Q1

2. Let X and Y be random variables with joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{ay}{x^2} & x \geq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is a?

$$f_X(x) = \int_0^1 \frac{ay}{x^2} dy = \left[\frac{ay^2}{2x^2}\right]_0^1 = \frac{a}{2x^2}$$

$$1 = \int_1^\infty \frac{a}{2x^2} dx = \left[-\frac{a}{2x}\right]_1^\infty = 1/2$$

so, $a = 2$

(b) What is the conditional PDF $f_{Y|X}(y|x)$ of Y given $X = x$?

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{ay/x^2}{a/2x^2} = 2y$$

(c) What is the conditional expectation of Y given X?

$$E[Y|X = x] = \int_0^1 2y^2 dy = 2/3$$

So, $E[Y|X] = 2/3$.

(d) What is the expected value of Y? $E[Y] = E[E[Y|X]] = 2/3$

Q2

4. Let X be the number of ice-cream cones a vendor sells on a day. If the average temperature of a summer day is a random variable Y (in Fahrenheit), where $Y \sim Uniform([95, 105])$. We also have $X \sim Poisson(Y^2/10 + Y/5 + 5)$.

(a) What is $E[X|Y]$? $E[X|Y = y] = y^2/10 + y/5 + 5$. So $E[X|Y] = Y^2/10 + Y/5 + 5$.

(b) What is $E[X]$? Remember $E[Y^2] = \text{var}(Y) + E[Y]^2 = (105 - 95)^2/12 + 100^2 = 1008$. $E[X] = E[Y^2]/10 + E[Y]/5 + 5 = 1008/10 + 100/5 + 5 = 100.8 + 20 + 5 = 125.8$.

Q3

5. If there are no distractions, it takes me 30 minutes to walk to the store. However, if I pass someone with a dog, I stop and pet the dog and chat to the owner. The number Y of dogs I pass is a Poisson random variable with mean 2. Each time I stop, the number of minutes I spend petting the dog and chatting is an exponential random variable with PDF:

$$f_X(x) = 0.5e^{-0.5x}$$

   (a) If I see a single dog, what is the expectation and variance of the time spent petting the dog and chatting to its owner? X is exponential, and so $E[X] = \int_0^\infty 0.5xe^{-0.5x}dx = 2$, $E[X^2] = \int_0^\infty 0.5x^2e^{-0.5x} = 8$, $\text{var}(X) = E[X^2] - E[X]^2 = 4$.

   (b) What is the conditional expectation of the total time spent petting dogs and chatting to their owners, as a function of Y? $E[X|Y = y] = 2y$, (expectation of sum), so $E[X|Y] = 2Y$

   (c) Using the law of iterated expectation calculate $E[X]$. $E[X] = E[E[X|Y]] = 2E[Y] = 2 \times 2 = 4$.

Q4

1. (20 points) Let X and Y be independent Poisson random variables with parameter 1. Compute the following. (Give a correct formula involving sums — does not need to be in closed form.)

   (a) The probability mass function for X given that $X + Y = 5$.

   ANSWER: Write $p(k) = P\{X = k\} = e^{-1}/k!$. Then suppose $x \in \{0, 1, 2, 3, 4, 5\}$. (Mass function is zero at other values.) $P\{X = x|X + Y = 5\} = \frac{P\{X=x,X+Y=5\}}{P\{X+Y=5\}} = \frac{P\{X=x\}P\{Y=5-x\}}{P\{X+Y=5\}}$. This is equal to $\frac{p(x)p(5-x)}{\sum_{j=0}^{5} p(x)p(5-x)}$.

   (b) The conditional expectation of Y^2 given that $X = 2Y$.

   ANSWER: Let $y \geq 0$ be an integer. First we compute $P\{Y = y|X = 2Y\} = \frac{P\{Y=y,X=2y\}}{P\{X=2Y\}} = \frac{p(2y)p(y)}{\sum_{k=0}^{\infty} p(2k)p(k)}$. Then we note that the $E[Y^2|X = 2Y] = \sum_{y=0}^{\infty} P\{Y = y|X = 2Y\}y^2$.

   (c) The probability mass function for $X - 2Y$ given that $X > 2Y$.

   ANSWER: Write $Z = X - 2Y$. Then

$$p_Z(z) = P\{Z = z\} = \sum_{y=-\infty}^{\infty} P\{Y = y\}P\{X = z+2y\} = \sum_{y=0}^{\infty} p(y)p(z+2y).$$

   Now for $z > 0$ we have $P\{Z = z|Z > 0\} = \frac{p_Z(z)}{P\{Z>0\}} = \frac{p_Z(z)}{\sum_{j=1}^{\infty} p_Z(j)}$.

