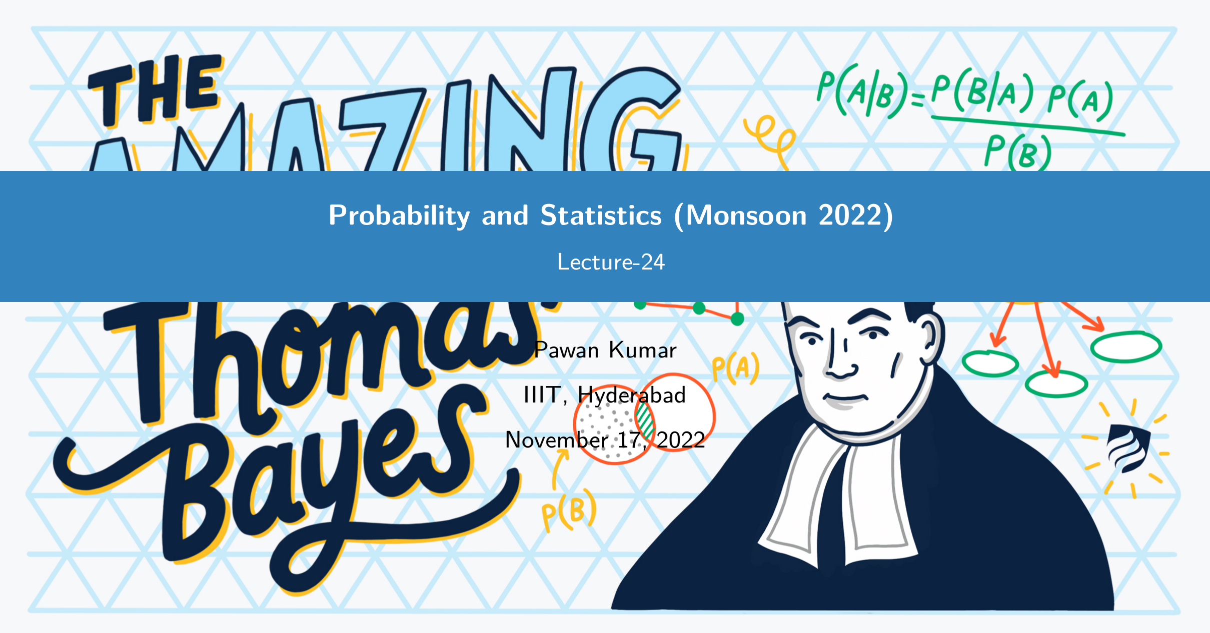

Probability and Statistics (Monsoon 2022)

Lecture-24

Pawan Kumar

IIIT, Hyderabad

November 17, 2022

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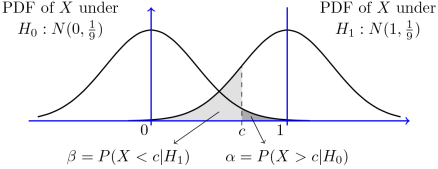

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Hypothesis Testing for Mean

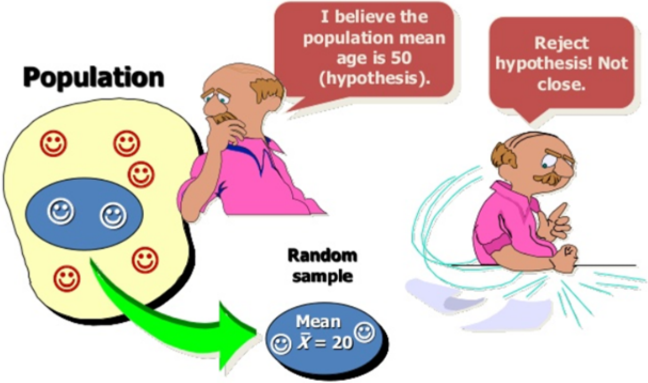

Figure: Hypothesis Testing for Mean.

