PnS Tutorial 7

Q1

Part1

Let $M_X(s)$ be finite for $s \in [-c, c]$, where $c > 0$. Assume $EX = 0$, and $\mathrm{Var}(X) = 1$. Prove

$$\lim_{n\to\infty}\left[M_X\left(\frac{s}{\sqrt{n}}\right)\right]^n = e^{\frac{s^2}{2}}.$$

Note: From this, we can prove the Central Limit Theorem (CLT) which is discussed in Section 7.1.

Solution

Equivalently, we show

$$\lim_{n\to\infty} n \ln\left(M_X(\frac{s}{\sqrt{n}})\right) = \frac{s^2}{2}.$$

We have

$$\begin{aligned}
\lim_{n\to\infty} n \ln\left(M_X(\frac{s}{\sqrt{n}})\right) &= \lim_{n\to\infty} \frac{\ln\left(M_X(\frac{s}{\sqrt{n}})\right)}{\frac{1}{n}} \quad (\text{let} \quad t = \frac{1}{\sqrt{n}}) \\
&= \lim_{t\to 0} \frac{\ln(M_X(ts))}{t^2} \\
&= \lim_{t\to 0} \frac{\frac{sM_X'(ts)}{M_X(ts)}}{2t} \quad (\text{by L'Hôpital's rule}) \\
&= \lim_{t\to 0} \frac{sM_X'(ts)}{2t} \quad (\text{again} \quad \frac{0}{0},) \\
&= \lim_{t\to 0} \frac{s^2 M_X''(ts)}{2} \quad (\text{by L'Hôpital's rule}) \\
&= \frac{s^2}{2} \quad (\text{since} \quad M_X''(0) = EX^2 = 1).
\end{aligned}$$

Part2

The goal in this problem is to prove the central limit theorem using MGFs. In particular, let X_1, X_2, ... , X_n be i.i.d. random variables with expected value $EX_i = \mu < \infty$, $\mathrm{Var}(X_i) = \sigma^2 < \infty$, and MGF $M_X(s)$ that is finite on some interval $[-c, c]$, where $c > 0$ is a constant. As usual, let

$$Z_n = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sqrt{n}\sigma}.$$

Prove

$$\lim_{n\to\infty} M_{Z_n}(s) = e^{\frac{s^2}{2}}, \qquad \text{for all } s \in [-c, c].$$

Since this is the MGF of a standard normal random variable, we conclude that the distribution of Z_n converges to the standard normal random variable.

Hint: Use the result of Problem 9 in Section 6.1.6: for a random variable Y with a well-defined MGF, $M_Y(s)$, and $EY = 0$, $\mathrm{Var}(Y) = 1$, we have

$$\lim_{n\to\infty}\left[M_Y\left(\frac{s}{\sqrt{n}}\right)\right]^n = e^{\frac{s^2}{2}}.$$

Solution

Let Y_i's be the normalized versions of the X_i's, i.e.,

$$Y_i = \frac{X_i - \mu}{\sigma}.$$

Then, Y_i's are i.i.d. and

$$\begin{aligned}
EY_i &= 0, \\
\mathrm{Var}(Y_i) &= 1.
\end{aligned}$$

We also have

$$\begin{aligned} Z_n &= \frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \\ &= \frac{Y_1+Y_2+\cdots+Y_n}{\sqrt{n}}. \end{aligned}$$

Thus, we have

$$\begin{aligned} M_{Z_n}(s) &= E[e^{s\frac{Y_1+Y_2+\cdots+Y_n}{\sqrt{n}}}] \\ &= E[e^{\frac{sY_1}{\sqrt{n}}}]\cdot E[e^{\frac{sY_2}{\sqrt{n}}}]\cdots E[e^{\frac{sY_n}{\sqrt{n}}}] \quad \text{(the since } Y_i\text{'s are independent)} \\ &= M_{Y_1}\left(\frac{s}{\sqrt{n}}\right)^n \quad (the Y_i\text{'s are identically distributed}). \end{aligned}$$

Thus, we conclude

$$\begin{aligned} \lim_{n\to\infty} M_{Z_n}(s) &= \lim_{n\to\infty} M_{Y_1}\left(\frac{s}{\sqrt{n}}\right)^n \\ &= e^{\frac{s^2}{2}} \quad \text{(by the hint)}. \end{aligned}$$

Since this is the MGF of a standard normal random variable, we conclude the CDF of Z_n converges to the standard normal CDF.

