

Probability & Statistics

Tutorial 5

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September 16, 2022

Problem 1

Let X and Y be random variables with joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{ay}{x^2} & x \geq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- (a) What is a ?
- (b) What is the conditional PDF $f_{Y|X}(y|x)$ of Y given $X = x$?
- (c) What is the conditional expectation of Y given X ?
- (d) What is the expected value of Y ?

Problem 2

Let X be the number of ice-cream cones a vendor sells on a day. If the average temperature of a summer day is a random variable Y (in Fahrenheit), where $Y \sim \text{Uniform}([95, 105])$. We also have $X \sim \text{Poisson}(Y^2/10 + Y/5 + 5)$.

- (a) What is $E[X|Y]$?
- (b) What is $E[X]$?

Problem 3

If there are no distractions, it takes me 30 minutes to walk to the store. However, if I pass someone with a dog, I stop and pet the dog and chat to the owner. The number Y of dogs I pass is a Poisson random variable with mean 2. Each time I stop, the number of minutes I spend petting the dog and chatting is an exponential random variable with PDF:

$$f_X(x) = 0.5e^{-0.5x}$$

- (a) If I see a single dog, what is the expectation and variance of the time spent petting the dog and chatting to its owner?
- (b) What is the conditional expectation of the total time spent petting dogs and chatting to their owners, as a function of Y ?
- (c) Using the law of iterated expectation calculate $E[X]$.

Problem 4

Let X and Y be independent Poisson random variables with parameter 1. Compute the following. (Give a correct formula involving sums — does not need to be in closed form.)

- (a) The probability mass function for X given that $X + Y = 5$.
- (b) The conditional expectation of Y^2 given that $X = 2Y$.
- (c) The probability mass function for $X - 2Y$ given that $X > 2Y$.
- (d) The probability that $X = Y$.

Problem 5

Suppose that the joint density of X and Y is given by

$$f(x, y) = \begin{cases} \frac{e^{-x/y} e^{-y}}{y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X > 1 \mid Y = y)$.

Problem 6

Show that

$$E[aY + bZ \mid X] = aE[Y \mid X] + bE[Z \mid X] \text{ for } a, b \in \mathbb{R}$$

Problem 7

Each child is equally likely to be male or female, independently of all other children.

- (a) Show that, in a family of predetermined size, the expected number of boys equals the expected number of girls. Was the assumption of independence necessary?
- (b) A randomly selected child is male; does the expected number of his brothers equal the expected number of his sisters? What happens if you do not require independence?

Problem 8

Consider two continuous random variables Y and Z , and a random variable X that is equal to Y with probability p and to Z with probability $1 - p$.

- (a) Show that the PDF of X is given by

$$f_X(x) = pf_Y(x) + (1 - p)f_Z(x).$$

- (b) Calculate the CDF of the two-sided exponential random variable that has PDF given by

$$f_X(x) = \begin{cases} p\lambda e^{\lambda x}, & \text{if } x < 0 \\ (1 - p)\lambda e^{-\lambda x}, & \text{if } x \geq 0 \end{cases}$$

where $\lambda > 0$ and $0 < p < 1$.

Problem 9

Let X and Y be independent and have PMFs given by

$$p_X(x) = \begin{cases} \frac{1}{3} & \text{if } x = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases} \quad p_Y(y) = \begin{cases} \frac{1}{2} & \text{if } y = 0, \\ \frac{1}{3} & \text{if } y = 1, \\ \frac{1}{6} & \text{if } y = 2, \\ 0 & \text{otherwise} \end{cases}$$

Calculate the PMF of $W = X + Y$ by convolution.

Problem 10

Let Q be a continuous random variable with PDF:

$$f_Q(q) = \begin{cases} 6q(1-q) & \text{if } 0 \leq q \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

This Q represents the probability of success of a Bernoulli random variable X , i.e.,

$$P(X = 1 \mid Q = q) = q$$

Find $f_{Q|X}(q|x)$ for $x \in \{0, 1\}$ and all q .

Problem 11

Shashank performs an experiment comprising a series of independent trials. On each trial, he simultaneously flips a set of Z fair coins.

1. Given that Shashank has just had a trial with Z tails, what is the probability that next two trials will also have this result?
2. Sandeep conducts an experiment like Shashank's, except that he uses M coins for the first trial, and then he obeys the following rule: Whenever all the coins land on the same side in a trial, Sandeep permanently removes one coin from the experiment and continues with the trials. He follows this rule until the $(M - 1)^{th}$ time he removes a coin, at which point the experiment ceases. Find $E[X]$, where X is the number of trials in Sandeep's experiment.

Problem 12

Let X be a continuous random variable with PDF:

$$f_X(x) = \begin{cases} x^2 (2x + \frac{3}{2}) & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $Y = \frac{2}{X} + 3$, find $Var(Y)$.