**Solution - 1**

$P(E \cap F) = P(E)\,P(F) = 1/6$
$P(E' \cap F') = (1 - P(E))(1 - P(F)) = 1/3$

$(P(E) - P(F))\,(1 - P(F)) > 0$
$0 < P(F) < 1 \Rightarrow (1 - P(F)) > 0$
$\Rightarrow P(E) > P(F)$

Solve the above equations

**Solution - 2**

(a) We have $P(A) = P(B) = P(C) = 1/2$. Writing the outcome of die 1 first, we can easily list all outcomes in the following intersections.

$$A \cap B = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5)\}$$
$$A \cap C = \{(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\}$$
$$B \cap C = \{(2,1),(4,1),(6,1),(2,3),(4,3),(6,3),(2,5),(4,5),(6,5)\}$$

By counting we see

$$P(A \cap B) = \frac{1}{4} = P(A)P(B).$$

Likewise,

$$P(A \cap C) = \frac{1}{4} = P(A)P(C) \quad \text{and} \quad P(B \cap C) = \frac{1}{4} = P(B)P(C).$$

So, we see that $A$, $B$, and $C$ are pairwise independent.

However, $A \cap B \cap C = \emptyset$, since if we roll an odd on die 1 and an odd on die 2, then the sum of the two will be even. So, in this case,

$$P(A \cap B \cap C) = 0 \neq P(A)P(B)P(C),$$

and we conclude that $A, B$ and $C$ are not mutually independent.

**Solution - 3**

$$P_k = k! \sum_{1 \leq i_1 < \cdots < i_k \leq k} p_{i_1} \ldots p_{i_k}.$$

(The sum is known as the $k$'th *symmetric polynomial* in $p_1, \ldots, p_n$.) This is obtained selecting some $k$ different birthdays and then deciding which of them belongs to which person.

For $i < j$, write

$$P_n = Ap_ip_j + B(p_i + p_j) + C,$$

where $A$, $B$ , and $C$ do not depend on either $p_i$ or $p_j$. Let $p'_i = p'_j = (p_i + p_j)/2$. Then (as it easy to verify by squaring out), $p'_ip'_j \geq p_ip_j$, with strict inequality unless $p_i = p_j$. Of course, $p'_i + p'_j = p_i + p_j$. Now if you replace $p_i$ and $p_j$ by $p'_i$ and $p'_j$, then $p'_ip'_j \geq p_ip_j$.

Now assume that $P_n$ is maximized while not all $p_i$ are equal, say $p_i \neq p_j$. We can then also assume that $P_n$ is nonzero (when it is zero it is obviously not maximal) and therefore that some $n$ of $p$'s are nonzero. Then $A \neq 0$ (even though $p_i$ or $p_j$ might be zero). Now replace $p_i$ and $p_j$ by $p'_i$ and $p'_j$; The sum of $p$'s is still 1, while $P_n$ has strictly increased. This contradiction shows that $p_i = p_j$ for all $i$ and $j$.

## Solution - 4

C be the event that coin B was chosen

P(A | B) = P(B | A) *(P(A) / P(B))

P(B) = P(B | A) * P(A) + P(B | C)*P(C)

= $(0.9)^{10} * 0.5 + (0.1)^{10} * 0.5$

= 0.17443392201 approx.

Hence

P(A | B) = $(0.9)^{10} * 0.5/((0.9)^{10} * 0.5 + (0.1)^{10} * 0.5)$

= 0.999457 Approx.

## Solution - 5

Let $A$ be the event that the first toss is a head and let $B$ be the event that the second toss is a head. We must compare the conditional probabilities $\mathbf{P}(A \cap B | A)$ and $\mathbf{P}(A \cap B | A \cup B)$. We have

$$\mathbf{P}(A \cap B | A) = \frac{\mathbf{P}((A \cap B) \cap A)}{\mathbf{P}(A)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)},$$

and

$$\mathbf{P}(A \cap B | A \cup B) = \frac{\mathbf{P}((A \cap B) \cap (A \cup B))}{\mathbf{P}(A \cup B)} = \frac{A \cap B}{A \cup B}.$$

Since $\mathbf{P}(A \cup B) \geq \mathbf{P}(A)$, the first conditional probability above is at least as large, so Alice is right, regardless of whether the coin is fair or not. In the case where the coin is fair, that is, if all four outcomes $HH, HT, TH, TT$ are equally likey, we have

$$\frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{1/4}{1/2} = \frac{1}{2}, \qquad \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A \cup B)} = \frac{1/4}{3/4} = 1/3.$$

A generalization of Alice's reasoning is that if $A$, $B$, and $C$ are events such that $B \subset C$ and $A \cap B = A \cap C$ (for example, if $A \subset B \subset C$), then the event $A$ is at least as likely if we know that $B$ has occurred than if we know that $C$ has occurred. Alice's reasoning corresponds to the special case where $C = A \cup B$.

