PNS Tutorial 4 Solutions

Solution 1

Solution: To find the pdf of Y (same as X), consider

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \frac{1}{\pi r^2}\int_{\{x:\, x^2+y^2 \le r^2\}} dx$$

$$= \frac{1}{\pi r^2}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx = \begin{cases} \frac{2}{\pi r^2}\sqrt{r^2-y^2}, & \text{if } |y| \le r \\ 0, & \text{otherwise} \end{cases}$$

Solution 2

We have

$$\begin{aligned} E[XY^2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy^2) f_{XY}(x,y) \;\; dxdy \\ &= \int_0^1 \int_0^1 xy^2(x+y) \;\; dxdy \\ &= \int_0^1 \int_0^1 x^2y^2 + xy^3 \;\; dxdy \\ &= \int_0^1 \left(\frac{1}{3}y^2 + \frac{1}{2}y^3\right) dy \\ &= \frac{17}{72}. \end{aligned}$$

Solution 3

<u>Solution:</u> For $0<x<1$, $0<y<1$ we have:

$$0<x<1,\quad f_X(x) = \int_0^1 \frac{15}{2} x(2-x-y)\,dy = \frac{15}{2} x\left(2y - xy - \frac{y^2}{2}\right)\Big|_0^1 = \underline{\frac{45}{4}x - \frac{15}{2}x^2}$$

$$E[X] = \int_0^1 x f_X(x)\,dx = \frac{15}{4}x^3 - \frac{15}{8}x^4\Big|_0^1 = \underline{\frac{15}{8}}$$

$$0<y<1,\quad f_Y(y) = \int_0^1 \frac{15}{2} x(2-x-y)\,dx = \frac{15}{2}\left(x^2 - \frac{x^3}{3} - \frac{x^2 y}{2}\right)\Big|_0^1 = \underline{5 - \frac{15}{4}y}$$

$$E[Y] = \int_0^1 y f_Y(y)\,dy = \frac{5}{2}y^2 - \frac{5}{4}y^3\Big|_0^1 = \underline{\frac{5}{4}}$$

$$\rightarrow E[XY] = \int_0^1\int_0^1 xy\, f_{XY}(x,y)\,dx\,dy = \int_0^1\int_0^1 y\,\frac{15}{2}\left(2x^2 - x^3 - yx^2\right)dx\,dy$$

$$\rightarrow E[XY] = \int_0^1 \frac{15}{2} y\left(\frac{5}{12} - \frac{y}{3}\right)dy = \frac{15}{2}\left[\frac{5y^2}{24} - \frac{y^3}{9}\right]_0^1 = \underline{\frac{35}{48}}$$

Solution 4

Solution

(i) $f_{X,Y}(x,y) = \dfrac{1}{\text{Area } \triangle} = 2, \quad (x,y)$ lies in the triangle.

(ii) $f_X(x) = \displaystyle\int_{-\infty}^{\infty} f_{X,Y}(x,y')\ dy' = \int_0^{1-x} 2\ dy = 2(1-x).$

$f_Y(y) = \displaystyle\int_{-\infty}^{\infty} f_{X,Y}(x',y)\ dx' = \int_0^{1-y} 2\ dx = 2(1-y).$

Hence, $E[X] = 2\displaystyle\int_0^1 x(1-x)\ dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

Solution 5

(a) Since $f(x)$ satisfies Property 1 if $c \geq 0$, it must satisfy Property 2 in order to be a density function. Now

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^3 cx^2\,dx = \left.\frac{cx^3}{3}\right|_0^3 = 9c$$

and since this must equal 1, we have $c = 1/9$.