   (d) The probability that $X = Y$.

   ANSWER: $\sum_{k=0}^{\infty} p(k)^2$.

Q5

EXAMPLE 5b

Suppose that the joint density of X and Y is given by

$$f(x, y) = \begin{cases} \dfrac{e^{-x/y}e^{-y}}{y} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find $P\{X > 1|Y = y\}$.

Solution. We first obtain the conditional density of X given that $Y = y$.

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x, y)}{f_Y(y)} \\ &= \frac{e^{-x/y}e^{-y}/y}{e^{-y}\int_0^\infty (1/y)e^{-x/y}\,dx} \\ &= \frac{1}{y}e^{-x/y} \end{aligned}$$

Hence,

$$\begin{aligned} P\{X > 1|Y = y\} &= \int_1^\infty \frac{1}{y}e^{-x/y}\,dx \\ &= -e^{-x/y}\Big|_1^\infty \\ &= e^{-1/y} \end{aligned}$$

■

Q6

1. Show the following:

(a) $\mathbb{E}(aY + bZ \mid X) = a\mathbb{E}(Y \mid X) + b\mathbb{E}(Z \mid X)$ for $a, b \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}(aY + bZ \mid X = x) &= \sum_{y,z}(ay + bz)\mathbb{P}(Y = y, Z = z \mid X = x) \\ &= a\sum_{y,z} y\mathbb{P}(Y = y, Z = z \mid X = x) + b\sum_{y,z} z\mathbb{P}(Y = y, Z = z \mid X = x) \\ &= a\sum_{y} y\mathbb{P}(Y = y \mid X = x) + b\sum_{z} z\mathbb{P}(Z = z \mid X = x). \end{aligned}$$

Q7

8. Families. Each child is equally likely to be male or female, independently of all other children.

(a) Show that, in a family of predetermined size, the expected number of boys equals the expected number of girls. Was the assumption of independence necessary?

(b) A randomly selected child is male; does the expected number of his brothers equal the expected number of his sisters? What happens if you do not require independence?

Solution:

8. (a) Let m be the family size, ϕ_r the indicator that the rth child is female, and μ_r the indicator of a male. The numbers G, B of girls and boys satisfy

$$G = \sum_{r=1}^{m} \phi_r, \quad B = \sum_{r=1}^{m} \mu_r, \quad \mathbb{E}(G) = \tfrac{1}{2}m = \mathbb{E}(B).$$

(It will be shown later that the result remains true for random m under reasonable conditions.) We have not used the property of independence.

(b) With M the event that the selected child is male,

$$\mathbb{E}(G \mid M) = \mathbb{E}\left(\sum_{r=1}^{m-1} \phi_r\right) = \tfrac{1}{2}(m-1) = \mathbb{E}(B).$$

The independence is necessary for this argument.

Q8

Problem 8. Consider two continuous random variables Y and Z, and a random variable X that is equal to Y with probability p and to Z with probability $1-p$.

(a) Show that the PDF of X is given by

$$f_X(x) = p f_Y(x) + (1-p) f_Z(x).$$

(b) Calculate the CDF of the two-sided exponential random variable that has PDF given by

$$f_X(x) = \begin{cases} p\lambda e^{\lambda x}, & \text{if } x < 0, \\ (1-p)\lambda e^{-\lambda x}, & \text{if } x \geq 0, \end{cases}$$

where $\lambda > 0$ and $0 < p < 1$.

Solution:

Solution to Problem 3.8. (a) By the total probability theorem, we have

$$F_X(x) = \mathbf{P}(X \le x) = p\mathbf{P}(Y \le x) + (1-p)\mathbf{P}(Z \le x) = pF_Y(x) + (1-p)F_Z(x).$$

By differentiating, we obtain

$$f_X(x) = pf_Y(x) + (1-p)f_Z(x).$$

(b) Consider the random variable Y that has PDF

$$f_Y(y) = \begin{cases} \lambda e^{\lambda y}, & \text{if } y < 0 \\ 0, & \text{otherwise,} \end{cases}$$

and the random variable Z that has PDF

$$f_Z(z) = \begin{cases} \lambda e^{-\lambda z}, & \text{if } y \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$

