PNS Tutorial 8 Solutions

Solution 1

Sample mean = 20/5 = 4
Sample variance = $(1^2+(-3)^2+(-1)^2+(-1)^2+4^2)/(5-1) = 7$
Sample standard deviation = √7

Solution 2

Solution

The only way that the maximum of the X_i will be less than or equal to 2 is if all of the X_i are less than or equal to 2. That is:

$$P(\max X_i \leq 2) = P(X_1 \leq 2, X_2 \leq 2, X_3 \leq 2)$$

Now, because X_1, X_2, X_3 are a random sample, we know that (1) X_i is independent of X_j, for all $i \neq j$, and (2) the X_i are identically distributed. Therefore:

$$P(\max X_i \leq 2) = P(X_1 \leq 2)P(X_2 \leq 2)P(X_3 \leq 2) = [P(X_1 \leq 2)]^3$$

The first equality comes from the independence of the X_i, and the second equality comes from the fact that the X_i are identically distributed. Now, the probability that X_1 is less than or equal to 2 is:

$$P(X \leq 2) = P(X=1) + P(X=2) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{1-1} + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{2-1} = \frac{3}{4} + \frac{3}{16} = \frac{15}{16}$$

Therefore, the probability that the maximum of the X_i is less than or equal to 2 is:

$$P(\max X_i \leq 2) = [P(X_1 \leq 2)]^3 = \left(\frac{15}{16}\right)^3 = 0.824$$

Solution 3

Recall that if X_i is a Bernoulli random variable with parameter p, then $E(X_i) = p$. Therefore:

$$E(\hat{p}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n} p = \frac{1}{n}(np) = p$$

The first equality holds because we've merely replaced $\hat{p}$ with its definition. The second equality holds by the rules of expectation for a linear combination. The third equality holds because $E(X_i) = p$. The fourth equality holds because when you add the value p up n times, you get np. And, of course, the last equality is simple algebra.

In summary, we have shown that:

$$E(\hat{p}) = p$$

Therefore, the maximum likelihood estimator is an unbiased estimator of p.

Solution 4

Recall that if X_i is a normally distributed random variable with mean μ and variance σ^2, then $E(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$. Therefore:

$$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1} \mu = \frac{1}{n}(n\mu) = \mu$$

The first equality holds because we've merely replaced $\bar{X}$ with its definition. Again, the second equality holds by the rules of expectation for a linear combination. The third equality holds because $E(X_i) = \mu$. The fourth equality holds because when you add the value μ up n times, you get $n\mu$. And, of course, the last equality is simple algebra.

In summary, we have shown that:

$$E(\bar{X}) = \mu$$

Therefore, the maximum likelihood estimator of μ is unbiased. Now, let's check the maximum likelihood estimator of σ^2. First, note that we can rewrite the formula for the MLE as:

$$\hat{\sigma}^2 = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) - \bar{X}^2$$

because:

$$\begin{aligned}\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x} + \bar{x}^2)\\ &= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\bar{x}\cdot\underbrace{\frac{1}{n}\sum x_i}_{\bar{x}} + \frac{1}{\cancel{n}}(\cancel{n}\bar{x}^2)\\ &= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2\end{aligned}$$

Then, taking the expectation of the MLE, we get:

$$E(\hat{\sigma}^2) = \frac{(n-1)\sigma^2}{n}$$

as illustrated here:

$$\begin{aligned}E(\hat{\sigma}^2) &= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right] = \left[\frac{1}{n}\sum_{i=1}^{n} E(X_i^2)\right] - E(\bar{X}^2)\\ &= \frac{1}{n}\sum_{i=1}^{n}(\sigma^2 + \mu^2) - \left(\frac{\sigma^2}{n} + \mu^2\right)\\ &= \frac{1}{n}(n\sigma^2 + n\mu^2) - \frac{\sigma^2}{n} - \mu^2\\ &= \sigma^2 - \frac{\sigma^2}{n} = \frac{n\sigma^2 - \sigma^2}{n} = \frac{(n-1)\sigma^2}{n}\end{aligned}$$

The first equality holds from the rewritten form of the MLE. The second equality holds from the properties of expectation. The third equality holds from manipulating the alternative formulas for the variance, namely:

$$Var(X) = \sigma^2 = E(X^2) - \mu^2 \text{ and } Var(\bar{X}) = \frac{\sigma^2}{n} = E(\bar{X}^2) - \mu^2$$

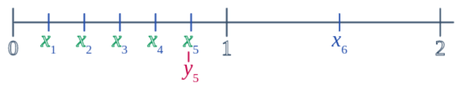
The remaining equalities hold from simple algebraic manipulation. Now, because we have shown:

$$E(\hat{\sigma}^2) \neq \sigma^2$$

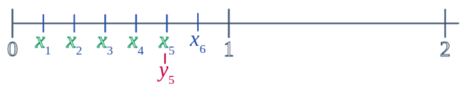
the maximum likelihood estimator of σ^2 is a biased estimator.