(b)
$$P(1 < X < 2) = \int_1^2 \frac{1}{9}x^2\,dx = \left.\frac{x^3}{27}\right|_1^2 = \frac{8}{27} - \frac{1}{27} = \frac{7}{27}$$

In case $f(x)$ is continuous, which we shall assume unless otherwise stated, the probability that X is equal to any particular value is zero. In such case we can replace either or both of the signs $<$ in (8) by $\leq$. Thus, in Example 2.5,

$$P(1 \leq X \leq 2) = P(1 \leq X < 2) = P(1 < X \leq 2) = P(1 < X < 2) = \frac{7}{27}$$

Solution 6

(i) Since $P[X \leq a_1] = P[X > a_1]$

$$P[X \leq a_1] = \frac{1}{2}$$

i.e., $$\int_0^{a_1} f(x)dx = \frac{1}{2}$$

i.e., $$\int_0^{a_1} 5x^4 dx = \frac{1}{2}$$

$$5\left[\frac{x^5}{5}\right]_0^{a_1} = \frac{1}{2}$$

$$a_1 = (0.5)^{\frac{1}{5}}$$

(ii) $$P[X > a_2] = 0.05$$

$$\int_{a_2}^{1} f(x)dx = 0.05$$

$$\int_{a_2}^{1} 5x^4 dx = 0.05$$

$$5\left[\frac{x^5}{5}\right]_{a_2}^{1} = 0.05$$

$$a_2 = [0.95]^{\frac{1}{5}}$$

Solution 7

Given: $Y = g(X) = aX + b$

1) Since, $g(x)$ is differentiable and continuously increasing $(a > 0)$ or continuously decreasing $(a < 0)$, we can apply Method of Transformation: [continuously increasing or continuously decreasing $\Rightarrow$ strictly monotonic $(a \neq 0)$]

I $(a \neq 0)$
(by MoT) $\Rightarrow$
$$f_Y(y) = \begin{cases} f_X(x)\left|\frac{dx}{dy}\right| & ; \quad (\text{where } y = g(x)) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$\Rightarrow y = ax + b \Rightarrow dy = a\,dx$

$$\Rightarrow f_Y(y) = \begin{cases} f_X\left(\frac{y-b}{a}\right)\left|\frac{1}{a}\right| & ; \quad y = ax + b \\ 0 & ; \quad \text{otherwise} \end{cases}$$

II If $a = 0$ [never, as ATQ, $a, b \neq 0$]

$\Rightarrow Y = b \quad \Rightarrow F_Y(y) = \begin{cases} 0 & ; \quad y < b \\ 1 & ; \quad y \geq b \end{cases}$

$\Rightarrow f_Y(y) = 0 \quad \forall y$

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} f_X\left(\frac{y-b}{a}\right)\left|\frac{1}{a}\right| & ; \quad y = ax + b \\ 0 & ; \quad \text{otherwise} \end{cases}} \quad -(1)$$

2) $X \sim \text{Exponential}(\lambda) \rightarrow f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; \; x > 0 \\ 0 & ; \; \text{otherwise} \end{cases}$

by ① $\rightarrow f_Y(y) = \begin{cases} \lambda e^{-\lambda(y-b)/a} \times \dfrac{1}{|a|} & ; \; y > b \text{ and } a \neq 0 \\ 0 & ; \; \text{otherwise} \end{cases}$

→ Now, if $Y \sim \text{Exponential}(\lambda')$ $\rightarrow f_Y(y) = \begin{cases} \lambda' e^{-\lambda' y} & ; \; y > 0 \\ 0 & ; \; \text{otherwise} \end{cases}$

→ Comparing 2 PDF of Y, we get $\underbrace{b = 0}_{\text{not possible}}$, $a = |a| \Rightarrow a \geq 0$, $a \neq 0$

→ Thus, Y will be a exponential RV if $\{b = 0 \text{ and } a > 0\}$ → $\boxed{\text{never}}$

3.) If X is a normal RV and let Z be a standard normal RV, then:

$\rightarrow X = \sigma Z + \mu \quad ; \; \sigma > 0$

$\rightarrow f_X(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad ; \; \forall x$

