Probability & Statistics
Tutorial I

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Problem 1

Consider the sample space $\Omega = \{a, b, c, d, e\}$. Given that $\{a, b, e\}$ and $\{b, c\}$ are both events (elements of the σ-algebra), what other events are implied by taking union, intersection and complement?

Problem 2

Let $\{A_i : i \in I\}$ be a collection of sets. Prove **De Morgan's Laws**:

$$\left(\bigcap_i A_i\right)^c = \bigcup_i A_i$$

and

$$\left(\bigcup_i A_i\right)^c = \bigcap_i A_i$$

Problem 3

Prove Boole's inequality

$$P\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \sum_{i=1}^{\infty} P(B_i)$$

Problem 4

State and prove the Inclusion-Exclusion principle for $P(\cup_{i=1}^{n} A_i)$.

Problem 5

Let $\mathcal{F}$ be a a σ-algebra of subsets of Ω, and suppose $P : \mathcal{F} \to [0, 1]$ satisfies: (i) $P(\Omega) = 1$, and (ii) P is additive, in that $P(A \cup B) = P(A) + P(B)$ whenever $A \cap B = \phi$. Show that $P(\phi) = 0$.

Problem 6

If $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$. Then show that

$$\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$$

Problem 7

Prove that

$$P(E \cup F \cup G) = P(E)+P(F)+P(G)-P(E^cFG)-P(EF^cG)-P(EFG^c)-2P(EFG)$$

Problem 8

Express $[a, b]$ as a countable union or intersection over open intervals.

Problem 9

Let A, B and C be three events in the sample space S. Suppose we know

- $A \cup B \cup C = S$
- $P(A) = 1/2$
- $P(B) = 2/3$
- $P(A \cup B) = 5/6$

Answer the following questions:

a) Find $P(A \cap B)$

b) Do A, B, C form a partition of S?

c) Find $P(C - (A \cup B))$

d) If $P(C \cap (A \cup B)) = 5/12$, find $P(C)$

Problem 10

Show that if $P(A_i) = 1$ for all $i \geq 1$, then

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1$$

Problem 11

Let $\mathcal{F}$ be a σ-algebra on Ω and suppose that $B \in \mathcal{F}$.

Show that $\mathcal{G} = \{A \cap B : A \in \mathcal{F}\}$ is a σ-algebra of subsets of B.

Problem 12

There are n urns of which the r^{th} contains $r-1$ red balls and $n-r$ magenta balls. You pick an urn at random and remove two balls at random without replacement. Find the probability that:

(a) the second ball is magenta

(b) the second ball is magenta, given that the first is magenta.

Problem 13

A box contains three coins: two regular coins and one fake two-headed coin ($P(H) = 1$),

(i) You pick a coin at random and toss it. What is the probability that it lands heads up?

(ii) You pick a coin at random and toss it, and get heads. What is the probability that it is the two-headed coin?

Problem 14

Given the following:

- 60% of all the emails is spam
- 20% of spam has the word 'DEA'
- 1% of non-spam has the word 'DEA'

You get an email with the word 'DEA'. Find the probability of it being spam.

Problem 15

I roll a fair die. Let A be the event that the outcome is an odd number, i.e. $A = \{1, 3, 5\}$. Also, let B be the event that the outcome is less than or equal to 3, i.e. $B = \{1, 2, 3\}$. What is $P(A)$? What is $P(A|B)$?

Problem 16

Show that for any events E and F,

$$P(E|E \cup F) \geq P(E|F)$$

Problem 17

Three coins are tossed at the same time. We say A as the event of receiving at least 2 heads. Likewise, B denotes the event of getting no heads and C is the event of getting heads on the second coin. Which of these is mutually exclusive?