CS 302.1 - Automata Theory

Lecture 02

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INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY

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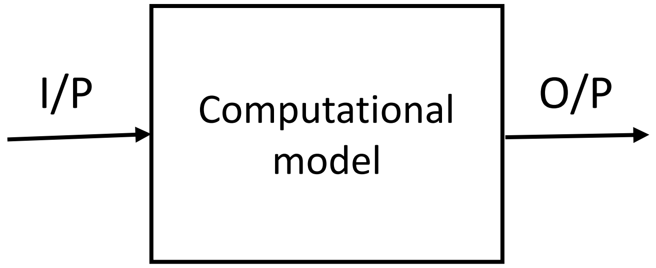
A quick recap

- Can a given problem be computed by a particular computational model?

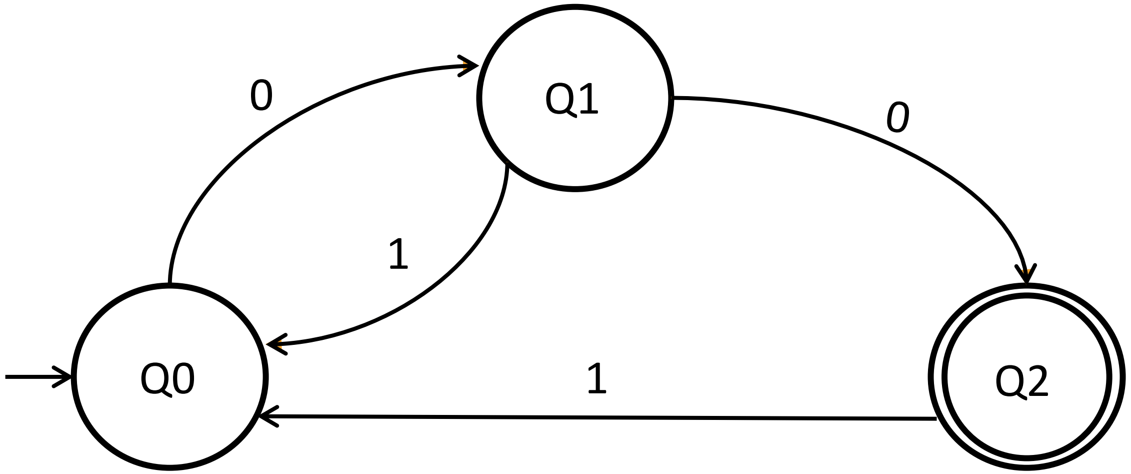


A computational model solves a problem P if,

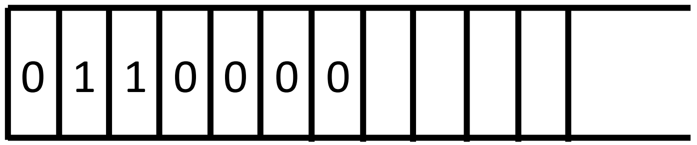
(i) For all inputs belonging to the YES instance of P, the device outputs **YES**

(ii) For all inputs belonging to the NO instance of P, the device outputs **NO**.

If (i) and (ii) hold, we say that the problem **P is computable** by this computational model.



Deterministic Finite Automata (DFA)



Run:

$$\boldsymbol{Q0} \xrightarrow{0} Q1 \xrightarrow{1} Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q1 \xrightarrow{0} Q2 \xrightarrow{0} Q2 \xrightarrow{0} \boldsymbol{Q2}$$

L(M) = {$\omega | \omega$ results in an accepting run}

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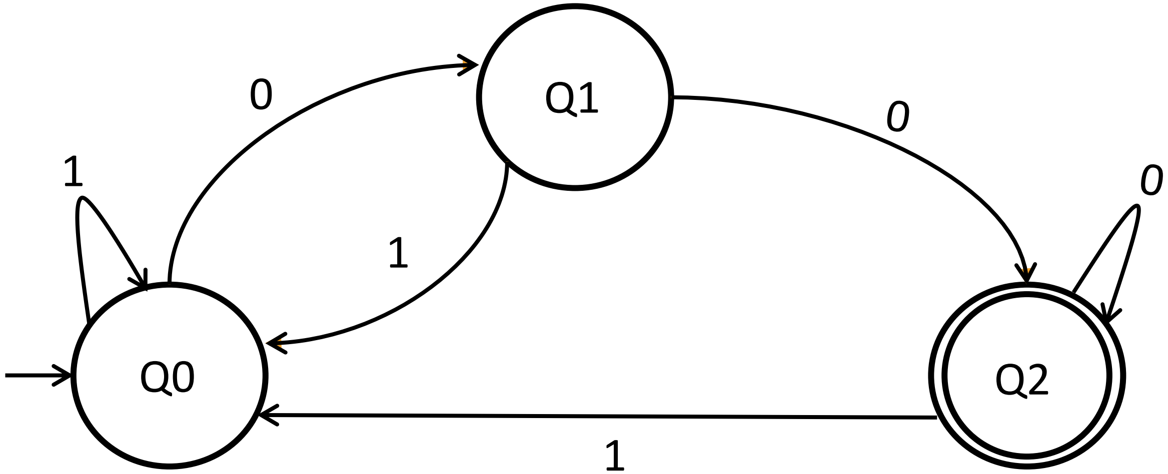
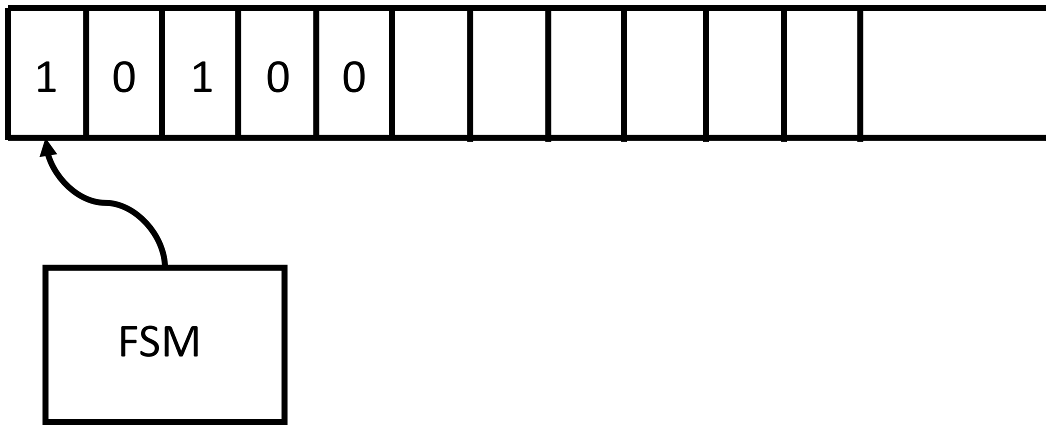


State transition diagram of the Finite State Machine

One-way infinite tape



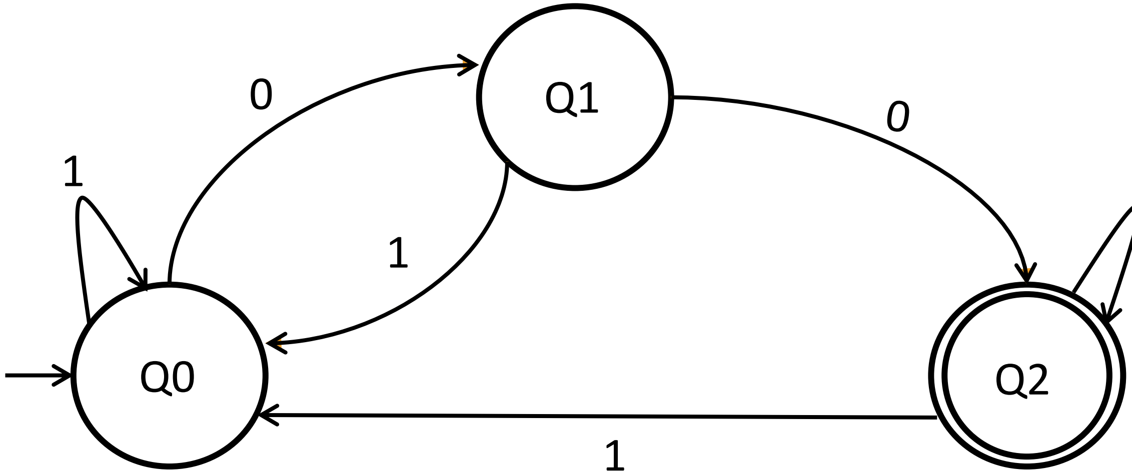
ACCEPT = {0111000, 10100, 0100, 00, 10000….}
REJECT = {11101, 0, 1, 11, 001,…….}

For any language L, we say **M recognizes L if**

$$\forall \boldsymbol{\omega} \in \boldsymbol{L}, \boldsymbol{M}(\boldsymbol{\omega})$$ **accepts**

For the example above, $\boldsymbol{M}$ **recognizes L= {$\omega|\omega$ ends in "00"}**

Deterministic Finite Automata (DFA)



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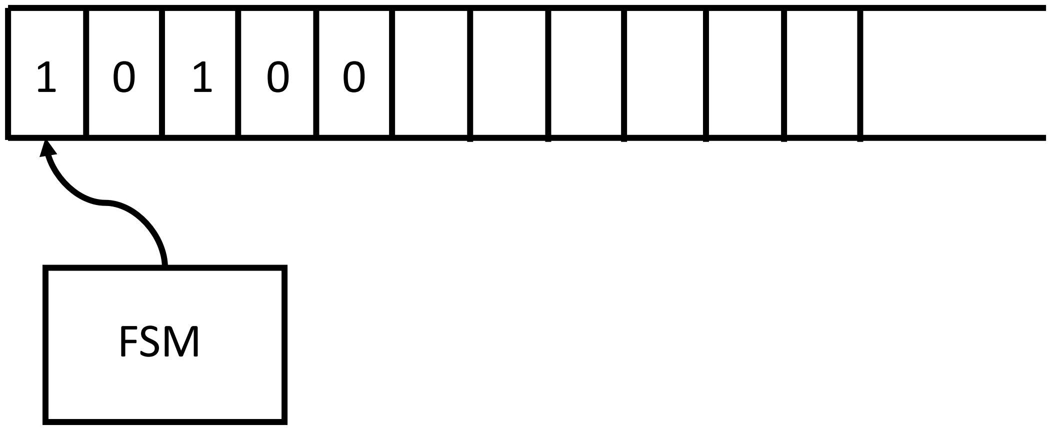


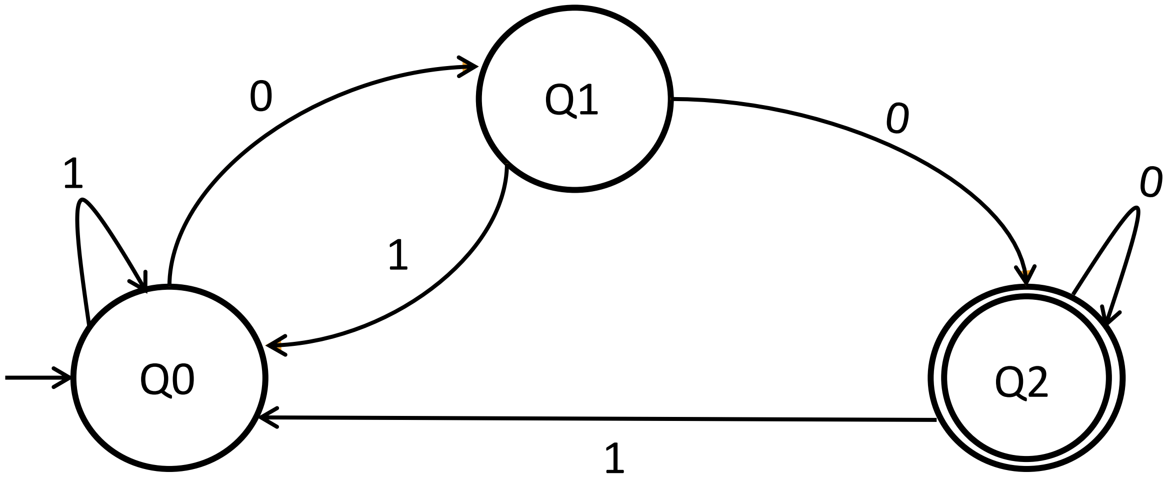
ACCEPT = {0111000, 10100, 0100, 00, 10000….}
REJECT = {11101, 0, 1, 11, 001,…….}

For any language L, we say **the problem M solves or M decides L if**

$$\forall \boldsymbol{\omega} \in \boldsymbol{L}, \boldsymbol{M}(\boldsymbol{\omega}) \textbf{ accepts}$$
$$\forall \boldsymbol{\omega} \notin \boldsymbol{L}, \boldsymbol{M}(\boldsymbol{\omega}) \textbf{ rejects}$$

For the example above, $\boldsymbol{M}$ **decides L= {$\omega|\omega$ ends in "00"}**

Deterministic Finite Automata (DFA)



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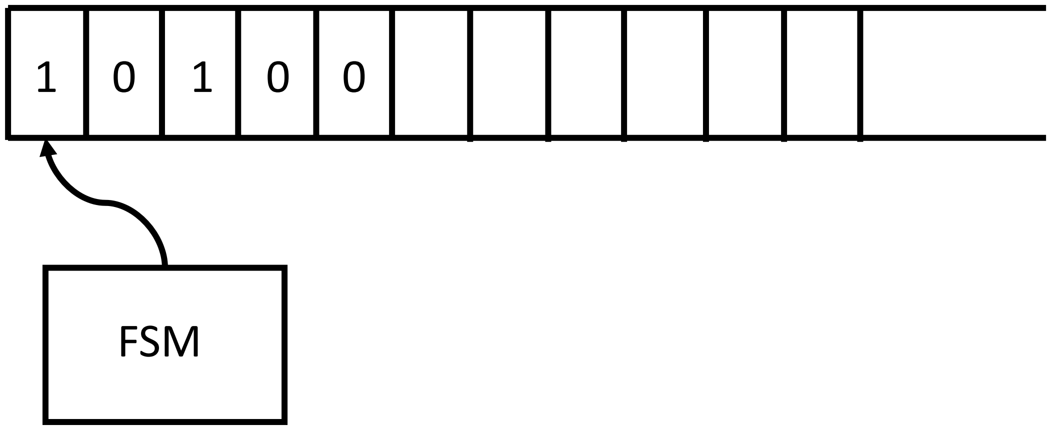

For any language $\boldsymbol{L}$, we say **M recognizes L if**

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For any language $\boldsymbol{L}$, we say **M decides L if**