→ by ① $\rightarrow f_Y(y) = \begin{cases} \dfrac{1}{\sigma\sqrt{2\pi}\,|a|} e^{-\frac{(y-(b+a\mu))^2}{2a^2\sigma^2}} & ; \; \forall y, \; a \neq 0 \\ 0 & ; \; \text{otherwise } (a = 0) \end{cases}$

→ Thus, $Y \sim \text{Normal}(b + a\mu, \sigma^2 a^2)$, if $a \neq 0$ → $\boxed{\text{always}}$

Solution 8

Ans. $X \sim N(\mu, \sigma^2) \Rightarrow F_X(x) = \phi\left(\frac{x-\mu}{\sigma}\right) = P(X \leq x)$

1) $\Rightarrow Y = e^X$, $X \in (-\infty, \infty) \Rightarrow e^X \in (0, \infty) \Rightarrow Y \in (0, \infty)$

$$\Rightarrow F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln y)$$
$$= F_X(\ln y)$$

$$\Rightarrow F_Y(y) = \begin{cases} \phi\left(\frac{\ln y - \mu}{\sigma}\right) & ; \ y > 0 \\ 0 & ; \ \text{otherwise} \end{cases} \quad -①$$

2) I Using method of Transformation:

$$\Rightarrow f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad ; \ \forall x$$

$\Rightarrow$ $\underline{Y = e^X = g(X)}$ $\Rightarrow$ g is differentiable and strictly increasing for $y > 0$

$\Rightarrow$ By method of Transformation:

$$\Rightarrow f_Y(y) = \begin{cases} f_X(x) \frac{dx}{dy} & ; \ \text{where } y = e^x \ (y > 0) \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}} \cdot \frac{d(\ln y)}{dy} & ; \ y > 0 \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$\Rightarrow \quad f_Y(y) = \begin{cases} \dfrac{1}{\sigma y\sqrt{2\pi}}\, e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}} & ; y>0 \\ 0 & ; \text{otherwise} \end{cases}$$

II Without using method of transformation.

$\Rightarrow$ by ① $\Rightarrow$

$$F_Y(y) = \begin{cases} \phi\left(\dfrac{\ln y-\mu}{\sigma}\right) & ; y>0 \\ 0 & ; \text{otherwise} \end{cases}$$

$$\Rightarrow \quad F_Y(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\frac{\ln y-\mu}{\sigma}} e^{-u^2/2}\, du & ; y>0 \\ 0 & ; \text{otherwise} \end{cases}$$

$$\Rightarrow \quad f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \left[e^{-u^2/2}\right]_{-\infty}^{\frac{\ln y-\mu}{\sigma}} \dfrac{d\left(\frac{\ln y-\mu}{\sigma}\right)}{dy} & ; y>0 \\ 0 & ; \text{otherwise} \end{cases}$$

$$\Rightarrow \quad f_Y(y) = \begin{cases} \dfrac{1}{\sigma y\sqrt{2\pi}}\, e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}} & ; y>0 \\ 0 & ; \text{otherwise} \end{cases}$$

Solution 9

A3: let X_i be a RV defined as:

$$\rightarrow X_i = \begin{cases} 1 & ; \; i^{th} \text{ dice roll gives } 2 \\ 0 & ; \; \text{otherwise} \end{cases}$$

→ Then $X_i \sim \text{Bernoulli}(p)$, where $p = P(2 \text{ on } i^{th} \text{ dice roll}) = \frac{1}{6}$
(independent $\forall i$)

→ Now, let Y denote the RV corresponding to number of 2's in n (12,000) dice rolls.

$$\rightarrow Y = X_1 + X_2 + X_3 + \cdots + X_n$$

→ Then, as we know, sum of Bernoulli's (independent) is a Binomial,
$Y \overset{n}{\sim} \text{Binomial}(n, p)$

$$\Rightarrow \mu_Y = np, \quad \sigma^2(y) = np(1-p) \quad -①$$

→ Here, $\left(np = 12000\left(\frac{1}{6}\right) = 2000\right) \geq 5$, and $\left(n(1-p) = 12000\left(\frac{5}{6}\right) = 10000\right) \geq 5$

⇒ Hence, we can use Normal approximation of Binomial Distribution (NABD).