**Solution - 6**

*Solution*: Let $A$ be the event of receiving an A, $W$ be the event of being a woman, and $M$ the event of being a man. We are given $\mathbb{P}(A \mid W) = 0.30, \mathbb{P}(A \mid M) = 0.25, \mathbb{P}(W) = 0.60$ and we want $\mathbb{P}(W \mid A)$. From the definition

$$\mathbb{P}(W \mid A) = \frac{\mathbb{P}(W \cap A)}{\mathbb{P}(A)}.$$

As in the previous example,

$$\mathbb{P}(W \cap A) = \mathbb{P}(A \mid W)\mathbb{P}(W) = (0.30)(0.60) = 0.18.$$

To find $\mathbb{P}(A)$, we write

$$\mathbb{P}(A) = \mathbb{P}(W \cap A) + \mathbb{P}(M \cap A).$$

Since the class is 40% men,

$$\mathbb{P}(M \cap A) = \mathbb{P}(A \mid M)\mathbb{P}(M) = (0.25)(0.40) = 0.10.$$

So

$$\mathbb{P}(A) = \mathbb{P}(W \cap A) + \mathbb{P}(M \cap A) = 0.18 + 0.10 = 0.28.$$

Finally,

$$\mathbb{P}(W \mid A) = \frac{\mathbb{P}(W \cap A)}{\mathbb{P}(A)} = \frac{0.18}{0.28}.$$

## Solution - 7

$A$ is independent of itself if and only if $\mathbf{P}(A \cap A) = \mathbf{P}(A)\mathbf{P}(A)$. Since $A \cap A = A$ then $A$ must satisfy $\mathbf{P}(A) = (\mathbf{P}(A))^2$. Therefore, $A$ is independent of itself if and only if $\mathbf{P}(A) = 1$ or $\mathbf{P}(A) = 0$.

## Solution - 8

P(knows answer) = 1 - ½ = ½

P(knew answer / correct answer) = P(knew answer ∩ correct) / P(knew answer ∩ correct) + P(guessed answer ∩ correct) + P(copied ∩ correct)

## Solution - 9

**Example 4.12.** A total of 500 married couples are polled about their salaries with the following results

| | husband makes less than \$25K | husband makes more than \$25K |
|---|---|---|
| wife makes less than \$25K | 212 | 198 |
| wife makes more than \$25K | 36 | 54 |

(a) Find the probability that a husband earns less than \$25K.
*Solution*:

$$\mathbb{P}(\text{husband makes } < \$25\text{K}) = \frac{212}{500} + \frac{36}{500} = \frac{248}{500} = 0.496.$$

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(b) Find the probability that a wife earns more than \$25K, given that the husband earns as that much as well.
*Solution*:

$$\mathbb{P}\left(\text{wife makes } > \$25\text{K} \mid \text{husband makes } > \$25\text{K}\right) = \frac{54/500}{(198+54)/500} = \frac{54}{252} = 0.214$$

(c) Find the probability that a wife earns more than \$25K, given that the husband makes less than \$ 25K.
*Solution*:

$$\mathbb{P}\left(\text{wife makes } > \$25\text{K} \mid \text{husband makes } < \$25\text{K}\right) = \frac{36/500}{248/500} = 0.145.$$

## Solution - 10

Let $A_i$ be the event that the result of first roll is $i$, and note that $\mathbf{P}(A_i) = 1/4$ for each $i$. Let $B$ be the event that the sum total is at least 4. Given the event $A_1$, the sum total will be at least 4 if the second roll results in 3 or 4, which happens with probability 1/2. Similarly, given the event $A_2$, the sum total will be at least 4 if the second roll results in 2, 3, or 4, which happens with probability 3/4. Also, given the event $A_3$, we stop and the sum total remains below 4. Therefore,

$$\mathbf{P}(B \mid A_1) = \frac{1}{2}, \qquad \mathbf{P}(B \mid A_2) = \frac{3}{4}, \qquad \mathbf{P}(B \mid A_3) = 0, \qquad \mathbf{P}(B \mid A_4) = 1.$$

By the total probability theorem,

$$\mathbf{P}(B) = \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{3}{4} + \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1 = \frac{9}{16}.$$

