Question 1: Solution sketch. (Ignoring the $\lceil\ \rceil$ or $\lfloor\ \rfloor$).

$$T_1(n) = a \cdot T_1\left(\frac{n}{b}\right) + bn$$

$$= a\left(a \cdot T_1\left(\frac{n}{b^2}\right) + bn\right) + bn$$

$$= a^2 \cdot T_1\left(\frac{n}{b^2}\right) + (1+a) \cdot bn.$$

Claim: $T_1(n) = a^k \cdot T_1\left(\frac{n}{b^k}\right) + \left(\sum_{i=0}^{k-1} a^i\right) \cdot bn.$ for any $1 \leq k \leq \log_b n$

↳ Easy to prove.

For $k = \log_b n$; $T(n) = a^{\log_b n} \cdot T(1) + \sum_{i=0}^{\log_b n - 1} a^i \cdot bn$

$$= a^{\log_b n} + \left(\frac{a^{\log_b n} - 1}{a-1}\right) bn.$$

$$= a^{\log_b n}\left[1 + \frac{bn}{a-1}\right] - \frac{bn}{a-1}.$$

$$= \frac{bn^{\log_b a + 1}}{a-1} + n^{\log_b a} - \frac{bn}{a-1}.$$

$$= \Theta\left(n^{\log_b a + 1}\right)$$

Similarly, $T_2(n) = \Theta\left(n^{\log_a b + 1}\right)$.

In case when $a \geq b \geq 1$, we have that $T_1(n) \geq T_2(n)$ asymptotically.

Question 2: Let T be a BFS tree of the graph G.

Claim: For any edge $(u,v) \in E(G)$ but $(u,v) \notin E(T)$, if u and v have a common ancestor in T then $\exists$ a cycle in G.

Proof: Let x be the least common ancestor of u and v in T. Running BFS from x gave us two distinct, disjoint paths from x to u and x to v (based on our assumption). Paths of u to x and u to v are also paths in G.

Thus these two distinct $x \rightsquigarrow u$ and $x \rightsquigarrow v$ and the edge (u,v) form a cycle. $x \rightsquigarrow u - v \rightsquigarrow x$.

For each edge not in T, look for a least common ancestor of its end points in T. If there is a common ancestor $\Rightarrow$ cycle.

Question 3: We would like to show that no. of nodes with 2 children = no. of leaves -1.

Proof by induction on no. of total nodes (leaves + non-leaves)

$$\left.\begin{array}{l} N_L(T) = \text{no. of leaves in } T \\ N_B(T) = \text{no. of nodes with two children in } T \\ N(T) = \text{Total no. of nodes in } T. \end{array}\right\} \begin{array}{l} \underline{\text{R.T.P:}} \\ N_B(T) = N_L(T) - 1. \end{array}$$

Base case: $N(T) = 1 \Rightarrow$ The only node is trivially a leaf $\Rightarrow N_L(T) = 1$

$$\Rightarrow \quad N_B(T) = 0 = N_L(T) - 1.$$

Induction step: Let u be an arbitrary leaf. Let T' be the binary tree obtained by deleting u.

→ If parent of u had just u as a child then parent(u) now becomes a leaf. Thus no. of leaves and no. of nodes with two children remains the same.

$$N_B(T) = N_B(T') \text{ and } N_L(T') = N_L(T).$$

From induction hypothesis and fact that $N(T') < N(T)$;

$$N_B(T') = N_L(T') - 1 \Rightarrow N_B(T) = N_L(T) - 1.$$

→ If parent of u was a binary node (i.e two children) then upon deletion of u, parent(u) becomes a node w/ one child. Further no. of leaves decreases by 1.

$$N_B(T) = N_B(T') + 1 \quad \text{and} \quad N_L(T) = N_L(T') + 1.$$

From I.H: $N_B(T') = N_L(T') - 1$.

$$N_B(T) = N_B(T') + 1 = N_L(T') + \cancel{1} - \cancel{1} = N_L(T) - 1.$$

Question 4: We are given that T is both a BFS tree and a DFS tree of G. Suppose $\exists$ an edge $(u,v) \in E(G)$ s.t $(u,v) \notin T$. Let the node u be visited before v in DFS. In BFS, nodes u and v are at a distance at most 1 from each other's layer. Thus, in BFS tree if u is an ancestor of v then (u,v) is in BFS tree. Where as in the DFS tree, edge (u,v) was not added to the tree because node v might have been visited before and v is at a distance > 1.

Putting both of these together, we get that if u is an ancestor of v in T, then (u,v) is in BFS tree which contradicts our assumption that $(u,v) \in T$. Else, (u,v) is not in T (the DFS tree) then $dist(u,v) > 1$ which contradicts our assumption that T is also a BFS tree in which $dist(u,v) = 1$.
Thus we cannot have any edge $(u,v) \in E(G) \setminus E(T)$ if T is both a BFS and a DFS tree.

Question 5: Let G have two connected components G_1 and G_2. Let G_1 have n_1 vertices and G_2 have n_2 vertices. It is easy to observe that $\min\{n_1, n_2\} \leq \frac{n}{2}$. Let $n_1 \leq \frac{n}{2}$. For every node in G_1 can have a degree of at most $\frac{n}{2}$. This contradicts the fact that every node in G has a min degree of $\frac{n}{2}$.

Assumption: Graph is simple.

We can generalize this argument to k connected components and get same implication.

Question 6: Let $L_0, L_1, \dots, L_{k'}$ be the layers of BFS when run from s. Note that $L_0 = \{s\}$ and $dist(s,t) > \frac{n}{2}$. Then t is in a layer L_k where $k > \frac{n}{2}$. Now we claim that

$\exists$ a layer with exactly one node in it. Assume for the sake of contradiction each layer L_i (for $1 \leq i \leq \frac{n}{2}$) has at least 2 nodes each. Then there are at least $2 \times \frac{n}{2}$ nodes in layers $L_1, \dots, L_{\frac{n}{2}}$ but there can at most be $n-2$ nodes in them as $s \in L_0$ and $t \in L_k$ for $k > \frac{n}{2}$. Thus we arrive at a contradiction that every layer in $L_1, \ldots, L_{\frac{n}{2}}$ has at least 2 nodes.

$\Rightarrow$ $\exists$ a node in $L_1, \ldots, L_{\frac{n}{2}}$ with exactly one node. Now the task is to delete that node. This can be found using BFS in $O(m+n)$ time.