→ We have to find $P(1900 < Y < 2150)$

→ by NABD, $P(1900 < Y < 2150) = P(1900.5 < Y < 2149.5)$

$$= P\left(\frac{1900.5 - \mu_Y}{\sigma_Y} < Z < \frac{2149.5 - \mu_Y}{\sigma_Y}\right)$$

→ here, $Z \sim N(0,1)$

→ from ①, $\mu_Y = np = \frac{12000}{6} \doteq 2000$, $\sigma_Y = \sqrt{np(1-p)} = \sqrt{\frac{5000}{3}}$

$$\rightarrow P(1900 < Y < 2150) = P\left(\frac{-99.5}{\sqrt{5000/3}} < z < \frac{149.5}{\sqrt{5000/3}} \right)$$

$$= \phi(3.66) - \phi(-2.44)$$

$$= \phi(3.66) + \phi(2.44) - 1$$

$$= 0.99987 + 0.99266 - 1 = 0.99253$$

$$\rightarrow P(1900 < Y < 2150) = 0.992$$

Solution 10

a. We can write

$$f_{XY}(x,y) = \left[e^{-x}u(x)\right]\left[2e^{-2y}u(y)\right],$$

where $u(x)$ is the unit step function:

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, we conclude that X and Y are independent.

b. For this case, it does not seem that we can write $f_{XY}(x,y)$ as a product of some $f_1(x)$ and $f_2(y)$. Note that the given region $0 < x < y < 1$ enforces that $x < y$. That is, we always have $X < Y$. Thus, we conclude that X and Y are not independent. To show this, we can obtain the marginal PDFs of X and Y and show that $f_{XY}(x,y) \neq f_X(x)f_Y(y), \text{ for some } x, y$. We have, for $0 \leq x \leq 1$,

$$\begin{aligned} f_X(x) &= \int_x^1 8xy\,dy \\ &= 4x(1-x^2). \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 4x(1-x^2) & 0 < x < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we obtain

$$f_Y(y) = \begin{cases} 4y^3 & 0 < y < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

As we see, $f_{XY}(x,y) \neq f_X(x)f_Y(y)$, thus X and Y are NOT independent.

Solution 11

5. Two components of a laptop computer have the following joint probability density function for their useful lifetimes X and Y (in years):

$$f(x,y)=\begin{cases} x\,e^{-x(1+y)} & x\ge 0,\ y\ge 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the marginal probability density function of X, $f_X(x)$.

$$f_X(x) = \int_0^\infty x\,e^{-x(1+y)}\,dy = x\,e^{-x}\int_0^\infty e^{-xy}\,dy = e^{-x}, \qquad x\ge 0.$$

$$F_X(x) = \int_0^x f_X(x)dx = 1 - e^{-x} \quad , \qquad x\ge 0$$

b) Find the marginal probability density function of Y, $f_Y(y)$.

$$f_Y(y) = \int_0^\infty x\,e^{-x(1+y)}\,dx = \frac{1}{(1+y)^2}, \qquad y\ge 0.$$

$$F_Y(y) = \int_0^y f_Y(y)dy = y/(1+y) \quad , \qquad y\ge 0$$

c) What is the probability that the lifetime of at least one component exceeds 1 year (when the manufacturer's warranty expires)?

$$P(X>1\cup Y>1) = 1-P(X\le 1\cap Y\le 1) = 1-\int_0^1\left(\int_0^1 x\,e^{-x(1+y)}\,dy\right)dx$$

$$= 1-\int_0^1 x\,e^{-x}\left(\int_0^1 e^{-xy}\,dy\right)dx = 1-\int_0^1 x\,e^{-x}\left(\frac{1}{x}-\frac{1}{x}e^{-x}\right)dx$$

$$= 1-\int_0^1\left(e^{-x}-e^{-2x}\right)dx = 1-\left(-e^{-x}+\frac{1}{2}e^{-2x}\right)\Big|_0^1$$