Solution 5

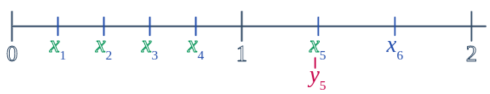
The key to finding the desired probability is to recognize that the only way that the fifth order statistic, Y_5, would be less than one is if at least 5 of the random variables X_1, X_2, X_3, X_4, X_5 are less than one.
For the sake of simplicity, let's suppose the first five observed values x_1, x_2, x_3, x_4, x_5 are less than one, but the sixth x_6 is not. In that case, the observed fifth-order statistic, y_5, would be less than one:



The observed fifth order statistic, y_5, would also be less than one if all six of the observed values x_1, x_2, x_3, x_4, x_5, x_6 are less than one:



The observed fifth order statistic, y_5, would not be less than one if the first four observed values x_1, x_2, x_3, x_4 are less than one, but the fifth x_5 and sixth x_6 is not:



Again, the only way that the fifth order statistic, Y_5, would be less than one is if 5 or 6... that is, at least 5... of the random variables $X_1, X_2, X_3, X_4, X_5, \text{ and } X_6$ are less than one. For the sake of simplicity, we considered just the first five or six random variables, but in reality, *any* five or six random variables less than one would do. We just have to do some "choosing" to count the number of ways that we can get any five or six of the random variables to be less than one.

If you think about it, then, we have a binomial probability calculation here. If the event $\{X_i < 1\}, i = 1, 2, \cdots, 5$ is considered a "success," and we let Z = the number of successes in six mutually independent trials, then Z is a binomial random variable with $n = 6$ and $p = 0.25$:

$$P(X_i \le 1) = \frac{1}{2}\int_0^1 x dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_{x=0}^{x=1} = \frac{1}{2}\left(\frac{1}{2} - 0\right) = \frac{1}{4}$$

Finding the probability that the fifth order statistic, Y_5, is less than one reduces to a binomial calculation then. That is:

$$P(Y_5 < 1) = P(Z = 5) + P(Z = 6) = \binom{6}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^1 + \binom{6}{6}\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^0 = 0.0046$$

The fact that the calculated probability is so small should make sense in light of the given p.d.f. of X. After all, each individual X_i has a greater chance of falling above, rather than below, one. For that reason, it would unusual to observe as many as five or six X's less than one.

Solution 6

Example 2: Suppose $X_1, X_2, \cdots, X_n$ are i.i.d. random variables with density function $f(x|\sigma) = \frac{1}{2\sigma}\exp\left(-\frac{|x|}{\sigma}\right)$, please find the maximum likelihood estimate of σ.

Solution: The log-likelihood function is

$$l(\sigma) = \sum_{i=1}^{n}\left[-\log 2 - \log\sigma - \frac{|X_i|}{\sigma}\right]$$

Let the derivative with respect to θ be zero:

$$l'(\sigma) = \sum_{i=1}^{n}\left[-\frac{1}{\sigma} + \frac{|X_i|}{\sigma^2}\right] = -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n}|X_i|}{\sigma^2} = 0$$

and this gives us the MLE for σ as

$$\hat{\sigma} = \frac{\sum_{i=1}^{n}|X_i|}{n}$$

Again this is different from the method of moment estimation which is

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}X_i^2}{2n}}$$

Solution 7

Example 3. Light bulbs

Suppose that the lifetime of *Badger* brand light bulbs is modeled by an exponential distribution with (unknown) parameter λ. We test 5 bulbs and find they have lifetimes of 2, 3, 1, 3, and 4 years, respectively. What is the MLE for λ?

