CS 302.1 - Automata Theory

Lecture 12

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Quick Recap

Recursive Language/Turing Decidable/Decidable: A language L is called Recursive or Turing decidable or Decidable if there exists a Total Turing Machine M and

$$\left.\begin{array}{l} \forall \omega \in L, M(\omega) \text{ accepts} \\ \forall \omega \notin L, M(\omega) \text{ rejects} \end{array}\right\} \text{Halts on all inputs}$$

The Church Turing thesis: An algorithm can be written for a problem if and only if it is decidable, i.e. there exists a Total Turing machine that solves the problem. **Total TM ⇔ Algorithms!**

Recursively Enumerable Language/Turing Recognizable (RE): A language L is called Recursively Enumerable (RE) or Turing Recognizable if

$$\forall \omega \in L, M(\omega) \textbf{ accepts}$$
$$\forall \omega \notin L, M(\omega) \textbf{ doesn't accept} \qquad \text{(rejects or runs infinitely)}$$

Co-Recursively Enumerable Language/co-Turing Recognizable (Co-RE/$\overline{RE}$/nRE): A language L is **Co-Recursively Enumerable (co-RE/$\overline{RE}$) or Co-Turing Recognizable** if

$$\forall \omega \in L, M(\omega) \textbf{ doesn't reject} \qquad \text{(accepts or loops)}$$
$$\forall \omega \notin L, M(\omega) \textbf{ rejects}$$

Quick Recap

There exists a one-one mapping (bijective relationship) between the set of binary strings and TMs.

Universal Turing Machine: A Universal Turing Machine, denoted as U_{TM} accepts as input (i) the encoding of a Turing Machine M and (ii) an input string w and **simulates M running on w**, i.e.

$$U_{TM}(\langle M, w \rangle) = \begin{cases} \textbf{ACCEPTS, if } M(w) \textbf{ accepts} \\ \textbf{REJECTS, if } M(w) \textbf{ rejects} \\ \textbf{LOOPS INFINITELY, if } M(w) \textbf{ loops infinitely} \end{cases}$$

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Some examples of languages that are recursive/decidable:

- $A_{DFA} = \{\langle DFA \rangle, w | w \in L(DFA)\}$
- $E_{DFA} = \{\langle DFA \rangle | L(DFA) = \Phi\}$
- $A_{CFG} = \{\langle CFG, w \rangle | w \in L(CFG)\}$
- $E_{CFG} = \{\langle CFG \rangle | L(CFG) = \Phi\}$

An undecidable language:

- $A_{TM} = \{\langle M, w \rangle | M \textbf{ accepts input } w\}$

Quick Recap

There exists a one-one mapping (bijective relationship) between the set of binary strings and TMs.
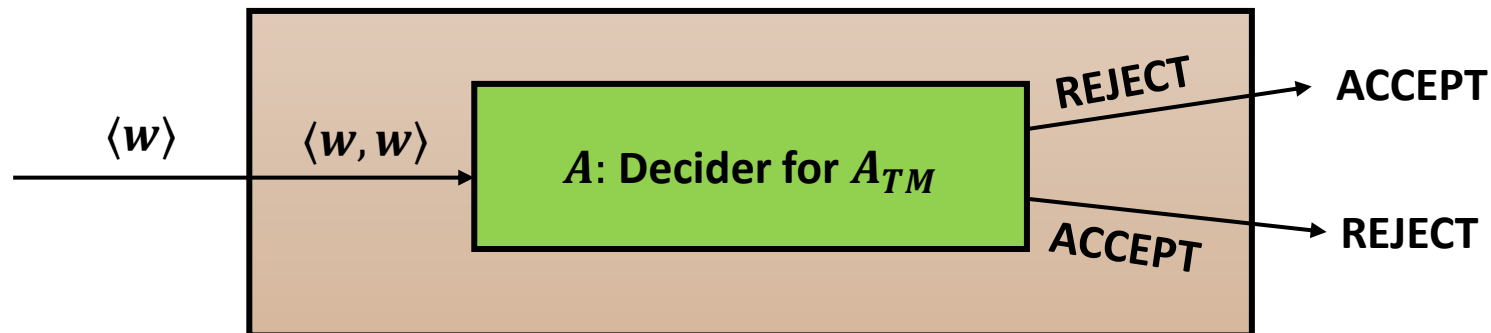
An undecidable language:

- $A_{TM} = \{\langle M, w\rangle | M \text{ accepts input } w\}$

- A_{TM} is undecidable
- $A_{TM} \in RE$ but not recursive
- A_{TM} is partially decidable

Proof strategy: By contradiction. We assume that a Total TM A exists that decides A_{TM}.
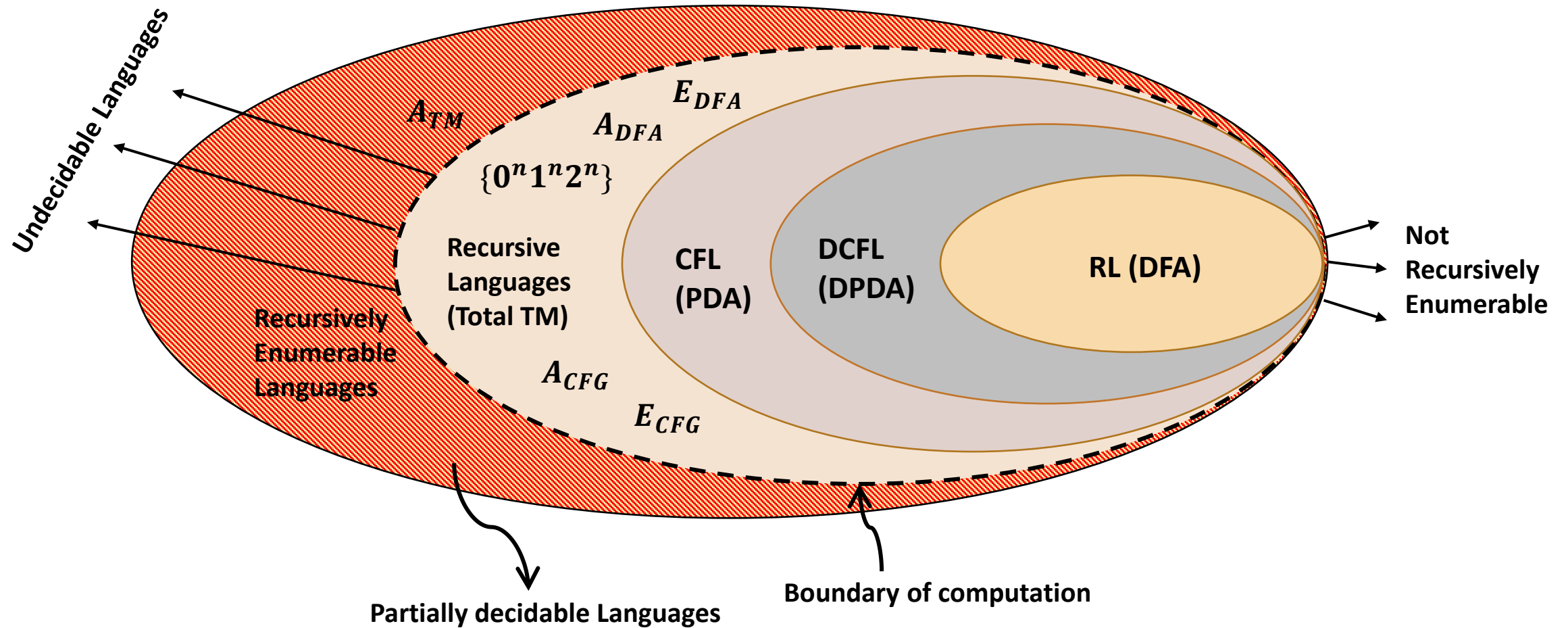
We build a Total TM D that accepts w as input and calls $A(\langle w, w\rangle)$ as a subroutine and outputs the opposite of A.