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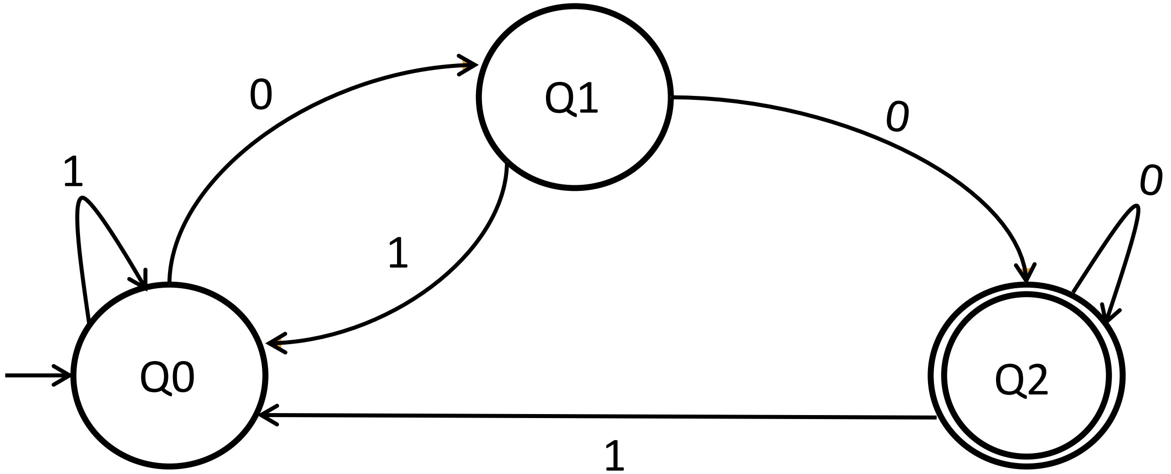
For a DFA, the notions of **deciding a language** and **recognizing a language** are equivalent, but this may not be true for other, more powerful computational models

Deterministic Finite Automata (DFA)



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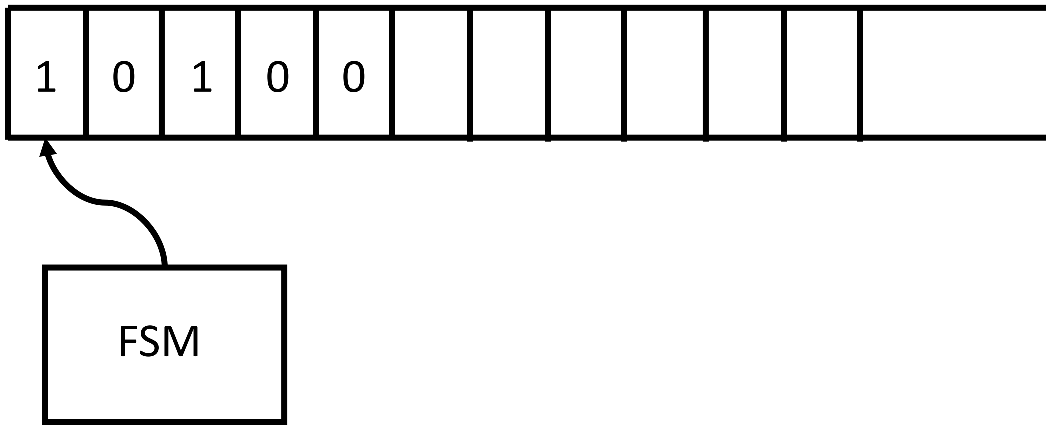

Characteristics of DFA : (i) Single start state (ii) Unique transitions (iii) Zero or more final states

Formally, a finite automaton M is a 5-tuple $(\mathrm{Q}, \Sigma, \delta, q_0, F)$ where

• Q is a finite set called the ***states.***	Q = {Q0, Q1, Q2}
• Σ is a finite set called the ***alphabet***.	$\Sigma = \{0,1\}$
• $\delta: Q \times \Sigma \mapsto Q$ is the ***transition function*** *(unique)*.	$(Q0,0) \mapsto Q1;\ (Q0,1) \mapsto Q0, \ldots\ldots, (Q2,1) \mapsto Q0$
• $q_0 \in Q$ is the ***start state***.	$q_0 = Q0$
• $F \subseteq Q$ are the ***final/accepting states***.	$F = Q2$

Constructing DFA for a language

Examples: $\boldsymbol{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\omega$|$\omega$ has an even number of 0's}**

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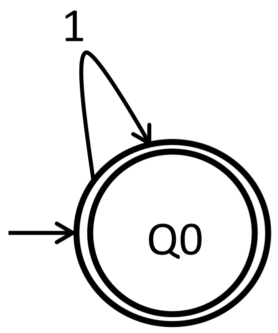

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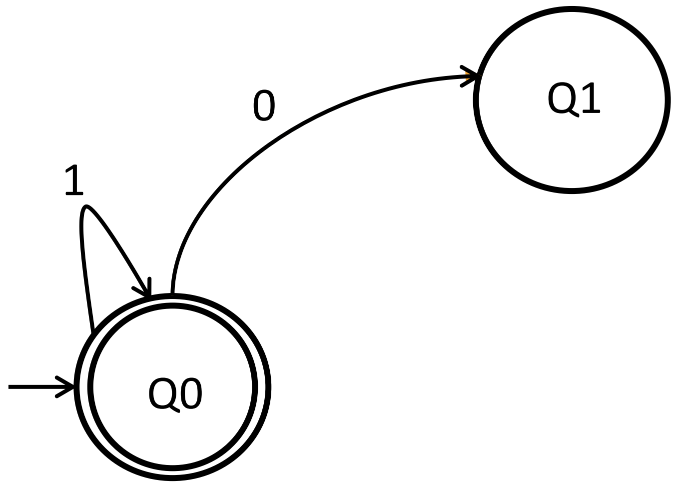

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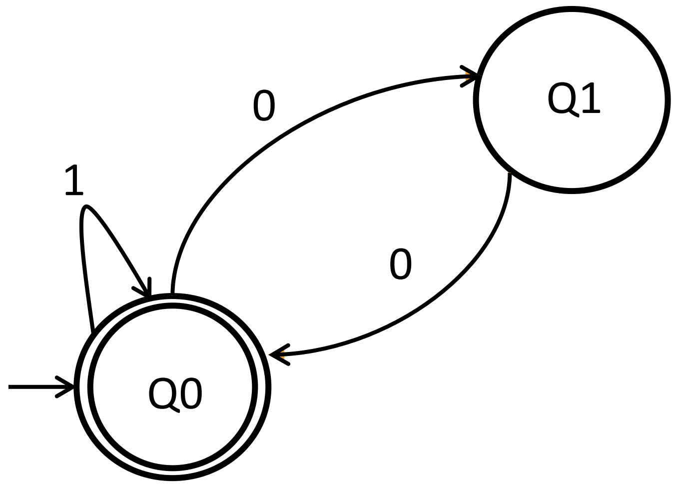

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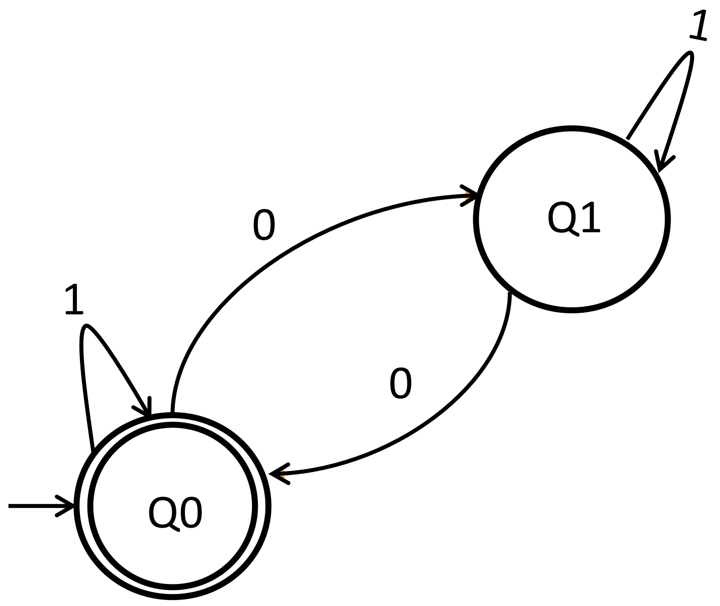

	0	**1**
Q0	**Q1**	**Q0**
Q1	**Q0**	**Q1**

Constructing DFA for a language

Examples: $\boldsymbol{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\boldsymbol{\omega}|\boldsymbol{\omega}$ is divisible by 3}**

Any input string would leave three remainders: $0, 1$ or 2.

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Any input string would leave three remainders: $0, 1$ or 2.

Intuition: Let ω be any substring of the input string divisible by 3, i.e. $\omega = 0(mod\ 3)$

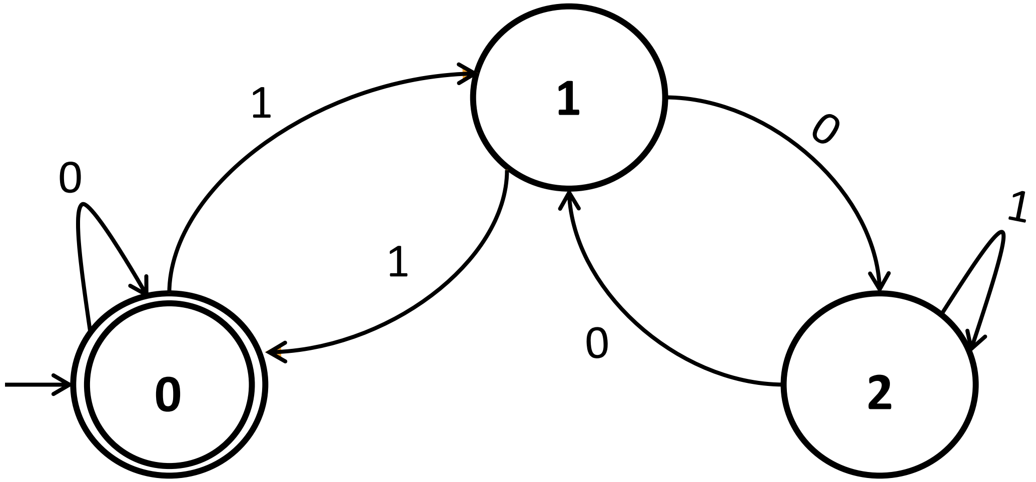
$$\omega\ 0 = 2 \times value\ (\omega) = 0\ (mod\ 3)$$
$$\omega\ 1 = 2 \times value(\omega) + 1 = 1(mod\ 3)$$
$$\omega\ 10 = 2 \times value(\omega 1) = 2(mod\ 3)$$
$$\omega\ 11 = 2 \times value(\omega 1) + 1 = 0(mod\ 3)$$

…. And so on

- The DFA will have three states, each corresponding to the remainder of $value(\omega)/3$.
- The final state $= 0(mod\ 3)$ – the string ω is accepted if after reading it, the DFA ends in this state.

Constructing DFA for a language

Examples: $\boldsymbol{\Sigma} = \{\mathbf{0}, \mathbf{1}\}$, **L(M)={$\boldsymbol{\omega}|\boldsymbol{\omega}$ is divisible by 3}**



Any input string would either leave remainders 0, 1 or 2.

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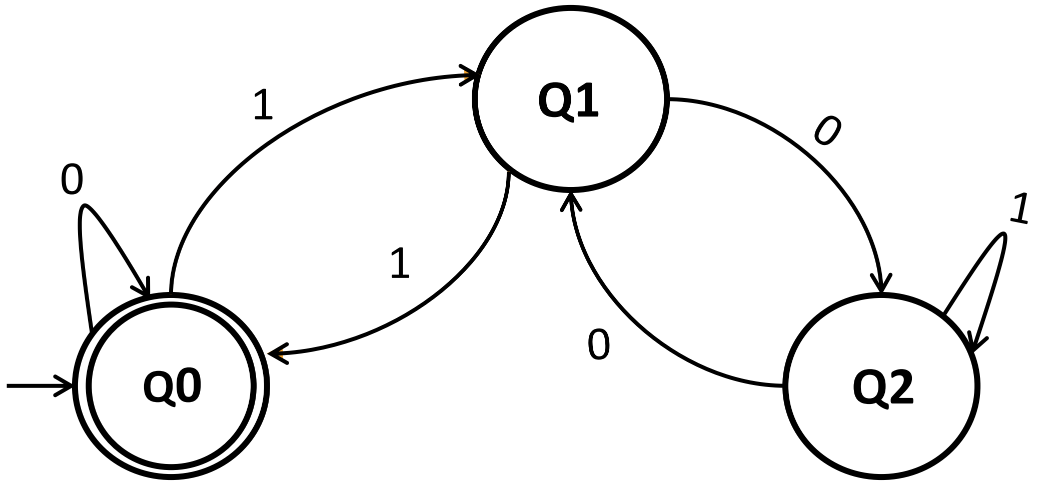

	0	**1**
Q0	**Q0**	**Q1**
Q1	**Q2**	**Q0**
Q2	**Q1**	**Q2**

Constructing DFA for a language

Examples: $\boldsymbol{\Sigma} = \{\boldsymbol{0}, \boldsymbol{1}\}$, **L(M)={$\boldsymbol{\omega}|\boldsymbol{\omega}$ is NOT divisible by 3}**

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Intuition - Construct a **Toggled DFA:** Toggle the final states and the non-final states!

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Intuition - Construct a **Toggled DFA:** Toggle the final states and the non-final states!

