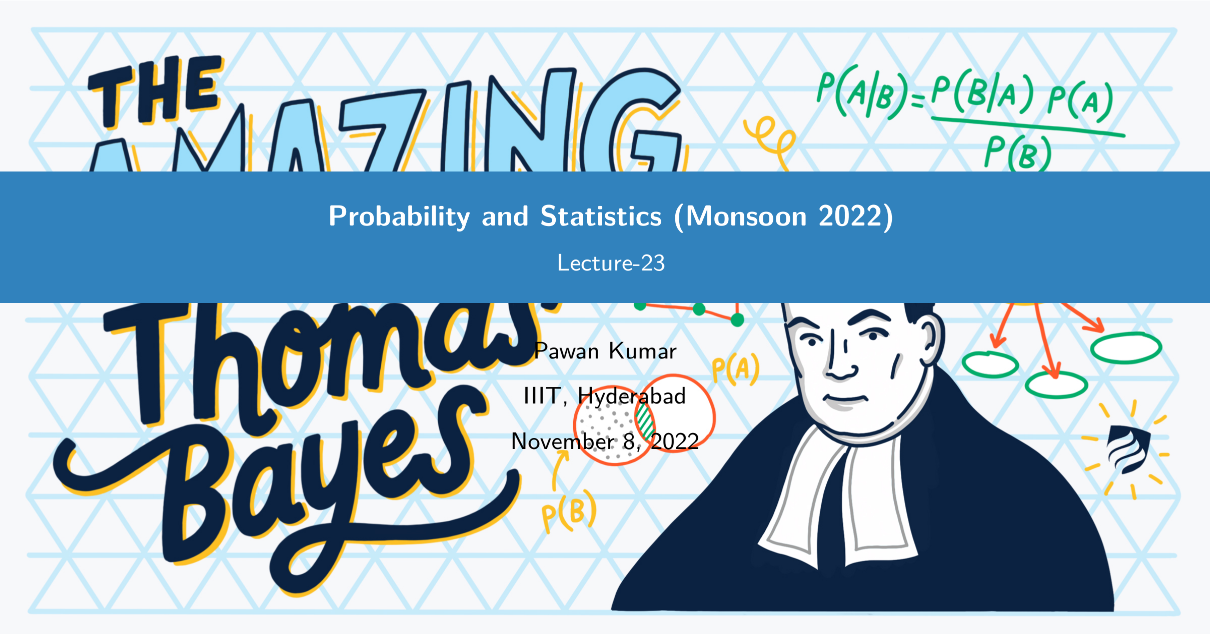

Probability and Statistics (Monsoon 2022)

Lecture-23

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Hypothesis Testing Problem

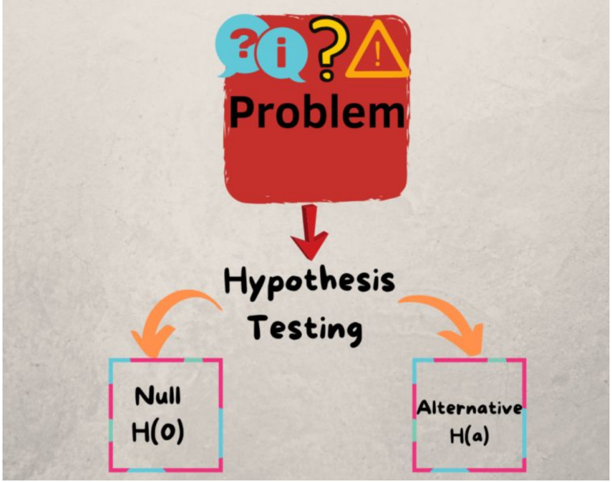

Figure: Illustration of Hypothesis Testing

Example (Drug is Effective or Not?)

- For example, a pharmaceutical company might be interested in knowing if a new drug is effective in treating a disease. Here, there are two hypotheses.
    1. drug is not effective.
    2. drug is effective.

Example (Aircraft Present of Not?)

- Consider a radar system that uses radio waves to detect aircraft. The system receives a signal and, based on the received signal, it needs to decide whether an aircraft is present or not. Here, there are again two opposing hypotheses:
    1. $H0$: No aircraft is present.
    2. $H1$: An aircraft is present.