Q2.

Say you have a new algorithm and you want to test its running time. You have an idea of the variance of the algorithm's run time: $\sigma^2 = 4\text{sec}^2$ but you want to estimate the mean: $\mu = t\text{sec}$. You can run the algorithm repeatedly (IID trials). How many trials do you have to run so that your estimated runtime $= t \pm 0.5$ with 95% certainty? Let X_i be the run time of the i-th run (for $1 \le i \le n$).

$$0.95 = P(-0.5 \le \frac{\sum_{i=1}^n X_i}{n} - t \le 0.5)$$

By the central limit theorem, the standard normal Z must be equal to:

$$\begin{aligned} Z &= \frac{(\sum_{i=1}^n X_i) - n\mu}{\sigma\sqrt{n}} \\ &= \frac{(\sum_{i=1}^n X_i) - nt}{2\sqrt{n}} \end{aligned}$$

Now we rewrite our probability inequality so that the central term is Z:

$$\begin{aligned} 0.95 &= P\left(-0.5 \le \frac{\sum_{i=1}^n X_i}{n} - t \le 0.5\right) = P\left(\frac{-0.5\sqrt{n}}{2} \le \frac{\sum_{i=1}^n X_i}{n} - t \le \frac{0.5\sqrt{n}}{2}\right) \\ &= P\left(\frac{-0.5\sqrt{n}}{2} \le \frac{\sqrt{n}}{2}\frac{\sum_{i=1}^n X_i}{n} - \frac{\sqrt{n}}{2}t \le \frac{0.5\sqrt{n}}{2}\right) = P\left(\frac{-0.5\sqrt{n}}{2} \le \frac{\sum_{i=1}^n X_i}{2\sqrt{n}} - \frac{\sqrt{n}}{\sqrt{n}}\frac{\sqrt{n}t}{2} \le \frac{0.5\sqrt{n}}{2}\right) \\ &= P\left(\frac{-0.5\sqrt{n}}{2} \le \frac{\sum_{i=1}^n X_i - nt}{2\sqrt{n}} \le \frac{0.5\sqrt{n}}{2}\right) \\ &= P\left(\frac{-0.5\sqrt{n}}{2} \le Z \le \frac{0.5\sqrt{n}}{2}\right) \end{aligned}$$

And now we can find the value of n that makes this equation hold.

$$\begin{aligned} 0.95 &= \Phi\left(\frac{\sqrt{n}}{4}\right) - \Phi\left(-\frac{\sqrt{n}}{4}\right) = \Phi\left(\frac{\sqrt{n}}{4}\right) - \left(1 - \Phi\left(\frac{\sqrt{n}}{4}\right)\right) \\ &= 2\Phi\left(\frac{\sqrt{n}}{4}\right) - 1 \\ 0.975 &= \Phi(\frac{\sqrt{n}}{4}) \\ \Phi^{-1}(0.975) &= \frac{\sqrt{n}}{4} \\ 1.96 &= \frac{\sqrt{n}}{4} \\ n &= 61.4 \end{aligned}$$

Thus it takes 62 runs. If you are interested in how this extends to cases where the variance is unknown, look into variations of the students' t-test.

Q3.