In fact if any DFA accepts $\boldsymbol{L}$, the toggled DFA accepts $\overline{\boldsymbol{L}}$, the complement of $\boldsymbol{L}$

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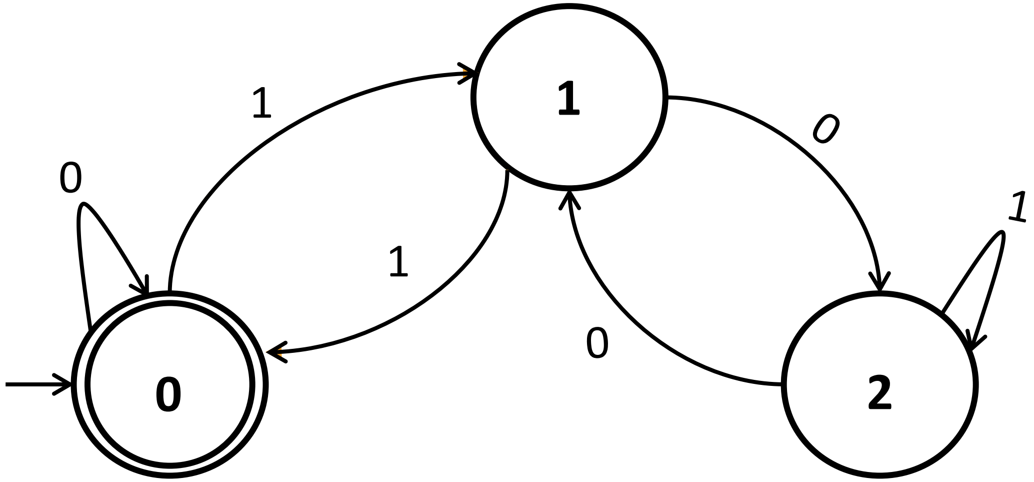

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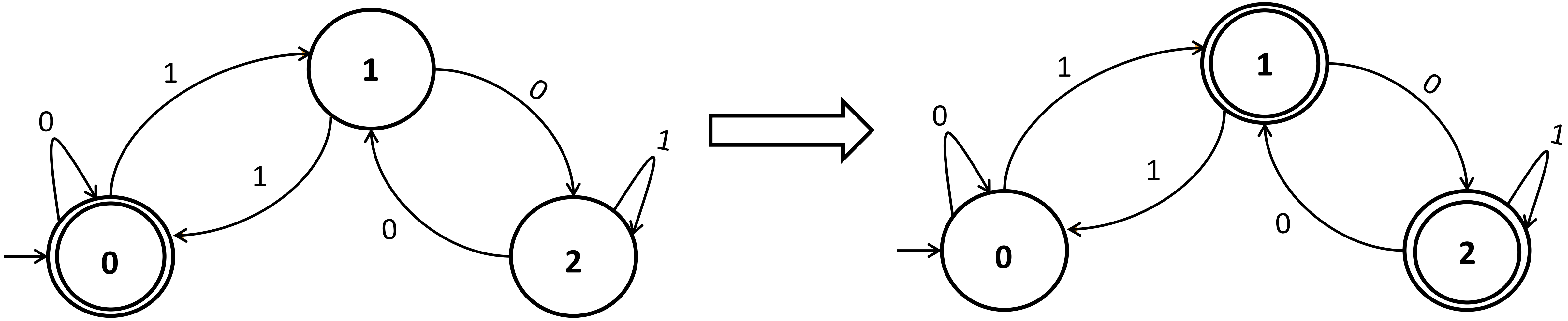

Non-deterministic Finite Automata (NFA)

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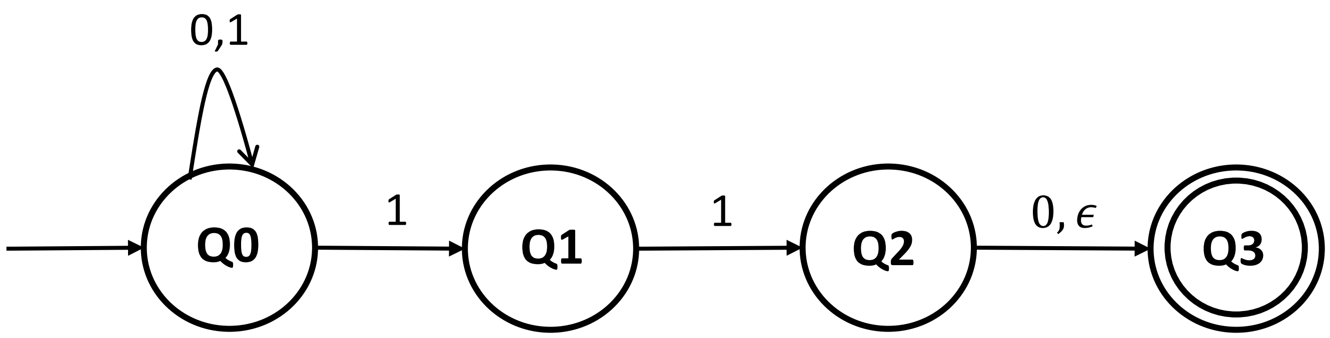
Characteristics of NFA : (i) Single start state (ii) Zero or more final states

(iii) Multiple transitions are possible on the same input for a state

(iv) Some transitions might be missing

(v) ϵ - transitions



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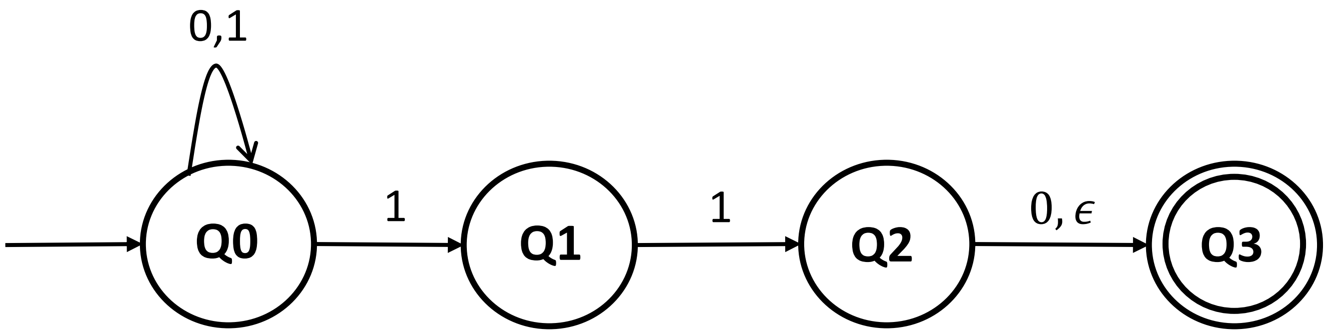
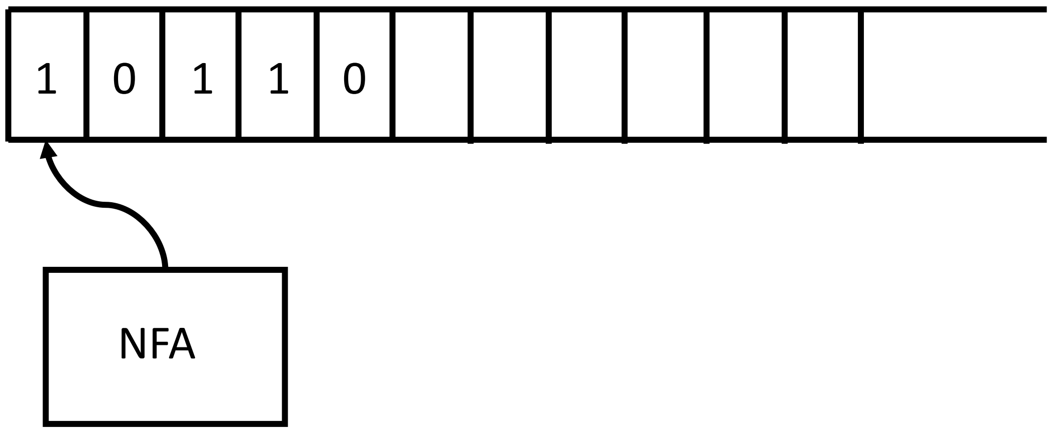


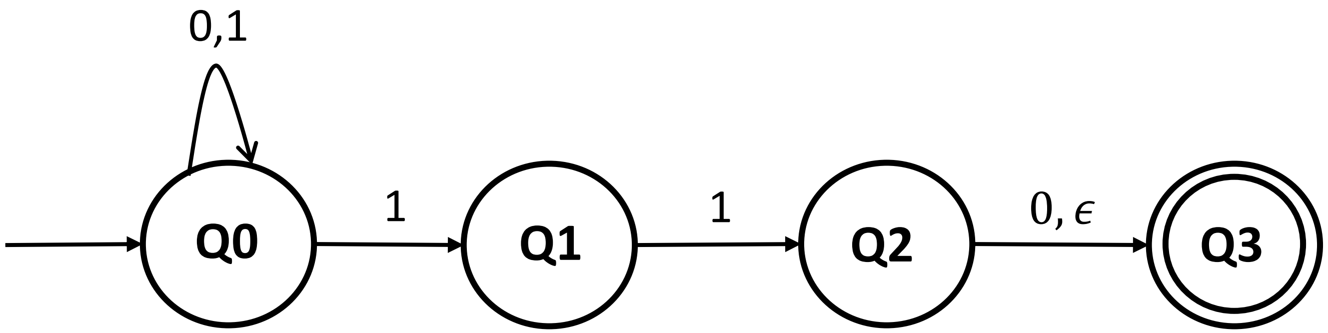


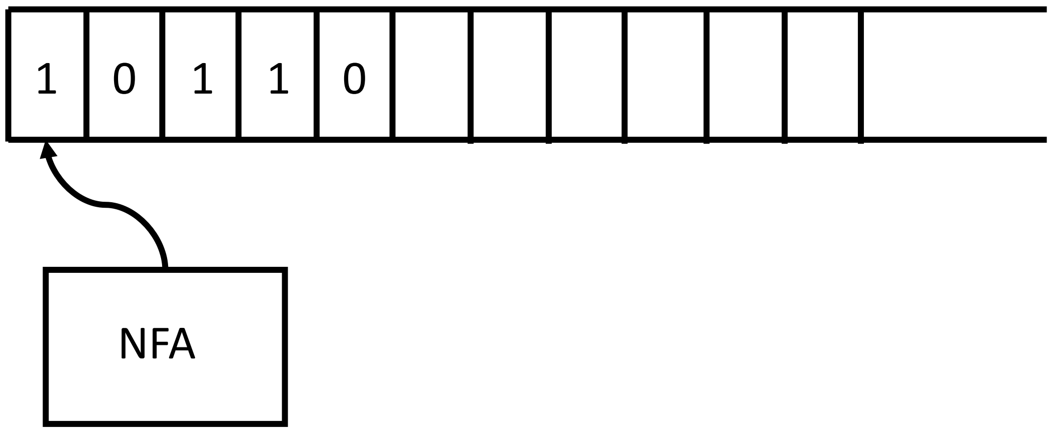
Run 1: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0$ (**REJECT**)

Run 2: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q1 \xrightarrow{1} Q2 \xrightarrow{0} Q3$ (**ACCEPT**)

Multiple **runs** per input is possible

Non-deterministic Finite Automata (NFA)





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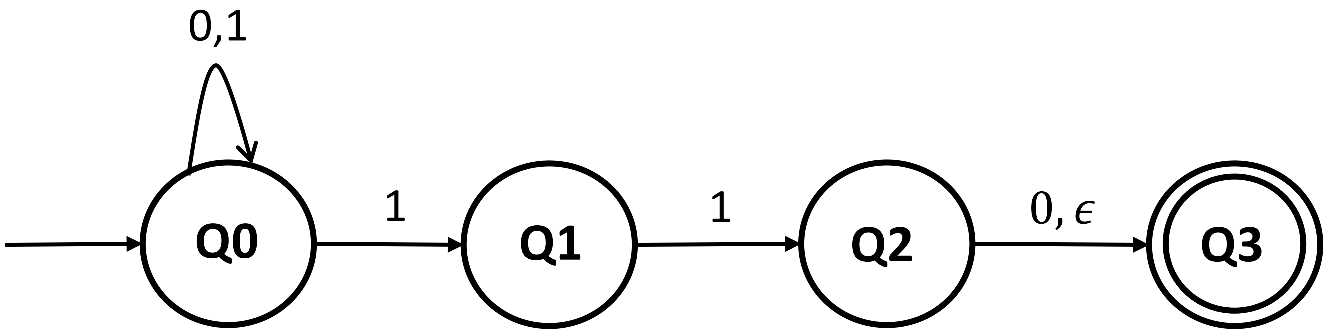
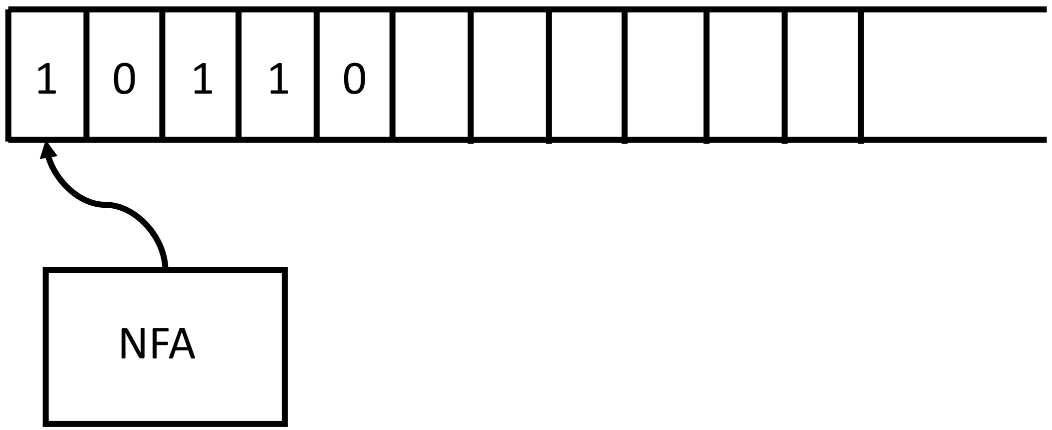
Run 3: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q0 \xrightarrow{1} Q1 \xrightarrow{0}$ **CRASH**

Run 4: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q1 \xrightarrow{1} Q2 \xrightarrow{\epsilon} Q3 \xrightarrow{0}$ **CRASH**

CRASH is a Rejecting Run

Non-deterministic Finite Automata (NFA)





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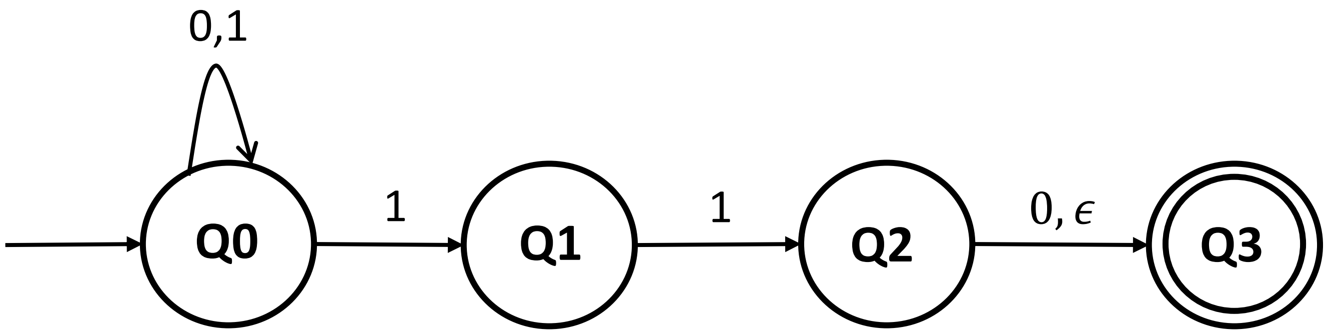
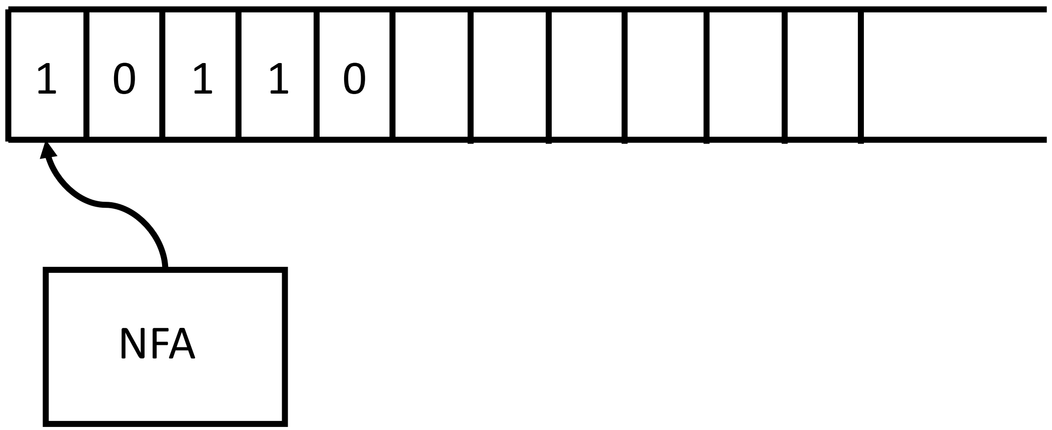
Run 4: $Q0 \xrightarrow{1} Q0 \xrightarrow{0} Q0 \xrightarrow{1} Q1 \xrightarrow{1} Q2 \xrightarrow{\epsilon} Q3 \xrightarrow{0}$ CRASH **(REJECT)**