The TM D cannot decide the input $w = \langle D\rangle$. So A is undecidable.

For $w = \langle D\rangle$, there is a contradiction!

Quick Recap

Undecidable Languages

A_{TM}

E_{DFA}

A_{DFA}

$\{0^n 1^n 2^n\}$

Recursive Languages (Total TM)

CFL (PDA)

DCFL (DPDA)

RL (DFA)

Recursively Enumerable Languages

A_{CFG}

E_{CFG}

Not Recursively Enumerable

Partially decidable Languages

Boundary of computation

An undecidable problem

$A_{TM} = \{\langle M, w \rangle | M \text{ accepts input } w\}$. A_{TM} **is undecidable**

- A_{TM} is undecidable
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$$A(\langle M, w \rangle) = \begin{cases} \textbf{ACCEPTS, if } M(w) \textbf{ accepts} \\[2em] \textbf{REJECTS, if } M(w) \textbf{ rejects or loops infinitely} \end{cases}$$

The proof uses a technique called **Diagonalization**.

First, recall that there exists a bijective map (one-one correspondence) between the set of all binary strings and Turing Machines.

We can list all the Turing Machines and write down the result of running any M_i on input $\langle M_j \rangle$.

An undecidable problem

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	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_0	Accept	Accept	Loops	Reject	Accept	$\cdots$
M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_2	Reject	Loops	Accept	Loops	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

First, recall that there exists a bijective map (one-one correspondence) between the set of all binary strings and Turing Machines.

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M_0	Accept	Accept	Loops	Reject	Accept	$\cdots$
M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_2	Reject	Loops	Accept	Loops	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

How would this Table look for A?

An undecidable problem

$A_{TM} = \{\langle M, w \rangle | M \text{ accepts input } w\}. \ A_{TM}$ **is undecidable**

$$A(\langle M, w \rangle) = \begin{cases} \textbf{ACCEPTS, if } M(w) \textbf{ accepts} \\ \\ \textbf{REJECTS, if } M(w) \textbf{ rejects or loops infinitely} \end{cases}$$

- A_{TM} **is undecidable**
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How would this Table look for A?

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M_0	Accept	Accept	Loops	Reject	Accept	$\cdots$
M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_2	Reject	Loops	Accept	Loops	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

A	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_0	Accept	Accept	**Reject**	Reject	Accept	$\cdots$
M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_2	Reject	**Reject**	Accept	**Reject**	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

An undecidable problem

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The proof uses a technique called **Diagonalization**.

A	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_0	Accept	Accept	**Reject**	Reject	Accept	$\cdots$	Accept	$\cdots$
M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$	Accept	$\cdots$
M_2	Reject	**Reject**	Accept	**Reject**	Accept	$\cdots$	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$
D						$\cdots$		$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$

$D(w) = \{$ **Run** $A(\langle w, w \rangle)$

If $A(\langle w, w \rangle)$ **accepts, then** $REJECT$
If $A(\langle w, w \rangle)$ **rejects, then** $ACCEPT$

$\}$

$$D(\langle M_w \rangle) = \begin{cases} \textbf{ACCEPTS, if } M_w(\langle M_w \rangle) \textbf{ doesn't accept} \\ \\ \textbf{REJECTS, if } M_w(\langle M_w \rangle) \textbf{ accepts} \end{cases}$$

- Somewhere we will also have the TM D.
- Note that D by definition **computes the opposite of the diagonal entries** of the table.

An undecidable problem

$A_{TM} = \{\langle M, w\rangle | M \text{ accepts input } w\}$. A_{TM} is undecidable

The proof uses a technique called **Diagonalization**.

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M_0	**Accept**	Accept	Reject	Reject	Accept	$\cdots$	Accept	$\cdots$
M_1	Accept	Reject	Reject	Accept	Reject	$\cdots$	Accept	$\cdots$
M_2	Reject	Reject	Accept	Reject	Accept	$\cdots$	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$
D	**Reject**					$\cdots$		$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$

An undecidable problem

$A_{TM} = \{\langle M, w \rangle | M \text{ accepts input } w\}. A_{TM}$ **is undecidable**

The proof uses a technique called **Diagonalization**.

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M_0	**Accept**	Accept	Reject	Reject	Accept	$\cdots$	Accept	$\cdots$
M_1	Accept	**Reject**	Reject	Accept	Reject	$\cdots$	Accept	$\cdots$
M_2	Reject	Reject	Accept	Reject	Accept	$\cdots$	Accept	$\cdots$
M_3	Accept	Reject	Reject	Accept	Reject	$\cdots$	Reject	$\cdots$
M_4	Accept	Accept	Accept	Accept	Reject	$\cdots$	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$
D	**Reject**	**Accept**				$\cdots$		$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$

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M_1	Accept	**Reject**	Reject	Accept	Reject	$\cdots$	Accept	$\cdots$
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M_4	Accept	Accept	Accept	Accept	**Reject**	$\cdots$	Reject	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$
D	**Reject**	**Accept**	**Reject**	**Reject**	**Accept**	$\cdots$		$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$

- A_{TM} **is undecidable**
- $A_{TM} \in RE$ **but not recursive**
- A_{TM} **is partially decidable**

Note that D by definition **computes the opposite of the diagonal entries** of the table.

What will be the Dth diagonal entry??

An undecidable problem

$A_{TM} = \{\langle M, w \rangle \mid M$ **accepts input** $w\}$. A_{TM} **is undecidable**

The proof uses a technique called **Diagonalization**.