We have

$$\begin{aligned} EX_i &= (0.6)(1) + (0.4)(-1) \\ &= \frac{1}{5}, \end{aligned}$$

$$\begin{aligned} EX_i^2 &= 0.6 + 0.4 \\ &= 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Var}(X_i) &= 1 - \frac{1}{25} \\ &= \frac{24}{25}; \\ \text{thus,} \quad \sigma_{X_i} &= \frac{2\sqrt{6}}{5}. \end{aligned}$$

Therefore,

$$\begin{aligned} EY &= 25 \times \frac{1}{5} \\ &= 5, \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= 25 \times \frac{24}{25} \\ &= 24; \\ \text{thus,} \quad \sigma_Y &= 2\sqrt{6}. \end{aligned}$$

$$\begin{aligned} P(4 \le Y \le 6) &= P(3.5 \le Y \le 6.5) \quad \text{(continuity correction)} \\ &= P\left(\frac{3.5-5}{2\sqrt{6}} \le \frac{Y-5}{2\sqrt{6}} \le \frac{6.5-5}{2\sqrt{6}}\right) \\ &= P\left(-0.3062 \le \frac{Y-5}{2\sqrt{6}} \le +0.3062\right) \\ &\approx \Phi(0.3062) - \Phi(-0.3062) \quad \text{(by the CLT)} \\ &= 2\Phi(0.3062) - 1 \\ &\approx 0.2405 \end{aligned}$$

Q4.

You will roll a 6 sided dice 10 times. Let X be the total value of all 10 dice = $X_1 + X_2 + \cdots + X_{10}$. You win the game if $X \le 25$ or $X \ge 45$. Use the central limit theorem to calculate the probability that you win.

Recall that $E[X_i] = 3.5$ and $\mathrm{Var}(X_i) = \frac{35}{12}$.

$$\begin{aligned} P(X \le 25 \text{ or } X \ge 45) &= 1 - P(25.5 \le X \le 44.5) \\ &= 1 - P\left(\frac{25.5 - 10(3.5)}{\sqrt{35/12}\sqrt{10}} \le \frac{X - 10(3.5)}{\sqrt{35/12}\sqrt{10}} \le \frac{44.5 - 10(3.5)}{\sqrt{35/12}\sqrt{10}}\right) \\ &\approx 1 - (2\Phi(1.76) - 1) \approx 2(1 - 0.9608) = 0.0784 \end{aligned}$$

Q5.

WE are given information on a random variable in the question. We can use this information to calcualte the expectation as shown below:

$$\mathbb{E}(X) = [(-0.05) \times 0.3] + [0.0 \times 0.2] + [0.05 \times 0.5] = 0.2 \times 0.05 = 0.01$$

We can also use this information to calcualte the variance.

$$\begin{aligned} \mathbb{E}[(X - \mu)^2] &= (-0.05 - 0.01)^2 \times 0.3 + (0.0 - 0.01)^2 \times 0.2 + (0.05 - 0.01)^2 \times 0.5 \\ &= 0.0019 \end{aligned}$$

And the square root of the variance.

$$\sigma = \sqrt{\mathbb{E}[(X - \mu)^2]} = \sqrt{0.0019} = 0.0436$$

We now use the central limit theorem to determine how the share price changes. We want to look at what happens after 3 hours. Our random variable tells us how the share price changes each minute. What we are thus calculating is the sum of 180 of these random variables. Substituting this into the central limit theorem we find that:

$$x = \frac{S_n/n - \mu}{\sigma/\sqrt{n}} = \frac{1.2/180 - 0.01}{0.0436/\sqrt{180}} \approx \frac{-0.00333}{0.00325} = -1.025$$

and thus:

$$P(s \le 1.2) = \Phi(x) = \Phi(-1.025) = 0.15625$$

We want the probability that $s > 1.2$, however, which is just:

$$P(s > 1.2) = 1 - P(s \le 1.2) = 1 - 0.15625 = 0.844$$

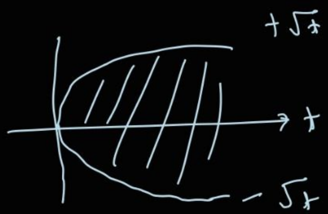
Part 2: Probabilities are already given, all that is needed is to formally write it.
Part 3: Markov - yes

Q6.