The NFA “accepts” an input string, if it at **least one *run* ends up in the final state**. **(Accepting Run)**

The NFA “rejects” an input string, if there are **no runs that end up in a final state**. **(Rejecting Run)**

Non-deterministic Finite Automata (NFA)





	0	**1**	ϵ
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			

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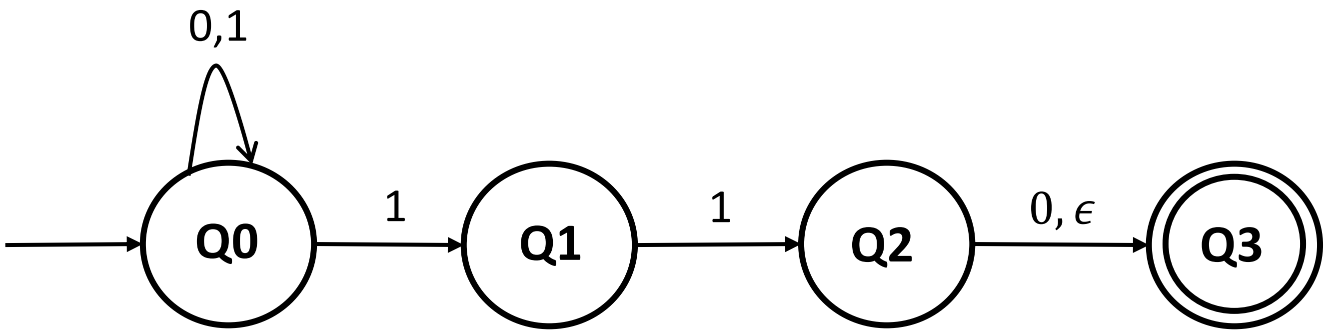

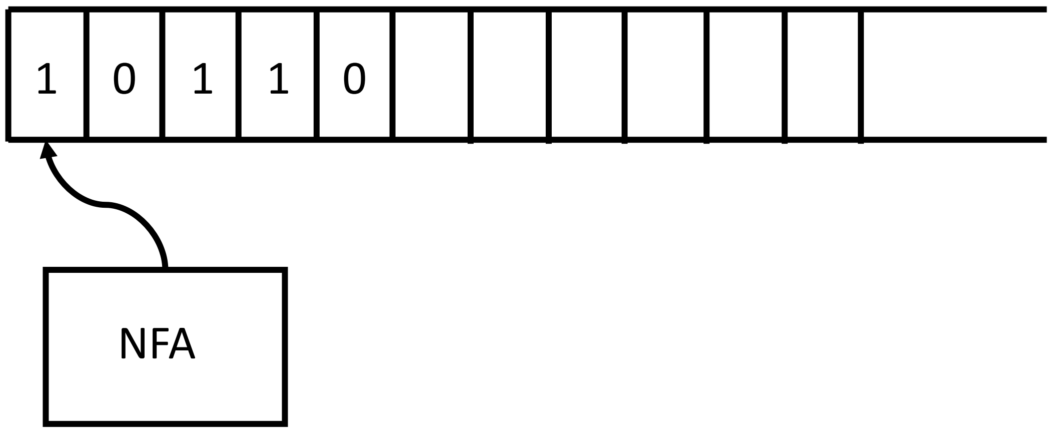

Formally, a finite automaton M is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set called the ***states.***
- Σ is a finite set called the ***alphabet***.
- $\delta: Q \times \Sigma \mapsto P(Q)$ is the ***transition function***. $P(Q)$ is the power set of Q
- $q_0 \in Q$ is the ***start state***.
- $F \subseteq Q$ is the set of ***final/accepting states***.

	0	**1**	ϵ
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			

NFA vs DFA

- Are NFAs more powerful than DFAs?
- Intuitively, non-determinism seems to be adding more “power”.

NFA vs DFA

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- Let L_1 be the language accepted by NFAs and L_2 be the language accepted by DFAs
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NFA vs DFA

- Are NFAs more powerful than DFAs?
- Intuitively, non-determinism seems to be adding more "power".
- Let L_1 be the language accepted by NFAs and L_2 be the language accepted by DFAs
- Is $L_2 \subseteq L_1$? Clearly true, because a DFA is just a special case of an NFA.
- Surprisingly, what we will show next is that $\boldsymbol{L_1 \subseteq L_2}$!
- That is, **given an NFA, we can convert it to a DFA that accepts the same language**.
- Such a DFA is called a "**Remembering DFA**".

Thus, DFAs and NFAs are completely equivalent and $\boldsymbol{L_1 = L_2}$!

Converting an NFA to a DFA

Intuitive idea for the construction of a Remembering DFA from an NFA:

- Let R be the Remembering DFA corresponding to an NFA N.
- R on an input enters a state that is labelled by all possible states that N can enter on that input.
- Note that this “trims away” the non-determinism of the NFA N without “losing” the language it accepts.
- Also note that if N has k states, then R has at most 2^k states. Why?

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- Also note that if N has k states, then R has at most 2^k states. Why?
- Any label in the Remembering DFA is a subset of $\{Q_0,\ Q_1,\ Q_2,\ldots\ldots,Q_{k-1}\}$, where Q_i = State of the NFA.
- There are at most 2^k labels for the DFA.

Converting an NFA to a DFA

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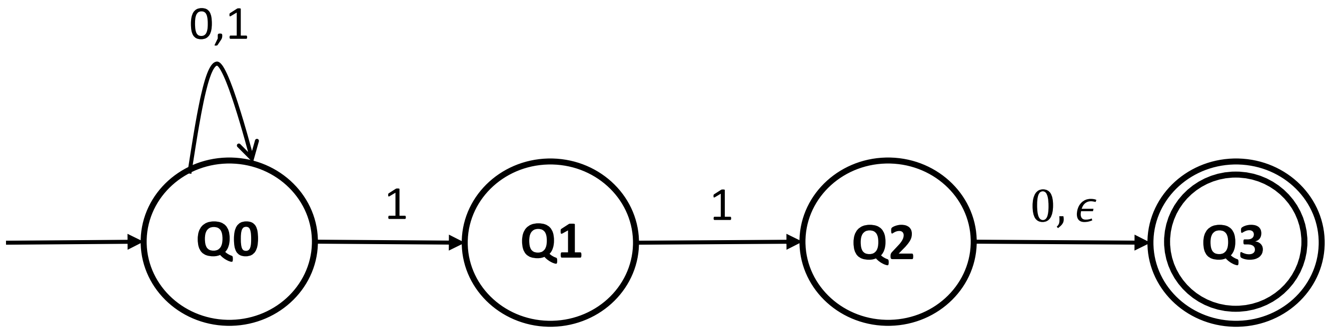

NFA N

	0	**1**	$\boldsymbol{\epsilon}$
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			

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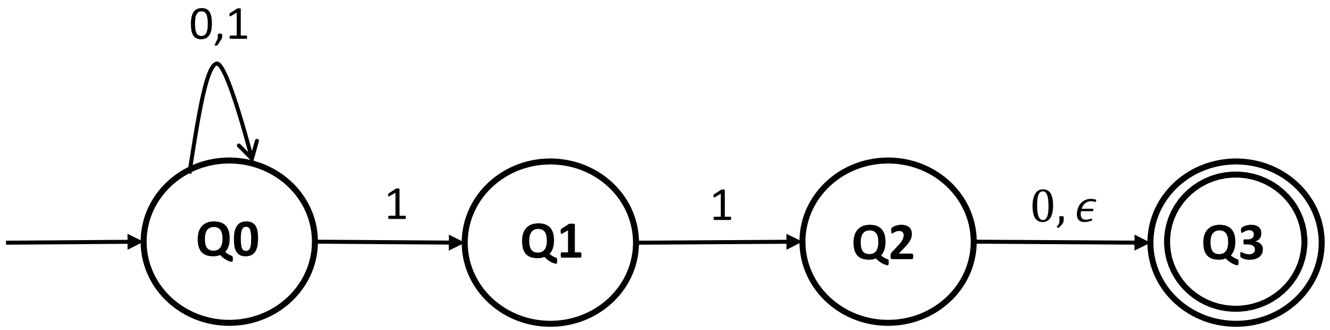
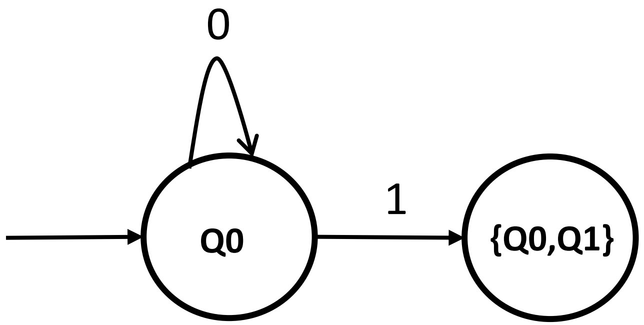


NFA N

	0	1	ϵ
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			



Remembering DFA R

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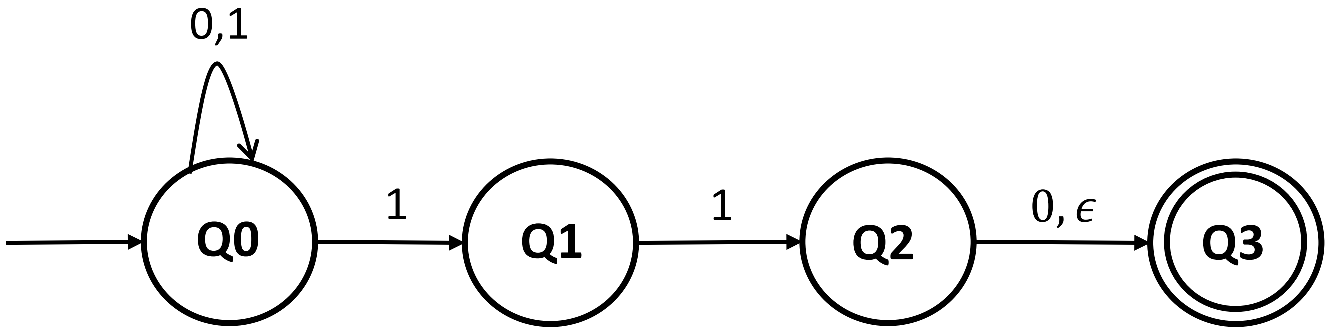

NFA N

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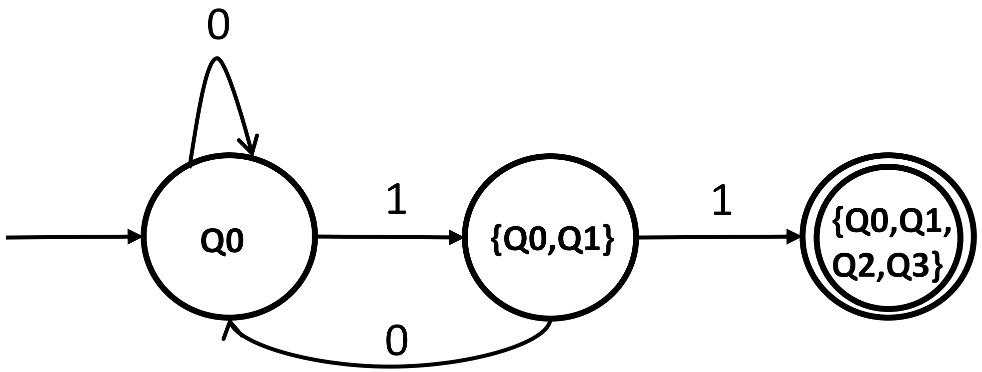
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			

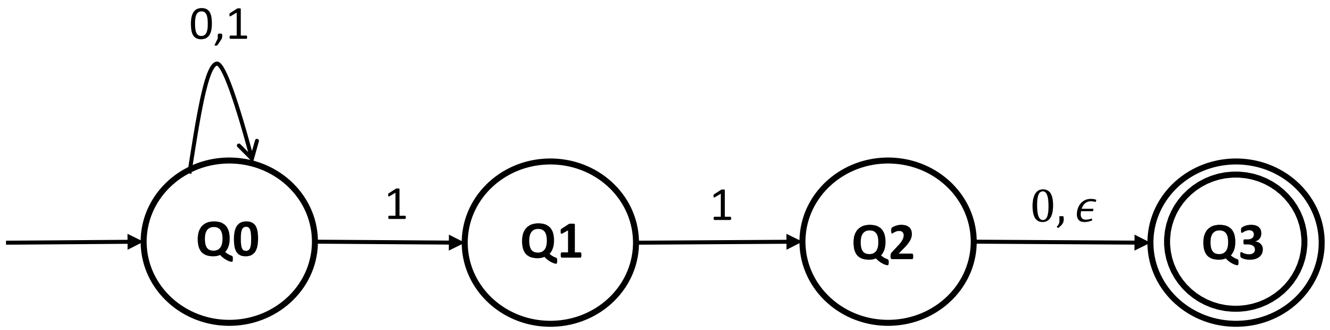


Remembering DFA R

Any state of R that contains in its label, an accepting state of N is an accepting state of R.

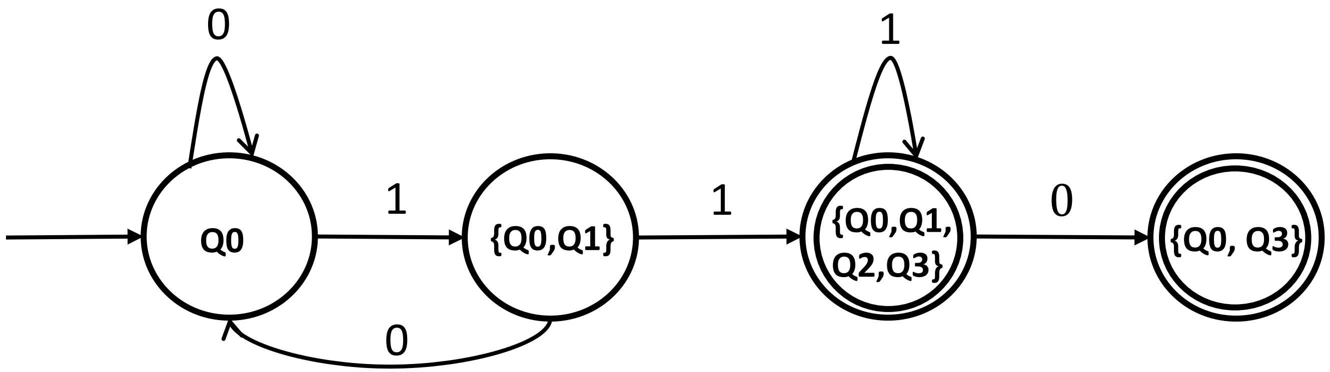
Converting an NFA to a DFA

- R on an input enters a state that is labelled by all possible states that N can enter on that input.