## Solution - 11

Since the $C_i$'s form a partition of the sample space, we can apply the law of total probability for $A \cap B$:

$$\begin{aligned} P(A \cap B) &= \textstyle\sum_{i=1}^{M} P(A \cap B|C_i)P(C_i) \\ &= \textstyle\sum_{i=1}^{M} P(A|C_i)P(B|C_i)P(C_i) && (A \text{ and } B \text{ are conditionally independent}) \\ &= \textstyle\sum_{i=1}^{M} P(A|C_i)P(B)P(C_i) && (B \text{ is independent of all } C_i\text{'s}) \\ &= P(B)\textstyle\sum_{i=1}^{M} P(A|C_i)P(C_i) \\ &= P(B)P(A) && (\text{law of total probability}). \end{aligned}$$

**Solution - 12**

Let A be the event of getting an A, B be the event of living on campus

P(A) = 120/600

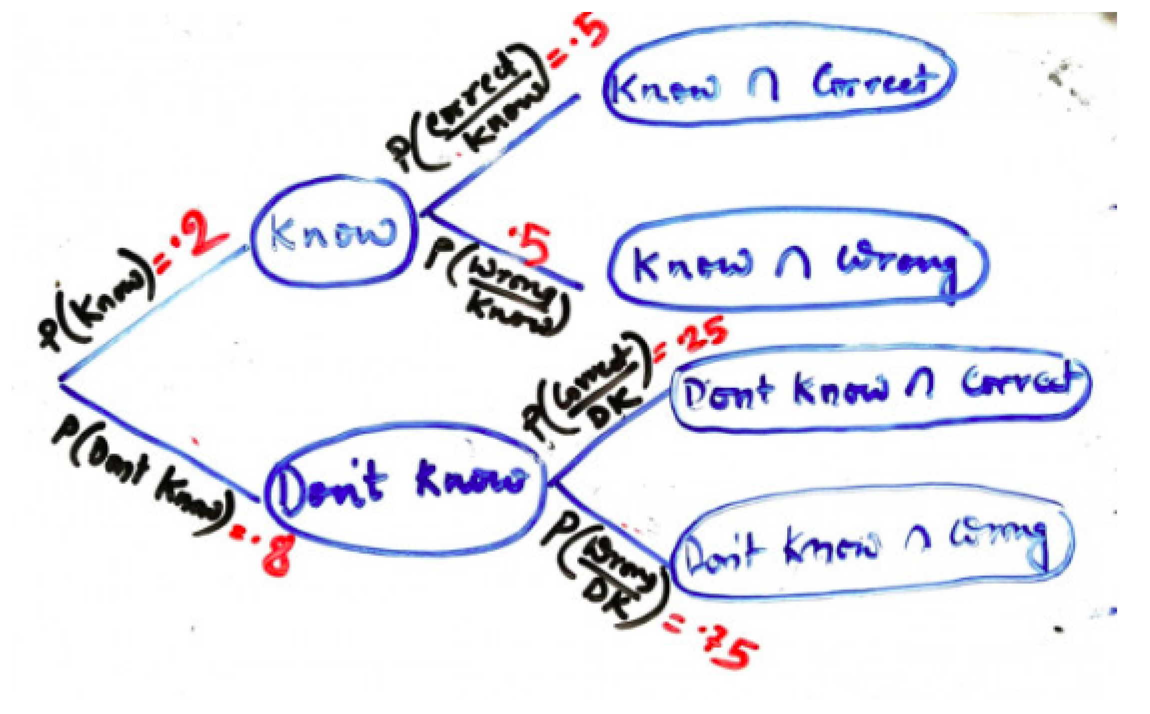
P(B) = (600-200)/600

P(A∩B) = 80/600

P(A∩B) = P(A) * P(B)

So, they are independent

**Solution - 13**




__Given that__

P(Know)=.2 , P(Don't know)=.8

P(Correct/Know)=.5 , P(Wrong/Know)=.5

P(Correct/ Don't Know)=.25 , P(Wrong/ Don't Know)=.75

**We have to find P(Know/Correct)**

By Bayes' Theorem,

$$P(Know/Correct) = \frac{P(Know \cap Correct)}{P(Correct)}$$

$$= \frac{P(Know) \times P(Correct/Know)}{P(Know \cap Correct) + P(Don'tKnow \cap Correct)}$$

$$= \frac{P(Know) \times P(Correct/Know)}{P(Know) \times P(Correct/Know) + P(Don'tKnow) \times P(Correct/(Don'tKnow)}$$

$$= \frac{.2 \times .5}{.2 \times .5 + .8 \times .25}$$

$$= \frac{.1}{.3}$$

$$= .333$$

## Solution - 14

For the first, simply use the basic definition of probability

$$\frac{Favourable}{Total} = \frac{1+\frac{3}{4}+\frac{1}{2}}{3}$$

For the second, use **Bayes Theorem**. Let the events be :

1. $U_1$: Coin with both heads flipped.
2. $U_2$: Coin which shows head 3/4 times flipped.
3. $F$: Fair coin flipped.
4. $H$ : Head is the result.