- A_{TM} is undecidable
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M_0	**Accept**	Accept	Reject	Reject	Accept	...	Accept	...
M_1	Accept	**Reject**	Reject	Accept	Reject	...	Accept	...
M_2	Reject	Reject	**Accept**	Reject	Accept	...	Accept	...
M_3	Accept	Reject	Reject	**Accept**	Reject	...	Reject	...
M_4	Accept	Accept	Accept	Accept	**Reject**	...	Reject	...
⋮	⋮	⋮	⋮	⋮	⋮	...	⋮	...
D	**Reject**	**Accept**	**Reject**	**Reject**	**Accept**	...	**??**	...
⋮	⋮	⋮	⋮	⋮	⋮	...	⋮	...

What will be the Dth diagonal entry??

Contradiction!

The Halting problem

$HALT_{TM} = \{\langle M, w \rangle | M$ **halts on input** $w\}$**. Is** $HALT_{TM}$ **decidable?**

The Halting Problem: Does there exist a Total Turing Machine H that accepts as input a Turing Machine M and an input string w and outputs YES, if $M(w)$ halts (accepts or rejects) and NO, if $M(w)$ does not halt (loops forever), i.e.

$$H(\langle M, w \rangle) = \begin{cases} \text{ACCEPTS, if } M(w) \text{ HALTS, i.e. accepts or rejects} \\ \\ \text{REJECTS, if } M(w) \text{ does not HALT, i.e. loops infinitely} \end{cases}$$

The Halting problem

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- Turing stated the Halting problem and demonstrated its undecidability in his famous 1936 paper.
- This provided a negative answer to Hilbert's Eintscheidungsproblem.
- **Proof strategy:** We will try to show that if we had such a Total TM H, we would be able to build a Total TM for A_{TM}
- But since we proved that A_{TM} is undecidable, this would mean that **H is undecidable**.

The Halting problem

$HALT_{TM} = \{\langle M, w \rangle | M \text{ halts on input } w\}$. Is $HALT_{TM}$ decidable?

$\langle M, w \rangle \longrightarrow \boxed{H} \longrightarrow ACCEPT$, if $M(w)$ halts

$\longrightarrow REJECT$, if $M(w)$ loops

Proof idea: We first assume that there exists such a Total Turing Machine H. Then, we use H as a subroutine to construct a Total Turing Machine A for A_{TM}

$\langle M, w \rangle \longrightarrow \boxed{A} \longrightarrow ACCEPT$, if $M(w)$ accepts

$\longrightarrow REJECT$, if $M(w)$ rejects or loops

But A cannot be Total and so a total TM that decides H cannot exist

The Halting problem

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$\langle M, w \rangle \rightarrow \boxed{H} \rightarrow$ **ACCEPT**, if $M(w)$ halts

REJECT, if $M(w)$ loops

Proof idea: We first assume that there exists such a Total Turing Machine H. Then, we use H as a subroutine to construct a Total Turing Machine A for A_{TM}

Outlining the steps for building A using H:

- A calls $H(\langle M, w \rangle)$
- If H rejects, then we know that $M(w)$ loops forever and so A would output $REJECT$.

$\langle M, w \rangle \rightarrow \boxed{A} \rightarrow$ **ACCEPT**, if $M(w)$ accepts
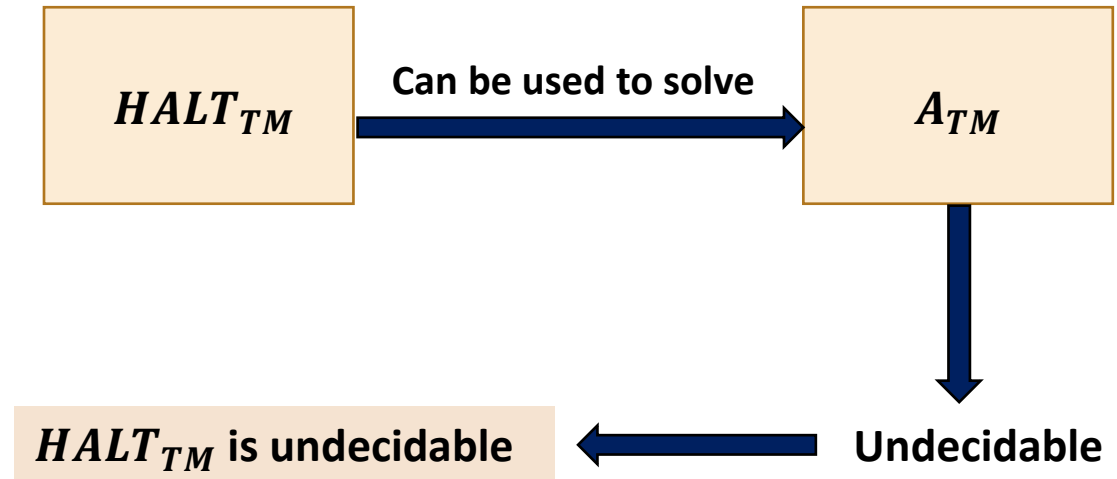
REJECT, if $M(w)$ rejects or loops

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Outlining the steps for building A using H:

- A calls $H(\langle M, w \rangle)$
- If H rejects, then we know that $M(w)$ loops forever and so A would output $REJECT$.
- If H accepts,
  - $M(w)$ surely halts (either accepts or rejects).

$\langle M, w \rangle \rightarrow \boxed{A} \rightarrow ACCEPT$, if $M(w)$ accepts

$\rightarrow REJECT$, if $M(w)$ rejects or loops

The Halting problem

$HALT_{TM} = \{\langle M, w \rangle | M$ **halts on input** $w\}$**. Is** $HALT_{TM}$ **decidable?**

$\langle M, w \rangle \rightarrow \boxed{H}$

$ACCEPT$, if $M(w)$ halts

$REJECT$, if $M(w)$ loops

Proof idea: We first assume that there exists such a Total Turing Machine H. Then, we use H as a subroutine to construct a Total Turing Machine A for A_{TM}

Outlining the steps for building A using H:

- A calls $H(\langle M, w \rangle)$
- If H rejects, then we know that $M(w)$ loops forever and so A would output $REJECT$.
- If H accepts,
  - $M(w)$ surely halts (either accepts or rejects).
  - Simply run $M(w)$ and
    - $ACCEPT$ if $M(w)$ accepts
    - $REJECT$ if $M(w)$ rejects

$\langle M, w \rangle \rightarrow \boxed{A}$

$ACCEPT$, if $M(w)$ accepts

$REJECT$, if $M(w)$ rejects or loops

The Halting problem

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Proof: Assume that there exists such a Total Turing Machine H. Then, we use H as a subroutine to construct a Total Turing Machine A for A_{TM} as follows:

$A =$ On input $\langle M, w \rangle$
       Run $H(\langle M, w \rangle)$
      If H rejects, output $REJECT$
      If H accepts,
            Run $M(w)$
         If $M(w)$ accepts, output $ACCEPT$
         If $M(w)$ rejects, output $REJECT$

https://www.udiprod.com/halting-problem/

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      If H rejects, output $REJECT$
      If H accepts,
            Run $M(w)$
          If $M(w)$ accepts, output $ACCEPT$
          If $M(w)$ rejects, output $REJECT$

$HALT_{TM}$ **Can be used to solve** → A_{TM}

↓ **Undecidable**

$HALT_{TM}$ **is undecidable** ←

The Halting problem

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$HALT_{TM} \in RE$ as H halts whenever M accepts or rejects w and so

$Q = $ On input $\langle M, w \rangle$:
- Simulate M on input w
- If M accepts w, $ACCEPT$; if M rejects w, $ACCEPT$

Q recognizes $HALT_{TM}$

The Halting problem

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Q recognizes $HALT_{TM}$

- $HALT_{TM}$ is undecidable
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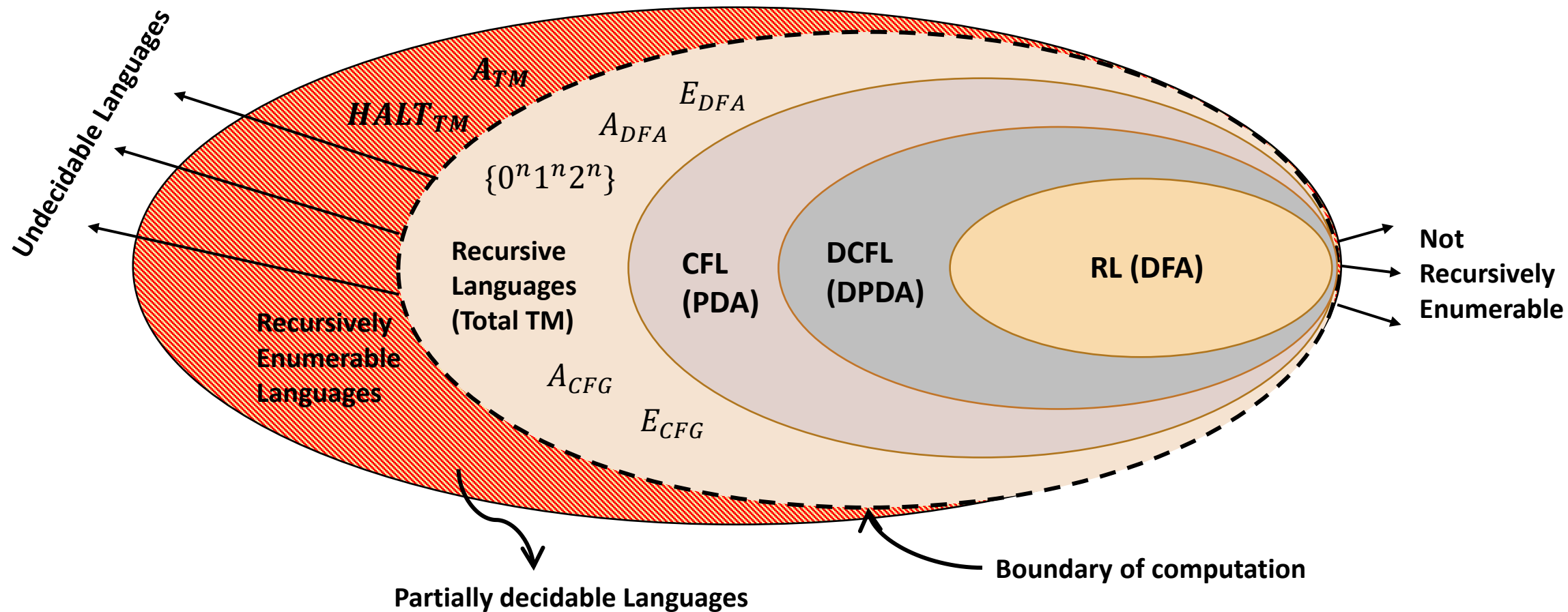
Reduction

Recall the proof of the undecidability of the Halting Problem

What did we do there?

- We used a (supposed) decider for the Halting Problem to build a decider for A_{TM}.
- This established that $HALT_{TM}$ can be used to solve A_{TM}.
- As A_{TM} is undecidable, this established that $HALT_{TM}$ is undecidable too.
- The key underlying concept is an idea called **Reduction.**
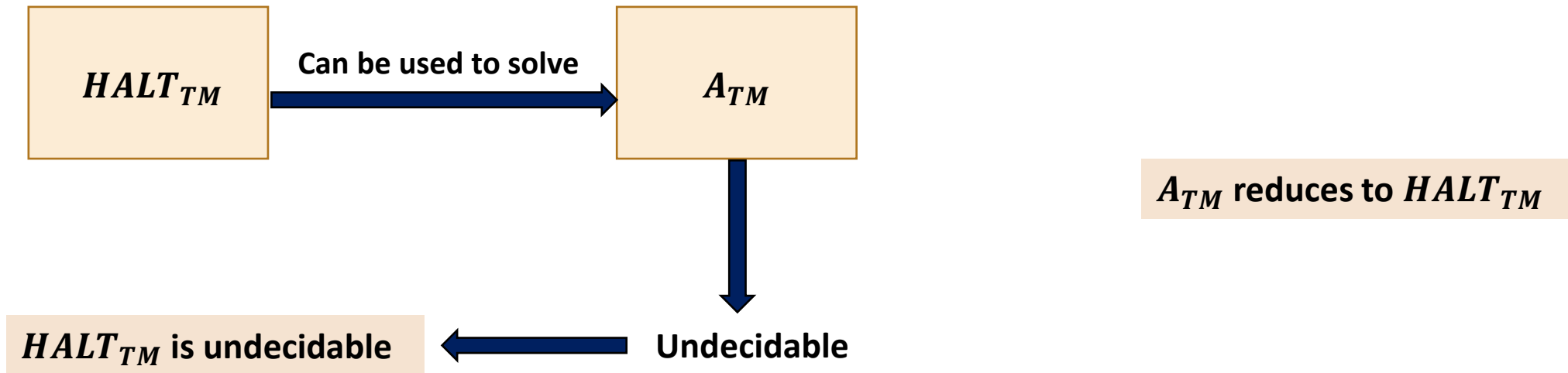
$$HALT_{TM} \xrightarrow{\textbf{Can be used to solve}} A_{TM}$$

A_{TM} **reduces to** $HALT_{TM}$

$HALT_{TM}$ **is undecidable** $\leftarrow$ **Undecidable**

Reduction

Generally,

- A language A **reduces to** another language **B** ($A \leqslant B$) iff we can build a **solver for** A using a **solver for** B

- In terms of computability, suppose using B we can compute A. Then, if A **is undecidable then so is** B.

