Probability And Statistics

Tutorial 6

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Question 1:

Generate an exponential random variable using the samples of a uniform random variable.

Question 2:

Give a method for generating a random variable having density function

$$f(x) = \begin{cases} \frac{x-2}{2} & \text{if} \quad 2 \leq x \leq 3 \\ 1 - \frac{x}{6} & \text{if} \quad 3 \leq x \leq 6 \end{cases}$$

Question 3:

Give a method for generating a random variable having density function

$$f(x) = \begin{cases} e^{2x} & \text{if} \quad -\infty \leq x < 0 \\ e^{-2x} & \text{if} \quad 0 \leq x \leq \infty \end{cases}$$

Question 4:

A binomial distribution with $n = 3$ and $p = 0.4$ is simulated by the inverse transform method with the uniform random numbers: $0.31, 0.71, 0.66, 0.48 and 0.19$. How many of the generated random variables are equal to 2?

Question 5:

Generate a Poisson random variable using the Inverse Transform method.

Question 6:

Let $\{X_n, n = 1, 2, \cdots\}$ and $\{Y_n, n = 1, 2, \cdots\}$ be two sequences of random variables, defined on the sample space S. Suppose that we know:

1. $$X_n \xrightarrow{a.s.} X$$ $$Y_n \xrightarrow{a.s.} Y$$

Prove that: $X_n + Y_n \xrightarrow{a.s.} X + Y$

2. $$X_n \xrightarrow{p} X$$ $$Y_n \xrightarrow{p} Y$$

Prove that: $X_n + Y_n \xrightarrow{p} X + Y$

Question 7:

Let $X_1,\ X_2,\ X_3,\ \cdots$ be a sequence of i.i.d. $Uniform(0,1)$ random variables. The sequence Y_n is defined as:

$$Y_n = min(X_1, X_2, \cdots, X_n)$$

Prove the following convergence results independently (i.e, do not conclude the weaker convergence modes from the stronger ones).

1. $Y_n \xrightarrow{d} 0$
2. $Y_n \xrightarrow{p} 0$
3. $Y_n \xrightarrow{L^r} 0$, for all $r \geq 1$
4. $Y_n \xrightarrow{a.s.} 0$

Question 8:

Consider a sequence $\{X_n, n = 1, 2, 3, ...\}$ such that,

$$X_n(s) = \begin{cases} n^2, & \text{with probability } \frac{1}{n} \\ 0, & \text{if with probability } 1 - \frac{1}{n} \end{cases} \tag{1}$$

Show that:

1. $X_n \xrightarrow{p} 0$
2. X_n does not converge in the r^{th} mean for any $r \geq 1$

Question 9:

Let $X_2, X_3, X_4, ...$ be a sequence of a random variable such that:

$$F_{X_n}(x) = \begin{cases} 1 - (1 - \frac{1}{n})^{nx}, & x > 0 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Show that X_n converges in distribution to $Exponential(1)$.

Question 10:

Let $X_1, X_2, X_3, ...$ be a sequence of random variables such that

X_n is a geometric random variable, $Geometric(\frac{\lambda}{n})$, for $n = 1, 2, 3, ...$

where $\lambda > 0$ is constant. Define a new sequence Y_n as

$$Y_n = \frac{1}{n} X_n,$$

for $n = 1, 2, 3, ...$

Show that Y_n converges in distribution to $Exponential(\lambda)$.

Question 11:

Catastrophes occur at time $T_1, T_2, T_3, ...$ where $T_i = X_1 + X_2 + ... + X_i$ and the X_i are independent identically distributed positive random variables. Let $N(t) = max\{n : T_n \leq t\}$ be the number of catastrophes which have occured by time t.

Prove that if $E(X_1) < \infty$ then $N(t) \to \infty$ and $N(t)/t \to 1/E(X_1)$ as $t \to \infty$, almost surely.