Question 1

Q1.

Given $\{a, b, e\} \in \mathcal{F}$ & $\{b, c\} \in \mathcal{F}$

On repeated ~~or~~ unions, complement & intersection $\mathcal{F}$ is found as

$$\mathcal{F} = \{ \{a, b, c, d, e\}, \phi,$$
$$\{a, b, e\}, \{b, c\}, \{b, d\}, \{a, e\},$$
$$\{c, d\} \quad \{a, d, e\} \quad \{a, c, e\}$$
$$\{a, b, c, e\} \quad \{a, b, d, e\}$$
$$\{b, c, d\} \quad \{a, c, d, e\}$$
$$\{b\} \{d\} \{c\}$$
$$\}$$

Question 2

**1.** (a) Let $a \in (\bigcup A_i)^c$. Then $a \notin \bigcup A_i$, so that $a \in A_i^c$ for all $i$. Hence $(\bigcup A_i)^c \subseteq \bigcap A_i^c$. Conversely, if $a \in \bigcap A_i^c$, then $a \notin A_i$ for every $i$. Hence $a \notin \bigcup A_i$, and so $\bigcap A_i^c \subseteq (\bigcup A_i)^c$. The first De Morgan law follows.

(b) Applying part (a) to the family $\{A_i^c : i \in I\}$, we obtain that $\left(\bigcup_i A_i^c\right)^c = \bigcap_i (A_i^c)^c = \bigcap_i A_i$. Taking the complement of each side yields the second law.

Question 3

1. Let $B_1 = A_1, B_i = A_i \left(\bigcup_{j=1}^{i-1} A_j\right)^c, i > 1$. Then

$$
\begin{aligned}
P\left(\bigcup_{i=1}^{\infty} A_i\right) &= P\left(\bigcup_{i=1}^{\infty} B_i\right) \\
&= \sum_{i=1}^{\infty} P(B_i) \\
&\leq \sum_{i=1}^{\infty} P(A_i)
\end{aligned}
$$

where the final equality uses the fact that the $B_i$ are mutually exclusive. The inequality then follows, since $B_i \subset A_i$.

Question 4

**4.** We prove this by induction on $n$, considering first the case $n = 2$. Certainly $B = (A \cap B) \cup (B \setminus A)$ is a union of disjoint sets, so that $\mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A)$. Similarly $A \cup B = A \cup (B \setminus A)$, and so

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) = \mathbb{P}(A) + \left\{\mathbb{P}(B) - \mathbb{P}(A \cap B)\right\}.$$

Hence the result is true for $n = 2$. Let $m \geq 2$ and suppose that the result is true for $n \leq m$. Then it is true for pairs of events, so that

$$
\begin{aligned}
\mathbb{P}\left(\bigcup_1^{m+1} A_i\right) &= \mathbb{P}\left(\bigcup_1^{m} A_i\right) + \mathbb{P}(A_{m+1}) - \mathbb{P}\left\{\left(\bigcup_1^{m} A_i\right) \cap A_{m+1}\right\} \\
&= \mathbb{P}\left(\bigcup_1^{m} A_i\right) + \mathbb{P}(A_{m+1}) - \mathbb{P}\left\{\bigcup_1^{m} (A_i \cap A_{m+1})\right\}.
\end{aligned}
$$

Using the induction hypothesis, we may expand the two relevant terms on the right-hand side to obtain the result.

Question 5

**8.** $\Omega \cup \varnothing = \Omega$ and $\Omega \cap \varnothing = \varnothing$, and therefore $1 = \mathbb{P}(\Omega \cup \varnothing) = \mathbb{P}(\Omega) + \mathbb{P}(\varnothing) = 1 + \mathbb{P}(\varnothing)$, implying that $\mathbb{P}(\varnothing) = 0$.

Question 6

(i) We have (using the fact that $\mathbb{P}$ is a non-decreasing set function) that

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1 = \tfrac{1}{12}.$$





Also, since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $\mathbb{P}(A \cap B) \leq \min\{\mathbb{P}(A), \mathbb{P}(B)\} = \frac{1}{3}$.

These bounds are attained in the following example. Pick a number at random from $\{1, 2, \ldots, 12\}$. Taking $A = \{1, 2, \ldots, 9\}$ and $B = \{9, 10, 11, 12\}$, we find that $A \cap B = \{9\}$, and so $\mathbb{P}(A) = \frac{3}{4}$, $\mathbb{P}(B) = \frac{1}{3}$, $\mathbb{P}(A \cap B) = \frac{1}{12}$. To attain the upper bound for $\mathbb{P}(A \cap B)$, take $A = \{1, 2, \ldots, 9\}$ and $B = \{1, 2, 3, 4\}$.