- So, in the last proof we showed: $A_{TM} \leqslant HALT_{TM}$ to prove that $HALT_{TM}$ is undecidable.

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- This is a common technique to show that certain problems are decidable/undecidable.

Suppose, $A \leqslant B$ and

- A is undecidable. Then B is undecidable.

For example, we can prove that a problem P is undecidable by reducing the Halting problem to P.

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- So, in the last proof we showed: $A_{TM} \leqslant HALT_{TM}$ to prove that $HALT_{TM}$ is **undecidable**.
- This is a common technique to show that certain problems are decidable/undecidable.

Suppose, $A \leqslant B$ and

- A is undecidable. Then B is undecidable.
- **B is decidable. Then A is decidable.**

For example, we can prove that a problem P is undecidable by reducing the Halting problem to P.

$$A_{TM} \leqslant HALT_{TM}$$

If there existed a decider for $HALT_{TM}$, then we would have a decider for A_{TM}.

Reduction

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- In terms of computability, suppose using B we can compute A. Then, if A **is undecidable then so is B**.

- So, in the last proof we showed: $A_{TM} \leqslant HALT_{TM}$ to prove that $HALT_{TM}$ is undecidable.

- If **B is decidable, then so is A.**

- This is a common technique to show that certain problems are decidable/undecidable.

We can prove that several problems are undecidable by the **reduction to some known undecidable problem.**

Examples:

- $E_{TM} = \{\langle M \rangle | M$ is a Turing Machine and $L(M) = \Phi\}$
- $EQ_{TM} = \{\langle M_1, M_2 \rangle | M_1$ and M_2 are Turing Machines having $L(M_1) = L(M_2)\}$
- $ALL_{TM} = \{\langle M \rangle | M$ halts on all inputs$\}$
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Reduction

We can prove that several problems are undecidable by the **reduction of these problems** to some known undecidable problem.

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Generic strategy for proof:

- Consider that a Total TM for the given problem exists (say T_M).
- Show that can build a total TM that can decide some undecidable problem (e.g. $HALT_{TM}$, A_{TM}) using T_M as a subroutine.

This reduction would prove that $\boldsymbol{E_{TM}}, \boldsymbol{EQ_{TM}}, \boldsymbol{ALL_{TM}}$ **are undecidable**.

Undecidability

$E_{TM} = \{\langle M \rangle | M$ is a Turing Machine and $L(M) = \Phi\}$

Proof: Let T_E be the Turing Machine that decides E_{TM}. We shall prove that $\overline{A_{TM}} \leqslant E_{TM}$ by constructing a Turing Machine N that decides $\overline{A_{TM}}$ using $\boldsymbol{T_E}$.

What is $\overline{A_{TM}}$?

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$A_{TM} = \{\langle M, w \rangle | M$ **accepts input** $w\}$

$\overline{A_{TM}} = \{\langle M, w \rangle | M$ **DOES NOT accept input** $w\}$

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$$A(\langle M, w \rangle) = \begin{cases} \textbf{ACCEPTS, if } M(w) \textbf{ accepts} \\ \\ \textbf{REJECTS, if } M(w) \textbf{ rejects/loops} \end{cases}$$

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Key idea:

- $N(\langle M, w\rangle)$ first builds the encoding of a TM $T_{\langle M,w\rangle}$
- $T_{\langle M,w\rangle}$ does not accept any string if and only if M does not accept w.
- That is, running $T_{\langle M,w\rangle}(x)$ on **ANY** input x, runs M on w.

What does this achieve??

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- This means, $L(T) = \Phi$ if M does not accept w and $L(T) \neq \Phi$ if M accepts w!
- This allows N to call $T_E(\langle T \rangle)$

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  - ACCEPT if $T_E(\langle T \rangle)$ accepts $\Leftrightarrow L(T) = \Phi$ [M does not accept w]
  - REJECT if $T_E(\langle T \rangle)$ rejects $\Leftrightarrow L(T) \neq \Phi$ [M accepts w]

Undecidability

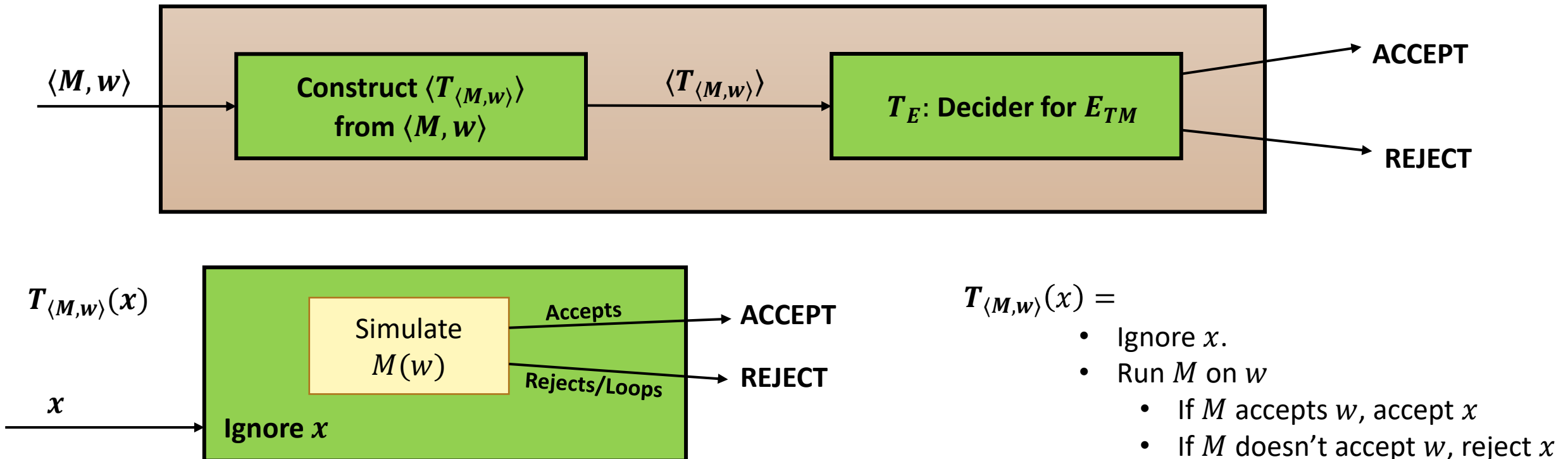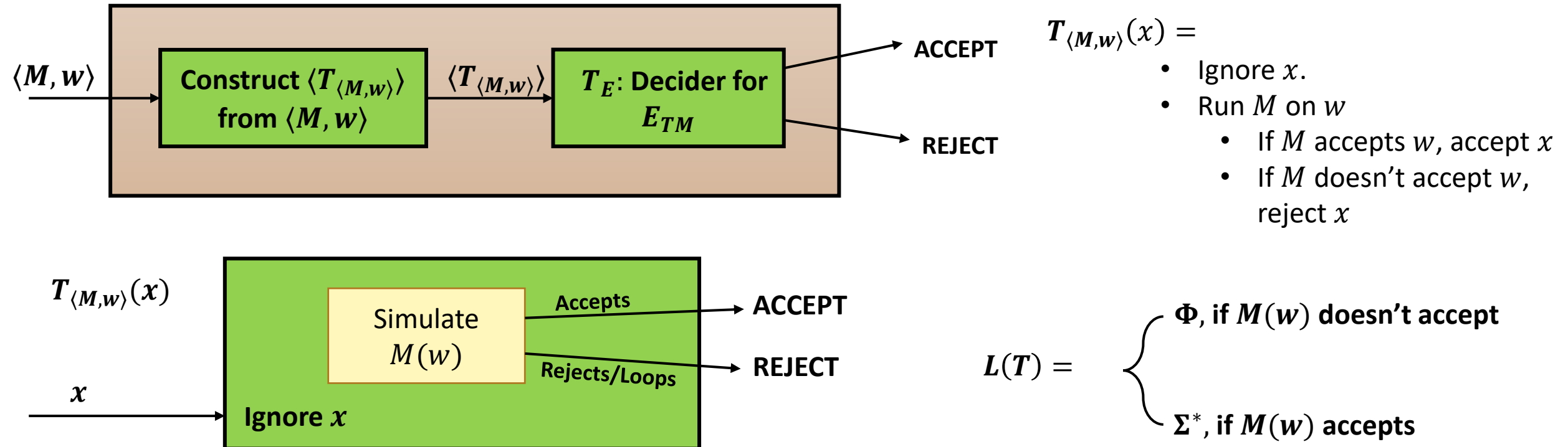$E_{TM} = \{\langle M \rangle | M \text{ is a Turing Machine and } L(M) = \Phi\}$