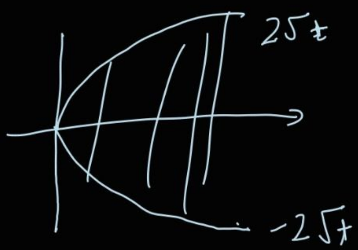
1. $$\frac{Y_1 + Y_2 \cdots Y_t - n\mu}{\sqrt{t}\,\sigma} \approx N(0,1)$$

$\mu = 0$, $\sigma = 1$

$$\Rightarrow \frac{X_t}{\sqrt{t}} \approx N(0,1) \quad \Rightarrow X_t \approx N(0, t)$$



The random walk will remain in this parabola with $\approx 68.27\%$ probability.



and in this parabola with $\approx 95.45\%$ prob.

2. Let $f(k) = \left\{ \begin{array}{l} \text{prob. that you reach } A \text{ before } -B \\ \text{if currently on } k. \end{array} \right\}$

Now $\forall\, k \in [A, -B]$, $k \subset \mathbb{Z}$

$$f(k) = \frac{1}{2} f(k+1) + \frac{1}{2} f(k-1)$$

Using the characteristic eqn. to solve this recursive function, we get only one root $r=1$

Thus $f(n) = a r^n + bn r^n$

$$\Rightarrow f(n) = a + bn$$

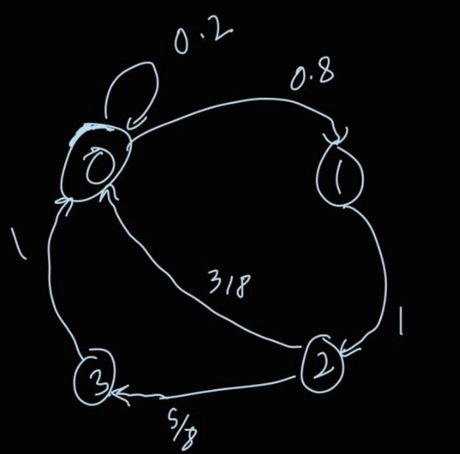
Also $f(A) = 1$, $f(-B) = 0$

using this to calculate a and b, we get

our answer $= f(0) = \dfrac{B}{A+B}$

3. Yes

Q7.



state i represents the length of time since the arrival of last customer.

Q8

1.

Let $U \sim \text{Unif}([0,1])$ and let $X_n = \frac{(-1)^n U}{n}$ for $n \geq 1$. Let the probability space be the standard unit-interval probability space

1. Show that $(X_n : n \in \mathbb{N})$ converges almost surely.
2. Show that the sequence converges in the mean squared sense.

Solution

part a)

Solution. Fix an $\epsilon > 0$. Our claim is that $X_n \xrightarrow[n\to\infty]{a.s.} 0$.

We wish to show that the set

$$A_\epsilon^0 \triangleq \bigcup_{N(\epsilon)\geq 1} \bigcap_{n \geq N(\epsilon)} \{\omega : |X_n(\omega)| \leq \epsilon\}$$

is of probability 1. To see this, note that

$$\begin{aligned} A_\epsilon^0 &= \bigcup_{N(\epsilon)\geq 1} \bigcap_{n \geq N(\epsilon)} \{\omega : U(\omega) \leq \epsilon\} \\ &= \bigcup_{N(\epsilon)\geq 1} \bigcap_{n \geq N(\epsilon)} [0, \min\{n\epsilon, 1\}] \\ &= [0,1] \quad \text{(Fill in the details)} \end{aligned}$$

Hence, $\mathbb{P}(A_\epsilon^0) = 1$, $\forall \epsilon > 0$, showing that $X_n \xrightarrow[n\to\infty]{\text{a.s.}} 0$.

part b)

Solution. Observe that

$$\begin{aligned} \mathbb{E}\left[|X_n^2 - 0|^2\right] &= \mathbb{E}\left[|X_n|^2\right] \\ &= \frac{1}{n^2}\mathbb{E}\left[U^2\right] \\ &= \frac{1}{3n^2}, \text{ and hence,} \\ \lim_{n\to\infty} \mathbb{E}\left[|X_n - 0|^2\right] &= 0. \end{aligned}$$

Thus $X_n \xrightarrow[n\to\infty]{\text{m.s.}} 0$.