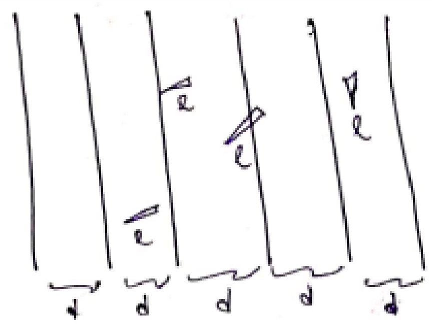
$$= 1-\left(-e^{-1}+\frac{1}{2}e^{-2}\right)+\left(-1+\frac{1}{2}\right) = \frac{1}{2}+e^{-1}-\frac{1}{2}e^{-2} \approx 0.800212.$$

OR

$$P(X>1\cup Y>1) = P(X>1) + P(Y>1) - P(X>1\cap Y>1) = \ldots$$

Solution 12

A12.



→ let x be the distance of center of needle to closest line and let θ be the acute angle b/w needle and the parallel lines.

→ Then; X, Θ will be uniform RV

→ Here,

$$f_X(x) = \begin{cases} \frac{2}{d} & ; \quad 0 \le x \le d/2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$f_\Theta(\theta) = \begin{cases} \frac{2}{\pi} & ; \quad 0 \le \theta \le \pi/2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

→ Since, X and Θ are independent, their joint PDF is;

$$\rightarrow \quad f_{X\Theta}(x,\theta) = \begin{cases} \frac{4}{d\pi} & ; \quad 0 \le x \le \frac{d}{2}, \quad 0 \le \theta \le \frac{\pi}{2} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

→ The needle will intersect a line, if $\underline{x \le \frac{l}{2}\sin\theta}$

→ If E is event corres. to needle intersecting a line, then;

$$\rightarrow \quad P(E) = \int_0^{\frac{\pi}{2}} \int_0^{\frac{l}{2}\sin\theta} \frac{4}{d\pi}\, dx\, d\theta = \int_0^{\pi/2} \frac{2l}{d\pi} \sin\theta\, d\theta = \frac{2l}{d\pi}\left[-\cos\theta\right]_0^{\pi/2}$$

$$\rightarrow \quad \boxed{P(E) = \frac{2l}{d\pi}}$$

Solution 13

Ans. let X denote position of ambulance and Y denote position of accident. (X and Y are independent)

→ Then

→ $f_X(x) = \begin{cases} 1/L & ; x \in [0, L] \\ 0 & ; \text{otherwise} \end{cases}$

→ Similarly, $f_Y(y) = \begin{cases} 1/L & ; y \in [0, L] \\ 0 & ; \text{otherwise} \end{cases}$

→ ∵, X and Y are independent random variables

$\Rightarrow f_{XY}(x, y) = f_X(x) f_Y(y) = \begin{cases} 1/L^2 & ; x \in [0, L], y \in [0, L] \\ 0 & ; \text{otherwise} \end{cases}$

→ let Z be the RV denoting distance between ambulance & accident. then $Z = |X - Y|$

$\Rightarrow F_Z(z) = P(Z \leq z) = P(|X - Y| \leq z) = P(-z \leq X - Y \leq z)$

$\Rightarrow F_Z(z) = P(Y - z \leq X \leq Y + z)$

$\rightarrow F_Z(z) = \int_0^L \int_{\max(y-z,0)}^{\min(y+z,L)} f_{XY}(x, y)\, dx\, dy$

$\Rightarrow F_Z(z) = \int_0^L \int_{\max(y-z,0)}^{\min(y+z,L)} \frac{1}{L^2}\, dx\, dy = \frac{1}{L^2} \int_0^L [\min(y+z, L) - \max(y-z, 0)]\, dy$