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- **Deciding $\overline{A_{TM}}$ is tied to whether $L(T) = \Phi$!**

Undecidability
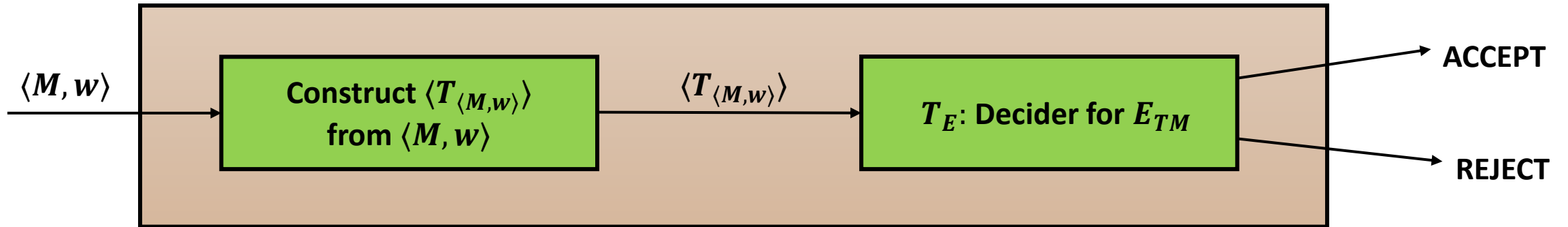
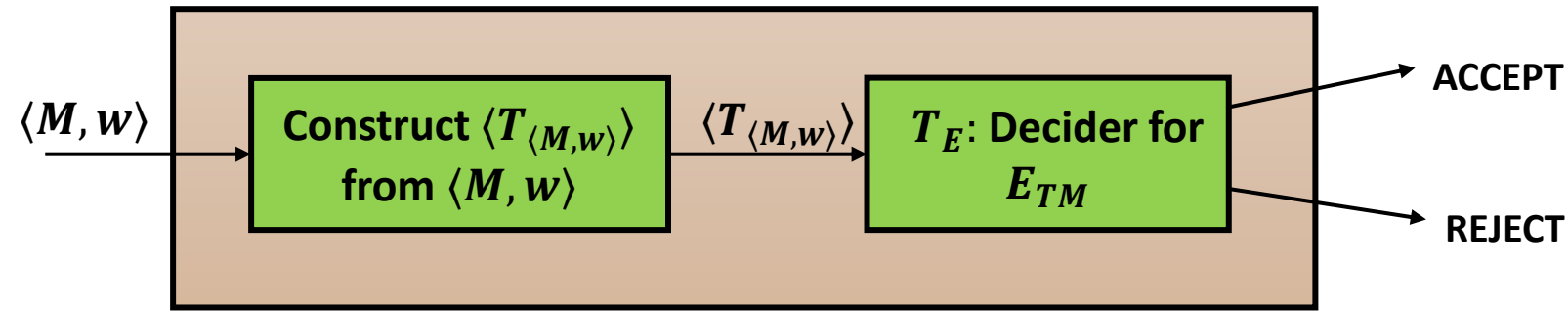$E_{TM} = \{\langle M \rangle | M \text{ is a Turing Machine and } L(M) = \Phi\}$

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$T_{\langle M,w \rangle}(x) =$
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$\langle M, w \rangle \longrightarrow$ **Construct $\langle T_{\langle M,w \rangle} \rangle$ from $\langle M, w \rangle$** $\xrightarrow{\langle T_{\langle M,w \rangle} \rangle}$ **T_E: Decider for E_{TM}** $\longrightarrow$ **ACCEPT** / **REJECT**

