## Question 1

$$F(x) = 1 - e^{-x} \qquad x > 0$$
$$U \sim Uniform(0,1)$$
$$X = F^{-1}(U)$$
$$= -\ln(1-U)$$
$$X \sim F$$

This formula can be simplified since

$$1 - U \sim \quad Uniform(0,1)$$

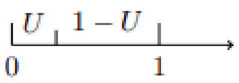


Figure 13.3: Symmetry of Uniform

Hence we can simulate $X$ using

$$X = -\ln(U)$$

$$U = runif(1, min = 0, max = 1);$$
$$X = -log(U)$$

## Question 2

Apply the inverse transform method. In this case

$$F(x) = \begin{cases} \frac{(x-2)^2}{4} & \text{if } 2 \le x \le 3 \\ \frac{1}{4} + (x-3)(\frac{3}{4} - \frac{x}{12}) & \text{if } 3 < x < 6 \end{cases}$$

Inverting this c.d.f. gives

$$F^{-1}(U) = \begin{cases} 2 + 2\sqrt{U} & \text{if } 0 \le U \le 1/4 \\ 6 - 2\sqrt{3 - 3U} & \text{if } U > 1/4 \end{cases}$$

## Question 3

$$F(x) = \begin{cases} \frac{e^{2x}}{2} \text{ if } x < 0 \\ \frac{2-e^{-2x}}{2} \text{ if } x > 0 \end{cases}$$

$$X = F^{-1}(U) = \begin{cases} \frac{\ln(2U)}{2} \text{ if } U < 1/2 \\ -\frac{\ln(2-2U)}{2} \text{ if } U > 1/2 \end{cases}$$

## Question 4

| s: | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $f$(s) | .216 | .432 | .288 | .064 |
| $F$(s) | .216 | .648 | .936 | 1.00 |

A binomial value of 2 will be simulated by a uniform (0,1) value that is both greater than or equal to .648 and less than 0.936 . Thus, the uniform numbers 0.71 and 0.66 result in simulated binomial values of 2.

## Question 5

https://math.stackexchange.com/questions/3693280/generate-a-poisson-random-variable-from-a-standard-uniform-random-variable

**Question 6:**
**Part 1**

Define the sets $A$ and $B$ as follows:

$$A = \left\{ s \in S : \lim_{n \to \infty} X_n(s) = X(s) \right\},$$
$$B = \left\{ s \in S : \lim_{n \to \infty} Y_n(s) = Y(s) \right\}.$$

By definition of almost sure convergence, we conclude $P(A) = P(B) = 1$. Therefore, $P(A^c) = P(B^c) = 0$. We conclude

$$\begin{aligned} P(A \cap B) &= 1 - P(A^c \cup B^c) \\ &\geq 1 - P(A^c) - P(B^c) \\ &= 1. \end{aligned}$$

Thus, $P(A \cap B) = 1$. Now, consider the sequence $\{Z_n, n = 1, 2, \cdots\}$, where $Z_n = X_n + Y_n$, and define the set $C$ as

$$C = \left\{ s \in S : \lim_{n \to \infty} Z_n(s) = X(s) + Y(s) \right\}.$$

We claim $A \cap B \subset C$. Specifically, if $s \in A \cap B$, then we have

$$\lim_{n \to \infty} X_n(s) = X(s), \qquad \lim_{n \to \infty} Y_n(s) = Y(s).$$

Therefore,

$$\begin{aligned} \lim_{n \to \infty} Z_n(s) &= \lim_{n \to \infty} \left[ X_n(s) + Y_n(s) \right] \\ &= \lim_{n \to \infty} X_n(s) + \lim_{n \to \infty} Y_n(s) \end{aligned}$$

We claim $A \cap B \subset C$. Specifically, if $s \in A \cap B$, then we have

$$\lim_{n \to \infty} X_n(s) = X(s), \qquad \lim_{n \to \infty} Y_n(s) = Y(s).$$

Therefore,

$$\begin{aligned} \lim_{n \to \infty} Z_n(s) &= \lim_{n \to \infty} \left[ X_n(s) + Y_n(s) \right] \\ &= \lim_{n \to \infty} X_n(s) + \lim_{n \to \infty} Y_n(s)
$$\\ &= X(s) + Y(s). \end{aligned}$$

Thus, $s \in C$. We conclude $A \cap B \subset C$. Thus,

$$P(C) \geq P(A \cap B) = 1,$$

which implies $P(C) = 1$. This means that $Z_n \xrightarrow{a.s.} X + Y$.