NFA N

	0	**1**	ϵ
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			



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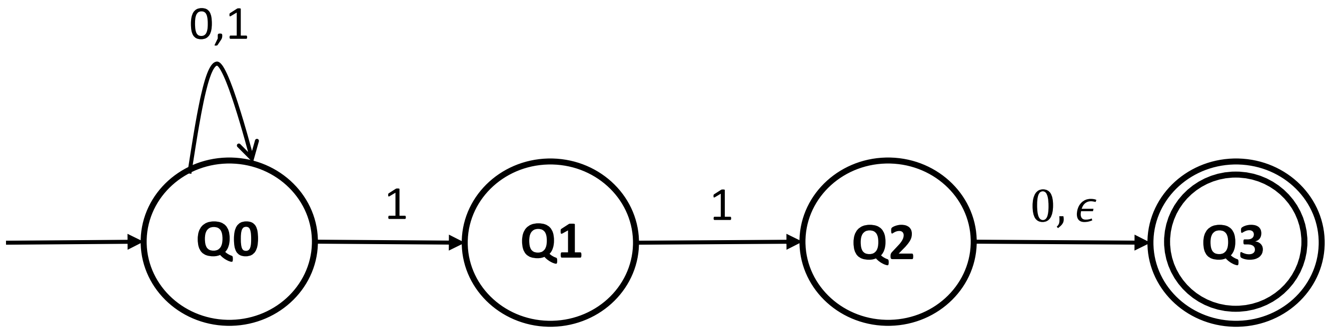
Converting an NFA to a DFA

- M_2 on an input enters a state that is labelled by all possible states that M_1 can enter on that input.



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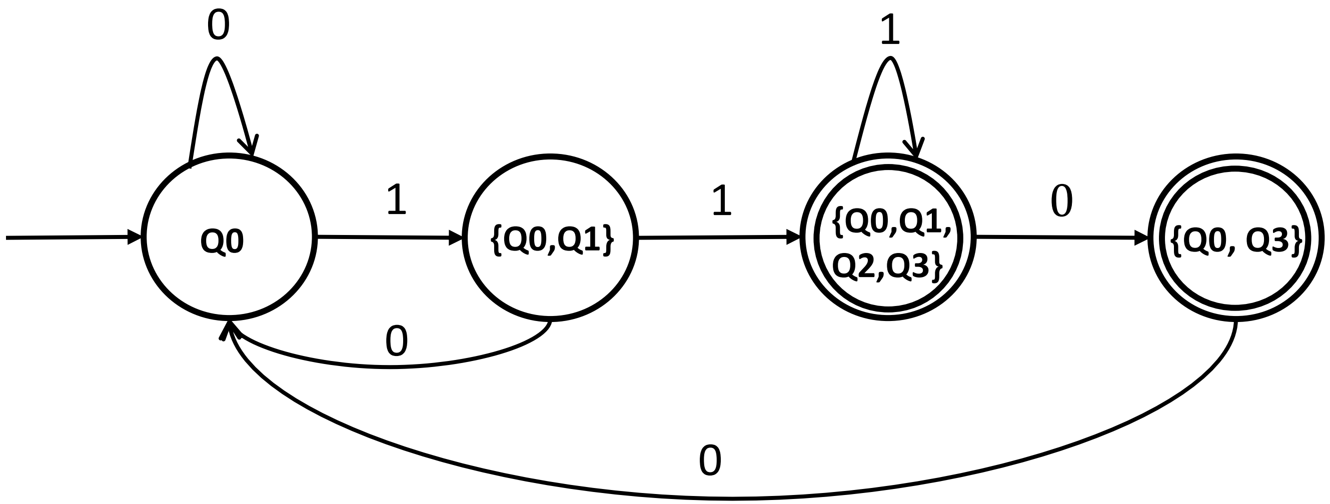
	0	**1**	ϵ
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			



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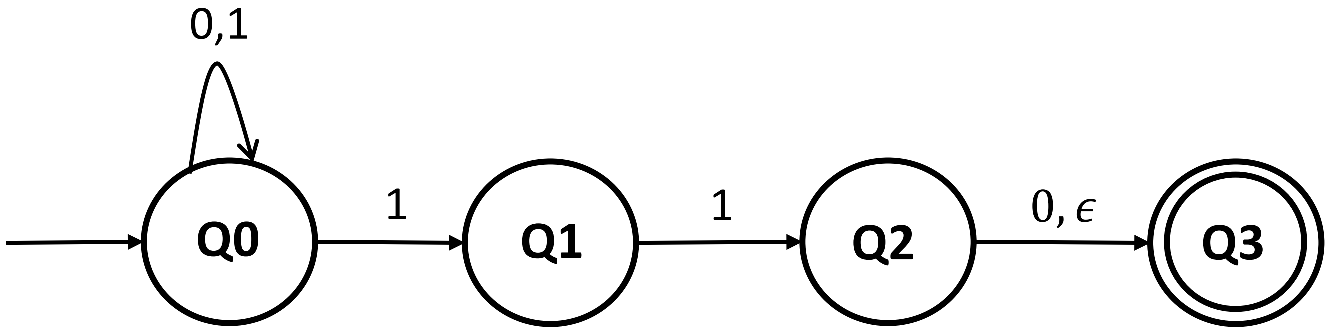
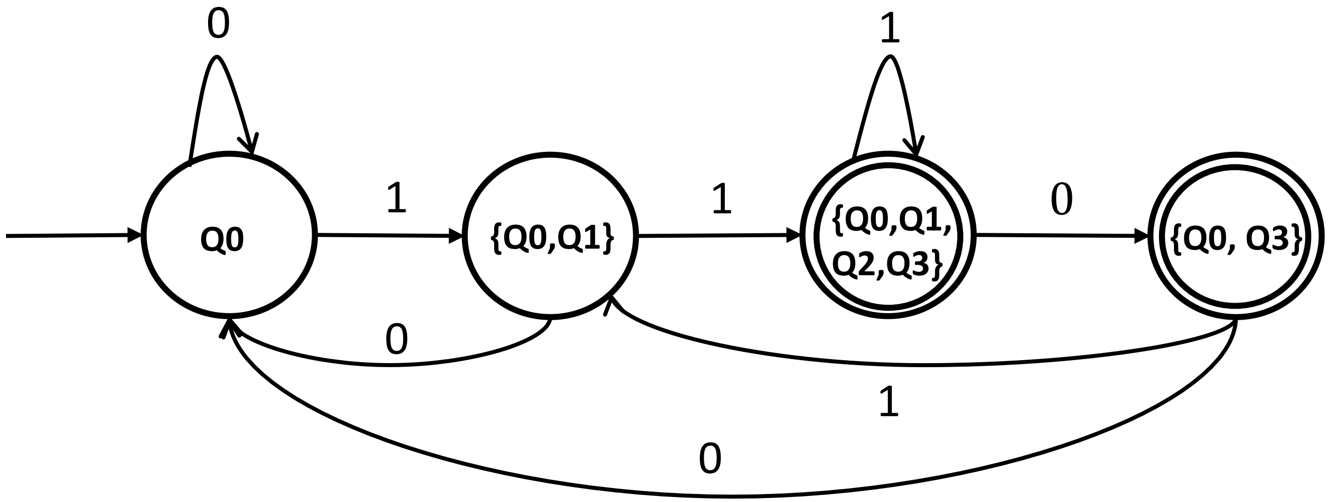


NFA N

	0	**1**	ϵ
Q0	**Q0**	**Q0, Q1**	
Q1		**Q2**	
Q2	**Q3**		**Q3**
Q3			



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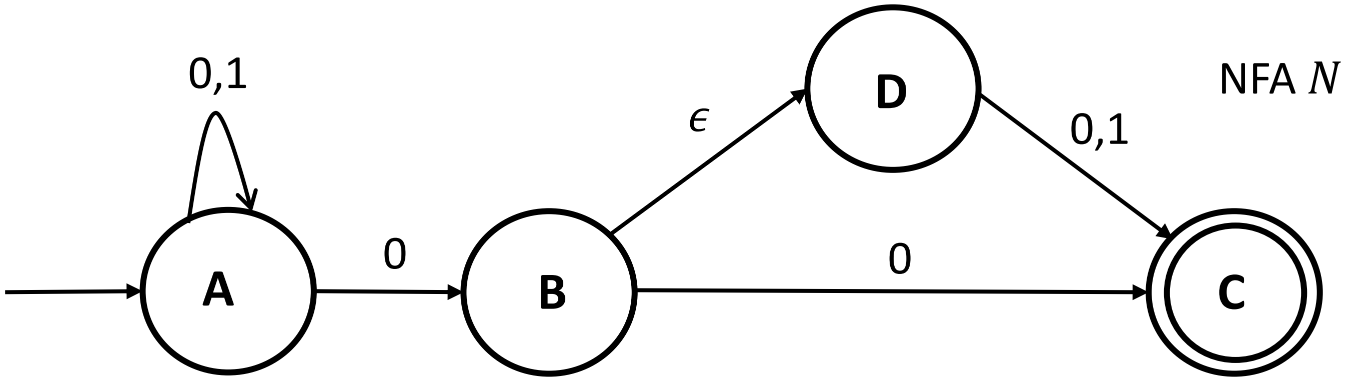
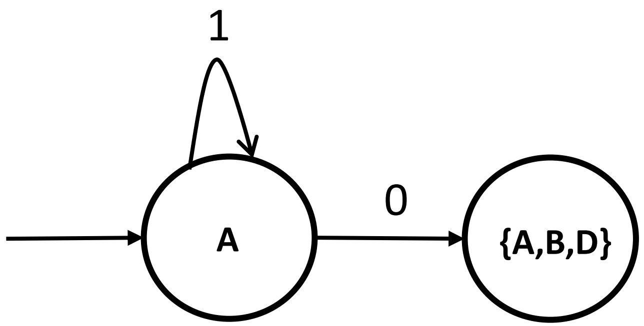


NFA N

	0	1	ϵ
A	A, B	A	
B	C		D
C			
D	C	C	



Remembering DFA R

Converting an NFA to a DFA

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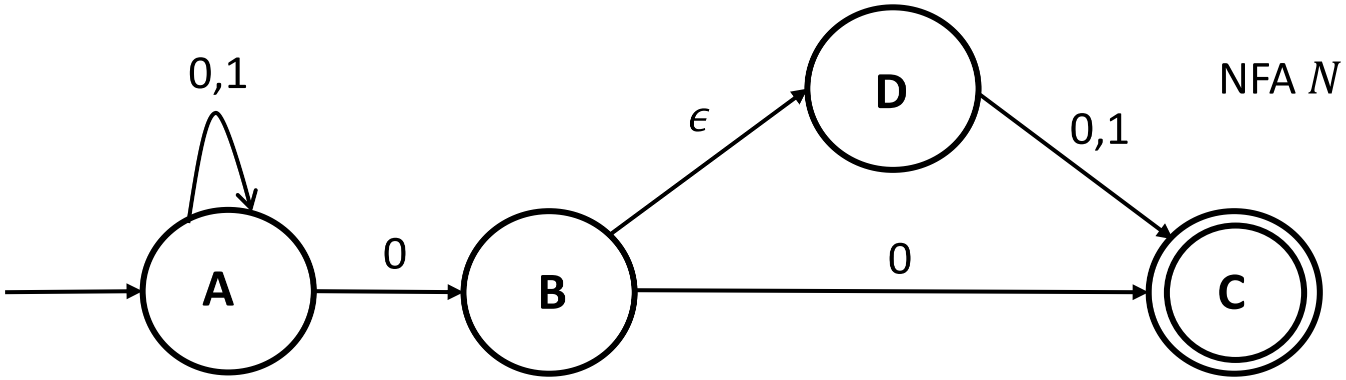
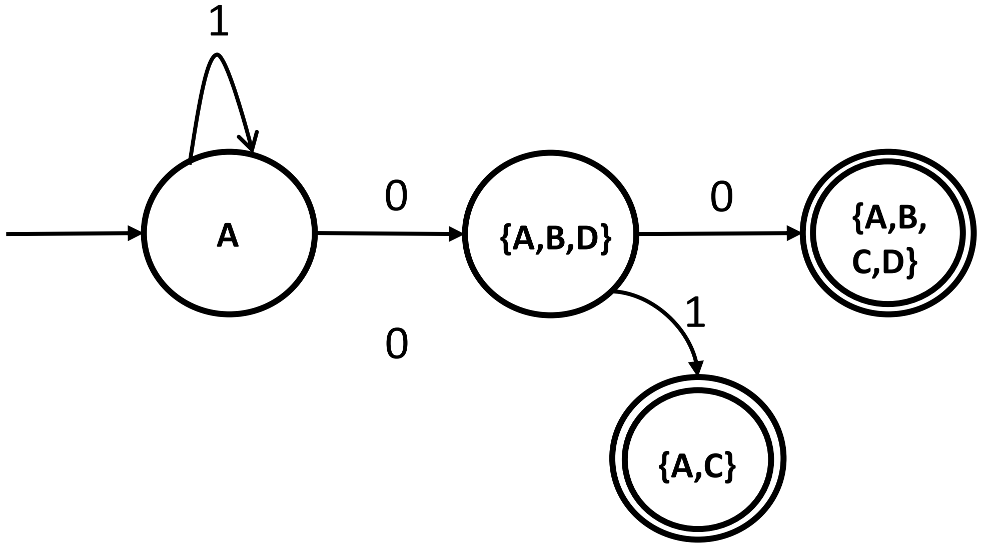