$\rightarrow F_Z(z) = \frac{1}{L^2} \left[\int_0^{L-z} \min(y+z, L)\, dy + \int_{L-z}^{L} \min(y+z, L)\, dy - \int_0^z \max(y-z, 0)\, dy - \int_z^L \max(y-z, 0)\, dy \right]$

$\rightarrow F_Z(z) = \frac{1}{L^2} \left[\int_0^{L-z} (y+z)\, dy + \int_{L-z}^{L} dy \cdot L - \int_0^z 0\, dy - \int_z^L (y-z)\, dy \right]$

$\rightarrow F_Z(z) = \frac{1}{L^2} \left[\left(\frac{y^2}{2} + yz\right)_0^{L-z} + L[y]_{L-z}^{L} - 0 - \left(\frac{y^2}{2} - yz\right)_z^L \right]$

$\rightarrow F_Z(z) = \frac{1}{L^2} \left[\left(\frac{L^2}{2} + \frac{z^2}{2} - Lz + Lz - z^2\right) + Lz - \left(\frac{L^2}{2} - Lz - \frac{z^2}{2} + z^2\right) \right]$

$$\Rightarrow F_Z(z) = \frac{1}{L^2}\left[\frac{L^2}{2} - \frac{z^2}{2} + Lz - \frac{L^2}{2} - \frac{z^2}{2} + Lz\right]$$

$$\Rightarrow \boxed{F_Z(z) = \frac{2Lz - z^2}{L^2}} \quad [\forall z \in (0, L)]$$

$$\Rightarrow F_Z(z) = \begin{cases} \frac{2Lz - z^2}{L^2} & ; 0 < z < L \\ 0 & ; z \leq 0 \\ 1 & ; z \geq L \end{cases}$$

→ Then, its PDF $f_Z(z)$ will be:

$$f_Z(z) = \begin{cases} \frac{2(L-z)}{L^2} & ; 0 < z < L \\ 0. & ; \text{otherwise} \end{cases}$$

→ Now, we need to find $T = \frac{Z}{v}$ (where v is scalar)

$$\Rightarrow F_T(t) = \begin{cases} 0 & ; t \leq 0 \\ \frac{2Ltv - t^2v^2}{L^2} & ; 0 < t < L/v \\ 1 & ; t \geq L/v \end{cases}$$

$$f_T(t) = \begin{cases} \frac{2(L - tv)v}{L^2} & ; 0 < t < L/v \\ 0 & ; \text{otherwise} \end{cases}$$

Solution 14

a. We have

$$\begin{aligned}1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy \\ &= \iint\limits_{D} c \;\; dxdy \\ &= c(\text{area of } D) \\ &= c(\pi).\end{aligned}$$

Thus, $c = \frac{1}{\pi}$.

b. For $-1 \leq x \leq 1$, we have

$$\begin{aligned}f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \;\; dy \\ &= \frac{2}{\pi}\sqrt{1-x^2}.\end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \quad -1 \leq x \leq 1 \\ & \\ 0 & \quad \text{otherwise} \end{cases}$$

Similarly,

$$f_Y(y) = \begin{cases} \frac{2}{\pi}\sqrt{1-y^2} & \quad -1 \leq y \leq 1 \\ & \\ 0 & \quad \text{otherwise} \end{cases}$$

c. We have

$$\begin{aligned}f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \begin{cases} \frac{1}{2\sqrt{1-y^2}} & \quad -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \\ & \\ 0 & \quad \text{otherwise} \end{cases}\end{aligned}$$

Note that the above equation indicates that, given $Y = y$, X is uniformly distributed on $[-\sqrt{1-y^2}, \sqrt{1-y^2}]$. We write

$$X|Y = y \quad \sim \quad Uniform(-\sqrt{1-y^2}, \sqrt{1-y^2}).$$

d. Are X and Y independent? No, because $f_{XY}(x,y) \neq f_X(x)f_Y(y)$.