(ii) Likewise we have in this case $\mathbb{P}(A \cup B) \leq \min\{\mathbb{P}(A) + \mathbb{P}(B), 1\} = 1$, and $\mathbb{P}(A \cup B) \geq \max\{\mathbb{P}(A), \mathbb{P}(B)\} = \frac{3}{4}$. These bounds are attained in the examples above.

Question 7

$\boxed{7}$

To Prove : $P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E^c F G) - P(E F^c G) - P(E F G^c) - 2P(EFG)$. ①

Consider $P(E^c F G)$,

$\cancel{P(E^c \cap F \cap G) =}$

$E^c \cap F \cap G = F \cap G \cap (1 - E)$

$= F \cap G - F \cap G \cap E$.

$\Longrightarrow P(E^c F G) = P(FG) - P(EFG)$. ②

( $FGE$ is a subset of $FG$ )

Similarly, $P(E F^c G) = P(EG) - P(EFG)$ ③

$P(E F G^c) = P(EF) - P(EFG)$. ④

Put ②, ③, ④ in ①:

$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EG) - P(FG) - P(EF) + 3P(EFG) - 2P(EFG)$.

$\Longrightarrow P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EG) - P(FG) - P(EF) + P(EFG)$.

We know, from Inclusion-Exclusion Principle, the above holds.
Hence, proved.

Question 8

$$[a,b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$$

Question 9

As before, it is always useful to draw a Venn diagram; however, here we provide the solution without using a Venn diagram.

a. Using the inclusion-exclusion principle, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Thus,

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= \tfrac{1}{2} + \tfrac{2}{3} - \tfrac{5}{6} \\ &= \tfrac{1}{3}. \end{aligned}$$

b. No, since $A \cap B \neq \emptyset$.

c. We can write

$$\begin{aligned} C - (A \cup B) &= \Big(C \cup (A \cup B)\Big) - (A \cup B) \\ &= S - (A \cup B) && \text{(since } A \cup B \cup C = S\text{)} \\ &= (A \cup B)^c. \end{aligned}$$

Thus

$$\begin{aligned} P(C - (A \cup B)) &= P\big((A \cup B)^c\big) \\ &= 1 - P(A \cup B) \\ &= \tfrac{1}{6}. \end{aligned}$$

d. We have

$$P(C) = P(C \cap (A \cup B)) + P(C - (A \cup B)) = \frac{5}{12} + \frac{1}{6} = \frac{7}{12}.$$

Question 10

By the continuity of $\mathbb{P}$, Exercise (1.2.1), and Problem (1.8.11),

$$\mathbb{P}\left(\bigcap_{r=1}^{\infty} A_r\right) = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{r=1}^{n} A_r\right) = \lim_{n\to\infty}\left[1 - \mathbb{P}\left(\left(\bigcap_{r=1}^{n} A_r\right)^{\mathrm{c}}\right)\right]$$
$$= 1 - \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{r=1}^{n} A_r^{\mathrm{c}}\right) \geq 1 - \lim_{n\to\infty} \sum_{r=1}^{n} \mathbb{P}(A_r^{\mathrm{c}}) = 1.$$

Question 11

**4.** We must check that $\mathcal{G}$ satisfies the definition of a $\sigma$-field:

(a) $\varnothing \in \mathcal{F}$, and therefore $\varnothing = \varnothing \cap B \in \mathcal{G}$,

(b) if $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_i (A_i \cap B) = \left(\bigcup_i A_i\right) \cap B \in \mathcal{G}$,

(c) if $A \in \mathcal{F}$, then $A^{\mathrm{c}} \in \mathcal{F}$ so that $B \setminus (A \cap B) = A^{\mathrm{c}} \cap B \in \mathcal{G}$.

Note that $\mathcal{G}$ is a $\sigma$-field *of subsets of* $B$ but not a $\sigma$-field of subsets of $\Omega$, since $C \in \mathcal{G}$ does not imply that $C^{\mathrm{c}} = \Omega \setminus C \in \mathcal{G}$.