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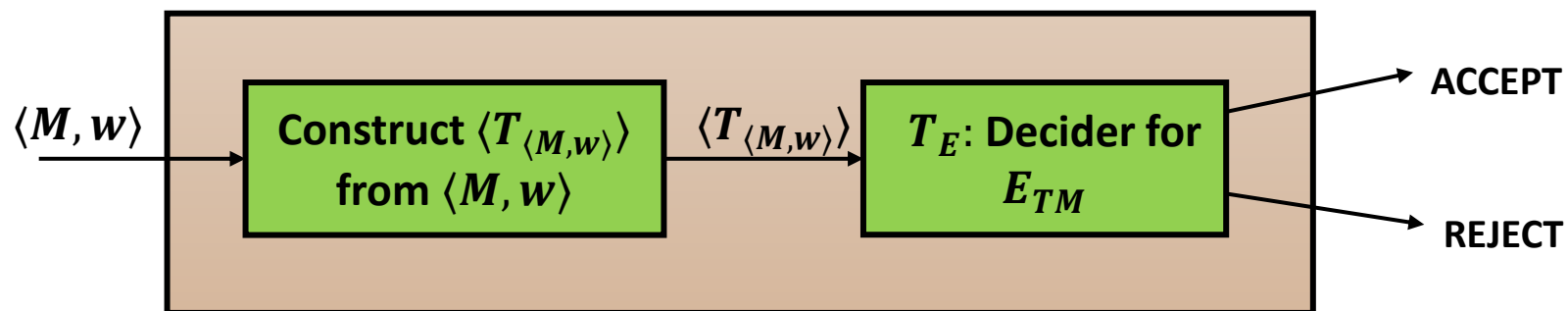
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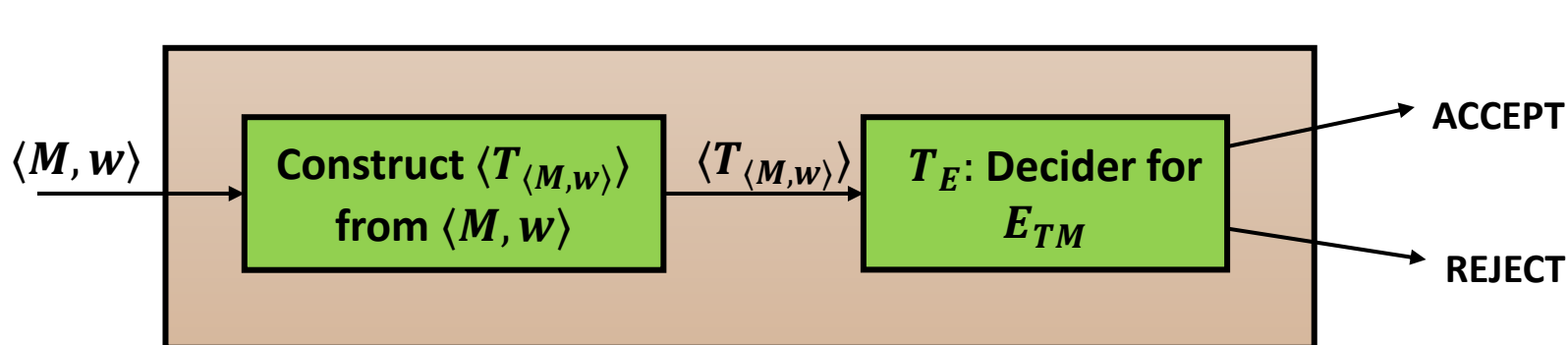
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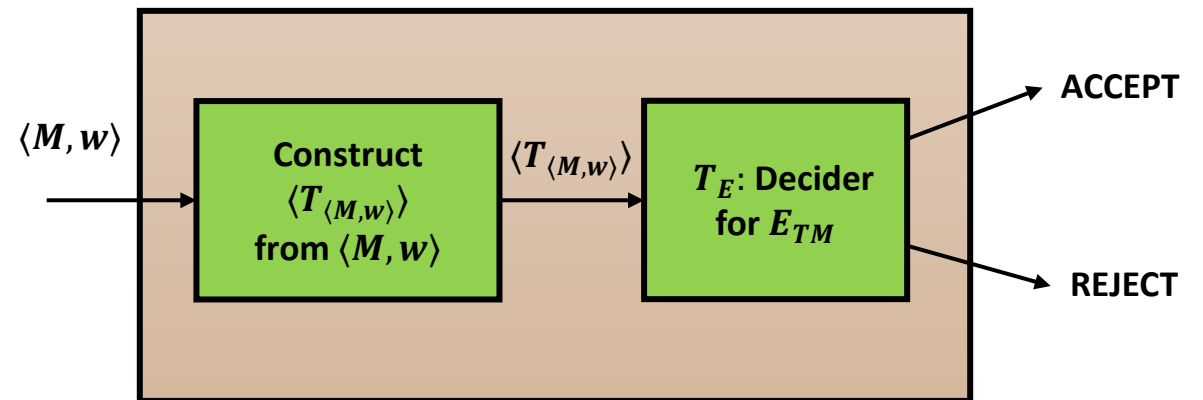- ACCEPT, **if** $L(T) = \Phi$
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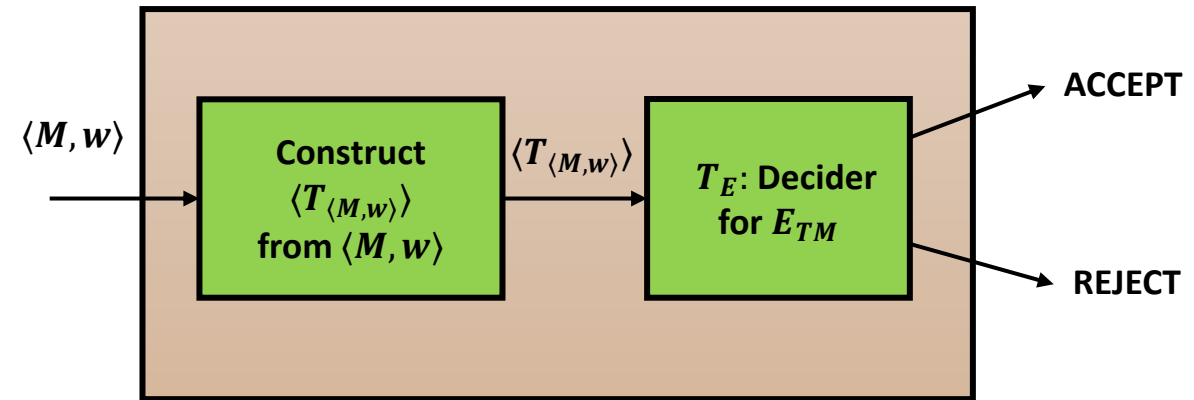- REJECT, **if** $L(T) \neq \Phi$ (if $M(w)$ accepts)

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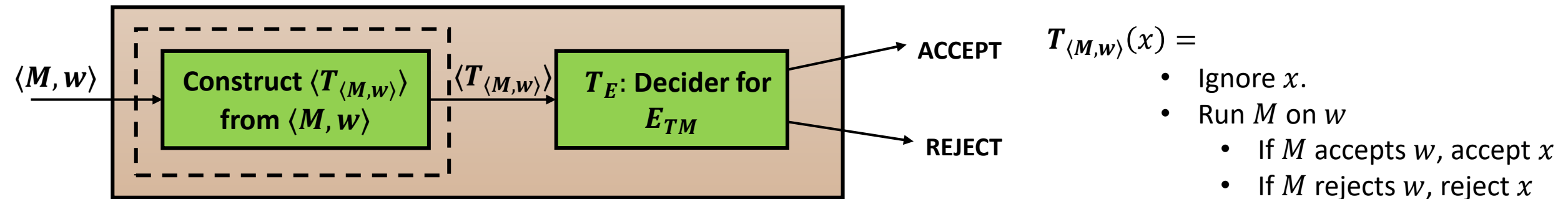