NFA N

	0	1	ϵ
A	**A, B**	**A**	
B	**C**		**D**
C			
D	**C**	**C**	



Remembering DFA R

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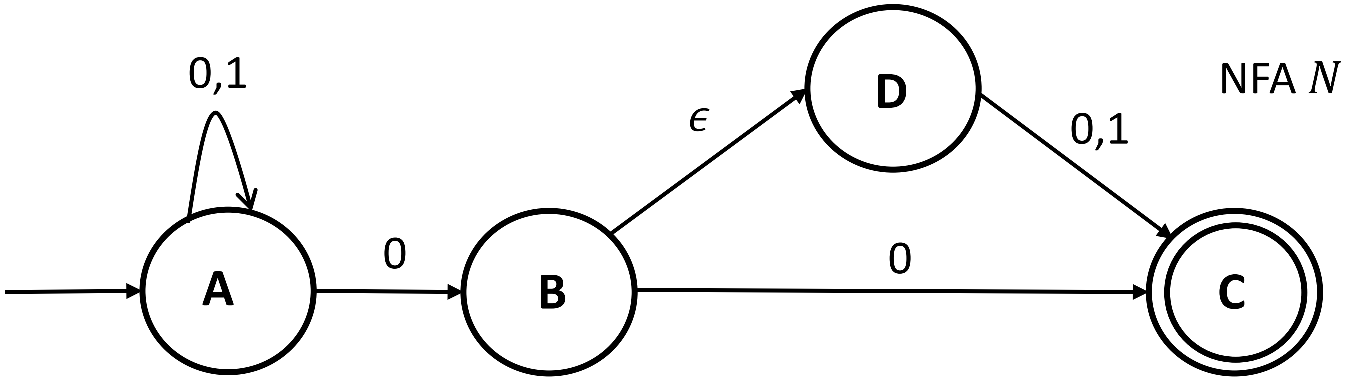
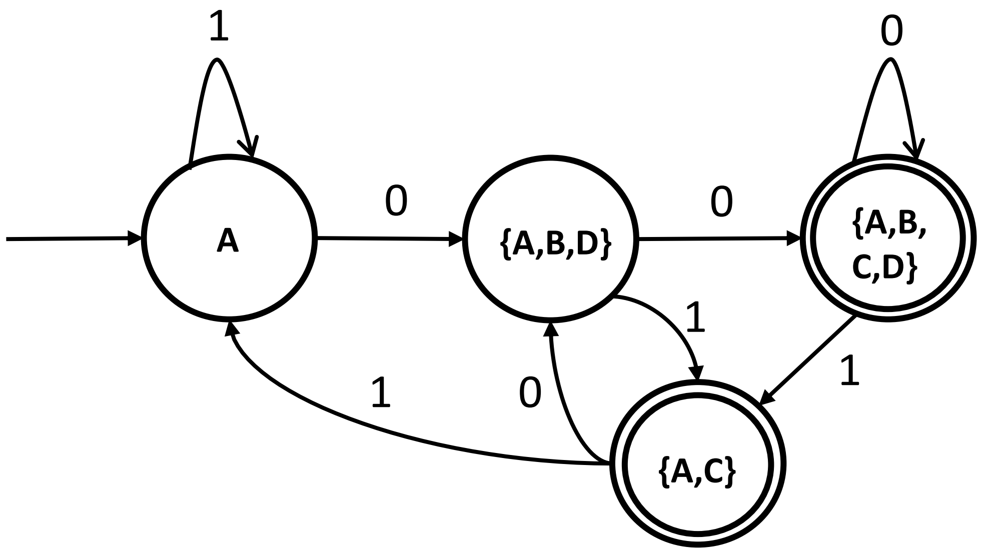


NFA N

	0	1	ϵ
A	A, B	A	
B	C		D
C			
D	C	C	



Remembering DFA R

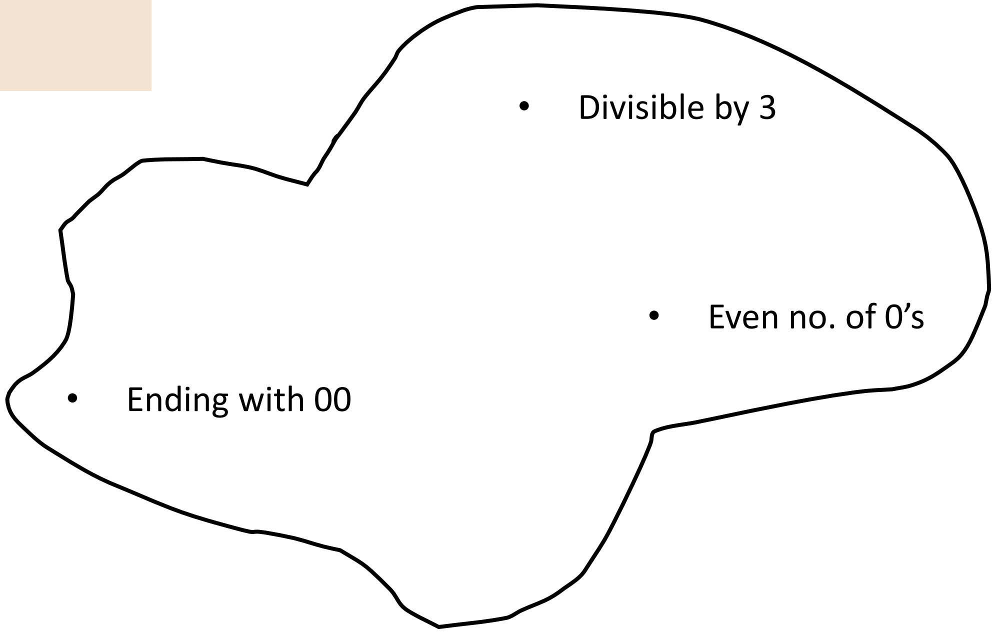
Regular Languages

A language is called a **Regular Language** if there exists some finite automata recognizing it.

If M be a finite automaton (DFA/NFA) and,

$$L(M) = \{\omega | \omega \text{ is accepted by } M\}$$

$\boldsymbol{L(M)}$ **is regular.**



Set of all regular Languages

Regular Languages

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If M be a finite automaton (DFA/NFA) and,

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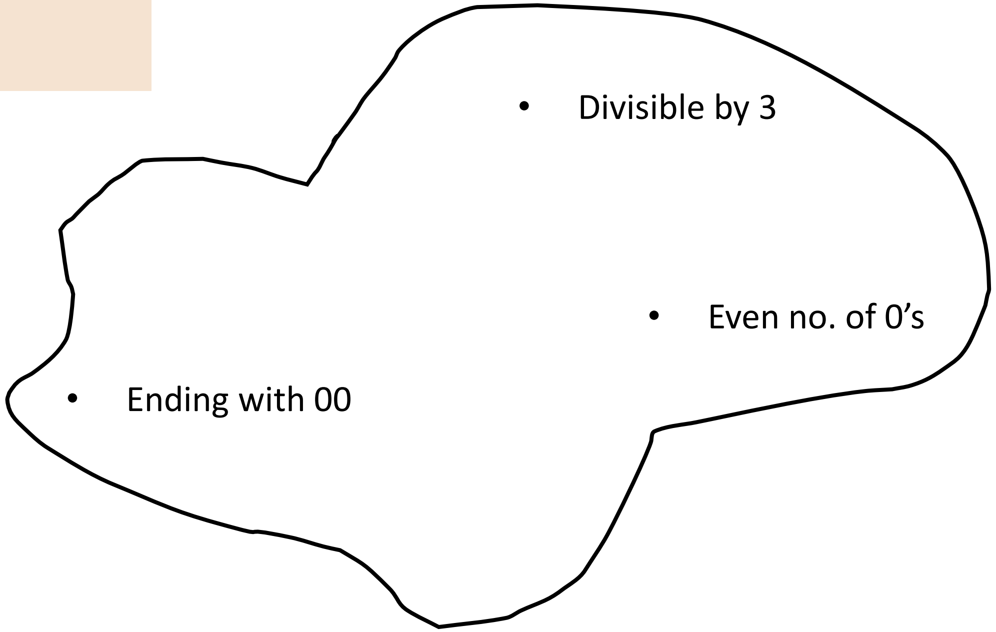
$L(M)$ is regular.

- Any language has associated with it, a set of operations that can be performed on it.
- These operations help us to understand the properties of that language, e.g. closure properties
- For regular languages, this will help us prove that certain languages are non-regular and hence we cannot hope to design a finite automaton for them



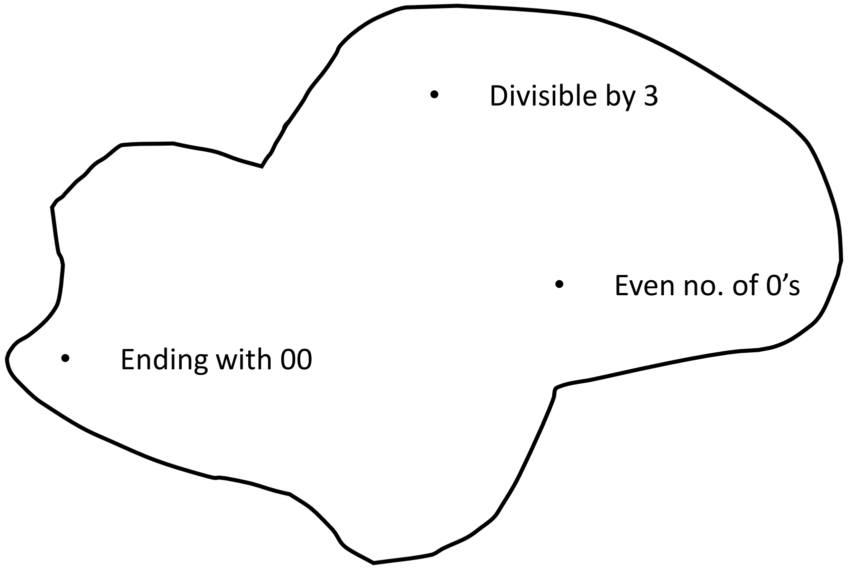
Set of all regular Languages

Regular Languages

Regular Operations:

Let L_1 and L_2 be languages. The following are the *regular operations*:

- **Union:** $L_1 \cup L_2 = \{x | x \in L_1 \text{ or } x \in L_2\}$
- **Concatenation:** $L_1 . L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$
- **Star:** $L_1^* = \{x_1 x_2 \cdots x_k |\ k \geq 0 \text{ and each } x_i \in L_1\}$



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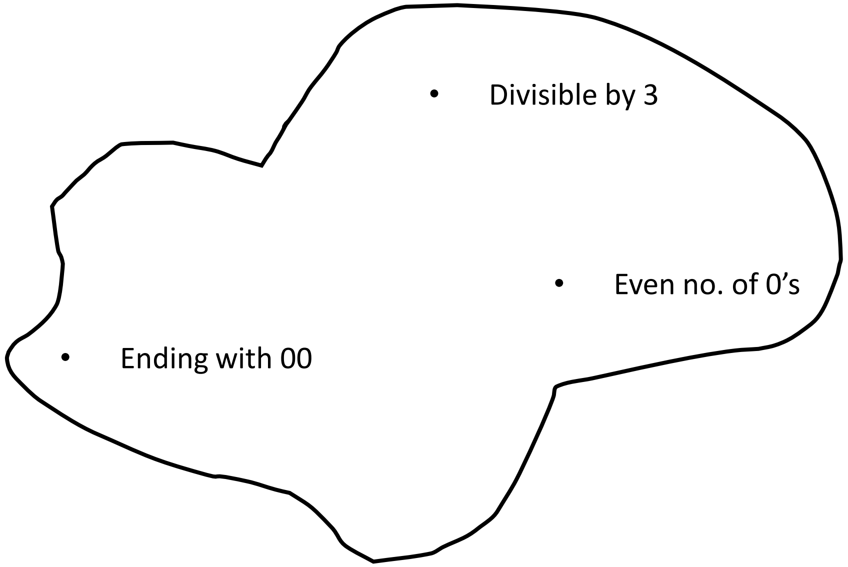

Set of all regular Languages

Star operation: It is an unary operation (unlike the other two) and involves putting together *any number of strings in L_1 together to obtain a new string.*

Note: Any number of strings includes "0" as a possibility and so the empty string ϵ is a member of L_1^***.**

If $\Sigma = \{a\}$, $\Sigma^* = \{\epsilon, a, aa, aaa, \ldots \ldots\}$; If $\Sigma = \{\Phi\}, \Sigma^* = \{\epsilon\}$

Regular Languages

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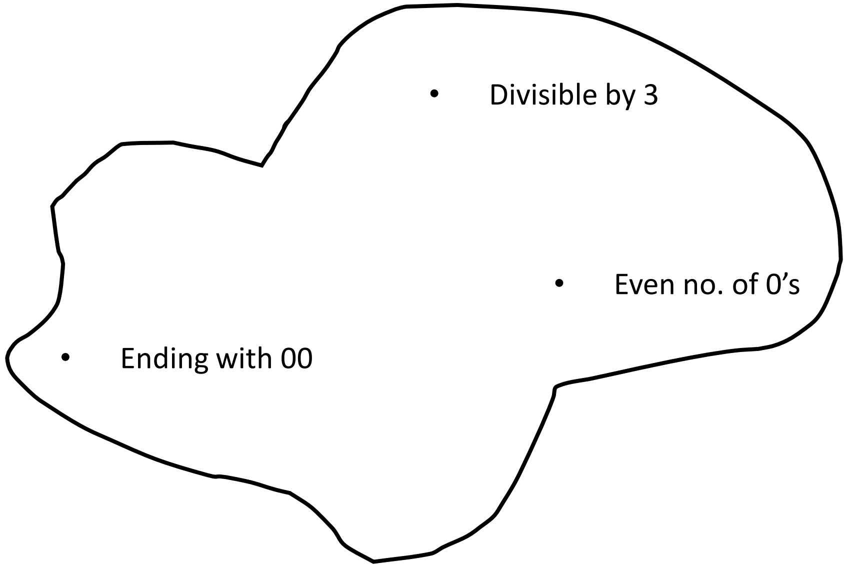

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If $\Sigma = \{0,1\}$, we have that $\Sigma^* = \{0,1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots \ldots\}$

Regular Languages

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Example: Let the alphabet $\Sigma = \{a, b, \cdots, z\}$. If $L_1 = \{\textit{social, economic}\}$ and $L_2 = \{\textit{justice, reform}\}$, then

- $L_1 \cup L_2 = \{\textit{social, economic, justice, reform}\}$

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Regular Languages

Regular Operations: Let L_1 and L_2 be languages.

- **Union:** $L_1 \cup L_2 = \{x | x \in L_1$ or $x \in L_2\}$
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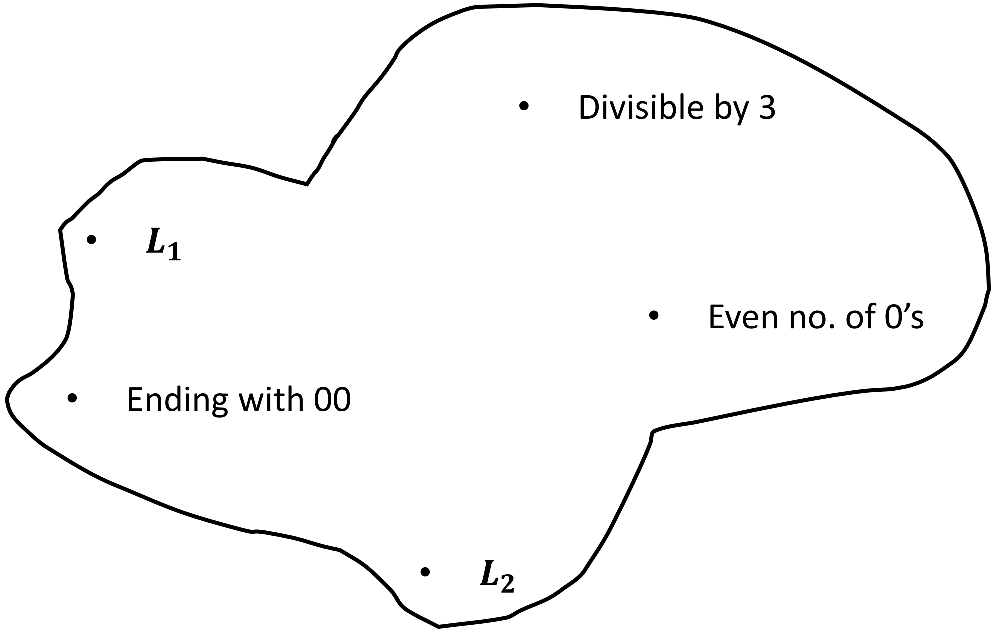
- $L_1 \cup L_2 = \{$*social, economic, justice, reform*$\}$
- $L_1 . L_2 = \{$*socialjustice, socialreform, economicjustice, economicreform*$\}$
- $L_1^* = \{\epsilon$, *social, economic, socialsocial, socialeconomic, economicsocial, economiceconomic, socialsocialsocial, socialsocialeconomic, socialeconomiceconomic,……*$\}$
- $L_2^* = \{\epsilon$, *justice, reform, justicejustice, justicereform, reformjustice, reformreform, justicejusticejustice, ……*$\}$

Closure of Regular Languages

We want to check whether the set of regular languages are **closed** under some operations.