answer: We need to be careful with our notation. With five different values it is best to use subscripts. Let X_j be the lifetime of the i^{th} bulb and let x_i be the value X_i takes. Then each X_i has pdf $f_{X_i}(x_i) = \lambda e^{-\lambda x_i}$. We assume the lifetimes of the bulbs are independent, so the joint pdf is the product of the individual densities:

$$f(x_1, x_2, x_3, x_4, x_5 \,|\, \lambda) = (\lambda e^{-\lambda x_1})(\lambda e^{-\lambda x_2})(\lambda e^{-\lambda x_3})(\lambda e^{-\lambda x_4})(\lambda e^{-\lambda x_5}) = \lambda^5 e^{-\lambda(x_1+x_2+x_3+x_4+x_5)}.$$

Note that we write this as a conditional density, since it depends on λ. Viewing the data as fixed and λ as variable, this density is the likelihood function. Our data had values

$$x_1 = 2,\ x_2 = 3,\ x_3 = 1,\ x_4 = 3,\ x_5 = 4.$$

So the likelihood and log likelihood functions with this data are

$$f(2,3,1,3,4 \,|\, \lambda) = \lambda^5 e^{-13\lambda}, \quad \ln(f(2,3,1,3,4 \,|\, \lambda) = 5\ln(\lambda) - 13\lambda$$

Finally we use calculus to find the MLE:

$$\frac{d}{d\lambda}(\text{log likelihood}) = \frac{5}{\lambda} - 13 = 0 \Rightarrow \boxed{\hat{\lambda} = \frac{5}{13}}.$$

Solution 8

Suppose our data $x_1, \dots x_n$ are independently drawn from a uniform distribution $U(a, b)$. Find the MLE estimate for a and b.

answer: This example is different from the previous ones in that we won't use calculus to find the MLE. The density for $U(a, b)$ is $\frac{1}{b-a}$ on $[a, b]$. Therefore our likelihood function is

$$f(x_1, \dots, x_n \mid a, b) = \begin{cases} \left(\frac{1}{b-a}\right)^n & \text{if all } x_i \text{ are in the interval } [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

This is maximized by making $b - a$ as small as possible. The only restriction is that the interval $[a, b]$ must include all the data. Thus the MLE for the pair (a, b) is

$$\hat{a} = \min(x_1, \dots, x_n) \qquad \hat{b} = \max(x_1, \dots, x_n).$$

Solution 9

Example 7. Hardy-Weinberg. Suppose that a particular gene occurs as one of two alleles (A and a), where allele A has frequency θ in the population. That is, a random copy of the gene is A with probability θ and a with probability $1 - \theta$. Since a diploid genotype consists of two genes, the probability of each genotype is given by:

genotype	AA	Aa	aa
probability	θ^2	$2\theta(1-\theta)$	$(1-\theta)^2$

Suppose we test a random sample of people and find that k_1 are AA, k_2 are Aa, and k_3 are aa. Find the MLE of θ.

answer: The likelihood function is given by

$$P(k_1, k_2, k_3 \mid \theta) = \binom{k_1 + k_2 + k_3}{k_1} \binom{k_2 + k_3}{k_2} \binom{k_3}{k_3} \theta^{2k_1} (2\theta(1-\theta))^{k_2} (1-\theta)^{2k_3}.$$

So the log likelihood is given by

$$\text{constant} + 2k_1 \ln(\theta) + k_2 \ln(\theta) + k_2 \ln(1-\theta) + 2k_3 \ln(1-\theta)$$

We set the derivative equal to zero:

$$\frac{2k_1 + k_2}{\theta} - \frac{k_2 + 2k_3}{1 - \theta} = 0$$

Solving for θ, we find the MLE is

$$\hat{\theta} = \frac{2k_1 + k_2}{2k_1 + 2k_2 + 2k_3},$$

which is simply the fraction of A alleles among all the genes in the sampled population.

Solution 10

Example 2.2.2 (Weibull with known α) *$\{Y_i\}$ are iid random variables, which follow a Weibull distribution, which has the density*

$$\frac{\alpha y^{\alpha-1}}{\theta^{\alpha}} \exp(-(y/\theta)^{\alpha}) \qquad \theta, \alpha > 0.$$

Suppose that α is known, but θ is unknown. Our aim is to fine the MLE of θ.

The log-likelihood is proportional to

$$\begin{aligned} \mathcal{L}_n(\underline{X}; \theta) &= \sum_{i=1}^{n} \left(\log \alpha + (\alpha - 1) \log Y_i - \alpha \log \theta - \left(\frac{Y_i}{\theta}\right)^{\alpha} \right) \\ &\propto \sum_{i=1}^{n} \left(-\alpha \log \theta - \left(\frac{Y_i}{\theta}\right)^{\alpha} \right). \end{aligned}$$

The derivative of the log-likelihood wrt to θ is

$$\frac{\partial \mathcal{L}_n}{\partial \theta} = -\frac{n\alpha}{\theta} + \frac{\alpha}{\theta^{\alpha+1}} \sum_{i=1}^{n} Y_i^{\alpha} = 0.$$

Solving the above gives $\widehat{\theta}_n = (\frac{1}{n} \sum_{i=1}^{n} Y_i^{\alpha})^{1/\alpha}$.