Then we need to find $P(F|H)$

$$P(F|H) = \frac{P(H|F)\cdot P(F)}{P(H|U_1)\cdot P(U_1) + P(H|U_2)\cdot P(U_2) + P(H|F)\cdot P(F)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{1}\cdot\frac{1}{3}+\frac{3}{4}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{3}} = 2/9$$

## Solution - 15

*Solution*: Let $A = Sarah\ has\ two\ aces$, and let $B = Bob\ has\ exactly\ one\ ace$. In order to compute $\mathbb{P}(B \mid A)$, we need to calculate $\mathbb{P}(A)$ and $\mathbb{P}(A \cap B)$. On the one hand, Sarah could have any of $\binom{52}{13}$ possible hands. Of these hands, $\binom{4}{2}\cdot\binom{48}{11}$ will have exactly two aces so that

$$\mathbb{P}(A) = \frac{\binom{4}{2}\cdot\binom{48}{11}}{\binom{52}{13}}.$$

On the other hand, the number of ways in which Sarah can pick a hand and Bob another (different) is $\binom{52}{13}\cdot\binom{39}{13}$. The the number of ways in which $A$ and $B$ can simultaneously occur is $\binom{4}{2}\cdot\binom{48}{11}\cdot\binom{2}{1}\cdot\binom{37}{12}$ and hence

$$\mathbb{P}(A \cap B) = \frac{\binom{4}{2}\cdot\binom{48}{11}\cdot\binom{2}{1}\cdot\binom{37}{12}}{\binom{52}{13}\cdot\binom{39}{13}}.$$

Applying the definition of conditional probability we finally get

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\binom{4}{2}\cdot\binom{48}{11}\cdot\binom{2}{1}\cdot\binom{37}{12}\Big/\binom{52}{13}\cdot\binom{39}{13}}{\binom{4}{2}\cdot\binom{48}{11}\Big/\binom{52}{13}} = \frac{2\cdot\binom{37}{12}}{\binom{39}{13}}$$

**Solution - 16**

P(1st head at kth toss) = (1-p)^(k-1) * p
First head at even => sum of all probabilities where k = 2, 4, ...
This is a GP

**Solution - 17**

(a) What is the probability that the three balls are blue?
*Solution*: In this case, we can define the sequence of events $B_1, B_2, B_3, \ldots$, where $B_i$ is the event that *the ith ball drawn is blue*. Applying the multiplication rule yields

$$\mathbb{P}(B_1 \cap B_2 \cap B_3) = \mathbb{P}(B_1)\mathbb{P}(B_2 \mid B_1)\mathbb{P}(B_3 \mid B_1 \cap B_2) = \frac{5}{13}\frac{6}{14}\frac{7}{15}.$$

(b) What is the probability that only 1 ball is blue?
*Solution*: denoting by $R_i$ = the even that *the ith ball drawn is red*, we have

$$\mathbb{P}\,(\text{only 1 blue ball}) = \mathbb{P}(B_1 \cap R_2 \cap R_3) + \mathbb{P}(R_1 \cap B_2 \cap R_3) + \mathbb{P}(R_1 \cap R_2 \cap B_3) = 3\frac{5 \cdot 8 \cdot 9}{13 \cdot 14 \cdot 15}.$$

**Solution - 18**

$$\mathbf{P}(A_1) = 0.5, \qquad \mathbf{P}(A_2) = 0.25, \qquad \mathbf{P}(A_3) = 0.25.$$

Also, $B$ is the event of winning, and

$$\mathbf{P}(B \mid A_1) = 0.3, \qquad \mathbf{P}(B \mid A_2) = 0.4, \qquad \mathbf{P}(B \mid A_3) = 0.5.$$

Suppose that you win. What is the probability $\mathbf{P}(A_1 \mid B)$ that you had an opponent of type 1?

Using Bayes' rule, we have

$$\begin{aligned}\mathbf{P}(A_1 \mid B) &= \frac{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1)}{\mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \mathbf{P}(A_2)\mathbf{P}(B \mid A_2) + \mathbf{P}(A_3)\mathbf{P}(B \mid A_3)} \\ &= \frac{0.5 \cdot 0.3}{0.5 \cdot 0.3 + 0.25 \cdot 0.4 + 0.25 \cdot 0.5} \\ &= 0.4.\end{aligned}$$