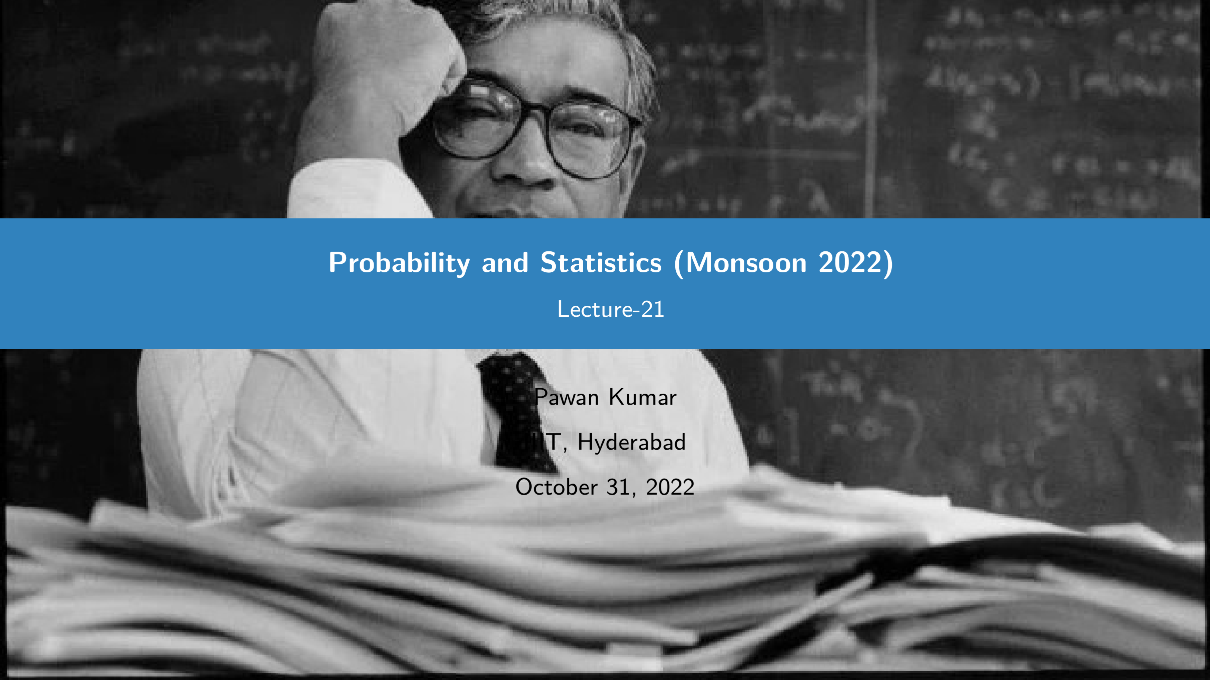

Probability and Statistics (Monsoon 2022)

Lecture-21

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Outline

Likelihood and log likelihood Function...

Definition of Likelihood and log likehood Function

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a distribution with a parameter θ. Suppose that we have observed $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$.

1. If X_i's are discrete, then the likelihood function is defined as

$$L(x_1, x_2, \ldots, x_n; \theta) = P_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; \theta)$$

2. If X_i's are jointly continuous, then the likelihood function is defined as

$$L(x_1, x_2, \ldots, x_n; \theta) = f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; \theta)$$

In some problems, it is easier to work with the log likelihood function given by

$$\ln L(x_1, x_2, \ldots, x_n; \theta)$$

Example (Example of maximum likelihood estimator)

Suppose that we have observed the random sample $X_1, X_2, X_3, ..., X_n$, where $X_i \sim N(\theta_1, \theta_2)$ so

$$f_{X_i}(x_i; \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i - \theta_1)^2}{2\theta_2}}$$

Find the maximum likelihood estimators for θ_1 and θ_2.

Answer to previous problem...

Answer to previous problem...

Asymptotic Properties of MLEs...

Asymptotic Properties of MLEs

[By asymptotic properties we mean properties that are true when the sample size becomes large.]
Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a distribution with a parameter θ. Let $\hat{\Theta}_{ML}$ denote the maximum likelihood estimator (MLE) of θ. Then, under some mild regularity conditions,

1. $\hat{\Theta}_{ML}$ is asymptotically consistent, i.e., $\lim_{n\to\infty} P(|\hat{\Theta}_{ML} - \theta| > \epsilon) = 0$
2. $\hat{\Theta}_{ML}$ is asymptotically unbiased, i.e., $\lim_{n\to\infty} E[\hat{\Theta}_{ML}] = \theta$
3. As n becomes large, $\hat{\Theta}_{ML}$ is approximately a normal random variable. More precisely, the random variable

$$\frac{\hat{\Theta}_{ML} - \theta}{\sqrt{\mathrm{Var}(\hat{\Theta}_{ML})}}$$

   converges in distribution to $N(0, 1)$.

Solved Example 1 ...