Definition (Two sided hypothesis test)

- Consider a random sample $X_1, X_2, \ldots, X_n$ from a distribution.
- Our goal is to make inference about the mean of the distribution μ.
- Two sided hypothesis test: Decide between the following hypotheses:
  - $H_0 : \mu = \mu_0$
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Definition (One-Sided Hypothesis)

- The second and the third cases are one-sided tests. More specifically

$$H_0 : \mu \leq \mu_0, \quad H_1 : \mu > \mu_0.$$

- Here, both H_0 and H_1 are one-sided, so we call this test a one-sided test.
- The third case is similar

$$H_0 : \mu \geq \mu_0, \quad H_1 : \mu < \mu_0.$$

Statistics and Test Statistics

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- If we don't know the variance, then use sample variance

$$W(X_1, X_2, \cdots, X_n) = \frac{\overline{X} - \mu_0}{S/\sqrt{n}},$$

  where S is sample standard deviation

$$S = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X})^2} = \sqrt{\frac{1}{n-1}\left(\sum_{k=1}^{n} X_k^2 - n\overline{X}^2\right)}. \tag{1}$$

Plan for two-sided tests for the mean

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- Thus, we can choose c such that $P(|W| > c|H_0) = \alpha$.

Example (level α hypothesis test for mean)

Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution, where μ is unknown but σ is known. Design a level α test to choose between

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https://www.probabilitycourse.com/chapter8/8_4_3_hypothesis_testing_for_mean.php

Hypothesis Testing is Related to Confidence Interval

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- Recall that It is the $(1-\alpha)100\%$ confidence interval for μ_0.
- Relationship between confidence interval problems and hypothesis testing problems.

Example

For the above example, find β, the probability of type II error, as a function of μ.

https://www.probabilitycourse.com/chapter8/8_4_3_hypothesis_testing_for_mean.php

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https://www.probabilitycourse.com/chapter8/8_4_3_hypothesis_testing_for_mean.php#example8_24

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https://www.probabilitycourse.com/chapter8/8_4_3_hypothesis_testing_for_mean.php#example8_24

- The above can be extended to any other distribution. Let

$$W(X_1, X_2, \cdots, X_n) = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}, \tag{2}$$

- If $W \leq c$, accept H_0, otherwise accept H_1. To choose c

$$\begin{aligned} P(\text{type I error}) &= P(\text{Reject } H_0 \mid H_0) \\ &= P(W > c \mid \mu \leq \mu_0) \\ &\leq P(W > c \mid \mu = \mu_0). \end{aligned}$$

- Last inequality is because we assume worst-case scenario.

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Definition ($P-$value)

$P-$value is the lowest significance level α that results in rejecting the null hypothesis.

- If the $P-$value is small, it means that the observed data is very unlikely to have occurred under H_0, so we are more confident in rejecting the null hypothesis.

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1. Can we reject H_0 at significance level $\alpha = 0.05$?
2. Can we reject H_0 at significance level $\alpha = 0.01$?
3. What is the $P-$value?

https://www.probabilitycourse.com/chapter8/8_4_4_p_vals.php

Computing P values

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- Assume H_0 is true.
- The $P-$value is $P(\text{type I error})$ when the test threshold c is chosen to be $c = w_1$.

- For the above example, we can consider

$$W = \frac{X - 50}{5},$$

which is approximately $N(0,1)$ under H_0. The observed value of W is

$$w_1 = \frac{60 - 50}{5} = 2.$$

- Thus,

$$\begin{aligned} P - \text{value} &= P(\text{type I error when } c = 2) \\ &= P(W > 2) = 1 - \Phi(2) = 0.023 \end{aligned}$$

Likelihood Ratio Test for Simple Hypotheses

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a distribution with a parameter θ. Suppose that we have observed $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$. To decide between two simple hypotheses

$$H_0 : \theta = \theta_0$$
$$H_1 : \theta = \theta_1$$

We define

$$\lambda(x_1, x_2, \cdots, x_n) = \frac{L(x_1, x_2, \cdots, x_n; \theta_0)}{L(x_1, x_2, \cdots, x_n; \theta_1)}.$$

To perform a likelihood ratio test (LRT), we choose a constant c. We reject H_0 if $\lambda < c$ and accept it if $\lambda \geq c$. The value of c can be chosen based on the desired α.

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$$H_0 : \theta = \theta_0 = 0,$$
$$H_1 : \theta = \theta_1 = 1.$$

Let $X = x$. Design a level 0.05 test ($\alpha = 0.05$) to decide between H_0 and H_1.

https://www.probabilitycourse.com/chapter8/8_4_5_likelihood_ratio_tests.php