Q9.

$(X_n : n \in \mathbb{N})$ is a sequence of independent random variables with marginal pmfs given by

$$\mathbb{P}\left(X_n = \frac{1}{2}\left(1 - \frac{1}{n}\right)\right) = \mathbb{P}\left(X_n = \frac{1}{2}\left(1 + \frac{1}{n}\right)\right) = \frac{1}{2}.$$

1. Show that the sequence converges almost surely.
2. Check if $(X_n : n \in \mathbb{N})$ converges in $\mathcal{L}^2$ (mean-squared convergence)

Solution

Part a)

Solution. First, we note that for a fixed $\epsilon > 0$,

$$\mathbb{P}\left(\left\{\omega : |X_n(\omega) - \frac{1}{2}| \le \epsilon\right\}\right) = \mathbb{P}\left(\frac{1}{2} - \epsilon \le X_n \le \frac{1}{2} + \epsilon\right)$$
$$= \begin{cases} 0, & \text{if } \epsilon < \frac{1}{2n} \text{ (or } n < \frac{1}{2\epsilon}) \\ 1, & \text{o.w.} \end{cases}$$

Hence, it is immediate that

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\omega : |X_n(\omega) - \frac{1}{2}| > \epsilon\right\}\right) = 0.$$
$$\Rightarrow X_n \xrightarrow[n\to\infty]{\text{P.}} \frac{1}{2}.$$

Now, observe that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|X_n - \frac{1}{2}| > \epsilon\right) = \frac{1}{2\epsilon} < \infty$$

Hence, $X_n \xrightarrow[n\to\infty]{\text{a.s.}} \frac{1}{2}$.

Part b)

Solution. We have that

$$\mathbb{E}\left[\left(X_n - \frac{1}{2}\right)^2\right] = \frac{1}{2}\frac{1}{4n^2} + \frac{1}{2}\frac{1}{4n^2}$$
$$= \frac{1}{4n^2} \xrightarrow[n\to\infty]{0}.$$

Hence, $X_n \xrightarrow[n\to\infty]{\text{m.s.}} 0$.

Q10.

A certain calculating machine uses only the digits 0 and 1. It is supposed to transmit one of these digits through several stages. However, at every stage, there is a probability p that the digit that enters this stage will be changed when it leaves and a probability $q = 1 - p$ that it won't. Form a Markov chain to represent the process of transmission by taking as states the digits 0 and 1. What is the matrix of transition probabilities?
Now draw a tree and assign probabilities assuming that the process begins in state 0 and moves through two stages of transmission. What is the probability that the machine, after two stages, produces the digit 0 (i.e., the correct digit)?

Solution. Taking as states the digits 0 and 1 we identify the following Markov chain (by specifying states and transition probabilities):

$$\begin{array}{ccc} & 0 & 1 \\ 0 & q & p \\ 1 & p & q \end{array}$$

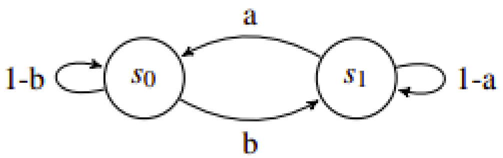
where $p + q = 1$. Thus, the transition matrix is as follows:

$$\mathsf{P} = \begin{pmatrix} q & p \\ p & q \end{pmatrix} = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix} = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}.$$

It is clear that the probability that that the machine will produce 0 if it starts with 0 is $p^2 + q^2$.

Q11

In the below figure, answer the following questions:-



a. For what values of a and b is the above Markov chain irreducible?
   Solution - The markov chain is irreducible if both a and b are non-zero.

b. For what values of a and b is the above Markov chain reducible?
   Solution - It is reducible if at least one is 0.

c. Construct a transition probability matrix using the above Markov chain.
   Solution -

$$\begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

Q12.