What does this mean?

- We pick up points within the set of all regular languages (say L_1 and L_2)
- Perform *set operations* such as Union, concatenation, Star, intersection, reversal, complement etc on them.
- Observe whether the resulting language still belongs to the set of all regular languages.
- If so, we say, regular languages are **closed** under that operation.



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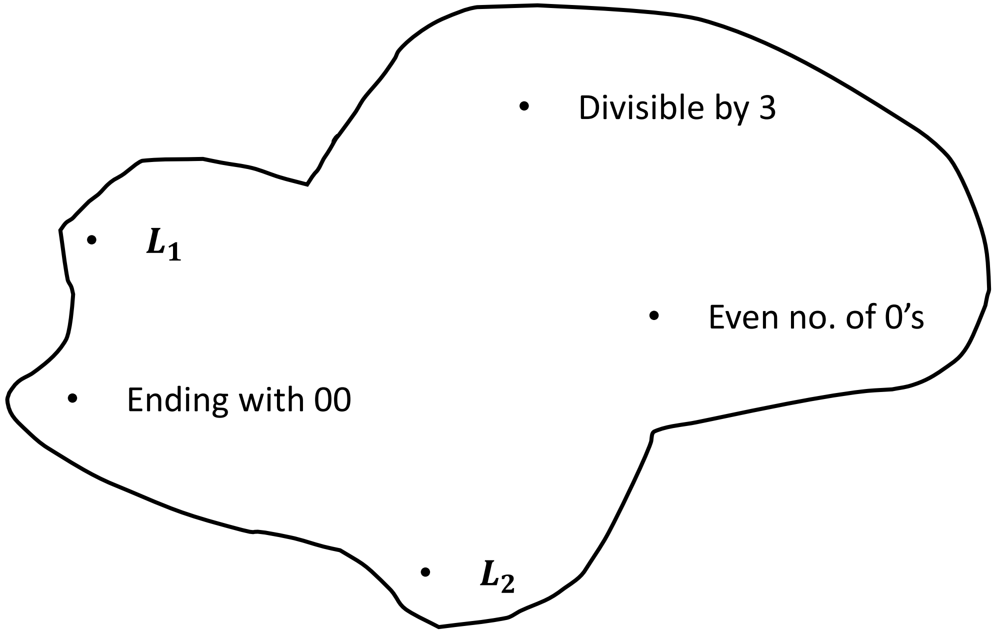


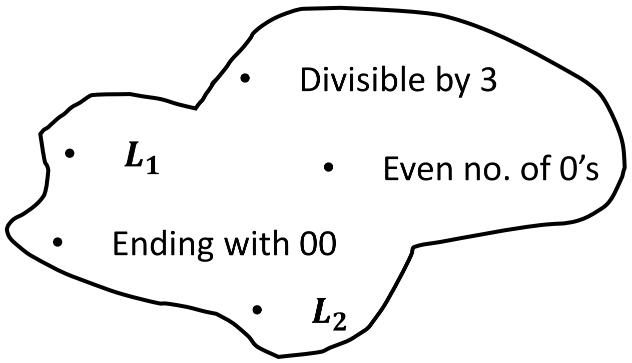
Set of all regular Languages

For example, the **natural numbers are closed under addition/multiplication** and **not under subtraction/division**.

Closure of Regular Languages

Q: Is the set of all regular languages **closed under union**?

Suppose L_1 and L_2 are regular languages. Is $L = L_1 \cup L_2$ also regular?



Set of all regular Languages

Closure of Regular Languages

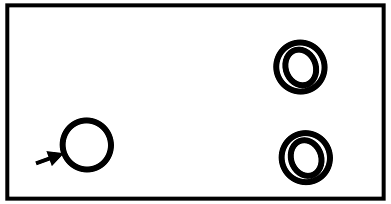
Q: Is the set of all regular languages **closed under union**?

Suppose L_1 and L_2 are regular languages. Is $L = L_1 \cup L_2$ also regular?

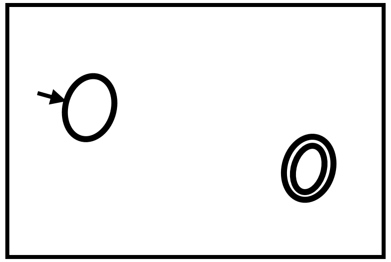
Proof: Since L_1 and L_2 are regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$ and a DFA M_2 that accepts L_2, i.e. $L(M_2) = L_2$.

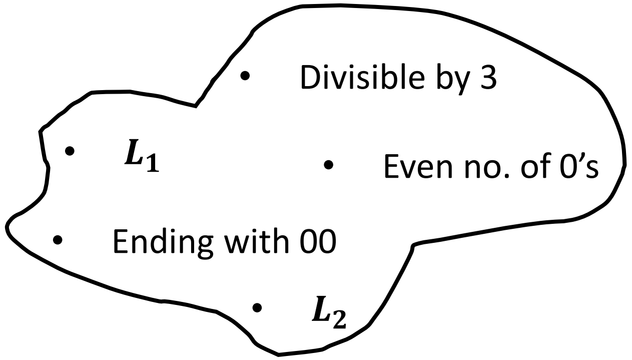
Using M_1 and M_2, we will show how to construct an NFA M that accepts $L = L_1 \cup L_2$, i.e. $L(M) = L_1 \cup L_2$.

Suppose the DFA for M_1 is



And the DFA for M_2 is





Set of all regular Languages

Closure of Regular Languages

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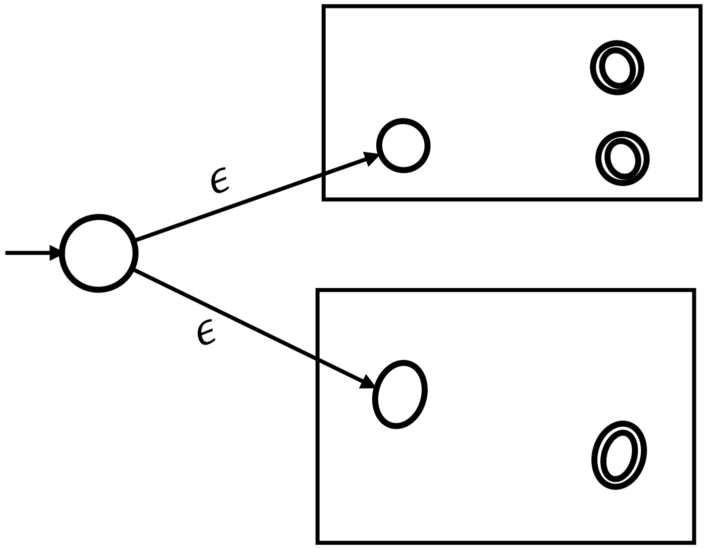
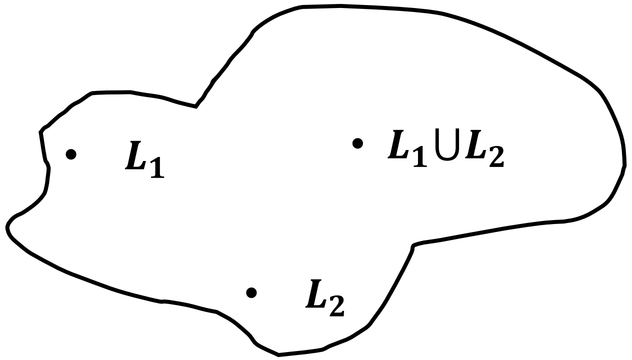
NFA M accepting $\boldsymbol{L = L_1 \cup L_2}$





Set of all regular Languages

Closure of Regular Languages

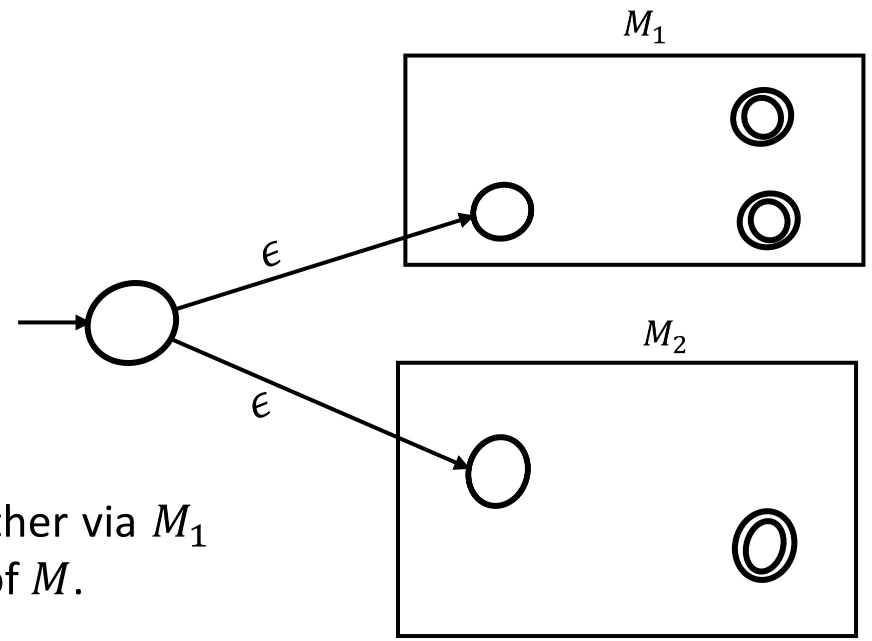
Q: Is the set of all regular languages **closed under union**?

Suppose L_1 and L_2 are regular languages. Is $L = L_1 \cup L_2$ also regular?

Proof: In order to prove that $L(M) = L_1 \cup L_2$, we show two things:

(i) $L \subseteq L_1 \cup L_2$

Let $\omega \in L$, i.e. ω is accepted by M. The final state for L can be reached either via M_1 or M_2. Thus ω must be accepted by either of them to reach the final state of M.



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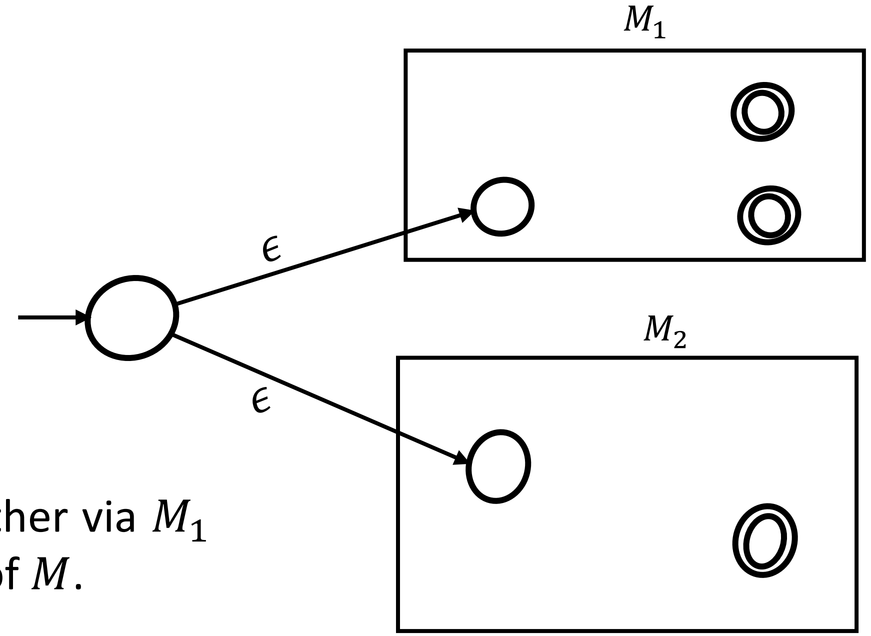
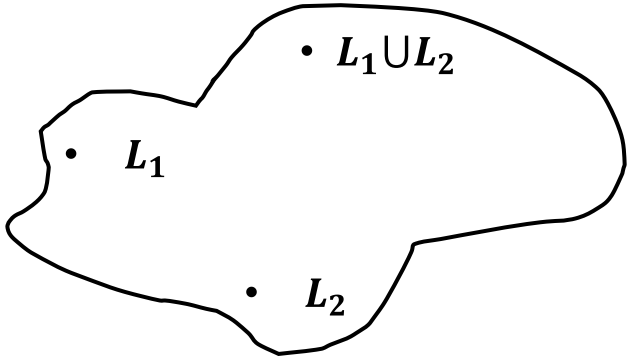
Let $\omega \in L$, i.e. ω is accepted by M. The final state for L can be reached either via M_1 or M_2. Thus ω must be accepted by either of them to reach the final state of M.

(ii) $L_1 \cup L_2 \subseteq L$

Let $\omega \in L_1 \cup L_2$. Then, $\omega \in L_1$ or $\omega \in L_2$.

Thus, ω must reach the final state of M_1 or M_2. But since the start state of M_1 or M_2 can be reached from the start state of M by taking an ϵ-transition, $\omega \in L$.





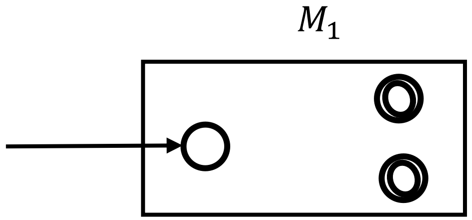
Set of all regular Languages

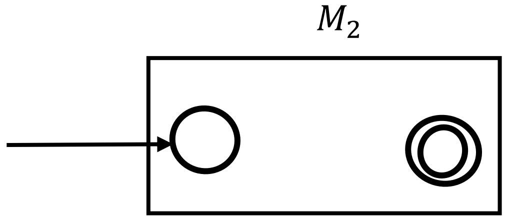
Closure of Regular Languages

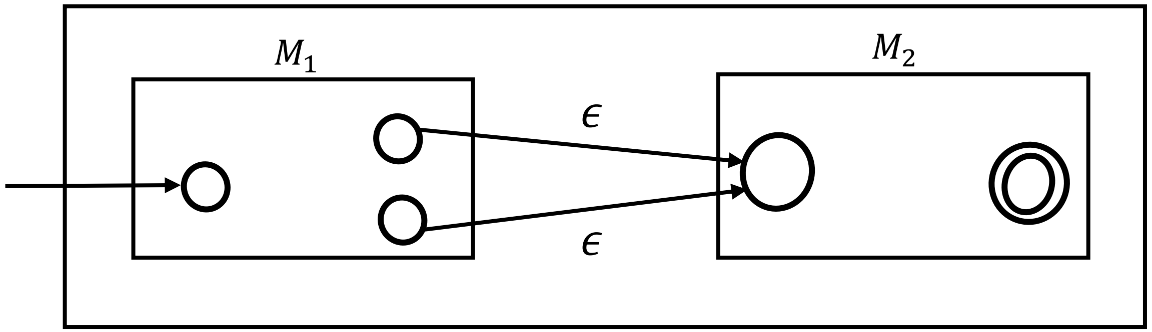
Q: Is the set of all regular languages **closed under concatenation**? Suppose L_1 and L_2 are regular languages. Is $L = L_1.L_2$ also regular?