Question 12

**7.** Let $C_i$ be the colour of the $i$th ball picked, and use the obvious notation.

(a) Since each urn contains the same number $n-1$ of balls, the second ball picked is equally likely to be any of the $n(n-1)$ available. One half of these balls are magenta, whence $\mathbb{P}(C_2 = \mathrm{M}) = \frac{1}{2}$.

(b) By conditioning on the choice of urn,

$$\mathbb{P}(C_2 = \mathrm{M} \mid C_1 = \mathrm{M}) = \frac{\mathbb{P}(C_1, C_2 = \mathrm{M})}{\mathbb{P}(C_1 = \mathrm{M})} = \sum_{r=1}^{n} \frac{(n-r)(n-r-1)}{n(n-1)(n-2)} \Big/ \frac{1}{2} = \frac{2}{3}.$$

Question 13

This is another typical problem for which the law of total probability is useful. Let $C_1$ be the event that you choose a regular coin, and let $C_2$ be the event that you choose the two-headed coin. Note that $C_1$ and $C_2$ form a partition of the sample space. We already know that

$$P(H|C_1) = 0.5,$$

$$P(H|C_2) = 1.$$

a. Thus, we can use the law of total probability to write

$$\begin{aligned} P(H) &= P(H|C_1)P(C_1) + P(H|C_2)P(C_2) \\ &= \tfrac{1}{2}\cdot\tfrac{2}{3} + 1.\tfrac{1}{3} \\ &= \tfrac{2}{3}. \end{aligned}$$

b. Now, for the second part of the problem, we are interested in $P(C_2|H)$. We use Bayes' rule

$$\begin{aligned} P(C_2|H) &= \frac{P(H|C_2)P(C_2)}{P(H)} \\ &= \frac{1.\frac{1}{3}}{\frac{2}{3}} \\ &= \frac{1}{2}. \end{aligned}$$

Question 14

14

We need to find $P(S/DEA)$.

$S$ = Spam

$$P(S/DEA) = \frac{P(DEA/S) \cdot P(S)}{P(DEA)} \quad (1)$$

Given:

$$P(S) = \frac{60}{100}, \quad P(DEA/S) = \frac{20}{100},$$

$$P(DEA/S^c) = \frac{1}{100}.$$

Also,

$$P(DEA) = P(DEA/S) \cdot P(S) + P(DEA/S^c) \cdot P(S^c)$$

(Law of total probability).

So, (1) becomes:

$$P(S/DEA) = \frac{\frac{20}{100} \times \frac{60}{100}}{\frac{20}{100} \times \frac{60}{100} + \frac{1}{100} \times \frac{40}{100}}$$

Question 15

This is a finite sample space, so

$$P(A) = \frac{|A|}{|S|} = \frac{|\{1,3,5\}|}{6} = \frac{1}{2}.$$

Now, let's find the conditional probability of $A$ given that $B$ occurred. If we know $B$ has occurred, the outcome must be among $\{1,2,3\}$. For $A$ to also happen the outcome must be in $A \cap B = \{1,3\}$. Since all die rolls are equally likely, we argue that $P(A|B)$ must be equal to

$$P(A|B) = \frac{|A \cap B|}{|B|} = \frac{2}{3}.$$

## Question 16

$$P(E|E \cup F) = P(E|F(E \cup F))P(F|E \cup F) + P(E|F^c(E \cup F))P(F^c|E \cup F)$$

Using

$$F(E \cup F) = F \quad \text{and} \quad F^c(E \cup F) = F^c E$$

gives

$$\begin{aligned}
P(E|E \cup F) &= P(E|F)P(F|E \cup F) + P(E|EF^c)P(F^c|E \cup F) \\
&= P(E|F)P(F|E \cup F) + P(F^c|E \cup F) \\
&\geq P(E|F)P(F|E \cup F) + P(E|F)P(F^c|E \cup F) \\
&= P(E|F)
\end{aligned}$$

## Question 17

Solution: Firstly, let us create a sample space for each event. For the event ‘A’ we have to get at least two head. Therefore, we have to include all the events that have two or more heads.

Or we can write:

A = {HHT, HTH, THH, HHH}.

This set A has 4 elements or events in it i.e. n(A) = 4

In the same way, for event B, we can write the sample as:

B = {TTT} and n(B) = 1

Again using the same logic, we can write;

C = {THT, HHH, HHT, THH} and n(C) = 4

So B & C and A & B are mutually exclusive since they have nothing in their intersection.