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Some definitions

- The hypothesis H_0 is called the null hypothesis and the hypothesis.
- H_1 is called the alternative hypothesis.
- The null hypothesis, H_0, is usually referred to as the default hypothesis, i.e., the hypothesis that is initially assumed to be true.
- The alternative hypothesis, H_1, is the statement contradictory to H_0.
- Based on the observed data, we need to decide either to accept H_0, or to reject it, in which case we say we accept H_1.

Motivation for Hypothesis Testing with an Example

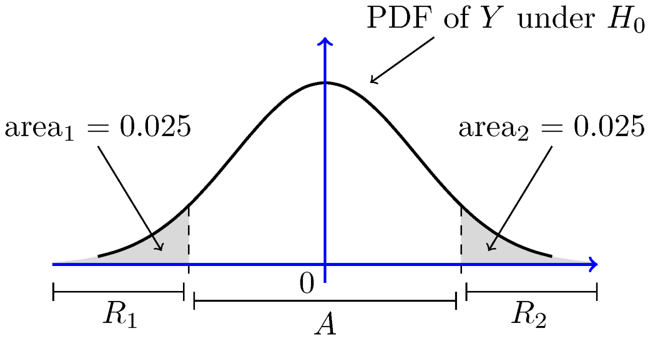
Example ((FairCoin))

You have a coin and you would like to check whether it is fair or not. More specifically, let θ be the probability of heads, $\theta = P(H)$. You have two hypotheses:

- $H0$ (the null hypothesis): The coin is fair, i.e. $\theta = \theta_0 = 1/2$.
- $H1$ (the alternative hypothesis): The coin is not fair, i.e., $\theta \neq 1/2$.

https://www.probabilitycourse.com/chapter8/8_4_1_intro.php

Example Illustration



$$A = \text{Acceptance Region}$$

$$R = R_1 \cup R_2 = \text{Rejection Region}$$

$$\alpha = \mathrm{P}(\text{type I error}) = \text{area}_1 + \text{area}_2 = 0.05$$

Figure: Illustration of previous example.

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Some definitions

Definition

(Hypothesis, Null, Alternate)

- Suppose θ be an unknown parameter. A hypothesis is a statement such as $\theta = 1, \theta > 1.3, \theta \neq 0.5$, etc.
- In hypothesis testing problem we need to decide between two contradictory hypotheses.
- More precisely, let S be the set of possible values for θ.
- Suppose that we can partition S into two disjoint sets S_0 and S_1.
- Let H_0 be the hypothesis that $\theta \in S_0$, and let H_1 be the hypothesis that $\theta \in S_1$.
  - H_0 (the null hypothesis): $\theta \in S_0$.
  - H_1 (the alternative hypothesis): $\theta \in S_1$.

Example

- In previous example (FairCoin), $S = [0, 1], S_0 = \{1/2\}$, and $S_1 = [0, 1] - \{1/2\}$.
- H_0 is an example of a simple hypothesis because S_0 contains *only one* value of θ.
- H_1 is an example of composite hypothesis since S_1 contains *more than one* element.
- Often the case that the null hypothesis is chosen to be a simple hypothesis.

Test Statistics

Remark

Often, to decide between H_0 and H_1, we look at a function of the observed data. For instance, in previous example, we looked at

$$Y = \frac{X - n\theta_0}{\sqrt{n\theta_0(1-\theta_0)}}, \tag{1}$$

where X was the total number of heads. Here, X is a function of the observed data (sequence of heads and tails), and thus Y is a function of the observed data. We call Y a statistic.

Definition (Test Statistics)

Let $X_1, X_2, \ldots, X_n$ be a random sample of interest. A statistic is a real-valued function of the data. For example, the sample mean, defined as

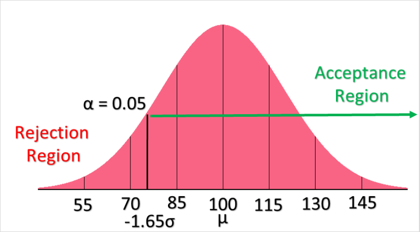
$$W(X_1, X_2, \cdots, X_n) = \frac{X_1 + X_2 + \ldots + X_n}{n}, \tag{2}$$

is a statistic. A test statistic is a statistic based on which we build our test.