Proof: Since L_1 and L_2 are regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$ and a DFA M_2 that accepts L_2, i.e. $L(M_2) = L_2$**.**

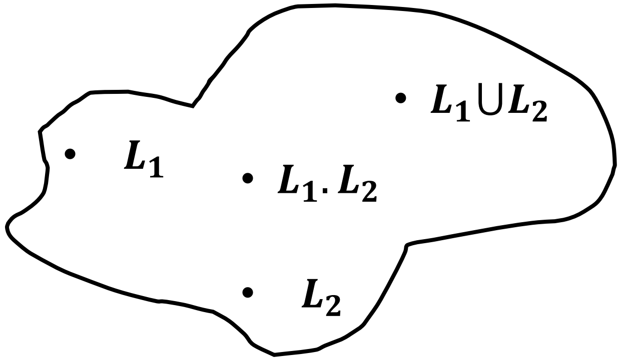
Using $M_1$and M_2, we will show how to construct an NFA M that accepts $L = L_1.L_2$.







NFA M accepting $\boldsymbol{L = L_1.L_2}$



Set of all regular Languages

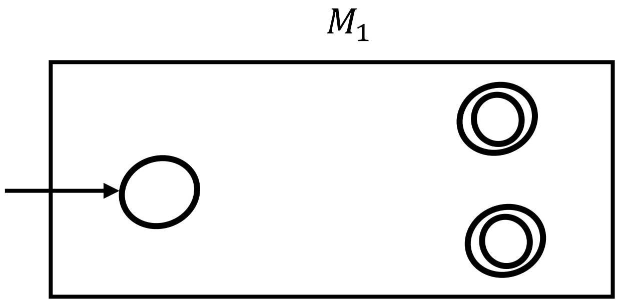
$L_1.L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$

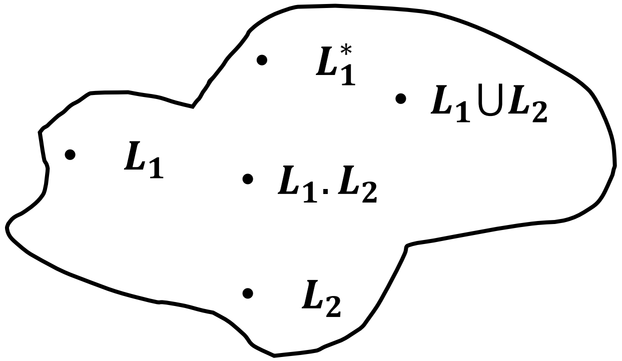
Closure of Regular Languages

Q: Is the set of all regular languages **closed under star**? Suppose L_1 is a regular language. Is L_1^* also regular?

Proof: Since L_1 is regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$**.** Using M_1, we will show how to construct an NFA M that accepts $L = L_1^*$.

$L_1^* = \{x_1 x_2 \cdots x_k \mid k \geq 0 \text{ and each } x_i \in L_1\}$





Set of all regular Languages

Closure of Regular Languages

Q: Is the set of all regular languages **closed under star**? Suppose L_1 is a regular language. Is L_1^* also regular?

Proof: Since L_1 is regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$**.** Using M_1, we will show how to construct an NFA M that accepts $L = L_1^*$.

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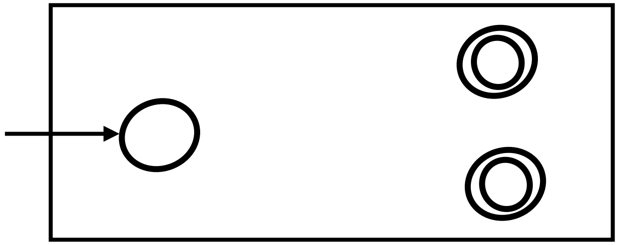
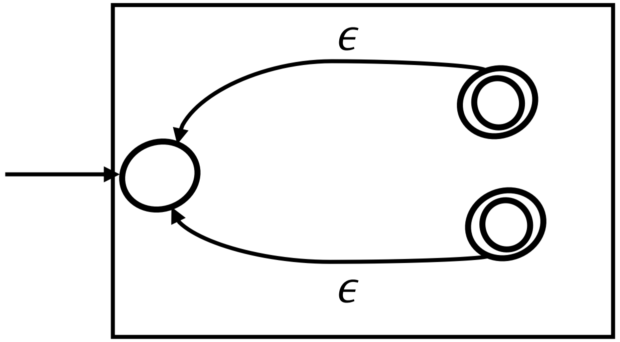
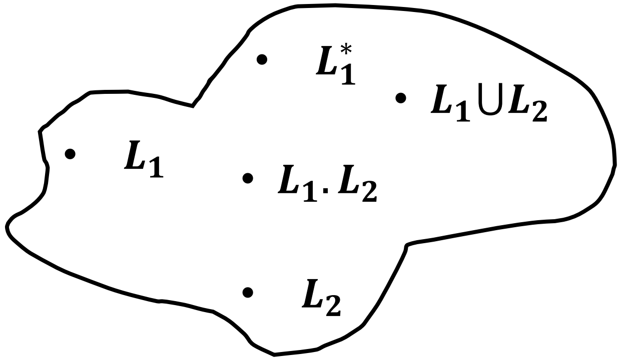
M_1





NFA accepting $\boldsymbol{L = L_1^*}$



Set of all regular Languages

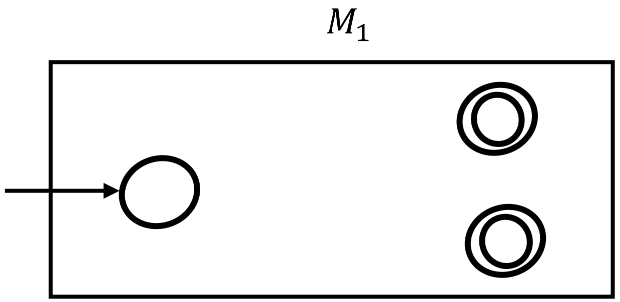
Steps:

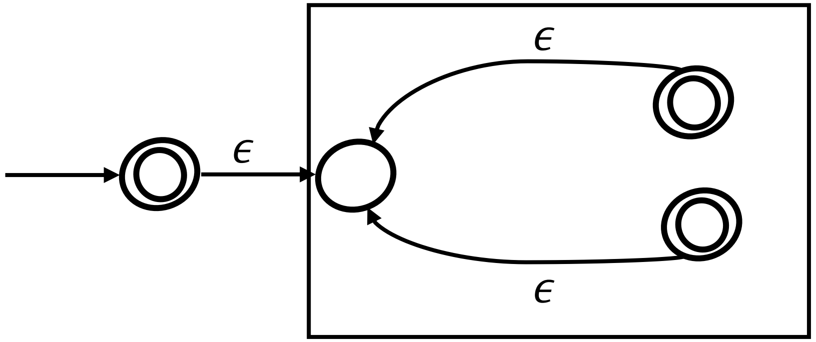
- Make ϵ-transitions from the final states of L_1 to the initial state of L_1.

Closure of Regular Languages

Q: Is the set of all regular languages **closed under star**? Suppose L_1 is a regular language. Is L_1^* also regular?

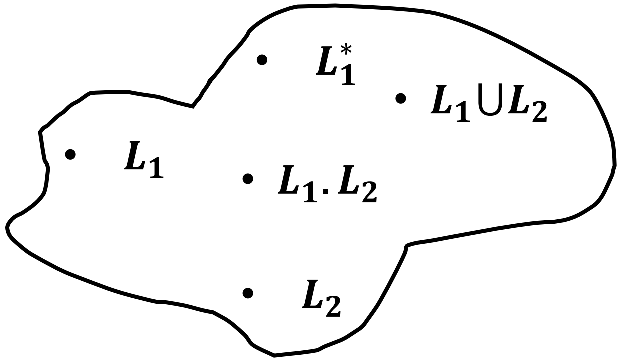
Proof: Since L_1 is regular, there must be a DFA M_1 that accepts L_1, i.e. $L(M_1) = L_1$**.** Using M_1, we will show how to construct an NFA M that accepts $L = L_1^*$.





NFA accepting $\boldsymbol{L = L_1^*}$

$L_1^* = \{x_1 x_2 \cdots x_k \mid k \geq 0$ and each $x_i \in L_1\}$



Set of all regular Languages

Steps:

- Make ϵ-transitions from the final states of L_1 to the initial state of L_1.
- Make a new final state as the start state and make an ϵ-transition from this state to the previous start state of L_1.

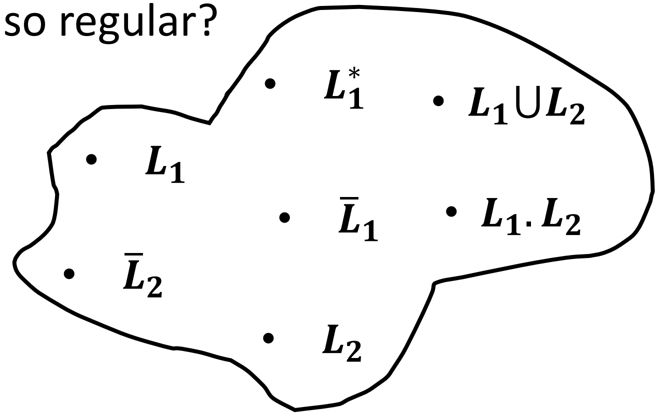
Closure of Regular Languages

Q: Is the set of all regular languages **closed under complement**? If L is regular, then is $\overline{L}$ also regular?

Proof: Given a DFA M, such that $L(M) = L$, construct the **toggled DFA** M' from M, by

(i) changing all the non-final states of M to be the final states of M' and
(ii) changing all the final states M to be the non-final states of M'.

$$\boldsymbol{L(M') = \overline{L}}$$



Set of all regular Languages

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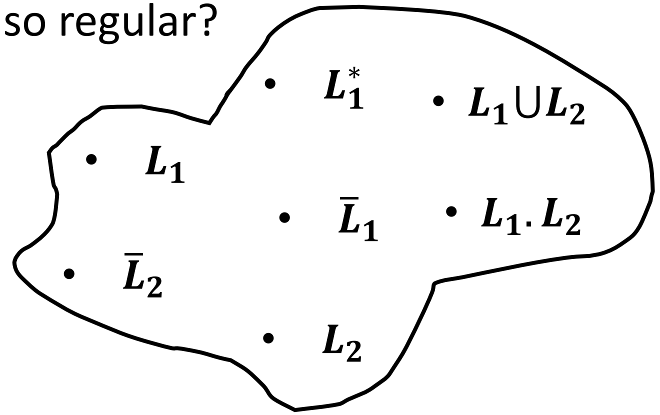

Set of all regular Languages

Q: If L is the language accepted by an NFA, does "toggling" its states result in an NFA that accepts $\overline{L}$?

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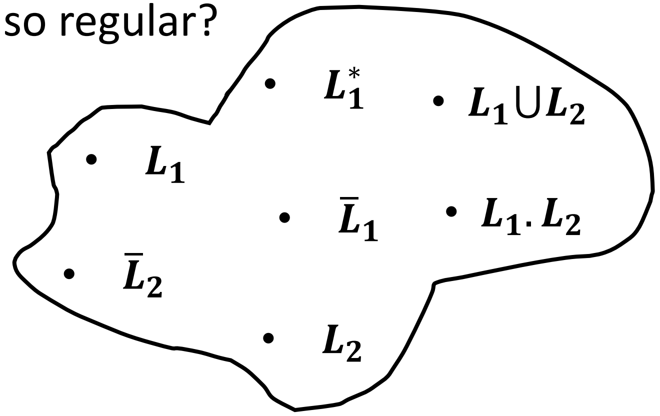

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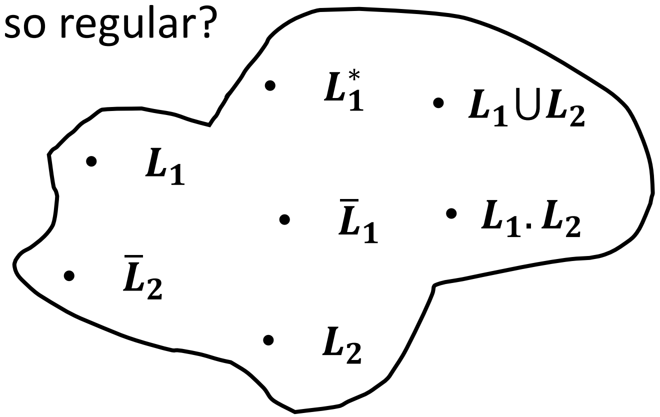
Q: If L is the language accepted by an NFA, does "toggling" its states result in an NFA that accepts $\overline{L}$?

Proof: Consider that for an input string $x \in L$**,** such that N accepts it. Suppose there is an rejecting run and an accepting run for input x. (See Table)

	NFA N	**Toggled NFA N'**
Run 1	Rejecting	
Run 2	Accepting	

Closure of Regular Languages

Q: Is the set of all regular languages **closed under complement**? If L is regular, then is $\overline{L}$ also regular?



Set of all regular Languages

Proof: Given a DFA M, such that $L(M) = L$, construct the **toggled DFA** M' from M, by

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For toggled NFA N' too, there are two runs for x. However, the rejecting run N is an accepting run for N'. Thus x is accepted by both N and N'.

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Run 1	Rejecting	Accepting
Run 2	Accepting	Rejecting

Closure of Regular Languages

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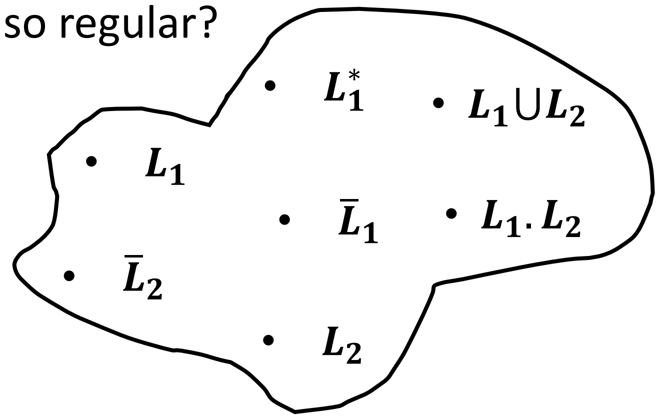

Set of all regular Languages

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Contradiction! So No, the **toggled NFA does not accept $\overline{L}$**.

	NFA N	**Toggled NFA N'**
Run 1	Rejecting	Accepting
Run 2	Accepting	Rejecting

Thank You!