For the following transition matrices, specify which are recurrent amd which are transient?

$$P_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad P_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

Assume states are $1, \ldots, 5$. For P_1: $\{1,2,3\}$ recurrent, $\{4\}$ transient, $\{5\}$ transient. For P_2: irreducible, so all states recurrent. For P_3: $\{1,2,3\}$ recurrent, $\{4,5\}$ recurrent. For P_4: $\{1,2\}$ recurrent, $\{3\}$ recurrent (absorbing), $\{4\}$ transient, $\{5\}$ transient.

Q13.

Consider the Markov chain with three states, $S = \{1, 2, 3\}$, that has the following transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

a. Draw the state transition diagram for this chain.
b. If we know $P(X_1 = 1) = P(X_1 = 2) = \frac{1}{4}$, find $P(X_1 = 3, X_2 = 2, X_3 = 1)$.

Solution

a. The state transition diagram is shown in Figure 11.6

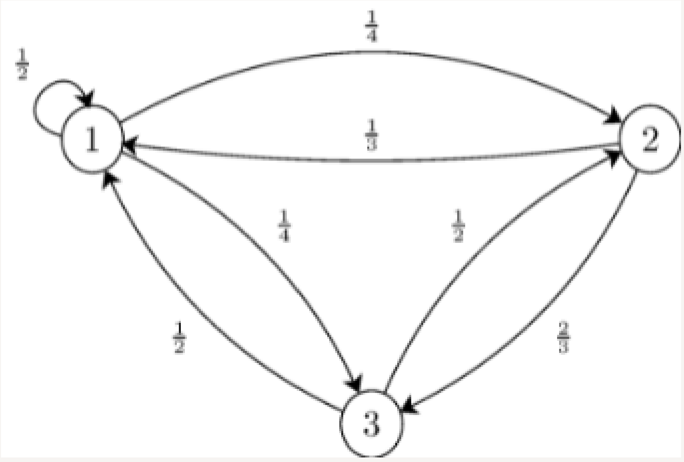


Figure 11.6 - A state transition diagram.

b. First, we obtain

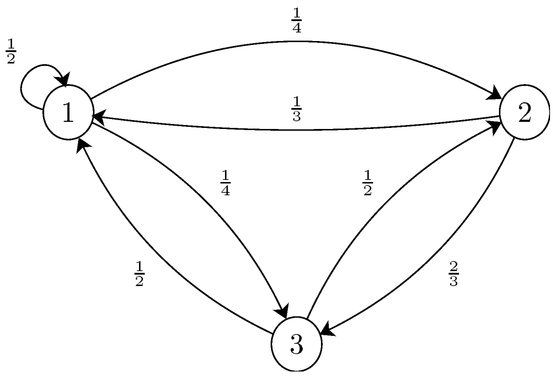
$$\begin{aligned} P(X_1 = 3) &= 1 - P(X_1 = 1) - P(X_1 = 2) \\ &= 1 - \frac{1}{4} - \frac{1}{4} \\ &= \frac{1}{2}. \end{aligned}$$

We can now write

$$\begin{aligned} P(X_1 = 3, X_2 = 2, X_3 = 1) &= P(X_1 = 3) \cdot p_{32} \cdot p_{21} \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{1}{12}. \end{aligned}$$

Q14

1. Consider the Markov chain shown in below image:-



   a. Is the chain irreducible?
   b. Find the stationary distribution for this chain.

Solution

Part a-

The chain is irreducible since we can go from any state to any other states in a finite number of steps.

Part b-

To find the stationary distribution, we need to solve

$$\begin{aligned}
\pi_1 &= \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3, \\
\pi_2 &= \frac{1}{4}\pi_1 + \frac{1}{2}\pi_3, \\
\pi_3 &= \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2, \\
\pi_1 &+ \pi_2 + \pi_3 = 1.
\end{aligned}$$

We find

$$\pi_1 \approx 0.457,\ \pi_2 \approx 0.257,\ \pi_3 \approx 0.286$$