**Part 2**

For $n \in \mathbb{N}$, define the following events

$$A_n = \left\{ |X_n - X| < \frac{\epsilon}{2} \right\},$$
$$B_n = \left\{ |Y_n - Y| < \frac{\epsilon}{2} \right\}.$$

Since $X_n \ \xrightarrow{p}\ X$ and $Y_n \ \xrightarrow{p}\ Y$, we have for all $\epsilon > 0$

$$\lim_{n \to \infty} P\big(A_n\big) = 1,$$
$$\lim_{n \to \infty} P\big(B_n\big) = 1.$$

We can also write

$$\begin{aligned} P(A_n \cap B_n) &= P(A_n) + P(B_n) - P(A_n \cup B_n) \\ &\geq P(A_n) + P(B_n) - 1. \end{aligned}$$

Therefore,

$$\lim_{n \to \infty} P\big(A_n \cap B_n\big) = 1.$$

Now, let us define the events $C_n$ and $D_n$ as follows:

$$C_n = \left\{ |X_n - X| + |Y_n - Y| < \epsilon \right\},$$
$$D_n = \left\{ |X_n + Y_n - X - Y| < \epsilon \right\}.$$

Now, let us define the events $C_n$ and $D_n$ as follows:

$$C_n = \Big\{|X_n - X| + |Y_n - Y| < \epsilon\Big\},$$
$$D_n = \Big\{|X_n + Y_n - X - Y| < \epsilon\Big\}.$$

Now, note that $(A_n \cap B_n) \subset C_n$, thus $P(A_n \cap B_n) \leq P(C_n)$. Also, by the triangle inequality for absolute values, we have

$$|(X_n - X) + (Y_n - Y)| \leq |X_n - X| + |Y_n - Y|.$$

Therefore, $C_n \subset D_n$, which implies

$$P(C_n) \leq P(D_n).$$

We conclude

$$P(A_n \cap B_n) \leq P(C_n) \leq P(D_n).$$

Since $\lim_{n\to\infty} P(A_n \cap B_n) = 1$, we conclude $\lim_{n\to\infty} P(D_n) = 1$. This by definition means that $X_n + Y_n \xrightarrow{p} X + Y$.

**Question 7:**

a. $Y_n \xrightarrow{d} 0$: Note that

$$F_{X_n}(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Also, note that $R_{Y_n} = [0, 1]$. For $0 \leq y \leq 1$, we can write

$$\begin{aligned} F_{Y_n}(y) &= P(Y_n \leq y) \\ &= 1 - P(Y_n > y) \\ &= 1 - P(X_1 > y, X_2 > y, \cdots, X_n > y) \\ &= 1 - P(X_1 > y)P(X_2 > y)\cdots P(X_n > y) \qquad \text{(since } X_i\text{'s are independent)} \\ &= 1 - (1 - F_{X_1}(y))(1 - F_{X_2}(y))\cdots(1 - F_{X_n}(y)) \\ &= 1 - (1 - y)^n. \end{aligned}$$

Therefore, we conclude

$$\lim_{n\to\infty} F_{Y_n}(y) = \begin{cases} 0 & y \leq 0 \\ 1 & y > 0 \end{cases}$$

Therefore, $Y_n \xrightarrow{d} 0$.

b. $Y_n \xrightarrow{p} 0$: Note that as we found in part (a)

$$F_{Y_n}(y) = \begin{cases} 0 & y < 0 \\ 1-(1-y)^n & 0 \leq y \leq 1 \\ 1 & y > 1 \end{cases}$$

In particular, note that $Y_n$ is a continuous random variable. To show $Y_n \xrightarrow{p} 0$, we need to show that

$$\lim_{n\to\infty} P\big(|Y_n| \geq \epsilon\big) = 0, \qquad \text{for all } \epsilon > 0.$$

Since $Y_n \geq 0$, it suffices to show that

$$\lim_{n\to\infty} P\big(Y_n \geq \epsilon\big) = 0, \qquad \text{for all } \epsilon > 0.$$

For $\epsilon \in (0, 1)$, we have

$$\begin{aligned} P\big(Y_n \geq \epsilon\big) &= 1 - P(Y_n < \epsilon) \\ &= 1 - P(Y_n \leq \epsilon) \qquad \text{(since } Y_n \text{ is a continuous random variable)} \\ &= 1 - F_{Y_n}(\epsilon) \\ &= (1-\epsilon)^n. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} P\big(|Y_n| \geq \epsilon\big) &= \lim_{n\to\infty} (1-\epsilon)^n \\ &= 0, \qquad \text{for all } \epsilon \in (0, 1]. \end{aligned}$$

c. $Y_n \xrightarrow{L^r} 0$, for all $r \geq 1$: By differentiating $F_{Y_n}(y)$, we obtain

$$f_{Y_n}(y) = \begin{cases} n(1-y)^{n-1} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, for $r \geq 1$, we can write

$$\begin{aligned} E|Y_n|^r &= \int_0^1 ny^r(1-y)^{n-1}dy \\ &\leq \int_0^1 ny(1-y)^{n-1}dy \qquad (\text{since } r \geq 1) \\ &= \Big[-y(1-y)^n\Big]_0^1 + \int_0^1 (1-y)^n dy \qquad (\text{integration by parts}) \\ &= \frac{1}{n+1}. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} E\left(|Y_n|^r\right) = 0.$$

d. $Y_n \xrightarrow{a.s} 0$: We will prove

d. $Y_n \xrightarrow{a.s} 0$: We will prove

$$\sum_{n=1}^{\infty} P\big(|Y_n| > \epsilon\big) < \infty,$$

which implies $Y_n \xrightarrow{a.s} 0$. By our discussion in part (b),

$$\begin{aligned} \sum_{n=1}^{\infty} P\big(|Y_n| > \epsilon\big) &= \sum_{n=1}^{\infty} (1-\epsilon)^n \\ &= \frac{1-\epsilon}{\epsilon} < \infty \qquad (\text{geometric series}). \end{aligned}$$