Definitions: Acceptance region, rejection region, etc.

Definition (Acceptance, Rejection Region)

- To decide whether to choose H_0 or H_1, we choose a test statistic, $W = W(X_1, X_2, \ldots, X_n)$.
- Assuming H_0, we can define the set $A \subset \mathbb{R}$ as the set of possible values of W for which we would accept H_0.
  - Set A is called the acceptance region
  - Set $R = \mathbb{R} - A$ is called the rejection region.
  - In the example (FairCoin) before, $A = [-1.96, 1.96]$ and $R = (-\infty, -1.96) \cup (1.96, \infty)$.



Definitions of Type-I Error and Significance level

Definition (Type-I Error)

- We define type I error as the event that we reject H_0 when H_0 is true.
- Probability of type I error in general depends on the real value of θ.

$$\begin{aligned} P(\text{type I error } \mid \theta) &= P(\text{Reject } H_0 \mid \theta) \\ &= P(W \in R \mid \theta), \quad \text{for } \theta \in S_0. \end{aligned}$$

Definition (Significance level)

If the probability of type I error satisfies

$$P(\text{type I error}) \leq \alpha, \quad \text{for all } \theta \in S_0, \tag{3}$$

then we say the test has significance level α or simply the test is a level α test.

Definition of Type-II Error

Definition (Type-II Error)

- The second possible error that we can make is to accept H_0 when H_0 is false. This is called the type II error.
- Since the alternative hypothesis, H_1, is usually a *composite* hypothesis (so it includes more than one value of θ), the probability of type II error is usually a function of θ.
- The probability of type II error is usually shown by β :

$$\beta(\theta) = P(\text{Accept } H_0 \mid \theta), \quad \text{for } \theta \in S_1. \tag{4}$$

Aircraft Present or Not?

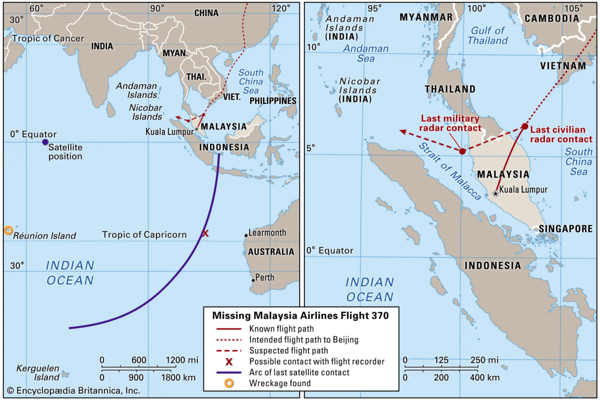


Figure: Disappearance of Flight 370, Malaysian Airlines, 8th March, 2014 from KL to Beijing

Example (RADAR)

Consider a radar system that uses radio waves to detect aircraft. The system receives a signal and based on the received signal, it needs to decide whether an aircraft is present or not. Let X be the received signal. Suppose that we know

$$X = W \quad \text{if no aircraft is present.}$$
$$X = 1 + W \quad \text{if an aircraft is present,}$$

where $W \sim N(0, \sigma^2 = \frac{1}{9})$. Thus, we can write $X = \theta + W$, where $\theta = 0$ if there is no aircraft, and $\theta = 1$ if there is an aircraft. Let H_0 (null hypothesis): No aircraft is present, and H1 (alternative hypothesis): An aircraft is present.

1. Write the null hypothesis, H_0, and the alternative hypothesis, H_1, in terms of possible values of θ.
2. Design a level 0.05 test ($\alpha = 0.05$) to decide between H_0 and H_1.
3. Find the probability of type II error, β, for the above test. Note that this is the probability of missing a present aircraft.
4. If we observe $X = 0.6$, is there enough evidence to reject H_0 at significance level $\alpha = 0.01$?
5. If we would like the probability of missing a present aircraft to be less than 5%, what is the smallest significance level that we can achieve?