- $\overline{A_{TM}} \leqslant E_{TM}$
- $\overline{A_{TM}}$ is undecidable
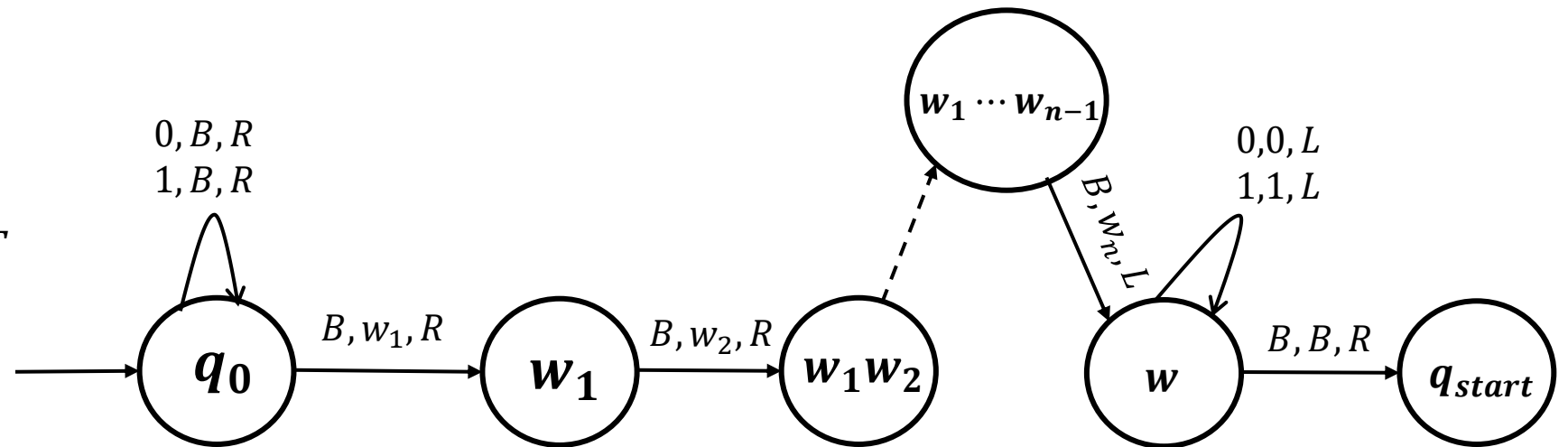- E_{TM} is undecidable!

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$T_{\langle M,w\rangle}$ from $\langle M,w\rangle$:

- Remove i/p x from the tape of T
- `Embed' w into the TM of M

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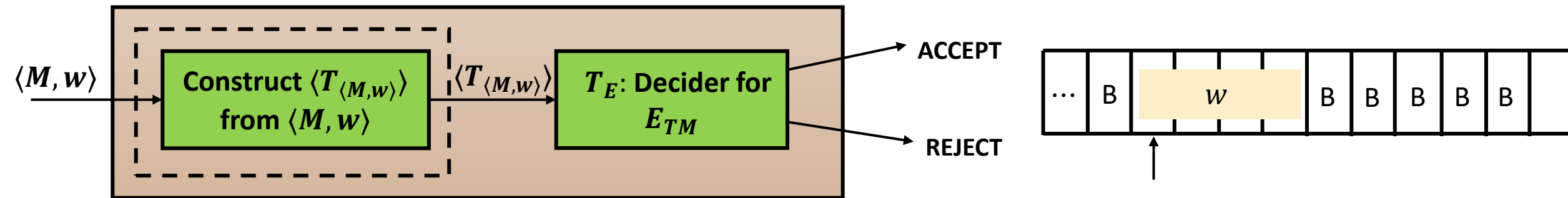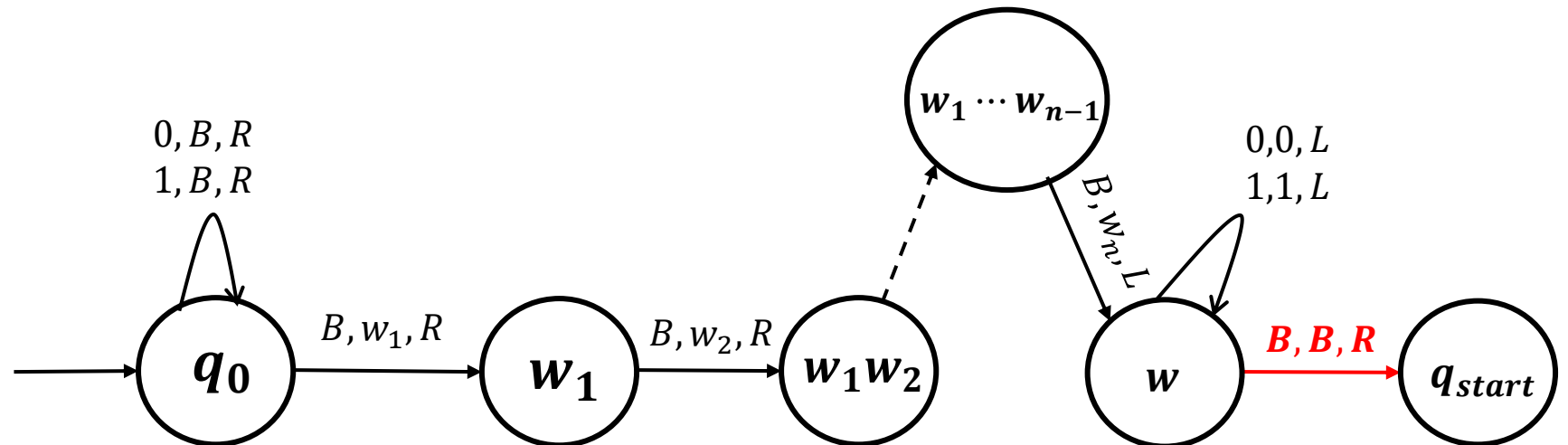$T_{\langle M,w \rangle}$ from $\langle M, w \rangle$:

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- $\overline{A_{TM}} \leqslant E_{TM}$
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- E_{TM} is undecidable!

Claim: $E_{TM} \in$ co-RE$-R$

WHY?

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$E_{TM} \in$ **co-RE**$-R$

Proof idea: We can build a co-recognizer for E_{TM}.

$C =$ On input $\langle M \rangle$
- For $i = 1, 2, 3, \cdots$
  - For $j = 1, 2, 3, \cdots i$

    Run M on s_j for i steps.

    If M accepts s_j, REJECT.

Thank You!