**Question 8:**

a. To show $X_n \xrightarrow{p} 0$, we can write, for any $\epsilon > 0$

$$\begin{aligned}\lim_{n\to\infty} P(|X_n| \geq \epsilon) &= \lim_{n\to\infty} P(X_n = n^2) \\ &= \lim_{n\to\infty} \frac{1}{n} \\ &= 0.\end{aligned}$$

We conclude that $X_n \xrightarrow{p} 0$.

b. For any $r \geq 1$, we can write

$$\begin{aligned}\lim_{n\to\infty} E\left(|X_n|^r\right) &= \lim_{n\to\infty} \left(n^{2r} \cdot \frac{1}{n} + 0 \cdot \left(1 - \frac{1}{n}\right)\right) \\ &= \lim_{n\to\infty} n^{2r-1} \\ &= \infty \qquad (\text{since } r \geq 1).\end{aligned}$$

Therefore, $X_n$ does not converge in the $r$th mean for any $r \geq 1$. In particular, it is interesting to note that, although $X_n \xrightarrow{p} 0$, the expected value of $X_n$ does not converge to $0$.

**Question 9:**

Let $X \sim Exponential(1)$. For $x \leq 0$, we have

$$F_{X_n}(x) = F_X(x) = 0, \qquad \text{for } n = 2, 3, 4, \cdots.$$

For $x \geq 0$, we have

$$\begin{aligned}\lim_{n\to\infty} F_{X_n}(x) &= \lim_{n\to\infty} \left(1 - \left(1 - \frac{1}{n}\right)^{nx}\right) \\ &= 1 - \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{nx} \\ &= 1 - e^{-x} \\ &= F_X(x), \qquad \text{for all } x.\end{aligned}$$

Thus, we conclude that $X_n \xrightarrow{d} X$.

**Question 10:**

where $\lfloor ny \rfloor$ is the largest integer less than or equal to $ny$. We then write

$$\begin{aligned}
\lim_{n\to\infty} F_{Y_n}(y) &= \lim_{n\to\infty} 1 - \left(1-\frac{\lambda}{n}\right)^{\lfloor ny \rfloor} \\
&= 1 - \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{\lfloor ny \rfloor} \\
&= 1 - e^{-\lambda y}.
\end{aligned}$$

The last equality holds because $ny - 1 \leq \lfloor ny \rfloor \leq ny$, and

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{ny} = e^{-\lambda y}.$$

Note that if $W \sim Geometric(p)$, then for any positive integer $l$, we have

$$\begin{aligned}
P(W \leq l) &= \sum_{k=1}^{l} (1-p)^{k-1} p \\
&= p \sum_{k=1}^{l} (1-p)^{k-1} \\
&= p \cdot \frac{1-(1-p)^l}{1-(1-p)} \\
&= 1-(1-p)^l.
\end{aligned}$$

Now, since $Y_n = \frac{1}{n} X_n$, for any positive real number, we can write

$$\begin{aligned}
P(Y_n \leq y) &= P(X_n \leq ny) \\
&= 1 - \left(1-\frac{\lambda}{n}\right)^{\lfloor ny \rfloor},
\end{aligned}$$

where $\lfloor ny \rfloor$ is the largest integer less than or equal to $ny$. We then write

$$\begin{aligned}
\lim_{n\to\infty} F_{Y_n}(y) &= \lim_{n\to\infty} 1 - \left(1-\frac{\lambda}{n}\right)^{\lfloor ny \rfloor} \\
&= 1 - \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{\lfloor ny \rfloor} \\
&= 1 - e^{-\lambda y}.
\end{aligned}$$

The last equality holds because $ny - 1 \leq \lfloor ny \rfloor \leq ny$, and

**Question 11:**

**2.** The following two observations are clear:

(a) $N(t) < n$ if and only if $T_n > t$,

(b) $T_{N(t)} \le t < T_{N(t)+1}$.

If $\mathbb{E}(X_1) < \infty$, then $\mathbb{E}(T_n) < \infty$, so that $\mathbb{P}(T_n > t) \to 0$ as $t \to \infty$. Therefore, by (a),

$$\mathbb{P}(N(t) < n) = \mathbb{P}(T_n > t) \to 0 \quad \text{as } t \to \infty,$$

implying that $N(t) \xrightarrow{\text{a.s.}} \infty$ as $t \to \infty$.

Secondly, by (b),

$$\frac{T_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{T_{N(t)+1}}{N(t)+1} \cdot (1 + N(t)^{-1}).$$

Take the limit as $t \to \infty$, using the fact that $T_n/n \xrightarrow{\text{a.s.}} \mathbb{E}(X_1)$ by the strong law, to deduce that $t/N(t) \xrightarrow{\text{a.s.}} \mathbb{E}(X_1)$.