Example

Show the following:

1. Let $\hat{\Theta}_1$ be an unbiased estimator for θ, and W is a zero mean random variable. Show that

$$\hat{\Theta}_2 = \hat{\Theta}_1 + W$$

is also an unbiased estimator for θ

2. Let $\hat{\Theta}_1$ be an estimator for θ such that $E[\hat{\Theta}_1] = a\theta + b$, where $a \neq 0$. Show that

$$\hat{\Theta}_2 = \frac{\hat{\Theta}_1 - b}{a}$$

is an unbiased estimator for θ

Answer to previous problem...

Solved Example...

Example

Let $X_1, X_2, \ldots, X_n$ be a random variable from a Uniform$(0, \theta)$ distribution, where θ is unknown. Consider the estimator

$$\hat{\Theta}_n = \max\{X_1, X_2, \cdots, X_n\}$$

1. Find the bias of $\hat{\Theta}_n$, $B(\hat{\Theta}_n)$
2. Find the MSE of $\hat{\Theta}_n$, $\mathrm{MSE}(\hat{\Theta}_n)$
3. Is $\hat{\Theta}_n$ a consistent estimator of θ?

Answer to previous problem...

Answer to previous problem...

Answer to previous problem...

Solved Example...

Example

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a Geometric(θ) distribution, where θ is unknown. Find the maximum likelihood estimator (MLE) of θ based on this random sample.

Answer to previous problem...

Solved Example...

Example

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a Uniform$(0, \theta)$ distribution, where θ is unknown. Find the maximum likelihood estimator (MLE) of θ based on this random sample.

Answer to previous problem...

Interval Estimation and Confidence Level...

Interval Estimation and Confidence Level

1. Let $X_1, X_2, \ldots, X_n$ be random sample from a distribution with a parameter θ to be estimated
2. Suppose we observed $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, and obtained point estimate $\hat{\theta}$ of θ
3. Without additional information, we don't know whether $\hat{\theta}$ is close to θ
4. In an interval estimation, instead of just one value $\hat{\theta}$, we produce an interval $[\hat{\theta}_\ell, \hat{\theta}_h]$ that is likely to include true value of θ
5. The confidence level is the probability that the interval that we construct includes the real value of θ
6. The smaller the interval, the higher the precision with which we can estimate θ, and higher the confidence level

Interval Estimation with Confidence Level...

Interval Estimation

- Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a distribution with a parameter θ that is to be estimated
- An interval estimator with confidence level $1-\alpha$ consists of two estimators $\hat{\Theta}_l(X_1, X_2, \ldots, X_n)$ and $\hat{\Theta}_h(X_1, X_2, \cdots, X_n)$ such that

$$P\left(\hat{\Theta}_l \leq \theta \text{ and } \hat{\Theta}_h \geq \theta\right) \geq 1-\alpha,$$

  for every possible value of θ
- Equivalently, we say that $[\hat{\Theta}_l, \hat{\Theta}_h]$ is a $(1-\alpha)100\%$ confidence interval for θ
- The randomness in these terms is due to $\hat{\Theta}_l$ and $\hat{\Theta}_h$, not θ
- Here $\hat{\Theta}_l$ and $\hat{\Theta}_h$ are random variables because they are functions of $X_1, \ldots, X_n$

Steps for Finding Interval Estimators...

1. Let X be a continuous random variable with CDF $F_X(x) = P(X \leq x)$
2. We are interested in finding two values x_l and x_h such that

$$P\left(x_l \leq X \leq x_h\right) = 1 - \alpha$$

3. We can choose this as follows

$$P(X \leq x_l) = \frac{\alpha}{2} \quad \text{and} \quad P(X \geq x_h) = \frac{\alpha}{2}$$

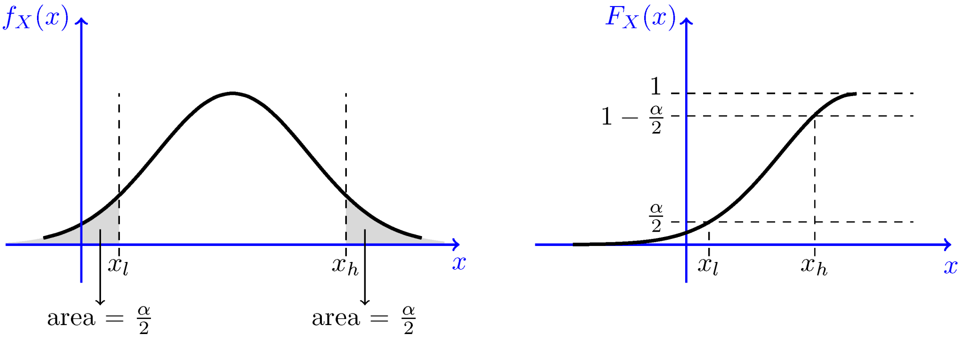
4. That is, we have from above

$$F_X(x_l) = \frac{\alpha}{2} \quad \text{and} \quad F_X(x_h) = 1 - \frac{\alpha}{2}$$

5. Rewriting these equations by using inverse, we have

$$x_l = F_X^{-1}\left(\frac{\alpha}{2}\right) \quad \text{and} \quad x_h = F_X^{-1}\left(1 - \frac{\alpha}{2}\right)$$

Plot of confidence Interval...



- $[x_l, x_h]$ is a $(1-\alpha)$ interval for X, that is, $P\left(x_l \leq X \leq x_h\right) = 1 - \alpha$