Solution 11

Solution: Let's first EX and $\mathrm{Var}(X)$ in terms of θ. We have

$$\begin{aligned} EX &= \int_0^1 x \left[\theta\left(x - \frac{1}{2}\right) + 1\right] \; dx \\ &= \frac{\theta + 6}{12}, \end{aligned}$$

$$\begin{aligned} EX^2 &= \int_0^1 x^2 \left[\theta\left(x - \frac{1}{2}\right) + 1\right] \; dx \\ &= \frac{\theta + 4}{12}, \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) &= EX^2 - EX^2 \\ &= \frac{12 - \theta^2}{144}. \end{aligned}$$

(a) Is $\hat{\Theta}_n$ an unbiased estimator of θ? To see this, we write

$$\begin{aligned} E[\hat{\Theta}_n] &= E[12\overline{X} - 6] \\ &= 12E[\overline{X}] - 6 \\ &= 12 \cdot \frac{\theta + 6}{12} - 6 \\ &= \theta. \end{aligned}$$

Thus, $\hat{\Theta}_n$ IS an unbiased estimator of θ.

(b) To show that $\hat{\Theta}_n$ is a consistent estimator of θ, we need to show

$$\lim_{n \to \infty} P\big(|\hat{\Theta}_n - \theta| \geq \epsilon\big) = 0, \qquad \text{for all } \epsilon > 0.$$

Since $\hat{\Theta}_n = 12\overline{X} - 6$ and $\theta = 12EX - 6$, we conclude

$$\begin{aligned} P\big(|\hat{\Theta}_n - \theta| \geq \epsilon\big) &= P\big(12|\overline{X} - EX| \geq \epsilon\big) \\ &= P\big(|\overline{X} - EX| \geq \frac{\epsilon}{12}\big) \end{aligned}$$

which goes to zero as $n \to \infty$ by the law of large numbers. Therefore, $\hat{\Theta}_n$ is a consistent estimator of θ.

(c) To find the mean squared error (MSE) of $\hat{\Theta}_n$, we write

$$\begin{aligned} MSE(\hat{\Theta}_n) &= \text{Var}(\hat{\Theta}_n) + B(\hat{\Theta}_n)^2 \\ &= \text{Var}(\hat{\Theta}_n) \\ &= \text{Var}(12\overline{X} - 6) \\ &= 144\text{Var}(\overline{X}) \\ &= 144\frac{\text{Var}(X)}{n} \\ &= 144 \cdot \frac{12 - \theta^2}{144n} \\ &= \frac{12 - \theta^2}{n}. \end{aligned}$$

Note that this gives us another way to argue that $\hat{\Theta}_n$ is a consistent estimator of θ. In particular, since

$$\lim_{n \to \infty} MSE(\hat{\Theta}_n) = 0,$$

we conclude that $\hat{\Theta}_n$ is a consistent estimator of θ.

Solution 12

Solution:

(a) Note that

$$E(X^2) = \text{Var}(X) + (EX)^2 = \sigma^2 + \mu^2 = \sigma^2.$$

Therefore $\hat{\sigma}(X) = X^2$ is an unbiased estimator of σ^2.

(b) The likelihood function is

$$L(x; \sigma^2) = f_X(x; \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}(x)^2}.$$

The log-likelihood function is

$$\ln L(x; \sigma^2) = -\ln(2\pi)^{\frac{1}{2}} - \ln \sigma - \frac{x^2}{2\sigma^2}.$$

(c) To find the MLE for σ, we differentiate $\ln L(x; \sigma^2)$ with respect to σ and set it equal to zero.

$$\begin{aligned} \frac{\partial}{\partial \sigma} \ln L &= -\frac{1}{\sigma} + \frac{x^2}{\sigma^3} \\ &= -\frac{1}{\sigma} + \frac{x^2}{\sigma^3} \overset{set}{=} 0. \end{aligned}$$

