# Probability and Statistics
# Tutorial 7

Pratyanshu Pandey and Lavisha Bhambri

October 13, 2022

## Question 1

1. Let the moment generating function $M_X(s)$ be finite for $s \in [c, c]$, where $c > 0$. Assume $E[X] = 0$, and $Var(X) = 1$. Prove that

$$\lim_{n\to\infty}\left[M_X\left(\frac{s}{\sqrt{n}}\right)\right]^n = e^{\frac{s^2}{2}}$$

2. Use the result in part 1 to prove the central limit theorem.

## Question 2

Say you have a new algorithm and you want to test its running time. You have an idea of the variance of the algorithm's run time: $\sigma^2 = 4sec^2$ but you want to estimate the mean: $\mu = tsec$. You can run the algorithm repeatedly (IID trials). How many trials do you have to run so that your estimated runtime = t ± 0.5 with 95% certainty?

## Question 3

**Continuity Correction for Discrete Random Variables**
Let $X_1, X_2...X_n$be independent discrete random variables and let $Y = X_1 + X_2 + ... + X_n$. Suppose that we are interested in finding $P(A) = P(l \leq Y \leq u)$ using the CLT, where l and u are integers. Since Y is an integer-valued random variable, we can write

$$P(A) = P(l - \frac{1}{2} \leq Y \leq u + \frac{1}{2})$$

It turns out that the above expression sometimes provides a better approximation for P(A) when applying the CLT.
Wikipedia and probabilitycourse.com.

Now, let $X_1, X_2 ... X_{25}$ be i.i.d. with PMF:

$$P_X(k) = \begin{cases} 0.6 & \text{if } k = 1 \\ 0.4 & \text{if } k = -1 \\ 0 & \text{otherwise} \end{cases}$$

and $Y = X_1 + X_2 + ... + X_n$, using the CLT and continuity correction, estimate $P(4 \leq Y \leq 6)$.

## Question 4

You will roll a 6 sided dice 10 times. Let X be the total value of all 10 dice = $X_1 + X_2 + ... + X_{10}$. You win the game if $X \leq 25$ or $X \geq 45$. Use the central limit theorem to calculate the probability that you win.

## Question 5

The share price of a stock varies in a random manner, such that the price increase each minute is described by a discrete random variable X, with the following probability mass function:

$$P_X(k) = \begin{cases} 0.5 & \text{if } k = 0.05 \\ 0.2 & \text{if } k = 0 \\ 0.3 & \text{if } k = -0.05 \end{cases}$$

1. Use the central limit theorem to estimate the probability that the price will increase by 1.20 , or more, after 3 hours.
2. Assume a starting share price and model the above as a stochastic process.
3. Is the above stochastic process a Markov chain?

## Question 6

**Simple Random Walk:** Let $Y_1, Y_2..$ be i.i.d. random variables such that $Y_i = 1$ with equal probability. Let $X_0 = 0$ and $X_k = Y_1 + ... + Y_k$, for all $k \geq 1$. The sequence of random variables $X_i$ form a discrete time stochastic process called 1d simple random walk.

1. What can you information can you gain by applying the CLT on this problem?
2. For two positive integers A and B, what is the probability that the random walk reaches A before it reaches $-B$?
3. Is simple random walk a Markov chain?

## Question 7

The times between successive customer arrivals at a facility are independent and identically distributed random variables with the following PMF:

$$p(k) = \begin{cases} 0.2 & \text{if } k = 1 \\ 0.3 & \text{if } k = 3 \\ 0.5 & \text{if } k = 4 \end{cases}$$

Construct a four-state Markov chain model that describes the arrival process.

## Question 8

Let $U \sim Unif([0,1])$ and let $X_n = ((-1)^n U)/n$ for $n \geq 1$. Let the probability space be the standard unit-interval probability space.

1. Show that $(X_n : n \in N)$ converges almost surely.
2. Show that the sequence converges in the mean squared sense.

## Question 9

$(X_n : n \in N)$ is a sequence of independent random variables with marginal pmfs given by $P(X_n = 1/2(1 - 1/n)) = P(X_n = 1/2(1 + 1/n)) = 1/2$.

1. Show that the sequence converges almost surely.
2. Check if $(X_n : n \in N)$ converges in the mean squared convergence.

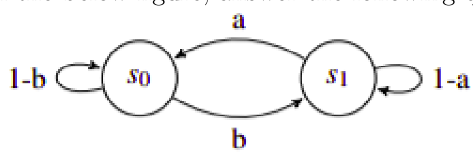
## Question 10

A certain calculating machine uses only the digits 0 and 1. It is supposed to transmit one of these digits through several stages. However, at every stage, there is a probability p that the digit that enters this stage will be changed when it leaves and a probability $q = 1$ - $p$ that it won't.

1. Form a Markov chain to represent the process of transmission by taking as states the digits 0 and 1.
2. What is the matrix of transition probabilities?
3. Now draw a tree and assign probabilities assuming that the process begins in state 0 and moves through two stages of transmission.
4. What is the probability that the machine, after two stages, produces the digit 0 (i.e., the correct digit)?

## Question 11

In the below figure, answer the following questions:-




1. For what values of a and b is the above Markov chain irreducible?
2. For what values of a and b is the above Markov chain reducible?
3. Construct a transition probability matrix using the above Markov chain.

## Question 12

For the following transition matrices, specify which are recurrent and which are transient?

$$P_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad P_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

## Question 13

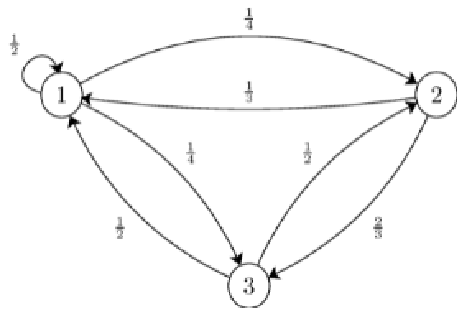
Consider the Markov chain with three states, S = 1, 2, 3, that has the following transition matrix.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

1. Draw the states transition diagram for this chain.
2. If we know $P(X_1 = 1) = P(X_1 = 2) = 1/4$, find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.

# Question 14

Consider the Markov chain shown in below figure:-



1. Is the chain irreducible?
2. Find the stationary distribution for this chain.