

Probability and Statistics (Monsoon 2022)

Lecture-20

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Maximum Likelihood Estimation

Outline

Example (Prediction Using Data)

I have a bag that contains 3 balls. Each ball is either red or blue, but I have no information in addition to this. Thus, the number of blue balls, call it θ, might be 0, 1, 2, or 3. I am allowed to choose 4 balls at random from the bag with replacement. We define the random variables X_1, X_2, X_3, and X_4 as follows

$$X_i = \begin{cases} 1 & \text{if the } ith \text{ chosen ball is blue} \\ 0 & \text{if the } ith \text{ chosen ball is red} \end{cases}$$

We observe here that X_i's are i.i.d. and $X_i \sim \text{Bernoulli}\left(\frac{\theta}{3}\right)$. After the experiment, we observe the values for X_i's

$$x_1 = 1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 1.$$

From above, we have 3 blue balls and 1 red ball. Answer the following

1. Find the probability of the observed sample $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ for each possible θ
2. Find the value of θ that maximizes the probability of the observed sample

Answer to previous problem...

Answer to previous problem...

Likelihood and log likelihood Function...

Definition of Likelihood and log likehood Function

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a distribution with a parameter θ. Suppose that we have observed $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

1. If X_i's are discrete, then the likelihood function is defined as

$$L(x_1, x_2, \ldots, x_n; \theta) = P_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; \theta)$$

2. If X_i's are jointly continuous, then the likelihood function is defined as

$$L(x_1, x_2, \ldots, x_n; \theta) = f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; \theta)$$

In some problems, it is easier to work with the log likelihood function given by

$$\ln L(x_1, x_2, \ldots, x_n; \theta)$$

Example

Example (Example)

Find the likelihood function for the following random sample

1. $X_i \sim \text{Binomial}(3, \theta)$ and we have observed $(x_1, x_2, x_3, x_4) = (1, 3, 2, 2)$
2. $X_i \sim \text{Exponential}(\theta)$ and we have observed $(x_1, x_2, x_3, x_4) = (1.23, 3.32, 1.98, 2.12)$

Answer to previous problem...

Answer to previous problem...

Maximum Likelihood Estimator...

Definition of maximum likelihood estimator

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a distribution with a parameter θ. Given that we have observed $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, a maximum likelihood estimate of θ, shown by $\hat{\theta}_{ML}$ is a value of θ that maximizes the likelihood function

$$L(x_1, x_2, \ldots, x_n; \theta)$$

A maximum likelihood estimator (MLE) of the parameter θ, shown by $\hat{\theta}_{ML}$ is a random variable $\hat{\theta}_{ML} = \hat{\theta}_{ML}(X_1, X_2, \ldots, X_n)$ whose value when $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$ is given by $\hat{\theta}_{ML}$.

Example of Maximum Likelihood Estimator...

Example

For the following examples, find the maximum likelihood estimator (MLE) of θ :

1. $X_i \sim \text{Binomial}(m, \theta)$, and we have observed $X_1, X_2, \ldots, X_n$
2. $X_i \sim \text{Exponential}(\theta)$ and we have observed $X_1, X_2, \ldots, X_n$

Answer to previous problem...

Answer to previous problem...

Example of Maximum Likelihood Estimators...

Example (Example of maximum likelihood estimator)

Suppose that we have observed the random sample $X_1, X_2, X_3, ..., X_n$, where $X_i \sim N(\theta_1, \theta_2)$ so

$$f_{X_i}(x_i; \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\dfrac{(x_i - \theta_1)^2}{2\theta_2}}$$

Find the maximum likelihood estimators for θ_1 and θ_2.

Answer to previous problem...

Answer to previous problem...