Therefore,

$$\hat{\sigma} X^2 = \hat{\sigma}^3 \rightarrow \hat{\sigma} = |X|.$$

Also, we can verify that the second derivative is negative to make sure that $\hat{\sigma} = |X|$ is actually the maximizing value:

$$\frac{\partial^2}{\partial \sigma^2} \ln L = \frac{1}{\sigma^2} - \frac{3x^2}{\sigma^4} < 0 \text{ when } \hat{\sigma} = |x|.$$

Solution 13

(a) Here we define the test statistic as

$$\begin{aligned} W &= \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \\ &= \frac{5.96 - 5}{1/\sqrt{5}} \\ &\approx 2.15. \end{aligned}$$

Here, $\alpha = .05$, so $z_{\frac{\alpha}{2}} = z_{0.025} = 1.96$. Since $|W| > z_{\frac{\alpha}{2}}$, we reject H_0 and accept H_1.

(b) The 95% CI is given by

$$\left(5.96 - 1.96 * \frac{1}{\sqrt{(5)}}, 5.96 + 1.96 * \frac{1}{\sqrt{(5)}}\right) = (5.09, 6.84).$$

Since μ_0 is not included in the interval, we are able to reject the null hypothesis and conclude that μ is not 5.

Solution 14

$$\begin{aligned} W &= \frac{\overline{X} - \mu_0}{S/\sqrt{n}} \\ &= \frac{52.28 - 50}{\sqrt{30.9/150}} \\ &= 5.03 \end{aligned}$$

Since $5.03 > 1.96$, we reject H_0.

Solution 15

Solution: We define the test statistic as

$$
\begin{aligned}
W &= \frac{\overline{X} - \mu_0}{S/\sqrt{n}} \\
&= \frac{29.25 - 30}{\sqrt{20.7}/\sqrt{121}} \\
&= -1.81
\end{aligned}
$$

and by Table 8.4 the test threshold is $-z_\alpha$. The P-value is $P(\text{type I error})$ when the test threshold c is chosen to be $c = -1.81$. Thus,

$$-z_\alpha = 1.81$$

Noting that by definition $z_\alpha = \Phi^{-1}(1-\alpha)$, we obtain $P(\text{type I error})$ as

$$\alpha = 1 - \Phi(1.81) \approx 0.035$$

Therefore,

$$P - \text{value} = 0.035$$

Solution 16

$$L(\alpha, \beta) = \beta^n e^{-\beta \sum_{i=1}^{n} (x_i - \alpha)}$$

$$\log_e L(\alpha, \beta) = n \log_e \beta - \beta \sum_{i=1}^{n} (x_i - \alpha)$$

$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - \sum (x_i - \alpha)$ and $\frac{\partial \log L}{\partial \alpha} = n\beta$.

Now, $\frac{\partial \log L}{\partial \alpha} = 0$ gives us $\beta = 0$ which is nonadmissible. Thus, the method of differentiation fails here.

Now, from the expression of $L(\alpha, \beta)$, it is clear that for fixed $\beta(>0)$, $L(\alpha, \beta)$ becomes maximum when α is the largest. The largest possible value of α is $X_{(1)}$ = Min x_i.

Now, we maximize $L\{X_{(1)}, \beta\}$ with respect to β. This can be done by considering the method of differentiation.

$$\frac{\partial \log L\{x_{(1)}, \beta\}}{\partial \beta} = 0 \Rightarrow \frac{n}{\beta} - \sum (x_i - \min x_i) = 0 \quad \Rightarrow \beta = \frac{n}{\sum (x_i - \min x_i)}$$

So, the MLE of (α, β) is $\left\{ \min x_i, \frac{n}{\sum (x_i - \min x_i)} \right\}$.

Solution 17

Proof

$$(a)L(\alpha,\beta) = \frac{1}{(\beta-\alpha)^n} \quad \text{if } \alpha \le \text{Min}x_i < \text{Max}x_i \le \beta \tag{2.1}$$

It is evident from (2.1), that the likelihood will be made as large as possible when $(\beta - \alpha)$ is made as small as possible. Clearly, α cannot be larger than Min x_i and β cannot be smaller than Max x_i; hence, the smallest possible value of $(\beta - \alpha)$ is (Max x_i − Min x_i). Then the MLE'S of α and β are $\hat{\alpha} = \text{Min}\, x_i$ and $\hat{\beta} = \text{Max}\, x_i$, respectively.

