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Tutorial 4 - Probability and Statistics

Problem 1

Throw a dart on a disk of radius r. Probability on the coordinates (X, Y) is described by a pdf on the disk:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2}, & \text{if } x^2 + y^2 \leq r^2 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal pdfs

Also compute the marginal CDFs of X and Y.

Problem 2

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x + y & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[XY^2]$.

Problem 3

The joint density of X and Y is given by:

$$f(x,y) = \begin{cases} \frac{15}{2}x(2-x-y) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Compute the marginal PDFs, CDFs of X and Y. Also, find the expectations E[X], E[Y], E[XY].

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Problem 4

A point is chosen uniformly at random from the triangle that is formed by joining the three points $(0, 0), (0, 1)$ and $(1, 0)$ (units measured in centimetre). Let X and Y be the co-ordinates of a randomly chosen point.

(i) What is the joint density of X and Y?

(ii) Calculate the expected value of X and Y, i.e., expected co-ordinates of a randomly chosen point.

Problem 5

(a) Find a constant c, such that the function:

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

is a density function, and (b) compute $P(1 < X < 2)$.

Problem 6

A continuous random variable X has p.d.f

$f(x) = 5x^4, 0 \le x \le 1$

Find a_1 and a_2 such that (i) $P[X \le a_1] = P[X > a_1]$ (ii) $P[X > a_2] = 0.05$

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Problem 7

You are given the probability distribution function (PDF) of a continuous random variable X is $f_X(x)$. Let Y be a continuous random variable such that $Y = aX + b$, where a and b are non-zero real constants.

1. Find the PDF of Y in terms of f_X, a, and b. [2 points]
2. Let X be an exponential random variable with parameter λ. When will Y also be an exponential random variable? [1 point]
3. Let X be a normal random variable with mean μ and variance σ^2 . When will Y also be a normal random variable? [1 point]

Note: Problems 8 and 9 can be answered using the formula derived in problem 7. You can use Method of Transforms in problem 7, however it is not necessary to use it.

Problem 8

Let X be a normal distribution with mean μ and variance σ^2 i.e. $X \sim N(\mu, \sigma^2)$. If $Y = e^X$, then find the

1. CDF of Y. You can leave the answer of CDF in terms of ϕ.
2. PDF of Y. Solve it in two ways: directly using Method of Transformations, and without using Method of Transformations.

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Problem 9

Find an approximation to the probability that the number of 2s obtained in 12,000 rolls of a die are between 1900 and 2150 (non-inclusive). Give the answer up to three decimal places correct. [Hint: Use Normal Approximation of Binomial Distribution – read here https://online.stat.psu.edu/stat414/lesson/28/28.1]

Problem 10

Determine whether X and Y are independent:

a. $f_{XY}(x,y) = \begin{cases} 2e^{-x-2y} & x,y>0 \\ 0 & \text{otherwise} \end{cases}$

b. $f_{XY}(x,y) = \begin{cases} 8xy & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$

Problem 11

Two components of a laptop have the following joint probability density function for their useful lifetimes X and Y (in years):

$$f_{XY}(x,y) = \begin{cases} x\,e^{-x(1+y)} & x \geq 0, y \geq 0 \\ 0 & otherwise \end{cases}$$

(a) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$, and the marginal CDFs $F_X(x)$ and $F_Y(y)$.
(b) What is the probability that the lifetime of at least one component exceeds 1 year ?

Problem 12

A surface has infinite parallel lines with equal spacing of length *d* between them. We have a needle of length *l* which we throw randomly on the surface. What is the probability that the needle intersects a line? Assume $l < d$, and that the needle lies in the same plane as the surface.

Problem 13

An ambulance travels back and forth, at a constant specific speed v, along a road of length L. We may model the location of the ambulance at any moment in time to be uniformly distributed over the interval $(0, L)$. Also at any moment in time, an accident (not involving the ambulance itself) occurs at a point uniformly distributed on the road; that is, the accidents distance from one of the fixed ends of the road is also uniformly distributed over the interval $(0, L)$. Assume the location of the accident and the location of the ambulance are independent. Supposing the ambulance is capable of immediate U-turns, compute the CDF and PDF of the ambulances travel time T to the location of the accident.

Problem 14

Consider the unit disc:

$$D = \left\{(x, y) \mid x^2 + y^2 \leq 1\right\}$$

Suppose that we choose a point (X,Y) uniformly at random in D. That is, the joint PDF of X and Y is given by:

$$f_{XY}(x, y) = \begin{cases} c & (x, y) \in D \\ 0 & otherwise \end{cases}$$

(a) Find the constant c.

(b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

(c) Find the conditional PDF of X given Y = y, where $-1 \leq y \leq 1$.

(d) Are X and Y independent?