We note that the random variables $-Y$ and Z are exponential. Using the CDF of the exponential random variable, we see that the CDFs of Y and Z are given by

$$F_Y(y) = \begin{cases} e^{\lambda y}, & \text{if } y < 0, \\ 1, & \text{if } y \ge 0, \end{cases}$$

$$F_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ 1 - e^{-\lambda z}, & \text{if } z \ge 0. \end{cases}$$

We have $f_X(x) = pf_Y(x) + (1-p)f_Z(x)$, and consequently $F_X(x) = pF_Y(x) + (1-p)F_Z(x)$. It follows that

$$\begin{aligned} F_X(x) &= \begin{cases} pe^{\lambda x}, & \text{if } x < 0, \\ p + (1-p)(1 - e^{-\lambda x}), & \text{if } x \ge 0, \end{cases} \\ &= \begin{cases} pe^{\lambda x}, & \text{if } x < 0, \\ 1 - (1-p)e^{-\lambda x}, & \text{if } x \ge 0. \end{cases} \end{aligned}$$

Q9

Example 4.13. Let X and Y be independent and have PMFs given by

$$p_X(x) = \begin{cases} \frac{1}{3} & \text{if } x = 1, 2, 3, \\ 0 & \text{otherwise,} \end{cases} \qquad p_Y(y) = \begin{cases} \frac{1}{2} & \text{if } x = 0, \\ \frac{1}{3} & \text{if } x = 1, \\ \frac{1}{6} & \text{if } x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

To calculate the PMF of $W = X + Y$ by convolution, we first note that the range of possible values of w are the integers from the range $[1, 5]$. Thus we have

$$p_W(w) = 0 \quad \text{if } w \neq 1, 2, 3, 4, 5.$$

We calculate $p_W(w)$ for each of the values $w = 1, 2, 3, 4, 5$ using the convolution formula. We have

$$p_W(1) = \sum_x p_X(x) p_Y(1 - x) = p_X(1) \cdot p_Y(0) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6},$$

where the second equality above is based on the fact that for $x \neq 1$ either $p_X(x)$ or $p_Y(1 - x)$ (or both) is zero. Similarly, we obtain

$$p_W(2) = p_X(1) \cdot p_Y(1) + p_X(2) \cdot p_Y(0) = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{18},$$

$$p_W(3) = p_X(1) \cdot p_Y(2) + p_X(2) \cdot p_Y(1) + p_X(3) \cdot p_Y(0) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3},$$

$$p_W(4) = p_X(2) \cdot p_Y(2) + p_X(3) \cdot p_Y(1) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{6},$$

$$p_W(5) = p_X(3) \cdot p_Y(2) = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}.$$

Q10

Problem 3

Let Q be a continuous random variable with PDF:

$$f_Q(q) = \begin{cases} 6q(1 - q) & \text{if } 0 \leq q \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

This Q represents the probability of success of a Bernoulli random variable X, i.e.,

$$P(X = 1 \mid Q = q) = q$$

Find $f_{Q|X}(q|x)$ for x $\in$ {0, 1} and all q. (2 marks)

$\boxed{3}$ $P(X=1 \mid Q=q) = q$

$\Longrightarrow P(X=0 \mid Q=q) = 1-q$

(As X is a Bernoulli R.V.)

$$f_{Q|X}(q|x) = \frac{P_{X|Q}(x|q) \cdot f_Q(q)}{P_X(x)} = \frac{P_{X|Q}(x|q) \cdot f_Q(q)}{\int_0^1 P_{X|Q}(x|q) \cdot f_Q(q)\,dq}$$

$$f_{Q|X}(q|0) = \frac{(1-q)(6q)(1-q)}{\int_0^1 6q(1-q)(1-q)\,dq} = \frac{6q(1-q)^2}{\frac{1}{2}}$$

$$= 12q(1-q)^2.$$

$$\left(\int_0^1 6q(1-q)^2\,dq = \left(\frac{3}{2}q^4 - 4q^3 + 3q^2\right)\Big|_0^1 = \frac{3}{2} - 4 + 3 = \frac{1}{2} \right).$$

Similarly,

$$f_{Q|X}(q|1) = \frac{6q(1-q) \cdot q}{\int_0^1 6q(1-q)q\,dq} = \frac{6q(1-q) \cdot q}{(1/2)}$$

$$= 12q^2(1-q)$$

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$$\left(\int_0^1 6q^2(1-q) = -6\left[\frac{q^4}{4} - \frac{q^3}{3}\right]_0^1 = -6\left(-\frac{1}{12}\right) = \frac{1}{2} \right)$$

$$\Longrightarrow f_{Q|X}(q|x) = \begin{cases} 12q(1-q)^2 & x=0,\ 0 \le q \le 1 \\ 12q^2(1-q) & x=1,\ 0 \le q \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Q11

Shashank performs an experiment comprising a series of independent trials. On each trial, he simultaneously flips a set of Z fair coins.