(b) We know $E(x) = \mu_1^1 = \frac{\alpha+\beta}{2}$ and $V(x) = \mu_2 = \frac{(\beta-\alpha)^2}{12}$

Hence, $\frac{\alpha+\beta}{2} = \bar{x}$ and $\frac{(\beta-\alpha)^2}{12} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$

By solving, we get $\hat{\alpha} = \bar{x} - \sqrt{\frac{3\sum(x_i-\bar{x})^2}{n}}$ and $\hat{\beta} = \bar{x} + \sqrt{\frac{3\sum(x_i-\bar{x})^2}{n}}$

Solution 18

Solution (i) Suppose μ is known.

We take $\psi(T,\theta) = \frac{\sum(x_i-\mu)^2}{\sigma^2}$ which is distributed as χ^2 with n d.f. Thus its distribution is independent of θ. We can choose k_1 and k_2 from χ^2 distribution with n d.f such that

$$P\left[\chi^2_{n,1-\alpha_1} \le \frac{\sum(x_i-\mu)^2}{\sigma^2} \le \chi^2_{n,\alpha_2}\right] = 1-(\alpha_1+\alpha_2) = 1-\alpha$$

$$\Rightarrow P\left[\frac{\sum(x_i-\mu)^2}{\chi^2_{n,\alpha_2}} \le \sigma^2 \le \frac{\sum(x_i-\mu)^2}{\chi^2_{n,1-\alpha_1}}\right] = 1-\alpha$$

Thus $\left(\frac{\sum(x_i-\mu)^2}{\chi^2_{n,\alpha_2}}, \frac{\sum(x_i-\mu)^2}{\chi^2_{n,1-\alpha_1}}\right)$ is $100(1-\alpha)\%$ confidence interval of σ^2 when μ is known. (ii) Suppose μ is unknown.

We take the function $\psi(T,\theta) = \frac{\sum(x_i-\bar{x})^2}{\sigma^2}$ which is distributed as χ^2 with (n − 1) d.f. This distribution is independent of θ. Proceeding as in (i), $\left(\frac{\sum(x_i-\bar{x})^2}{\chi^2_{n-1,\alpha_2}}, \frac{\sum(x_i-\bar{x})^2}{\chi^2_{n-1,1-\alpha_1}}\right)$ is $100(1-\alpha)\%$ confidence interval of σ^2 when μ is unknown.

Solution 19

Solution The likelihood function is $L = \frac{1}{\theta^n}$. This is maximum when θ is the smallest; but θ cannot be less than $x_{(n)}$, the maximum of sample observations. Thus $\widehat{\theta} = x_{(n)}$.

The p.d.f of $\widehat{\theta}$ is given by

$$h\left(\widehat{\theta}\right) = \frac{n\widehat{\theta}^{n-1}}{\theta^n}, 0 < \widehat{\theta} < \theta.$$

Let $u = \frac{x_{(n)}}{\theta} = \frac{\widehat{\theta}}{\theta}$. so that $g(u) = nu^{n-1}$, $0 < u < 1$.
Thus the distribution of u is independent of θ.
We find u_1 and u_2 such that

$$P[u_1 < u < u_2] \quad = \quad 1 - (\alpha_1 + \alpha_2) \quad = \quad 1 - \alpha$$

where $\int_0^{u_1} g(u)du = \alpha_1$, $\int_{u_2}^{1} g(u)du = \alpha_2$

i.e. $P\left[u_1 < \frac{\widehat{\theta}}{\theta} < u_2\right] = 1 - \alpha. \Rightarrow P\left[\frac{\widehat{\theta}}{u_2} < \theta < \frac{\widehat{\theta}}{u_1}\right] = 1 - \alpha$

Thus, $\left(\frac{\max X_i}{u_2}, \frac{\max X_i}{u_1}\right)$ is a $100(1-\alpha)\%$ confidence interval for θ.

Solution 20

Based on the above discussion,

$$\left[\bar{X} - \frac{z_{\frac{\alpha}{2}}}{2\sqrt{n}}, \bar{X} + \frac{z_{\frac{\alpha}{2}}}{2\sqrt{n}}\right]$$

is a valid $(1-\alpha)100\%$ confidence interval for θ. Therefore, we need to have

$$\frac{z_{\frac{\alpha}{2}}}{2\sqrt{n}} = 0.03$$

Here, $\alpha = 0.05$, so $z_{\frac{\alpha}{2}} = z_{0.025} = 1.96$. Therefore, we obtain

$$n = \left(\frac{1.96}{2 \times 0.03}\right)^2.$$

We conclude $n \geq 1068$ is enough. The above calculation provides a reason why most polls before elections are conducted with a sample size of around one thousand.