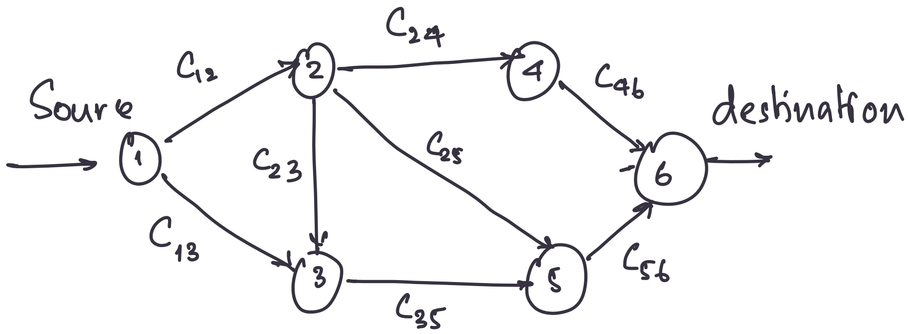
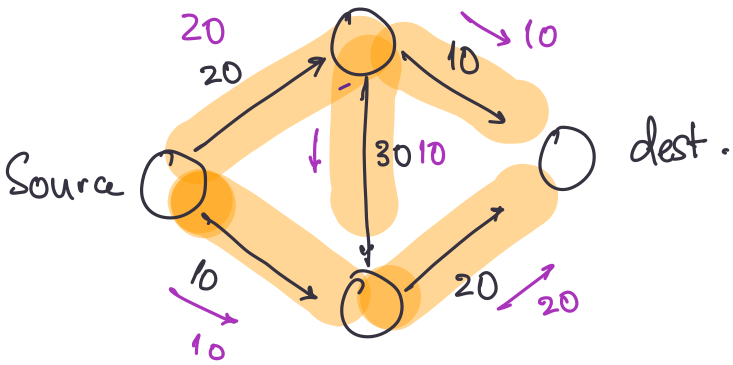
Network Flows.





Conservation of flow:

Any flow that reaches the node is exactly the flow that leaves it.

Flow problem: Given inputs

→ Graph $G = (V, E)$ and for every edge we have a capacity defined.

Properties of capacity: $c(u \to v) \geq 0 \quad \forall \; u \to v \in E$

and any flow $f(u \to v)$ must be s.t

$$0 \leq f(u \to v) \leq c(u \to v).$$

$\underline{\text{Cut}}\,(S, \underbrace{V \setminus S}_{T})$:

Let S contain source and T contain dest.

$$\text{Capacity}(S, T) = \sum_{u \in S} \sum_{v \in T} c(u \to v)$$

$$\text{Min cut}(G) = \min_{S \subseteq V} \text{capacity}(S, T).$$

1. For every edge $u \to v \in E$, we can define $f(u \to v)$

flow along the edge $u \to v$.

$0 \le f(u \to v) \le c(u \to v)$ ← constraints over edges

$f(u \to v) = 0$ if $u \to v \notin E$.

2. For any vertex $v \in G$,

$$\sum_w f(w \to v) = \sum_u f(v \to u)$$ ← constraints over nodes.

→ flow/Throughput → $\underbrace{\sum_w f(s \to w)}_{\text{maximize}}$ subject to 1 and 2

// $\sum_u f(u \to t)$

Any flow that constraints 1 and 2 is called "feasible".

<u>Thm</u>: If f is any feasible flow and (S,T) is any (source, dest)-cut, then total flow $|f|$ is at most the capacity of the cut.

$$|f| = \sum_w f(s \to w)$$

$$\sum_w f(s \to w) - \sum_u \cancel{f(u \to s)}^{\,0}$$

<u>Pf</u>: $$|f| = \sum_{w \in V} f(s \to w) = \sum_{w \in V} f(s \to w) + \underbrace{\left(\sum_{v \in S \setminus \{s\}} \left(\sum_w f(v \to w) - \sum_u f(u \to v) \right) \right)}_{0}$$

$$= \sum_{v \in S} \sum_w f(v \to w) - \sum_{v \in S} \sum_u f(u \to v)$$

$$= \sum_{v \in S'} \sum_{w \notin S'} f(v \to w) - \sum_{v \in S'} \sum_{u \notin S'} f(u \to v)$$

$$= \sum_{v \in S'} \sum_{w \in T} f(v \to w) - \underbrace{\sum_{v \in S} \sum_{u \in T} f(u \to v)}_{\geq 0} .$$

$A - B \leq A$
if $B \geq 0$

$$\leq \sum_{v \in S} \sum_{w \in T} f(v \to w)$$

$$\leq \sum_{v \in S} \sum_{w \in T} c(v \to w) \quad \leftarrow \text{Constraint 1.}$$

$$= \text{Capacity}(S,T) \quad \leftarrow \text{defn.}$$

$$\text{any feasible Flow} \underbrace{\leq}_{} \text{capacity of any cut}$$

$$\text{maximum flow} = \max \left\{ \begin{array}{l} \text{flow value over all} \\ \text{feasible flows} \end{array} \right\}$$

$$\text{min-cut} = \min_{S,T} \{ \text{capacity of the } S,T \text{ cut} \}.$$

max-flow $\leq$ min cut.

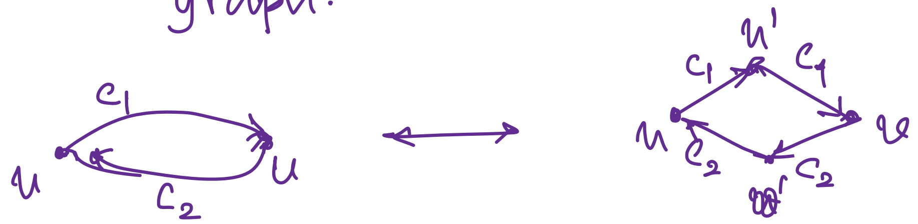
Also, we can show that max flow $\geq$ min cut.

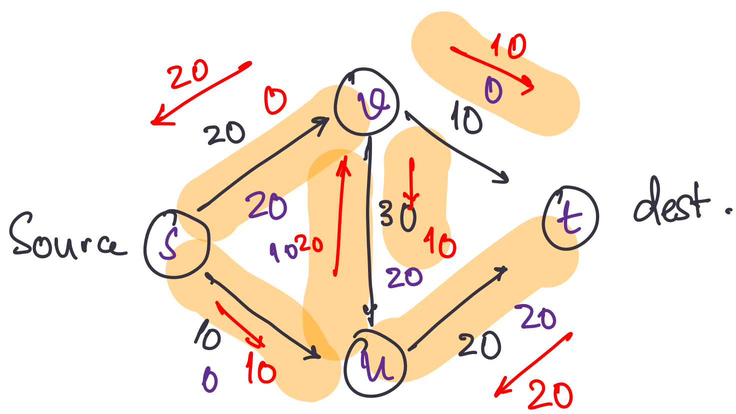
maxflow = min cut.

Residual capacity:

$$c_f(u\to v) = \begin{cases} c(u\to v) - f(u\to v) & \text{if } u\to v\in E \\ f(v\to u) & \text{if } v\to u \in E \\ 0 & \text{o/w.}\end{cases}$$

Assumption: No parallel edges in the underlying dir. graph.





$c_f(s,v) = 20-20=0$

$c_f(v,u) = 30-20=10$

$c_f(s,u) = 10-0=10$

$c_f(u,t) = 0$

$c_f(v,t)=10$

$c_f(u\to v) = f(v\to u) = 20$

$v\to u \in E$