1. Given that Shashank has just had a trial with Z tails, what is the probability that next two trials will also have this result?
2. Sandeep conducts an experiment like Shashank's, except that he uses M coins for the first trial, and then he obeys the following rule: Whenever all the coins land on the same side in a trial, Sandeep permanently removes one coin from the experiment and continues with the trials. He follows this rule until the $(M-1)^{th}$ time he removes a coin, at which point the experiment ceases. Find $E[X]$, where X is the number of trials in Sandeep's experiment.

Solution:

Given, Shashank performs an experiment of a series of *independent* trials. On each trial he flips a set of Z fair coins.

9.1 Next two trials have same result

Since, trials are independent of each other.

Probability that next two trials will have same result (of Z tails) is the same as probability that two successive trials will have the same result (of Z tails).

Therefore, Probability of Z tails in one trial $= \frac{1}{2^Z}$.

Therefore for two trails

$$= \frac{1}{2^Z} \times \frac{1}{2^Z} = \frac{1}{2^{2Z}}$$

9.2 Sandeep's Experiment

Given, X is the number of trials in Sandeep's Experiment.

Let Y_i denote the number of trials such that i coins are left.

$$\therefore Y_i \sim \text{Geometric}(p)$$

where p is the probability of all coins landing on the same side (at which point we remove a coin and experiment with $i-1$ coins begins).

Thus,

$$X = Y_M + Y_{M-1} + \dots Y_2$$

(since only $M-1$ coins are removed, last set of trials will have two coins).

Parameter p for each Y_i is $= \frac{2}{2^i} = \frac{1}{2^{i-1}}$.

Therefore Expectation of X

$$\begin{aligned}
E[X] &= E\left[\sum_{i=2}^{M} Y_i\right] \\
&= \sum_{i=2}^{M} E[Y_i] \\
&= \sum_{i=2}^{M} \frac{1}{\frac{1}{2^{i-1}}} \\
&= \sum_{i=2}^{M} 2^{i-1} \\
&= \frac{2^{M-1} \times 2 - 2}{2-1} \\
&= 2^M - 2
\end{aligned}$$

Q12

Let X be a continuous random variable with PDF:

$$f_X(x) = \begin{cases} x^2\left(2x + \frac{3}{2}\right) & ; \quad 0<x\leq 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

If $Y = \frac{2}{X} + 3$, find Var(Y).

8

$$Y = \frac{2}{X} + 3$$

$$Var(Y) = Var\left(\frac{2}{X} + 3\right)$$

$$Var(Y) = 4 \cdot Var\left(\frac{1}{X}\right)$$

(As $Var(aX+b) = a^2 Var(X)$, X is a random variable, a, b are constants)

$$Var\left(\frac{1}{X}\right) = E\left(\frac{1}{X^2}\right) - E\left(\frac{1}{X}\right)^2$$

$$E\left(\frac{1}{X}\right) = \int_0^1 \left(\frac{1}{x}\right) x^2 \left(2x + \frac{3}{2}\right) dx$$

$$= 2\int_0^1 x^2 dx + \frac{3}{2}\int_0^1 x\, dx$$

$$= \frac{2}{3}\left([x^3]_0^1\right) + \frac{3}{4}[x^2]_0^1$$

$$= \frac{2}{3} + \frac{3}{4} = \frac{8+9}{12} = \frac{17}{12}.$$

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$$E\left(\frac{1}{X^2}\right) = \int_0^1 \left(\frac{1}{x^2}\right) x^2 \left(2x + \frac{3}{2}\right) dx$$

$$= 2\int_0^1 x\, dx + \frac{3}{2}\int_0^1 dx$$

$$= [x^2]_0^1 + \frac{3}{2}[x]_0^1$$

$$= 1 + \frac{3}{2} = \frac{5}{2}$$

$$\Rightarrow Var(Y) = 4\left(\frac{5}{2} - \left(\frac{17}{12}\right)^2\right)$$

$$= 4\left(\frac{5}{2} - \frac{289}{144}\right) = \left(\frac{360 - 289}{144}\right) 4$$

$$= \frac{71}{144} \times 4 = \frac{71}{36}$$